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RM/06/031

JEL code: C71

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December 8, 2005

Abstract

We present a new solution to partition function form games that is novel in at least two ways. Firstly, the solution exploits the consistency of the partition function form, namely that the response to a deviation is established as the same solution applied to the residual game, itself a partition function form game. This consistency allows us to model residual behaviour in a natural, intuitive way. Secondly, we consider a pair of solutions as the extrema of an interval for set inclusion. Taking the whole interval rather than just one of the extremes enables us to include or exclude outcomes with certainty.

JEL subject classification: C71

Keywords: Core; Externalities; Partition function Games; Optimism; Pessimism

1 Introduction

Its intuitive, straightforward definition makes the *core* one of the most popular solution concepts in coalition formation games. Peleg (1992) claims that a solution is "acceptable" only if its axiomatisation is similar to that of the core. The original definition, however, relies on the assumption that the value of a coalition can be given independently of other coalitions. Game theory is more and more often used in models where this assumption is simplistic, where the externalities of coalition formation form a fundamental part of the model. Examples include trade blocks (Yi, 1996), common pool resource games (Funaki and Yamato, 1999), and international environmental agreements (Eyckmans and Tulkens, 2003).

Such effects are captured by the more general *partition function form* (Thrall and Lucas, 1963) we consider here, where the value of a coalition depends on the entire *coalition structure* or *partition.* While this game form specifies the value of a coalition *given* all possible partitions of the remaining players it does not specify and it is not clear which of those partitions would emerge should this coalition form.

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In the case of the core the stability of imputations is tested against possible coalitional deviations. In a partition function form game the value of this deviation depends on the potential reaction of the remaining or *residual* players. Different assumptions about this reaction can make the deviation profitable or not; accurate modelling is essential to understand deviation and hence the core. The present paper develops a model that unifies a number of existing theories.

When von Neumann and Morgenstern (1944) defined the characteristic function form they used the minimax rule. This definition equips deviators with excessive pessimism: they expect that residual players hurt them to the maximum possible extent. While this is perfectly appropriate to assume in the constant-sum case, generally they might hurt their own interests. Rosenthal (1971) argues that any reaction should be *reasonable* and already such a mild restriction has a substantial effect on the core (Richter, 1974). Using pessimism (or optimisim for that matter) to evaluate the residual reaction is still common practice. As a major step Chander and Tulkens (1997) introduced a model that considers *individual best responses* of the residual players. While per definition the reaction of the residual players is "best" it is restricted in the sense that no coalitions are allowed to form and the existing ones must break apart. As externalities are often either uniformly positive or negative reducing this model respectively to the pessimistic or optimistic case. Our model generalises this approach to allow coalitions to form.

A similar enterprise was –independently– undertaken by Huang and Sjöström (2003). Their r-core allows also arbitrary reactions, but as an intermediate step it uses a characteristic function that is, for larger games especially, seldom defined. Our approach is new in the sense that we do not attempt to generate a characteristic function, but to define the core directly to the game.

We must also mention the equilibrium binding agreements of Ray and Vohra (1997), a farsighted concept, where only credible, refining deviations are permitted. Our concept allows for arbitrary deviations and includes some limited farsighted flavour while being a fundamentally myopic concept; it offers the same rational behaviour to residuals as to deviators.

The structure of the paper is as follows. After the fundamentals we present the motivation and the key definitions. Then we show some properties and give comparisons to existing concepts.

2 Preliminaries

Let N be a finite set of *players*. Subsets of N are called *coalitions*. A partition $\mathcal{P}$ of N is a breaking up of N into disjoint coalitions. $\Pi(S)$ is the set of partitions of any set S of the players with $\mathcal{S}$ denoting a typical element. A *characteristic function* $v : 2^N \longrightarrow \mathbb{R}$ assigns a value to each coalition. A *partition function* $V : \Pi \longrightarrow (2^N \to \mathbb{R})$ assigns a characteristic function to each partition. In essence $V : \Pi \times 2^N \longrightarrow \mathbb{R}$ and the payoff of coalition C in partition $\mathcal{P}$ is denoted $V(C,\mathcal{P})$.[1] A *partition function form game* is a pair (N,V). Payoffs are

[1]The assumption $V(C,\mathcal{P}) = 0$ if $C \notin \mathcal{P}$ is usual, though not necessary.

transferable.

In a partition function form game strategies correspond to coalitions and the details of the bargaining to form coalitions is hidden. An *outcome* is then an imputation-partition pair $(x, \mathcal{P})$, with x in $\mathbb{R}^N$ and $\mathcal{P}$ in Π such that the vector $x = (x_1, x_2, \ldots, x_n)$ satisfies *participation rationality* $\forall i \in N : x_i \geq 0$ and *feasibility and efficiency* $\forall C \in \mathcal{P} : x(C) \equiv \sum_C x_i = v(C)$. The first condition expresses that a player should be interested in taking part in the game, the second that a coalition can only share the payoff available to it $(x(C) \leq v(C))$, but it will do this without waste $(x(C) \geq v(C))$. The set of outcomes of game (N, V) is denoted $\Omega(N, V)$.

Outcome $(x, \mathcal{P})$ *dominates* $(y, \mathcal{Q})$ if there exists a coalition C in $\mathcal{P}$ such that $x >_C y$.[2] Then we say that $(y, \mathcal{Q})$ is dominated *by* $(x, \mathcal{P})$ *via* C. Also C dominates $(y, \mathcal{Q})$ if such an $(x, \mathcal{P}) \in \Omega(N, V)$ exists. Outcome $(y, \mathcal{Q})$ is dominated if some coalition C dominates it. The *coalition structure core* or simply *core* collects undominated outcomes.

We also say that coalition C *deviates* if its members reject the current outcome and form coalition C. A deviation is profitable if the abandoned partition is dominated via the coalition. We adopt the terminology of calling this set the *deviating coalition*, the players in it the *deviators*, and the remaining set of players the *residuals*, who then form the residual set.

2.1 Existing models

In the following we return in more detail to the approaches already mentioned in the introduction.

2.1.1 Optimism/pessimism

Pessimism corresponds to the maximin-minimax principle of von Neumann and Morgenstern (1944), although their equivalence is not true in general. Following Aumann and Peleg (1960) we denote the characteristic functions and cores derived from these approaches as α- and β- characteristic function and core. Here it is the deviating coalition who makes the first move and therefore it is natural to focus on the α-approach. Further variants are discussed by Cornet (1998). Shenoy (1979) assumes that residual players maximise the payoff of the deviating players. We refer to this as the ω-approach to stress being at the other extreme. Consider an outcome $(x, \mathcal{P})$. Coalition C α-dominates $(x, \mathcal{P})$ if $x(C) \leq \min_{\mathcal{R} \in \Pi(N \setminus C)} \{V(C, \{C\} \cup \mathcal{R})\}$. It ω-dominates $(x, \mathcal{P})$ if $x(C) \leq \max_{\mathcal{R} \in \Pi(N \setminus C)} \{V(C, \{C\} \cup \mathcal{R})\}$.

The α-core (ω-core) collects α-(ω-)undominated outcomes. It is denoted by $C_\alpha(N, V)$ and $C_\omega(N, V)$, respectively.

The following proposition requires no proof:

Proposition 1. *The α-core contains the ω-core: $C_\omega(N, V) \subseteq C_\alpha(N, V)$.*

[2] We write $x > y$ if $x_i \geq y_i$ for all $i \in N$ and there exists $i \in N$ such that $x_i > y_i$. We write $x >_C y$ if $x_i \geq y_i$ for all $i \in C$ and there exists $i \in C$ such that $x_i > y_i$.

Our solution concept refines this approach and uses the extra "information", namely that the residual players are also rational, and that based on this knowledge some behaviours may be expected from them others not. This enables us to predict the core much better.

2.1.2 Structural approaches

Hart and Kurz (1983) proposed two approaches where the residual partition is explicitly determined. In the γ model, coalitions which are left by some member, dissolve. In the δ model they stay together and form smaller coalitions. These are used with slight modifications as follows.

Chander and Tulkens (1997) study the (naive) pessimistic and optimistic approaches and find that it is not reasonable to assume that the residual players act to hurt or help deviating players and not to maximise their own payoffs. Therefore they introduce the γ-approach that is *individually reasonable* in the sense that residual players choose their strategies according to a Nash behaviour, essentially breaking up to singletons. Let $\mathcal{S}(C)$ denote the partition consisting of C and $N \setminus C$ partitioned into singletons.

Consider an outcome $(x, \mathcal{P})$. Coalition C γ-dominates $(x, \mathcal{P})$ if $x(C) \leq V(C, \mathcal{S}(C))$. The γ-*core*, denoted $C_\gamma(N, V)$, collects γ-undominated outcomes.

In a partition function form game where strategies correspond to coalitions the γ-approach leaves the residual players defenseless.

The δ- or *status quo* approach first appeared in studies of cartel games (d'Aspremont et al., 1983). Here deviators assume that residuals do not react, completely overlooking the externalities arising from a deviation. For partition $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ and coalition C let $\mathcal{P}^{-C}$ denote the partition $\{P_1 \setminus C, P_2 \setminus C, \ldots, P_k \setminus C\}$.

Consider an outcome $(x, \mathcal{P})$. Coalition C δ-dominates $(x, \mathcal{P})$ if $x(C) \leq V(C, \{C\} \cup \mathcal{P}^{-C})$. The δ-*core*, denoted by $C_\delta(N, V)$ collects δ-undominated outcomes.

The core $C_\delta(N, V)$ is very similar to the set of Strong Nash equilibria (Aumann, 1959), but in the Strong Nash setting nonprofitable deviations are also possible and so the set of Strong Nash equilibria may not contain payoff-equivalent outcomes.

The δ- is the opposite of the γ-approach in the sense that there the smallest number of 'links' break up between players. Our solution generalises these two approaches: As special cases residual players do not react or break up to singletons, but they can form any other partition as well if they find that more attractive.

2.1.3 A recursive theory

Huang and Sjöström (2003) introduce the r-theory, a recursive approach that translates a normal form game into the characteristic function form and solves it using the usual techniques for characteristic function games. The normal form game in question has two stages: first a partition is formed cooperatively and we look for the Nash equilibria of the noncooperative play among these coalitions. Our interest lies in the first, cooperative stage. The main difference between this and the previous approaches is that it uses the "same concept" to evaluate the residual game as is used in the original game. While this start is rather attractive and intuitive the actual application of this "same concept" is rather indirect and while when one uses familiar tools one expects an easy journey the definition is actually rather technical so that we prefer not to reproduce it here.

Apart from the complexity of the presentation, the r-theory has a number of limitations. Firstly, the characteristic function used to define the core is seldom defined. Its existence requires all residual cores to be non-empty - a condition which becomes very demanding as the number of players increases. The authors also acknowledge a problem inherent to the normal form they work with: superadditivity. A translation of the r-theory to partition function form games can solve this problem. As soon as we allow for subadditivity in a game we can give characteristic function games with a nonempty core, where the appropriately defined r-core is undefined. Hence the r-core is *not* an extension of the core of a characteristic function form game.

Example 2. Consider a 4-player game with a characteristic function that gives a payoff of 2 for each singleton, 6 for each pair and 0 for every other coalition. The corresponding partition function form game is rather easy to define as the partition function assigns the same characteristic function to each partition. The vector of coalitional payoffs for partition $\{1, 2, 34\}$ is then $(2, 2, 6)$. Huang and Sjöström (2003) define a characteristic function by checking how much a coalition would obtain by deviating. Suppose player 1 deviates (as a singleton). It is easy to see that the remaining players will not settle to any outcome as a triple will not form and a singleton can always convince one of the other players to form a pair. It is clear without a formal argument that the core is empty, making the characteristic function undefined despite the fact that player 1 will obtain a payoff of 1 irrespective of the partition players 2,3 and 4 form.

Finally, authors take the pessimistic approach: while this is not pessimism in the naive α-sense it may produce a core too large.

3 Our model

3.1 Motivation for a new concept

The listed core concepts for partition-function games (or in general for games with externalities) exhibit certain shortcomings. A new concept should ideally answer all these criticisms. In the following we elaborate these criticisms, some of which have already been mentioned in the introduction

3.1.1 Rational residual reaction

Unless explicitly mentioned otherwise game theory deals with rational, that is, payoff-maximising players. Surprisingly the widely used approach of pessimism (or *conservativism*) as well as optimism assumes that the residual players look at the deviators' payoff when choosing their course of action rather than maximising their own as rational players ought to do. "Why should we expect that residual players act in such a bloodthirsty fashion as to hurt deviators to the maximum extent?" Ray and Vohra (1997) Why should residual players ever choose (or be assumed to choose) an 'inferior' strategy? (What is inferior is to be made precise soon.) In sum we can state the following property:

Property 1. Residuals do not (are not assumed to) choose an inferior strategy.

While both the optimistic and the pessimistic approaches fail the γ- and r-core satisfy this property. In the case of the δ-core we cannot really talk about a reaction.

3.1.2 Consistent choice of equilibria

If residual players discard some partitions by the very same act they embrace others. Their selection should be consistent with their selection in the original game.

Property 2. Residuals (are assumed to) exhibit "consistence" in their choice of partitions. In particular, if their reaction can be modelled by a game similar to the initial game, they select outcomes that belong to the core of this residual game.

There is only a point in discussing approaches that have passed the previous test, and of these only the r-core exhibits this property.

3.1.3 Well-defined

The solution of a game should be unambigous and should always exist.

Property 3. The solution is well-defined.

The r-core is only defined for games with non-empty residual cores. (In Example 2 we have seen a game with an empty residual core, where the r-core was undefined.) For other models existence is guaranteed, but of course existence does not mean non-emptiness, which we do not require here.

3.1.4 Efficient

The solution should not contain outcomes such that another outcome is preferred by all players. In other words: a Pareto improvement should always be possible to make.

Property 4. The solution contains only efficient outcomes.

If we want to meet this condition we must allow deviations by multiple coalitions: in essence deviations that end up with the deviating coalition splitting up. Consequently, pure characteristic function approaches fail this property.

In the following we present the new core concept and show that it satisfies these properties.

3.1.5 Generalisation of the core

If we want to call our concept "core" then, when applied to a game with a well-defined characteristic function form, it should coincide with the (coalition structure) core.

Property 5. When applied to a game that has a well-defined characteristic function form, the solution should return the solution as defined by the original concept for characteristic function form games.

All listed solutions satisfy this property except the r-core, which may be undefined for such games, as in Example 2.

3.1.6 Robust to behavioural assumptions

In general these conditions do not uniquely determine the residual behaviour. The remaining alternatives still leave room for making an optimistic or pessimistic choice as well as something in between. For instance, Huang and Sjöström (2003), as many others, choose pessimism. This choice will typically affect the solution. Pessimism results in a larger, optimism in a smaller solution (Theorem 11). Mathematically the chance that the deviators behaviour can be described by perfect optimism or pessimism is very small. The "empirical solution" would therefore lie somewhere in between. By choosing optimism we would not get some of these solutions, but with the pessimistic approach we would get too many. A situation similar to hypothesis testing in statistics and to making type I or type II errors. It is, of course the best to take the two approaches together to see which outcomes are solutions for sure and which are not for sure. [3]

3.2 Recursive cores

Before we proceed to the formal definitions we provide the intuition for the concept.

When players deviate they break away from the rest and start to live their own life. First of all they form a partition. The particular partition formed influences the payoff of residual players and this triggers a reaction: the residual players find themselves in a game that is embedded in the original game, but with part of the partition (the deviating players' partition) already fixed. One of the attractive features of the partition function form is its consistency: this embedded game is also a partition function game and hence we can apply the same solution to this game as to the original one. Notably, this game is smaller, which fact hints the possibility of an inductive definition.

[3] Our choice of using the word pessimistic instead of conservativie is not per chance. A player is conservative if it is not willing to take chances, if it considers the worst cases. Often it seems more appropriate to "be conservative" in this sense and use conservative players. Yet, if we are looking for the stable outcomes in an international environmental agreement, the conservative approach is to use optimistic players to get a small, hopefully nonempty core to see which are the agreements (if any) that are "very" stable. Using pessimistic players can give completely the wrong answer, although nonemptiness is more likely.

What is a “solution?” Irrespective of why do we pick a particular solution, if we did, we assume that in “equilibrium play” the outcomes collected in that solution will emerge. The core has the unattractive property that it can be empty. When this is the case, the core does not define such a preferred set, instead, *based on our solution*, any outcome might emerge equally likely. In particular, if a residual core is empty, the corresponding deviating coalition will still have an idea about its best and worst case scenarios, except that these will be rather extreme and might offer little support in its decision.

Before moving on to the main definitions we formalise the treatment of residual players: After a deviation the residual players play a game embedded in the initial one. The set of players is given; the residual partition function has to be consistent with the partition function of the initial game, and, at the same time, take account of the deviation. Then the residual game is defined as follows:

Definition 3 (Residual Game)**.** Let (N, V) be a game. Let S be the set of deviators and $\mathcal{S}$ their partition. Then $R \equiv N \setminus S$ denotes the set of residual players. Given the deviation $\mathcal{S}$ the residual game $(R, V_{\mathcal{S}})$ is the partition function form game over the player set R and with the partition function

$$\begin{aligned} V_{\mathcal{S}} &: \Pi(R) \times 2^R \longrightarrow \mathbb{R} \\ V_{\mathcal{S}} &: \mathcal{R}, C \longmapsto V(C, \mathcal{R} \cup \mathcal{S}) \end{aligned}$$

Once the residual game is defined it can be solved without reference to the way it was created or to the fact that it is residual. As a very trivial, but crucial property this game has fewer players than the initial one.

Now we can move on to our main definition.

Once we know this preferred set of the residual players (the core or if the core is empty, the set of all outcomes), we can give the post-deviation payoffs for the deviating coalitions. This payoff is nothing but the payoff of the deviating coalitions in the partition completed with the preferred residual partitions. Then a deviation is only profitable if it is profitable for *all* deviating coalitions.

In general the preferred set contains outcomes with different partitions, thus giving different values for the coalitions, some of which may be attractive, some may be very bad for the deviating coalitions. It is up to the deviating players to decide which ones to take into account. We consider both an optimistic and a pessimistic scenario.

The definition of the core is inductive and is done if four steps. For a trivial single-player game we can give the core explicitly. Given the definition for all at most $k-1$ player games we can give our definition of dominance for k player games. Once dominance is defined, we may define the core.

Definition 4. *Pessimistic recursive core.* The definition consists of four steps.

1. *Trivial game.* Let (N, V) be a game. The *core* of a game with $N = \{1\}$ is the only outcome with the trivial partition: $C_{-}(\{1\}, V) = \left\{ \left(V\left(1, (1)\right), (1) \right) \right\}$.

2. *Inductive assumption.* Given the definition of the core $C_-(R,V)$ for every game with at most $k-1$ players we can define dominance for a game of k players. Let $A_-(R,V)$ denote the *pessimistic assumption about the game* (R,V). If $C_-(R,V) \neq \varnothing$ then $A_-(R,V) = C_-(R,V)$, otherwise $A_-(R,V) = \Omega(R,V)$.

3. *Dominance.* The outcome $(x,\mathcal{P})$ is *dominated via the coalition S forming partition $\mathcal{S}$* if **for all** assumptions $(y_{N\setminus S},\mathcal{R}) \in A_-(N\setminus S, V_{\mathcal{S}})$ there exists an outcome $((y_S, y_{N\setminus S}), \mathcal{S}\cup\mathcal{R}) \in \Omega(N,V)$ such that $y >_S x$.
The outcome $(x,\mathcal{P})$ is *dominated* if it is dominated via a coalition.

4. *Core.* The *core* of a game of k players is the set of undominated outcomes and we denote it by $C_-(N,V)$.

Optimistic recursive core. The definition consists of four steps.

1. *Trivial game.* Let (N,V) be a game. The *core* of a game with $N=\{1\}$ is the only outcome with the trivial partition: $C_+(\{1\},V) = \left\{\left(V\left(1,(1)\right),(1)\right)\right\}$.

2. *Inductive assumption.* Given the definition of the core $C_+(R,V)$ for every game with at most $k+1$ players we can define dominance for a game of k players. Let $A_+(R,V)$ denote the *optimistic assumption about the game* (R,V). If $C_+(R,V) \neq \varnothing$ then $A_+(R,V) = C_+(R,V)$, otherwise $A_+(R,V) = \Omega(R,V)$.

3. *Dominance.* The outcome $(x,\mathcal{P})$ is *dominated via the coalition S forming partition $\mathcal{S}$* if **there exists** an assumption $(y_{N\setminus S},\mathcal{R}) \in A_+(N\setminus S, V_{\mathcal{S}})$ and an outcome $((y_S, y_{N\setminus S}), \mathcal{S}\cup\mathcal{R})$ such that $y >_S x$.
The outcome $(x,\mathcal{P})$ is *dominated* if it is dominated via a coalition.

4. *Core.* The *core* of a game of k players is the set of undominated outcomes and we denote it by $C_+(N,V)$.

The difference between the optimistic and pessimistic cores is solely in the definition of domination: while an optimistic player deviates if the deviation *may* represent an improvement, a pessimistic player deviates only if an improvement is inevitable.

In the definition of deviation we discuss two cases. While our contribution is to look at the residual core outcomes a residual core may be empty. We can see this by considering a game with an empty core, and from this game we can build up a larger game containing this as a residual game. Considering this option makes our concept generally applicable. We return to this issue in Section 2.1.3.

Having nonempty residual cores may still lead to very different optimistic and pessimistic cores. Indeed, unlike in a characteristic function form game, here the core may contain, in their effect, substantially different outcomes. Already having a 0 residual game (a game where all payoffs are 0) the residual core contains all residual outcomes. While the residual players are indifferent among these the deviating players may experience very different payoffs depending on the partition of the chosen residual core outcome. Moreover we have the following proposition.

Proposition 5. *In partition function games we may have core outcomes with different partitions that are not only not payoff equivalent, but even where the total payoffs are different.*

Proof. Consider the following 5-player game:

$$\begin{aligned}
V(123,4,5) &= (10,1,1)\\
V(123,45) &= (0,5)\\
V(12,3,4,5) &= (0,1,0,0)\\
V(12,3,45) &= (0,1,0)\\
V(1,2,3,45) &= (0,0,1,0)\\
V(12,345) &= (5,0)\\
V(1,2,345) &= (1,1,8)
\end{aligned}$$

In order to simplify our notation we use some abbreviations. Instead of writing out the coalitional value for each coalition in each partition, we write the coalitional values for a partition as a vector. Payoffs not indicated here are all zero. When we look at possible deviations it is clear that only deviations by collections of coalitions that form a subset of one of the above partitions may lead to non-zero payoffs. After an inspection of the game we find that deviations by (123), (12), (45) and (345) all result in a zero-payoff and that deviations by (12,3) and (3,45) obtain (0,1) and (1,0) respectively. Moreover the outcomes with (12,3,4,5), (12,3,45) and (1,2,3,45) are not undominated. Thus the core outcomes have either (123,4,5) or (1,2,345) as partitions. The total payoff is 12 and 10 respectively. This proves our proposition. Note also that this game is not a unique example, we have relied on the actual numbers very little, and for larger games it becomes even easier to devise such examples. □

What remains is to see to what extent are the two approaches different.

3.3 Properties

First we verify that the optimistic/pessimistic recursive cores satisfy the properties listed in Section 3.1.

Proposition 6. *Residuals do not choose inferior strategies.*

Firstly we must clarify the term "inferior." While it is clear that the term captures the fact that something gives a lower payoff than some alternatives we must stress that these alternatives must be attainable. In particular a player or coalition cannot have only inferior strategies. In this context it is natural to link *inferior* to *dominated.* Then the recursive cores meet this requirement in the sense that, if possible, dominated residual outcomes are discarded.

Proposition 7. *The solution is consistent with itself.*

Once again there is no need for a formal proof: By construction the solution of the residual game is established just as the solution of the original game.

Proposition 8. *The solution is well defined.*

The property holds for trivial games and then an inductive argument shows that both in the optimistic and the pessimistic cases the profitability of deviations is well defined, hence undominated outcomes and the core are well defined. Of course the solution can be empty, just as it is required by Property 5.

Proposition 9. *The solution is efficient.*

Proof. Suppose not: there exists an outcome $(x, \mathcal{P})$ belonging to a recursive core, such that there is another outcome $(y, \mathcal{Q})$ with $y > x$. Then consider a deviation by coalition N forming partition $\mathcal{Q}$. The deviation is profitable making $(x, \mathcal{P}$ a dominated outcome. Hence it cannot belong to the core. Note that here there was no need to make a reference to optimism or pessimism. □

Proposition 10. *The optimistic and pessimistic cores are generalisations of the (coalition structure) core.*

Proof. Property 5 refers to games with well-defined characteristic functions. The characteristic function is well-defined if a coalition's payoff is independent of the strategies, in this case: of the partition of the remaining players. If this is the case, the assumed payoff of a deviating coalition is the same in the optimistic and the pessimistic core and also coincides with the payoffs in the α-, ω-, γ- and δ- approaches as well as the payoff the coalition would get in the characteristic function form game. Hence the corresponding cores also coincide. □

Finally we turn to the issue of behavioural assumptions. For the reasons explained in Section 3.1.6 we prefer to see the two cores as the two extremes of the same solution. In order to be able to refer to the recursive core as an interval for inclusion, we need to prove the following theorem:

Theorem 11. *Given a game (N, V) the pessimistic core contains the optimistic core, $C_+(N, V) \subseteq C_-(N, V)$.*

Proof. There are two key ideas in the proof. Firstly, if deviating players are less hesitant (optimistic) they deviate even if they would not otherwise (being pessimistic) making the core smaller. Secondly, expanding the set of options makes hesitant players even more hesitant.

The proof is by induction on the player set N.

For a single-player game $C_+(\{1\}, V) = C_-(\{1\}, V)$ and so the result holds.

Assuming that $C_+(N_{k-1}, V) \subseteq C_-(N_{k-1}, V)$ for all games where $|N_{k-1}| \leq k-1$, we consider a deviation $\mathcal{S}$ from an outcome $(x, \mathcal{P})$ in a game of k players. As the deviation includes at least one player, the residual game consists of at most $k-1$ players. Since the residual game is a game in partition function form with a player set $R = N \setminus \bigcup_{S \in \mathcal{S}} S$, and a partition function $V_{\mathcal{S}}$, and $|R| \leq k-1$, we have $C_+(R, V_{\mathcal{S}}) \subseteq C_-(R, V_{\mathcal{S}})$.

We discuss four cases, depending on the emptiness of the residual cores. In each of these cases we consider residual outcomes that make the deviation profitable or not, and show that profitability in the pessimistic case implies profitability in the optimistic case.

1. Both residual cores are non-empty
Deviating players form their expectations with respect to the residual cores $C_+(R, V_{\mathcal{S}})$ and $C_-(R, V_{\mathcal{S}})$. If under pessimistic assumptions the deviation is profitable, it is profitable for all outcomes in the set $C_-(R, V_{\mathcal{S}})$. By our assumption $C_+(R, V_{\mathcal{S}}) \subseteq C_-(R, V_{\mathcal{S}})$ and is non-empty, all outcomes in $C_+(R, V_{\mathcal{S}})$ make the deviation profitable, so it is profitable in the best case as well. On the other hand profitability in the best case has no implications on the worst case.

2. Both residual cores are empty
Deviating players form their expectations with respect to the entire residual outcome set $\Omega(R, V_{\mathcal{S}})$. If in the worst case the deviation is profitable, it is profitable for all residual outcomes, and therefore in the best case as well. Again, profitability in the best case has no implication on the worst case.

3. The optimistic residual core is empty, the pessimistic is non-empty
Deviating players form their expectations with respect to the entire residual outcome set $\Omega(R, V_{\mathcal{S}})$ in the optimistic approach and $C_-(R, V_{\mathcal{S}})$ in the pessimistic approach. If in the worst case the deviation is profitable, it is profitable in the entire set $C_-(R, V_{\mathcal{S}})$. Since $C_-(R, V_{\mathcal{S}})$ is contained in $\Omega(R, V_{\mathcal{S}})$, the set $\Omega(R, V_{\mathcal{S}})$ contains residual behaviours for which the deviation is profitable, and hence it is profitable in the best case.

4. The optimistic residual core is non-empty, the pessimistic is empty
Formally we have $\emptyset = C_-(R, V_{\mathcal{S}}) \subsetneq C_+(R, V_{\mathcal{S}})$, which contradicts to our inductive assumption, and hence this case does not arise.

If a deviation is profitable in the pessimistic case, it is profitable in the optimistic one as well. Hence if an outcome does not belong to the pessimistic core $C_-(N, V)$ it does not belong to the optimistic core $C_+(N, V)$ either. $\square$

Corollary 12.
$$\emptyset \subseteq C_+(N, V) \subseteq C_-(N, V) \subseteq \Omega(N, V),$$
and we can talk about intervals (where size is defined for inclusion).

In the following we further reduce the interval in which the recursive core (also an interval) is confined to.

Lemma 13. *The optimistic core contains the ω-core, that is, $C_\omega(N, V) \subseteq C_+(N, V)$. The pessimistic core is contained in the α-core, that is, $C_-(N, V) \subseteq C_\alpha(N, V)$.*

We prove the part for the optimistic approaches; the corresponding result for the pessimistic approaches is proven in a like manner.

Proof. In the given game, consider an outcome and a deviation from it. We want to check the profitability of this deviation. The two concepts, the ω and the optimistic approach look at two different residual outcome sets. The ω-approach considers the entire residual outcome set $\Omega(R, V_{\mathcal{S}})$, our optimistic approach considers $C_+(R, V_{\mathcal{S}})$ only, *provided* this set is not empty.

If the residual core is empty the two approaches both look at $\Omega(R, V_{\mathcal{S}})$ and expect the same post-deviation payoff. In particular, if the deviation is profitable in the optimistic approach it is also profitable for the ω-approach.

Now we look at the case when the residual core is not empty. Since $C_+(R, V_{\mathcal{S}})$ is contained in $\Omega(R, V_{\mathcal{S}})$ the best case in the ω approach is at least as good as in our optimistic approach looking only at residual core outcomes. Therefore if the deviation is profitable in the optimistic approach, it is also profitable in our ω-approach.

If an outcome is not in the optimistic core $C_+(N, V)$, there exists a profitable deviation from it in the optimistic approach. As the same deviation is also profitable in the ω-approach the outcome is neither in the $C_\omega(N, V)$. $\square$

Corollary 14. *The proposed new concepts, as a pair are a refinement of the α-core, ω-core pair. $C_\omega(N, V) \subseteq C_+(N, V) \subseteq C_-(N, V) \subseteq C_\alpha(N, V)$.*

This corollary expresses that the recursive core is less sensitive to behavioural assumptions than the α-/ω-pair.

Corollary 15.

$$\emptyset \subseteq C_\omega(N, V) \subseteq C_+(N, V) \subseteq C_-(N, V) \subseteq C_\alpha(N, V) \subseteq \Omega(N, V), \qquad (1)$$

and we can talk about intervals (where size is defined for inclusion).

Can we further reduce the "distance" of the optimistic and pessimistic cores? There is no general answer to this question. Their distance can be maximal (the optimistic core is empty, the pessimistic contains all outcomes), and they might also coincide. We think that the latter happens more often, and that the interval is, in general, rather small. While no formal proof exists, we can provide some evidence.

Our main argument is that the inductive definition diminishes differences before going to higher levels. Even large differences of the optimistic and pessimistic approaches in one of the residual cores may or may not lead to differences at the top level. Differences disappear due to the following:

- As long as the optimistic and pessimistic expectations are not too different the resulting cores will be the same.
- Should the two cores be non-empty even if they are different in most of the cases they support only one partition. Then for deviations in a larger game the optimistic and pessimistic scenarios coincide.

- If the partition function satisfies some regularity, the externalities exerted by "similar" residual outcomes will also be similar. Since the core collects outcomes that are stable against the same possible deviations, and hence the individual payoffs can vary only to a limited extent, the deviating payoffs will not be very different when two residual *core* outcomes are considered even if they support different partitions. In other words, in such games even if the core supports multiple partitions there are no substantial differences between the outcomes realising them.

Where things can go wrong are residual cores that are empty. Here the residual behaviour is unpredictable and optimistic and pessimistic expectations can indeed be rather different. We must note however that it is not expected that the solution will work here the best, but even here thanks to the diminishing effect during the induction we may expect to get far better results than with other concepts that are explained in Section 2.1.

4 Summary

We present a new solution to partition function form games that is novel in at least two ways. Firstly the solution exploits the consistency of the partition function form, namely that a residual game is also a partition function form game. This consistency allows us to model residual behaviour in a natural, intuitive way. Secondly, we consider a pair of solutions as the extrema of an interval for set inclusion. Taking the whole interval rather than just one of the extremes enables us to include or exclude some outcomes with certainty.

The solution captures some of the best elements of the existing concepts. The recursive core, although uses optimism/pessimism, it is less sensitive to such behavioural assumptions than the classical α-/ω-approaches. It allows for any residual partition, and as special cases it covers the γ- and δ-approaches. It implements a similar logic as the r-core, but in our view, in a more natural environment and making sure that the solution is defined for all games.

A few questions are left open. Firstly, in case a residual core is empty it is somewhat naive to take the entire residual set: some outcomes can often be discarded with certainty. We may, for instance, only consider Pareto-efficient outcomes. One possibility is to consider a dynamic version of the game: starting from the status quo in the residual game, residual players start to make deviations and only outcomes emerging in these negotiations are considered. In characteristic function form games such a coalitional bargaining process will reach the core if it is nonempty (Kóczy and Lauwers, 2004a), moreover this dynamic bargaining process defines a non-empty core extension (Kóczy and Lauwers, 2004b) allowing us to treat the empty/nonempty core cases together. Unfortunately we suspect a similar result does not hold when externalities are introduced and hence the corresponding solution concept would not be attractive for partition function form games.

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