

# Regret Aversion and Annuity Risk in Defined Contribution Pension Plans

Citation for published version (APA):

Hoevenaars, R. P. P. M., Frehen, R. G. P., Palm, F. C., & Schotman, P. C. (2008). Regret Aversion and Annuity Risk in Defined Contribution Pension Plans. *Insurance: Mathematics and Economics*, 42(3), 1050-1061. <https://doi.org/10.1016/j.insmatheco.2008.01.001>

## Document status and date:

Published: 01/01/2008

## DOI:

[10.1016/j.insmatheco.2008.01.001](https://doi.org/10.1016/j.insmatheco.2008.01.001)

## Document Version:

Publisher's PDF, also known as Version of record

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# Regret aversion and annuity risk in defined contribution pension plans

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Received April 2007; received in revised form January 2008; accepted 9 January 2008

## Abstract

The high value of the implicit option to choose a retirement date at which interest rates are particularly high and life annuities relatively cheap, leads to the possibility to introduce regret aversion in the retirement investment decision of defined contribution plan participants. As a remedy for regret aversion in retirement investment decisions, this paper develops and prices a lookback option on a life annuity contract. We determine a closed-form option value under the restriction that the option holder invests risklessly during the time to maturity of the option and without the guarantee that the exact amount of retirement wealth is converted into a life annuity at retirement. Thereafter the investment restriction is relaxed and the guarantee of exact conversion is imposed and the option is priced via Monte Carlo simulations in an economic environment with a stochastic discount factor. Option price sensitivities are determined via the pricing of alternative options. We find that the price of a lookback option, with a maturity of three years, amounts to 8%–9% of the wealth at the option issuance date. The option price is highly sensitive to the exercise price of the option, i.e. pricing alternative options (e.g. Asian) substantially lowers the price. Time to maturity and interest rate volatility are other important option price drivers. Asset allocation decisions and initial interest rates hardly affect the option price.

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*JEL classification:* C15; G12; G13; G23; IB13; IE50; IE43; IM00

*Keywords:* Lookback option; Life annuity; Annuity risk; Defined contribution; Stochastic discount factor

*Don't look back in anger*

## 1. Introduction

In recent years an increasing number of pension funds shifted from defined benefit (DB) to defined contribution (DC) plans. Particularly in Anglo-Saxon countries DC plans have gained popularity. Boulier et al. (2001) attribute this shift to two major advantages of DC plans over traditional DB plans. Participants can observe their pension claim at any moment in time and can transfer this claim more easily when changing employer. Moreover, DC plan sponsors are unburdened from

risk associated with the pension plan. This risk is passed on to participants. As a direct result of the change in risk bearer, large numbers of uninformed participants face a difficult decision at retirement: Whether and when should they invest their retirement wealth irreversibly in a life annuity? The high price sensitivity of the life annuity with respect to the long-term interest rate complicates the retirement investment decision. In times of increasing interest rates, plan participants may be forced to postpone their retirement date. Beside the shorter payout period and longer contribution period, the high price sensitivity of the life annuity can offer substantial gains in times of increasing interest rates. Vice versa, plan participants may also decide to retire early in times of high interest rates. Hence, the high price sensitivity of the life annuity makes the implicit option to time the retirement date valuable (see Milevsky and Young (2002)).

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Fig. 1. Conversion rate. This figure displays the time series evolution of the 10 year bond yield from January 1952 to January 2004. The yields are annualized.

The large value of the timing option creates fertile ground for regret aversion. For instance, the pension income of a participant who retired at the end of our data set (December 2003) was, *ceteris paribus*, 6.1% higher than a colleague's income who retired half a year earlier. The 6.1% difference is even more remarkable if we take into account that long-term interest rate differences at the sample end are small relative to differences earlier in the sample (see Fig. 1).

Numerous experiments document that regret aversion plays a role in human decision making (e.g. Kahneman and Tversky (1979)). Bell (1982) and Loomes and Sugden (1982) are the first to extend standard utility functions by incorporating regret aversion components. Since this development, the effect of regret aversion has been explored in many different areas. Braun and Muermann (2004) document that regret aversion has a mitigating effect on extreme demands for insurance. Michenaud and Solnik (2006) show that even if the currency risk premium is zero, regret averse investors hold positive currency positions. Gollier and Salanié (2006) document that the introduction of regret aversion in a complete market with Arrow-Debreu securities, shifts the optimal asset allocation more to states with low probabilities. Muermann et al. (2006) analyze the role of regret aversion in pension investment decisions. They evaluate the effect of regret aversion on the willingness to pay for a pension guarantee. One of their main findings is that the willingness to pay for a pension guarantee, of an investor with a risky portfolio, increases with the introduction of regret aversion.

This paper prices an option that protects regret averse participants against annuity risk at retirement. At retirement, many retirees convert their pension wealth into a life annuity and regret aversion plays an important role in the timing of the life annuity purchase. Merton and Bodie (2005) argue that lookback options are a valuable asset for regret averse individuals. We address regret aversion and annuity risk at retirement by developing and pricing a lookback option on a life annuity. The option holder buys an insurance against regret aversion some years before the retirement date. Using a replicating portfolio strategy, we derive the closed-form price for the lookback life annuity option under some simplifying assumptions. We also determine the option price sensitivity with respect to the volatility of the underlying interest rate

process. In the closed-form setting, the option holder is obliged to invest in a riskless product during the time to maturity of the option. Furthermore, the closed-form solution cannot guarantee that the exact amount of retirement wealth is converted into a life annuity for the minimum forward price. As a result, the closed-form solution does not eliminate annuity risk completely.

To allow risky investment strategies and eliminate all annuity risk, we also price the lookback life annuity option in an extended economic environment. In this more elaborate setting, we price the option under the restriction that the complete retirement wealth is converted into a life annuity and allow stock and bond investments during the time to maturity of the option. Monte Carlo simulations and a stochastic discount factor are used to price the option. Finally, we determine the option Greeks by changing different parameters in the pricing environment.

We find that the closed-form price of the lookback life annuity option amounts to 7%–9% of the wealth at the option issuance date and that interest rate volatility and time to maturity are important price drivers. A relaxation of some simplifying assumptions and the application of a stochastic discount factor results in a comparable option price of 8% for men and 9% for women. This relatively high price indicates that the insurance against regret aversion is expensive. It is therefore unlikely that optimization of life cycle functions ignoring regret aversion leads to life annuity lookback option purchases. Hence accounting for regret aversion results in fundamentally different investment strategies than observed in standard non-regret averse investor strategies. Furthermore, we document that the lookback feature is an important driver of the option price. If the participant is offered the right to buy the life annuity for the average forward life annuity price (Asian option), the option price decreases to slightly more than 2% of the wealth at the option issuance date. Extending the time to maturity of the option by one year increases the option price to 11%–12%. We also find that the option price is insensitive to the initial interest rate and hardly affected by the asset allocation during the time to maturity.

The presence of regret aversion in retirement investment decisions contrasts our paper sharply with most earlier literature on the topic. The lookback life annuity option, which offers protection against regret aversion, is not available in financial markets yet. Earlier literature addresses the problem of annuity risk at retirement effectively using standard utility functions and asset allocation approaches. Assuming power utility for participants, Campbell and Viceira (2002) derive the optimal pre-retirement asset allocation. Furthermore, Yaari (1965) points out that it is optimal for participants without bequest motives to convert retirement wealth into a life annuity. Koijen et al. (2007) determine the optimal allocation to nominal, real and equity-linked annuities at retirement. Subsequently, they determine the optimal pre-retirement hedging strategy in stocks, nominal and inflation-linked bonds and cash, for the risk created by the optimal annuity mix. These optimal investments differ strongly from a portfolio replicating a lookback life annuity option. Boulier et al. (2001) optimize the

asset allocation in a DC pension plan. They impose a minimum guarantee on the benefits in the form of a life annuity. Deelstra et al. (2003) generalize the problem to allocation optimization in the presence of a lower bound on the retirement benefits (not necessarily converted into a life annuity). These solutions often impose long-term restrictions on the asset allocation and therefore imply a serious reduction in the freedom of choice of the DC plans. A solution that suffers less from the loss in freedom of choice is the possibility to buy an option that protects participants against annuity risk. Lachance et al. (2003) develop an option to buy back a defined benefit claim. However, this option turns out to be extremely expensive and also fails to recognize possible regret aversion in participant investment decisions. The lookback life annuity option creates only a minor loss in freedom of choice and provides a possibility to insure participants against regret aversion.

In Section 2 we develop the lookback life annuity option and determine a closed-form price in a simplified environment. Section 3 describes the option pricing via Monte Carlo techniques in an extended valuation setting. Our empirical results are discussed in Section 4.

## 2. Defined contribution and the lookback option

In this section we develop and price the lookback option on a life annuity contract in a continuous time framework. The lookback option is offered to plan participants three years before the retirement age of 65. We assume that participants who buy the option survive until their retirement date. Moreover, we price the option under the assumption of exogenous and constant mortality rates and fairly priced life annuities. As a result, a life annuity is nothing more than a portfolio of zero-coupon bonds. The pricing of the option can therefore be reduced to pricing a series of lookback options and each is on a pure discount bond. We therefore start with the pricing of a lookback option on a zero coupon bond with fixed maturity date  $s$ . We consider a fixed option issuance date  $t_a$ . This allows us to suppress the dependence of mortality rates and realized forward bond prices on  $t_a$ . Bond option formulas developed by Jamshidian (1989) and Goldman et al. (1979) are used as a guideline to price the lookback option. We assume that the instantaneous spot rate follows an Ornstein–Uhlenbeck interest rate process characterized as

$$dr = \alpha (\gamma - r) dt + \sigma_r dW, \tag{1}$$

with  $r$  being the instantaneous spot rate,  $\alpha$  the mean-reversion parameter,  $\gamma$  the long-term average interest rate,  $\sigma_r$  the interest rate volatility and  $dW$  a standard Brownian motion under the risk neutral measure.

To characterize the lookback option on a discount bond we specify the option payoff. Since we price the option via a replicating portfolio technique this characterization is sufficient to determine a closed-form option price. The lookback bond option provides the right at the fixed retirement date  $T$  to buy a discount bond with maturity date  $s$  for the minimum forward price. The lookback bond option payoff  $U_T$  can consequently

be characterized as

$$U_T = P(r, T, s) - Q(\bar{f}_T, T, s), \tag{2}$$

with  $P(r, T, s)$  the time  $T$  price of a bond maturing at  $s$  and  $Q(\bar{f}_T, T, s)$  the minimum forward bond price in the time span  $[t_a, T]$ , where the dependence of  $Q$  on  $t_a$  is suppressed. The minimum forward bond price is a function of the maximum forward rate  $\bar{f}_T$  in the time interval  $[t_a, T]$ , retirement date  $T$ , and bond maturity date  $s$ .

An investor who wants to replicate the lookback option on an  $s$ -maturity discount bond must buy a straddle on the  $s$ -maturity bond with exercise price equal to the initial (time  $t_a$ ) forward bond price at time  $T$ . Once the time  $T$  forward bond price attains a new minimum, the replicating investor sells the old position and buys a new straddle with the new minimum as exercise price. He repeats this exercise until the straddle matures. The time  $t$  exercise price  $Q(\bar{f}_t, t, s) = Q$  of the straddle is consequently the minimum forward bond price up till time  $t$ .  $Q$  thus summarizes the history of the lookback option and therefore contains the lookback-element. The bond prices and therefore also the bond option prices are direct contingents of the time time  $t$  interest rate. Moreover, the option and bond maturity dates  $T$  and  $s$  are fixed. The hedging portfolio value can consequently be characterized as a function of the time  $t$  interest rate  $r$ , exercise price  $Q$ , and time  $t$ , and follows

$$H(r, Q, t) = \text{Call}^B + \text{Put}^B, \tag{3}$$

with  $\text{Call}^B$  and  $\text{Put}^B$  respectively the time  $t$  prices of bond call and put options maturing at time  $T$  on pure discount bonds with maturity  $s$ . A simple application of the put-call-parity leads to

$$H(r, Q, t) = 2\text{Call}^B + P(r, t, T) Q - P(r, t, s). \tag{4}$$

Substitution of bond option formulas by Jamshidian (1989) shows that the bond straddle that replicates the lookback option is composed of investments in discount bonds with maturities  $T$  and  $s$ . This is a result of the fact that standard put and call bond options can be replicated with investments in bonds maturing at times  $T$  and  $s$ . The hedging portfolio for the lookback option on an  $s$ -maturity bond can therefore be simplified to two bond investments and can be expressed as

$$H(r, t, Q) = \phi_t^{(s)} P(r, t, s) + \phi_t^{(T)} P(r, t, T), \tag{5}$$

with

$$\phi_t^{(T)} = [1 - 2N(h - \sigma_p)] Q, \tag{6}$$

$$\phi_t^{(s)} = 2N(h) - 1, \tag{7}$$

where  $N$  represents the normal cumulative distribution function and Jamshidian (1989) characterizes  $h$  and  $\sigma_p$  as

$$h = \frac{\ln\left(\frac{P(r,t,s)}{P(r,t,T)Q}\right)}{\sigma_p} + \frac{1}{2}\sigma_p, \tag{8}$$

$$\sigma_p = \frac{v(t, T) (1 - e^{-\alpha(s-T)})}{\alpha}, \tag{9}$$



Table 1  
Base closed-form option prices

|       | 2    | 3    | 4     |
|-------|------|------|-------|
| Men   | 5.93 | 7.64 | 8.90  |
| Women | 7.59 | 9.60 | 11.10 |

This table presents prices for the lookback life annuity option with times to maturity of respectively two, three and four years and a yearly payoff of 10,000. Prices for both men and women are expressed in percentages of the wealth at the issuance date of the option ( $W_{t_a}$ ).

$$v^2(t, T) = \sigma_r^2 \frac{1 - e^{-2\alpha(T-t)}}{2\alpha}. \tag{10}$$

In the Appendix we prove that the hedging portfolio is indeed a replicating strategy for the lookback bond option. Hence, the price of the hedging portfolio must equal the price of the lookback bond option. The time  $t_a$  price of the lookback option on a zero coupon bond maturing at time  $s$  can therefore be characterized as

$$C_L^B(P(r, t_a, s), Q, T - t_a, s) = \phi_{t_a}^{(s)} P(r, t_a, s) + \phi_{t_a}^{(T)} P(r, t_a, T). \tag{11}$$

Under certain conditions Jamshidian (1989) proves that the value of a portfolio of bond options equals the value of an option on a bond portfolio. Consequently, the price of a lookback call option on a life annuity is defined as

$$C_L^L(r, \bar{f}, t_a, T) = \int_T^\infty C_L^B(P(r, t_a, s), Q, T - t_a, s) \zeta_s ds, \tag{12}$$

with  $\zeta_s$  the time  $t$  conditional survival density, i.e. if the time  $t$  probability that the participant is still alive at time  $s$  is modeled as  $\pi_s$ , then its derivative with respect to  $s$  is denoted  $\zeta_s = \frac{\partial \pi_s}{\partial s}$ .

This closed-form option pricing formula enables us to price a lookback option on a life annuity with a monthly nominal payoff equal to 1. Table 1 displays the percentage of wealth at the option issuance date ( $t_a$ ) that a participant pays for the right to buy a yearly pension income of 10,000 for the minimum price. The numbers in Table 1 are obtained by discretizing Eq. (12) on a monthly basis and estimating Eq. (1) with monthly US one-month interest rate data from January 1952 to December 2003 and substituting the estimates for the parameters in Eq. (12).

Table 1 shows that time to maturity is an important driver of the lookback life annuity option price. Furthermore the table shows that for a time to maturity of three years a plan participant pays 7–9% of the wealth at the option issuance date for the right to convert his retirement wealth against the minimum forward life annuity price during the time to maturity of the option.

A closed-form solution also enables us to estimate the effect of a change in standard deviation of the underlying interest rate model on the option price. Fig. 2 displays option prices for volatilities of the interest rate process ranging from 50% to 150% of the estimated volatility.

Fig. 2 shows that the volatility of the instantaneous spot rate is an important driver of the option price. Decreasing

the volatility with 50% leads to a decrease in option price of approximately 50%. Vice versa, increases in the volatility of the spot rate of 50% lead to option price increases of approximately 50%.

Fig. 3 displays the effects of the changes in hedging portfolio holdings, due to a rise in interest rate from 2% to 11%. We zoom in on the time  $t$  holdings of a three ( $\phi_t^{(T)}$ ) and four ( $\phi_t^{(s)}$ ) year maturity bond, assuming a fixed minimum interest rate of 2% and fixed time to maturity of three years.

Fig. 3 shows that, ceteris paribus, a rise in interest rate leads to larger long positions in the bond with three year maturity and larger short positions in a four year maturity bond. The figure also shows an upper- and lowerbound on bond holdings. These upper- and lowerbounds can easily be explained by Eqs. (6) and (7). The cumulative normal distribution implies a range of  $[-Q, Q]$  for  $\phi_T$  and a range of  $[-1, 1]$  for  $\phi_s$ , as clearly depicted in Fig. 3.

One limitation of the closed form model is that it works with an option payoff that is per unit of annuity income after retirement. The participant buys a fixed number of options, and each option provides the payoff (2). When the participant invests his wealth in risky assets, the final wealth at age 65 is unknown, and therefore he does not know a priori how many option he needs to convert his entire pension wealth at the retirement date. An alternative design of the option is that it gives the right to convert all wealth in the pension account at the retirement.

### 3. Extended environment and price drivers

In this section we describe the pricing of a lookback life annuity option that guarantees conversion of final wealth in a pension account. The pricing environment generalizes the one-factor Ornstein–Uhlenbeck assumption from the previous section to a multi-factor setting. We also allow the participant to invest part of his pension wealth in stocks. Closed-form solutions are no longer available in this extended setting and we therefore price the option using Monte Carlo methods and a stochastic discount factor that is consistent with market prices for stocks and bonds. If the participant can choose his investment portfolio (stocks, bonds of different maturities), he also controls the expected value and volatility of his final pension wealth. This in turn implies that the price of the option depends on the investments strategy chosen by the participant. As we work with simulation, we also switch from continuous to discrete time.

#### 3.1. Pricing environment

This subsection describes the economic environment and pricing techniques applied to determine the lookback life annuity price. We price the option using Monte Carlo methods, again assuming that mortality rates are exogenous and constant, fair pricing of life annuities and neglecting mortality risk and contributions during the time to maturity of the option (as in the previous section). The probability that a participant dies during the time to maturity of the option is approximately 3%.

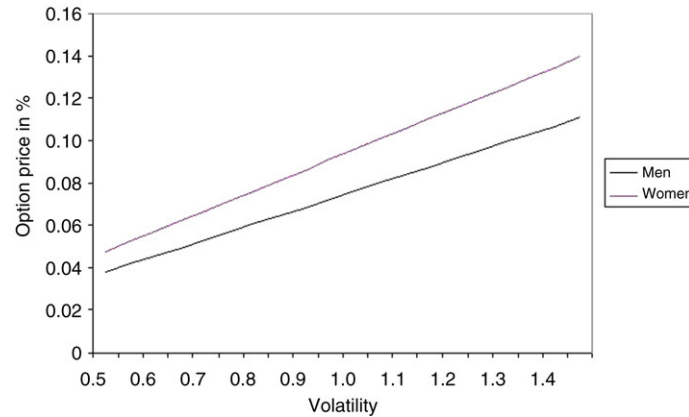


Fig. 2. Changes in interest rate volatility. This figure displays the effect of a change interest rate volatility on the option price. Option prices are expressed in a percentage of wealth at the option issuance date, for both men and women. Prices are provided for volatilities ranging from 50% to 150% of the estimated volatility.

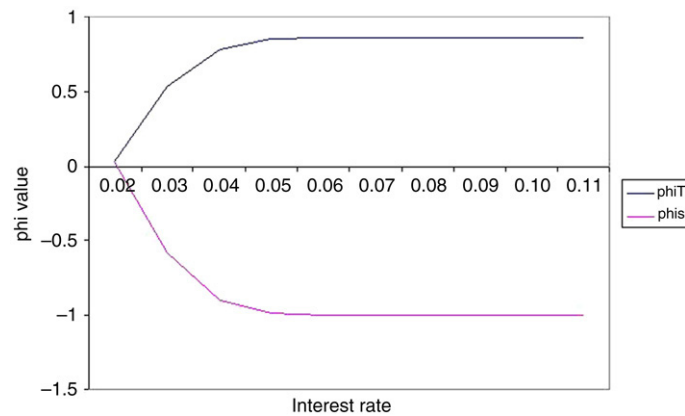


Fig. 3. Changes in hedging portfolio holdings. This figure displays the effect of a change in three and four year bond holdings due to a rise in interest rate from 2% to 11%. We consider a fixed time to maturity of three years and fixed minimum interest rate of 2%. Option prices are expressed in a percentage of wealth at the option issuance date, for both men and women.

Including mortality risk during the time to maturity of the option, would therefore decrease the option price by approximately 3%. We neglect the effect of the lookback option purchase on the pre-retirement wealth. Despite its implausible character, this assumption is necessary. Otherwise, the retirement wealth and thus the option price would be a function of the option price. This self-dependency is typically hard to model. In the simulation setting the retirement wealth  $W_T$  is a function of the investment weights, specified by the option holder and the accompanying future returns which are uncertain at time  $t_a$ ,

$$W_T = A_{t_a} \prod_{t=t_a}^T (w_s (1 + R_{s,t}) + w_{2b} (1 + R_{2b,t}) + w_{10b} (1 + R_{10b,t})), \tag{13}$$

with  $w_s$ ,  $w_{2b}$  and  $w_{10b}$  respectively the investment weights for the stock portfolio, two-year and ten-year bond and  $R_{s,t}$ ,  $R_{2b,t}$  and  $R_{10b,t}$  the corresponding time  $t$  returns.  $A_{t_a}$  represents the participant's initial wealth. In the base simulation setting, we set  $A_{t_a}$  equal to 100,000 and specify equal investment weights ( $w_s = w_{2b} = w_{10b}$ ), whereas the returns are simulated using a VAR-model. We first determine the option payoff and

specify the processes for the stock and bond returns in the investment universe with the corresponding stochastic discount factor. Subsequently, 100,000 scenarios with discounted option payoffs are generated on a monthly frequency. The option price is then approximated by the average discounted option payoff.

To ultimately determine the option price, the option payoff, investment universe and discount factor need to be specified. In this setting option holders have the right to buy a life annuity for the minimum forward price attained during the time to maturity of the option. Furthermore, we require that both the option holder and non-option holder convert the exact amount of retirement wealth  $W_T$  into a life annuity. As a result, the right to buy the life annuity for minimum price is equivalent to the right to convert retirement wealth against the maximum interest rate during time to maturity of the option. We specify the guarantee of exact conversion of retirement wealth for non-option holders as

$$W_T = L^N \sum_{s=T+1}^{\infty} \frac{\pi_s}{(1 + r_T)^{s-T}}, \tag{14}$$

where  $\pi_s$  is the time  $T$  probability that the participant is still alive at time  $s$  and  $L^N$  the after retirement income of the non-option holder (i.e. the nominal monthly life annuity payoff

for the non-option holder). The option holder has the right to convert against the maximum rate. The guarantee for the option holder is therefore expressed as

$$W_T = L^H \sum_{s=T+1}^{\infty} \frac{\pi_s}{(1+r_{\max})^{s-T}}, \tag{15}$$

where  $L^H$  is the after retirement income of the option holder (i.e. the nominal monthly life annuity payoff of the option holder) and  $r_{\max}$  the maximum interest rate during the time to maturity of the option. The option payoff is the difference between the present value of the option holder's and non-option holder's benefits

$$\begin{aligned} U_T &= \sum_{s=T+1}^{\infty} \frac{\pi_s [L^H - L^N]}{(1+r_T)^{s-T}} \\ &= \sum_{s=T+1}^{\infty} \frac{\pi_s L^H}{(1+r_T)^{s-T}} - W_T. \end{aligned} \tag{16}$$

Eqs. (15) and (16) show that the option payoff is a function of the retirement conversion rate  $r_T$ , maximum conversion rate  $r_{\max}$ , terminal wealth  $W_T$  and mortality rates  $\pi_s$ .

In a DC pension plan, participants have more freedom with regard to the asset allocation than in a DB plan. DC plan participants have to specify weights ( $w_s$ ,  $w_{2b}$  and  $w_{10b}$ ) for each product (portfolio of stocks, two-year bond and ten-year bond) in the investment universe. The investment universe with stocks and different bonds implies that investment risk during the time to maturity of the option is contained in the option price, in contrast to the closed-form analysis. Analogous to Ang and Piazzesi (2003) and Cochrane and Piazzesi (2005) we model the log returns of the different investment alternatives as a VAR(1)

$$X_{t+1} = \mu + \phi X_t + \eta_{t+1} \quad \eta_{t+1} \sim N(0, \Sigma), \tag{17}$$

where  $X_{t+1}$  contains log stock, 2 year and 10 year bond returns at time  $t + 1$ . As Campbell et al. (1997), we approximate  $n$ -maturity log bond returns by

$$r_{n,t+1} = D_{n,t} y_{n,t} - (D_{n,t} - 1) y_{n-1,t+1}, \tag{18}$$

with  $D_{n,t}$  and  $y_{n,t}$  respectively the time  $t$  duration and yield of an  $n$ -maturity bond. We approximate  $y_{n-1,t+1}$  by  $y_{n,t+1}$  and consider zero coupon bonds, for which maturity equals duration.

The VAR(1) model is estimated using monthly U.S. data on bond yields and stock returns for the period ranging from January 1952 to December 2003. Value weighted stock returns are obtained from the CRSP database. Bond yields are provided by Bliss (1997). Summary statistics and parameter estimates are reported in Tables 2 and 3.

In order to price the option, we specify a stochastic discount factor. The stochastic nature of the discount factor allows it to discount payoffs in relatively “good” scenarios stronger than payoffs in relatively “bad” scenarios. In line with Cochrane and Piazzesi (2005) we assume for the discount factor the following

Table 2  
Return summary statistics

|                    | $r_s$  | $r_{2b}$ | $r_{10b}$ |
|--------------------|--------|----------|-----------|
| Mean               | 0.108  | 0.059    | 0.065     |
| Standard deviation | 0.149  | 0.100    | 0.093     |
| Skewness           | -0.708 | 0.840    | 0.713     |
| Kurtosis           | 2.790  | 0.762    | 0.188     |

This table presents summary statistics of the annualized stock, two and ten-year bond returns. For each return series, different rows display the mean, standard deviation, skewness and kurtosis. The statistics are computed on a data period ranging from January 1952 until January 2004.

Table 3  
VAR estimation results

|                  | $c$               | $r_{s,t-1}$        | $r_{2b,t-1}$      | $r_{10b,t-1}$      |
|------------------|-------------------|--------------------|-------------------|--------------------|
| $r_{s,t}$        | 0.006<br>(2.560)  | 0.049<br>(1.220)   | 0.581<br>(1.616)  | 0.071<br>(0.492)   |
| $r_{2b,t}$       | 0.004<br>(10.784) | -0.033<br>(-4.717) | 0.210<br>(3.359)  | 0.012<br>(0.463)   |
| $r_{10b,t}$      | 0.004<br>(3.890)  | -0.065<br>(-3.642) | 0.421<br>(2.622)  | -0.049<br>(-0.757) |
| $y_{1m_t}^{(1)}$ | 0.004<br>(35.043) | -0.004<br>(-2.266) | 0.191<br>(11.033) | -0.046<br>(-6.583) |

This table presents parameter estimates and the corresponding T-statistics of the VAR(1)-model for stock and bond returns and for  $\delta_0$  and  $\delta_1$  as specified in Section 3. Log monthly stock, two-year and ten-year bond returns are respectively denoted by  $r_{s,t}$ ,  $r_{2b,t}$  and  $r_{10b,t}$  and  $c$  represents a constant. The one month log bond yield is represented by  $y_{1m_t}^{(1)}$ . We approximate the one month maturity bond yield by the monthly yield on a three month maturity bond, for data reasons. T-statistics are reported in parentheses. The estimations are performed on a data period ranging from January 1952 until January 2004.

process

$$M_{t+1} = \exp \left( -\delta_0 - \delta'_1 X_t - \frac{1}{2} \lambda'_t \Sigma \lambda_t - \lambda'_t \eta_{t+1} \right), \tag{19}$$

$$\lambda_t = \lambda_0 + \lambda_1 X_t, \tag{20}$$

where  $M_{t+1}$  denotes the time  $t + 1$  discount factor and  $\lambda_t > 0$ . By definition, the expected deflator value should equal the price of a zero coupon bond maturing in the same period. Considering a one period horizon, this yields

$$p_t^{(1)} = \log (E_t [M_{t+1}]), \tag{21}$$

with  $p_t^{(1)}$  the log time  $t$  price of a zero coupon bond with maturity date  $t + 1$ . Substitution of Eqs. (19) and (20) in Eq. (21) leads to

$$-\delta_0 - \delta'_1 X_t = -y_{1m_t}, \tag{22}$$

with  $y_{1m_t}$  the log yield of a one month maturity bond at time  $t$ . To determine values for  $\lambda_0$  and  $\lambda_1$ , we note that the expected value of the discounted return of any asset in the economic environment should be equal to 1, represented as

$$\iota = E_t [M_{t+1} R_{t+1}], \tag{23}$$

with  $\iota$  a  $3 \times 1$  unit vector and  $R_{t+1}$  a  $3 \times 1$  vector containing the returns of the different products in the investment universe.

Rewriting and taking logs allows us to rewrite Eq. (23) as

$$0 = \iota E_t [\log (M_{t+1})] + E_t [\log (R_{t+1})] + \frac{1}{2} \iota \sigma_m^2 + \frac{1}{2} \text{diag}(\Sigma) + \sigma_{mr}, \quad (24)$$

with  $\sigma_m^2$  the variance of the log discount factor,  $\text{diag}(\Sigma)$  a  $3 \times 1$  vector containing the diagonal elements of  $\Sigma$  and  $\sigma_{mr}$  a  $3 \times 1$  vector containing the covariances between the log discount factor and the log returns of respectively the stock, 2 year bond and 10 year bond. Substitution and some more rewriting then gives

$$\lambda_t = \Sigma^{-1} \left( \mu + \phi X_t + \iota (-\delta_0 - \delta'_1 X_t) + \frac{1}{2} \text{diag}(\Sigma) \right), \quad (25)$$

$\lambda_0$  and  $\lambda_1$  can consequently be defined as

$$\lambda_0 = \Sigma^{-1} \left( \mu - \delta_0 \iota + \frac{1}{2} \text{diag}(\Sigma) \right), \quad (26)$$

$$\lambda_1 = \Sigma^{-1} (\phi - \iota \delta'_1), \quad (27)$$

where values for  $\delta_0$  and  $\delta_1$  have been obtained by simply regressing the log short-term interest rate yield on the VAR-variables  $X_t$  and  $\mu$  and  $\phi$  and  $\Sigma$  from the VAR-regression.

We have specified the option payoff in Eq. (16) and discount factor in Eq. (19) in VAR parameters, mortality rates and terminal wealth. VAR parameters have been obtained from standard regressions and mortality rates are readily available. However terminal wealth depends on the investment weights and corresponding returns in each investment category for the investor. Hence, by specifying weights ( $w_s$ ,  $w_{2b}$  and  $w_{10b}$ ) for each investment category and repetitively, randomly drawing values for  $\eta$  from a normal distribution, scenarios-specific values for the terminal wealth and thus for the option payoff and discount factor can be determined. The option price can subsequently be approximated by averaging the discounted option payoffs across all scenarios.

Any  $n$ -maturity bond can be priced consistently with the VAR specified before. This allows us to determine the exact life annuity price and easily extend the investment universe. The exact price is obtained by discounting each pension payoff with the corresponding maturity bond rate. The investment universe as specified contains two bonds, a two-year and ten-year maturity zero coupon bond. This may seem a restrictive universe. However, [Cochrane and Piazzesi \(2005\)](#) document that a complete term structure of bonds can be defined recursively. As a result, it is straightforward to extend the universe with bonds of any maturity. However, we have to note that the term structure in our investment universe is based on only two maturities. As a result, the empirical term structure could differ (even in shape) from ours. If we assume that the  $n$ -maturity bond yield is a linear function of the VAR-variables

$$y_t^{(n)} = a_n + b'_n X_t, \quad (28)$$

and use the fact that  $A_0 = 0$  and  $B_0 = 0$ ,  $A_n$  and  $B_n$  can be defined recursively using

$$A_{n+1} = -\delta_0 + A_n + B'_n \mu - B'_n \Sigma \lambda_0 + \frac{1}{2} B'_n \Sigma B_n, \quad (29)$$

$$B'_{n+1} = -\delta'_1 + B'_n \phi - B'_n \Sigma \lambda_1. \quad (30)$$

In line with [Cochrane and Piazzesi \(2005\)](#)  $a_n$  and  $b'_n$  are obtained as

$$a_{n+1} = -\frac{1}{n+1} \times \left( -\delta_0 + A_n + B'_n \mu - B'_n \Sigma \lambda_0 + \frac{1}{2} B'_n \Sigma B_n \right), \quad (31)$$

$$b'_{n+1} = -\frac{1}{n+1} (-\delta'_1 + B'_n \phi - B'_n \Sigma \lambda_1). \quad (32)$$

### 3.2. Measuring price drivers with alternative options

This subsection describes the different alternative options priced to quantify the Greeks of the lookback life annuity option. We start with a base option and first modify the exercise price (i.e. the option version) to determine the effect of the lookback characteristic. Subsequently the parameters time to maturity, initial interest rate and investment weights are changed to resolve the price sensitivities of the option price.

#### 3.2.1. Base option

The base option is a lookback life annuity option with a time to maturity of three years. This option provides plan participants the right to buy a life annuity at retirement for the minimum forward price during the last three years before retirement. In addition to the time horizon, the option price is a function of the retirement wealth  $W_T$ .  $W_T$  is determined by the investment weights ( $w_s$ ,  $w_{2b}$  and  $w_{10b}$ ) and their corresponding returns. We assume that contributions do not take place in the last years before retirement and assume that participants that buy the option survive until retirement. For the base option we assume that investment weights are equal ( $w_s = w_{2b} = w_{10b}$ ) and that the portfolio is monthly rebalanced. As a final characterization, we set the initial value of the conversion rate equal to its last value in the dataset (4.42%).

#### 3.2.2. Option greeks

As a first sensitivity test, we change the exercise price to determine the impact of the lookback feature on the price. We price an Asian option with the same characteristics as the lookback option. The Asian option provides the right to buy the life annuity for the average forward price during the time to maturity. This option only has a positive payoff if the time  $T$  life annuity price exceeds the average forward life annuity price. The second alternative option we price has a predetermined and fixed exercise price. This option offers holders the right to convert their retirement wealth against a predetermined rate (e.g. 6%). The option holder exercises the option if the retirement rate does not exceed the guaranteed rate (6%). For both alternatives the probability of having a payoff of



zero is substantial, whereas the lookback option has a negligible chance of ending at-the-money (and generating a zero payoff). Since the option price is the expected value of the discounted payoff, the effect of changing the option version is large.

Furthermore, we measure the effect of a change in time to maturity. We price the lookback life annuity option on a horizon of two, four and five years. By providing the right to buy for the minimum price, the time to maturity is an important driver of the option price. An extension in time horizon with one year, gives the life annuity price 12 additional possibilities of attaining a new minimum. Hence, the longer (shorter) the time to maturity, the larger (smaller) the expected option payoff and thus the option price.

As a further possible price driver we select the interest rate at the time the option is issued. Particularly if the interest rate three years before retirement is high, a large demand for the lookback option is likely to arise. Participants then want to protect themselves against a fall in interest rate shortly before the retirement date. We quantify the impact of a change in initial rate. However, we point out in the closed-form valuation (see Appendix) that changes in the lookback option price are not affected by changes in minimum price of the life annuity. On the issuance date  $t_a$ , the life annuity price is at a minimum by construction. Hence, the option price is unlikely to be affected by changes in the initial rate. The guarantee that the exact amount of retirement wealth is converted does not alter this result.

Finally, we measure the impact of the investment weights on the option price. In addition to the equally weighted portfolio of the base option, we construct three pure asset portfolios. Investments are restricted to one of the three assets (stock portfolio, two-year and ten-year bond) in the pure portfolios. The portfolios are re-balanced on a monthly basis. Since we consider the ten-year bond yield as the conversion rate of the life annuity, especially the pure ten-year bond portfolio is an interesting investment strategy. This portfolio is particularly risky. When the interest rate increases during the time to maturity of the option, the difference in interest rates ( $r_T - r_{\max}$ ) is likely to be small and the option payoff consequently low. Additionally, interest rate increases lead to a relatively low retirement wealth  $W_T$ , via low bond returns. On the other hand, interest rate decreases lead to large differences between  $r_T$  and  $r_{\max}$  and thus high option payoffs. The high payoff is amplified by a relatively large retirement wealth. The large retirement wealth is a result of high bond returns. The higher risk of the pure ten-year bond investment strategy is partially offset by the stochastic discount factor. Nevertheless, investing in the pure ten-year bond portfolio is a risky strategy.

## 4. Empirical results

This section reports the prices of the previously specified options. We start with pricing a base option and then provide prices for different alternatives, to determine the option Greeks.

### 4.1. Option prices

The base option as described in the previous section is a lookback life annuity option with a time to maturity of three

Table 4  
Base Monte Carlo option prices

|               | 90,000 | 100,000 | 110,000 |
|---------------|--------|---------|---------|
| Men           | 8.02   | 7.98    | 7.97    |
| Women         | 9.22   | 9.17    | 9.17    |
| $E[MR_{s,T}]$ | 0.997  | 0.999   | 0.995   |

This table presents prices for the lookback life annuity option with a time to maturity of three years. Prices for both men and women are expressed in percentages of the wealth at the issuance date of the option ( $W_{t_a}$ ). We present option prices based on 90,000, 100,000 and 110,000 scenarios. Prices are significant up to less than 2 basispoints. The last row reports the average across all scenarios of the deflated cumulative stock returns.

years. Since men and women have different mortality rates, we determine option prices for men and women separately.

Table 4 reveals that the price of the base option is 8% for male participants and slightly higher than 9% for female participants, due to better life expectancies for women. Furthermore, the simulation results show that the standard deviation across all scenarios of the discounted option payoff is high, approximately 55% for men and 63% for women. This is an indication that the annuity risk that participants encounter at the conversion date is considerable. Table 4 also shows convergence in option prices at 100,000 simulations. Furthermore, 100,000 simulations ensure significance up to the second decimal. We therefore hereafter generate prices based on 100,000 simulations. Finally, the table with base option prices displays the average across all scenarios of the deflated cumulative stock returns. As expressed by Eq. (23) the expected value of discounted cumulative stock returns should be equal to one. We approximate the expected value by the average across all scenarios and the results shows that the theoretical condition is satisfied.

### 4.2. Alternative option prices

We first determine the sensitivity of the option price with respect to the exercise price. First an Asian option with the same time to maturity is priced. The Asian option holder has the right to convert his retirement wealth into a life annuity for the average interest rate during time to maturity.

Fig. 4 shows that transforming the option into an Asian one, considerably lowers the price. For both men and women the price difference between the Asian and lookback option is approximately 6% points. The Asian option has fewer scenarios with extremely high payoffs and many scenarios with a zero payoff. Only if the terminal rate  $r_T$  is substantially lower than the average rate, high payoffs are realized.

As a second exercise price sensitivity test, we price an option that guarantees participants a certain conversion rate. Option prices are determined for guaranteed rates of 5% and 6%. Fig. 4 again shows that the impact of changing the option version is large. Price differences between the guaranteed rate option and the base option range from 2.5% to 6% points.

Beside the exercise price, we consider the sensitivity of the option price with respect to the time to maturity. The base option has a time to maturity of three years. We determine the

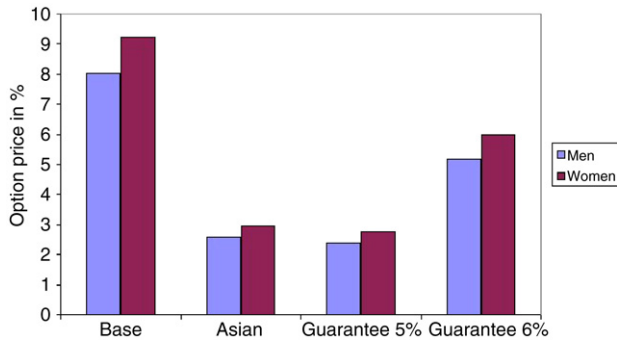


Fig. 4. Variation in option design. This figure displays the effect of a change in option design. Option prices are expressed in a percentage of wealth at the option issuance date, for both men and women. Prices are provided for the base option (3 year lookback life annuity) and an Asian option and Guaranteed Rate option with equal characteristics. Conversion rates of 5% and 6% are guaranteed in the Guaranteed Rate option.

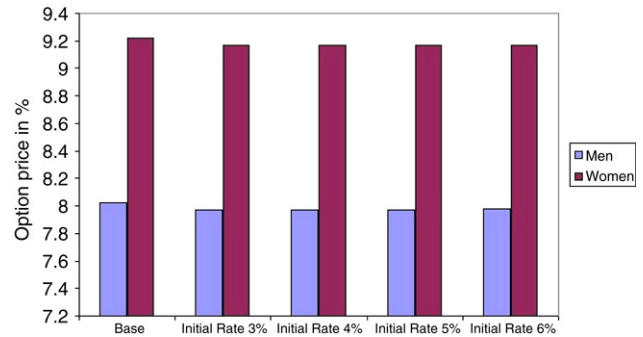


Fig. 6. Variation in initial rate. This figure displays the effect of a change in initial conversion rate. Option prices are expressed in a percentage of wealth at the option issuance date, for both men and women. Prices are provided for the base option (3 year lookback life annuity with initial rate of 4.42%) and for a lookback life annuity option with initial rates of 4%, 5% and 6% respectively.

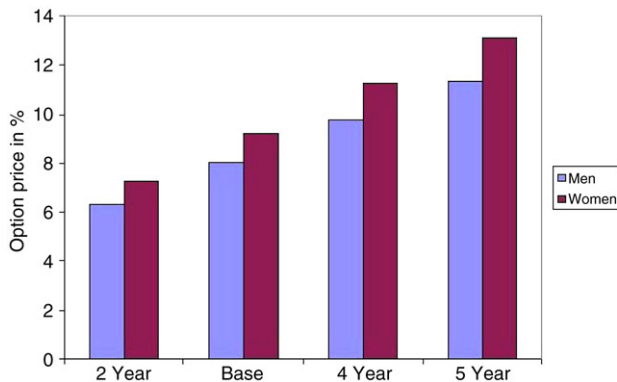


Fig. 5. Variation in time to maturity. This figure displays the effect of a change in time to maturity. Option prices are expressed in a percentage of wealth at the option issuance date, for both men and women. Prices are provided for the lookback life annuity option with times to maturity of respectively 2 year, 3 year (base), 4 and 5 year.

effect of a change in time to maturity by pricing lookback life annuity option with times to maturity of 2, 4 and 5 years.

Fig. 5 affirms the relatively large sensitivity with respect to the time to maturity documented in Section 2. Decreasing the lookback period with one year leads to a price decrease of 2% points, whereas an increase of one year causes a price increase of 2 percentage points. A lookback life annuity with a time to maturity of five years would cost a participant approximately 12% of the wealth at the option issuance date. Compared to the base option this implies a price increase of 4 percentage points.

Furthermore, we document the effect of a change in initial interest rate. The base option has an initial rate equal to the last data point in our sample (4.42%). Fig. 6 shows that the initial rate does not affect the option price. These findings support the proof in the Appendix that the option payoff is not affected by changes in the minimum price.

As a final sensitivity test, we price the lookback life annuity option with different asset allocations. We restrict the investments during the lifetime of the option to one of the three available products: stock portfolio, two-year bond and ten-year bond. Fig. 7 shows that the asset allocation hardly affects the option price. A pure stock investment lowers the

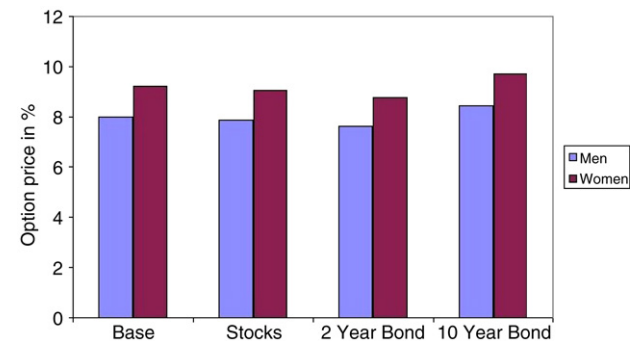


Fig. 7. Variation in asset allocation. This figure displays the effect of a change in asset allocation. Option prices are expressed in a percentage of wealth at the option issuance date, for both men and women. Prices are provided for the base option (3 year lookback life annuity with equal allocation in the three investment categories (stocks, two-year bond and 10 year bond)) and for a lookback life annuity option with investments restricted to one of the three categories.

price compared to the base option by 1% point. Restricting the investments to the two-year bond reduces the price by 4% points. Since stocks have a higher risk premium than two-year bonds, the “stock-only” strategy is more expensive than the two-year bond strategy. However, when investments are restricted to the ten-year bond, we observe an increase in price. The price increase is a result of the risky character of the ten-year bond. Since we consider the ten-year bond rate as the conversion rate, restricting the investments to this category creates a multiplier effect. In relatively good scenarios this strategy leads to very high option payoffs via the difference in terminal and maximum interest rate and via high bond returns. In relatively bad scenarios this strategy leads to extremely poor payoffs vice versa. Nevertheless, we conclude that the asset mix drives the price of the option, but only to a minor extent.

### 5. Conclusion

Large numbers of uninformed plan participants face difficult decisions at retirement. Should they make an irreversible and risky investment in a life annuity or risk the possibility of outliving their money? If they buy a life annuity, when should they do so? These decisions affect their income for the rest of

their lives. Especially the high value of timing (i.e. early retiring if the long-term interest rate is decreasing and postponing retirement in increasing interest rate states) the investment decision provokes regret aversion.

As a remedy for the regret aversion in that decision making process, we develop and price a lookback option on a life annuity. Participants buy this option in the last years before retirement. It provides them the right to lookback at retirement and buy a life annuity for the minimum forward price in the lookback period. First, we determine a closed-form lookback life annuity option price without the guarantee that the exact amount of retirement wealth is converted into a life annuity and under the assumption that the option holder invests risklessly during the time to maturity of the option. The closed-form option price is 7%–9% of the plan participant’s wealth at the option issuance date. Then we price the option in an extended pension environment conditional on the guarantee that participants convert the exact amount of retirement wealth into a life annuity, with the possibility to invest in stocks and bonds during the time to maturity of the option and using a stochastic discount factor. The price for a DC plan participant to insure against regret aversion at retirement, whilst retaining investment freedom, is 8%–9% of the wealth three years before retirement. Hence, closed-form and simulated prices are comparable.

In the closed-form setting we document that interest rate volatility and time to maturity are important option price drivers. In the simulation setting, we determine the option Greeks by pricing alternative options. Another important driver of the option price is the lookback feature (i.e. the exercise price). If participants are offered the right to buy the life annuity for the average forward price (Asian option) or for a predetermined price, the option price decreases substantially. We find that the option price is neither sensitive to the initial interest rate, nor to the asset allocation during the time to maturity.

Future research could extend our analysis in multiple ways. Participants could be offered real instead of nominal pension income. Furthermore, mortality risk and life annuity risk premia could be added in the option pricing environment.

**Acknowledgements**

We thank Fons Quix as instigator of this paper, and an anonymous referee, Mathijs Cosemans, Frank de Jong, Roderick Molenaar and Roald Versteeg for helpful comments and suggestions. The views expressed in this paper do not necessarily represent the views of ABP.

**Appendix**

This appendix proves that the hedging strategy as specified in Section 2 is a replicating and self-financing strategy for a lookback option on a discount bond. To formally prove that the hedging portfolio is indeed replicating and self-financing, we check the following two conditions (see Baxter and Rennie (1996)):

**Condition A.1.** If  $(\phi_t^{(T)}, \phi_t^{(s)})$  is a portfolio and  $(P(r, t, T), P(r, t, s))$  are the time  $t$  prices of bonds maturing at  $T$  and  $s$  respectively, then

$$(\phi_t^{(T)}, \phi_t^{(s)}) \text{ is self-financing if and only if } dH = \phi_t^{(T)} dP(r, t, T) + \phi_t^{(s)} dP(r, t, s).$$

**Condition A.2.** A replicating strategy for  $U_T$  is a self-financing portfolio  $(\phi_T^{(T)}, \phi_T^{(s)})$  such that  $H(r, T, Q) = \phi_T^{(T)} P(r, T, T) + \phi_T^{(s)} P(r, T, s) = U_T$ .

It is straightforward to check that the payoff of the hedging portfolio replicates the option payoff  $U_T$ . When the maturity date of the option is approaching, Eqs. (5)–(10) show that the portfolio has limit value

$$\lim_{t \rightarrow T} H(r, t, Q) = P(r, T, s) - Q(\bar{f}_T, T, s), \tag{33}$$

which is identical to the option payoff as specified in Eq. (2).

To prove that the hedging portfolio is self-financing, we have to prove that

$$dH = \phi_t^{(T)} dP(r, t, T) + \phi_t^{(s)} dP(r, t, s). \tag{34}$$

An application of Itô’s lemma to expression (5) shows that changes in hedging portfolio value can be characterized as

$$dH = H_r dr + H_Q dQ - H_t dt + \frac{1}{2} H_{rr} (dr)^2. \tag{35}$$

Since the hedging portfolio is a function of the time to maturity  $T - t$ , the dependence of the hedging portfolio on  $t$  is negative, as indicated by the minus in Eq. (35).

We start with proving that  $H_Q dQ = 0$ . To prove this we can restrict ourselves to the cases where the forward bond price  $F^B(r, t, T, s) = \frac{P(r, t, s)}{P(r, t, T)}$  attains a minimum

$$F^B(r, t, T, s) \rightarrow Q(\bar{f}_t, t, s).$$

If the forward bond price is not at a minimum  $dQ = 0$  and hence  $H_Q dQ = 0$ . We therefore have to prove that  $H_Q = 0$  if the forward bond price attains a minimum. The methodology to prove this is introduced by Goldman et al. (1979). We prove that the moments of  $Q(\bar{f}_T, T, s)$ , conditional on  $P(r, t, s)$ ,  $t$  and  $Q(\bar{f}_t, t, s)$  are independent of  $Q(\bar{f}_t, t, s)$  for any  $t_a < t < T$ . Since the hedging portfolio payoff and therefore also its value is a function of  $Q(\bar{f}_T, T, s)$ , it suffices to show that the distribution of  $Q(\bar{f}_T, T, s)$  is unaffected by changes in  $Q(\bar{f}_t, t, s)$ . To see the dependence of the option price on  $Q(\bar{f}_T, T, s)$  we note that

$$C_L^B(P(r, t, s), Q(\bar{f}_t, t, s), T - t, s) = e^{-rt} E[P(r, T, s) - Q(\bar{f}_T, T, s)]. \tag{36}$$

Let

$$\psi = \begin{cases} \frac{Q(\bar{f}_t, t, s)}{F^B(r, t, T, s)} & \text{if } Q(\bar{f}_t, t, s) < Q_T(\bar{f}_T, T, s) \\ \frac{Q_T(\bar{f}_T, T, s)}{F^B(r, t, T, s)} \equiv Z & \text{otherwise,} \end{cases} \tag{37}$$

and let  $Q_T(\bar{f}_{i,T}, t, T, s)$  and  $f_{i,T}$  respectively be the minimum bond price and maximum forward rate in the period  $[t, T]$ . By construction  $Q(\bar{f}_T, T, s) = F^B(r, t, T, s)\psi$ . The  $n$ th moment of  $Q(\bar{f}_T, T, s)$  is defined as

$$G(n) = \int_{-\infty}^{F^B(r,t,T,s)} [Q(\bar{f}_T, T, s)]^n d\Phi_Q [Q(\bar{f}_T, T, s)], \quad (38)$$

with  $\Phi_Q$  the conditional CDF of  $Q(\bar{f}_T, T, s)$  given  $F^B(r, t, T, s)$ . Substitution transforms the  $n$ th raw moment into

$$G(n) = \int_0^1 [F^B(r, t, T, s)\psi]^n d\Phi_Q[\psi]. \quad (39)$$

Decomposing  $\psi$  as in the definition and noting that we condition on the information set at time  $t$  leads to

$$G(n) = \left[ Q(\bar{f}_t, t, s)^n \int_{\frac{Q(\bar{f}_t,t,s)}{F^B(r,t,T,s)}}^1 d\Phi_Z(Z) \right] + \left[ Q(\bar{f}_t, t, s)^n \int_0^{\frac{Q(\bar{f}_t,t,s)}{F^B(r,t,T,s)}} Z^n d\Phi_Z(Z) \right], \quad (40)$$

with  $\Phi_Z$  the distribution function of  $Z$ .  $Z$  is like a return and  $F^B(r, t, T, s)$  follows a random walk (see Jamshidian (1989)). Therefore the derivative of the second part towards  $Q(\bar{f}_t, t, s)$  is zero. Differentiation of the first part with respect to  $Q(\bar{f}_t, t, s)$  yields

$$\frac{\partial G(n)}{\partial Q(\bar{f}_t, t, s)} = nQ(\bar{f}_t, t, s)^{n-1} \int_{\frac{Q(\bar{f}_t,t,s)}{F^B(r,t,T,s)}}^1 d\Phi_Z(Z). \quad (41)$$

Since we only consider the case where  $F^B(r, t, T, s) \rightarrow Q(\bar{f}_t, t, s)$  and we assumed an Ornstein–Uhlenbeck process for the underlying interest rate, the probability measure at the point  $\frac{Q(\bar{f}_t,t,s)}{F^B(r,t,T,s)}$  is zero. Hence, the  $n$ th moment of  $Q(\bar{f}_T, T, s)$  is independent of  $Q(\bar{f}_t, t, s)$  for any  $n$  and therefore  $H_Q = 0$  if the forward bond price attains a minimum.

After proving that  $H_Q dQ = 0$  and substituting Eq. (1) for  $dr$ , the change in hedging portfolio value is characterized as

$$dH = H_r \sigma_r dW + \left[ H_r \alpha (\gamma - r) + \frac{1}{2} H_{rr} \sigma_r^2 - H_t \right] dt. \quad (42)$$

Differentiating the hedging portfolio value twice towards  $r$ , and towards  $t$  leads to:

$$H_r = -\phi_t^s \frac{1 - e^{-\alpha(s-t)}}{\alpha} P(r, t, s) - \phi_t^T \frac{1 - e^{-\alpha(T-t)}}{\alpha} P(r, t, T), \quad (43)$$

$$H_{rr} = \phi_t^s \frac{(1 - e^{-\alpha(s-t)})^2}{\alpha^2} P(r, t, s) + \phi_t^T \frac{(1 - e^{-\alpha(T-t)})^2}{\alpha^2} P(r, t, T) - 2n(h)P(r, t, s) \frac{1}{\sigma_p \alpha^2} \times \left[ e^{-2\alpha(T-t)} \left( 1 - 2e^{-\alpha(s-T)} + e^{-2\alpha(s-T)} \right) \right], \quad (44)$$

$$H_t = -\phi_t^s P(r, t, s) \left( e^{-\alpha(s-t)} r + (1 - e^{-\alpha(s-t)}) \gamma - \sigma_r^2 \frac{(1 - e^{-\alpha(s-t)})^2}{2\alpha^2} \right) - \phi_t^T P(r, t, T) \times \left( e^{-\alpha(T-t)} r + (1 - e^{-\alpha(T-t)}) \gamma - \sigma_r^2 \frac{(1 - e^{-\alpha(T-t)})^2}{2\alpha^2} \right) - n(h)P(r, t, s) \frac{\sigma_r^2}{\sigma_p \alpha^2} e^{-2\alpha(T-t)} \times \left[ 1 - 2e^{-\alpha(s-T)} + e^{-2\alpha(s-T)} \right]. \quad (45)$$

Substitution of the expressions derived for  $H_t$ ,  $H_r$ , and  $H_{rr}$  gives

$$dH = \left[ \phi_t^{(T)} P(r, t, T) + \phi_t^{(s)} P(r, t, s) \right] r dt \quad (46)$$

$$- \left[ \phi_t^{(T)} P(r, t, T) \frac{1 - e^{-\alpha(T-t)}}{\alpha} + \phi_t^{(s)} P(r, t, s) \frac{1 - e^{-\alpha(s-t)}}{\alpha} \right] \sigma_r dW. \quad (47)$$

$$= \phi_t^{(T)} dP(r, t, T) + \phi_t^{(s)} dP(r, t, s) \quad (48)$$

Hence the hedging portfolio  $H_t$  is self-financing and consequently a replicating portfolio.

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