Essays on the economics of social networks

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Dissertation

To obtain the degree of Doctor at Maastricht University
on the authority of the Rector Magnificus Prof. dr. Rianne M. Letschert,
in accordance with the decision of the Board of Deans,
to be defended in public on
Wednesday, June 16th 2021 at 10:00

by

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- Невольо! Невольо!...

...

На Неволята

Acknowledgements

It takes a village to raise a child and in my case it took the combined efforts of people from a whole continent to produce a PhD student. Therefore, I want to recognise the role of the key figures who have been directly or indirectly responsible for my ending up here. All mistakes on the way to the completion of the PhD are mine, but many people had a role in my relative successes.

I have to start with my supervisors, whose expertise, critical advice and supervision methods have shaped much of the content of this small book and my approach towards research. Over the years we had many meetings and exchanged hundreds of e-mails. Working with them has taught me patience and perseverance, even in the face of abject failure. I am obstinate and before I go along with something I need to be convinced of its value, plus I often come across as too serious, grumpy or unfriendly and I am an extreme pessimist, so I am sure it was not always very easy or even pleasant to work with me on common projects. The three of them deserve much credit for pulling through. That is why I wish to thank Jean-Jacques Herings for his composure in supervising me and for finding the right time to push me towards independent research. Frank Thuijsman was always receptive to my relentless requests for meetings and could usually connect me with the right people for any technical research problems for which I am truly grateful. Ronald Peeters has accompanied me in my research endeavours since the time I became his assistant during my Research Master. He oversaw my research visit to his new home university which was a very enjoyable and productive time and much of that was due to him. I thank him for his limitless optimism, wise and insightful advice, and for always spindoctoring my ideas and doing his best to convince me they are not all that terrible. I hope to prove to you that the time during our meetings was well spent.

This is the right place to thank the person whose acumen, unyielding commitment to truth and real critical thinking never once let me stray off the path of logic in conversations and debates. Jasmine Maes is a formidable opponent in discussion and the harshest of critics, but has the humility to be an impartial judge of evidence and has always been my unequivocal supporter. To me she was my unofficial fourth supervisor. Sharing an office with her for three years made me a much better thinker, greatly enriched my knowledge of the world and was an invaluable education in and of itself. I have enjoyed her unorthodox sense of humour and have greatly benefited from her lucid reasoning but also from her immense creative talent which produced the cover of this dissertation.

The years of the PhD would not have passed so fast had I not worked in such an enjoyable environment. I would like to thank my former teachers at Maastricht University who later treated me as a colleague, conveniently forgetting all of my embarrassing moments as a student.

Additionally, I would like to thank: the people from TS6, who created a social, enjoyable and relaxing workplace; my fellow PhD students from AE1/MPE with whom I shared laughter, good food and my all-time favourite, cheap gossip; Dominik Karos who was a source of great advice and feedback in time of need; Riccardo Saulle, who became an example of discipline and achievement; Shashwat Khare for his help in organising the MLSE; Anna Zseleva, who did everything she could for my successful integration into the SBE community; Lars Wittrock and David Albrecht for their professional approach to teaching; Elke Lucas for her valuable help and addictively positive attitude. Péter Bayer blazed the trail for my own academic development with his fearless approach to research. I hope I am brave enough to follow in his footsteps. Henrik Zaunbrecher has been one of my main sources of interesting and entertaining news for years. His easygoing attitude to life and willingness to engage in discussions always refreshed me during the drudgery of draft rewriting. Toygar Kerman is the first co-author of mine who does not formally supervise me. I have never laughed so much while writing a paper - bitterly, ironically, at my own failures, but mostly because it was funny. We also finished a paper in record time for my standards. The time spent with Frauke Stehr made my boorish self feel like a sophisticated person during the many cultural events we attended. I am truly grateful that I was able to share the joy of music with a real connoisseur.

I am a generally inert person and I always appreciate it when people consider knowing me to be worth the effort of talking to me. Giang Tran and Raginee Baruah have accompanied and supported me throughout my time in Maastricht. They stoically endured my endless whining and have often given me motivation to go on, even when they did not realise this was the case for which I am deeply grateful.

And last, but not in any way least, I thank my family who supported my decisions in life even when they did not agree with them, who always encouraged me to persevere and who trust in me even when I do not.

January 2021
Haskovo, Bulgaria

Contents

Introduction

Theoretical models are not a precise representation of reality. They help to shed light on specific factors, so that researchers can think about them more clearly and inspect their effects by means of an extended thought experiment. Given the fact that they investigate outcomes under a magnifying glass and in a pure isolated environment, their conclusions should be taken with a grain of salt. Nevertheless, they are important in providing a reference point, a benchmark, an extreme case or a standard by which to judge real life phenomena.

Apart from their theoretical nature, the chapters in this dissertation have in common the fact that they try to study behaviour in the presence of a (social) network. With the increased ability of people in recent years to connect and communicate (virtually or otherwise) there has also been a growing interest in the structures and effects of social networks, how they are shaped by reality and how they help shape reality through their influence on human behaviour. The three chapters of this thesis provide an economic perspective of social networks and the driving forces in them through two distinct lenses, each representing a theoretical extreme of (possible) human behaviour and cognitive sophistication.

The first two chapters consider naive agents who employ decision rules/ heuristics instead of complicated optimisation. They make decisions every period and are assumed to have no foresight or memory of the past which can inform their decision-making. Chapter Three takes the opposite approach and rather than modelling agents with cognitive limitations, it presents highly sophisticated agents, who faultlessly update their beliefs with all available information.

Chapter One has its main focus in the investigation of altruistic and cooperative behaviour. It builds on the literature considering the interplay between altruistic and egoistic agents (Nowak and May, 1992; Ellison, 1993; Fosco and Mengel, 2011; García-Martínez and Vega-Redondo, 2015) and importantly on the public goods game of Eshel et al. (1998) in which the stable states of a local interaction model between non-cooperators and cooperators are investigated. The chapter expands the framework by adding a third possible type of behaviour. Rather than treating all of their neighbours similarly (with either full cooperation or none), the agents can now cooperate with only half of their neighbours, i.e. *partially.* Within this framework, partial cooperation does not coexist with any of the other strategies in an absorbing state and it limits the diffusion of altruistic behaviour in the population.

Chapter Two starts with the basic premise that if people employ heuristics in taking purely economic decisions, it is reasonable to investigate the effects of heuristic decision-making on social interactions and in particular how they shape the resulting (stable) network structures. The chapter relies on the seminal article by Jackson and Wolinsky (1996) introducing pairwise stable networks along with the often cited co-author model. Pairwise stability is the notion that links in a (social) network will exist if both parties agree to that, i.e. the people involved are not worse off *with* the link than *without* it. Taking the results from one of the few laboratory experiments on network formation, Harmsen-van Hout et al. (2016), the chapter examines the pairwise stable networks which result from applying a particular heuristic for allocating the agents' scarce resources amongst their neighbours. The results demonstrate that heuristics can be a decisive factor in social network formation.

Chapter Three considers a sender-receiver game with a single sender and multiple receivers following the pathbreaking article of Kamenica and Gentzkow (2011) and building on the work by Kerman et al. (2020). The sender is an entity which is trying to persuade the receivers by carefully structuring the information they get. The most extreme case of such an entity is a propaganda machine optimising its communication strategy to achieve the most influence. One remarkable feature of the model is that the receivers are aware that they are being targeted in this way. Moreover, they have at their disposal the best tool logic can provide, Bayes' rule, in order to sort out the information they obtain and get an idea about the state of the world. They have knowledge of the exact strategy the sender is going to follow and are capable of faultlessly interpreting the available information. Theoretical models have shown that even under circumstances so heavily skewed in favour of the receivers, they can still be manipulated to an extent according to the sender's desires. Chapter Three addresses the problem of persuading agents in the presence of a network which links their information flow and gives connected agents access to each other's personal signals from the sender. In this way the chapter combines the literature on Bayesian persuasion with the one on networks. The setup significantly increases the difficulty of the sender's persuasion problem compared to the case of no information flow between receivers. However, there are many cases in which the sender's persuasion probability is not affected by the additional information of the receivers' neighbourhoods.

Much of the literature on networks considers standard network structures or networks which have some degree of regularity. Chapters Two and Three are not an exception to this rule, but they attempt to push this boundary wherever possible.

Finally, it is important to make a remark about the methodology used. Besides the theoretical analysis, the first two chapters employ computer simulations to a great extent. This is not necessarily standard in theoretical economic research, while being widely used in other fields. A great deal of the work in this dissertation has been underpinned by an extensive use of software like Mathematica and Matlab not only to produce some of the results but also to check many of the proofs or simply to help in finding and developing the right examples to illustrate the concepts at hand.

Chapter 1

Naïve Imitation and Partial Cooperation in a Local Public Goods Model

1.1 Introduction

A lot of economic thought has been devoted to the factors and systems that are conducive to altruism, so that public goods can be sufficiently financed and the systems in place can be upheld efficiently and sustainably. While theoretical work has identified that altruistic behaviour has slim chances of survival if agents use best responses, many results point to the fact that altruism can be sustained in a local interaction framework where the agents are imitators (Bergstrom and Stark, 1993; Eshel et al., 1998; Young, 1998; Bergstrom, 2002; Levine and Pesendorfer, 2007). A common feature of these models is that they often include two possible strategies for the agents involved: they are either cooperators or non-cooperators, i.e. altruists or egoists (Nowak and May, 1992; Ellison, 1993; Fosco and Mengel, 2011; García-Martínez and Vega-Redondo, 2015). However, there is evidence that at least three general types of people can be distinguished in an organisation: givers, takers and matchers (Grant, 2013). Givers are cooperating altruists, takers are non-cooperating egoists, and matchers strive for a balance in their giving and receiving (Mäthner and Lanwehr, 2017).

Attempting to bring the models closer to observed reality, this chapter adds a third possible strategy to the public goods game of Eshel et al. (1998) by allowing the agents to treat their neighbours differently, cooperating with only some of them, i.e. they exhibit *partial cooperation* and contribute to one of their neighbours and are egoistic to the other one. This reflects the intuition that one social interaction is not necessarily indicative of all others, i.e. altruists

This chapter is co-authored with Jean-Jacques Herings, Ronald Peeters and Frank Thuijsman. It is based on Herings, Peeters, Tenev, and Thuijsman (2019).

may not be altruistic to everyone. Which of the neighbours a partial cooperator cooperates with depends on a probabilistic realization. The agents are situated on a fixed simplified network structure, a circle, and use naïve imitation such that at each period agents imitate the strategy with the highest average payoff in their observed neighbourhood. The range that agents immediately affect with their choices and about which they have full information with regards to obtained payoffs and chosen strategies consists of only their two direct neighbours.

The local interaction intrinsic in the model allows its interpretation as a local public goods model. Local public goods are public goods that can be enjoyed in a particular geographic area. In a broader context, this means that a selection of all agents work together towards a common goal and, if achieved, the goal would benefit everyone in the neighbourhood equally. It is possible to conceptualise the local communities as clubs formed to provide local public goods (Hindriks and Myles, 2013). The concept of local public goods can be used in contexts involving various scales, from considering the interactions of different countries or states on local environmental issues like water pollution, etc. to very small scale interaction involving a few participants, for example, pair programming.[1]

It is difficult to foresee how altruism would fare in such an environment, especially if the partial cooperators have an equal chance of cooperating or not with a particular neighbour. The current chapter addresses this main research question by investigating the absorbing states of the resulting process, their frequency of occurrence, times to absorption, and initial configurations of agents' strategies that are most supportive of outcomes with a high proportion of altruist acts.

One of the main results of this chapter is that the partially cooperative strategy cannot coexist with the other strategies in an absorbing state. This is a non-trivial observation, because even though partial cooperation as specified here stochastically determines the neighbour an agent cooperates with, the total amount of collaboration is deterministic. One could therefore reasonably expect stationary states where only some agents use the partially cooperative strategy. Moreover, theoretically the partially egoistic strategy can be conducive to fully eliminating the egoists from the population and in states in which altruists and egoists coexist, the partial strategy does decrease the number of free-riders. However, overall the partial strategy actually *hinders* the spread of altruism by enabling the egoists' strategy even more than it favours its own proliferation in the population. Somewhat in contrast to other models, due to this underlying dynamic, relatively big groups of altruists at the beginning stage are beneficial for the partial cooperators, who in turn facilitate the egoists. Even though clustering of altruists is assumed to be best for cooperative outcomes, here a middle range of segregation of altruists helps the spread of altruism the most.

[1] This is a way of developing software whereby two programmers work on one work station – one of them writing the code and focusing on the technical aspects of the current code, while the other one reviews the code and considers the strategic direction of the work. The two participants often change roles.

This chapter contributes to the strand of literature on public goods in networks. Since models that rely on a best response dynamic typically produce results dominated by egoistic behaviour (Young, 1998), many models have assumed boundedly rational behaviour, expressed by a variety of decision rules. These span from imitating the strategy with the highest *average* payoff in the observed neighbourhood (Eshel et al., 1998), to imitating the strategy with the highest payoff (Matros, 2012), additionally taking the linking choices of the imitated agents into account (Fosco and Mengel, 2011), etc. These behavioural decision rules imply little oversight of the agents over the game or system that they are part of and little or no knowledge of the available strategies and their potential payoffs.

To elaborate on the latter two contributions, Matros (2012) investigates whether altruism can survive when agents can choose between naïve imitation and myopic best-reply rules. He finds that altruistic behaviour can be sustained if local neighbourhoods are not too small or too large both in short-run and in long-run settings. Moreover, when the agents can use both decision rules, altruism survives in all cases except in those in which the decision rules are introduced into the population randomly. In a setting with naïve imitation and endogenous link formation on a more complex graph, Fosco and Mengel (2011) find that coexistence of cooperators and defectors is possible. However, it results in either their full separation or the cooperators taking a central position in the graph and the defectors being marginalised. Unlike the setting in the current model, not all agents revise their choices simultaneously at every period, but only a random sample of them do so. Moreover, their model includes different information and interaction neighbourhoods and shows that the results are robust towards variations in these neighbourhoods. This is in contrast to Mengel (2009), where under the framework of Eshel et al. (1998), the polymorphic states featuring altruists and egoists are not stochastically stable if the agents use information from beyond their interaction neighbourhood. Mengel (2009) also finds that the co-existence result from Eshel et al. (1998) does not extend to general networks. However, introducing conformist bias (if an action is more popular, the agent is more likely to use it) stabilizes the cooperation results. Also Eshel et al. (1999) and Eshel et al. (2000) find that, even with a bigger size of the information neighbourhood as compared to the interaction neighbourhood, altruism is sustainable. In this study the finding crucially depends on a conservative learning assumption where an individual adopts a specific strategy only if it is used by at least one of his immediate neighbours. Kirchkamp (2000) introduces strategies in the form of two-state automata which can discriminate between specific neighbours (e.g. kin and others) in a spatial model and investigates the outcomes of extensive simulations. The author finds a much larger range of payoffs for which cooperation is sustained in the presence of discriminative strategies than otherwise. Unlike the current model, the distinction between the neighbours is not stochastic.

The chapter is organised as follows. The next section describes the specifics of the model. Afterwards, the absorbing states are outlined analytically and investigated by means of computer simulations. The last section discusses the findings and contains the conclusion.

1.2 The Model

This model extends the model of local public goods provision of Eshel et al. (1998) by adding the possibility for agents to be partially altruistic. Within this extended context it investigates and showcases the dynamics of behaviour when agents follow the naïve heuristic to imitate the strategy that performed on average the best among the strategies observed within their neighbourhoods.

1.2.1 Local Public Goods Game

There are $n \geq 3$ agents/players on a circle and $N = \{1, 2, 3, \ldots, n\}$ denotes the set of agents. Every agent interacts with his two direct neighbours and exhibits either egoistic or altruistic behaviour towards each of them. All agents have three possible strategies at their disposal: the set of strategies is $S = \{A, P, E\}$, where A represents the altruistic strategy, E represents the egoistic strategy, and P represents the partially altruistic strategy.

Every two neighbours share a local public good (LPG). Altruistic contributions to the LPG benefit the contributors and the neighbour with whom they share the good but they also impose costs to the contributors. Agents who behave egoistically do not invest in a particular LPG and thus bear no costs. However, they benefit from the LPG provision if their neighbours have contributed. Altruists (A) contribute to *both* LPGs they can support (i.e., the one they share with their left-hand and the one they share with their right-hand neighbour), while egoists contribute to *neither* of them. The partially altruistic strategy (P) enables agents to be altruistic to only one of their neighbours. Strategy P manifests itself in two possible decisions: L, representing altruistic behaviour towards the left-hand neighbour and egoistic behaviour towards the right-hand neighbour, and R, representing altruistic behaviour towards the right-hand neighbour and egoistic behaviour towards the left-hand neighbour.[2] If a player uses the P strategy, then in every period he chooses each of the possible decisions, L or R, with probability 1/2.

A state $x = (x_1, x_2, \ldots, x_n)$ is given by a profile of strategies, so for all agents $i \in N$ it holds that $x_i \in S$. Hence, the state space is $X = S^n$. The vector $d = (d_1, d_2, \ldots, d_n)$ specifies the decisions for every player in a particular state. Given a player's strategy $x_i \in S$, his

[2] Agents on the circle are numbered in clockwise direction such that agent $i-1$ is left of agent $i \geq 2$. Due to the circular structure agent n is left of agent 1.

decision d_i belongs to the set of decisions $D_i(x_i)$ given by:

$$D_i(x_i) = \begin{cases} \{A\} & \text{if } x_i = A \\ \{L, R\} & \text{if } x_i = P \\ \{E\} & \text{if } x_i = E. \end{cases}$$

The decision space is $D = \{A, L, R, E\}^n$. The feasible realizations in state $x \in X$ belong to $D(x) = \prod_{i \in N} D_i(x_i)$.

Let $\pi_i : D \to [-2c, 2]$ denote the payoff function for agent i, where $c \in (0, 1/4)$ is a fixed constant representing the cost of an altruist act. The function can be decomposed as $\pi_i(d) = b_i(d) - c_i(d)$, given that the benefits for every agent are

$$b_i(d) = \begin{cases} 2 & \text{if } d_{i-1} \in \{R, A\} \text{ and } d_{i+1} \in \{L, A\} \\ 1 & \text{if either } d_{i-1} \in \{R, A\} \text{ or } d_{i+1} \in \{L, A\} \\ 0 & \text{otherwise} \end{cases}$$

and the costs are

$$c_i(d) = \begin{cases} 2c & \text{if } d_i = A \\ c & \text{if } d_i \in \{L, R\} \\ 0 & \text{if } d_i = E. \end{cases}$$

Every interaction (with left-hand or right-hand neighbour) in which the agent is altruistic costs him c and it brings him a benefit of N_A, where $N_A \in \{0, 1, 2\}$ is the number of altruistic acts towards him.[3] For example, if an agent uses strategy A and is surrounded from the left by a decision R and from the right by a decision L, then there are 2 altruist acts by his neighbours towards him, 1 by each of his neighbours, while he commits 2 altruist acts towards his neighbours. Therefore his payoff is $2 - 2c$. Even though this payoff structure renders E the sole survivor in a best response dynamic and the only rationalizable strategy, altruistic behaviour can survive when a naïve imitation rule is adopted.

The partial strategy bridges the two extremes of behaviour following from strategies A and E. In reality people often prefer the middle ground and do not go for the extremes. Moreover, irrespective of the naïveté of an agent, it is reasonable to assume that altruistic behaviour in one interaction would not immediately imply altruism in other interactions, even within the same time frame. Therefore, a partial strategy implemented as a modelling assumption is warranted.

[3]The values for c are within $(0, 1/4)$ because in case $c > 1/4$ altruism becomes too difficult to sustain and even E and A strategies cannot coexist. It follows logically that the more expensive altruism is, the more difficult it would be to sustain. In light of this observation, costs c in the interval $(0, 1/4)$ are the most favourable to altruism and therefore the most interesting to investigate. When $c = 1/4$ there are payoff ties and for this reason the inequality is taken to be strict.

There are many conceivable ways to define a partial strategy. Examples could be a reciprocal partial strategy, which responds to every neighbour with the action he took in the previous period, or a fully stochastic partial strategy characterised by a coin toss every period to determine if one would play an A or E with every one of his neighbours. The current chapter's definition of the partial strategy has several attractive features as a modelling choice. It satisfies history-independence, generates exactly one altruistic act with probability one, and treats the two neighbours symmetrically.

1.2.2 Naïve Imitation Dynamics

The agents interact repeatedly and can re-evaluate their strategies once per period. It is important to stress that by observing the *payoff* of a neighbour an agent can also infer the *strategy* that the neighbour uses. Therefore assuming that every agent has full information about the payoffs of his direct neighbours, he will also know their strategies. The strategies of the agents are based on *naïve imitation* which is described by the following rule: *Every agent considers his direct neighbours' payoffs in the current state and in the next period imitates the strategy yielding the highest average payoff from the ones he observes, including his own.* This does not exclude that the agent plays his current strategy in the next period. It also implies that if an agent and *all* his neighbours use the same strategy the agent will continue employing this strategy. More formally, for some given agent $i \in N$ at a decision profile $d \in D$, let $N_s^i(d) = \{j \in \{i-1, i, i+1\} \mid d_j \in D_j(s)\}$ be the set of neighbours of agent i (including agent i himself) that employ strategy $s \in S$ and define:

$$\mu_s^i(d) = \begin{cases} \displaystyle\sum_{k \in N_s^i(d)} \frac{\pi_k(d)}{|N_s^i(d)|} & \text{if } |N_s^i(d)| \geq 1 \\ -1 & \text{otherwise.} \end{cases}$$

Here $\mu_s^i(d)$ gives the average observed payoff per strategy s for an agent i at decision profile d. If a strategy is not observed in the neighbourhood of the agent, then, for technical reasons, it gets a value of -1, which ensures it will not be implemented. Then the naïve imitation decision rule articulated above can be summarized as $f : D \to X^n$, where for $i \in N$ it holds that $f_i(d) = \text{argmax}_{s \in S}\ \mu_s^i(d)$. The particular payoff functions ensure that the maximum average payoff is uniquely determined.

Next, for $x \in X$, let

$$F(x) = \{y \in X \mid \text{there exists a } d \in D(x) \text{ such that } y = f(d)\}$$

denote the set of states that, starting from x, can be reached with positive probability in one iteration. In the presence of the P strategy, the set $F(x)$ can contain more than one element.

Further, let $F^m(x)$ denote the states that are reached with positive probability from x in m steps (with $F(x) = F^1(x)$).

In the process outlined so far, agents follow a decision rule that requires only information from the current state to fully define the next one – the crucial property for a Markovian process. Let Q_{xy} denote the probability to change from state x to state y after one single iteration of the imitation process. It holds that $Q_{xy} > 0$ if and only if $y = f(d)$ with $d \in D(x)$. The matrix of transition probabilities $[Q_{xy}]_{x,y \in X}$, along with a specification of the initial state $x^0 \in X$, determines a Markov process on the state space X. A set of states $\mathcal{X}$ is absorbing if it is a minimal set of states such that once a state in the set is reached, the Markov process cannot leave this set. In other words, the system cannot transition into another minimal set of states once it has reached this particular set of states (Ross, 2000). The absorbing sets of the current imitation process will be examined in detail in Section 1.3.

If every agent in a state uses the same strategy, this produces a singleton absorbing set. Therefore the model has at least three singleton absorbing sets where all agents use strategies A, P, or E respectively. In the deterministic version of this model where only strategies A and E are possible, once the E strategy appears in a state it can only be isolated in small pockets but will never fully disappear from the population. In contrast, in small group interactions the P strategy in the current model can potentially completely eliminate E strategies from states with a predominance of the A's, further increasing cooperation. Besides that, the introduction of the P strategy gives the model an overarching "rock-paper-scissors" structure, whereby each strategy is doing better against one other strategy when considering interactions of bigger homogeneous groups. Section 1.3 will first address these questions analytically, while Section 2.4 will explore further using simulations.

1.2.3 Preliminaries

A number of definitions are necessary to clarify the content of the next parts.

Definition 1.1 (segments)**.** *For $a, b \in N$, the **segment** $[a, b]$ is the set of agents defined by:*

$$[a,b] = \begin{cases} \{i \in N \mid a \leq i \leq b\} & \text{if } a \leq b \\ \{i \in N \mid a \leq i \text{ or } i \leq b\} & \text{if } a > b. \end{cases}$$

Hence, a segment consists of a number of consecutive agents on the circle, irrespective of their strategies.

Definition 1.2 (strings)**.** *For $k \in N$ and strategy $s \in S$, a ks-**string** of the state $x \in X$ is a segment $[a, b]$ such that the cardinality of $[a, b]$ is equal to k, for every $i \in [a, b]$, $x_i = s$ and if $k < n$ then $x_{a-1} \neq s$ and $x_{b+1} \neq s$.*

In other words, a ks-string is a maximal segment of agents who employ the same strategy in a particular iteration of the imitation process. Based on Definition 1.2, the notation *≥ms-string* is used as shorthand for ks-strings with $k \geq m$. The generic name *s-string* will be used for a ks-string of any length.

For a state $x \in X$, let $p_1(x)$ be the number of $1P$-strings of x and $p_{\geq 2}(x)$ be the number of $\geq 2P$-strings of x. By analogy $e_2(x)$ is the number of $2E$-strings of x, $a_{\geq 2}(x)$ is the number of $\geq 2A$-strings of x, etc. The notation $a(x) = a_{\geq 1}(x)$, $e(x) = e_{\geq 1}(x)$ and $p(x) = p_{\geq 1}(x)$ is used as shorthand for the total number of A-strings, E-strings, and P-strings, respectively.

In the model of Eshel et al. (1998) all agents employ the strategies A and E and all state transitions are deterministic. The following lemma presents one of their results, where absorbing sets consisting of two states with deterministic transitions between them are called **blinkers**.

Lemma 1.1 (Eshel et al., 1998)**.** *Let $x, y \in X$ be such that, for every $i \in N$, $x_i \in \{A, E\}$ and $y \in F(x)$. The set $\{x, y\}$ is a blinker if and only if: (i) $e_1(x) + e_3(x) \geq 1$; (ii) for all $k \geq 4$, $e_k(x) = 0$; (iii) in x the length of an A-string between any two consecutive E-strings is as specified in Table 1.1.*

	(E)	(E,E)	(E,E,E)
(E)	$\geq 5A$	$\geq 4A$	$\geq 3A$
(E,E)		$\geq 3A$	$\geq 3A$
(E,E,E)			$\geq 3A$

Table 1.1: Length of an A-string between E-strings of different lengths within blinkers. (Due to symmetry only half of the table is filled.)

If there is only one E-string in a blinker, then the number of A's counted is the number of A's between one end of the E-string and the other. Note that no $\geq 4E$-strings can exist in a blinker. Blinkers can only occur when $n \geq 6$. An example of a blinker is presented in the next subsection.

1.2.4 Two Detailed Examples

This section provides two examples. The first one constitutes a blinker and the second one illustrates how, given the partial strategy, starting from a specific state more than one absorbing state can be reached.

First, consider the situation with $n = 6$ agents with the state being $(x_1, x_2, x_3, x_4, x_5, x_6) = (A, A, E, A, A, A)$. Consider $y \in F(x)$. Since agent 1 adopts the A strategy and is surrounded by neighbours also adopting the A strategy, he does not experience the use of any other strategy and will continue to play the A strategy: $y_1 = A$. The same applies to agents 5 and 6: $y_5 = y_6 = A$. Both agents 2 and 4 receive a payoff of $1 - 2c$ and observe one A playing neighbour who receives a payoff of $2 - 2c$ and one E playing neighbour who receives a payoff of 2. On average, they find the E strategy to be more rewarding than the A strategy ($2 > 3/2 - 2c$) and both shift to the E strategy: $y_2 = y_4 = E$. Agent 3 sticks to the E strategy as it yields a better payoff than the A strategies of his neighbours ($2 > 1 - 2c$): $y_3 = E$. Hence, as also illustrated in Table 1.2, after one iteration the state changes to $(y_1, y_2, y_3, y_4, y_5, y_6) = (A, E, E, E, A, A)$. From here, agents 3 and 6 stick to their strategies as they do not observe any other strategy: $z_3 = E$ and $z_6 = A$. Agents 1 and 5 stick to the A strategy, since $((1 - 2c) + (2 - 2c))/2 = 3/2 - 2c > 1$: $z \in F(y)$ satisfies $z_1 = z_5 = A$. It is already clear that the E strategy does not spread further, this being a consequence of the boundary A playing agents observing an A player that is fully surrounded by A players. This underlines the importance of having a sufficiently large string of A players in between the E-strings (Table 1.1). Finally, agents 2 and 4 both observe agent 3 being very unsuccessful with the E strategy such that on average the A strategy they observe is more productive ($(1 + 0)/2 = 1/2 < 1 - 2c$) and this leads them back to it: $z_2 = z_4 = A$. The process is back at the starting state: $(z_1, z_2, z_3, z_4, z_5, z_6) = (A, A, E, A, A, A) = (x_1, x_2, x_3, x_4, x_5, x_6)$.

Current state (x)	Payoffs in current state ($\pi(d)$)	Next state (y)
(A, A, E, A, A, A)	$(2 - 2c, 1 - 2c, 2, 1 - 2c, 2 - 2c, 2 - 2c)$	(A, E, E, E, A, A)
(A, E, E, E, A, A)	$(1 - 2c, 1, 0, 1, 1 - 2c, 2 - 2c)$	(A, A, E, A, A, A)

Table 1.2: Example of a blinker. The two segments of the table comprise a blinker for the smallest possible circle length for which blinkers can exist, $n = 6$.

Second, consider the situation with $n = 4$ agents with the state being $(x_1, x_2, x_3, x_4) = (A, P, P, E)$. In this state there are four possible realizations for the pair of P strategies: d equals (A, R, R, E), (A, L, L, E), (A, L, R, E) or (A, R, L, E), each with equal probability. These realizations induce a transition to the next state being (E, P, E, E), (P, P, E, E), (E, A, E, E) or (P, P, P, E), respectively; again, each with equal probability. While from states (E, P, E, E) and (E, A, E, E) the process moves towards the absorbing state (E, E, E, E), from the other two states further continuation of the process again depends on the realizations of the P strategies. Table 1.3 presents the states that can be visited in the process starting from state x. Figure 1.1 illustrates the full process of state transitions and shows that, in each of these states the probability to reach the absorbing state (P, P, P, P) equals 1/4, and that in case the process does not move to this state it will ultimately end up in the egoistic absorbing state. In short, given that the process starts in state x, it evolves to the absorbing state (E, E, E, E) with probability 7/8 and to the absorbing state (P, P, P, P) with probability 1/8.

This illustrates the possibility for the process to reach different absorbing states with different payoff levels: in (P, P, P, P) the average payoff is $1 - c$, while it equals 0 in (E, E, E, E).

Current state	Prob.	Current decisions	Current payoffs	Next state
(A, P, P, E)	1/4	(A, R, R, E)	$(-2c, 1-c, 1-c, 2)$	$(E, P, E, E)^a$
	1/4	(A, L, L, E)	$(1-2c, 2-c, -c, 1)$	(P, P, E, E)
	1/4	(A, L, R, E)	$(1-2c, 1-c, -c, 2)$	$(E, A, E, E)^a$
	1/4	(A, R, L, E)	$(-2c, 2-c, 1-c, 1)$	(P, P, P, E)
(P, P, E, E)	1/4	(R, R, E, E)	$(-c, 1-c, 1, 0)$	$(P, E, P, E)^a$
	1/4	(L, L, E, E)	$(1-c, -c, 0, 1)$	$(E, P, E, P)^a$
	1/4	(L, R, E, E)	$(-c, -c, 1, 1)$	(E, E, E, E)
	1/4	(R, L, E, E)	$(1-c, 1-c, 0, 0)$	(P, P, P, P)
(P, P, P, E)	1/8	(L, L, L, E)	$(1-c, 1-c, -c, 1)$	$(E, P, E, E)^a$
	1/8	(L, L, R, E)	$(1-c, -c, -c, 2)$	$(E, P, E, E)^a$
	1/8	(L, R, L, E)	$(-c, 1-c, 1-c, 1)$	$(E, P, E, E)^a$
	1/8	(R, L, L, E)	$(1-c, 2-c, -c, 0)$	(P, P, P, P)
	1/8	(L, R, R, E)	$(-c, -c, 1-c, 2)$	$(E, P, E, E)^a$
	1/8	(R, R, L, E)	$(-c, 2-c, 1-c, 0)$	(P, P, P, P)
	1/8	(R, L, R, E)	$(1-c, 1-c, -c, 1)$	$(E, P, E, E)^a$
	1/8	(R, R, R, E)	$(-c, 1-c, 1-c, 1)$	$(E, P, E, E)^a$

Table 1.3: Example of the development of the process. It reaches two types of absorbing states, given a specific starting state. The states marked with an ^{a} reach absorption in the following period with probability 1.

1.3 Theoretical Results

This section describes the different absorbing sets that the process can reach. There are two general types of absorbing sets that will be considered separately: stationary states (singleton absorbing sets) and non-singleton absorbing sets. After characterizing the two types of absorbing sets, the section continues with some further analytical observations.

Proposition 1.1. *A state $x \in X$ is a stationary state if and only if: (i) all agents play the same strategy; or (ii) $a_{\leq 2}(x) = 0$, $a_{\geq 3}(x) \geq 1$, $e_1(x) = 0$, $e_2(x) \geq 1$, $e_{\geq 3}(x) = 0$ and $p(x) = 0$.*

Proof. Eshel et al. (1998) demonstrate that states consisting of only A's, only E's or states which consist of 2E-strings and ≥3A-strings are stationary. Introducing the P strategy does not affect this result. What is left to prove is that in the case of three possible strategies for

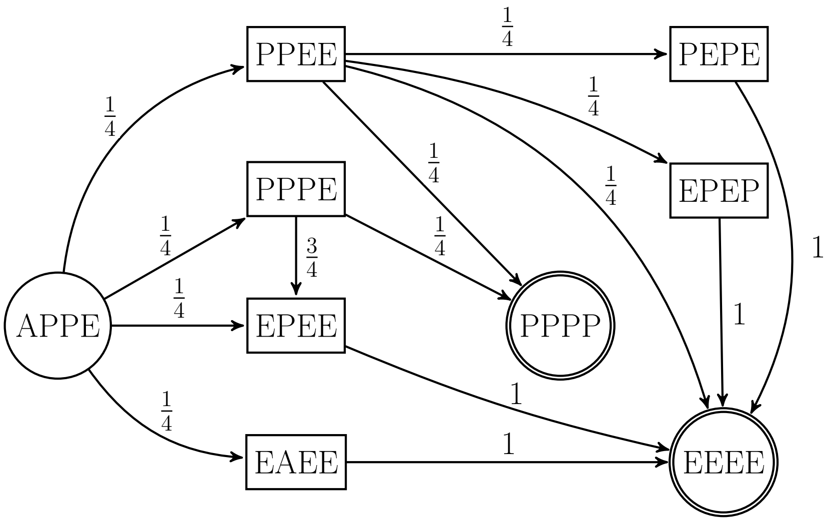


Figure 1.1: Example of the development of the imitation dynamics starting from a state (A, P, P, E) in the circle on the left and reaching one of two possible absorbing sets (in double circles) within 2 or 3 iterations. The transition probabilities between the different states are given above the arrows pointing in the direction of the transitions.

the agents, the only additional stationary state is the one where all agents play the strategy P. In this case all agents are surrounded by the same strategy P and no agent will change his strategy according to the naïve imitation rule. Hence a state with only P strategies is also a stationary state. It remains to be shown that there are no stationary states in which some but not all players use strategy P.

Let $x \in X$ be a state such that $x_i = P$ for some $i \in N$ and at least one of his immediate neighbours uses strategy A or E. Let $y \in F(x)$ be a state reached from x in one iteration.

Case 1.3.1: $(x_1, x_2) = (E, P)$
Take $d \in D(x)$ such that $d_2 = L$. It follows that $\pi_1(d) \geq 1$ and $\pi_2(d) \in \{-c, 1-c\}$. There are three possibilities for the strategies of the third agent: (i) $x_3 = E$: $\pi_2(d) = -c$ and $\pi_3(d) \geq 0$. (ii) $x_3 = P$: $\pi_3(d) \leq 1 - c$. (iii) $x_3 = A$: $\pi_3(d) \leq 1 - 2c$. For all possibilities it holds that $y_2 = E \neq x_2$.

Case 1.3.2: $(x_1, x_2) = (P, A)$
For every $d \in D(x)$, $\pi_1(d) \geq 1 - c$. Two subcases can be distinguished:

(i) $x_3 \in \{E, P\}$: If $d_1 = L$, then $\pi_2(d) \leq 1 - 2c$ and $\pi_3(d) \geq 1 - c$, so $x_2 \neq y_2 \in \{E, P\}$.

(ii) $x_3 = A$: In case $n = 3$, $y = (P, P, P)$ for every $d \in D(x)$. Next consider $n \geq 4$. Two cases are distinguished, since if $d_0 = E$ this is identical to Case 1.3.1(iii).

  (a) $d_0 = L$: Take $d_1 = R$, then $\pi_0(d), \pi_1(d) \leq 1 - c$, $\pi_2(d) = 2 - 2c$ and $y_1 = A \neq x_1$.

  (b) $d_0 \in \{R, A\}$. It holds that $\pi_1(d) = 2 - c$, $\pi_2(d), \pi_3(d) \leq 2 - 2c$ and $y_2 = P \neq x_2$.

Therefore, with positive probability $d \in D(x)$ is such that $y(d) \neq x$, so x is not a stationary state. □

To approach the non-singleton absorbing sets with the right tools, some terminology is introduced in Definitions 1.3–1.8 below. While the statements of the definitions seem rather complex, they describe natural concepts for which the reader can reliably use their intuition. For ease of exposition, the examples from the previous section will be used to illustrate these definitions afterwards.

Definition 1.3 (splitting)**.** *For $k \in \{1, 2, \ldots, n-1\}$, $s \in \{A, P, E\}$ and a state $x \in X$ that transitions into a state $y \in F(x)$, a ks-string $[a, b]$ of x* ***splits*** *if at least one of the following three conditions holds:*

(i) for $k \geq 2$, at least one of the following two conditions holds:

(a) $y_{b-1} = y_{b+1} = s$ and $y_b \neq s$;

(b) $y_{a-1} = y_{a+1} = s$ and $y_a \neq s$;

(ii) for $k = 2$, $y_{a-1} = y_{b+1} = s$, $y_a \neq s$ and $y_b \neq s$;

(iii) for $k = 1$, $y_{a-1} = y_{a+1} = s$ and $y_a \neq s$.

Definition 1.4 (growing and shrinking)**.**

(i) For $k \in \{1, 2, \ldots, n-1\}$, $s \in \{A, P, E\}$ and a state $x \in X$ that transitions into a state $y \in F(x)$, a ks-string $[a, b]$ of x ***grows*** *if for every $i \in [a, b]$, $y_i = s$ and ($y_{a-1} = s$ or $y_{b+1} = s$).*

(ii) For $k \in \{2, 3, \ldots, n-1\}$, $s \in \{A, P, E\}$ and a state $x \in X$ that transitions into a state $y \in F(x)$, a ks-string $[a, b]$ of x ***shrinks*** *if $y_{a-1} \neq s$, $y_{b+1} \neq s$ and ($y_a \neq s$ or $y_b \neq s$).*

Remark: Under the imitation process, if the whole circle is one ns-string it can neither grow any further, nor can it shrink, because no agent will change his strategy.

Definition 1.5 (disappearing)**.** *For $k \in \{1, 2\}$, $s \in \{A, P, E\}$ and a state $x \in X$ that transitions into a state $y \in F(x)$, a ks-string $[a, b]$ of x* ***disappears*** *if for every $i \in [a, b]$, $y_i \neq s$ and: (i) if $y_{a-1} = s$, then $y_{a-2} = s$ or (ii) if $y_{b+1} = s$, then $y_{b+2} = s$.*

It is worth noting that if in Definition 1.5 $y_{a-1} \neq s$ and $y_{b+1} \neq s$, then for every $i \in [a-1, b+1]$, $y_i \neq s$.

The example with four agents in the previous section can illustrate the concepts that have been introduced so far. For instance, between (A, P, P, E) and (E, P, E, E) the $2P$-string shrinks in length to a $1P$-string; the $1E$-string grows in length to a $3E$-string; the $1A$-string disappears.

To analyse the absorbing sets of the imitation process, it is convenient to introduce a pre-order on the set of states X. This motivates the following definition of improvement.

Definition 1.6 (improvement)**.** *For $x, y \in X$, state y **improves** upon x, denoted by $y \succ x$, if one of the following two conditions holds:*

(i) $p(y) < p(x)$

(ii) $p(y) = p(x)$ *and* $[p_n(y), p_{n-1}(y), \ldots, p_1(y)] >_{lex} [p_n(x), p_{n-1}(x), \ldots, p_1(x)]$.

In other words, a state $y \in X$ is an improvement upon a state $x \in X$ if it has a smaller overall number of P-strings or, if it has the same number of P-strings, then it has more longer P-strings in a lexicographic sense. This is considered an improvement because it implies that the state y is closer to an absorbing state with only P's or to an absorbing state with no P's.

Decisions of agents within a P-string which cause a specific P-string not to split will be referred to as "no-splitting decisions". These no-splitting decisions are provided explicitly by means of Definition 1.7.

Definition 1.7 (no splitting)**.** *Take a state $x \in X$ and $i \in N$. The **no-splitting decision** d_i^* of agent i at x is defined by:*

(i) if i belongs to a $1P$-string, then $d_i^* = R$ if $x_{i-1} = E$ and $x_{i+1} = A$, and $d_i^* = L$ otherwise;

(ii) if i belongs to a $\geq 2P$-string $[a, b]$, then $d_i^* = R$ if $i \in \{a, b-1\}$, and $d_i^* = L$ otherwise;

(iii) if $x_i \in \{A, E\}$, then d_i^* is trivially defined as $d_i^* = x_i$.

In other words, the no-splitting decisions for a $2P$-string are $(d_1^*, d_2^*) = (R, L)$; for $3P$-strings they are $(d_1^*, d_2^*, d_3^*) = (R, R, L)$, for $4P$-strings they are $(d_1^*, d_2^*, d_3^*, d_4^*) = (R, L, R, L)$, etc. The next lemma states that the no-splitting decisions indeed result in no splitting of a specific P-string.

Lemma 1.2. *Let $[a, a+k-1]$ be a kP-string of the state $x \in X$, $d \in D(x)$ be such that, for every $i \in [a, a+k-1]$, $d_i = d_i^*$ and $y = f(d)$. Then $[a, a+k-1]$ **does not split**.*

Proof. See Appendix A. □

If all agents take the no-splitting decision, then the number of P-strings does not increase in the next iteration of the imitation process. Moreover, if a particular P-string disappears whereas all agents outside the P-string take the no-splitting decision, then the number of P-string has gone down in this iteration of the imitation process. To formalize this, the additional notion of a P-section is needed.

Definition 1.8 (P-section)**.** *For $x \in X$, let $\mathcal{P}(x) = \{[a_1, b_1], [a_2, b_2], \ldots, [a_r, b_r]\}$ be the collection of P-strings of x with $a_1 < a_2 < \ldots < a_r$ and $r = p(x)$. The set $J \subseteq N$ is called a P-**section** of size $k \in \{1, \ldots, p(x)\}$ of state x if there is $q \in \{1, \ldots, r\}$ such that $J = [a_q - 1, b_{q+k-1} + 1]$.*

Finally, for $x \in X$, $p_{[a,b]}(x)$ is used to denote the number of P-strings intersecting the interval $[a, b]$.

Lemma 1.3 states that if all agents in a P-section take the no-splitting decision, then the number of P-strings in this section will not go up.

Lemma 1.3. *Let $x \in X$ and J be a P-section of size k of the state x. Let $d \in D(x)$ be such that, for every $j \in J$, $d_j = d_j^*$. Then for $y = f(d)$ it holds that $p_J(y) \leq p_J(x) = k$. Moreover, if $k = p(x) - 1$ and the P-string in $N \setminus J$ disappears, then $p(y) \leq p(x) - 1$.*

Proof. By Lemma 1.2 it holds that none of the P-strings in J splits. The definition of splitting implies that if $[a, b] \subset J$ is a P-string of state x, then y has at most one P-string intersecting the interval $[a - 1, b + 1]$. It follows that $p_J(y) \leq p_J(x)$.

Assume $k = p(x) - 1$ and the P-string in $N \setminus J$, say $[a_q, b_q]$, disappears. The result follows immediately if $y_{a_q - 1} \neq P$ and $y_{b_q + 1} \neq P$. If $y_{a_q - 1} = P$, then by the definition of disappearing it holds that $y_{a_q - 2} = P$. Since $[a_{q-1}, b_{q-1}]$ does not split, it follows that $a_q - 1$ and $a_q - 2$ are part of the unique P-string intersecting $[a_{q-1} - 1, b_{q-1} + 1]$. It follows that $p(y) = p_J(y) \leq p_J(x) = p(x) - 1$. A symmetric argument applies if $y_{b_q + 1} = P$. □

The inequalities in Lemma 1.3 are strict if a P-string in J disappears or if two or more P-strings in J merge.

The next lemma makes clear that at states where some but not all agents use strategy P, it is always possible to find an improvement in a finite number of iterations.

Lemma 1.4. *Let $k \in \{1, \ldots, n - 1\}$ be the cardinality of the longest P-string of state $x \in X$. Then there is $m \in \{1, \ldots, \lfloor k/2 \rfloor + 2\}$ and $y \in F^m(x)$ such that $y \succ x$.*

Proof. Let $[a, b]$ be any of the longest P-strings of x.

If $k = 1$ and $x_{a-2}, x_{a+2} \in \{A, E\}$, then it holds by Lemma A.1 in Appendix A.2 that there is $y \in F(x)$ such that $p(y) \leq p(x)$ and y contains a $2P$-string or there is $y \in F(x) \cup F^2(x)$ such that $p(y) \leq p(x) - 1$. Since $[a, b]$ is one of the longest P-strings of x, it follows in all cases that $y \succ x$.

If $k = 1$ and $x_{a-2} = P$ or $x_{a+2} = P$, then it holds by Lemma A.2 in Appendix A.2 that there $y \in F(x) \cup F^2(x)$ such that $p(y) \leq p(x)$ and y contains a $\geq 2P$-string or there is $y \in F(x) \cup F^2(x) \cup F^3(x)$ such that $p(y) \leq p(x) - 1$. Since $[a, b]$ is one of the longest P-strings of x, it follows in all cases that $y \succ x$.

If $k = 2$, then it holds by Lemma A.3 in Appendix A.2 that there is $y \in F(x) \cup F^2(x)$ such that $p(y) \leq p(x)$ and y contains a $\geq 3P$-string or there is $y \in F(x)$ such that $p(y) \leq p(x) - 1$. Since $[a, b]$ is one of the longest P-strings of x, it follows in all cases that $y \succ x$.

If $k = 3$, then it holds by Lemma A.4 in Appendix A.2 that there is $y \in F(x) \cup F^2(x)$ such that $p(y) \leq p(x)$ and y contains a $\geq 4P$-string or there is $y \in F^2(x) \cup F^3(x)$ such that $p(y) \leq p(x) - 1$. Since $[a, b]$ is one of the longest P-strings of x, it follows in all cases that $y \succ x$.

If $k \geq 4$, then it holds by Lemma A.5 in Appendix A.2 that there is $y \in F(x) \cup F^2(x) \cup F^3(x)$ such that $p(y) \leq p(x)$ and y contains a $\geq (k+1)P$-string or k is even and there is $y \in F^{k/2}(x) \cup F^{k/2+1}(x) \cup F^{k/2+2}(x)$ such that $p(y) \leq p(x) - 1$ or k is odd and there is $y \in F^{(k+1)/2}(x) \cup F^{(k+3)/2}(x)$ such that $p(y) \leq p(x) - 1$. Since $[a, b]$ is one of the longest P-strings of x, it follows in all cases that $y \succ x$. □

The proofs of these lemmas involve constructing explicit paths to improvement for every possible case; that is, all P-strings of the initial state are traced to eventual disappearance or growth (also through merging with other P-strings) so that after a number of iterations the state has either only P's or no P's at all. Proposition 1.2 then concludes that the only non-singleton absorbing sets of the model are the ones in Lemma 1.1.

Proposition 1.2. *A set of states is a non-singleton absorbing set if and only if it is a blinker as presented in Lemma 1.1.*

Proof. Let $x \in X$ be any state which contains the strategy P and at least one other strategy. By Lemma 1.4 there exists $m^1 \in \mathbb{N}$ and $y \in F^{m^1}(x)$ such that $y \succ x$. Either y contains no P's at all, after which the results follow from Eshel et al. (1998), or y is a stationary state with only P's (Proposition 1.1), or we can apply Lemma 1.4 again and there exists $m^2 \in \mathbb{N}$ and $z \in F^{m^2}(y)$ such that $z \succ y$. Since the set of states is finite and $\succ$ is a pre-order, we reach a state that contains no P's at all or a stationary state with only P's in a finite number of steps. □

The section concludes with a few analytical results, complementing the simulations in the following section. The aim of these results is to show the relative advantage some strategies have over others under specific circumstances. The first series of observations deals with three specific structures of states, consisting of only $1s$-strings, for $s \in \{A, P, E\}$:

(i) Let $n \geq 4$ be even and let x be such that all odd agents take the same strategy A and all even agents take strategy P. Then, all strategy A agents will adopt strategy P and all strategy P agents will continue playing P.

(ii) Let $n \geq 4$ be even and let x be such that all odd agents take the same strategy E and all even agents take strategy P. Then, all strategy P agents will adopt strategy E and all strategy E agents will continue playing E.

(iii) Let $n \geq 3$ be divisible by three and let x be such that players alternately play A, P and E (i.e., $x = \cdots APEAPE \cdots$). Then, all agents will adopt strategy E in the next period.

The second series of observations deals with longer strings and concerns what happens at the borders of these strings, via two cases: one where a P-string meets an A-string (Case 1.3.3) and one where a P-string meets an E-string (Case 1.3.4).

Case 1.3.3: $n \geq 4$ and $(x_1, x_2, x_3, x_4) = (P, P, A, A)$
Let $d \in D(x)$ and $y = f(d)$. If $(d_1, d_2) = (R, R)$, then $(y_2, y_3) = (A, P)$. If $(d_1, d_2) = (R, L)$, then $(y_2, y_3) = (P, P)$. If $(d_1, d_2) = (L, R)$, then $(y_2, y_3) = (A, A)$. If $(d_1, d_2) = (L, L)$, then $(y_2, y_3) \in \{(P, A), (P, P)\}$.

Case 1.3.4: $n \geq 4$ and $(x_1, x_2, x_3, x_4) = (P, P, E, E)$
Let $d \in D(x)$ and $y = f(d)$. If $(d_1, d_2) = (R, R)$, then $(y_2, y_3) \in \{(E, P), (E, E)\}$. If $(d_1, d_2) = (R, L)$, then $(y_2, y_3) = (P, P)$. If $(d_1, d_2) = (L, R)$, then $(y_2, y_3) = (E, E)$. If $(d_1, d_2) = (L, L)$, then $(y_2, y_3) = (P, E)$.

While this is certainly not an exhaustive list of all possibilities, some trends can be observed, such as: when A's are pitted against the P's the P strategy is more likely to propagate ((i) and Case 1.3.3). In contrast, when P's are facing E's, the E's are more likely to take over ((ii) and Case 1.3.4). Finally, Eshel et al. (1998) show that many clustered A's can limit the E's to small pockets of length 2 on average. This gives a general "rock-paper-scissors" overarching structure of the setup in which any of the strategies can be defeated by one of the other strategies. This broad observation proves to be important when analysing the computer simulations in the next section.

1.4 Simulation Results

This section presents the results from the computer simulations of the model. Several characteristics of the initial states have been investigated to establish which of them influence the probability of a particular absorbing set being reached. Every simulation run started with selecting an initial state based on specific parameters, including the length of the circle n, the probability of using the P strategy and the probability of segregation for the strategies. Next, the decision rule was implemented until absorption was reached. For every choice of parameters the model has been simulated 10,000 times. The graphs in this section plot the averages of the specific outcome variables over these 10,000 runs. The situation where initially each agent plays one of the three strategies by uniform random choice is used as default and applies unless otherwise stated.

1.4.1 Varying the Probability the Agents Start with the P Strategy

Let ρ denote the probability by which agents use strategy P in the starting state. In the default case this probability is 1/3. In the extreme case when $\rho = 0$ the setup is equivalent to the model by Eshel et al. (1998) and only A and E strategies appear in the starting state. Four other intermediate options are considered: $\rho \in \{1/6, 1/2, 2/3, 5/6\}$. In all cases the probabilities for A and E in the initial state were kept equal to each other.

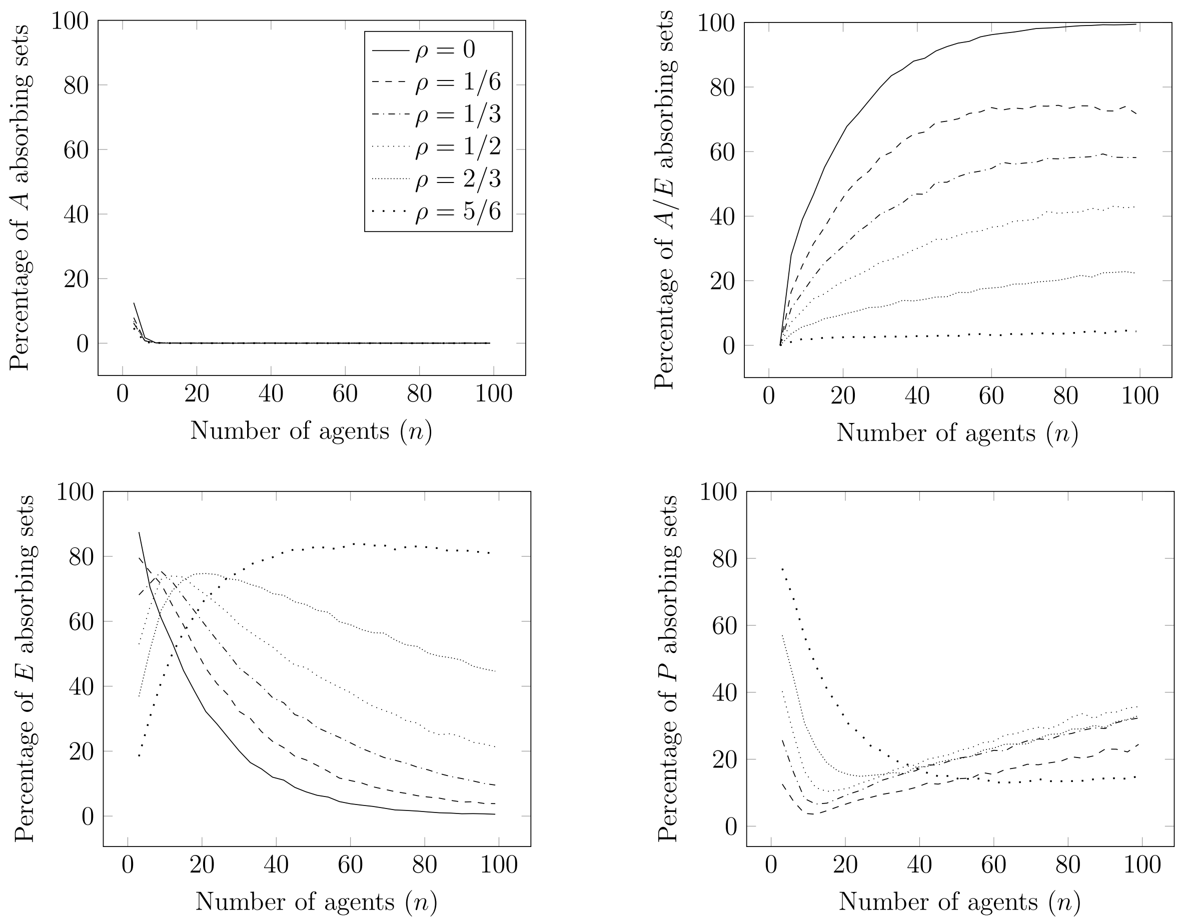


Figure 1.2: Percentages of specific absorbing sets resulting for different numbers of agents, varying the probability of agents starting with the P strategy.

Absorbing sets Figure 1.2 presents the frequencies with which every possible absorbing set is reached. It is immediately obvious that the probability that an only A stationary state appears decreases sharply when n increases (top-left graph). This is due to the fact that, apart from starting at this absorbing state, it can only be reached if first all E's disappear after which the remaining P's are driven out by the A's. However, the probability for this development is very low because strategy A does worse on average when pitted against strategy P. The percentage of A/E absorbing sets increases steadily as n increases for any value of ρ

(top-right graph). This happens at the expense of the E stationary states, which follow the opposite pattern. The A/E absorbing sets are negatively affected by an increased presence of the P strategy, while the E's benefit from that (bottom-left graph). At first glance it is unexpected that the P stationary state is not positively influenced by more P's in the initial state (bottom-right graph). However, this can be explained by them being taken over by E's. It is interesting to note that increasing ρ also increases the probability of ending in a P stationary state only for small n, while for bigger n increasing ρ works in the same way only up to about $\rho = 1/2$. After that, increasing ρ further clearly decreases the probability of ending in a P absorbing state, which is consistent with the explanation above. On a bigger circle, the chances that some E-string succeeds overturning the P's get bigger.

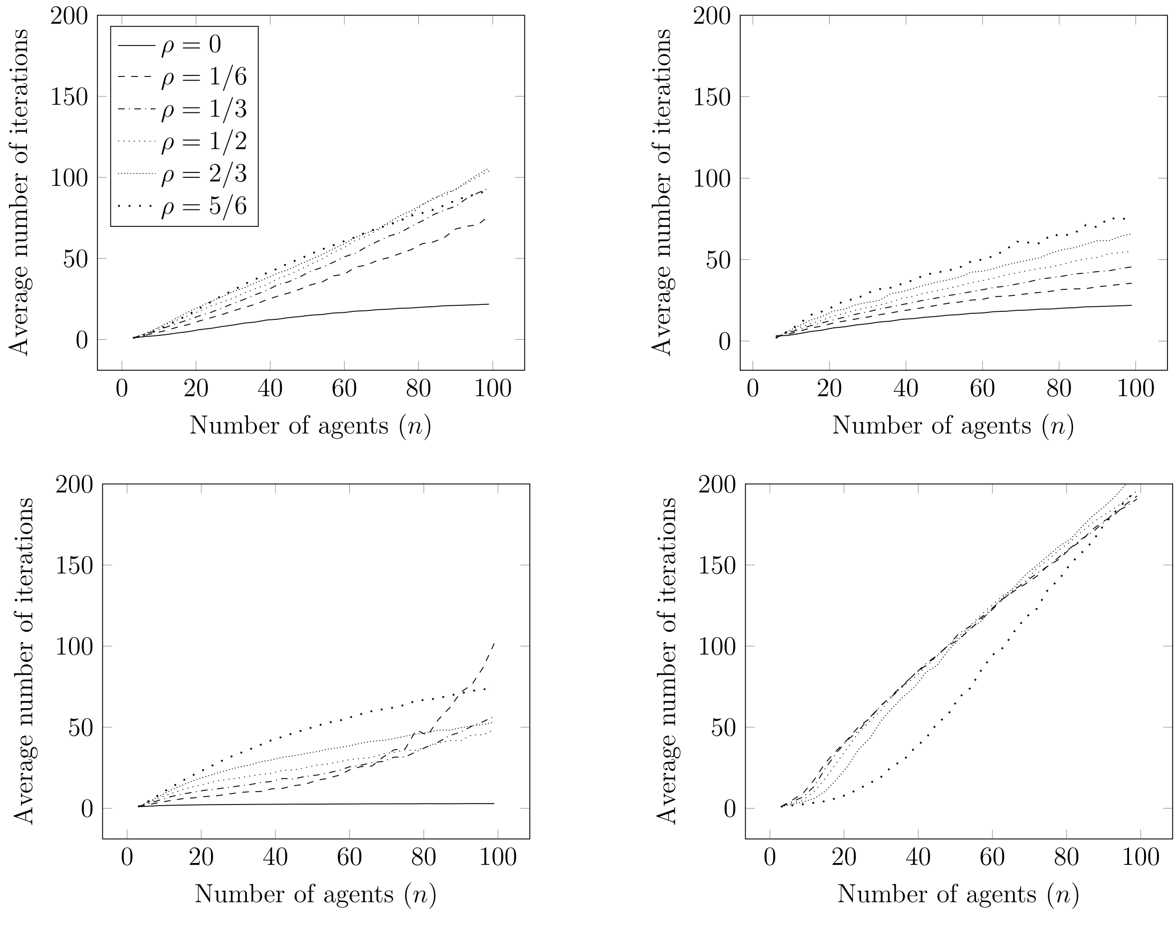


Figure 1.3: Average number of iterations to absorption to any absorbing state (top-left), to A/E states (top-right), to E states (bottom-left), and to P states (bottom-right).

Times to absorption Figure 1.3 presents the absorption times for the process in general and conditional on a specific absorbing set being reached. Even the sets which take most iterations to reach do not require a number of iterations that exceeds 2.5 times the length of the circle. The disparate patterns of the absorption times are due to the various ways in which different absorbing states are generally reached. To get to an A/E set, all P's have to disappear. While in general the P's could help eliminate all E strategies, leaving even one E before the P's themselves are removed suffices to reach an A/E set. Absorbing to a state containing only P's always takes the most iterations, because P-strings can fluctuate between growing and shrinking much longer when surrounded by A- or E-strings than strings of any of the two other strategies can when pitted against each other. Figure 1.3 does not feature absorbing times to A states because they are increasingly rare as an occurrence and therefore susceptible to greater influence of outlier absorption times (see top-left graph of Figure 1.2).

It is noteworthy that even for higher values of ρ the average absorption times are almost linear. Having more P's at the outset causes absorption to A/E states to take increasingly longer (top-right), while the opposite effect is observed in the absorption to the E stationary state (bottom-left).[4] This is consistent with P's impeding the A's from spreading fast, while being conducive to the spread of E's.

Strategies in the absorbing sets Figure 1.4 presents the collapsed results from Figure 1.2 divided by the three strategies rather than by absorbing sets. The results for the A-strategy are quite intuitive because, given the relative advantage that the P strategy has over the A strategy, increasing the overall number of P's in the initial state naturally leads to fewer A's in the absorbing states. Nevertheless, a circle with a higher number of agents always increases the probability that the altruists will increase in number in the absorbing states. The reverse of this is apparent for the P-strategy. The initially surprising dip as the number of P's in the initial states increases can be explained by the fact that small numbers of P strategies, even if densely distributed, can easily be overpowered by neighbouring E's. This explanation is corroborated by the data presented for the E-strategy, showing that E's thrive when the initial number of P's increases. The downward trend in the number of E's in absorbing states as n increases can therefore be explained by a prevalence of the effect of increasing the number of agents. This is due to the fact that on a larger circle a string of E's has a higher chance of colliding with a $\geq 3A$-string, in which case the E's inevitably shrink.

Efficiency The left graph in Figure 1.5 presents the average numbers of altruist acts as a proportion of the possible maximum number of altruist acts in the absorbing states. Each agent can employ two altruistic acts: one for each of the two neighbours. Strategy A counts for two altruist acts, strategy P for one, and strategy E for zero. Since this number reflects

[4] The flat line for $\rho = 0$ in the graph for absorption to E states (bottom-left) is due to the fact that when there are no P's, the setup is very similar to the neat example in the previous section and this explains the short absorption times of about 1–2 iterations. The line corresponding to the $\rho = 1/6$ in the same graph behaves somewhat erratically at first glance. However, this is due to the fact that there are very few observations in this range.

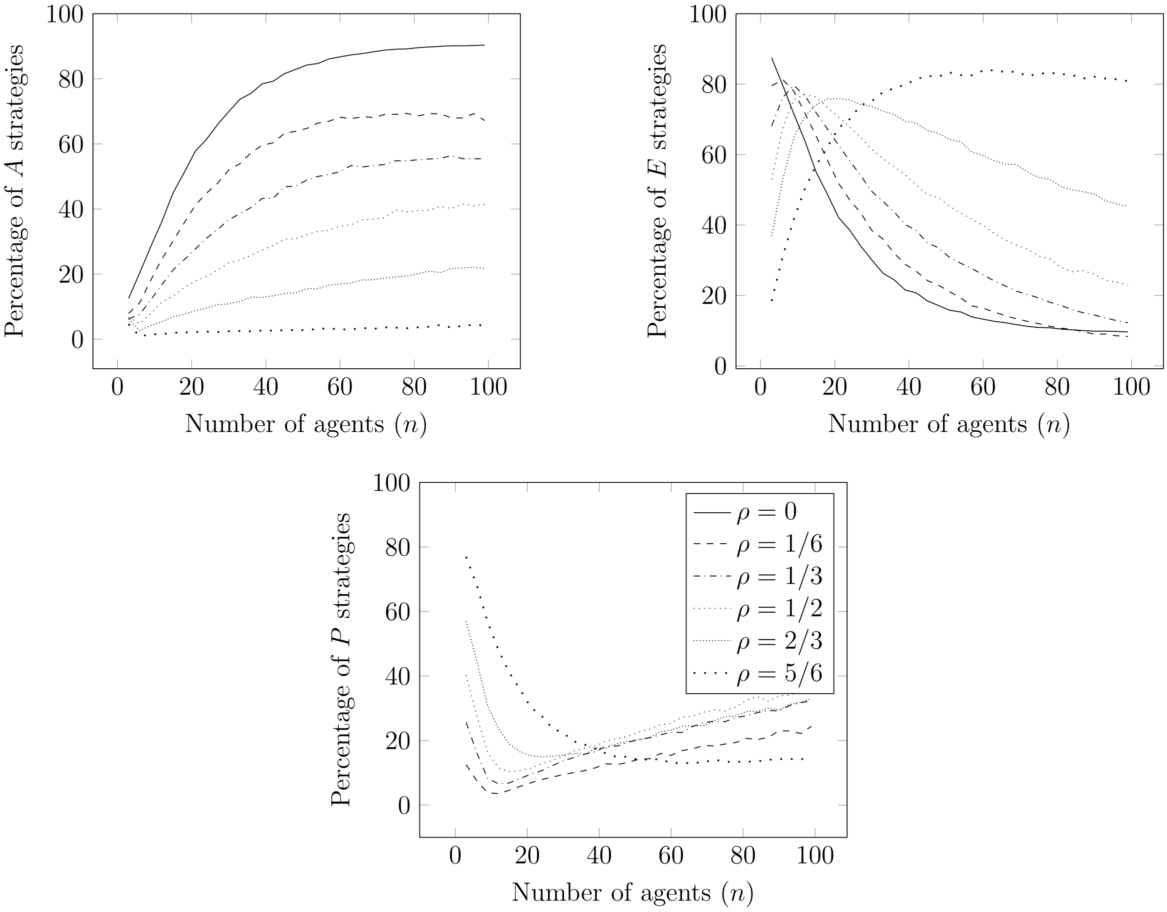


Figure 1.4: Percentages of specific strategies in the absorbing states resulting for different numbers of agents, varying the probability of agents starting with the P strategy.

the total value that is created in absorbing states, it measures efficiency.

Within the current setup the average percentage of altruist acts in the starting state equals 50% for all values of ρ. The graph shows that while efficiency improves with a bigger number of agents on the circle, the increased prevalence of the P strategy in the starting state actually counters this effect. However, in states in which A and E coexist, P marginally decreases the number of free-riding E's (right graph). This happens because P is the strategy which can fully dispose of E's. In the model with only A and E strategies the A's can only significantly shrink long strings of E's, but once there is an E strategy in the initial state it can never fully disappear, i.e. it can never absorb to a stationary state of only A's.

1.4.2 Development of the Process

Using the data gathered over 10,000 simulation runs for a network with $n = 99$ agents under default starting conditions for the strategies, the left graph of Figure 1.6 presents the development of the distribution over strategies used by the agents throughout the process from

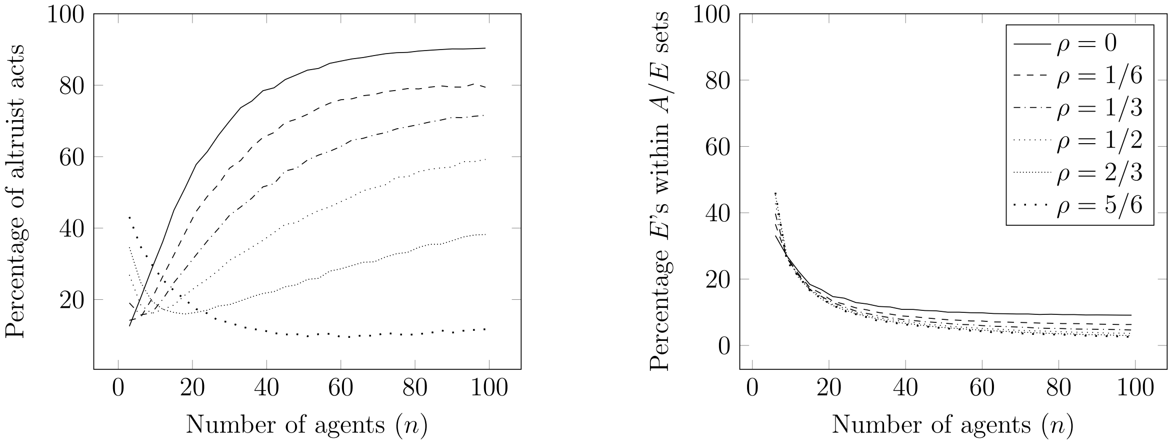


Figure 1.5: Efficiency as measured by the number of altruist acts (left) and percentage of E strategies within A/E absorbing sets (right).

start to absorption, irrespective of the particular absorbing set that is reached. Note that in order to disentangle short-run, medium-run and long-run effects, the iterations on the x-axis follow a square root scale. By default, in the starting state all three strategies have an equal probability of occurrence. The graph ends at the 469$^{\text{th}}$ iteration, when the longest simulation run reached absorption. Information on the absorbing states that were reached earlier is taken into account when deriving the percentages at later iterations. The distribution of absorption times in the right graph explains the overall smoothness after the 200$^{\text{th}}$ iteration, with very few simulations reaching absorption after this number of iterations.

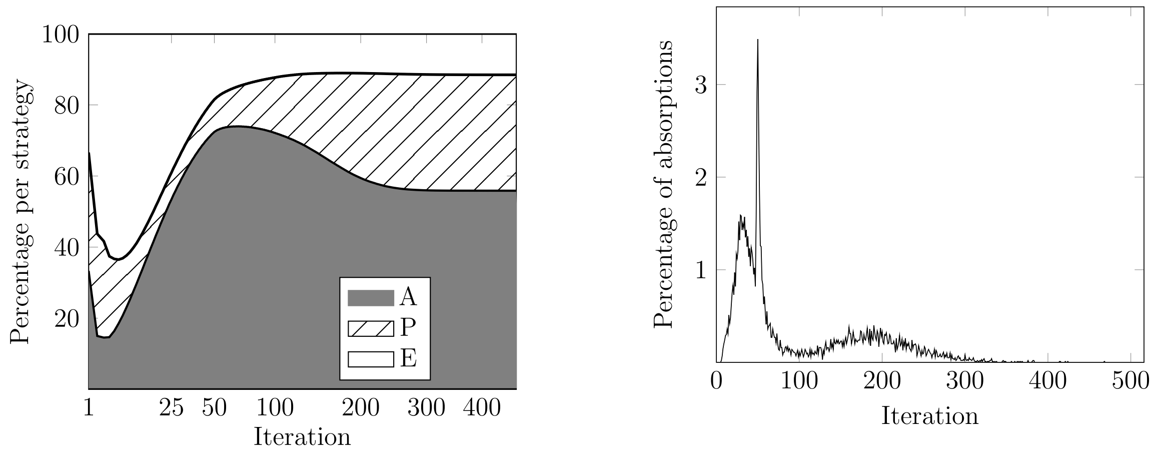


Figure 1.6: The left graph presents the development of the distribution over strategies used by the agents throughout the process from start to absorption. The right graph presents for each iteration the probability of absorption. Both graphs are based on $n = 99$ agents and 10,000 simulation runs.

A striking feature of the distribution over strategies presented in the left graph is the sharp decrease in A and P strategies in the first few periods. Strategy A suffers a great loss in numbers from the first to the second iteration, which is consistent with strategy E replacing A and on occasion reaching an all-E absorbing state when the A's are not clustered. After segregated groups of A's have been formed they are able to overpower the E's, after which only small highly resistant pockets of E's are left within lengthy A-strings. At this point the relatively dormant P's that are left in states which have not reached absorption can take advantage of the A's. This development is also suggested by the distribution over absorption times (right graph), which exhibits peaks at roughly the average absorption times for the three strategies as suggested by Figure 1.3, namely at around period 40 for A/E sets, around 50 for E states and around 200 for P states.

1.4.3 Varying the Probability of Segregation

One of the characteristics used to analyse the initial states and how they influence the probability of reaching a particular absorbing set is the index of segregation introduced by Ballester and Vorsatz (2014). This measure, defined for general networks, captures the likelihood that a node of a certain type (in this case, the agent holding a particular strategy) ends his random walk along the links of the network at a node of the same type (in this case, an agent holding the same strategy), where the walk could be terminated at every step with an exogenously given probability.[5]

A gradual increase in the probability of segregation of all three strategies is simulated by increasing the probability (denoted here by α) that an agent with one strategy would be followed by an agent with the same strategy when constructing the starting state. That is, after the first agent is seeded with one of the strategies, every following agent is assigned the same strategy as the previous agent with probability α and each of the other strategies with probability $\frac{1}{2}(1-\alpha)$. The value of α is varied between $\alpha = 0$ (i.e., agents with the same strategy cannot be next to each other, except possibly the first and last agent) to $\alpha = 5/6$ in steps of $1/6$. Figure 1.7 presents the distributions over strategies in the absorbing state when designing the starting state as just described, for the various values of α.

The figure shows that that increasing the probability of segregation increases the chances of the P strategy to take over the circle and overall decreases this probability for the E strategy. This can be explained by clusters of A's predictably overpowering the E's after which they give in to the P's. For $\alpha = 5/6$ this effect is offset by the fact that too many P's are susceptible to free-riding by the E's. The effect for the A strategy is more complex in the

[5] In the simple setting of this model, counting the number of >1-strings of different strategies in the starting state gives similar indications to the index of segregation about the probability that a certain absorbing state is reached.

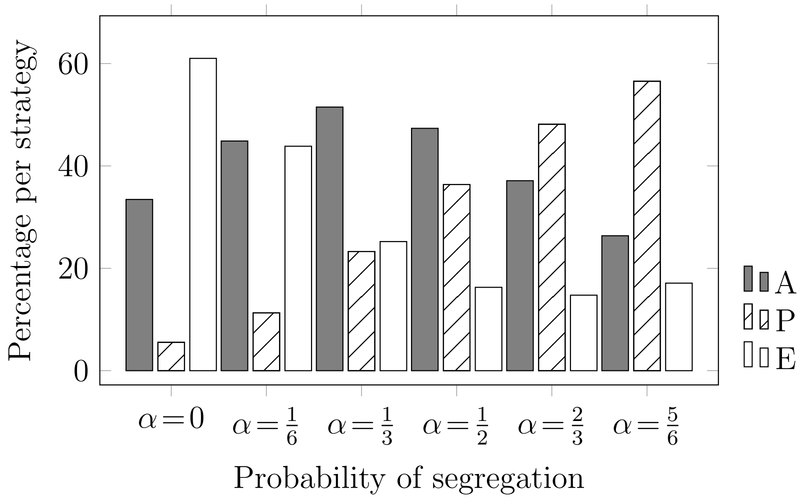


Figure 1.7: Strategies in the absorbing state varying the segregation probability ($n = 60$).

sense that if too scattered, the A's cannot benefit from cooperating with each other, while if they are too clustered, they could be taken over by the P's, leaving the circle more vulnerable to the E's.

1.5 Conclusion

This chapter presents a local interaction model where agents on a circle use a naïve imitation decision rule, by adopting in each following period the strategy that has provided the highest average payoff from the ones in their observed neighbourhood in the current period. The new element in this model is its relative departure from the dichotomous "altruist/egoist" framework that is frequently implemented to study cooperation (and is at least in part imposed by requirements of analytical tractability) by introducing a partially altruistic strategy. It allows a middle option between the two extremes, giving the agents the opportunity to cooperate with only one of their neighbours. This feature gives the model a "rock-paper-scissors" flavour, whereby every strategy's propagation in the population is inhibited by one other strategy and propped up by the third one, especially when interacting in big homogeneous groups. As such the introduction of the partial strategy fundamentally changes the dynamics in the model and poses interesting questions about its development and potential outcomes.

The absorbing sets of the Markov process resulting from the imitation dynamic are theoretically characterized and further analysed by means of computer simulations. The partially cooperative strategy does not coexist with the other two strategies in any absorbing states. While it limits the relative number of stable states with both altruists and egoists, it does not eliminate them. This provides an interesting comparison with Matros (2012), where the random use of one out of two decision rules, imitation or best response, was fatal for altruism.

Kirchkamp (2000) shows that cooperation can be sustained with non-random preference for certain neighbours and the current model builds on that result by proving that cooperation survives even in the presence of random discrimination between neighbours. However, the introduction of the partially cooperative strategy has interesting consequences for the development of the process: it hinders the propagation of the altruist strategy and enables the spread of the egoists, and this effect is much stronger than the influence it has on its own diffusion in the population.

This chapter also provides insights about the temporal evolution of the specific strategies. On average, the process of reaching absorption could be divided into several phases. Firstly, there is a swift elimination of isolated altruists, which solidifies them in bigger blocks. Secondly, they slowly start gaining ground at the expense of big neighbouring clusters of egoists. Once this stage is completed, the remaining groups of agents with the partial strategy slowly take over the remaining non-absorbed altruists.

Finally, somewhat contrary to the usual conditions for sustaining altruism, where clusters of altruists in a local interaction model are generally seen as favouring altruism, this model suggests that while it is true that very isolated altruists cannot survive, if they are in too big groups at the beginning of the process, this would benefit the partial strategy and in time even enable the egoists.

Appendix A

Supplementing Chapter 1

To ease the notation, π denotes $\pi(d)$ throughout Appendix A.

A.1 No splitting of P-strings

Lemma 1.2. *Let $[a, a+k-1]$ be a kP-string of the state $x \in X$, $d \in D(x)$ such that, for every $i \in [a, a+k-1]$, $d_i = d_i^*$ and $y = f(d)$. Then $[a, a+k-1]$* ***does not split****.*

Proof. First consider the case $k = 1$. Without loss of generality, let $x \in X$ be such that $a = 2$, so $x_2 = P$ and $x_1, x_3 \neq P$. The statement of the lemma is true if one of the following two cases occurs:

(i) $y_2 = P$.

(ii) $y_2 \neq P$ and ($y_1 \neq P$ or $y_3 \neq P$).

Up to symmetries, Cases A.1.1–A.1.3 cover all possible segments of 3 agents that contain a $1P$-string.

Case A.1.1: $(x_1, x_2, x_3) = (A, P, A)$
It holds that $d_2^* = L$, $\pi_1 \leq 2-2c$, $\pi_2 = 2-c$ and $\pi_3 \leq 1-2c$, so $y_2 = P$.

Case A.1.2: $(x_1, x_2, x_3) = (E, P, E)$
It holds that $d_2^* = L$, $\pi_1 \geq 1$, $\pi_2 = -c$ and $\pi_3 \geq 0$, so $y_1 \neq P$ and $y_2 = E$.

Case A.1.3: $(x_1, x_2, x_3) = (A, P, E)$
It holds that $d_2^* = L$, $\pi_1 \in \{1-2c, 2-2c\}$, $\pi_2 = 1-c$ and $\pi_3 \in \{0, 1\}$. Three cases can be distinguished:

(i) $\pi_1 = 1-2c$ and $\pi_3 = 0$: $y_2 = P$.

(ii) $\pi_1 = 1-2c$ and $\pi_3 = 1$: $y_2 = E$ and $d_4 \in \{A, L\}$. It follows that $\pi_4 \leq 1-c$ and $y_3 = E$.

(iii) $\pi_1 = 2-2c$ and $\pi_3 \in \{0,1\}$: $y_2 = A$ and $d_0 \in \{A, R\}$. It follows that $\pi_0 \geq 1-2c$ and $y_1 = A$.

Next, consider the cases $k \neq 1, 3$. Without loss of generality, let $x \in X$ be such that $a = 2$, so $x_1 \neq P$, $x_2 = \cdots = x_{k+1} = P$ and $x_{k+2} \neq P$. Then, $y_2 = P$ or ($y_2 \neq P$ and $y_1 \neq P$). The same argument can be used to show that $y_{k+1} = P$ or ($y_{k+1} \neq P$ and $y_{k+2} \neq P$). It holds that $d_2^* = R$, $d_3^* = L$, $\pi_2 \geq 1-c$ and $\pi_3 \geq 1-c$. If $x_1 = A$, then $\pi_1 \leq 1-2c$, so $y_2 = P$. Assume $x_1 = E$ and $y_2 \neq P$. It follows that $\pi_1 = 1$, so $y_1 = E$.

Finally, consider the case $k = 3$. Without loss of generality, let $x \in X$ be such that $a = 2$, so $x_1 \neq P$, $x_2 = x_3 = x_4 = P$ and $x_5 \neq P$. It holds that $d_2^* = R$, $d_3^* = R$, and $d_4^* = L$. Since $d_3^* = R$ and $d_4^* = L$, the conclusion that $y_4 = P$ or ($y_4 \neq P$ and $y_5 \neq P$) follows as for the case $k \neq 1, 3$. It suffices to show that $y_2 = P$ or ($y_2 \neq P$ and $y_1 \neq P$). If $x_1 = A$, then $\pi_1 \leq 1-2c$, $\pi_2 = 1-c$ and $\pi_3 = 2-c$, so $y_2 = P$. Assume $x_1 = E$ and $y_2 \neq P$. Since $\pi_2 = -c$ and $\pi_3 = 2-c$, $y_2 \neq P$ implies $\pi_1 = 1$. It then follows that $y_1 \neq P$. □

A.2 $1P$-strings

Let $\{a\}$ be a $1P$-string of the state $x \in X$, so agents $a-1$ and $a+1$ do not use strategy P. In case also agents $a-2$ and $a+2$ do not use strategy P, we call the segment $[a-2, a+2]$ a $1P$-segment. Lemma A.1 deals with this situation.

Lemma A.1. *Let $\{a\}$ be a $1P$-string within a $1P$-segment of the state $x \in X$. Then one of the following two cases occurs:*

(i) there is $y \in F(x)$ such that $p(y) \leq p(x)$ and y contains a $2P$-string;

(ii) there is $y \in F(x) \cup F^2(x)$ such that $p(y) \leq p(x) - 1$.

Proof. Table A.1 lists all possible $1P$-segments, omitting symmetric equivalent possibilities. Cases A.2.1–A.2.5 below cover all these possible $1P$-segments and the table makes clear which possibilities are covered by which of these cases. Moreover, the table makes explicit for each possible $1P$-segment what can happen to the $1P$-string in this segment. The development of the $1P$-segments is illustrated in Figure A.1.

Case A.2.1: $(x_1, x_2, x_3, x_4) = (A, A, P, A)$
Take $d_3 = L$, for $i \neq 3$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \leq 2-2c$, $\pi_2 = 2-2c$, $\pi_3 = 2-c$ and $\pi_4 \leq 1-2c$, so $(y_2, y_3) = (P, P)$ and the $1P$-string $\{3\}$ grows. Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.2.2: $(x_1, x_2, x_3, x_4) = (E, E, P, E)$
Take $d_3 = R$, for $i \neq 3$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 0$, $\pi_2 = 0$, $\pi_3 = -c$, $\pi_4 \in \{1, 2\}$

and $\pi_5 < 1$ if $x_5 \neq E$ and $\pi_5 \in \{0,1\}$ if $x_5 = E$. In any case $(y_2, y_3, y_4) = (E, E, E)$ and the $1P$-string $\{3\}$ disappears. Lemma 1.3 implies $p(y) \leq p(x) - 1$.

Case A.2.3: $(x_1, x_2, x_3) = (A, E, P)$
Take $d_3 = R$, for $i \neq 3$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \leq 1 - 2c$ and $\pi_2 = 1$.

(i) $(x_4, x_5) = (A, A)$: then $\pi_3 = 1-c$, $\pi_4 = 2-2c$ and $\pi_5 \geq 1-2c$, so $(y_2, y_3, y_4) = (E, A, A)$.

(ii) $(x_4, x_5) = (A, E)$: then $\pi_3 = 1 - c$, $\pi_4 = 1 - 2c$ and $\pi_5 \geq 1$, so $(y_2, y_3, y_4) = (E, E, E)$.

(iii) $x_4 = E$: then $\pi_3 = -c$, $\pi_4 \geq 1$ and $\pi_5 \leq 1$, so $(y_2, y_3, y_4) = (E, E, E)$.

In all three subcases above the $1P$-string $\{3\}$ disappears. Lemma 1.3 implies $p(y) \leq p(x) - 1$.

Case A.2.4: $(x_1, x_2, x_3) = (E, A, P)$
Take $d_3 = L$, for $i \neq 3$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_0 \leq 1$, $\pi_1 \geq 1$ and $\pi_2 = 1 - 2c$.

(i) $(x_4, x_5) = (A, E)$: then $\pi_3 = 2-c$, $\pi_4 = -2c$, $\pi_5 \geq 1$ and $\pi_6 \leq 1$, so $y_1 \neq P$, $y_2 \in \{P, E\}$, $y_3 = P$, $y_4 \in \{P, E\}$ and $y_5 = E$. Lemma 1.3 implies $p(y) \leq p(x)$. Either the $1P$-string $\{3\}$ grows or $[1, 5]$ is a $1P$-segment of y that satisfies the conditions of Case A.2.2, so there is $z \in F(y)$ such that the $1P$-string $\{3\}$ disappears and $p(z) \leq p(y) - 1$.

(ii) $(x_4, x_5) = (E, E)$: then $\pi_3 = 1 - c$, $\pi_4 = 0$ and $\pi_5 \leq 1$, so $(y_2, y_3, y_4) = (E, P, P)$ and the $1P$-string $\{3\}$ grows. Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.2.5: $(x_1, x_2, x_3, x_4, x_5) = (A, A, P, E, E)$
Take $d_3 = R$, for $i \neq 3$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \in \{1 - 2c, 2 - 2c\}$, $\pi_2 = 1 - 2c$, $\pi_3 = 1 - c$, $\pi_4 = 1$ and $\pi_5 \in \{0, 1\}$. It is useful to distinguish a few cases here:

(i) $\pi_1 = 1 - 2c$: then $d_0 \in \{E, L\}$, so $\pi_0 \geq 1 - c$ and $y_1 \in \{E, P\}$, $(y_2, y_3) = (P, E)$ and $(y_4, y_5) \in \{(E, E), (P, P), (P, E)\}$.
If $(y_1, y_2, y_3, y_4, y_5) = (P, P, E, P, P)$, then the $1P$-string $\{3\}$ disappears and it follows by Lemma 1.3 that $p(y) \leq p(x) - 1$.
If $(y_1, y_2, y_3, y_4, y_5) = (E, P, E, P, P)$, then it holds by Lemma 1.3 that $p(y) \leq p(x)$. We can proceed as in Case A.2.2. Take $d'_2 = L$ and, for $i \neq 2$, $d'_i = d_i^*$. For $z = f(d')$ it holds that $(z_1, z_2, z_3, z_4) = (E, E, P, P)$, so the $1P$-string $\{2\}$ disappears and it follows by Lemma 1.3 that $p(z) \leq p(y) - 1$. The case $(y_1, y_2, y_3, y_4, y_5) = (P, P, E, P, E)$ follows by a symmetric argument.
If $(y_1, y_2, y_3, y_4, y_5) = (E, P, E, P, E)$, then it holds by Lemma 1.3 that $p(y) \leq p(x) + 1$. The construction that follows is also used in Case A.2.7. Take $d'_2 = L, d'_4 = R$, and, for $i \neq 2, 4$, $d'_i = d_i^*$. For $z = f(d')$ it holds that $(z_1, z_2, z_3, z_4, z_5) = (E, E, E, E, E)$, so the $1P$-strings $\{2\}$ and $\{4\}$ disappear and by Lemma 1.3 it holds that $p(z) \leq p(y) - 2$.
If $(y_1, y_2, y_3, y_4) = (E, P, E, E)$, then it follows by Lemma 1.3 that $p(y) \leq p(x)$. Since

$x_1 = x_2 \neq E$, we have $x_0 = E$, $\pi_0 \geq 1$, and $y_0 \neq P$. Now Case A.2.2 applies and there is $z \in F(y)$ such that the $1P$-string $\{2\}$ disappears and $p(z) \leq p(y) - 1$.

(ii) $\pi_1 = 2 - 2c$, $\pi_5 = 0$: then $(y_2, y_3, y_4) = (A, E, P)$, $y_5 \in \{P, E\}$. Lemma 1.3 implies $p(y) \leq p(x)$. Either the $1P$-string $\{3\}$ grows or $(y_2, y_3, y_4, y_5) = (A, E, P, E)$. We can proceed as in Case A.2.3. Take $d'_4 = R$, for $i \neq 4$, $d_i = d_i^*$, and $z = f(d')$. It holds that $\pi_2(d') \leq 1 - 2c$, $\pi_3(d') = 1$, $\pi_4(d') = -c$, $\pi_5(d') \geq 1$ and $\pi_6(d') \leq 1$, so $(z_3, z_4, z_5) = (E, E, E)$ and the $1P$-string $\{4\}$ disappears. Lemma 1.3 implies $p(z) \leq p(y) - 1$.

(iii) $\pi_1 = 2 - 2c$, $\pi_5 = 1$: then $(y_2, y_3, y_4) = (A, E, E)$. The $1P$-string $\{3\}$ disappears and Lemma 1.3 implies $p(y) \leq p(x) - 1$.

□

Case	Segment $(x_1, x_2, x_3, x_4, x_5)$	Decision d_3	Results
A.2.1	(A, A, P, A, A)	L	$\geq 2P$
A.2.1	(A, A, P, A, E)	L	$\geq 2P$
A.2.3	(A, A, P, E, A)	L	$0P$
A.2.5	(A, A, P, E, E)	R	$0P$; $\geq 2P$
A.2.3	(A, E, P, A, E)	R	$0P$
A.2.3	(A, E, P, E, A)	R	$0P$
A.2.2	(A, E, P, E, E)	L	$0P$
A.2.4	(E, A, P, A, E)	L	$0P$; $\geq 2P$
A.2.4	(E, A, P, E, E)	L	$\geq 2P$
A.2.2	(E, E, P, E, E)	R	$0P$

Table A.1: All possible $1P$-segments, omitting symmetric equivalent possibilities, and what can happen with the $1P$-string within the segment.

Let $\{a\}$ and $\{b\}$ be $1P$-strings of the state $x \in X$, where $b = a + 2$. We call the segment $[a - 1, b + 1]$ a $1P{+}1P$-segment. Notice that $x_{a+1} = x_{b-1} \neq P$. Lemma A.2 deals with this situation. The case where a $1P$-string does not border another $1P$-string, but does border a $\geq 2P$-string from at least one side (i.e. there is exactly one agent between the $1P$-string and a neighbouring $\geq 2P$-string, is addressed in Appendices A.3–A.5, where $\geq 2P$-strings are studied.

Lemma A.2. *Let $\{a\}$ and $\{b\}$ be $1P$-strings within a $1P{+}1P$-segment of the state $x \in X$. Then one of the following two cases occurs:*

(i) there is $y \in F(x) \cup F^2(x)$ such that $p(y) \leq p(x)$ and y contains a $\geq 2P$-string;

(ii) there is $y \in F(x) \cup F^2(x) \cup F^3(x)$ such that $p(y) \leq p(x) - 1$.

Proof. Table A.2 lists all possible $1P+1P$-segments, omitting symmetric equivalent possibilities. Cases A.2.6–A.2.10 below cover all these possible $1P+1P$-segments and the table makes clear which possibilities are covered by which of these cases. Moreover, the table makes explicit for each possible $1P+1P$-segment what can happen to the $1P$-strings in this segment. The development of the $1P+1P$-segments is illustrated in Figure A.1.

Case A.2.6: $(x_1, x_2, x_3, x_4) = (A, P, A, P)$
Take $d_2 = L$, $d_4 = R$, for $i \neq 2, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1 - 2c$, $\pi_2 = 2 - c$, $\pi_3 = -2c$ and $\pi_4 \geq 1 - c$, so $(y_2, y_3) = (P, P)$ and the $1P$-string $\{2\}$ grows. Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.2.7: $(x_1, x_2, x_3, x_4, x_5) = (E, P, E, P, E)$
Take $d_2 = L$, $d_4 = R$, for $i \neq 2, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1, \pi_5 \geq 1$, $\pi_2 = \pi_4 = -c$ and $\pi_3 = 0$, so $(y_1, y_2, y_3, y_4, y_5) = (E, E, E, E, E)$ and the $1P$-strings $\{2\}$ and $\{4\}$ disappear. Lemma 1.3 implies $p(y) \leq p(x) - 2$.

Case A.2.8: $(x_1, x_2, x_3, x_4, x_5) = (A, P, E, P, E)$
Take $d_2 = L$, $d_4 = R$, for $i \neq 2, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_2 = 1 - c$, $\pi_3 = 0$, $\pi_4 = -c$ and $\pi_5 \geq 1$.

(i) $\pi_1 = 1 - 2c$: then $(y_2, y_3, y_4, y_5) = (P, P, E, E)$. The $1P$-string $\{2\}$ grows and Lemma 1.3 implies $p(y) \leq p(x) - 1$.

(ii) $\pi_1 = 2 - 2c$: then $(y_1, y_2, y_3, y_4, y_5) = (A, A, P, E, E)$. Lemma 1.3 implies $p(y) \leq p(x) - 1$.[1]

Case A.2.9: $(x_1, x_2, x_3, x_4, x_5) = (A, P, E, P, A)$
Take $d_2 = L$, $d_4 = R$, for $i \neq 2, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \in \{1-2c, 2-2c\}$, $\pi_2 = 1-c$, $\pi_3 = 0$, $\pi_4 = 1 - c$ and $\pi_5 \in \{1 - 2c, 2 - 2c\}$. We consider the following cases concerning π_5 (bearing in mind that the cases for π_1 are symmetric):

(i) $\pi_5 = 1 - 2c$: then $d_6 \in \{R, E\}$, $(y_3, y_4) = (P, P), y_5 \in \{P, E\}$.

(ii) $\pi_5 = 2 - 2c$: then $d_6 \in \{A, L\}$, $(y_3, y_4, y_5) = (P, A, A)$.

Keeping the symmetric structure in mind, the segment results in a $\geq 2P$-string and, by Lemma 1.3, $p(y) \leq p(x) - 1$, or a $1P$-segment and, by Lemma 1.3, $p(y) \leq p(x) - 1$.[2]

Case A.2.10: $(x_1, x_2, x_3, x_4, x_5) = (E, P, A, P, E)$
Take $d_2 = L$, $d_4 = L$, for $i \neq 2, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \in \{1, 2\}$, $\pi_2 = 1 - c$, $\pi_3 = 1 - 2c$, $\pi_4 = 1 - c$ and $\pi_5 \in \{0, 1\}$. It holds that $y_1 \in \{P, E\}$ and $(y_2, y_3) = (E, P)$.

[1] From y we can continue as in Case A.2.5 and find a state such that the $1P$-string $\{3\}$ disappears or grows.
[2] Now Case A.2.1 applies, leading to a $\geq 2P$-string in the next iteration.

If $\pi_5 = 0$, then $d_6 \in \{E, R\}$ and $\pi_6 \leq 1$, so $(y_4, y_5) = (P, P)$. The $1P$-string $\{4\}$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.
If $\pi_5 = 1$, then $d_6 \in \{A, L\}$ and $\pi_6 \leq 1 - c$, so $(y_4, y_5) = (E, E)$. Lemma 1.3 implies $p(y) \leq p(x)$. We can proceed as in Case A.2.2. Take $d'_2 = L$ and, for $i \neq 2$, $d'_i = d^*_i$. For $z = f(d')$ it holds that $(z_2, z_3, z_4) = (E, E, E)$, so the $1P$-string $\{3\}$ disappears and Lemma 1.3 implies $p(z) \leq p(y) - 1$. □

Case	Segment $(x_1, x_2, x_3, x_4, x_5)$	Decisions d_2, d_4	Results
A.2.6	(A, P, A, P, A)	L, R	$\geq 2P$
A.2.6	(A, P, A, P, E)	L, R	$\geq 2P$
A.2.9	(A, P, E, P, A)	L, R	$\geq 2P$
A.2.8	(A, P, E, P, E)	L, R	$0P$; $\geq 2P$
A.2.10	(E, P, A, P, E)	L, L	$0P$; $\geq 2P$
A.2.7	(E, P, E, P, E)	L, R	$0P$

Table A.2: All possible $1P{+}1P$-segments, omitting symmetric equivalent possibilities.

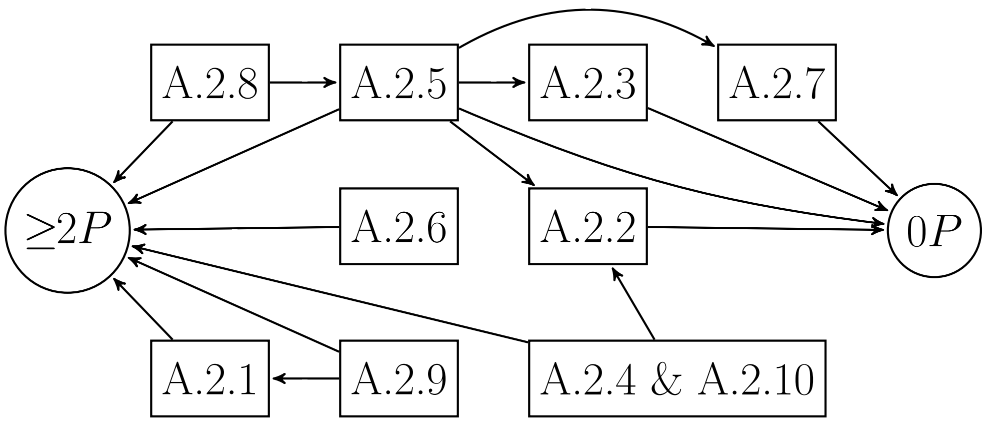


Figure A.1: Transforming $1P$-segments and $1P{+}1P$-segments based on the cases specified in the squares results in the P-strings in the circles.

A.3 $2P$-strings

Lemma A.3. *Let $[a, a+1]$ be a $2P$-string of the state $x \in X$. Then one of the following two cases occurs:*

(i) there is $y \in F(x) \cup F^2(x)$ such that $p(y) \leq p(x)$ and y contains a $\geq 3p$-string;

(ii) there is $y \in F(x)$ such that $p(y) \leq p(x) - 1$.

Proof. Cases A.3.1–A.3.10 below cover all these possible $2P$-strings. Table A.3 lists all possible cases and makes explicit for each case what can happen to the $2P$-strings. The development of the $2P$-strings is illustrated in Figure A.2.

Case A.3.1: $d_1^* \in \{A, R, L\}$, $(x_2, x_3, x_4, x_5) = (A, P, P, A)$ and $d_6^* \in \{A, R, L\}$
Take $(d_3, d_4) = (R, L)$, for $i \neq 3, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1 - 2c$, $\pi_2 \leq 1 - 2c$, $\pi_3 = 2 - c$, $\pi_4 = 2 - c$, $\pi_5 \leq 1 - 2c$ and $\pi_6 \geq 1 - 2c$ and hence $(y_2, y_3, y_4, y_5) = (P, P, P, P)$. The $2P$-string $[3, 4]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.3.2: $d_1^* \in \{A, R, L\}$, $(x_2, x_3, x_4, x_5) = (A, P, P, A)$ and $d_6^* = E$
Take $(d_3, d_4) = (R, L)$, for $i \neq 3, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1 - 2c$, $\pi_2 \leq 1 - 2c$, $\pi_3 = 2 - c$, $\pi_4 = 2 - c$, $\pi_5 = -2c$ and $\pi_6 \geq 1$ and hence $(y_2, y_3, y_4) = (P, P, P), (y_5, y_6) \in \{(P, E), (E, E)\}$. The $2P$-string $[3, 4]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.3.3: $d_1^* \in \{E, L\}$, $(x_2, x_3, x_4, x_5) = (E, P, P, E)$ and $d_6^* \in \{E, R\}$
Take $(d_3, d_4) = (R, L)$, for $i \neq 3, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \leq 1$, $\pi_2 = 0$, $\pi_3 = 1 - c$, $\pi_4 = 1 - c$, $\pi_5 = 0$ and $\pi_6 \leq 1$ and hence $(y_2, y_3, y_4, y_5) = (P, P, P, P)$. The $2P$-string $[3, 4]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.3.4: $d_1^* = E$, $(x_2, x_3, x_4, x_5) = (A, P, P, A)$ and $d_6^* = E$
Take $(d_3, d_4) = (R, L)$, for $i \neq 3, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1$, $\pi_2 = -2c$, $\pi_3 = 2 - c$, $\pi_4 = 2 - c$, $\pi_5 = -2c$ and $\pi_6 \geq 1$ and hence $(y_1, y_2) \in \{(E, P), (E, E)\}$, $(y_3, y_4) = (P, P)$ and $(y_5, y_6) \in \{(P, E), (E, E)\}$. Lemma 1.3 implies $p(y) \leq p(x)$. If $(y_1, y_2, y_3, y_4, y_5, y_6) \neq (E, E, P, P, E, E)$, then the $2P$-string $[3, 4]$ grows. Otherwise, we can apply Case A.3.3 and find $z \in F(y)$ such that $p(z) \leq p(y)$ and the $2P$-string $[3, 4]$ grows.

Case A.3.5: $d_1^* \in \{A, R, L\}$, $(x_2, x_3, x_4, x_5) = (A, P, P, E)$ and $d_6^* \in \{E, R\}$
Take $(d_3, d_4) = (R, L)$, for $i \neq 3, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1 - 2c$, $\pi_2 \leq 1 - 2c$, $\pi_3 = 2 - c$, $\pi_4 = 1 - c$, $\pi_5 = 0$ and $\pi_6 \leq 1$ and hence $(y_2, y_3, y_4, y_5) = (P, P, P, P)$. The $2P$-string $[3, 4]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.3.6: $d_1^* = E$, $(x_2, x_3, x_4, x_5) = (A, P, P, E)$ and $d_6^* \in \{E, R\}$
Take $(d_3, d_4) = (R, L)$, for $i \neq 3, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1$, $\pi_2 = -2c$, $\pi_3 = 2 - c$, $\pi_4 = 1 - c$, $\pi_5 = 0$ and $\pi_6 \leq 1$ and hence $(y_1, y_2) \in \{(E, P), (E, E)\}$ and $(y_3, y_4, y_5) = (P, P, P)$. The $2P$-string $[3, 4]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.3.7: $d_1^* \in \{A, R, L\}$, $(x_2, x_3, x_4, x_5) = (A, P, P, E)$ and $d_6^* \in \{A, L\}$
Take $(d_3, d_4) = (R, L)$, for $i \neq 3, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1 - 2c$, $\pi_2 \leq 1 - 2c$, $\pi_3 = 2 - c$, $\pi_4 = 1 - c$, $\pi_5 = 1$ and $\pi_6 \leq 1 - c$ and hence $(y_2, y_3, y_4, y_5) = (P, P, P, E)$. The $2P$-string $[3, 4]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.3.8: $d_1^* = E$, $(x_2, x_3, x_4, x_5) = (A, P, P, E)$ and $d_6^* \in \{A, L\}$

Take $(d_3, d_4) = (L, R)$, for $i \neq 3, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1$, $\pi_2 = 1 - 2c$, $\pi_3 = 1 - c$, $\pi_4 = -c$, $\pi_5 = 2$ and $\pi_6 \leq 1 - 2c$ and $(y_2, y_3, y_4, y_5) = (E, A, E, E)$. The $2P$-string $[3, 4]$ disappears and Lemma 1.3 implies $p(y) \leq p(x) - 1$.

Case A.3.9: $d_1^* \in \{E, L\}$, $(x_2, x_3, x_4, x_5) = (E, P, P, E)$ and $d_6^* \in \{A, L\}$
Take $(d_3, d_4) = (L, R)$, for $i \neq 3, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \leq 1$, $\pi_2 = 1$, $\pi_3 = -c$, $\pi_4 = -c$, $\pi_5 = 2$ and $\pi_6 \leq 1 - c$ and $(y_2, y_3, y_4, y_5) = (E, E, E, E)$. The $2P$-string $[3, 4]$ disappears and Lemma 1.3 implies $p(y) \leq p(x) - 1$.

Case A.3.10: $d_1^* \in \{A, R\}$, $(x_2, x_3, x_4, x_5) = (E, P, P, E)$ and $d_6^* \in \{A, L\}$
Take $(d_3, d_4) = (L, R)$, for $i \neq 3, 4$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \leq 1 - c$, $\pi_2 = 2$, $\pi_3 = -c$, $\pi_4 = -c$, $\pi_5 = 2$, $\pi_6 \leq 1 - c$ and $(y_2, y_3, y_4, y_5) = (E, E, E, E)$. The $2P$-string $[3, 4]$ disappears and Lemma 1.3 implies $p(y) \leq p(x) - 1$. □

$(x_3, x_4) = (P, P)$		$(d_5^*, d_6^*) =$							
		$(A, A/R/L)$		(A, E)		$(E, E/R)$		$(E, A/L)$	
$(d_1^*, d_2^*) =$	$(A/R/L, A)$	g	(A.3.1)	g	(A.3.2)	g	(A.3.5)	g	(A.3.7)
	(E, A)			g	(A.3.4)	g	(A.3.6)	d	(A.3.8)
	$(E/L, E)$					g	(A.3.3)	d	(A.3.9)
	$(A/R, E)$							d	(A.3.10)

Table A.3: Strings resulting from $2P$-strings surrounded from the left by the decisions in the columns and from the right by the decisions in the rows. Abbreviations 'g' and 'd' in the cells are used to identify whether the P-string will be growing or disappearing. Relevant cases are in parentheses.

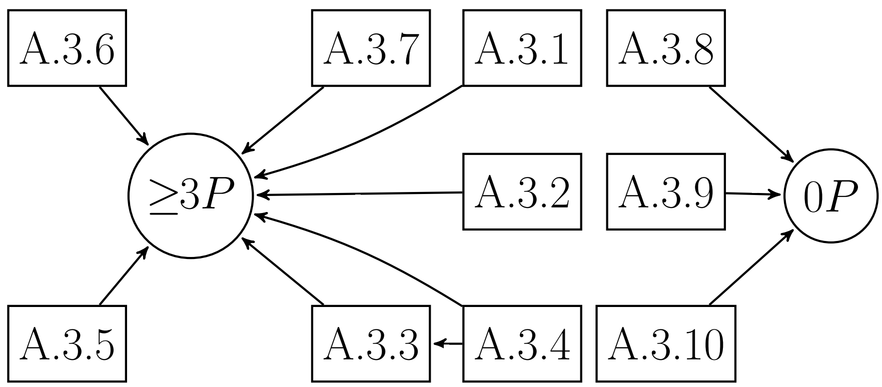


Figure A.2: Transforming $2P$-strings based on the cases specified in the squares results in the P-strings in the circles.

A.4 $3P$-strings

Lemma A.4. *Let $n \geq 4$ and let $[a, a+2]$ be a $3P$-string of the state $x \in X$. Then one of the following two cases occurs:*

(i) there is $y \in F(x) \cup F^2(x)$ such that $p(y) \leq p(x)$ and y contains a $\geq 4P$-string;

(ii) there is $y \in F^2(x) \cup F^3(x)$ such that $p(y) \leq p(x) - 1$.

Proof. Cases A.4.1–A.4.10 describe all possible $3P$-strings. Table A.4 lists all possible cases and makes explicit for each possibility what can happen to the $3P$-strings. The development of the $3P$-strings is illustrated in Figure A.3.

Case A.4.1: $d_1^* \in \{A, R, L\}$, $(x_2, x_3, x_4, x_5, x_6) = (A, P, P, P, A)$ and $d_7^* \in \{A, R, L\}$
Take $(d_3, d_4, d_5) = (R, R, L)$, for $i \neq 3, 4, 5$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1 - 2c$, $\pi_2 \leq 1 - 2c$, $\pi_3 = 1 - c$, $\pi_4 = 2 - c$, $\pi_5 = 2 - c$, $\pi_6 \leq 1 - 2c$ and $\pi_7 \geq 1 - 2c$ and hence $(y_3, y_4, y_5, y_6) = (P, P, P, P)$. The $3P$-string $[3, 5]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.4.2: $d_1^* \in \{A, R, L\}$, $(x_2, x_3, x_4, x_5, x_6) = (A, P, P, P, A)$ and $d_7^* = E$
Take $(d_3, d_4, d_5) = (R, L, L)$, for $i \neq 3, 4, 5$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1-2c$, $\pi_2 \leq 1-2c$, $\pi_3 = 2-c$, $\pi_4 = 2-c$, $\pi_5 = 1-c$, $\pi_6 = -2c$ and $\pi_7 \geq 1$ and hence $(y_2, y_3, y_4, y_5) = (P, P, P, P)$. The $3P$-string $[3, 5]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.4.3: $d_1^* \in \{E, L\}$, $(x_2, x_3, x_4, x_5, x_6) = (E, P, P, P, E)$ and $d_7^* \in \{E, R\}$
Take $(d_3, d_4, d_5) = (R, R, L)$, for $i \neq 3, 4, 5$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \leq 1$, $\pi_2 = 0$, $\pi_3 = -c$, $\pi_4 = 2 - c$, $\pi_5 = 1 - c$, $\pi_6 = 0$ and $\pi_7 \leq 1$ and hence $(y_3, y_4, y_5, y_6) = (P, P, P, P)$. The $3P$-string $[3, 5]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.4.4: $d_1^* = E$, $(x_2, x_3, x_4, x_5, x_6) = (A, P, P, P, A)$ and $d_7^* = E$
Take $(d_3, d_4, d_5) = (R, R, L)$, for $i \neq 3, 4, 5$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1$, $\pi_2 = -2c$, $\pi_3 = 1 - c$, $\pi_4 = 2 - c$, $\pi_5 = 2 - c$, $\pi_6 = -2c$ and $\pi_7 \geq 1$ and hence $(y_1, y_2, y_3, y_4, y_5) = (E, E, P, P, P)$, $y_6 \in \{P, E\}$ and $y_7 = E$. Lemma 1.3 implies $p(y) \leq p(x)$. If $(y_1, y_2, y_3, y_4, y_5, y_6, y_7) = (E, E, P, P, P, P, E)$, then the $3P$-string $[3, 5]$ grows. Otherwise, $(y_1, y_2, y_3, y_4, y_5, y_6, y_7) = (E, E, P, P, P, E, E)$ and Case A.4.3 implies that we can find $z \in F(y)$ such that $p(z) \leq p(y)$ and the $3P$-string $[3, 5]$ grows.

Case A.4.5: $d_1^* \in \{A, R, L\}$, $(x_2, x_3, x_4, x_5, x_6) = (A, P, P, P, E)$ and $d_7^* \in \{E, R\}$
Take $(d_3, d_4, d_5) = (R, L, L)$, for $i \neq 3, 4, 5$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1-2c$, $\pi_2 \leq 1-2c$, $\pi_3 = 2 - c$, $\pi_4 = 2 - c$, $\pi_5 = -c$, $\pi_6 = 0$ and $\pi_7 \leq 1$ and hence $(y_2, y_3, y_4, y_5) = (P, P, P, P)$. The $3P$-string $[3, 5]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.4.6: $d_1^* = E$, $(x_2, x_3, x_4, x_5, x_6) = (A, P, P, P, E)$ and $d_7^* \in \{E, R\}$
Take $(d_3, d_4, d_5) = (R, R, L)$, for $i \neq 3, 4, 5$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1$, $\pi_2 = -2c$,

$\pi_3 = 1-c$, $\pi_4 = 2-c$, $\pi_5 = 1-c$, $\pi_6 = 0$ and $\pi_7 \leq 1$ and hence $(y_1, y_2, y_3, y_4, y_5, y_6) = (E, E, P, P, P, P)$. The $3P$-string $[3,5]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.4.7: $d_1^* \in \{E, L\}$, $(x_2, x_3, x_4, x_5, x_6) = (E, P, P, P, E)$ and $d_7^* \in \{A, L\}$
Take $(d_3, d_4, d_5) = (L, R, R)$, for $i \neq 3, 4, 5$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \leq 1$, $\pi_2 = 1$, $\pi_3 = -c$, $\pi_4 = -c$, $\pi_5 = 1-c$, $\pi_6 = 2$ and $\pi_7 \leq 1-c$ and $(y_2, y_3, y_4, y_5, y_6) = (E, E, P, E, E)$. Lemma 1.3 implies $p(y) \leq p(x)$. By Case A.2.2 we can find $z \in F(y)$ such that $p(z) \leq p(y) - 1$ and the $1P$-string $\{4\}$ disappears.

Case A.4.8: $d_1^* \in \{A, R, L\}$, $(x_2, x_3, x_4, x_5, x_6) = (A, P, P, P, E)$ and $d_7^* \in \{A, L\}$
Take $(d_3, d_4, d_5) = (R, L, L)$, for $i \neq 3, 4, 5$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1-2c$, $\pi_2 \leq 1-2c$, $\pi_3 = 2-c$, $\pi_4 = 2-c$, $\pi_5 = -c$, $\pi_6 = 1$ and $\pi_7 \leq 1-c$ and hence $(y_2, y_3, y_4, y_5, y_6) = (P, P, P, E, E)$.
If $y_1 = P$, then Lemma 1.3 implies $p(y) \leq p(x) - 1$ and y contains a $\geq 4P$-string.
If $y_1 = A$, then Lemma 1.3 implies $p(y) \leq p(x)$. Moreover, either Case A.4.5 or Case A.4.6 applies, so there is $z \in F(y)$ such that the $3P$-string $[2,4]$ grows and $p(z) \leq p(y)$.
If $y_1 = E$, then Lemma 1.3 implies $p(y) \leq p(x)$. Either Case A.4.3 applies and there is $z \in F(y)$ such that $p(z) \leq p(y)$ and the $3P$-string $[2,4]$ grows or Case A.4.7 applies and there is $z \in F(y)$ such that $p(z) \leq p(y)$ and $(z_1, z_2, z_3, z_4, z_5) = (E, E, P, E, E)$ and there is $w \in F(z)$ such that $p(w) \leq p(z) - 1$ and the $1P$-string $\{3\}$ disappears.

Case A.4.9: $d_1^* = E$, $(x_2, x_3, x_4, x_5, x_6) = (A, P, P, P, E)$ and $d_7^* \in \{A, L\}$
Take $(d_3, d_4, d_5) = (R, R, L)$, for $i \neq 3, 4, 5$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \geq 1$, $\pi_2 = -2c$, $\pi_3 = 1-c$, $\pi_4 = 2-c$, $\pi_5 = 1-c$, $\pi_6 = 1$ and $\pi_7 \leq 1$ and hence $(y_1, y_2, y_3, y_4, y_5, y_6) = (E, E, P, P, P, E)$. Lemma 1.3 implies $p(y) \leq p(x)$. Either Case A.4.3 applies and there is $z \in F(y)$ such that the $3P$-string $[3,5]$ grows and $p(z) \leq p(y)$ or Case A.4.7 applies and there is $z \in F(y)$ such that $p(z) \leq p(y)$ and $(z_2, z_3, z_4, z_5, z_6) = (E, E, P, E, E)$ and there is $w \in F(z)$ such that $p(w) \leq p(z) - 1$ and the $1P$-string $\{4\}$ disappears.

Case A.4.10: $d_1^* \in \{A, R\}$, $(x_2, x_3, x_4, x_5, x_6) = (E, P, P, P, E)$ and $d_7^* \in \{A, L\}$
Take $(d_3, d_4, d_5) = (L, R, R)$, for $i \neq 3, 4, 5$, $d_i = d_i^*$, and $y = f(d)$. Then $\pi_1 \leq 1-c$, $\pi_2 = 2$, $\pi_3 = -c$, $\pi_4 = -c$, $\pi_5 = 1-c$, $\pi_6 = 2$ and $\pi_7 \leq 1-c$ and $(y_2, y_3, y_4, y_5, y_6) = (E, E, P, E, E)$. Lemma 1.3 implies $p(y) \leq p(x)$. By Case A.2.2 we can find $z \in F(y)$ such that $p(z) \leq p(y) - 1$ and the $1P$-string $\{4\}$ disappears. □

A.5 $\geq 4P$-strings

Lemma A.5. *Let $4 \leq k < n$ and let $[a, a+k-1]$ be a kP-string of the state $x \in X$. Then one of the following three cases occurs:*

$(x_3, x_4, x_5) = (P, P, P)$		$(d_6^*, d_7^*) =$			
		$(A, A/R/L)$	(A, E)	$(E, E/R)$	$(E, A/L)$
$(d_1^*, d_2^*) =$	$(A/R/L, A)$	g (A.4.1)	g (A.4.2)	g (A.4.5)	g; d (A.4.8)
	(E, A)		g (A.4.4)	g (A.4.6)	g; d (A.4.9)
	$(E/L, E)$			g (A.4.3)	d (A.4.7)
	$(A/R, E)$				d (A.4.10)

Table A.4: Strings resulting from $3P$-strings surrounded from the left by the decisions in the columns and from the right by the decisions in the rows. Abbreviations 'g' and 'd' in the cells are used to identify whether the P-string will be growing or disappearing. Relevant cases are in parentheses.

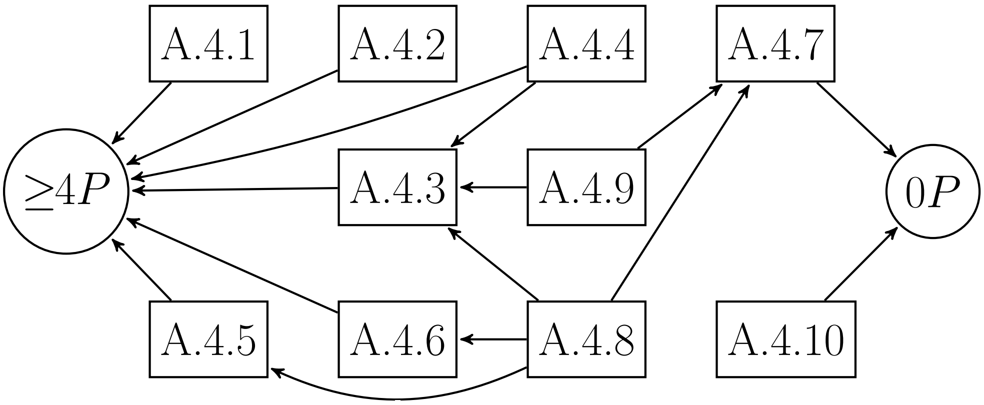


Figure A.3: Transforming $3P$-strings based on the cases specified in the squares results in the P-strings in the circles.

(i) there is $y \in F(x) \cup F^2(x) \cup F^3(x)$ *such that* $p(y) \leq p(x)$ *and* y *contains a* $\geq(k+1)P$*-string;*

(ii) k *is even and there is* $y \in F^{k/2}(x) \cup F^{k/2+1}(x) \cup F^{k/2+2}(x)$ *such that* $p(y) \leq p(x) - 1$*;*

(iii) k *is odd and there is* $y \in F^{(k+1)/2}(x) \cup F^{(k+3)/2}(x)$ *such that* $p(y) \leq p(x) - 1$.

Proof. An agent who is surrounded by agents using the same strategy as him will not change his strategy in one iteration. Therefore, an agent who is not at the very edge of a string cannot change his strategy within one iteration. That is why looking only at the ends of a $\geq 4P$-string in a particular state provides enough information about the string's development in the next iteration. Cases A.5.1–A.5.3 below look at the ends of a $\geq 4P$-string by considering the outermost two agents in the P-string who implement the P-strategy and their closest neighbours who do not implement it. These three cases along with the auxiliary Cases A.5.4–A.5.5 constitute the preliminary cases that are needed to construct and describe all possible $\geq 4P$-strings. This is done in Cases A.5.6–A.5.11. Cases A.5.1 and A.5.2 will be used in the construction of $\geq 4P$-strings which grow, while Case A.5.3 and Cases A.5.4–A.5.5 will be used for the construction of $\geq 4P$-strings which shrink.

Case A.5.1: $(x_1, x_2, x_3) = (P, P, A)$
Let d be such that $(d_1, d_2) = (R, L)$ and $y = f(d)$. Then $\pi_1 \geq 1-c$, $\pi_2 = 2-c$ and $\pi_3 \leq 1-2c$.

(a) $d_4 \in \{A, R, L, E\}$ and $\pi_4 \leq 2 - c$: then $(y_1, y_2, y_3) = (P, P, P)$.

(b) $d_4 = E$ and $\pi_4 = 2$: then $(y_1, y_2, y_3, y_4) = (P, P, E, E)$.

Case A.5.2: $(x_1, x_2, x_3) = (P, P, E)$
Let d be such that $(d_1, d_2) = (R, L)$ and $y = f(d)$. Then $\pi_1 \in \{1 - c; 2 - c\}$ and $\pi_2 = 1 - c$.

(a) $d_4 \in \{R, E\}$: then $\pi_3 = 0$ and $\pi_4 \leq 1$, hence $(y_1, y_2, y_3) = (P, P, P)$.

(b) $d_4 \in \{A, L\}$ and $\pi_1 = 2 - c$: then $\pi_3 = 1$ and $\pi_4 \leq 1 - c$, hence $(y_1, y_2, y_3) = (P, P, E)$.

(c) $d_4 \in \{A, L\}$ and $\pi_1 = 1 - c$: then $\pi_3 = 1$ and $\pi_4 \leq 1 - c$, hence $(y_1, y_2, y_3) = (P, E, E)$.

Case A.5.3: $(x_1, x_2, x_3) = (P, P, E)$
Let d be such that $(d_1, d_2) = (L, R)$ and $y = f(d)$. Then $\pi_1 \leq 1 - c, \pi_2 = -c$.

(a) $d_4 \in \{A, L\}$: then $\pi_3 = 2$ and $\pi_4 \leq 1 - c$, hence $(y_1, y_2, y_3, y_4) = (P, E, E, E)$.

(b) $d_4 \in \{E, R\}$: then $\pi_3 = 1$ and $\pi_4 \geq 0$, hence $(y_1, y_2, y_3) = (P, E, E)$.

(c) $(d_4, d_5) = (E, E)$: then $\pi_3 = 1$ and $\pi_4 = 0$, hence $(y_1, y_2, y_3, y_4) = (P, E, E, E)$.

Case A.5.4: $(x_1, x_2, x_3, x_4, x_5, x_6) = (E, E, P, P, E, E)$
Let d be such that $(d_3, d_4) = (L, R)$ and $y = f(d)$. Then $\pi_1 \leq 1$, $\pi_2 = 1$, $\pi_3 = -c$, $\pi_4 = -c$, $\pi_5 = 1$ and $\pi_6 \leq 1$, hence $(y_2, y_3, y_4, y_5) = (E, E, E, E)$.

Case A.5.5: $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (E, E, P, P, P, E, E, E)$
Let d be such that $(d_3, d_4, d_5) = (L, R, R)$ and $y = f(d)$. Then $\pi_1 \geq 0$, $\pi_2 = 1$, $\pi_3 = -c$, $\pi_4 = -c$, $\pi_5 = 1-c$, $\pi_6 = 1$, $\pi_7 = 0$ and $\pi_8 \geq 0$, hence $(y_2, y_3, y_4, y_5, y_6, y_7) = (E, E, P, E, P, E)$. Now Case A.2.7 applies and there is $z \in F(y)$ such that $(z_3, z_4, z_5, z_6, z_7) = (E, E, E, E, E)$ and $p(z) \leq p(y) - 2$.

This concludes the preliminary cases which are needed to handle the ones outlined below. Cases A.5.6–A.5.11 that follow explicitly describe the development of every $\geq 4P$-string that can occur in the current framework using the preliminary cases presented above. Table A.5 summarizes the outcomes. It is important to note that in some instances considering the closest *one* neighbouring agent to a $\geq 4P$-string is sufficient, while in other cases the closest *two* neighbours are needed. This is specified in every separate case.

Case A.5.6: $(x_1, x_2, \ldots, x_{k+1}, x_{k+2}) = (A, P, \ldots, P, A)$
Take $d_2 = R$, $d_3 = L$, $d_k = R$, $d_{k+1} = L$, for $i \neq 2, 3, k, k+1$, $d_i = d_i^*$, and $y = f(d)$. If Case A.5.1(a) applies on at least one of the sides, then the kP-string $[2, k+1]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$. Otherwise, Case A.5.1(b) applies for both sides, $(y_0, y_1, y_2, \ldots, y_k, y_{k+1}, y_{k+2}) = (E, E, P, \ldots, P, E, E)$, and Lemma 1.3 implies $p(y) \leq p(x)$. Take $d'_2 = R$, $d'_3 = L$, $d'_k = R$, $d'_{k+1} = L$, for $i \neq 2, 3, k, k+1$, $d'_i = d_i^*$, and $z = f(d')$. According to Case A.5.2(a), $(z_1, \ldots, z_{k+1}) = (P, \ldots, P)$, the kP-string $[2, k+1]$ grows, and Lemma 1.3 implies $p(z) \leq p(y)$.

Case A.5.7: $(x_1, x_2, \ldots, x_{k+1}, x_{k+2}) = (A, P, \ldots, P, E)$ and $d_{k+3}^* \in \{E, R\}$
Take $d_2 = R$, $d_3 = L$, $d_k = R$, $d_{k+1} = L$, for $i \neq 2, 3, k, k+1$, $d_i = d_i^*$, and $y = f(d)$. By Case A.5.1 and Case A.5.2(a), the kP-string $[2, k+1]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.5.8: $d_0^* \in \{E, L\}$, $(x_1, x_2, \ldots, x_{k+1}, x_{k+2}) = (E, P, \ldots, P, E)$ and $d_{k+3}^* \in \{E, R\}$
Take $d_2 = R$, $d_3 = L$, $d_k = R$, $d_{k+1} = L$, for $i \neq 2, 3, k, k+1$, $d_i = d_i^*$, and $y = f(d)$. According to Case A.5.2(a), the kP-string $[2, k+1]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

Case A.5.9: $d_0^* \in \{E, L\}$, $(x_1, x_2, \ldots, x_{k+1}, x_{k+2}) = (E, P, \ldots, P, E)$ and $d_{k+3}^* \in \{A, L\}$
Define $x^0 = x$. Take $d_2^0 = L$, $d_3^0 = R$, $d_k^0 = L$, $d_{k+1}^0 = R$, for $i \neq 2, 3, k, k+1$, $d_i^0 = d_i^*$, and $x^1 = f(d^0)$. According to Case A.5.3(a) and (b), the kP-string shrinks from both sides. It holds that $(x_1^1, x_2^1, x_3^1, \ldots, x_k^1, x_{k+1}^1, x_{k+2}^1, x_{k+3}^1) = (E, E, P, \ldots, P, E, E, E)$. Lemma 1.3 implies $p(x^1) \leq p(x^0)$. For $m = 1, \ldots, \lfloor (k-4)/2 \rfloor$, define $d_{2+m}^m = L$, $d_{3+m}^m = R$, $d_{k-m}^m = L$, $d_{k-m+1}^m = R$, for $i \neq 2+m, 3+m, k-m, k-m+1$, $d_i^m = d_i^*$, and $x^{m+1} = f(d^m)$. According to Case A.5.3(c), it holds that x^{m+1} has a $(k-2m-2)P$-string $[3+m, k-m]$ and $x_{1+m}^{m+1} = x_{2+m}^{m+1} = x_{k-m+1}^{m+1} = x_{k-m+2}^{m+1} = x_{k-m+3}^{m+1} = E$. Lemma 1.3 implies that $p(x^{m+1}) \leq p(x^m)$. If k is even, then take $d_{k/2+1}^{k/2-1} = L$, $d_{k/2+1}^{k/2-1} = R$, for $i \neq k/2+1, k/2+2$, $d_i^{k/2-1} = d_i^*$, and $x^{k/2} = f(d^{k/2} - 1)$. According to Case A.5.4, it holds that the $2P$-string $[k/2+1, k/2+2]$ disappears and Lemma 1.3 implies that $p(x^{k/2}) \leq p(x^{k/2-1}) - 1$. If k is odd, then take $d_{(k+1)/2}^{(k-3)/2} = L$, $d_{(k+3)/2}^{(k-3)/2} = R$, $d_{(k+5)/2}^{(k-3)/2} = R$, for $i \neq (k+1)/2, (k+3)/2, (k+5)/2$, $d_i^{(k-3)/2} = d_i^*$, and $x^{(k-1)/2} = f(d^{(k-3)/2})$. According to Case A.5.5, it holds that $x^{(k-1)/2}$ has two $1P$-strings $\{(k+1)/2\}$ and $\{(k+5)/2\}$. Lemma 1.3 implies that $p(x^{(k-1)/2}) \leq p(x^{(k-3)/2}) + 1$. Moreover, according to Case A.5.5, there is $x^{(k+1)/2} \in F(x^{(k-1)/2})$ such that $p(x^{(k+1)/2}) \leq p(x^{(k-1)/2}) - 2$ and the $1P$-strings $\{(k+1)/2\}$ and $\{(k+5)/2\}$ disappear.

Case A.5.10: $d_0^* \in \{A, R\}$, $(x_1, x_2, \ldots, x_{k+1}, x_{k+2}) = (E, P, \ldots, P, E)$ and $d_{k+3}^* \in \{A, L\}$
Take $d_2 = L$, $d_3 = R$, $d_k = L$, $d_{k+1} = R$, for $i \neq 2, 3, k, k+1$, $d_i = d_i^*$, and $y = f(d)$.

According to Case A.5.3(a) the kP-string shrinks from both sides. It holds that $(y_1, y_2, y_3, \ldots, y_k, y_{k+1}, y_{k+2}, y_{k+3}) = (E, E, P, \ldots, P, E, E, E)$. Lemma 1.3 implies $p(y) \leq p(x)$. From here, the argument is identical to the one in Case A.5.9.

Case A.5.11: $(x_1, x_2, \ldots, x_{k+1}, x_{k+2}) = (A, P, \ldots, P, E)$ and $d^*_{k+3} \in \{A, L\}$
On the left-hand side either Case A.5.1(a) or Case A.5.1(b) applies and on the right-hand side either Case A.5.2(b) or Case A.5.2(c) applies. This give rise to the following four possibilities:

(i) Case A.5.1(a) and Case A.5.2(b). Take $d_2 = R$, $d_3 = L$, $d_k = R$, $d_{k+1} = L$, for $i \neq 2, 3, k, k+1$, $d_i = d^*_i$, and $y = f(d)$. The kP-string $[2, k+1]$ grows and Lemma 1.3 implies $p(y) \leq p(x)$.

(ii) Case A.5.1(b) and Case A.5.2(b). Take $d_2 = R$, $d_3 = L$, $d_k = R$, $d_{k+1} = L$, for $i \neq 2, 3, k, k+1$, $d_i = d^*_i$, and $y = f(d)$. It holds that $(y_0, y_1, y_2, \ldots, y_{k+1}, y_{k+2}) = (E, E, P, \ldots, P, E)$ and Lemma 1.3 implies $p(y) \leq p(x)$. Either Case A.5.8 or Case A.5.9 applies. In the former case there is $z \in F(y)$ such that the kP-string $[2, k+1]$ grows and Lemma 1.3 implies $p(z) \leq p(y)$. In the latter case, if k is even there is $x^{k/2} \in F^{k/2}(y)$ such that $p(x^{k/2}) \leq p(y) - 1$ and the kP-string $[2, k+1]$ shrinks until it disappears, and if k is odd there is $x^{(k+1)/2} \in F^{(k+1)/2}(y)$ such that $p(x^{(k+1)/2}) \leq p(y) - 1$ and the kP-string $[2, k+1]$ shrinks until it disappears.

(iii) Case A.5.1(a) and Case A.5.2(c). Take $d_2 = R$, $d_3 = L$, $d_k = R$, $d_{k+1} = L$, for $i \neq 2, 3, k, k+1$, $d_i = d^*_i$, and $y = f(d)$. It holds that $(y_1, \ldots, y_k, y_{k+1}, y_{k+2}) = (P, \ldots, P, E, E)$ and Lemma 1.3 implies $p(y) \leq p(x)$. If $y_0 = P$ then y contains a $\geq(k+1)P$-string and we are done, so consider the case where $y_0 \neq P$. Either Case A.5.7, Case A.5.8 or Case A.5.9 applies. If Case A.5.7 applies there is $z \in F(y)$ such that the kP-string $[1, k]$ grows and Lemma 1.3 implies $p(z) \leq p(y)$. For the other two cases, the argument is identical to the one in (ii).

(iv) Case A.5.1(b) and Case A.5.2(c). Take $d_2 = R$, $d_3 = L$, $d_k = R$, $d_{k+1} = L$, for $i \neq 2, 3, k, k+1$, $d_i = d^*_i$, and $y = f(d)$. It holds that $(y_0, y_1, y_2, \ldots, y_k, y_{k+1}, y_{k+2}) = (E, E, P, \ldots, P, E, E)$ and Lemma 1.3 implies $p(y) \leq p(x)$.
We first consider the case $k \geq 5$. Take $d'_2 = R$, $d'_3 = L$, $d'_{k-1} = R$, $d'_k = L$, for $i \neq 2, 3, k-1, k$, $d'_i = d^*_i$, and $z = f(d')$. It holds by Case A.5.2(a) that $(z_1, \ldots, z_{k+1}) = (P, \ldots, P)$, so the kP-string $[2, \ldots, k+1]$ grows and Lemma 1.3 implies $p(z) \leq p(y)$.
Now consider the case $k = 4$. Case A.5.1(b) implies $\pi_0 = 2$, so $d_{-1} \in \{A, R\}$ and $\pi_{-1} \leq 1 - c$, and hence $y_{-1} = E$. Take $d'_2 = R$, $d'_3 = R$, $d'_4 = L$, for $i \neq 2, 3, 4$, $d'_i = d^*_i$, and $z = f(d')$. It holds that $(z_0, z_1, z_2, z_3, z_4, z_5) = (E, E, P, P, P, P)$ and Lemma 1.3 implies $p(z) \leq p(y)$. The argument now continues as in (iii). □

$(d_3, \ldots, d_{k+2}) =$		$(d^*_{k+3}, d^*_{k+4}) =$		
$(P, \ldots, P)$		$(A, A/R/L/E)$	$(E, E/R)$	$(E, A/L)$
(d^*_1, d^*_2)	$(A/R/L/E, A)$	g (A.5.6)	g (A.5.7)	g; d (A.5.11)
	$(E/L, E)$		g (A.5.8)	d (A.5.9)
	$(A/R, E)$			d (A.5.10)

Table A.5: Resulting structures for $\geq 4P$-strings surrounded from the left by the decisions in the columns and from the right by the decisions in the rows. Abbreviations 'g' and 'd' in the cells are used to identify whether the P-string will be growing or disappearing. Relevant cases are in parentheses.

Chapter 2

"Friends Are Thieves of Time": Heuristic Attention Sharing in Stable Friendship Networks

2.1 Introduction and Motivation

This chapter studies the decentrally emerging stable networks when all agents employ a simple decision rule and invest proportionally more in neighbours who have a lower degree. In the absence of complex optimisation at the level of every network link (cf. Brueckner (2006), Deroian (2009), Bloch and Dutta (2009), Salonen (2016), So (2016), Bourlès et al. (2017), Baumann (2021)) as opposed to the node/agent level, a common assumption in the literature has been that agents divide their limited resources equally amongst their neighbours. The Jackson and Wolinsky (1996) co-author model is a prime example of such a setup.[1] While a proportional allocation of resources, e.g. time, is a fair assumption in such a context, other social interactions could be driven by a different dynamic. Take a friendship network, for example. If one is interested in developing an enduring and deep friendship, this requires big investment from both parties and people do not usually allocate their time and effort equally amongst everyone they know – they have friends they see every day but they also have acquaintances with whom they have just minute-long conversations. Since no one wants to be on the more giving end of a "one-sided" friendship where the attention given is much more than the attention one gets, a rational friend-seeker would try to give more time/attention to people who are expected to return it. Even within the co-author context it would be wise to expect a smaller contribution from a co-author who is involved in many projects. In

This chapter is based on Tenev (2020).

[1]Consider also Albornoz et al. (2019) and Harmsen-van Hout et al. (2013) which have similar assumptions. Bala and Goyal (2000) and Galeotti et al. (2010) assume that links are of intensity either 0 or 1.

the presence of synergies and assuming that time and effort are the most important factors determining the quality of a paper one would be better off investing more in co-authors who are involved in fewer projects and less in co-authors who are involved in more projects to have higher utility, ceteris paribus. In the extreme cases, if author A connects with author B who has only one project, the common project AB would be finished much faster compared to a situation in which A connects to author C who has ten projects. If author C wants to allocate any time to even one of the remaining nine projects, he would have to spend strictly less than all of his research time on the common project AC.

Harmsen-van Hout et al. (2016) show experimentally that when faced with linking choices involving increased complexity of payoffs, experimental subjects resort to simplified decision rules, i.e. heuristics.[2] The authors identify two factors that are related to payoffs only qualitatively, but have been used by the experimental subjects: whether or not the choice option involves a deviation from the status quo and the number of direct neighbours of the (potential) linking partner. They call for future models to include the human tendencies[3] "to base complex linking decisions on heuristic cues like... node degree rather than exact payoff" (Harmsen-van Hout et al., 2016), as found by their analysis.

Addressing these findings, the current chapter investigates what happens if agents decide to form stronger connections with neighbours who have fewer links. More precisely, the setup assumes that link investments are proportionally greater when they are formed with nodes of a lower degree (since the connections are bilateral, in-degree is the same as out-degree). The heuristic in the current setup is intuitively appealing, since a self-interested agent would try to maximise his potential payoff and agents who are less connected have on average more resources to allocate to a specific link. The benefit is higher if the degree of the potential neighbour is lower, since he would have to spend his resources on fewer people.[4] If the agents are unaware of the exact functional form of the utility per link or if they cannot have a complete overview of the system they are part of (i.e. they have cognitive limitations), the degree gives an indication of how much one can benefit from a specific potential neighbour (on average). Using such a "rule of thumb", the model diverges from the literature on best responses to specific investment that the agents could get from their neighbours and enters the realm of heuristics-driven decisions.

One of the main objectives of the current study is to show that using a heuristic different than the equal split but equally plausible and intuitively appealing results in a highly non-trivial change in the predictions of the model, bringing it closer to observed reality. Namely,

[2]A related finding is discussed by Kovářík et al. (2018) in the context of learning where "people facing more complex environments... seem to resort to simpler learning rules."

[3]Consider also Hämäläinen et al. (2013) who urge researchers to include behavioural effects in OR processes.

[4]A similar logic is also captured in the Horse race betting model where betting on each horse proportional to its probability of winning is log-optimal (Cover and Thomas, 2006).

the pairwise stable networks in the current setup can have irregular components and clusters, something characteristic of social networks (Jackson and Rogers, 2007). Moreover, they have close to optimal welfare. This all is achieved at a very low cost – a decision rule that does not require complicated computations or complex strategic considerations like best responses or farsightedness,[5] for instance. There are also no assumptions like homophily as all nodes are identical at the start and their potential matchings are random.[6] Here, it is crucial that a distinction is made between the social and the informational value of links (Harmsen-van Hout et al., 2013). In particular, this setup assumes that social (direct) value is the source of utility as would be the case in personal relationships between friends, rather than the informational (indirect) value. In this sense, people wanting to be connected to the most popular nodes (in school, for example) want that for the indirect value (getting some of the popularity), not because of the direct impact that would have (support in difficult times, for instance).

This model bears a crucial similarity with the co-author model (Jackson and Wolinsky, 1996) in that it depends on the degree of the agents. There the indirect connections, the co-author's co-authors, influence the payoff of an agent because they take from the co-author's time and so impose an externality on the co-author's co-authors. The setup of the current chapter captures a more elaborate model of indirect (positive) externalities. For example, in a situation in which A is connected to B and X, while B is connected to A and Y, if B decides to create a new connection (say, to Z), this affects not only B's contribution to A and A's investment in B. It is interesting to note that through the second effect A's investment in X will also increase. In fact, B's decision to form a new link positively influences his neighbours' neighbour, while affecting his direct neighbours negatively. In its core idea of nodes differentiating the investment they commit to their neighbours this chapter is closest to Baumann (2021). She analyses a model of weighted network formation and characterises the types of equilibria that occur. However, there players can invest in others or in themselves and the equilibria are neither pairwise stable nor efficient.

The remainder of the chapter starts by outlining the model and considering some of its welfare properties (Section 2.2). It proceeds to analyse regular and commonly used network structures that can be approached analytically within the current framework (Section 2.3), contrasting the outcomes with the results seen when the resources are equally split amongst all neighbours. Section 2.4 is devoted to the analysis of the types of stable networks with irregular components that emerge from simulations of the current model. The last section

[5]Morbitzer et al. (2014) show through simulations that the co-author model can produce a number of irregular networks when the agents exhibit farsightedness.

[6]Homophily, whereby similar types of nodes are more likely to be connected than dissimilar ones, tends to characterise all social networks (McPherson et al., 2001), and produces clustering. Consider Bramoullé et al. (2012) who, building on the work of Jackson and Rogers (2007), introduce individual heterogeneity to investigate homophily as a result of biased meeting processes.

concludes the discussion.

2.2 Model

There is a set of agents $N \in \{1, 2, ..., n\}$, who form a network. A connection between two nodes can be sustained only with a positive investment from both parties. The payoff from a specific link depends positively on the bilateral investment of the involved agents. For all nodes i of the network, if t_{ij} is the contribution that agent i has in his link with agent j, investments in links are constrained to $t_{ij} > 0$ (if investments are 0 there is no link). Additionally, if N_i is the set of node i's neighbours, $\sum_{j \in N_i} t_{ij} = 1$ for all i, one being the highest contribution a node can make on a single link. Nodes cannot invest in themselves and therefore they always invest all their resources (a node with no links will have zero utility). There are no additional explicit costs for maintaining links except the contribution to the link, which is an opportunity cost of not investing in others. The investment in connections with neighbours is based on the following heuristic: *links are proportionally stronger when they are formed with nodes of a lower degree.* This means that in an equal-degree network every node spreads its resource of 1 equally amongst its neighbours, but also that if a node has two neighbours, one with degree 2 and the other one with degree 1, the neighbour with degree 1 will get an investment twice as big as the other neighbour. Therefore, the investment per link is inversely proportional to the degree of the node it connects to, so $t_{ik} = \frac{1/d_k}{\sum_{j \in N_i} 1/d_j}$, with d_i being node i's degree. The payoff of a specific link is given by $\sqrt{t_{ij} t_{ji}}$, which exhibits constant returns to scale.[7][8] Such a functional form ensures that spreading resources between neighbours can be more beneficial than investing in only one neighbour. Following the specified heuristic for spreading resources between links, the payoff of every node i can be expressed and rearranged in the following way:

$$u_i = \sum_{k \in N_i} \sqrt{\frac{\frac{1}{d_k}}{\sum\limits_{j \in N_i} \frac{1}{d_j}} * \frac{\frac{1}{d_i}}{\sum\limits_{\ell \in N_k} \frac{1}{d_\ell}}} = \frac{1}{\sqrt{\sum\limits_{j \in N_i} \frac{d_i}{d_j}}} \left(\sum_{k \in N_i} \frac{1}{\sqrt{\sum\limits_{\ell \in N_k} \frac{d_k}{d_\ell}}} \right).$$

In other words, according to the heuristic, the payoff of an agent is a function of the ratio of the degree of the agent compared to all his neighbours' degrees and the ratio between the degree of each one of the agent's neighbours and the neighbours' neighbours.

[7]The results are qualitatively similar for other concave functions.

[8]A similar payoff function is used in the occupational choice model with spillovers by Albornoz et al. (2019).

Efficiency

As usually, a regular network refers to a network where every node has the same degree. Additionally, in this chapter the term regular components/networks will only be used for components/networks with degrees greater than or equal to one, i.e. ones which have no isolated nodes.

Lemma 2.1. *The payoff for every node in a regular network is* 1.

Proof. Follows directly from the payoff expression and the fact that the degree of all nodes is the same, i.e. $d_i = d_j = d_k = d_\ell$ and $|N_i| = |N_k|$.

$$\frac{1}{\sqrt{\sum_{j \in N_i} \frac{d_i}{d_j}}} \left(\sum_{k \in N_i} \frac{1}{\sqrt{\sum_{\ell \in N_k} \frac{d_k}{d_\ell}}} \right) = \frac{1}{\sqrt{|N_i|}} \frac{|N_i|}{\sqrt{|N_k|}} = 1.$$

□

Proposition 2.1. *The maximum utilitarian welfare of a network with n nodes is n.*

Proof. It follows from Lemma 2.1 that the welfare is n when the network is complete. This means that the sum of all nodes' potential contributions $n * 1$ has been recovered after the investment in the network. In order to get welfare of more than n in a stable state some nodes must get more than their overall payoff of 1. Moreover, this has to compensate for the ones that get less than 1 (if there are such nodes). For a node to get more than 1 overall payoff, it has to get more than its investment in a link from at least one link. In other words, for a link between node i with contribution t_{ij} and node j with contribution t_{ji} to provide a higher return for i, the following needs to hold: $t_{ij} < \sqrt{t_{ij}t_{ji}}$, or $t_{ij} < t_{ji}$. This implies that the return for j would be strictly smaller. If the net gain of every link (the combined gains of both sides i and j compared to their investments t_{ij} and t_{ji} in the link) in the network is negative, then the welfare will be strictly smaller than n. So, for utilitarian welfare greater than or equal to n at least one link should have a non-negative net gain. In other words, the sum of the two investments in the link t_{ij} and t_{ji} and the return to the two players $2\sqrt{t_{ij}t_{ji}}$ should be positive:

$$-t_{ij} - t_{ji} + 2\sqrt{t_{ij}t_{ji}} \geq 0 \Leftrightarrow (\sqrt{t_{ij}} - \sqrt{t_{ji}})^2 \leq 0.$$

This is only possible for $t_{ij} = t_{ji}$ and then all investments are recovered, so there is a payoff of 1 for every node, for a total welfare of n. □

Corollary 2.1. *A network consisting of regular components has maximum utilitarian welfare.*

Utilitarian welfare is not the only notion of welfare, another prominent measure is Rawlsian welfare, measured by the agent who is worst-off.

Proposition 2.2. *The maximum Rawlsian welfare of a network with n nodes is* 1.

Proof. Rawlsian welfare has an upper bound that is implied by the reasoning suggested in Proposition 2.1. In order for it to be bigger, all nodes need to have payoff higher than 1, which would contradict Proposition 2.1. □

The propositions above establish that in the current model the most efficient networks are the ones with regular components. Since this chapter focuses on social interactions and they rarely if ever produce fully regular networks, the results on efficiency present merely a benchmark against which all stable networks can be measured.

2.3 Analytical Observations

This section looks into networks frequently appearing in the literature to identify which ones are pairwise stable in the current setup. This chapter uses the notion of pairwise stability (Jackson and Wolinsky, 1996), whereby links do not exist if one of the parties is strictly worse off with the link (they are severed unilaterally) and both nodes need to agree to form a link (bilateral consent). More formally, in a network if u_i, u_j are the utilities of nodes i and j for $i \in N_j$ and $j \in N_i$ and u'_i, u'_j are the utilities of the same nodes i and j for $i \notin N_j$ and $j \notin N_i$, keeping all other links constant, a network is pairwise stable if it holds that: (i) $u_i \geq u'_i$ and $u_j \geq u'_j$, for all existing links ij; and (ii) if $u'_i < u_i$, then $u'_j > u_j$ for all non-existing links ij.

Complete Networks

Proposition 2.3. *The complete network is stable.*

Proof. In a fully connected graph every node has payoff 1. In order for the complete graph to be stable, removing a link should be equally good or worse than the status quo for both nodes that are connected so that they decide not to disconnect. For a complete graph with n nodes this implies that after disconnecting the two nodes have degrees $n-2$, while the other $n-2$ nodes keep their degrees of $n-1$. The two nodes with degrees $n-2$ each have $n-2$ neighbours with degrees $n-1$, while the nodes with degrees $n-1$ have two neighbours with degrees $n-2$ and $n-3$ neighbours with degrees $n-1$. This is equivalent to:

$$\frac{1}{\sqrt{\frac{(n-2)^2}{n-1}}}\left(\frac{n-2}{\sqrt{2\frac{n-1}{n-2}+\frac{n-1}{n-1}(n-3)}}\right) \leq 1 \Leftrightarrow \frac{\sqrt{n-1}}{n-2}\left(\frac{n-2}{\sqrt{\frac{2n-2+(n-3)(n-2)}{n-2}}}\right) \leq 1 \Leftrightarrow$$

$$(n-1)(n-2) \leq 2n-2+(n-3)(n-2).$$

This always holds for $n > 2$. Therefore, no two nodes would decide to disconnect in a complete graph with $n > 2$. If the network has $n = 2$, it is clearly stable as disconnecting would bring both nodes 0. □

This establishes existence of stable networks since for any n there is a stable network that is also efficient. While the complete network is stable and it achieves maximum welfare, it might not be possible to always reach it. Fortunately, it is possible to construct paths for reaching it from the empty network as long as there is a positive probability of adding a link at any step of the process.

Proposition 2.4. *The complete network is reachable from the empty network.*

Proof. See Appendix B.1. □

The following proposition condenses what is known from the previous observations about regular networks and introduces a short discussion of the networks which are not pairwise stable under the current setup.

Proposition 2.5. *A regular network is stable if and only if it is complete.*

Proof. See Appendix B.1. □

Proposition 2.5 shows that some of the potentially welfare maximising networks, like networks with incomplete regular components, are not stable. This is an interesting result as high regularity in components is indeed *not* characteristic of social networks.[9]

Unstable Incomplete Networks

Proposition 2.5 gives the following corollary.

Corollary 2.2. *The circle network with $n \geq 4$ nodes is not stable.*

Proposition 2.6 below, just like the corollary above, shows that often used and investigated network types are all unstable in the current setting.

Proposition 2.6. *The line, star with $n \geq 3$ peripheral nodes, wheel with $n \geq 4$ peripheral nodes and biregular graphs are not stable.*[10]

Proof. See Appendix B.1. □

[9]Cf. for example the discussion in Jackson and Rogers (2007) of the relatively small distance between any pair of nodes and the fact that there are more nodes with relatively high and low degrees in social networks.

[10]Biregular graphs are bipartite graphs in which vertices in the same subset of nodes of the given bipartition have the same degree.

Networks with Complete Components

Besides complete networks, networks with complete components can also be pairwise stable. As observed above, such networks would also exhibit maximum welfare.

Proposition 2.7. *A network consisting of two disconnected complete components with sizes* $m_1, m_2 \geq 2$ *such that* $m_1 + m_2 = n$ *is stable if and only if* $|m_1 - m_2| \geq 2$.

Proof. See Appendix B.1. □

Corollary 2.3. *A network consisting of* k *disconnected complete components of respective sizes* $m_1, \ldots, m_k \geq 2$ *such that* $\sum_{j=1}^{k} m_j = n$ *is stable if and only if* $|m_j - m_{j'}| \geq 2$ *for all* $j, j' = 1, \ldots, k$ *with* $j \neq j'$.

To interpret the result above in the context of a friendship network, one can imagine that having a few good friends is as good as having many marginal friends in terms of social value. In this case they form a group in which everyone gets equal attention from their friends which is a qualitatively different result from Proposition 2.5. A graph with regular components is not sufficient for stability, the components need to be complete – getting equal treatment from your friends is only stable when you are a part of a tight social group.

The following statement shows that the observations made so far are the only ones needed to analyse the stable states for small networks with $n \leq 11$ nodes. After that the model also produces stable networks with irregular components. They are discussed in the next section. This proposition is given without an analytical proof since it has been verified by checking all possible options with a computer.

Proposition 2.8. *In graphs with:*

(a) $n \leq 5$ *nodes the complete network is the only stable network;*

(b) $5 \leq n \leq 11$ *nodes the only stable networks are the complete network and networks consisting of two disconnected complete components with sizes* $m_1, m_2 \geq 2$ *such that* $m_1 + m_2 = n$ *and* $|m_1 - m_2| \geq 2$.

Proof. a) The propositions above exclude many potential networks. Others are excluded for having a node unconnected to any other nodes. These cases are not stable because the isolated node would want to connect and the others would want to connect with it. The rest can be verified by a case-by-case check. b) Verified on a case-by-case basis.[11] □

[11] The proposition has been fully verified by explicitly checking all possible networks. The data used have been taken from https://users.cecs.anu.edu.au/∼ bdm/data/graphs.html and for the case of $n = 11$ they were additionally generated by the software nauty (McKay and Piperno, 2014). The author of this chapter owes special gratitude to Matúš Mihalák from the Department of Data Science and Knowledge Engineering at Maastricht University for his technical assistance with this task.

In other words, for $n \leq 11$ Propositions 2.5 and 2.7 describe all stable networks, since there can be no more than two fully connected disjoint parts for these cases.

At this stage it is instructive to contrast the results of the current setup with the situation in which everything is the same (in particular the payoff function) except that the heuristic applied is equal split of resources amongst all neighbours, i.e. $t_{ij} = 1/d_i$ for all $i \in N_i$.

Proposition 2.9. *Under equal split of resources the stable networks have no incomplete or irregular components.*[12]

Proof. See Appendix B.1. □

Stated alternatively, under equal split the stable networks can only be the complete network or networks consisting of disjoint complete components. In light of Proposition 2.9, the results described above mirror the ones for the case when resources are spread equally amongst the neighbours. The networks under equal split also exhibit maximum efficiency as per Corollary 2.1. There is, however, one major difference between the current setup and the equal split. Under equal splitting of resources there are no other stable networks but ones consisting of complete subgraphs. It is important to note that only changing the way resources are split between a node's neighbours from equal to the current heuristic produces a much bigger and diverse set of stable networks.[13] They are the subject of investigation in the next section.

2.4 Stable Networks with Irregular Components: Simulations

Since the model produces a plethora of possible stable networks which are not regular and have no regular components, computer simulations were employed to see the range of potential

[12]While this proposition does not appear in their work, this setup is investigated in Proposition 2 in Harmsen-van Hout et al. (2013), corresponding to their case of $\rho = 1/2$. They observe that regular networks with degrees $d < n - 1$ are not stable and networks consisting of fully connected components are stable if and only if $m \geq 4\ell - 2$ for m and ℓ being the number of nodes in every component. Moreover, they also find that the star, circle and wheel are not stable.

[13]Moreover, substituting the current heuristic for the equal split used in Jackson and Wolinsky (1996) also qualitatively changes the set of stable networks that the co-author model produces, even though the payoff function it uses is different in a few dimensions from the one in this chapter. Under equal split a pairwise stable network can be partitioned into fully intraconnected components with a different number of members (as specified in Proposition 4 of Jackson and Wolinsky (1996)), while the current heuristic once again produces stable networks which can consist of irregular components. One can speculate that uniform treatment like equal splitting of resources is the main force producing only regular stable networks in this case.

outcomes. Cycles are possible with a specific non-random matching procedure.[14] One simple cycle that occurs under this protocol is described in Appendix B.3.[15] However, under random matching of potential neighbours all simulations converged to a stable network. These results are investigated below with some examples which show the main observed trends.

Examples

Example 2.1 below gives an in-depth look into one of the simplest irregular stable networks of the model. Example 2.2 shows the variability of individual payoffs within the same stable network consisting of one component. Example 2.3 zooms out and shows that multiple components, regular and irregular, can form a stable network.

Example 2.1. ***Stability and Welfare***

Consider the stable network in Figure 2.1a. It consists of two cliques – an upper and a lower one which are directly connected. The members of the upper clique all have degree 5*, while the members of the lower one have degrees* 8*. The payoff of all nodes in the upper clique is:*[16]

$$\frac{1}{\sqrt{\frac{5*3}{5}+\frac{5*2}{8}}}\left(\frac{2}{\sqrt{\frac{8}{5}+\frac{8*7}{8}}}+\frac{3}{\sqrt{\frac{5*3}{5}+\frac{5*2}{8}}}\right)=1.037$$

while the nodes in the lower clique have only:

$$\frac{1}{\sqrt{\frac{8}{5}+\frac{8*7}{8}}}\left(\frac{7}{\sqrt{\frac{8*7}{8}+\frac{8}{5}}}+\frac{1}{\sqrt{\frac{5*3}{5}+\frac{5*2}{8}}}\right)=0.979$$

leading to overall welfare of 11.982*. The network is stable and this is shown explicitly below. There are three types of links – between the nodes in the upper clique, between the ones in the*

[14]Using the conventional ordering of the adjacency matrix, it is possible to find a cycle in the following way: first, check if a link can be formed. If a link is formed, then the stability of all possible links is tested (once again following the specified order). This procedure stops if no links (could be more than one) can be severed. After no more links can be cut, the check for adding links is implemented again. The procedure is repeated until no changes (adding or severing links) are possible.

[15]This serves as testimony that proving convergence for all cases even under random matching is not a trivial matter. The cycle illustrates that utilitarian welfare can vary in both directions as unstable networks add or sever links. This is telling since utilitarian welfare is the usual suspect for a function (similar to a potential function) a la Jackson and Watts (2001) which behaves in a predictable way, always increasing or always decreasing between two adjacent networks to show no cycles are possible in the process. Further, Hellmann and Staudigl (2014) survey different techniques which have been applied to show existence of pairwise stable networks, none of which involve properties exhibited by the current model. In particular Hellmann (2013) identifies common network properties for which there is no closed improving network cycle. One of them is ordinal convexity in own links, i.e. if a link is desirable for a player at some point, it stays desirable when adding new links. This is clearly not the case in the current setup.

[16]All numbers are rounded up to the third digit.

lower clique and between the two cliques. Severing one of the links would make no agent(s) strictly better off. For example, cutting the link between a node in the upper clique and a node in the lower clique would bring the node in the upper clique:

$$\frac{1}{\sqrt{\frac{4*3}{5}+\frac{4}{8}}}\left(\frac{1}{\sqrt{\frac{8}{4}+\frac{8*6}{8}+\frac{8}{7}}}+\frac{3}{\sqrt{\frac{5*2}{5}+\frac{5}{4}+\frac{5*2}{8}}}\right)=1.025<1.037$$

and it would bring to the node in the lower clique:

$$\frac{1}{\sqrt{\frac{7*7}{8}}}\frac{7}{\sqrt{\frac{8*6}{8}+\frac{8}{5}+\frac{8}{7}}}=0.957<0.979.$$

Similarly, none of the links between members of the upper clique would be cut:

$$\frac{1}{\sqrt{\frac{4*2}{5}+\frac{4*2}{8}}}\left(\frac{2}{\sqrt{\frac{8}{4}+\frac{8*7}{8}}}+\frac{2}{\sqrt{\frac{5}{5}+\frac{5*2}{4}+\frac{5*2}{8}}}\right)=0.983<1.037.$$

None of the links between members of the lower clique would be cut:

$$\frac{1}{\sqrt{\frac{7}{5}+\frac{7*6}{8}}}\left(\frac{6}{\sqrt{\frac{8*6}{8}+\frac{8}{5}+\frac{8}{7}}}+\frac{1}{\sqrt{\frac{5*3}{5}+\frac{5}{8}+\frac{5}{7}}}\right)=0.973<0.979.$$

Finally, the only type of link that could be formed is between the upper and lower clique. For instance, this would bring the node in the upper clique:

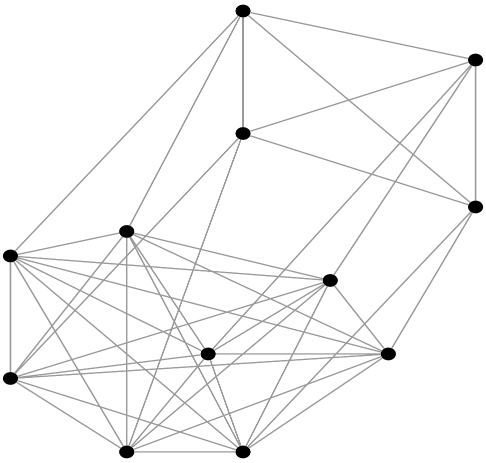
$$\frac{1}{\sqrt{\frac{6*3}{5}+\frac{6*2}{8}+\frac{6}{9}}}\left(\frac{1}{\sqrt{\frac{9}{5}+\frac{9*7}{8}+\frac{9}{6}}}+\frac{1}{\sqrt{\frac{5}{6}+\frac{5*2}{5}+\frac{5}{9}+\frac{5}{8}}}+\frac{2}{\sqrt{\frac{8}{6}+\frac{8*6}{8}+\frac{8}{9}}}+\frac{2}{\sqrt{\frac{5*2}{5}+\frac{5*2}{8}+\frac{5}{6}}}\right)=$$
$$=1.035<1.037$$

and therefore the link would not be formed. In contrast, nodes in the lower clique can potentially benefit from this link:

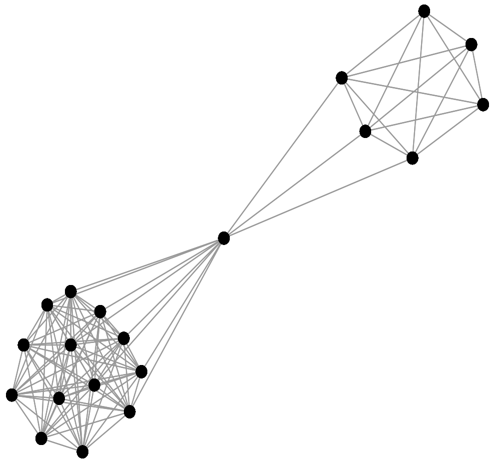
$$\frac{1}{\sqrt{\frac{9*7}{8}+\frac{9}{6}+\frac{9}{5}}}\left(\frac{1}{\sqrt{\frac{6*3}{5}+\frac{6*2}{8}+\frac{6}{9}}}+\frac{1}{\sqrt{\frac{5}{6}+\frac{5*2}{5}+\frac{5}{9}+\frac{5}{8}}}+\frac{2}{\sqrt{\frac{8}{9}+\frac{8}{6}+\frac{8*6}{8}}}+\frac{5}{\sqrt{\frac{8}{9}+\frac{8}{5}+\frac{8*6}{8}}}\right)=$$
$$=0.996>0.979.$$

Example 2.2. ***Degrees and Individual payoffs***

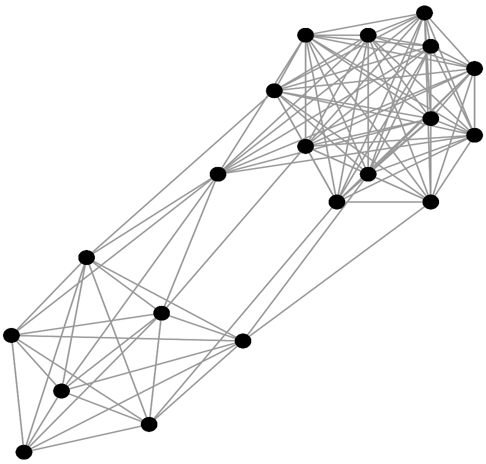
Figure 2.1b once again has two cliques but in each of them some members are also connected to an intermediary node and as a result they have one degree higher than the rest (degree 6 *in the case of the three nodes in the upper clique and degree* 13 *in the case of nodes from*



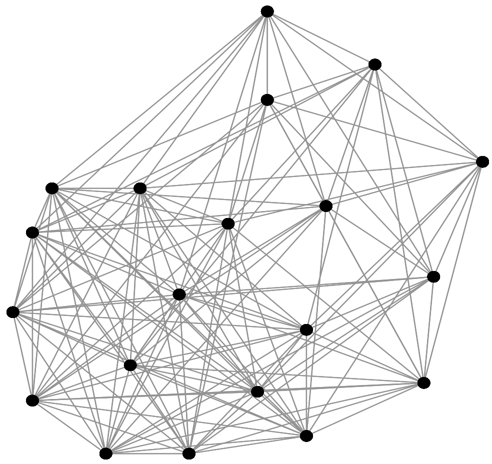
(a) The smallest simulated irregular stable network: $n = 12$



(b) Upper and lower cliques linked by an intermediary agent: $n = 20$



(c) Two cliques linked by an intermediary, but also with separate links: $n = 20$



(d) A network with a core-periphery structure: $n = 20$

Figure 2.1: Irregular stable networks

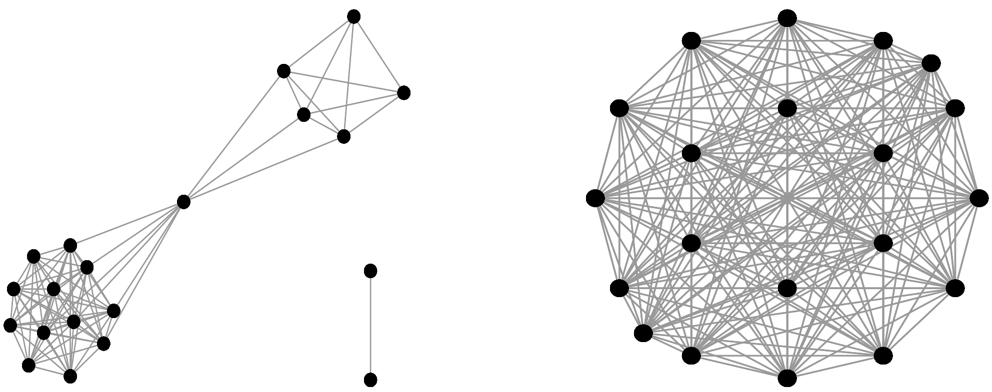


Figure 2.2: A stable network with regular and irregular components: $n = 40$

the lower clique). In this setting the highest payoffs 1.015 *are reaped by the members of the smaller clique who are not connected to anyone else, while the intermediary has the smallest payoff* 0.96*. The overall welfare is* 19.966*.*

Figure 2.1c shows a very similar structure, but in it the two clearly delineated cliques also have direct connections between each other. Interestingly, the node which is connected to the upper clique directly and through the intermediary has payoff 1.019*, another node has two direct connections to the upper clique and the highest payoff of* 1.021*, while the node which is connected only through the intermediary node has payoff* 0.998*. The lowest payoff in the case,* 0.984 *is for the members of the upper clique who are not in any way connected to the lower one. Moreover, nodes with the same degree, can have different payoffs, e.g. three different nodes in the upper clique have the same degree of 12, but one of them has* 0.993*, the second one has* 0.984*, while the third one has* 0.988*. The overall welfare is* 19.968*.*

Figure 2.1d has overall welfare of 19.995*. It has* 13 *nodes with degrees* 15*,* 11 *with degree* 4*, and one with degree* $12, 13$ *and* 14*.*

Example 2.3. ***Multiple components***

Figure 2.2 illustrates another interesting feature of the stable networks in the model – they can consist of disconnected irregular and regular components. The irregular component in such a structure is stable on its own. However, it cannot be verified that every stable irregular component can be integrated in such a structure with multiple complete components and still stay stable. In fact Figure 2.1b presents a counterexample. Combined with regular complete components with sizes between 6-8 and between 11-18 in one network the resulting whole in not stable. However, this is not the case for complete components with sizes between 2-5 or 9 and 10. This particular result is most likely related to the sizes of the cliques. The other subparts of the figure seem to be more resistant to such events, since combining Figures 2.1a,2.1c,2.1d with complete regular components of sizes between 2-60 still results in stable networks.

As illustrated by Figure 2.2 stable networks can consist of one irregular component and multiple regular ones. While combining any two of the irregular networks in Figure 2.1 does not yield a stable network, stable networks can consist of more than one irregular component.

Network Structure

Stable networks consisting of irregular components start appearing for $n \geq 12$. The components fall into three qualitative categories: networks with cliques and direct links between them (Figure 2.1a, 2.1b), networks with cliques and intermediaries (Figure 2.1c) between them and networks with a core-periphery structure (Figure 2.1d). The cliques are reminiscent of the disjoint complete components that were described in the previous section, since they are connected relatively regularly and moreover, they are never of the same (or very similar) sizes.

However, the irregular stable networks occasionally have a core and some relatively peripheral nodes like in Figure 2.1d.

It would be extremely unusual if this simple model perfectly reproduced all stylised facts about social networks. However, qualitatively some curious features can be identified. For example, in the simulated networks the distance between pairs of nodes in components is relatively small. The networks exhibit a high degree of clustering, high degree nodes tend to be related to other high-degree nodes and low-degree nodes also tend to be related to lower-degree nodes. There are fewer nodes with medium degrees and relatively more with high and low degrees as compared to networks where links are formed uniformly at random. These are all observed features of social networks (cf. Jackson and Rogers (2007)).

Welfare

While no formal results are derived for this, the price of anarchy in simulated pairwise stable networks is smaller than $\frac{n}{n-1}$, which implies that the loss of welfare in stable networks is not big, i.e. observing clique behaviour and even apparent separation is not necessarily significantly harming welfare and therefore might not warrant an intervention. According to the simulations the payoffs are very close to the theoretical maximum n, despite non-negligible differences between nodes' payoffs. One possible explanation for this occurrence is the implicit correction for externalities inherent in the current decision rule. As a node creates another link it imposes an externality on its direct neighbours but by becoming a less-desirable neighbour it immediately attracts smaller investments. In this way it partially internalises the externality it has imposed on others. This is not a feature of the equal split rule, where one only imposes an externality on one's current neighbours by acquiring new ones.

Extensions

It is evident that the outcomes under the current model greatly diverge from a similar setting where agents spread their resources equally between their neighbours. In fact it is possible to construe the two setups as two extreme cases, depending on how much weight (denoted by the parameter α) the degree of the node's neighbours has on the node's investments: the equal split attributes no weight ($\alpha = 0$) while the current heuristic gives weight inversely proportional to the degree ($\alpha = 1$).[17] Taking this broader perspective for completeness, it is immediately possible to note some interesting outcomes. Networks consisting of complete components are stable. Importantly, the possible size of complete components within the same stable network varies depending on the particular α. Moreover, for any fixed n it seems to be possible to find an α close enough to 0 (but not equal) for which the stable networks

[17] Appendix B.2 partially addresses this question analytically.

would only consist of complete components. Taking the opposite perspective, for any fixed $\alpha \neq 0$ it seems to be possible to find an n after which the stable networks would not consist of only complete components.[18]

The line, star and circle networks which were shown to be unstable in Proposition 2.6 and Corollary 2.2 (for $\alpha = 1$) can also be shown to be unstable for the intermediate values of α. Finally, taking values of α sufficiently close to 1 (bigger and smaller) results in (qualitatively) similar stable networks to the ones explored with the simulations in this section.

A natural addition to this model is to consider the case with fixed linking costs incurred for every link one makes. Clearly, if the linking costs are relatively high, the set of stable networks is going to shrink and allow only the ones with the highest welfare be stable (e.g. networks consisting of complete components). In the opposite case, for relatively low linking costs, it is possible that for a specific n the set of stable networks is preserved. High and low linking costs are determined by the relative size of the network. A network with *more* nodes would have a *lower* threshold for the linking costs for which the set of stable networks is preserved compared to the case of no linking costs, and vice versa. This is due to the fact that in a network with more nodes an additional link would bring less on average (within a connected component) and compared to the fixed linking cost this would sometimes not justify creating the link. Intermediate linking costs would logically change the nature of the stable networks for any specific n, but it would still be possible to observe pairwise stable networks consisting or regular and irregular components.

2.5 Conclusion

Heuristics have been a relatively overlooked possibility for network formation, while being widely explored in other areas of economics. This chapter takes first steps in this direction by investigating the possible outcomes of a simple heuristic, investing more in people who are deemed more likely to invest in you, based on their degree in the network. This is an intuitively appealing idea for modelling social value but it is also supported by experimental evidence. However, an important test for the credibility of such an approach is also considering the outcomes it produces. This is the main objective of this chapter. It looked into the pairwise stable networks that emerge from applying such a heuristic. They are non-trivial and possess interesting properties. On the one hand one starting state can yield many possible outcomes. On the other hand, especially with larger networks, the outcomes frequently exhibit descriptive features that are observed in real-life social networks. They often have cliques, connected by several intermediaries, but also sometimes have a well connected core and periphery struc-

[18]One intuitive way to explain this is that the proof of Proposition 2.9 includes strict rather than weak inequalities in its end.

ture. It is interesting to note that while on the individual level the potential losses that are implied by following the heuristic are non-negligible, they do not result in big losses in welfare when looking at the aggregate level, since the externalities the heuristic imposes are partially internalised. It is also worth noting that it is possible for people to be in the periphery of the network and to benefit more than the ones in more central positions – having a few good friends could be better than having many superficial acquaintances. Moreover, agents with the same degree can have different payoffs in the same network i.e. the correlation between the number of connections and payoff is not one-to-one and results in a diverse set of possible outcomes. One of the most appealing features of the current setup is that the rich set of stable networks results from a minimal number of simple starting assumptions – the agents are not initially differentiated, they are matched randomly, use myopic decisions and do not calculate complicated best responses but still end up in particular identifiable structures.

Building on the current results, a promising possible extension of the model could be adding an informational dimension to it. This would undoubtedly change the set of stable networks depending on the specific payoffs from information and can be an additional robustness check for the networks identified in the current chapter. Looking at an even larger set of different payoff functions for the links, e.g. functions with multiple additive elements, can give an indication to what extent the stable network structures are determined by the allocation of resources at the link level. Finally, it would be interesting to see if there are functional forms that can produce best response results (in terms of the allocation of resources) which come qualitatively close to the allocations prescribed by the heuristic in this chapter.

Appendix B

Supplementing Chapter 2

B.1 Proofs

Proof of Proposition 2.4: To prove that a fully connected network is reachable starting from an empty graph with n nodes it is sufficient to show that:

(i) it can form a complete subgraph with $n \geq m \geq 2$ nodes;

(ii) that a new node could be connected to it;

(iii) when a new node is added to the complete subgraph it triggers a process leading to a new complete (sub)graph with $m+1$ nodes.

(iv) the process is repeated until $m = n$.

Step (i) follows directly for $m = 2$ since any two loose nodes prefer to be connected to being isolated. Step (ii) implicitly includes two conditions:

(a) at least one node from the complete subgraph wants to connect to one of the isolated nodes;

(b) the isolated node should also be willing to create the link.

As noted above, isolated nodes are always willing to form a link as any link brings them more than 0. Condition (a) is captured in (B.1) below.

$$\frac{1}{\sqrt{m\frac{m-1}{m-1}+\frac{m}{1}}}\left(\frac{m-1}{\sqrt{\frac{m-1}{m}+(m-2)\frac{m-1}{m-1}}}+\frac{1}{\sqrt{\frac{1}{m}}}\right)>1 \tag{B.1}$$

Simplified:

$$\frac{1}{\sqrt{2m}}\left(\frac{m-1}{\sqrt{\frac{m-1+m^2-2m}{m}}}+\sqrt{m}\right)>1 \Leftrightarrow \frac{m-1}{\sqrt{2(m^2-m-1)}}+\frac{1}{\sqrt{2}}>1$$

$$m-1+\sqrt{m^2-m-1} > \sqrt{2(m^2-m-1)} \Leftrightarrow m-1 > \sqrt{m^2-m-1}(\sqrt{2}-1)$$

$$m^2-2m+1 > (m^2-m-1)(3-2\sqrt{2}) \Leftrightarrow m^2(2\sqrt{2}-2)+m(1-2\sqrt{2})-2\sqrt{2}+4 > 0$$

This is always true for $m \geq 2$ because the discriminant of the left-hand side is negative. Hence, step (ii) is always possible.

Step (iii) also includes two parts:

(a) at least one node from the *formerly* complete subgraph of m nodes wants to connect the *formerly* isolated node. This is captured in condition (B.3) where c refers to the number of nodes in the complete graph that are connected to the *formerly* isolated node.

(b) the *formerly* isolated node should also be willing to create the link. This is captured in condition (B.2) below.

$$\frac{1}{\sqrt{\frac{(c+1)(c+1)}{m}}}\frac{c+1}{\sqrt{\frac{m}{c+1}+\frac{m(m-c-1)}{m-1}+\frac{(c+1-1)m}{m}}} - \frac{1}{\sqrt{\frac{cc}{m}}}\frac{c}{\sqrt{\frac{m}{c}+\frac{m(m-c)}{m-1}+\frac{(c-1)m}{m}}} \geq 0 \tag{B.2}$$

Simplifies to:

$$\frac{1}{\sqrt{\frac{(m-c-1)}{m-1}+\frac{1}{c+1}+\frac{c}{m}}} - \frac{1}{\sqrt{\frac{m-c}{m-1}+\frac{1}{c}+\frac{c-1}{m}}} \geq 0$$

$$\frac{m-c}{m-1}+\frac{1}{c}+\frac{c-1}{m} \geq \frac{(m-c-1)}{m-1}+\frac{1}{c+1}+\frac{c}{m} \Leftrightarrow \frac{1}{m-1}-\frac{1}{m}+\frac{1}{c}-\frac{1}{c+1} \geq 0$$

This holds for $m \geq 2, c \geq 1, c < m$. Regarding (a) the following needs to hold:

$$\frac{1}{\sqrt{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+\frac{m}{m}c}}\left(\frac{1}{\sqrt{\frac{(c+1)^2}{m}}}+\frac{c}{\sqrt{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+\frac{m}{m}c}}+\frac{m-c-1}{\sqrt{\frac{(c+1)(m-1)}{m}+\frac{m-1}{m-1}(m-c-2)}}\right)-$$
$$\frac{1}{\sqrt{\frac{c(m-1)}{m}+\frac{m-1}{m-1}(m-c-1)}}\left(\frac{c}{\sqrt{\frac{m(m-c)}{m-1}+\frac{m}{c}+\frac{m}{m}(c-1)}}+\frac{m-c-1}{\sqrt{\frac{c(m-1)}{m}+\frac{m-1}{m-1}(m-c-1)}}\right) > 0 \tag{B.3}$$

In order to show that (B.3) holds, it is separated in two parts. The first part (first line of (B.3)) will be shown to always be bigger than or equal to 1 (see (B.4) below) while the second part (second line of (B.3)) is always strictly smaller than 1 (see (B.11) below).

$$\frac{1}{\sqrt{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c}}\left(\frac{1}{\sqrt{\frac{(c+1)(c+1)}{m}}}+\frac{c}{\sqrt{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c}}+\frac{m-c-1}{\sqrt{m-c-2+\frac{(m-1)(c+1)}{m}}}\right) \geq 1 \tag{B.4}$$

$$\frac{\sqrt{m}}{(c+1)\sqrt{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c}}+\frac{c}{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c}+$$
$$\frac{m-c-1}{\sqrt{(m-c-2+\frac{(m-1)(c+1)}{m})(\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c)}} \geq 1$$

$$\sqrt{m}\sqrt{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c}\sqrt{m-c-2+\frac{(m-1)(c+1)}{m}}+$$
$$c(c+1)\sqrt{m-c-2+\frac{(m-1)(c+1)}{m}}+(m-c-1)(c+1)\sqrt{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c}\geq$$
$$(c+1)\left(\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c\right)\sqrt{m-c-2+\frac{(m-1)(c+1)}{m}}$$

For the conditions for the terms under the square root, see (B.9) and (B.10).

$$\sqrt{m}\sqrt{\frac{m^2(c+2)-m(c+2)-c^2-c}{(m-1)(c+1)}}\sqrt{\frac{m^2-m-c-1}{m}}+$$
$$c(c+1)\sqrt{\frac{m^2-m-c-1}{m}}+(m-c-1)(c+1)\sqrt{\frac{m^2(c+2)-m(c+2)-c^2-c}{(m-1)(c+1)}}\geq$$
$$(c+1)\left(\frac{m^2(c+2)-m(c+2)-c^2-c}{(m-1)(c+1)}\right)\sqrt{\frac{m^2-m-c-1}{m}}$$

Group the first and third term together and the second and last term together.

$$\sqrt{\frac{m^2(c+2)-m(c+2)-c^2-c}{(m-1)(c+1)}}\left(\sqrt{m^2-m-c-1}+(m-c-1)(c+1)\right)\geq$$
$$(c+1)\sqrt{\frac{m^2-m-c-1}{m}}\left(\frac{m^2(c+2)-m(c+2)-c^2-c}{(m-1)(c+1)}-c\right)$$

Simplify the right-hand side.

$$\sqrt{\frac{m^2(c+2)-m(c+2)-c^2-c}{(m-1)(c+1)}}\left(\sqrt{m^2-m-c-1}+(m-c-1)(c+1)\right)\geq$$
$$\sqrt{m(m^2-m-c-1)}\left(\frac{m(c+2)-(c^2+2c+2)}{m-1}\right)$$

Remove the denominators.

$$\sqrt{(m-1)\left(m^2(c+2)-m(c+2)-c^2-c\right)}\left(\sqrt{m^2-m-c-1}+(c+1)(m-c-1)\right)$$
$$\geq\sqrt{m(c+1)\left(m^2-m-c-1\right)}\left(m(c+2)-(c^2+2c+2)\right)\quad\text{(B.5)}$$

(B.5) holds if both parts (under the square root and what is in the brackets) of the left-hand side of the inequality are bigger than the corresponding parts of the right-hand side. This is checked separately – first the square root parts (see (B.6) below) of both sides and then the rest (see (B.8)).

$$\sqrt{(m-1)\left(m^2(c+2)-m(c+2)-c^2-c\right)}\geq\sqrt{m(c+1)\left(m^2-m-c-1\right)}\quad\text{(B.6)}$$

$m^3(c+2)-m^2(c+2)-m(c^2+c)-m^2(c+2)+m(c+2)+c^2+c\geq m^3(c+1)-m^2(c+1)-mc(c+1)-m(c+1)$

$$m^3 - m^2(2c + 4 - c - 1) - m(c^2 + c - c - 2 - c^2 - c - c - 1) + c^2 + c \geq 0$$

$$m^3(c + 1) - m^2(c + 3) + m(2c + 3) + c^2 + c \geq 0 \tag{B.7}$$

(B.7) is true if $m^3(c+1) - m^2(c+3) \geq 0$ which is equivalent to $m \geq \frac{c+3}{c+1}$. $\frac{c+3}{c+1}$ decreases as c increases, its highest value for $c \geq 1$ being 2 and hence $m \geq 2$ always.

Now the second part of (B.5):

$$\sqrt{m^2 - m - c - 1} + (c + 1)(m - c - 1) \geq m(c + 2) - (c^2 + 2c + 2) \tag{B.8}$$

$$\sqrt{m^2 - m - c - 1} + m(c + 1) - (c + 1)^2 \geq m(c + 2) - (c + 1)^2 - 1$$

$$\sqrt{m^2 - m - c - 1} \geq m - 1 \Leftrightarrow m^2 - m - c - 1 \geq m^2 - 2m + 1 \Leftrightarrow m \geq c + 2$$

Therefore (B.6) and (B.8) hold for $m \geq c - 2$ and so does (B.5). What is left is to check is if (B.5) also holds for $m = c + 1$. Substituting $c = m - 1$ in (B.5) yields:

$$\sqrt{(m-1)\left(m^2(m+1) - m(m+1) - m(m-1)\right)}\left(\sqrt{m^2 - m - m} + m(m - m)\right)$$
$$\geq \sqrt{m^2\left(m^2 - m - m\right)}\left(m(m+1) - ((m-1)(m+1) + 2)\right)$$

$$\sqrt{m^2(m-1)^2}\sqrt{m^2 - 2m} \geq \sqrt{m^3(m-2)}(m-1)$$

This holds with equality. So, (B.5) and hence (B.4) holds also for $m = c+1$. Therefore, (B.4) is true for $m \geq c + 1$. Here,

$$m - c - 2 + \frac{(m-1)(c+1)}{m} > 0 \tag{B.9}$$

$$(\frac{m}{m-1})(m - c - 1) + \frac{m}{c+1} + c > 0 \tag{B.10}$$

need to hold. (B.10) always holds for $m \geq c + 1$, $c \geq 1$ and $m \geq 2$. Now consider (B.9).

$$m^2 - mc - 2m + mc - c + m - 1 \geq 0 \Leftrightarrow m^2 - m - c - 1 \geq 0$$

which holds if $m \geq 2, c \geq 1$ when $m \geq \frac{1+\sqrt{4c+5}}{2}$ which is always true if $m \geq c + 1$. Therefore (B.4) is true for $m \geq c + 1$, $c \geq 1$ and $m \geq 2$.

Consider the second part of (B.3):

$$\frac{1}{\sqrt{\frac{m-1}{m-1}(m-c-1) + \frac{c(m-1)}{m}}}\left(\frac{c}{\sqrt{\frac{m(m-c)}{m-1} + \frac{m}{m}(c-1) + \frac{m}{c}}} + \frac{m-c-1}{\sqrt{\frac{m-1}{m-1}(m-c-1) + \frac{(m-1)c}{m}}}\right) < 1 \tag{B.11}$$

$$\frac{m-c-1}{m-1-\frac{c}{m}} + \frac{c}{\sqrt{(\frac{m(m-c)}{m-1} + c - 1 + \frac{m}{c})(m - 1 - \frac{c}{m})}} < 1$$

$$\frac{m-c-1-m+1+\frac{c}{m}}{m-1-\frac{c}{m}} + \frac{c}{\sqrt{(\frac{m(m-c)}{m-1} + c - 1 + \frac{m}{c})(m - 1 - \frac{c}{m})}} < 0$$

$$\frac{\frac{c}{m}-c}{m-1-\frac{c}{m}}+\frac{c}{\sqrt{(\frac{m(m-c)}{m-1}+c-1+\frac{m}{c})(m-1-\frac{c}{m})}}<0$$

Simplify.

$$\frac{\frac{1}{m}-1}{\sqrt{m-1-\frac{c}{m}}}+\frac{1}{\sqrt{\frac{m(m-c)}{m-1}+c-1+\frac{m}{c}}}<0 \Leftrightarrow \frac{\frac{1-m}{m}}{\sqrt{\frac{m^2-m-c}{m}}}+\frac{1}{\sqrt{\frac{m(m-c)}{m-1}+c-1+\frac{m}{c}}}<0$$

$$\frac{1}{\sqrt{\frac{m^2(c+1)-m(c+1)-c^2+c}{(m-1)c}}}<\frac{\frac{m-1}{m}}{\sqrt{\frac{m^2-m-c}{m}}} \Leftrightarrow \frac{\sqrt{(m-1)c}}{\sqrt{m^2(c+1)-m(c+1)-c^2+c}}<\frac{\frac{m-1}{m}\sqrt{m}}{\sqrt{m^2-m-c}}$$

$$\frac{\sqrt{c}}{\sqrt{m^2(c+1)-m(c+1)-c^2+c}}<\frac{\sqrt{m-1}}{\sqrt{m(m^2-m-c)}}$$

This is equivalent to:

$$\sqrt{\frac{cm(m^2-m-c)}{(m-1)(m^2(c+1)-m(c+1)-c^2+c)}}<1$$

which holds if the numerator is smaller than the denominator, or:

$$cm(m^2-m-c)<(m-1)(m^2(c+1)-m(c+1)-c^2+c)$$

$$cm^3-cm^2-c^2m<m^3(c+1)-m^2(c+1)-mc^2+mc-m^2(c+1)+m(c+1)+c^2-c$$

$$0<m^3-m^2(c+2)+m(2c+1)+c^2-c$$

This always holds true for $m \geq c+2$. What is left is to check for $m = c+1, m \geq 2$. Then the expression becomes:

$$m^3-m^2(m+1)+m(2m-2+1)+(m-1)(m-2)>0$$

$$m^3-m^3-m^2+2m^2-m+m^2-m-2m+2>0 \Leftrightarrow (m-1)^2>0$$

This is correct for $m \geq 2$. Therefore (B.11) is always < 1, while (B.4) is always ≥ 1. Therefore (B.3) is always true. Hence steps (iii) and (iv) are always possible. □

Proof of Proposition 2.5: To prove the statement it is sufficient to show that in any regular network with degree k which is not complete two of its nodes would want to form a connection. This is equivalent to the condition:

$$\frac{1}{\sqrt{\frac{k+1}{k+1}+k\frac{k+1}{k}}}\left(\frac{1}{\sqrt{\frac{k+1}{k+1}+k\frac{k+1}{k}}}+\frac{\alpha}{\sqrt{2\frac{k}{k+1}+(k-2)\frac{k}{k}}}+\frac{k-\alpha}{\sqrt{\frac{k}{k+1}+(k-1)\frac{k}{k}}}\right)>1$$

where $\alpha \geq 1$ is the number of agents that are mutual neighbours of the two connecting nodes. The condition can be simplified.

$$\frac{1}{\sqrt{k+2}}\left(\frac{1}{\sqrt{k+2}}+\frac{\alpha\sqrt{k+1}}{\sqrt{k^2+k-2}}+\frac{(k-\alpha)\sqrt{k+1}}{\sqrt{k^2+k-1}}\right)>1 \quad \text{(B.12)}$$

Since

$$\frac{\sqrt{k+1}}{\sqrt{k^2+k-2}} \geq \frac{\sqrt{k+1}}{\sqrt{k^2+k-1}}$$

holds for every $k \geq 1$, taking $\alpha = 0$ presents the worst-case scenario for Inequality (B.12), i.e. the case in which its left-hand side has the lowest possible value, presenting the lowest incentive for the two nodes to connect. This leaves:

$$\frac{1}{\sqrt{k+2}}\left(\frac{1}{\sqrt{k+2}} + \frac{k\sqrt{k+1}}{\sqrt{k^2+k-1}}\right) > 1 \Leftrightarrow \frac{1}{k+2} + \frac{k\sqrt{k+1}}{\sqrt{(k^2+k-1)(k+2)}} > 1$$

$$\frac{k\sqrt{k+1}}{\sqrt{(k^2+k-1)(k+2)}} > \frac{k+1}{k+2}$$

$$k\sqrt{k+2} > \sqrt{(k+1)(k^2+k-1)} \Leftrightarrow k^3 + 2k^2 > k^3 + k^2 - k + k^2 + k - 1$$

which is always true for $k \geq 3$. $\square$

Proof of Proposition 2.6: This proof contains four parts.

A) Line: The two ends would want to connect, since:

$$\frac{1}{\sqrt{\frac{1}{2}}}\frac{1}{\sqrt{\frac{2}{1}+\frac{2}{1}}} < 1; \frac{1}{\sqrt{\frac{1}{2}}}\frac{1}{\sqrt{\frac{2}{1}+\frac{2}{2}}} < 1$$

The first inequality refers to a line of length 3, while the second one covers all cases of longer lines.

B) Star: It is sufficient to show that two of the periphery nodes of a star network with $n \geq 3$ would want to form a link. The condition is:

$$\frac{1}{\sqrt{\frac{2}{n-1}+\frac{2}{2}}}\left(\frac{1}{\sqrt{\frac{n-1}{2}2+\frac{n-1}{1}(n-3)}} + \frac{1}{\sqrt{\frac{2}{n-1}+\frac{2}{2}}}\right) \geq \frac{1}{\sqrt{\frac{1}{n-1}}}\frac{1}{\sqrt{(n-1)\frac{n-1}{1}}}$$

Simplified:

$$\frac{1}{\sqrt{\frac{2}{n-1}+1}}\left(\frac{1}{\sqrt{n-1+(n-1)(n-3)}} + \frac{1}{\sqrt{\frac{2}{n-1}+1}}\right) \geq \frac{1}{\sqrt{n-1}}$$

$$\frac{1}{\sqrt{\frac{n+1}{n-1}}}\left(\frac{1}{\sqrt{(n-1)(n-2)}} + \frac{1}{\sqrt{\frac{n+1}{n-1}}}\right) \geq \frac{1}{\sqrt{n-1}} \Leftrightarrow \frac{1}{\sqrt{(n+1)(n-2)}} + \frac{n-1}{n+1} \geq \frac{1}{\sqrt{n-1}}$$

$$\frac{\sqrt{n-1}}{\sqrt{(n+1)(n-2)}} + \frac{(n-1)\sqrt{n-1}}{n+1} \geq 1 \tag{B.13}$$

$\frac{(n-1)\sqrt{n-1}}{n+1}$ is bigger than 1 for $n \geq 4$ and it has a positive first derivative. The first term of (B.13) is always positive. Finally, a specific check for $n = 3$ shows that (B.13) holds for integers $n \geq 3$.

C) Wheel: Here it is sufficient to prove that at least one node would want to *disconnect* from the centre. Taking a peripheral node the following inequality should hold:

$$\frac{1}{\sqrt{2*\frac{3}{3}+\frac{3}{n}}}\left(\frac{2}{\sqrt{2*\frac{3}{3}+\frac{3}{n}}}+\frac{1}{\sqrt{n\frac{n}{3}}}\right)<\frac{1}{\sqrt{2*\frac{2}{3}}}\frac{2}{\sqrt{\frac{3}{2}+\frac{3}{3}+\frac{3}{n}}} \tag{B.14}$$

where the right-hand side expresses the payoff of disconnecting from the centre. Simplifying:

$$\frac{1}{\sqrt{\frac{2n+3}{n}}}\left(\frac{2}{\sqrt{\frac{2n+3}{n}}}+\frac{\sqrt{3}}{n}\right)<\frac{\sqrt{3}}{\sqrt{\frac{5}{2}+\frac{3}{n}}}\Leftrightarrow\frac{1}{\sqrt{\frac{2n+3}{n}}}\left(\frac{2}{\sqrt{\frac{2n+3}{n}}}+\frac{\sqrt{3}}{n}\right)<\frac{\sqrt{3}}{\sqrt{\frac{5n+6}{2n}}}$$

$$\frac{2n}{2n+3}+\frac{\sqrt{3}}{\sqrt{n(2n+3)}}<\sqrt{\frac{6n}{5n+6}} \tag{B.15}$$

The right-hand side of condition (B.15) is greater than or equal to 1 for $n \geq 6$. The left-hand side is strictly smaller than 1 in this range of values for n:

$$\frac{2n}{2n+3}+\frac{\sqrt{3}}{\sqrt{n(2n+3)}}<1\Leftrightarrow\frac{\sqrt{3}}{\sqrt{n(2n+3)}}<\frac{3}{2n+3}\Leftrightarrow\frac{1}{\sqrt{n}}<\frac{\sqrt{3}}{\sqrt{2n+3}}$$

$$2n+3<3n$$

Therefore, condition (B.14) holds for $n \geq 6$ and in these cases the wheel is not stable. To show that when $n \in \{4,5\}$ the wheel is also not stable, the conditions for two peripheral nodes to *connect* are checked separately. For $n=4$:

$$\frac{1}{\sqrt{2*\frac{3}{3}+\frac{3}{n}}}\left(\frac{2}{\sqrt{2*\frac{3}{3}+\frac{3}{n}}}+\frac{1}{\sqrt{n\frac{n}{3}}}\right)<$$
$$\frac{1}{\sqrt{\frac{4}{n}+2*\frac{4}{3}+\frac{4}{4}}}\left(\frac{1}{\sqrt{\frac{4}{n}+2*\frac{4}{3}+\frac{4}{4}}}+\frac{1}{\sqrt{2*\frac{n}{4}+(n-2)\frac{n}{3}}}+\frac{2}{\sqrt{2*\frac{3}{4}+\frac{3}{n}}}\right)$$

And for $n=5$:

$$\frac{1}{\sqrt{2*\frac{3}{3}+\frac{3}{n}}}\left(\frac{2}{\sqrt{2*\frac{3}{3}+\frac{3}{n}}}+\frac{1}{\sqrt{n\frac{n}{3}}}\right)<$$
$$\frac{1}{\sqrt{\frac{4}{n}+2*\frac{4}{3}+\frac{4}{4}}}\left(\frac{1}{\sqrt{\frac{4}{n}+2*\frac{4}{3}+\frac{4}{4}}}+\frac{1}{\sqrt{2*\frac{n}{4}+(n-2)\frac{n}{3}}}+\frac{1}{\sqrt{2*\frac{3}{4}+\frac{3}{n}}}+\frac{1}{\sqrt{\frac{3}{n}+\frac{3}{4}+\frac{3}{3}}}\right)$$

Both conditions hold for the specific values of n. Therefore, the wheel with $n \geq 4$ in the periphery is not stable.

D) This proof will be presented in two parts – D1) which deals with complete biregular graphs and D2) which deal with incomplete biregular graphs.

D1) In a bipartite graph with m and $k = n - m$ nodes in the two sets such that $m \leq k$ at least one set of nodes would want to connect and therefore the network would not be stable, since the payoff of a connection between two nodes in the m-set would be always preferred to the status quo:

$$\frac{1}{\sqrt{\frac{m^2}{k}}} * \frac{m}{\sqrt{\frac{k^2}{m}}} \leq \frac{1}{\sqrt{\frac{m(m+1)}{k} + \frac{m+1}{m+1}}} \left(\frac{1}{\sqrt{\frac{m(m+1)}{k} + \frac{m+1}{m+1}}} + \frac{m}{\sqrt{\frac{k(k-2)}{m} + \frac{2k}{m+1}}} \right)$$

Simplify:

$$\frac{1}{\sqrt{\frac{(m+1)m+n}{k}}} \left(\frac{1}{\sqrt{\frac{(m+1)m+k}{k}}} + \frac{m}{\sqrt{\frac{k(k-2)(m+1)+2mk}{m(m+1)}}} \right) \geq \sqrt{\frac{m}{k}} \tag{B.16}$$

$$\frac{k}{(m+1)m+k} + \frac{m\sqrt{m(m+1)}}{\sqrt{((m+1)m+k)((k-2)(m+1)+2m)}} \geq \sqrt{\frac{m}{k}}$$

$$n\sqrt{k((k-2)(m+1)+2m)} + m\sqrt{mk(m+1)((m+1)m+k)} \geq ((m+1)m+k)\sqrt{m((k-2)(m+1)+2m)}$$

$$k\sqrt{k(mk+k-2)} + m\sqrt{mk(m+1)(m^2+m+k)} \geq (m^2+m+n)\sqrt{m(mk+k-2)}$$

$$k\sqrt{k(mk+k-2)} \geq \sqrt{m^2+m+k}(\sqrt{m(m^2+m+k)(mk+k-2)} - m\sqrt{mk(m+1)})$$

Take the first part of the left-hand side and the second part of the right-hand side.

$$k\sqrt{k} \geq \sqrt{m(m^2+m+k)(mk+k-2)} - m\sqrt{mk(m+1)}$$

Rearrange:

$$k\sqrt{k} + m\sqrt{mk(m+1)} \geq \sqrt{m(m^2+m+k)(mk+k-2)}$$

Square both sides of the condition.

$$k^3 + 2k^2m\sqrt{m(m+1)} + m^3k(m+1) \geq (m^2(m+1)+km)(k(m+1)-2)$$

$$k^3 + 2k^2m\sqrt{m(m+1)} + m^3k(m+1) \geq km^2(m+1)^2 + k^2m(m+1) - 2m^2(m+1) - 2km$$

$$k^3 + 2k^2m\sqrt{m(m+1)} + m^2k(m+1)(m-m-1) \geq k^2m(m+1) - 2m^2(m+1) - 2km$$

$$k^3 + 2m^2(m+1) + 2km + 2k^2m\sqrt{m(m+1)} - m^2k(m+1) - k^2m(m+1) \geq 0$$

Consider only the last three terms (the first of them is split) $k^2m\sqrt{m(m+1)} - m^2k(m+1) + k^2m\sqrt{m(m+1)} - k^2m(m+1)$. Each of them is positive separately, except for $k = m+1$. This requires a separate check.

Now take the second part of the left-hand side and the first part of the right-hand side.

$$\sqrt{mk+k-2} \geq \sqrt{m^2+m+k} \tag{B.17}$$

Inequality (B.17) is true for $k \geq m + 1 + \frac{2}{m}$, so what is left is to see that condition (B.16) holds for $m \in \{k, k-1, k-2\}$. A manual check shows this is also true. Therefore a complete bipartite graph is not stable in this setting.

D2) In a biregular graph which is not complete at least two different nodes which are not in the same subgroup of nodes (with degrees k and ℓ) would want to connect.

$$\frac{1}{\sqrt{\frac{k+1}{\ell+1} + \frac{(k+1)k}{\ell}}} \left(\frac{1}{\sqrt{\frac{\ell+1}{k+1} + \frac{(\ell+1)\ell}{k}}} + \frac{k}{\sqrt{\frac{\ell(\ell-1)}{k} + \frac{\ell}{k+1}}} \right) \geq \frac{1}{\sqrt{\frac{k*k}{\ell}}} \frac{k}{\sqrt{\frac{\ell*\ell}{k}}}$$

Simplify:

$$\frac{1}{\sqrt{\frac{(k+1)(\ell+k\ell+k)}{(\ell+1)\ell}}} \left(\frac{1}{\sqrt{\frac{(\ell+1)(\ell+k\ell+k)}{(k+1)k}}} + \frac{k}{\sqrt{\frac{\ell(k\ell+\ell-1)}{k(k+1)}}} \right) \geq \sqrt{\frac{k}{\ell}}$$

$$\frac{\sqrt{\ell k}}{\ell + k\ell + k} + \frac{k\sqrt{k(\ell+1)}}{\sqrt{(\ell + k\ell + k)(k\ell + \ell - 1)}} \geq \sqrt{\frac{k}{\ell}}$$

$$\frac{\ell}{\ell + k\ell + k} + \frac{k\sqrt{\ell(\ell+1)}}{\sqrt{(\ell + k\ell + k)(k\ell + \ell - 1)}} \geq 1 \Leftrightarrow \frac{k\sqrt{\ell(\ell+1)}}{\sqrt{(\ell + k\ell + k)(k\ell + \ell - 1)}} \geq \frac{k\ell + k}{\ell + k\ell + k}$$

$$\frac{\sqrt{\ell}}{\sqrt{k\ell + \ell - 1}} \geq \frac{\sqrt{\ell + 1}}{\sqrt{\ell + k\ell + k}} \Leftrightarrow \frac{\ell}{k\ell + \ell - 1} \geq \frac{\ell + 1}{\ell + k\ell + k}$$

$$\ell(\ell + k\ell + k) \geq \ell(k\ell + \ell - 1) + k\ell + \ell - 1 \Leftrightarrow \ell(k+1) \geq k\ell + \ell - 1$$

This is always true. □

Proof of Proposition 2.7: As already showed in Proposition 2.3 no node in a complete graph would disconnect from the rest. Therefore, in order to prove the statement it is sufficient to show that the two subgraphs would not form links for $m_1 \geq m_2 + 2$ (here it is assumed that $m_1 > m_2$ without loss of generality). The conditions for two nodes from the subgraphs to want to form a link are:

$$\frac{1}{\sqrt{\frac{m_2}{m_1} + \frac{(m_2-1)m_2}{m_2-1}}} \left(\frac{1}{\sqrt{\frac{m_1}{m_2} + \frac{(m_1-1)m_1}{m_1-1}}} + \frac{m_2 - 1}{\sqrt{\frac{m_2-1}{m_2} + \frac{(m_2-2)(m_2-1)}{m_2-1}}} \right) \geq 1$$

$$\frac{1}{\sqrt{\frac{m_1}{m_2} + \frac{(m_1-1)m_1}{m_1-1}}} \left(\frac{1}{\sqrt{\frac{m_2}{m_1} + \frac{(m_2-1)m_2}{m_2-1}}} + \frac{m_1 - 1}{\sqrt{\frac{m_1-1}{m_1} + \frac{(m_1-2)(m_1-1)}{m_1-1}}} \right) \geq 1$$

where one of them needs to hold strictly. They could be simplified to:

$$\frac{1}{\sqrt{\frac{m_2(m_1+1)}{m_1}}} \left(\frac{1}{\sqrt{\frac{m_1(m_2+1)}{m_2}}} + \frac{m_2 - 1}{\sqrt{\frac{m_2^2-m_2-1}{m_2}}} \right) \geq 1 \qquad \text{(B.18)}$$

$$\frac{1}{\sqrt{\frac{m_1(m_2+1)}{m_2}}}\left(\frac{1}{\sqrt{\frac{m_2(m_1+1)}{m_1}}}+\frac{m_1-1}{\sqrt{\frac{m_1^2-m_1-1}{m_1}}}\right)\geq 1 \tag{B.19}$$

The first parts of the expressions on the left-hand side of the inequalities, $\frac{1}{\sqrt{\frac{m_1(m_2+1)}{m_2}}}\frac{1}{\sqrt{\frac{m_2(m_1+1)}{m_1}}}$, are the same. Comparing the second parts, it is true that:

$$\frac{(m_2-1)\sqrt{m_1}}{\sqrt{(m_1+1)(m_2^2-m_2-1)}}\geq\frac{(m_1-1)\sqrt{m_2}}{\sqrt{(m_2+1)(m_1^2-m_1-1)}}$$

because, for $m_1 \geq m_2$, it always holds that:

$$\frac{\sqrt{m_1(m_1^2-m_1-1)}}{(m_1-1)\sqrt{m_1+1}}\geq\frac{\sqrt{m_2(m_2^2-m_2-1)}}{(m_2-1)\sqrt{m_2+1}}.$$

Therefore, condition (B.19) is binding and it is sufficient to show that it holds for both (B.18) and (B.19) to hold. For $m_1 \geq 4$, condition (B.19) is equivalent to:

$$\sqrt{m_1^2-m_1-1}+(m_1-1)\sqrt{m_2(m_1+1)}\geq\sqrt{(m_1+1)(m_2+1)(m_1^2-m_1-1)}$$

$$m_1^2-m_1-1+m_2(m_1-1)^2(m_1+1)+2(m_1-1)\sqrt{m_2(m_1+1)(m_1^2-m_1-1)}\geq (m_1+1)(m_2+1)(m_1^2-m_1-1)$$

$$m_2(m_1^3-m_1^2-m_1-1-m_1^3+2m_1+1)+2\sqrt{m_2(m_1+1)(m_1^2-m_1-1)}(m_1-1)-m_1(m_1^2-m_1-1)\geq 0$$

This yields: $\frac{m_1^2-m_1-1}{m_1+1}\leq m_2\leq\frac{m_1\sqrt{(m_1+1)(m_1^2-m_1-1)}}{(m_1-2)(m_1+1)}$, but since $m_1 \geq m_2$, it simplifies to $\frac{m_1^2-m_1-1}{m_1+1}\leq m_2\leq m_1$ or $m_1-1-\frac{m_1}{m_1+1}\leq m_2\leq m_1$. Therefore, conditions (B.18) and (B.19) hold for only $m_2 \in \{m_1-1, m\}$. □

Proof of Proposition 2.9: It is sufficient to show that any other networks would have (at least) two nodes which would always want to make a connection. The strategy in this proof is excluding all networks in which it is clear that there are two nodes which want to form a connection and considering what this implies for all other networks.

The payoff of a node i with N_i being the set of its neighbours and d_i being its degree can be expressed and rearranged in the following way:

$$u_i=\sum_{j\in N_i}\sqrt{\frac{1}{d_i}*\frac{1}{d_j}}=\sum_{j\in N_i}\frac{1}{\sqrt{d_i*d_j}}=\frac{1}{\sqrt{d_i}}\left(\sum_{j\in N_i}\frac{1}{\sqrt{d_j}}\right).$$

In order for two nodes i and k to want to connect the following inequalities should hold with at least one of them being strict.

$$\frac{1}{\sqrt{d_i}}\left(\sum_{j\in N_i}\frac{1}{\sqrt{d_j}}\right)\leq\frac{1}{\sqrt{d_i+1}}\left(\sum_{j\in N_i}\frac{1}{\sqrt{d_j}}+\frac{1}{\sqrt{d_k+1}}\right) \tag{B.20}$$

$$\frac{1}{\sqrt{d_k}}\left(\sum_{j\in N_k}\frac{1}{\sqrt{d_j}}\right)\leq\frac{1}{\sqrt{d_k+1}}\left(\sum_{j\in N_k}\frac{1}{\sqrt{d_j}}+\frac{1}{\sqrt{d_i+1}}\right)\tag{B.21}$$

Note that d_i and N_i refer to the situation before a connection has been made and so $k\notin N_i$. Inequality (B.20) expresses that agent i would be better off connecting to agent k, because his current payoff (left-hand side) is smaller than the payoff he would have if agent k was his direct neighbour. In this case (right-hand side) he would split his resources in d_i+1 equal parts and get $1/(d_k+1)$ of k's resources as investment.[1] Consider inequality (B.20), which can be rewritten as:

$$\left(\frac{\sqrt{d_i+1}-\sqrt{d_i}}{\sqrt{d_i}*\sqrt{d_i+1}}\right)\left(\sum_{j\in N_i}\frac{1}{\sqrt{d_j}}\right)=\left(\frac{1}{\sqrt{d_i}}-\frac{1}{\sqrt{d_i+1}}\right)\left(\sum_{j\in N_i}\frac{1}{\sqrt{d_j}}\right)\leq\frac{1}{\sqrt{d_i+1}}*\frac{1}{\sqrt{d_k+1}}$$

$$\sum_{j\in N_i}\sqrt{\frac{d_k+1}{d_j}}\leq\frac{\sqrt{d_i}}{\sqrt{d_i+1}-\sqrt{d_i}}\Leftrightarrow\sum_{j\in N_i}\sqrt{\frac{d_k+1}{d_j}}\leq\sqrt{d_i}(\sqrt{d_i+1}+\sqrt{d_i})\Leftrightarrow$$

$$\sum_{j\in N_i}\sqrt{\frac{d_k+1}{d_j}}\leq d_i+\sqrt{d_i(d_i+1)}\tag{B.22}$$

Clearly, inequality (B.21) can be rewritten in a similar fashion.

Lemma B.1. *In a stable network if nodes i,x and k are such that $d_i\leq d_k, d_x\leq d_k$ and $i\in N_k$, then node i wants to connect to node x.*

Proof of Lemma B.1: In a stable network if i is connected to k this implies that it also wants to be connected to it, otherwise the network would not be stable. Here it is useful to distinguish two cases: (i) $d_x<d_k$ and (ii) $d_x=d_k$.

Since the network is stable i does not want to delete the link with k, so:

$$\sum_{j\in N_i}\sqrt{\frac{d_k}{d_j}}\leq d_i-1+\sqrt{d_i(d_i-1)}\tag{B.23}$$

In case $d_x<d_k$, Inequality (B.23) implies:

$$\sum_{j\in N_i}\sqrt{\frac{d_x+1}{d_j}}\leq\sum_{j\in N_i}\sqrt{\frac{d_k}{d_j}}\leq d_i-1+\sqrt{d_i(d_i-1)}$$

Excluding the middle part and adding $\sqrt{\frac{d_x+1}{d_k}}\leq 1$ on both sides implies:[2]

$$\sum_{j\in N_i}\sqrt{\frac{d_x+1}{d_j}}+\sqrt{\frac{d_x+1}{d_k}}\leq d_i-1+\sqrt{d_i(d_i-1)}+\sqrt{\frac{d_x+1}{d_k}}<d_i+\sqrt{d_i(d_i+1)}$$

[1]Inequality (B.21) expresses the analogous idea for agent k connecting to agent i.

[2]The strict inequality comes from the change of signs under the square root.

$$\sum_{j\in N_i\cup\{k\}}\sqrt{\frac{d_x+1}{d_j}} < d_i+\sqrt{d_i(d_i+1)}$$

Therefore, i is willing to connect to x.

In case $d_x = d_k$, multiply both sides of Inequality (B.23) with $\sqrt{\frac{d_k+1}{d_k}} > 1$ to get:

$$\sum_{j\in N_i}\sqrt{\frac{d_k+1}{d_j}} \le d_i\sqrt{\frac{d_k+1}{d_k}} - \sqrt{\frac{d_k+1}{d_k}} + \sqrt{\frac{d_i(d_i-1)(d_k+1)}{d_k}}$$

$$\sum_{j\in N_i}\sqrt{\frac{d_k+1}{d_j}} + \sqrt{\frac{d_k+1}{d_k}} \le d_i\sqrt{\frac{d_k+1}{d_k}} + \sqrt{\frac{d_i(d_i-1)(d_k+1)}{d_k}}$$

$$\sum_{j\in N_i\cup\{k\}}\sqrt{\frac{d_x+1}{d_j}} = \sum_{j\in N_i}\sqrt{\frac{d_k+1}{d_j}} + \sqrt{\frac{d_k+1}{d_k}} \le d_i\sqrt{\frac{d_k+1}{d_k}} + \sqrt{\frac{d_i(d_i-1)(d_k+1)}{d_k}} \tag{B.24}$$

In order for i to want to connect to x, the following is sufficient for the right-hand side:

$$d_i\sqrt{\frac{d_k+1}{d_k}} + \sqrt{\frac{d_i(d_i-1)(d_k+1)}{d_k}} < d_i+\sqrt{d_i(d_i+1)}$$

$$\sqrt{d_i}\sqrt{\frac{d_k+1}{d_k}}(\sqrt{d_i}+\sqrt{d_i-1}) < \sqrt{d_i}(\sqrt{d_i}+\sqrt{d_i+1})$$

$$(\sqrt{d_i}+\sqrt{d_i-1})\sqrt{d_k+1} < \sqrt{d_k}(\sqrt{d_i}+\sqrt{d_i+1})$$

$$\sqrt{d_i(d_k+1)}+\sqrt{(d_i-1)(d_k+1)} < \sqrt{d_k(d_i+1)}+\sqrt{d_id_k}$$

$$\sqrt{d_id_k+d_i}+\sqrt{d_id_k-d_k+d_i-1} < \sqrt{d_id_k+d_k}+\sqrt{d_id_k}$$

The condition holds since $d_k \ge d_i \ge 1$. Therefore, Inequality (B.24) is equivalent to Condition (B.22) and i wants to connect to x in both cases outlined above. □

Lemma B.2. *In a stable network if nodes i, x and k are such that $d_i \le d_k, d_x \le d_k, i \in N_k$ and $x \in N_k$, then $x \in N_i$.*

Proof of Lemma B.2: By Lemma B.1 if k has a neighbour i with $d_i \le d_k$, that means i is willing to connect to all nodes with degrees $\le d_k$, which includes x. By the same token, x wants to be connected to all nodes with degrees $\le d_k$, which includes i. Therefore, in a stable network all neighbours x of k with $d_x \le d_k$ will be also i's neighbours. □

By Lemma B.2 in a stable network all neighbours of the node with the highest degree within a component, say h, form a clique, i.e. they are fully interconnected and connected to h. Hence, they must all have degrees $\ge d_h$. They cannot have degrees strictly bigger than d_h, because that would contradict the assumption that d_h has the highest degree in the component. Therefore, they must have equal degrees. In other words, in a stable network any component forms a regular subgraph. Moreover, the subgraphs are complete as every node is connected to all other nodes in the component as per Lemma B.2. □

B.2 Extensions

One can view the heuristic investigated in this chapter and the equal split of resources between different neighbours as extreme cases of a common model, which just varies the weight that a node puts of the degree of its neighbours. In the case of equal split the degree of a neighbour plays no role, while for the heuristic used in this chapter it is inversely proportional to the resources allocation. A more general version of the payoff could be expressed in the following way:

$$u_i = \sum_{n \in N_i} \sqrt{\frac{\frac{1}{d_n^\alpha}}{\sum_{j \in N_i} \frac{1}{d_j^\alpha}} \frac{\frac{1}{d_i^\alpha}}{\sum_{l \in N_n} \frac{1}{d_l^\alpha}}} = \frac{1}{\sqrt{\sum_{j \in N_i} \left(\frac{d_i}{d_j}\right)^\alpha}} \left(\sum_{n \in N_i} \frac{1}{\sqrt{\sum_{l \in N_n} \left(\frac{d_n}{d_l}\right)^\alpha}} \right)$$

where $\alpha = 0$ corresponds to the case in which the resources are spread equally amongst all neighbours and $\alpha = 1$ is the current heuristic. For $0 \leq \alpha \leq 1$ Propositions 2.3 and 2.5 hold.

Proof of Proposition 2.3EXT: In a fully connected graph every node has payoff 1. In order for the complete graph to be stable, removing a link should be equally good or worse than the status quo for both nodes that are connected so that they decide not to disconnect. For a complete graph with n nodes, given that $1 \geq \alpha \geq 0$ this is equivalent to:

$$\frac{1}{\sqrt{(n-2)\frac{(n-2)^\alpha}{(n-1)^\alpha}}} \left(\frac{n-2}{\sqrt{2\frac{(n-1)^\alpha}{(n-2)^\alpha} + \frac{(n-1)^\alpha}{(n-1)^\alpha}(n-3)}} \right) \leq 1 \Leftrightarrow \frac{n-2}{\sqrt{\frac{(n-2)(2(n-1)^\alpha + (n-3)(n-2)^\alpha)}{(n-1)^\alpha}}} \leq 1$$

$$(n-2)(n-1)^\alpha \leq 2(n-1)^\alpha + (n-3)(n-2)^\alpha \Leftrightarrow (n-4)(n-1)^\alpha \leq (n-3)(n-2)^\alpha$$

$$\left(\frac{n-1}{n-2}\right)^\alpha \leq \frac{n-3}{n-4}$$

Both parts of the inequality are bigger than 1 for all positive $n > 4$. Moreover, as α comes closer to 1, the left-hand side grows. Therefore, the biggest value for the left-hand side is $\frac{n-1}{n-2}$. In this case the inequality still holds for $n > 4$. What is left is to check if this holds for $n \in \{3, 4\}$. For $n = 3$:

$$\frac{1}{\sqrt{1\frac{1^\alpha}{2^\alpha}}} \frac{1}{\sqrt{2\frac{2^\alpha}{1^\alpha}}} \leq 1$$

For $n = 4$:

$$\frac{1}{\sqrt{2\frac{2^\alpha}{3^\alpha}}} \frac{2}{\sqrt{2\frac{3^\alpha}{2^\alpha} + 1}} \leq 1$$

Both conditions hold. Therefore, no two nodes would decide to disconnect in a complete graph. □

Proof of Proposition 2.5EXT: To prove the statement it is sufficient to show that in any regular network with degrees n and $1 \geq \alpha \geq 0$ which is not complete two of its nodes would want to form a connection. This is equivalent to the condition:

$$\frac{1}{\sqrt{\frac{(n+1)^\alpha}{(n+1)^\alpha}+n\frac{(n+1)^\alpha}{n^\alpha}}}\left(\frac{1}{\sqrt{\frac{(n+1)^\alpha}{(n+1)^\alpha}+n\frac{(n+1)^\alpha}{n^\alpha}}}+\frac{\beta}{\sqrt{2\frac{n^\alpha}{(n+1)^\alpha}+(n-2)\frac{n^\alpha}{n^\alpha}}}+\frac{n-\beta}{\sqrt{\frac{n^\alpha}{(n+1)^\alpha}+(n-1)\frac{n^\alpha}{n^\alpha}}}\right)>1$$

where $\beta \geq 0$ is the number of agents that are mutual neighbours of the two connecting nodes. The condition can be simplified.

$$\frac{\sqrt{n^\alpha}}{\sqrt{n^\alpha+n(n+1)^\alpha}}\left(\frac{\sqrt{n^\alpha}}{\sqrt{n^\alpha+n(n+1)^\alpha}}+\frac{\beta\sqrt{(n+1)^\alpha}}{\sqrt{2n^\alpha+(n-2)(n+1)^\alpha}}+\frac{(n-\beta)\sqrt{(n+1)^\alpha}}{\sqrt{n^\alpha+(n-1)(n+1)^\alpha}}\right)>1 \tag{B.25}$$

Since

$$\frac{\sqrt{(n+1)^\alpha}}{\sqrt{2n^\alpha+(n-2)(n+1)^\alpha}} \geq \frac{\sqrt{(n+1)^\alpha}}{\sqrt{n^\alpha+(n-1)(n+1)^\alpha}}$$

holds for every $n \geq 1, 1 \geq \alpha \geq 0$, taking $\beta = 0$ presents the worst-case scenario for Inequality (B.25), i.e. the case in which its left-hand side has the lowest possible value, presenting the lowest incentive for the two nodes to connect. This leaves:

$$\frac{\sqrt{n^\alpha}}{\sqrt{n^\alpha+n(n+1)^\alpha}}\left(\frac{\sqrt{n^\alpha}}{\sqrt{n^\alpha+n(n+1)^\alpha}}+\frac{n\sqrt{(n+1)^\alpha}}{\sqrt{n^\alpha+(n-1)(n+1)^\alpha}}\right)>1$$

$$\frac{n^\alpha}{n^\alpha+n(n+1)^\alpha}+\frac{n\sqrt{n^\alpha(n+1)^\alpha}}{\sqrt{(n^\alpha+(n-1)(n+1)^\alpha)(n^\alpha+n(n+1)^\alpha)}}>1$$

$$\frac{n\sqrt{n^\alpha(n+1)^\alpha}}{\sqrt{(n^\alpha+(n-1)(n+1)^\alpha)(n^\alpha+n(n+1)^\alpha)}}>\frac{n(n+1)^\alpha}{n^\alpha+n(n+1)^\alpha}$$

$$\frac{\sqrt{n^\alpha}}{\sqrt{n^\alpha+(n-1)(n+1)^\alpha}}>\frac{\sqrt{(n+1)^\alpha}}{\sqrt{n^\alpha+n(n+1)^\alpha}}$$

$$n^{2\alpha}+n^{\alpha+1}(n+1)^\alpha>n^\alpha(n+1)^\alpha+(n-1)(n+1)^{2\alpha}$$

$$\left(\frac{n}{n+1}\right)^{2\alpha}+\left(\frac{n}{n+1}\right)^{\alpha}(n-1)-(n-1)>0$$

This expression is at its lowest for α as big as possible. Taking $\alpha = 1$, the expression is always true for $n \geq 3$. □

Proof of Proposition 2.6EXT: This proof looks at the first two parts of Proposition 2.6.

A) Line: The two ends would want to connect, since:

$$\frac{1}{\sqrt{\frac{1^\alpha}{2^\alpha}}}\frac{1}{\sqrt{\frac{2^\alpha}{1^\alpha}+\frac{2^\alpha}{1^\alpha}}}<1;\ \frac{1}{\sqrt{\frac{1^\alpha}{2^\alpha}}}\frac{1}{\sqrt{\frac{2^\alpha}{1^\alpha}+\frac{2^\alpha}{2^\alpha}}}<1$$

The first inequality refers to a line of length 3, while the second one covers all other cases.

B) Star: It is sufficient to show that two of the periphery nodes of a star network with $n \geq 3$ would want to form a link. The condition is:

$$\frac{1}{\sqrt{\frac{2^\alpha}{(n-1)^\alpha}+\frac{2^\alpha}{2^\alpha}}}\left(\frac{1}{\sqrt{\frac{(n-1)^\alpha}{2^\alpha}2+\frac{(n-1)^\alpha}{1^\alpha}(n-3)}}+\frac{1}{\sqrt{\frac{2^\alpha}{(n-1)^\alpha}+\frac{2^\alpha}{2^\alpha}}}\right) \geq \frac{1}{\sqrt{\frac{1^\alpha}{(n-1)^\alpha}}}\frac{1}{\sqrt{(n-1)\frac{(n-1)^\alpha}{1^\alpha}}}$$

Simplified:

$$\frac{1}{\sqrt{\frac{2^\alpha+(n-1)^\alpha}{(n-1)^\alpha}}}\left(\frac{1}{\sqrt{(n-1)^\alpha\left(\frac{2+2^\alpha(n-3)}{2^\alpha}\right)}}+\frac{1}{\sqrt{\frac{2^\alpha+(n-1)^\alpha}{(n-1)^\alpha}}}\right) \geq \frac{1}{\sqrt{n-1}}$$

$$\frac{2^\alpha}{\sqrt{(2^\alpha+(n-1)^\alpha)(2+2^\alpha(n-3))}}+\frac{(n-1)^\alpha}{2^\alpha+(n-1)^\alpha} \geq \frac{1}{\sqrt{n-1}}$$

Taking the second part:

$$\frac{(n-1)^\alpha}{2^\alpha+(n-1)^\alpha} \geq \frac{1}{\sqrt{n-1}} \tag{B.26}$$

$\frac{(n-1)^\alpha}{2^\alpha+(n-1)^\alpha}$ is bigger than $1/2$ for $n=4$ and it has a positive first derivative w.r.t. n so as n increases the term becomes bigger. The derivative is:

$$\frac{\alpha(n-1)^\alpha}{2^\alpha+(n-1)^\alpha}-\frac{\alpha(n-1)^{2\alpha}}{(2^\alpha+(n-1)^\alpha)^2}$$

The first term of (B.26) is always positive. Finally, a specific check for $n=3$ shows that (B.26) holds for integers $n \geq 3$.

□

B.3 Cycle with Fixed Order

This is a description of a short cycle that can occur within this setup if a specific order of operations is followed. There are two procedures – addition and deletion of a link. The addition checks if *one* link could be added. The deletion checks if *any number* of links could be gradually removed. First is the addition procedure after which the deletion procedure begins. The deletion cuts links step by step (restarting the checking procedure after every change) until no links can be removed and only then can the addition procedure start again. The deletion and addition procedures alternate until no links can be added or removed and then the process stops.

Importantly, there is a fixed order followed for every check after the adjacency matrix has been changed (this includes every time a link has been severed) – always starting from the cell $(1,1)$, continuing along the row and going to the next row after checking the whole row. Since the adjacency matrix is symmetric, only the values above the main diagonal are checked.

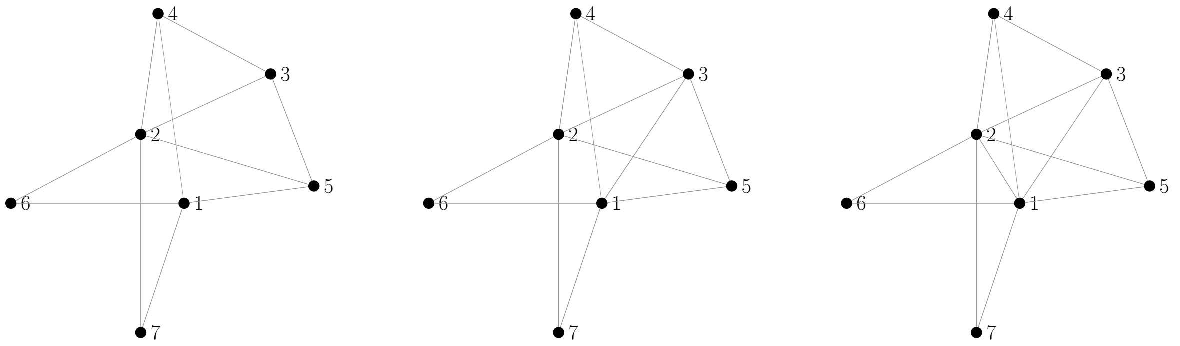
Consider Figure B.1 (a). The overall welfare is $W = 6.813$. The contributions of each node to each node and the payoffs at each node are: (i) for 1: $(\frac{1}{5}, \frac{1}{5}, \frac{3}{10}, \frac{3}{10})$, payoff 1.322; (ii) for 2: $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4})$, payoff 1.275; (iii) for 3: $\frac{3}{13}, \frac{5}{13}, \frac{5}{13}$, payoff 1.005; (iv) for $4, 5$: $\frac{15}{47}, \frac{12}{47}, \frac{20}{47}$, payoff 0.863; (v) for $6, 7$: $\frac{5}{9}, \frac{4}{9}$, payoff 0.742.

In Figure B.1 (b) overall welfare is $W = 6.752$ and the difference to (a) is that nodes 1 and 3 are connected. To see why consider the corresponding payoffs: (i) for $1, 2$: $\frac{1}{\sqrt{\frac{5\ 2}{3}+\frac{5}{4}+\frac{5\ 2}{2}}} \left(\frac{2}{\sqrt{\frac{3}{4}+\frac{2\ 3}{5}}} + \frac{1}{\sqrt{\frac{4\ 2}{3}+\frac{2\ 4}{5}}} + \frac{2}{\sqrt{\frac{2\ 2}{5}}}\right) = 1.341$; (ii) for 3: $\frac{1}{\sqrt{\frac{4\ 2}{3}+\frac{4\ 2}{5}}} \left(\frac{2}{\sqrt{\frac{3}{4}+\frac{2\ 3}{5}}} + \frac{2}{\sqrt{\frac{5}{4}+\frac{2\ 5}{3}+\frac{5\ 2}{2}}}\right) = 1.006$; (iii) for $4, 5$: $\frac{1}{\sqrt{\frac{3}{4}+\frac{3\ 2}{5}}} \left(\frac{1}{\sqrt{\frac{4\ 2}{3}+\frac{2\ 4}{5}}} + \frac{2}{\sqrt{\frac{5}{4}+\frac{2\ 5}{3}+\frac{5\ 2}{2}}}\right) = 0.809$; (iv) for $6, 7$: $\frac{1}{\sqrt{\frac{2\ 2}{5}}} \frac{2}{\sqrt{\frac{5}{4}+\frac{2\ 5}{3}+\frac{5\ 2}{2}}} = 0.722$.

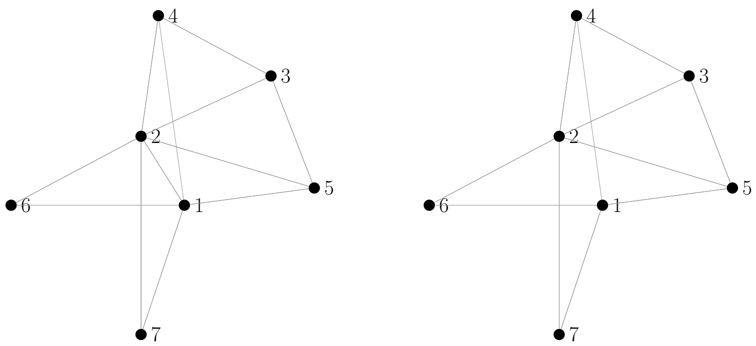
A connection between 1 and 2 would make 1 worse off (cf. Figure B.1 (d)), since the payoff of 1 would be $\frac{1}{\sqrt{\frac{5\ 2}{2}+\frac{5}{6}+\frac{5\ 2}{3}}} \left(\frac{2}{\sqrt{\frac{2}{5}+\frac{2}{6}}} + \frac{2}{\sqrt{\frac{3}{3}+\frac{3}{5}+\frac{3}{6}}} + \frac{1}{\sqrt{\frac{6\ 2}{2}+\frac{6}{5}+\frac{6\ 3}{3}}}\right) = 1.318$. And as a result nodes 1 and 3 connect, since both are better off connected. However, now no two nodes should be willing to sever their link. Clearly, 1 and 3 would not want to disconnect.

If 1 and 4 disconnect their corresponding payoffs would be: (i) for 1: $\frac{1}{\sqrt{\frac{4}{4}+\frac{4}{3}+\frac{4\ 2}{2}}} \left(\frac{1}{\sqrt{\frac{3}{5}+\frac{2\ 3}{4}}} + \frac{1}{\sqrt{\frac{4}{2}+\frac{4}{4}+\frac{4}{3}+\frac{4}{5}}} + \frac{2}{\sqrt{\frac{2}{5}+\frac{2}{4}}}\right) = 1.287$; (ii) for 4: $\frac{1}{\sqrt{\frac{2}{4}+\frac{2}{5}}} \left(\frac{1}{\sqrt{\frac{4}{2}+\frac{4}{4}+\frac{4}{3}+\frac{4}{5}}} + \frac{1}{\sqrt{\frac{5}{4}+\frac{5}{3}+\frac{5\ 3}{2}}}\right) = 0.792$. So, they would not disconnect. Same reasoning holds for 1 and 5.

If 1 and 6 disconnect their corresponding payoffs would be: (i) for 1: $\frac{1}{\sqrt{\frac{4\ 2}{3}+\frac{4}{4}+\frac{4}{2}}} \left(\frac{1}{\sqrt{\frac{2}{4}+\frac{2}{5}}} + \frac{1}{\sqrt{\frac{4}{4}+\frac{4\ 2}{3}+\frac{4}{5}}} + \frac{2}{\sqrt{\frac{3}{5}+\frac{2\ 3}{4}}}\right) = 1.221$; (ii) for 6: $\frac{1}{\sqrt{\frac{1}{5}}\sqrt{\frac{5}{4}+\frac{2\ 5}{3}+\frac{5}{2}+5\frac{1}{1}}} = 0.643$ So, they would not disconnect. Same reasoning holds for 1 and 7 and 2 and all its neighbours. If 3 and 4 disconnect their corresponding payoffs would be: (i) for 3: $\frac{1}{\sqrt{\frac{3}{3}+\frac{3\ 2}{5}}} \left(\frac{1}{\sqrt{\frac{3}{3}+\frac{2\ 3}{5}}} + \frac{2}{\sqrt{\frac{3\ 5}{2}+\frac{2\ 5}{2}}}\right) = 0.836$; (ii) for 4: $\frac{2}{\sqrt{\frac{4\ 2}{5}}\sqrt{\frac{3\ 5}{2}+\frac{2\ 5}{3}}} = 0.480$. So, they would not disconnect. Same reasoning holds for 3 and 5. Therefore, no nodes would want to disconnect in Figure B.1 (b). This leads to the next check if additional nodes want to form a connection.



(a) Initial stage (b) 1 and 3 connect (c) 1 and 2 connect



(d) 1 and 3 disconnect (e) 1 and 2 disconnect ⇒ (a).

Node	1	2	3	W
(a/e)	1.322	1.275	1.005	6.813
(b)	1.341	1.341	1.006	6.752
(c)	1.342	1.342	1.039	6.719
(d)	1.318	1.288	1.047	6.776

(f) Node payoffs and welfare (W)

Figure B.1: Cycle with fixed order for $n = 7$

In Figure B.1 (c) the overall welfare is $W = 6.719$ and nodes 1 and 2 have a link. To see why, consider the corresponding payoffs of $1, 2$ which are $\frac{1}{\sqrt{\frac{6\ 2}{2}+\frac{6}{6}+\frac{6}{4}+\frac{6\ 2}{3}}}\left(\frac{1}{\sqrt{\frac{4\ 2}{6}+\frac{4\ 2}{3}}}+\frac{1}{\sqrt{\frac{6\ 2}{2}+\frac{6}{6}+\frac{6}{4}+\frac{6\ 2}{3}}}+\frac{2}{\sqrt{\frac{3}{4}+\frac{3\ 2}{6}}}+\frac{2}{\sqrt{\frac{2\ 2}{6}}}\right) = 1.342$. So, they would connect since it improves both of them strictly.

At this point, 3 would choose to disconnect from 1 (Figure B.1 (d)) since it gets a payoff of $\frac{1}{\sqrt{\frac{2\ 3}{3}+\frac{3}{5}}}\left(\frac{2}{\sqrt{\frac{3}{3}+\frac{3}{6}+\frac{3}{5}}}+\frac{1}{\sqrt{\frac{3\ 6}{6}+\frac{6}{5}+\frac{6}{2}}}\right) = 1.087$ as compared to the payoff in (c), $\frac{1}{\sqrt{\frac{2\ 4}{3}+\frac{4\ 2}{6}}}\left(\frac{2}{\sqrt{\frac{3\ 2}{6}+\frac{3}{4}}}+\frac{2}{\sqrt{\frac{6\ 2}{2}+\frac{6}{6}+\frac{6}{4}+\frac{6\ 2}{3}}}\right) = 1.039$. There the overall welfare is $W = 6.776$.

In Figure B.1 (e), 2 would choose to disconnect from 1 since it gets a payoff of 1.341 (same as in (a)) as compared to the payoff in (d), $\frac{1}{\sqrt{\frac{5\ 2}{2}+\frac{5}{6}+\frac{5\ 2}{3}}}\left(\frac{2}{\sqrt{\frac{2}{5}+\frac{2}{6}}}+\frac{2}{\sqrt{\frac{3}{3}+\frac{3}{5}+\frac{3}{6}}}+\frac{1}{\sqrt{\frac{6\ 2}{2}+\frac{6}{5}+\frac{6\ 3}{3}}}\right) = 1.318$. This completes the cycle. It must be noted that the overall welfare in the different instances first decreases and then increases.

Chapter 3

Persuading Communicating Voters

3.1 Introduction

Multiple-receiver Bayesian persuasion models with private communication often assume that receivers do not exchange information with each other between receiving signals from the sender and taking their action. In reality, however, people usually deliberate before voting in favour of a political party or simply before buying a product, and might consult friends and acquaintances in search of additional opinions and information. We model such communication among receivers prior to making a decision with a simple setup: we assume that receivers are in a fixed network that is commonly known and that neighbours can observe each other's private messages. An application of such communication are social networks like Facebook, where parties can target political adverts at specific (potential) voter groups and such use of social media has become a common feature of election campaigns in recent years. In this setting, if a person likes or shares an ad or a video, it is visible to all of their friends on the social network. In this chapter we consider the limit case, where the sender sends perfectly custom-tailored messages to the receivers and the messages are observed by the receivers and their direct neighbours with probability one.

This setup significantly complicates the sender's problem of optimal persuasion as he must also take into account the intricacies of the information flow between receivers when deciding how to design his communication strategy. In the *absence* of a network structure, the sender can improve upon public communication by employing private messages. Hence, the immediate question arises whether private communication is still (strictly) beneficial for the sender when receivers communicate within a network. If so, are some networks *more* manipulable than others? Is the most manipulable network empty? Can the sender to *benefit* from a *denser* network?

This chapter is co-authored with Toygar Kerman. It is based on Kerman and Tenev (2021).

3.1.1 An Illustrative Example

Suppose that a company, which is either profitable (P) or not profitable (N), has two potential investors, I_1 and I_2, who initially believe it to be profitable with probability 1/3. Investors will invest in the company only if they consider it is profitable with probability at least 1/2. Both parties can potentially give the same amount, and the investment from only one of them is sufficient to cover the company's cash flow problems. Two financial reports are prepared regarding the company's finances, where each report is randomly assigned (with equal probability) to one of the investors. While one report provides true findings, the other one always favours investment.

First, assume that investors are not communicating, i.e. they are in the empty network. The *communication strategy* of the CEO can be formalized by distributions $\pi(\cdot|P)$ and $\pi(\cdot|N)$ on some set of signals. Let (p, n) denote the *signal* in which I_1 receives a *message* to invest and I_2 a *message* not to invest. The CEO's communication strategy above can be represented as follows.

π	P	N
(p, p)	1	0
(p, n)	0	$\frac{1}{2}$
(n, p)	0	$\frac{1}{2}$

After observing p, any investor's belief that the company is profitable is 1/2. Hence, after any realization at least one of them invests. The CEO guarantees at least one investment with probability 1 and the communication strategy above is indeed optimal.

Now, assume that I_1 and I_2 know each other and exchange the information they got from the reports before making their decisions. This can be presented with the network:

Since investors can observe each other's messages, the communication strategy above is no longer optimal: when the signal realization is (p, n) or (n, p), both investors know that the company is not profitable, i.e. the true state is N. In this case, the probability of obtaining an investment is determined by the probability of (p, p), i.e. 1/3. Since receivers can observe the signal realization (as the network is complete), the CEO cannot communicate privately and alternatively chooses the following communication strategy:

π'	P	N
(p, p)	1	$\frac{1}{2}$
(n, n)	0	$\frac{1}{2}$

Note that since the communication strategy is public, either both investors invest or none of them does. The probability of obtaining an investment under π' is $1/3 \cdot 1 + 2/3 \cdot 1/2 = 2/3$.

Since it is sufficient for the CEO to guarantee only one investment, the CEO is worse off when the investors are communicating.

The example illustrates how the additional exchange of information affects the Sender's gain from persuasion in two extreme cases: the *empty network* when there is no link between I_1 and I_2, and the *complete network* in the latter case. In a complete network, it is clear that the sender's optimal communication strategy is public. This chapter shows that in some networks that are neither empty nor complete (and for some voting quotas) the sender's optimal communication strategy is indeed public. However, in others he can still benefit from communicating privately, or even achieve the optimal probability of success as under the empty network.

3.1.2 Overview of Results

We consider an exogenous network that is common knowledge, a binary state space, and a sender who commits to a private communication strategy. Receivers know the joint distribution of signals (vectors of messages), but only observe their own and their neighbours' private messages from the signal realization. If the network is empty, then our model reduces to the model of Kerman, Herings, and Karos (2020), which is used as a benchmark.

First, we show that for non-empty networks the "traditional" assumptions for multiple-receiver Bayesian persuasion models are no longer without loss of generality. The most common of these assumptions, *straightforwardness*, under which the sender sends "recommendations" to vote for an alternative and receivers follow the recommendations, does not hold since receivers take into account not only their own messages, but also their neighbours'. The shared information limits the possible message combinations the sender can use for optimal persuasion as the recommendations are no longer fully private.

Another such assumption that is not without loss of generality is "truth-telling" in the sender's preferred state. In other words, it is not always in the sender's best interest to generate a perfectly informative message whenever the true state is his preferred one; there exist networks (e.g. the line network) where the sender is better off garbling information in such cases. This is due to the fact that, in addition to different messages, the variability between the signals in the sender's preferred state is used to limit the information flow between neighbours.

Given a number of receivers and a quota, the sender cannot achieve a higher probability of success in a non-empty network than under the empty network of the same size. The reason for this is that given any non-empty network and any communication strategy π, there exists a communication strategy π' which reveals exactly the same information in the empty network as π does in the non-empty network. In other words, any communication strategy in a non-empty network can be replicated in its corresponding empty network. In contrast, when the network is complete, the optimal communication strategy is *public*, as illustrated in the motivating example. For networks that lie in between the empty network and the complete network, we consider network structures which are often discussed in the literature (star, wheel, circle and other regular networks) and identify conditions under which the sender can achieve the probability of success under the empty network. Interestingly, if there are sufficiently many isolated nodes in a network, this can be achieved regardless of how many or between which of the remaining receivers the links are.

Finally, perhaps surprisingly, while it seems intuitive that adding a link to a given network would (weakly) decrease the optimal value (as receivers gain more information), it is possible that the optimal value is *higher* in a *denser* network. The reasoning behind this is that while any communication strategy in a non-empty network can be replicated in the empty network of the same size, it is not always the case that it can be replicated in a less dense network which is *non-empty*. Due to this fact, it is possible that an optimal communication strategy under a denser network has a higher value than an optimal communication strategy under a less dense network.

The rest of the chapter is organized as follows. Subsection 3.1.3 discusses related literature. Section 3.2 provides notation and preliminary definitions. Section 3.3 discusses the benchmark case and shows preliminary results. Section 3.4 gives observations which hold for network structures in general. Section 3.5 focuses on the optimal values in specific networks. Section 3.6 shows that a denser network is not always worse for the sender. Section 3.7 concludes.

3.1.3 Related Literature

The current model comes closest to and is an extension of Kerman et al. (2020), which builds upon Kamenica and Gentzkow (2011) and considers a sender communicating privately with multiple receivers. The authors derive an optimal communication strategy under the assumption that receivers vote sincerely. While we also assume sincere voting, a crucial difference to the current setup is that in their model a receiver only has access to information revealed to them by the sender, whereas in our setup directly connected voters perfectly exchange information. So, their model is a special case of ours, applied to the empty network setup. While they focus on finding an optimal communication strategy that yields an equilibrium under sincere voting, we assume sincere voting and determine optimal communication strategies for different types of networks.

The current chapter also relates to the literature on public communication and collective decision making. Schnakenberg (2015) considers an expert who is privately informed about the state and can conceal information from the receivers. One important distinction from this chapter is that the expert does not have to commit to a communication strategy, which in effect reduces the ex ante expected utilities of the voters. Alonso and Câmara (2016) study public communication where receivers have heterogeneous preferences which the sender can exploit by targeting minimal winning coalitions. Kosterina (2018) considers a model with a continuous state space in which receiver's prior is unknown and shows that the solution to the problem is the same as the one of persuading receivers with heterogeneous priors. She proves that even when the receiver's prior is unknown, the optimal communication strategy always includes a signal that tells the truth with probability 1, whereas in our setup this is not generally true.

This chapter also relates to the literature on voting games and private communication. Wang (2013) compares the outcomes from public and private communication strategies but also incorporates strategic voter behaviour in the analysis and concludes that the sender is weakly worse off under private communication when messages are uncorrelated. In contrast, we consider private correlated messages and show that this often improves upon public communication. Bardhi and Guo (2018) also focus on collective decision making and in particular the unanimity voting rule, whereas we allow for general monotonic and anonymous voting rules. Chan, Gupta, Li, and Wang (2019) consider private communication as well and assume that voting is costly. In their model the optimal communication strategy targets the receivers with the lowest costs who are easiest to persuade.

This chapter is also related to the literature on more general games in information design. Bergemann and Morris (2016) consider a game of incomplete information, characterize Bayesian Nash equilibria and demonstrate that this corresponds to Bayes correlated equilibria. They show that in equilibrium receivers are "obedient", they follow the recommendation by the sender. Taneva (2019) derives the optimal information structure in finite environments and characterizes it in a symmetric binary setting. She shows that a related notion dubbed "directness" (which also corresponds to straightforwardness in the sense of Kamenica and Gentzkow (2011)) can be used without loss of generality. One important difference of this chapter to these is that we do not consider strategic receivers. Moreover, our setting does not allow for straightforwardness.

One paper that considers Bayesian persuasion in networks is Egorov and Sonin (2019). In their model, a sender communicates publicly with the receivers who are in a fixed network by choosing the level of propaganda. A receiver might choose to get the information from the signal of the sender for a cost, or rely on his neighbours and get to the same information from them. The authors find that adding or removing links from the network can have different effects on the level or propaganda, depending on whether the probability that information passes through two agents is high or low. Unlike their paper, in our model the sender does not give identical information to receivers, but rather sends private messages to each receiver which are observed by his neighbours at no cost.

This chapter also shares some characteristics with Buechel and Mechtenberg (2019), where agents are in a network and experts can share with their direct neighbours a policy recommendation on the basis of private signals they receive. The authors find that the outcomes crucially depend on the network structure, and more precisely on whether there are experts with disproportionally large neighbourhoods. In this case, it might be beneficial for their neighbours (the non-experts) to ignore such recommendations. While their main model includes only bipartite networks, the authors extend it to general network structures and analyse equilibrium strategies. Unlike our setup, they have strategic receivers and agents decide whether or not to share information with their neighbourhood, e.g. they can ignore a connection with their neighbour.

Candogan and Drakopoulos (2020) consider a model of social network interactions in which agents choose whether or not to engage with some (possibly inaccurate) content on the social network and the agents' payoffs depend on the engagement of their neighbours. The platform tries to design a signalling mechanism which maximizes engagement or minimizes misinformation by sending recommendations to its users to engage or not. They find that in order to maximize engagement the platform needs to issue a recommendation to engage based on an agent-specific threshold determined their position in the network, while to minimize misinformation common thresholds are used. In another model with local strategic complementarities Candogan (2019) finds that when the degrees of some nodes in the network increase, this can reduce the information designer's payoff.

3.2 Notation

3.2.1 Communication Strategy

Let $N = \{1, \dots, n\}$ be the set of receivers and $\Omega = \{X, Y\}$ the set of states of the world. Sender and receivers share a common prior belief $\lambda^0 \in \Delta^o(\Omega)$ about the true state of the world, where $\Delta^o(\Omega)$ denotes the set of strictly positive probability distributions on Ω.

Let S_i be a finite set of *messages* the sender can send to receiver i, and let $S = \prod_{i \in N} S_i$, where the elements of S are called *signals*. A *communication strategy* is a function $\pi : \Omega \to \Delta(S)$ which maps each state of the world to a joint probability distribution over signal realizations. Denote the set of all communication strategies by Π.

For each signal $s \in S$, let $s_i \in S_i$ denote the message for receiver i. For each $s_i \in S_i$ and $\omega \in \Omega$, let $\pi_i(s_i|\omega) = \sum_{t \in S : t_i = s_i} \pi(t|\omega)$, which is the probability that receiver i observes s_i given ω. Define $S^\pi = \{s \in S | \exists \omega \in \Omega : \pi(s|\omega) > 0\}$. That is, S^π consists of signals in S which are sent with positive probability by π. Similarly, for each π and $i \in N$, define $S_i^\pi = \{s_i \in S_i | \exists \omega \in \Omega : \pi_i(s_i|\omega) > 0\}$, which is the set of messages receiver i observes with positive probability under π.

3.2.2 Networks

An *undirected network* is a map $g : N \times N \to \{0, 1\}$ with $g_{ij} = g(i, j)$ and $g_{ij} = g_{ji}$. Given a set of receivers N, let $G(N)$ be the set of all such networks. We assume that receivers are in a fixed network, which is common knowledge among the sender and receivers. Each receiver in the network can observe his neighbours' message realizations. Thus, in a non-empty network a receiver gathers more information about the true state than he would from the same communication strategy under the empty network.

A network $g \in G(N)$ is *complete* if for all $i, j \in N$ with $i \neq j$ it holds that $g_{ij} = 1$, i.e. every two nodes have a link. In this case each receiver knows the signal realization, so all communication strategies are public on the complete network. For any network $g \in G(N)$, we denote the empty network with the *same number* of receivers by g_0. Observe that an empty network corresponds to a standard multiple-receiver Bayesian persuasion model.

Let $N_i(g) = \{j \in N | g_{ij} = 1\}$ be the neighbourhood of receiver i in g and let $\delta_i^g = |N_i(g)|$ denote the *degree* of i in g. Let $\bar{N}_i(g) = N_i(g) \cup \{i\}$. For any $\pi \in \Pi$, $s \in S^\pi$, $i \in N$, and $j \in N_i(g)$, let s_{ij} be the message i observes from j in s, that is $s_{ij} = s_j$. Let $s_i(g) = (s_{ij})_{j \in \bar{N}_i(g)}$ be the *information neighbourhood* of receiver i in s, that is, $s_i(g)$ is the vector of messages (with length $\delta_i^g + 1$) receiver i observes upon signal realization s. Let $A_i^\pi(g, s) = \{t \in S^\pi | t_i(g) = s_i(g)\}$ be the set of signals i considers possible upon signal realization s, or in other words, the set of signals i *associates* with s. Given $s, t \in S^\pi$, we say that t *is associated with* s if there exists an agent $i \in N$ such that $t \in A_i^\pi(g, s)$. Let $A^\pi(g, s) = \cup_{i \in N} A_i^\pi(g, s)$ be the set of all signals associated with s.

For any $g \in G(N)$, $\pi \in \Pi$, and $s \in S^\pi$, the posterior belief vector $\lambda^{s,g} \in \Delta(\Omega)^n$ is defined by:

$$\lambda_i^{s,g}(\omega) = \frac{\sum_{t \in A_i^\pi(g,s)} \pi(t|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega} \sum_{t \in A_i^\pi(g,s)} \pi(t|\omega')\lambda^0(\omega')}, \quad i \in N, \omega \in \Omega. \tag{3.1}$$

That is, $\lambda_i^{s,g}(\omega)$ is receiver i's posterior belief that the state is ω upon observing $s_i(g)$.

A communication strategy $\pi \in \Pi$ *induces* $\sigma^g \in \Delta(\Delta(\Omega)^n)$ under network g if for all $\lambda \in \Delta(\Omega)^n$ it holds that:

$$\sigma^g(\lambda) = \sum_{s \in S^\pi : \lambda^{s,g} = \lambda} \sum_{\omega \in \Omega} \pi(s|\omega)\lambda^0(\omega). \tag{3.2}$$

In words, $\sigma^g(\lambda)$ is the probability of posterior vector λ under network g.[1]

3.2.3 Voting

For each $i \in N$, let $B_i = \{x, y\}$ be the set of actions of receiver i. Let $B = \prod_{i \in N} B_i$ denote the space of action profiles and let $Z = \{x, y\}$ be the set of voting outcomes. Following the signal realization, each receiver chooses an action according to his posterior belief.

[1]If $\lambda \notin \text{supp}(\sigma^g)$, then the right hand side of (3.2) is 0. Moreover, it is well-defined since S^π is finite.

Let $z^k : B \to Z$ be a map, where $z^k(a)$ is the outcome of the vote when the action profile is a and is defined by:

$$z^k(a) = \begin{cases} x & \text{if } |\{i \in N : a_i = x\}| \geq k, \\ y & \text{otherwise.} \end{cases}$$

We assume that the sender's utility function $v : Z \to \{0,1\}$ has value 1 if x is implemented and 0 otherwise.

For any $g \in G(N)$, $\pi \in \Pi$, and $i \in N$, let $S_i^\pi(g) = \prod_{j \in \bar{N}_i(g)} S_j^\pi$ be the space of vectors of length $\delta_i^g + 1$ that i can observe under g and π. Let $\alpha_i^{\pi,g} : S_i^\pi(g) \to B_i$ be agent i's *sincere action function*, such that for any realization $s \in S^\pi$ it holds that:

$$\alpha_i^{\pi,g}(s_i(g)) = \begin{cases} x & \text{if } \lambda_i^{s,g}(X) \geq \frac{1}{2}, \\ y & \text{otherwise.} \end{cases} \tag{3.3}$$

Throughout the chapter we assume that $\lambda^0(X) < \lambda^0(Y)$, since otherwise there is no need for persuasion. Define $Z_x^g(\pi) = \left\{s \in S^\pi | z^k\left(\alpha^{\pi,g}(s)\right) = x\right\}$. That is, $Z_x^g(\pi)$ is the set of signals which implement x in g under π and sincere voting.

Receiver i is *pivotal in* $s \in S^\pi$ if for any $a_i \in B_i$, $z^k(a_i, \alpha_{-i}^{\pi,g}(s_{-i}(g)) = a_i$. That is, i is pivotal in the voting following s if i's vote determines the voting outcome given that all $j \neq i$ vote sincerely.

Let $a \in B$ be an action profile and $z = z^k(a)$ be a voting outcome. The *value* of a communication strategy $\pi \in \Pi$ for quota k is defined as the sender's expected utility under distribution σ^g induced by π in network g. As we fix λ^0 and $\alpha_i^{\pi,g}$ throughout the chapter, we write $V_k^\pi(g) = V_k^\pi(\lambda^0, g, \alpha^{\pi,g})$. The value of the sender is:

$$V_k^\pi(g) = \mathbb{E}_{\lambda^0}\left[\mathbb{E}_\pi\left[v(z^k\left(\alpha^{\pi,g}(s)\right))\right]\right] = \lambda^0(X) \sum_{s \in Z_x^g(\pi)} \pi(s|X) + \lambda^0(Y) \sum_{s \in Z_x^g(\pi)} \pi(s|Y).$$

Thus, the value of a communication strategy is equal to the probability of x being implemented under π and g and quota k, given that receivers vote sincerely.

A communication strategy π^* is *optimal* in Π *under* g for *quota* k if $V_k^{\pi^*}(g) = \sup_{\pi \in \Pi} V_k^\pi(g)$. The value of an optimal communication strategy *on the empty network* with n nodes and quota k is denoted by V_k^n.

3.3 Preliminaries

This section demonstrates how the information-sharing feature of our model produces a non-trivial change in the overall setup of multiple-receiver Bayesian persuasion as many of the common assumptions in standard models cease to hold in general. First, we briefly consider the model in Kerman et al. (2020), which corresponds to the empty network case in our framework and is used as a benchmark. Second, we demonstrate that the network structure can affect the agents' posteriors and the (optimal) value. Third, we show that the notion of straightforwardness as defined in Kamenica and Gentzkow (2011) does not hold for optimal strategies in our model. Finally, sending a recommendation with probability 1 in the sender's preferred state, an assumption that is made without loss of generality for optimal strategies in many Bayesian persuasion models, is also no longer optimal.

3.3.1 Optimal Communication Strategy on the Empty Network and with Public Signals

In Kerman et al. (2020), it is without loss of generality to restrict attention to straightforward (in the sense of Kamenica and Gentzkow (2011)) and anonymous communication strategies. Given a binary state space $\Omega = \{X, Y\}$, it is optimal for the sender to send "recommendations", x or y, as messages. In this case, the probability that x is sent to exactly ℓ receivers if the state is X and the probability that x is sent to exactly ℓ receivers if the state is Y can be represented by q_ℓ and r_ℓ, respectively, where each signal in which the same number of receivers observe x has the same probability. An optimal communication strategy (in the empty network) is then given by the following theorem.

Theorem 3.1. *(Kerman et al., 2020) Let $\pi^* \in \Pi$ with representation (q^*, r^*) be:*

$$(q_n^*; r_0^*, r_k^*) = \begin{cases} (1; 0, 1) & \text{if } \lambda^0(X) \geq \frac{k}{n+k}, \\ \left(1; 1 - \frac{\lambda^0(X)}{\lambda^0(Y)}\frac{n}{k}, \frac{\lambda^0(X)}{\lambda^0(Y)}\frac{n}{k}\right) & \text{if } \lambda^0(X) < \frac{k}{n+k}. \end{cases}$$

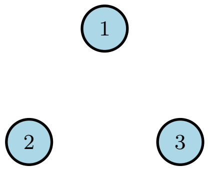
Then π^* is optimal at λ^0. In particular, $V_k^n = \min\left\{\frac{n+k}{k}\lambda^0(X), 1\right\}$.

Note that the optimal *public* communication strategy's value is always independent of the network and the quota since all agents observe the signal realization. In this case, since receivers have homogeneous prior beliefs, the situation is the same as persuading a single receiver. Thus, as in Kamenica and Gentzkow (2011), x is implemented with probability 1 in state X and with probability $\lambda^0(X)/\lambda^0(Y)$ in state Y. The value of the optimal public communication strategy is then given by $V^p = \lambda^0(X) \cdot 1 + \lambda^0(Y) \cdot \lambda^0(X)/\lambda^0(Y) = 2\lambda^0(X)$.

3.3.2 The Effects of the Network

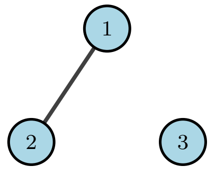
One important observation in our framework is that the set of posteriors which can be induced differs from the one under a model *without* a network structure. Given a communication strategy $\pi \in \Pi$, $s \in S^\pi$ and $g, g' \in G(N)$, let $\lambda^{s,g}$ and $\lambda^{s,g'}$ be the posterior vectors under g and g', respectively. It is easy to see that for $g \neq g'$, it is possible that $\lambda^{s,g} \neq \lambda^{s,g'}$. If $g \subseteq g'$, an agent gathers more information upon a signal realization under g' than under g by also observing the messages of his new neighbours, which leads to different posterior beliefs in the two models. This is shown in Example 3.1.

Example 3.1. *Let* $|N|= 3$, $\lambda^0(X) = 1/3$, *and* $k = 2$. *The optimal communication strategy* π *prescribed by Theorem 3.1 for the empty network* g_0 *is given by* $q_3^* = 1$, $r_2^* = 3/4$, *and* $r_0^* = 1/4$, *since* $\lambda^0(X) < k/(n + k)$. *This can be represented by:*



Network g_0

π	$\omega = X$	$\omega = Y$
(x, x, x)	1	0
(x, x, y)	0	$\frac{1}{4}$
(x, y, x)	0	$\frac{1}{4}$
(y, x, x)	0	$\frac{1}{4}$
(y, y, y)	0	$\frac{1}{4}$



Network g

The support of the distribution that π *induces under the empty network* g_0 *is given by:*

$$supp\,(\sigma^{g_0}) = \left\{ \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right), \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right), \left(0, \tfrac{1}{2}, \tfrac{1}{2}\right), (0, 0, 0) \right\},$$

and the value is $V_2^3 = V_2^\pi(g_0) = 5/6$.

Now consider network g *on the right-hand side, which has one added link compared to its counterpart* g_0, *so that receivers 1 and 2 can observe each other's messages. The set of posteriors induced by* π *under* g *is:*

$$supp\,(\sigma^{g}) = \left\{ \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{2}\right), \left(\tfrac{2}{3}, \tfrac{2}{3}, 0\right), \left(0, 0, \tfrac{1}{2}\right), (0, 0, 0) \right\}.$$

The value is $V_2^\pi(g) = 1/2 < 5/6 = V_2^\pi(g_0)$. *There are fewer posterior vectors under* g *than under the empty network* g_0. *Since receivers 1 and 2 can observe each other's messages, different signals lead to the same posterior vector. Note that 1 and 2 would have the same posterior belief in any communication strategy. Since they have the same information neighbourhood, they always update their beliefs in the same way. It follows that the sender can send the same message to 1 and 2 without loss of generality.*[2] *Thus, since* $k = 2$ *the problem is equivalent to persuading either receivers 1 and 2 together, or persuading all receivers. However, this is tantamount to employing a public communication strategy in which case the network does not have an effect since all agents observe the same message. Therefore, the optimal communication strategy in* g *is public and given by:*

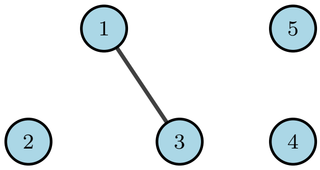
π'	$\omega = X$	$\omega = Y$
(x, x, x)	*1*	$\frac{1}{2}$
(y, y, y)	*0*	$\frac{1}{2}$

Thus, the value is $V^{\pi'}(g) = 1/3 \cdot 1 + 2/3 \cdot 1/2 = 2/3 = V^p$. *So, if* $|N|= 3$ *and* $k = 2$, *having even one link between agents reduces the optimal value from the one under the empty network,* V_2^3, *to the value of the optimal public communication strategy,* V^p. △

[2] A generalized version of this statement is proved later on in Proposition 3.3.

An intuitive observation in Example 3.1 is that adding a link to the network while keeping π (which is optimal on g_0) fixed decreases the value of π. However, it is also possible that the sender *benefits* from a *denser* network under a *fixed* π. We show this next.

Example 3.2. *Let* $|N|= 5$, $\lambda^0(X) = 1/3$ *and* $k = 3$. *Consider the following network* g *and communication strategy* π*:*



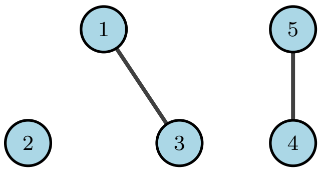
π	$\omega = X$	$\omega = Y$
(x, y, x, x, y)	$\frac{1}{3}$	$\frac{1}{6}$
(x, y, x, y, x)	$\frac{1}{3}$	$\frac{1}{6}$
(y, y, y, x, x)	$\frac{1}{6}$	$\frac{1}{6}$
(y, y, y, y, y)	$\frac{1}{6}$	$\frac{1}{2}$

The support of σ^g *is given by:*

$$supp\,(\sigma^g) = \left\{\left(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{3}{7}, \tfrac{3}{11}\right), \left(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{3}{11}, \tfrac{3}{7}\right), \left(\tfrac{1}{5}, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{3}{7}, \tfrac{3}{7}\right), \left(\tfrac{1}{5}, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{3}{11}, \tfrac{3}{11}\right)\right\}.$$

Thus, x *is not implemented after any realization, i.e.* $V_3^\pi(g) = 0$.

Now, consider the following network g'.



The beliefs of receivers 1, 2 and 3 are the same under g' *as under* g *but receivers 4 and 5 have different beliefs. The support under the same communication strategy* π *is:*

$$supp(\sigma^{g'}) = \left\{\left(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{1}{5}, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{3}, \tfrac{1}{3}\right), \left(\tfrac{1}{5}, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \tfrac{1}{7}\right)\right\}.$$

Thus, x *is implemented after realizations* (x, y, x, x, y) *and* (x, y, x, y, x) *as both lead to the posterior vector* $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. *Hence:*

$$V_3^\pi\,(g') = \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{9} > 0 = V_3^\pi(g),$$

so, the sender benefits from the same communication strategy in a denser network. △

While the sender benefits from the additional connection provided in g', a notable aspect of Example 3.2 is that π is not optimal on g. This leaves the question how adding a link generally affects the value of the *optimal* strategy. In Example 3.1, the optimal communication strategy π under g_0 has a strictly higher value than the optimal communication strategy π' under the denser network g. One might expect this to be a general observation. However, Section 3.6 shows that this is not always true.

3.3.3 Straightforwardness

Translating the straightforwardness definition of Kamenica and Gentzkow (2011) to multiple receivers, a communication strategy $\pi \in \Pi$ is *straightforward* if $S_i^\pi \subseteq B_i$ and if for the sincere action function $\alpha_i^\pi : S_i^\pi \to B_i$ and $a_i \in S_i^\pi$ it holds that $\alpha_i^\pi(a_i) = a_i$. First, observe that under the sincere action function α_i^π receivers vote only according to their own message, whereas in our model the sincere action function $\alpha_i^{\pi,g}$ also takes into account their neighbours' messages. Hence, using the standard definition of straightforwardness is not without loss of generality. This observation is illustrated in Example 3.3 below. However, first we need to introduce a definition and a lemma.

Upon a signal realization, a receiver votes for x if the probability of observing their specific message vector is sufficiently higher in state X than in state Y, given their prior belief and a specified communication strategy. In order for this to hold, there must exist at least one signal which includes this message vector and has higher probability in state X. Such signals are instrumental for increasing the probability of implementing x and will be referred to as "anchors", because the x votes are dependent on them.

Definition 3.1. *For any $\pi \in \Pi$ and $s \in S^\pi$, the signal s is an* anchor *if $\pi(s|X)\lambda^0(X) \geq \pi(s|Y)\lambda^0(Y)$. The set of all anchors is denoted by $An(\pi)$.*

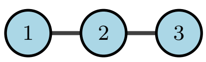
Given $\pi \in \Pi$, $s \in An(\pi)$ and $i \in N$, if for every $t \in S^\pi$ with $t \neq s$ it holds that $s_i(g) \neq t_i(g)$, then $\alpha_i^{\pi,g}(s_i(g)) = x$. That is, if a receiver i can uniquely identify the signal realisation as an anchor, he votes for x.

Lemma 3.1. *Let $g \in G(N)$. For any $\pi \in \Pi$ with $V_k^\pi(g) > 0$ it holds that $An(\pi) \neq \emptyset$.*

The proofs to all statements can be found in Appendix C. Lemma 3.1 shows that every communication strategy with a *non-zero* value has at least one anchor. Its proof further implies that every x-vote in such a strategy is associated with at least one anchor.

Now we can show that assuming straightforwardness is not without loss of generality.

Example 3.3. *Let $|N|= 3$, $\lambda^0(X) = 1/3$, and $k = 1$. Consider the following network g:*



π'	$\omega = X$	$\omega = Y$
(x, x, x)	1	0
(y, x, x)	0	$\frac{1}{2}$
(x, x, y)	0	$\frac{1}{2}$

Let us try to construct a straightforward communication strategy π. First, note that $\lambda_2^{s,g}(X) \geq 1/2$ if and only if $s \in An(\pi)$, so receiver 2 never votes for x in any non-anchor signal. This is the case since for any $s \in S^\pi$, $s_2(g) = s$, that is, receiver 2 always observes the whole signal as he has access to the messages of everyone in the network. Therefore, for any straightforward *communication strategy* $\pi \in \Pi$ *it must hold that: (i)* $t_2 = y$ *for all* $t \notin An(\pi)$ *and (ii)* $s_2 = x$ *for all* $s \in An(\pi)$. *However, since for any such* s *and* t *it holds that* $s_2 \neq t_2$, *we have* $\lambda_1^{t,g}(X), \lambda_3^{t,g}(X) < 1/2$, *as observing* 2 *would induce* 1 *and* 3 *to also vote* y *in non-anchor signals. Thus, it must also hold that: (i)* $t = (y, y, y)$ *for all* $t \notin An(\pi)$ *and (ii)* $s_2 = x$ *for all* $s \in An(\pi)$. *This implies that for any possible quota* $k = 1, 2, 3$, $V_k^{\pi}(g) = V^p = 2/3$, *so the maximum value of a straightforward communication strategy would be equal to the value of the optimal public communication strategy.*

In contrast, the non-straightforward communication strategy π' *above can improve on this value since:*

$$supp\,(\sigma^g) = \left\{ \left(\tfrac{1}{2}, 1, \tfrac{1}{2}\right), \left(0, 0, \tfrac{1}{2}\right), \left(\tfrac{1}{2}, 0, 0\right) \right\}$$

and $V_1^{\pi}(g) = 1$, *which is clearly optimal as it is the maximal possible value. Here it is important to note that under* π' *after observing the same message* (x), *receiver* 2 *votes for* x *in some signals and for* y *in others.*[3] $\triangle$

A naive extension of straightforwardness to the network setting would suggest that strategies send recommendations to neighbourhoods rather than individual receivers, where a receiver votes for x if he observes only x in his information neighbourhood and votes for y otherwise. Unfortunately, this is also not a viable option as shown in Example 3.4 below.

The process of finding the optimal communication strategy under the *empty* network is additionally simplified by the fact that restricting attention to anonymous communication strategies is without loss of generality (Kerman et al., 2020). For a general network structure it follows trivially that assuming anonymity is no longer without loss.

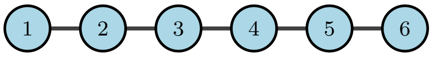
3.3.4 Truth-Telling in the Sender's Preferred State

In the standard single-receiver case, as well as most multiple-receiver models, telling the truth to the receiver(s) with probability 1 in the sender's preferred state (X) is optimal. This common result ceases to hold in general in the current setup, and this is illustrated by Example 3.4. To help with the analysis we introduce a lemma, which establishes that the maximum number of messages a communication strategy requires to achieve the optimal value is at most equal to its number of anchors plus one.

Lemma 3.2. *Let* $g \in G(N)$ *and* $\pi \in \Pi$. *There exists* $\pi' \in \Pi$ *with* $|An(\pi')| = |An(\pi)|$ *such that for each* $i \in N$ *it holds that* $|S_i^{\pi'}| \leq |An(\pi')| + 1$ *and* $V_k^{\pi'}(g) = V_k^{\pi}(g)$.

Example 3.4. *Let* $|N| = 6$, $\lambda^0(X) = 1/3$, *and* $k = 4$. *Consider the following network* g *and the communication strategy* π *which has one anchor and sends* x *to all receivers with probability 1 in state* X:

[3] Proposition 3.5 in the next section shows that in fact the structure of the communication strategy π' is optimal and results in the optimal value which can be achieved in any star network.



π	$\omega = X$	$\omega = Y$
(x,x,x,x,x,x)	1	0
(y,x,x,x,x,x)	0	$\frac{1}{4}$
(x,x,x,x,x,y)	0	$\frac{1}{4}$
(y,y,y,y,y,y)	0	$\frac{1}{2}$

The value of π *is given by* $V_4^\pi(g) = 2/3$*. This is equal to* V^p*, the value of the optimal public communication strategy.*

Now consider communication strategy π'*, which has two anchor signals and support:*

π'	$\omega = X$	$\omega = Y$
(x,x,x,x,x,x)	$\frac{1}{2}$	0
(y,y,y,y,y,y)	$\frac{1}{2}$	0
(x,x,x,y,y,y)	0	$\frac{1}{4}$
(y,x,x,x,x,x)	0	$\frac{1}{4}$
(y,y,y,y,y,x)	0	$\frac{1}{4}$
(y,x,y,x,y,x)	0	$\frac{1}{4}$

$$\begin{aligned}\text{supp}\,(\sigma^g) = \big\{ &\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \\ &\left(\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2}\right), \left(0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \\ &\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0\right), (0,0,0,0,0,0) \big\}.\end{aligned}$$

The value is $V_4^{\pi'}(g) = 1/3 \cdot 1 + 2/3 \cdot 3/4 = 5/6 > 2/3$*, which shows that the sender can improve upon* π *by garbling information in state* X*.*[4]

It is important that $V_4^{\pi'}(g)$ *cannot be achieved in a strategy with only one anchor. To prove this, assume the contrary. By Lemma 3.2, there exists* $\pi'' \in \Pi$ *such that for every* $i \in N$ *it holds that* $|S_i^{\pi''}| = 2$ *and* $V_4^{\pi''}(g) = V_4^{\pi'}(g) = 5/6$*. Assume without loss of generality that for each* $i \in N$*,* $S_i^{\pi''} = \{x, y\}$*. As truth-telling in state* X *requires, let* $\bar{x} = (x, \ldots, x)$ *be the anchor and let* $\pi''(\bar{x}|X) = 1$*. Let* $t \in S^{\pi''}$ *with* $t \neq \bar{x}$*. If for any* $i \in N$ *it holds that* $t_i \neq x$*, then for any* $j \in \bar{N}_i(g)$ *we have* $\alpha_j^{\pi'',g}(t_j(g)) = y$ *and thus, if* $i \in \{2,3,4,5\}$*, then* $\alpha^{\pi'',g}(t) = y$*, i.e.* t *does not implement* x*. Thus, in signals that implement* x*, only receivers 1 and 6 can observe a message different than* x*. In this case, an optimal communication strategy is* π *given above, which has value 2/3, contradicting that* $V_4^{\pi''}(g) = 5/6$*. Therefore, truth-telling in state* X *is not optimal in general.*

Recall the naive straightforwardness extension in the previous section; it is not violated by π*. Moreover,* π *is optimal among communication strategies which satisfy this definition. Since we can improve upon* π *with the "non-naively straightforward"* π'*, it follows that the naive extension is not optimal either. The feature which makes* π' *better than* π *is that apart*

[4]Note that π' is optimal on g (as implied by Proposition 3.1 in the next section). The same value could have been achieved with at most three messages (cf. Lemma 3.2) by sending a different message in the signal which does not implement x. However, it is possible to create this signal such that it is distinguished from both anchors by all nodes with only two messages.

from the different personal messages within signals, it uses the differences in the messages between the two anchors to create more variation in the strategy. △

3.4 General Observations

As shown, many of the usual simplifying assumptions in the empty network Bayesian persuasion model do not generally extend to non-empty networks. Nevertheless, there are optimality results which can be recovered and they are the subject of this section.

3.4.1 The Network Does not Benefit the Sender

We start by showing that having a network structure cannot be strictly beneficial for the sender. More precisely, for any communication strategy the optimal value under the empty network g_0 is at least as good as the optimal value under any *other* network g with the same number of nodes. This insight was already hinted at in Example 3.1. The simple logic behind this observation is that in a non-empty network the vote of every node is determined not by the individual messages of every receiver but by his whole neighbourhood. However, the information a receiver gathers from a communication strategy in a non-empty network can be replicated in an empty network of the same size.

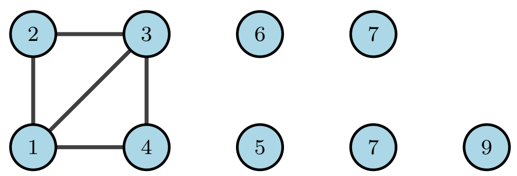
Proposition 3.1. *Let $g \in G(N)$. For any $\pi \in \Pi$ it holds that $V_k^{\pi}(g) \leq V_k^{n}$.*

Intuitively, the reason the empty network value weakly dominates the value of any other network with the same number of nodes is that the sender can target at least as many winning coalitions under the empty network as under any non-empty network. However, this might not hold when comparing two non-empty networks, as shown in Section 3.6.

3.4.2 Networks with Isolated Nodes

This section shows that with sufficiently many isolated nodes the network has no effect on what the sender can achieve. In particular, if $k \geq n/2$ and if the number of singletons in a network is at least k, then the sender can achieve the empty network optimal value, irrespective of the connections between the remaining nodes.

Example 3.5. *Let $|N|= 9$, $\lambda^0(X) = 1/3$, and $k = 5$. Consider network g below, the communication strategy π and the support of its induced distribution. Note that $2k = 10 > 9 = n$, so the singleton nodes are equal to k and are a simple majority.*



π	$\omega = X$	$\omega = Y$
(x,x,x,x,x,x,x,x,x)	1	0
(y,y,y,y,x,x,x,x,x)	0	$\frac{4}{10}$
(x,x,x,x,x,y,y,y,y)	0	$\frac{1}{10}$
(x,x,x,x,y,x,y,y,y)	0	$\frac{1}{10}$
(x,x,x,x,y,y,x,y,y)	0	$\frac{1}{10}$
(x,x,x,x,y,y,y,x,y)	0	$\frac{1}{10}$
(x,x,x,x,y,y,y,y,x)	0	$\frac{1}{10}$
(y,y,y,y,y,y,y,y,y)	0	$\frac{1}{10}$

$$supp\,(\sigma^g) = \big\{ \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \left(0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0\right), \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,0\right), \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},0,0\right), \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,\tfrac{1}{2},0\right), \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2}\right), (0,0,0,0,0,0,0,0,0) \big\}.$$

The value is $V_5^\pi(g) = 1/3 \cdot 1 + 2/3 \cdot 9/10 = 14/15 = V_5^9$, the value for persuading $k = 5$ out of $n = 9$ in an empty network. Observe that this value can be achieved irrespective of the connections between nodes 1-4, as these nodes are always treated uniformly. △

Proposition 3.2. *Let $g \in G(N)$ and $|\{i \in N : \delta_i^g = 0\}| \geq k$. If $k \geq n/2$, then there exists $\pi \in \Pi$ such that $V_k^\pi(g) = V_k^n$.*

3.4.3 Networks Consisting of Complete Components

If two nodes i and j in a network g have *exactly* the same neighbourhood, $\bar{N}_i(g) = \bar{N}_j(g)$, then the sets of signals they consider possible under any communication strategy π are the same, i.e. $A_i^\pi(g,s) = A_j^\pi(g,s)$. This implies that they *always* have the same posterior belief and vote for the *same* alternative, i.e. for every $s \in S^\pi$ it holds that $\alpha_i^{\pi,g}(s_i(g)) = \alpha_j^{\pi,g}(s_j(g))$. Therefore, in such cases we can restrict attention to communication strategies which send the same message to i and j. In particular, the information that the sender wants to provide to the neighbours of i and j by sending different messages to i and j can also be provided to the neighbours by sending the same message to i and j *within* the same signal. Note that in this case i and j can still observe different messages *between* signals. This logic is formalized in Proposition 3.3, which is particularly relevant for members of a complete component because they always share the same information neighbourhood.

Proposition 3.3. *Let $\pi \in \Pi$ and let $g \in G(N)$ and $i, j \in N$ be such that $\bar{N}_i(g) = \bar{N}_j(g)$. Then there exists $\pi' \in \Pi$ such that for any $s \in S^{\pi'}$, $s_i = s_j$ and $V_k^{\pi'}(g) = V_k^\pi(g)$.*

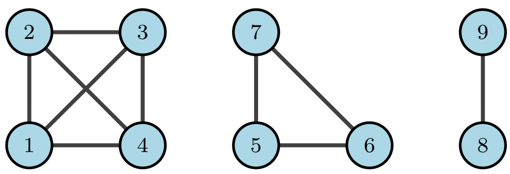
As every receiver in a complete network has the same information neighbourhood, an immediate corollary to Proposition 3.3 follows.

Corollary 3.1. *Let $g \in G(N)$ be a complete network. Then there exists an optimal $\pi \in \Pi$ such that $V_k^\pi(g) = V^p$.*

So, the optimal communication strategy in a complete network is public as it is optimal for the sender to treat *all* receivers within a signal uniformly by sending them same message.

Proposition 3.3 naturally extends to networks of disjoint complete components. Treating all nodes within a component uniformly in every signal makes this setup similar to an empty network with fewer *nodes* where every node has a *different* number of votes.

Example 3.6. *Let $|N| = 9$, $\lambda^0(X) = 1/3$, and $k = 5$. Consider the following network g with three complete components and the communication strategy π:*



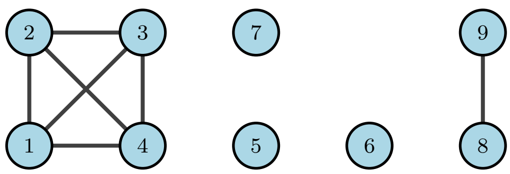
π	$\omega = X$	$\omega = Y$
(x,x,x,x,x,x,x,x,x)	1	0
(y,y,y,y,x,x,x,x,x)	0	$\frac{1}{4}$
(x,x,x,x,y,y,y,x,x)	0	$\frac{1}{4}$
(x,x,x,x,x,x,x,y,y)	0	$\frac{1}{4}$
(y,y,y,y,y,y,y,y,y)	0	$\frac{1}{4}$

Here, selecting any component to receive a y message leaves enough nodes (5, 6, or 7) to implement x. This results in:

$$supp\,(\sigma^g) = \left\{ \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \left(0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,\tfrac{1}{2},\tfrac{1}{2}\right), \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0\right), (0,0,0,0,0,0,0,0,0) \right\}.$$

The value is $V_5^\pi(g) = 1/3 \cdot 1 + 2/3 \cdot 3/4 = 5/6 = V_2^3$, the value for persuading $k = 2$ out of $n = 3$ nodes in the empty network and it is clear that no higher value can be achieved.

However, this is not necessarily the case if selecting a different number of components in different signals can still fulfil the quota, as this allows for much more variation. For example, when nodes $5, 6, 7$ are singletons strategy π' can be applied.



π'	$\omega = X$	$\omega = Y$
(x,x,x,x,x,x,x,x,x)	1	0
(y,y,y,y,x,x,x,x,x)	0	$\frac{1}{3}$
(x,x,x,x,y,x,y,y,y)	0	$\frac{1}{6}$
(x,x,x,x,x,y,y,y,y)	0	$\frac{1}{6}$
(x,x,x,x,y,y,x,y,y)	0	$\frac{1}{6}$
(y,y,y,y,y,y,y,y,y)	0	$\frac{1}{6}$

In this case the support of π' *is:*

$$supp\left(\sigma^{g}\right)=\left\{\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{5},\tfrac{3}{5}\right),\left(0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{5},\tfrac{3}{5}\right),\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,0\right),\right.$$
$$\left.\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0\right),\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},0,0\right),\left(0,0,0,0,0,0,0,0,0\right)\right\}$$

The value is $V_5^{\pi'}(g)=1/3\cdot 1+2/3\cdot 5/6=8/9>5/6.$ △

To formalize the logic of the example, given a network g, let $C(g)$ be the set of all components of g. For $|C(g)|=\ell$, let $\mathfrak{C}^q$ be the set of all subsets of components, where each subset has cardinality q. That is, $\mathfrak{C}^q=\{\mathcal{C}'\subseteq C(g):|\mathcal{C}'|=q\}$.

A network $g\in G(N)$ is *connected* if for any $i,j\in N$ there is a path between i and j.

Proposition 3.4. *Let* $g\in G(N)$ *be a disconnected network consisting of* ℓ *complete components of respective sizes* $c_1,c_2,\ldots,c_\ell\in C(g)$*. If* $q\in\mathbb{N}$ *is such that for each* $\mathcal{C}'\in\mathfrak{C}^q$ *it holds that* $\sum_{C\in\mathcal{C}'}|C|\geq k$*, then there exists an optimal* $\pi\in\Pi$ *such that* $V_k^{\pi}(g)=V_q^{\ell}$.

So, if a network consists of ℓ complete disjoint components *and* combining the *same number* of components q fulfils the quota, it is possible to construct a strategy with value equal to the optimal value of persuading q out of ℓ agents. This follows directly from the uniform treatment of all nodes within the same component as per Proposition 3.3.

3.5 Optimality in Specific Network Structures

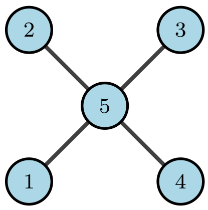
This section provides optimal strategies for networks which frequently appear in the literature. Every subsection deals with different network types and starts with an example conveying the intuition which drives the more general result and the main mechanisms to construct the optimal strategies. This is followed by a formal statement of the result.

3.5.1 Star Networks

A network $g\in G(N)$ is a *star* if there exists $j\in N$ such that for any $i\in N$ with $i\neq j$ it holds that $g_{ij}=1$ and for any $\ell\in N$ with $\ell\neq j$ it holds that $g_{i\ell}=0$. The star presents a situation in which the optimal value under the empty network *cannot* be achieved.

Before proceeding to the formal statement about star networks, consider the example below, which bears similarity to Example 3.3. The communication strategy employed keeps the centre's message the same across all signals which implement x while varying the periphery nodes' messages in the same way as in the empty network and thus achieves value equal to V_k^{n-1}. This is less than V_k^n and so the optimal value in the empty network with private communication is not achieved in this case.

Example 3.7. *Let* $|N|= 5$, $\lambda^0(X) = 1/3$, *and* $k = 3$. *Consider the following network* g, *the communication strategy* π *and its support:*



π	$\omega = X$	$\omega = Y$
(x, x, x, x, x)	1	0
(y, x, x, x, x)	0	$\frac{1}{6}$
(x, y, x, x, x)	0	$\frac{1}{6}$
(x, x, y, x, x)	0	$\frac{1}{6}$
(x, x, x, y, x)	0	$\frac{1}{6}$
(y, y, y, y, y)	0	$\frac{1}{3}$

$$supp\left(\sigma^g\right) = \left\{ \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1\right), \left(0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0\right), \left(\tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0\right), \left(\tfrac{1}{2}, \tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right), \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right), \left(0, 0, 0, 0, 0\right) \right\}.$$

The value is $V_3^\pi(g) = 1/3 \cdot 1 + 2/3 \cdot 2/3 = 7/9 = V_3^4$. △

Proposition 3.5. *Let* $g \in G(N)$ *be a star and let* $k < n$. *Then there exists an optimal* $\pi \in \Pi$ *such that* $V_k^\pi(g) = V_k^{n-1}$.

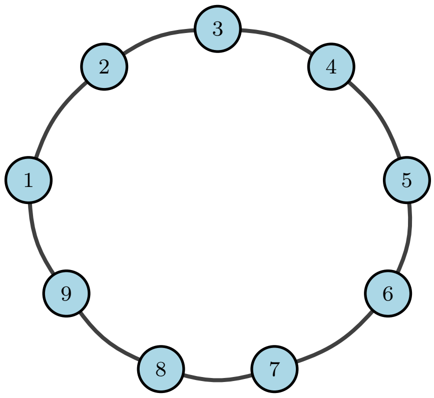
In the optimal communication strategy specified in the proof of Proposition 3.5 the center node observes the same message in all signals which implement x. It is possible to extrapolate this logic to all networks which have nodes that are connected to *all* other nodes. In the limit case (the complete network), this corresponds to sending the same messages to all receivers within a signal for every signal, i.e. a public communication strategy.[5]

3.5.2 Regular Networks

A network g is *regular* if for any $i, j \in N$ it holds that $\delta_i^g = \delta_j^g = \delta$. Regular networks are a natural extension of the situation described by Theorem 3.1 and Corollary 3.1, as the empty network is also regular. A regular connected network with $\delta = 2$ is a *circle*.

Example 3.8. *Let* $|N|= 9$, $\lambda^0(X) = 1/3$, *and* $k = 6$. *Consider the following network* g, *the communication strategy* π *and its support:*

[5] Similarly, when $n = k$ in a star, the optimal communication is public.



π	$\omega = X$	$\omega = Y$
(x,x,x,x,x,x,x,x,x)	1	0
(y,x,x,x,x,x,x,x,x)	0	$\frac{1}{12}$
(x,y,x,x,x,x,x,x,x)	0	$\frac{1}{12}$
(x,x,y,x,x,x,x,x,x)	0	$\frac{1}{12}$
(x,x,x,y,x,x,x,x,x)	0	$\frac{1}{12}$
(x,x,x,x,y,x,x,x,x)	0	$\frac{1}{12}$
(x,x,x,x,x,y,x,x,x)	0	$\frac{1}{12}$
(x,x,x,x,x,x,y,x,x)	0	$\frac{1}{12}$
(x,x,x,x,x,x,x,y,x)	0	$\frac{1}{12}$
(x,x,x,x,x,x,x,x,y)	0	$\frac{1}{12}$
(y,y,y,y,y,y,y,y,y)	0	$\frac{1}{4}$

$$\begin{aligned} supp\,(\sigma^g) = \big\{ &\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \left(0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0\right), \\ &\left(0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \left(\tfrac{1}{2},0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \left(\tfrac{1}{2},\tfrac{1}{2},0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \\ &\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,\tfrac{1}{2},\tfrac{1}{2}\right), \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,\tfrac{1}{2}\right), \\ &\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0\right), \left(0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0\right), (0,0,0,0,0,0,0,0,0)\big\}. \end{aligned}$$

Each receiver observes x with probability 1 in state X. Hence, sending y to a receiver in state Y implies that 3 receivers vote for y. So, in a signal implementing x in state Y exactly 6 receivers are persuaded and $V_6^\pi(g) = 1/3 \cdot 1 + 2/3 \cdot 9/12 = 5/6 = V_6^9$. △

Proposition 3.6. *Let $g \in G(N)$ be a circle and let $k < n-2$. Then there exists $\pi \in \Pi$ such that $V_k^\pi(g) = V_k^n$.*

Proposition 3.6 formalizes the observation in Example 3.8 and establishes that if the quota is less than $n-2$, the optimal value under the empty network can be achieved on a circle.

A network which combines many properties of circle and star networks is the wheel: formally, a network $g \in G(N)$ is a *wheel* if there exists $j \in N$ such that for any $i \in N$ with $i \neq j$ it holds that: *(i)* $g_{ij} = 1$, *(ii)* $\delta_i^g = 3$, and *(iii)* for any $\ell \in N, \ell \neq j$, and $\ell \neq i$ there is a path between i and ℓ which does not include j.

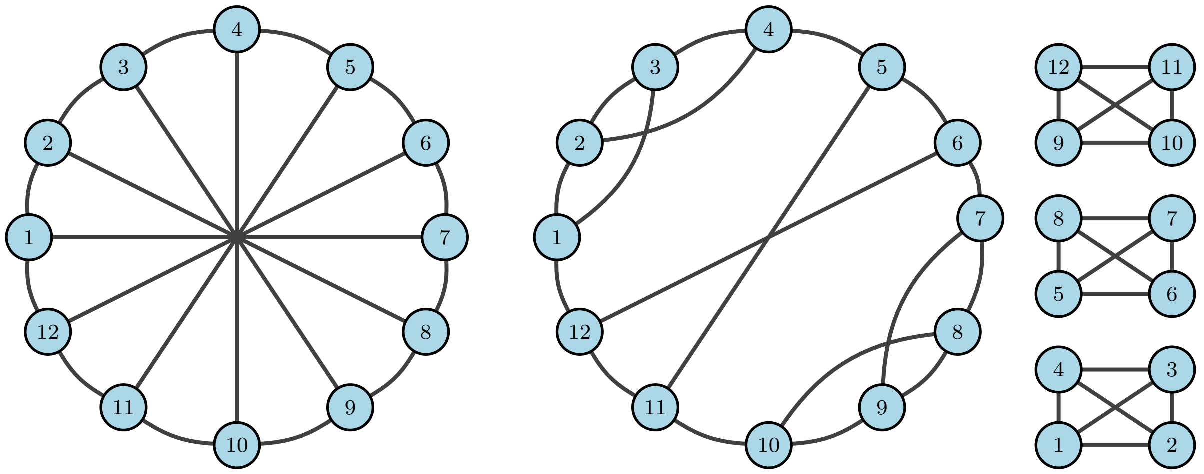
The reasoning behind sending the same message to the center node in all signals which implement x used for the star also applies to the wheel. In this case, the sender can treat the network as if it is a circle to achieve the optimal value provided that the quota is less than $n-3$. This observation is stated as a corollary to Propositions 3.5 and 3.6.

Corollary 3.2. *Let $g \in G(N)$ be a wheel and let $k < n-3$. Then there exists an optimal $\pi \in \Pi$ such that $V_k^\pi(g) = V_k^{n-1}$.*

Proposition 3.7. *Let $g \in G(N)$ be a regular network with degree δ and let $k = n - 1 - \delta$. Then there exists $\pi \in \Pi$ such that $V_k^{\pi}(g) = V_k^n$.*

Proposition 3.7 shows that if $k = n-1-\delta$ the logic used for the circle can be applied to every regular network to obtain the optimal value. The following example illustrates that extending the reasoning in the proposition to $k < n - 1 - \delta$ is a non-trivial task.

Example 3.9. *Consider the following regular networks g_1, g_2 and g_3, all with degree 3.*



Network g_1 Network g_2 Network g_3

Let $\lambda^0(X) = 1/3$. By Proposition 3.7, the optimal value under the empty network can be achieved in all three networks for $k = 8$. However, if $k = 6$ we can easily construct a communication strategy with value equal to $V_6^{12} = V_1^2 = 1$ for g_1 and g_2 (similar to the construction in Example 3.8), but not for g_3. Because the network is disconnected and each component is complete, persuading exactly 6 receivers is impossible due to Proposition 3.3 as receivers in a complete component all vote for the same alternative. In particular, any signal that implements x has at least 8 receivers who vote for x, implying that the value 1 cannot be achieved.

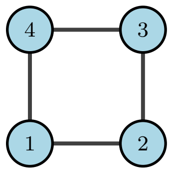
Decreasing the quota to $k = 4$, it is once again easy to construct optimal strategies for g_1 and g_3, but an optimal strategy for g_2 will clearly have a different structure. $\triangle$

It is important to note that the conditions in Proposition 3.6 and 3.7 are sufficient, but not necessary. The next example provides a network which satisfies neither of them but the optimal communication strategy on it still achieves the maximum value.

Example 3.10. *Let $|N| = 4$, $\lambda^0(X) = 1/3$. Consider the following network g and the communication strategies π for $k = 2$ and π' for $k = 3$ with corresponding supports:*

π	$\omega = X$	$\omega = Y$
(x,x,y,y)	$\frac{1}{4}$	0
(x,y,y,x)	$\frac{1}{4}$	0
(y,y,x,x)	$\frac{1}{4}$	0
(y,x,x,y)	$\frac{1}{4}$	0
(y,y,y,x)	0	$\frac{1}{8}$
(y,y,x,y)	0	$\frac{1}{8}$
(y,x,y,y)	0	$\frac{1}{8}$
(x,y,y,y)	0	$\frac{1}{8}$
(x,x,x,y)	0	$\frac{1}{8}$
(x,x,y,x)	0	$\frac{1}{8}$
(x,y,x,x)	0	$\frac{1}{8}$
(y,x,x,x)	0	$\frac{1}{8}$

$$\text{supp}\left(\sigma^g\right) = \left\{\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \left(\tfrac{1}{2},0,\tfrac{1}{2},0\right), \left(0,\tfrac{1}{2},0,\tfrac{1}{2}\right)\right\}$$



$$\text{supp}\left(\sigma^g\right) = \left\{\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0\right), \left(0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \left(\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2}\right), \left(\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2}\right)\right\}$$

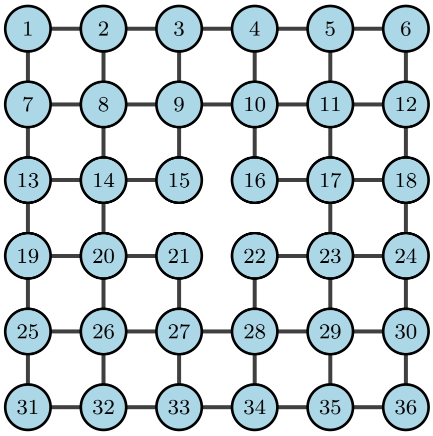
π'	$\omega = X$	$\omega = Y$
(x,x,x,x)	$\frac{1}{6}$	0
(y,x,x,y)	$\frac{1}{6}$	0
(y,y,x,x)	$\frac{1}{6}$	0
(y,x,y,y)	$\frac{1}{6}$	0
(x,y,y,x)	$\frac{1}{6}$	0
(y,y,y,y)	$\frac{1}{6}$	0
(x,x,x,y)	0	$\frac{1}{12}$
(y,x,x,x)	0	$\frac{1}{12}$
(x,x,y,x)	0	$\frac{1}{12}$
(x,y,x,x)	0	$\frac{1}{12}$
(y,y,y,x)	0	$\frac{1}{12}$
(x,y,y,y)	0	$\frac{1}{12}$
(y,y,x,y)	0	$\frac{1}{12}$
(y,x,y,y)	0	$\frac{1}{12}$
(x,y,x,y)	0	$\frac{1}{3}$

where in the first support the first four signals in π produce the first posterior, the odd-numbered remaining signals have the second one and the even-numbered remaining signals have the third posterior. As all signals in π implement x, the value is $V_2^{\pi}(g) = 1 = V_2^4$, while for π' it is $V_3^{\pi'}(g) = 7/9 = V_3^4$. Following the same logic as in Example 3.4 it is possible to see that in both cases the values cannot be achieved by simpler strategies, i.e. ones with fewer total signals or fewer anchors. The underlying structures also cannot be readily extrapolated to optimal strategies for the general case of $k = n-2$ or $k = n-1$ on a circle, as such an extrapolation would require using more than two messages. △

3.5.3 Irregular Networks

Moving away from regular networks, it is possible to derive intuitive results for irregular networks which have some sufficient degree of symmetry like in the example below.

Example 3.11. *Let $|N|= 36$, $\lambda^0(X) = 1/3$, and $k = 27$. Consider network g below.*



π	$\omega = X$	$\omega = Y$
$\bar{x}$	1	0
s^1	0	$\frac{1}{6}$
s^2	0	$\frac{1}{6}$
s^3	0	$\frac{1}{6}$
s^4	0	$\frac{1}{6}$
$\bar{y}$	0	$\frac{1}{3}$

Notice that there are no links between nodes 15 and 16, 15 and 21, 16 and 22, and 21 and 22. Further, it is important that while the network shows some symmetry, it is not regular as there are nodes with degrees 2, 3, and 4. Additionally, exactly $1/4$ *of it can be covered by sending a message* y *to specific agents in one of the quadrants. In particular, sending* y *to receivers 2, 7, 8, and 15 makes receivers in the NW quadrant (and no one else) observe at least one* y *in their information neighbourhood. Define signals:*

(i) $s^1 \in S$ *such that for each* $i \in N$, $s^1_i = y$ *if* $i \in \{2, 7, 8, 15\}$ *and* $s^1_i = x$ *otherwise,*

(ii) $s^2 \in S$ *such that for each* $i \in N$, $s^2_i = y$ *if* $i \in \{5, 11, 12, 16\}$ *and* $s^1_i = x$ *otherwise,*

(iii) $s^3 \in S$ *such that for each* $i \in N$, $s^3_i = y$ *if* $i \in \{21, 25, 26, 32\}$ *and* $s^1_i = x$ *otherwise,*

(iv) $s^4 \in S$ *such that for each* $i \in N$, $s^4_i = y$ *if* $i \in \{22, 29, 30, 35\}$ *and* $s^1_i = x$ *otherwise.*

Consider communication strategy π *and note that* s^1, s^2, s^3, *and* s^4 *make receivers in the NW, NE, SW, and SE quadrant respectively vote for* y. $V^{\pi}_{27}(g) = V^{36}_{27} = V^4_3 = 7/9$. △

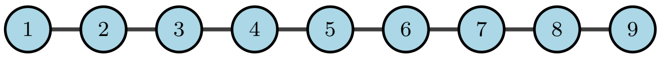
Proposition 3.8. *Let* $g \in G(N)$ *be a network and let* $q \in \mathbb{N}$ *be a common factor of* n *and* k. *If there exist* n/q *disjoint sets of nodes* $O_1, \ldots, O_{n/q} \subseteq N$ *such that for each* $j \in \{1, \ldots, n/q\}$ *it holds that* $|O_j| = q$ *and for each* $O_j \subseteq N$ *there exists* $L_j \subseteq O_j$ *with* $\cup_{i \in L_j} \bar{N}_i(g) = O_j$, *then there exists* $\pi \in \Pi$ *with* $V^{\pi}_k(g) = V^n_k$.

It is important to notice that the proposition presents only a sufficient condition. It generalizes the observation from Example 3.11 that if equal-sized parts of the network can observe at least one y message in their information neighbourhood in different signals this can be leveraged to get the optimal value, irrespective of regularity. Finally, it also extends the underlying logic behind Proposition 3.6 and 3.7 to irregular networks.

Line Networks

The line network is a notable case of a network which in some cases can be divided in the way described in Proposition 3.8.

Example 3.12. *Let* $|N|= 9$, $\lambda^0(X) = 1/3$, *and* $k = 6$. *Consider the following network* g, *the communication strategy* $\pi \in \Pi$ *and its support:*



π	$\omega = X$	$\omega = Y$
$(x, x, x, x, x, x, x, x, x)$	1	0
$(x, x, x, x, x, x, x, y, x)$	0	$\frac{1}{4}$
$(x, x, x, x, y, x, x, x, x)$	0	$\frac{1}{4}$
$(x, y, x, x, x, x, x, x, x)$	0	$\frac{1}{4}$
$(y, y, y, y, y, y, y, y, y)$	0	$\frac{1}{4}$

$$supp\,(\sigma^g) = \Big\{ \big(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\big), \big(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0\big), \big(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \big), \big(0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\big), (0, 0, 0, 0, 0, 0, 0, 0, 0) \Big\}.$$

The value is $V_6^\pi(g) = 1/3 \cdot 1 + 2/3 \cdot 3/4 = 5/6 = V_6^9$. *Interestingly, by Proposition 3.8,* π *is a simpler alternative to the communication strategy in Example 3.8. It preserves the value* V_6^9 *also in the case of a circle (which has an added link with respect to* g*).* △

The following corollary to Proposition 3.8 states that line networks with a common factor 2 or 3 for n and k can achieve the optimal value by building optimal strategies following the pattern of Example 3.4 or 3.12 and hence it is presented without proof.

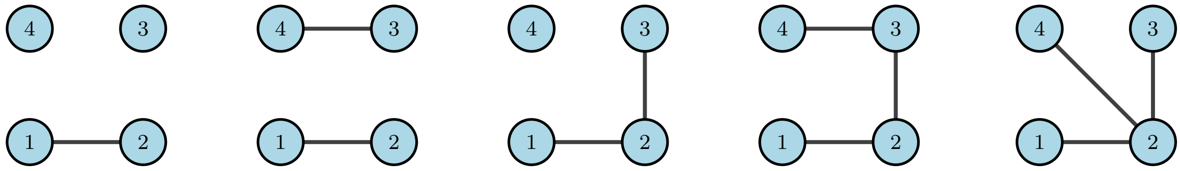
Corollary 3.3. *If* $g \in G(N)$ *is a line and if for* $\alpha, \beta \in \mathbb{N}$: *(i)* $k = 3\alpha, n = 3\beta$ *or (ii)* $k = 2\alpha, n = 2\beta$, *then there exists* $\pi \in \Pi$ *such that* $V_k^\pi(g) = V_k^n$.

3.6 Adding Links

Section 3 showed that adding a link to the network while keeping the communication strategy fixed can leave the sender better or worse off. Example 3.2 demonstrates how for a suboptimal communication strategy adding a link benefits the sender. Additionally, Example 3.12 illustrates that making a circle from a line can preserve the *optimal* value.

When the communication strategy is not fixed, however, one might expect that the *optimal* value (weakly) decreases when a link is added. Surprisingly, the optimal communication strategy might in fact have a *higher* value after adding a link, so that the sender benefits from a *denser* network. This is illustrated by means of an extended example.

Example 3.13. *Let* $|N|= 4$, $\lambda^0(X) = 1/3$, *and* $k = 2$. *Consider the following set of networks starting with* g, *obtained by adding a single link:* g_1 *and* g_2 *from* g*; the line* g_3 *from either* g_1 *or* g_2*; the star* g_4 *from* g_2.



$g : V_2^{\pi}(g) = 1$ $\quad g_1 : V_2^{\pi}(g_1) = 1$ $\quad$ Network g_2 $\quad g_3 : V_2^{\pi'}(g_3) = 1$ $\quad g_4 : V_2^{\pi''}(g_4) = 5/6$

Consider the maximum values which can be achieved in each case.

π	$\omega = X$	$\omega = Y$
(x, x, x, x)	1	0
(y, y, x, x)	0	$\frac{1}{2}$
(x, x, y, y)	0	$\frac{1}{2}$

π'	$\omega = X$	$\omega = Y$
(x, x, x, x)	1	0
(y, x, x, x)	0	$\frac{1}{2}$
(x, x, x, y)	0	$\frac{1}{2}$

Using communication strategy π for g and g_1 and π' for g_3, produces support:

$$supp\left(\sigma^g\right) = \left\{\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right), \left(0, 0, \tfrac{1}{2}, \tfrac{1}{2}\right)\right\}.$$

and therefore the maximum possible value of 1 in these cases.[6] *In the case of g_4, it follows from Proposition 3.5 that the optimal value is $V_2^3 = 5/6$.*

Adding a link between 1 and 4 in g_3 results in the circle network (call it g_5) analysed in Example 3.10 with value 1, so starting from g_0 and adding links in the following way: $g_0 \to g \to g_1 \to g_3 \to g_5$ keeps the optimal value at 1.

Considering a different sequence like $g \to g_2 \to g_4$, adding a link decreases the optimal value. This is not surprising given that the value of the optimal communication strategy with private messages on the empty network dominates (weakly for $n = k$, strongly for $n > k$) the value of the public communication strategy (i.e. on the complete network).

What is left to see is if adding a link can strictly increase the maximum value.

Claim 3.1. *For any $\hat{\pi} \in \Pi$, it holds that $V_2^{\hat{\pi}}(g_2) < 1$.*

The proof can be found in Appendix C. Claim 3.1 implies that adding a link between g and g_2 strictly decreases the value of the optimal communication strategy, but adding a link between g_2 and g_3 strictly increases it. The sequence $g \to g_2 \to g_3$ shows that the sender's value may be higher in a denser network. △

[6] Using Proposition 3.3, π treats identically pairs of nodes which observe the same information within a signal (1 and 2, 3 and 4 for g_1; 1 and 2 for g). Additionally, g_3 is a network which fulfils the conditions of Proposition 3.8 as it is possible to cover precisely k out of the n nodes in every non-anchor signal.

3.7 Discussion and Conclusion

Equilibrium

It is natural to ask whether the optimal communication strategies under sincere voting form an equilibrium. As per Kerman et al. (2020), if in a network with n nodes and quota k a communication strategy achieves the optimal value as under the empty network, V_k^n, then the strategy does *not* form a Bayesian Nash equilibrium (BNE) under sincere voting. This is because in optimal strategies agents are never pivotal in state X and are pivotal with positive probability in state Y. So, if all *other* agents vote sincerely, an agent has an incentive to vote *against* his belief *whenever* he considers the true state to be X: as he is not pivotal in state X, voting against his belief does not change the outcome of the vote if the true state is indeed X, while if the true state is Y he is pivotal and should vote against his belief. This exemplifies the *swing voter's curse*.

One remedy to the swing voter's curse is *never* making agents pivotal by having at least $k + 1$ agents vote for x in state Y (Kerman et al., 2020). A setting where an optimal communication strategy yields a BNE under sincere voting is given in g_3 in Example 3.9. By Proposition 3.3, if $k = 6$ an optimal communication strategy prescribes making all agents vote for x in state X and 8 (not 6) out of 12 agents vote for x in state Y. Hence, no agent is ever pivotal and sincere voting is a BNE. Similarly, the optimal communication strategy in a complete network is public and yields a BNE under sincere voting.

Concluding Remarks

This chapter investigates the optimal persuasion of voters who exchange private information with each other. This is modelled as a fixed network where direct neighbours can perfectly observe each other's private messages sent by a centralized body. The sender wants to implement a certain proposal and commits in advance to a communication strategy which sends correlated messages to the receivers. This presents several difficulties as most of the assumptions that hold under standard multiple-receiver Bayesian persuasion models fail. Crucially, while there are parallels to the empty network model, straightforward or anonymous strategies are not generally optimal, and neither are strategies which are truth-telling in state X.

The chapter tests the naive intuition that more information provided to the receivers through the network would make them less manipulable. This is true in some cases. However, the presence of a network structure does not *always* impede the persuasion abilities of the sender. In fact, the value of the optimal communication strategy is not monotone in terms of network density. Several network structures are identified where the sender can achieve the optimal value with private messages as under the empty network and their corresponding optimal strategies are outlined. While many of these situations rely on some form of symmetry or regularity of the network, this can also be achieved on general network structures, e.g. in networks with sufficiently many isolated nodes.

Appendix C

Supplementing Chapter 3

Proof of Lemma 3.1. Let $\pi \in \Pi$ be such that $V_k^\pi(g) > 0$. Assume to the contrary that $An(\pi) = \emptyset$. Then for every $s' \in S^\pi$ it holds that $\pi(s'|X)\lambda^0(X) < \pi(s'|Y)\lambda^0(Y)$. Since $V_k^\pi(g) > 0$, there exists $i \in N$ and $s \in S^\pi$ such that $\alpha_i^{\pi,g}(s_i(g)) = x$. That is,

$$\lambda_i^{s,g}(X) = \frac{\sum_{t\in A_i^\pi(g,s)} \pi(t|X)\lambda^0(X)}{\sum_{t\in A_i^\pi(g,s)} \pi(t|X)\lambda^0(X) + \sum_{t\in A_i^\pi(g,s)} \pi(t|Y)\lambda^0(Y)} \geq \frac{1}{2}. \tag{C.1}$$

Rearranging the terms in (C.1), we obtain:

$$\sum_{t\in A_i^\pi(g,s)} \pi(t|X)\lambda^0(X) \geq \sum_{t\in A_i^\pi(g,s)} \pi(t|Y)\lambda^0(Y),$$

a contradiction, since $\pi(s'|X)\lambda^0(X) < \pi(s'|Y)\lambda^0(Y)$ for every $s' \in S^\pi$.

Notice that this observation implies that every x vote under a strategy with $V_k^\pi > 0$ must be associated with at least one anchor. □

Proof of Lemma 3.2. Fix $i \in N$ and let $|S_i^\pi| \geq 2$. Note that if $\alpha_i^{\pi,g}(s_i(g)) = x$ for some $s \in S^\pi$, then $s \in \cup_{t\in An(\pi)} A^\pi(g,t)$, that is, whenever i votes in favour of x, his observation is associated with some anchor(s). Moreover, for each $t \in An(\pi)$ and $s, s' \in A_i^\pi(g,t)$ it holds that $s_i = s'_i = t_i$. Note that if $s'' \notin \cup_{t\in An(\pi)} A_i^\pi(g,t)$, then $\alpha_i^{\pi,g}(s''_i(g)) \neq x$, that is, whenever a receiver observes a *private* message not associated with any anchor, he does not vote in favour of x and neither do any of his neighbours. That is, for any $j \in N_i(g)$ it holds that $\alpha_j^{\pi,g}(s''_j(g)) \neq x$.

Define $\pi' : \Omega \to \Delta(\Omega)^n$ such that in π we replace all messages $s_i \in S_i^\pi$ such that $s \notin \cup_{t\in An(\pi)} A_i^\pi(g,t)$ with a message $m_i^y \in S_i \setminus S_i^\pi$. It follows that $An(\pi') = An(\pi)$ and i observes at most $|An(\pi')|+1$ messages under π', i.e. $|S_i^{\pi'}| \leq |An(\pi')|+1$. Let $s \in S^\pi$ be such that $s \notin \cup_{t\in An(\pi)} A_i^\pi(g,t)$ and therefore, $\alpha_i^{\pi,g}(s_i(g)) = y$. If $s' \in S^{\pi'}$ is obtained from $s \in S^\pi$ via replacing s_i with m_i^y, then $\alpha_i^{\pi',g}(s'_i(g)) = x$ if and only if $\alpha_i^{\pi,g}(s_i(g)) = x$. That is, the number of x votes are the same under s and s'. Thus, $V_k^{\pi'}(g) = V_k^\pi(g)$. □

Proof of Proposition 3.1. Let $\pi \in \Pi$. For each $i \in N$, assume that $|S_i^\pi(g)|= c(i)$. Let $R(i) = \left\{m_i^1, \dots, m_i^{c(i)}\right\} \subseteq S_i$ be a set of distinct messages for i. Moreover for any $j \in N$, $q \in \{1, \dots, c(i)\}$, and $q' \in \{1, \dots, c(j)\}$ let $m_i^q \neq m_j^{q'}$.

For each $i \in N$, let $\phi_i : S_i^\pi(g) \to R(i)$ be a bijection, so each *information neighbourhood* of i is mapped to a *unique message* in $R(i)$. For each $\omega \in \Omega$ and $s' \in S$, define $\pi' \in \Pi$:

$$\pi'(s'|\omega) = \begin{cases} \pi(s|\omega) & \text{if } \phi_i(s_i(g)) = s'_i, \quad \forall i \in N, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the definition of π' implies that there is a bijection $\phi : S^\pi \to S^{\pi'}$ such that for each $i \in N$, $\phi(s) = s'$ if and only if $\phi_i(s_i(g)) = s'_i$. Hence, π' is a communication strategy.

We want to show that the value of π' under the empty network is equal to the value of π under g, i.e., $V_k^{\pi'}(g_0) = V_k^\pi(g)$. What remains to be shown is that each receiver i has the same posterior belief upon observing $s_i(g)$ under π and upon observing $\phi_i(s_i(g))$ under π'. Let $s' \in S^{\pi'}$ be such that $s'_i \in \left\{m_i^1, \dots, m_i^{c(i)}\right\}$. For any $\omega \in \Omega$, we have

$$\lambda_i^{s'}(\omega) = \frac{\sum_{s \in S^{\pi'}: s_i = s'_i} \pi'(s|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega}\sum_{s \in S^{\pi'}: s_i = s'_i} \pi'(s|\omega')\lambda^0(\omega')} = \frac{\sum_{s \in S^{\pi}: s_i(g) = \phi^{-1}(s'_i)} \pi(s|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega}\sum_{s \in S^{\pi}: s_i(g) = \phi^{-1}(s'_i)} \pi'(s|\omega')\lambda^0(\omega')}$$
$$= \frac{\sum_{s \in A_i^\pi(g, \phi^{-1}(s'))} \pi(s|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega}\sum_{s \in A_i^\pi(g, \phi^{-1}(s'))} \pi(s|\omega')\lambda^0(\omega')} = \lambda_i^{\phi^{-1}(s'), g}(\omega).$$

Thus, for each $s \in S^\pi$ it holds that $\alpha^{\pi, g}(s) = \alpha^{\pi', g_0}(\phi(s))$. Hence, $V_k^{\pi'}(g_0) = V_k^\pi(g)$. Since any $\pi \in \Pi$ on some network g can be replicated on the empty network, $V_k^n \geq V_k^\pi(g)$. □

Proof of Proposition 3.2. Assume that $\lambda^0(Y)/\lambda^0(X) = \ell$, $|\{i \in N : \delta_i^g = 0\}| = q \geq k$ and $2k \geq n$. So, there are q singleton receivers and $n - q$ connected receivers. Denote the set of singleton receivers by N^q and the set of connected receivers by N^c.

Let $S' = \{x, y\}^n$. Define:

$$R = \{s \in S' : \forall i \in N^c, s_i = x \text{ and } |\{j \in N^q : s_j = x\}| = k - (n - q)\}.$$

In words, R is the set of signals in which all connected receivers and $k - n + q$ of the singleton receivers observe x. Note that $k - (n - q)$ is the required amount of x votes to fulfil the quota given that all connected receivers vote for x. Moreover, $|R| = \binom{q}{k-n+q}$.

Finally, define:

$$T = \{t \in S' : \forall i \in N^c, t_i = y \text{ and } |\{j \in N^q : t_j = x\}| = k\}.$$

So, T is the set of signals in which all $n - q$ connected receivers and $q - k$ singleton receivers observe y, while k singleton receivers observe x. Here $|T| = \binom{q}{k}$.

Let $\bar{x}$ be such that $\bar{x}_i = x$ for all $i \in N$ and define $\bar{y}$ analogously. Define π as follows:

$$\pi(s|\omega) = \begin{cases} 1 & \text{if } s = \bar{x} \text{ and } \omega = X, \\ 1 - \frac{n}{k\ell} & \text{if } s = \bar{y} \text{ and } \omega = Y, \\ \frac{n-k}{q\ell}\binom{q-1}{k-1}^{-1} & \text{if } s \in T \text{ and } \omega = Y, \\ \frac{1}{\binom{q}{k-n+q}\ell} & \text{if } s \in R \text{ and } \omega = Y. \end{cases}$$

It can be easily checked that π is a communication strategy:

$$\sum_{s\in S^\pi} \pi(s|Y) = 1 - \frac{n}{k\ell} + \binom{q}{k}\frac{n-k}{q\ell}\binom{q-1}{k-1}^{-1} + \binom{q}{k-n+q}\frac{1}{\binom{q}{k-n+q}\ell}$$
$$= 1 - \frac{n}{k\ell} + \frac{n-k}{q\ell}\frac{q}{k} + \frac{1}{\ell} = 1 - \frac{n}{k\ell} + \frac{n}{k\ell} = 1.$$

We will show that $V_k^\pi(g) = V_k^n = \lambda^0(X)(n+k)/k$. Under π, the connected agents always observe the same message. For any $s \in S^\pi$ with $s_i = x$ for all $i \in N^c$, we denote the information neighbourhood $s_i(g)$ of a connected receiver by $\tilde{x}(i)$. Note that for any $i \in N^c$, we have $\pi_i(\tilde{x}(i)|Y) = \frac{\binom{q}{k-n+q}}{\binom{q}{k-n+q}\ell} = 1/\ell$. Hence, for any $i \in N^c$ and $s \in S^\pi$ with $s_i(g) = \tilde{x}(i)$ it holds that:

$$\lambda_i^{s,g}(X) = \frac{\pi_i(\tilde{x}(i)|X)\lambda^0(X)}{\pi_i(\tilde{x}(i)|X)\lambda^0(X) + \pi_i(\tilde{x}(i)|Y)\lambda^0(Y)} = \frac{\lambda^0(X)}{\lambda^0(X) + \frac{1}{\ell}\lambda^0(Y)} = \frac{1}{2}.$$

Thus, a connected receiver i votes in favour of x upon observing $\tilde{x}(i)$.

Now, let $i \in N^q$. The probability of i observing x in state Y is given by:

$$\pi_i(x|Y) = \sum_{s\in S^\pi : s_i = x} \pi(s|Y) = \sum_{s\in R : s_i = x} \pi(s|Y) + \sum_{t\in T : t_i = x} \pi(t|Y)$$
$$= \frac{\binom{q-1}{k-n+q-1}}{\binom{q}{k-n+q}\ell} + \binom{q-1}{k-1}\frac{n-k}{q\ell}\binom{q-1}{k-1}^{-1} = \frac{k-n+q}{q\ell} + \frac{n-k}{q\ell} = \frac{1}{\ell}.$$

Similar calculations as in the connected receiver case follow and thus, each singleton receiver has posterior 1/2 that the state is X upon observing x. The value of π is then:

$$V_k^\pi(g) = \lambda^0(X)\cdot 1 + \lambda^0(Y)\left(\frac{n-k}{k\ell} + \frac{1}{\ell}\right) = \lambda^0(X) + \lambda^0(Y)\frac{n}{k\ell}$$
$$= \lambda^0(X) + \lambda^0(Y)\frac{n}{k}\frac{\lambda^0(X)}{\lambda^0(Y)} = \frac{n+k}{k}\lambda^0(X) = V_k^n. \qquad \square$$

Proof of Proposition 3.3. First, note that since $\bar{N}_i(g) = \bar{N}_j(g)$, we have $A_i^\pi(g,s) = A_j^\pi(g,s)$. Hence, i and j have the same posterior belief, i.e. for any $\omega \in \Omega$ and any $s \in S^\pi$:

$$\lambda_i^{s,g}(\omega) = \frac{\sum_{t\in A_i^\pi(g,s)} \pi(t|\omega)\lambda^0(\omega)}{\sum_{\omega'\in\Omega}\sum_{t\in A_i^\pi(g,s)} \pi(t|\omega')\lambda^0(\omega')} = \frac{\sum_{t\in A_j^\pi(g,s)} \pi(t|\omega)\lambda^0(\omega)}{\sum_{\omega'\in\Omega}\sum_{t\in A_j^\pi(g,s)} \pi(t|\omega')\lambda^0(\omega')} = \lambda_j^{s,g}(\omega).$$

Let $|S_i^\pi \times S_j^\pi| = c$. Let $R = \{m^1, \dots, m^c\}$ be a set of distinct messages. Define a bijection $\phi : S_i^\pi \times S_j^\pi \to R$. That is, for any *tuple* $(s_i, s_j), (t_i, t_j) \in S_i^\pi \times S_j^\pi$ it holds that $\phi(s_i, s_j) = \phi(t_i, t_j)$ if and only if $(s_i, s_j) = (t_i, t_j)$, so that each distinct *combination of messages* of i and j (and not every distinct neighbourhood) is mapped to a distinct message in R.

Define $S' = \{s' \in S |\, s \in S^\pi, s'_{-ij} = s_{-ij} \text{ and } \phi(s_i, s_j) = s'_i = s'_j \in R\}$. In words, S' consists of signals obtained by replacing the messages of i and j with distinct messages in R (for each distinct message combination) and leaving the other receivers' messages unchanged, in each signal in S^π. Let $\tau : S^\pi \to S'$ be a bijection such that for any $s \in S^\pi$ we have $\tau(s) = s'$ if $\tau(s_i, s_j) = s'_i = s'_j$ and $s'_{-ij} = s_{-ij}$.

For every $s \in S^\pi$ and $\omega \in \Omega$, define $\pi'(\tau(s)|\omega) = \pi(s|\omega)$. It is clear that π' is a communication strategy. Note that since the probability weights are the same under π and π', receivers i and j still have the same posterior belief under π', i.e. for any $\omega \in \Omega$ and $s \in S^{\pi'}$ it holds that $\lambda_i^{s,g}(\omega) = \lambda_j^{s,g}(\omega)$.

Next, we show that for any $r \in \bar{N}_i(g)$, $\omega \in \Omega$, and $s \in S^\pi$ we have $\lambda_r^{s,g}(\omega) = \lambda_r^{\tau(s),g}(\omega)$.

$$\lambda_r^{s,g}(\omega) = \frac{\sum_{t \in A_r^\pi(g,s)} \pi(t|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega}\sum_{t \in A_r^\pi(g,s)} \pi(t|\omega')\lambda^0(\omega')} = \frac{\sum_{t \in A_r^\pi(g,s)} \pi'(\tau(t)|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega}\sum_{t \in A_r^\pi(g,s)} \pi'(\tau(t)|\omega')\lambda^0(\omega')}$$
$$= \frac{\sum_{t' \in A_r^{\pi'}(g,\tau(s))} \pi'(t'|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega}\sum_{t' \in A_r^{\pi'}(g,\tau(s))} \pi'(t'|\omega')\lambda^0(\omega')} = \lambda_r^{\tau(s),g}(\omega).$$

Finally, any $r \notin \bar{N}_i(g)$ has the same posterior belief under π and π', as it is not affected by the transformation. Hence, $V_k^{\pi'}(g) = V_k^\pi(g)$. □

Proof of Proposition 3.4. As all components are complete, all of their elements can be sent the same private message within every signal by Proposition 3.3. Let $q \in \mathbb{N}$ be such that for each $\mathcal{C}' \in \mathfrak{C}^q$ it holds that $\sum_{C \in \mathcal{C}'} |C| \geq k$. Note that there are $\binom{\ell}{q}$ many ways to choose q components such that the total number of receivers in the components is at least k. Then, by Theorem 3.1 it follows that there exists $\pi \in \Pi$ such that $V_k^\pi(g) = V_q^\ell$. □

Proof of Proposition 3.5. First, Lemma C.1 shows that without loss of generality the center node is *not pivotal* whenever it votes for x. Denote the center node in g by $c \in N$.

Lemma C.1. *Let $g \in G(N)$ be a star, $k < n$, and let $\pi \in \Pi$ be a communication strategy such that there exists $s \in S^\pi$ with $\alpha_c^{\pi,g}(s) = x$ and c is pivotal in s. Then there exists $\pi' \in \Pi$ such that for any $s' \in S^{\pi'}$ with $\alpha_c^{\pi',g}(s') = x$, the center node is not pivotal in s' and $V_k^{\pi'}(g) = V_k^\pi(g)$.*

Proof of Lemma C.1. Note that $t_c(g) = t$ for all $t \in S^\pi$ (the information neighbourhood of c is the *whole* g). Therefore, $\lambda_c^{s,g}(X) \geq 1/2$ if and only if s is an anchor, so if $\alpha_c^{\pi,g}(s) = x$ for some $s \in S^\pi$, it follows that $s \in An(\pi)$. Moreover, if c is pivotal in this s and $k < n$, there exists a *peripheral* $i \in N$ (i.e. $i \neq c$) such that $\alpha_i^{\pi,g}(s_i(g)) = y$. Furthermore, i votes for y in any other signal he associates with s, i.e. for any $t \in A_i^\pi(g,s)$ it holds that $\alpha_i^{\pi,g}(t_i(g)) = y$. Thus, replacing i's message in the anchor s with a *unique* message would enable i to uniquely identify the anchor and hence *reverse* i's vote from y to x in s (see the remark after Definition 3.1). Since c votes for x if and only if the signal is an anchor, c's vote would *not change* if the probabilities of the communication strategy do not change. It would also keep everyone else's vote the same, as i is only observed by c.

To this end, let $S' \subseteq S^{\pi}$ such that $S' = \{s \in S^{\pi} |\, \alpha_c^{\pi,g}(s) = x \text{ and } c \text{ is pivotal in } s\}$. In particular, let $S' = \{s^1, \ldots, s^r\}$. Fix $i \in N$ such that for some $t \in S'$ it holds that $\alpha_i^{\pi,g}(t) = y$ and notice that such i exists as per the discussion above. Let $R = \{m^1, \ldots, m^r\}$ be a set of distinct messages such that for any $j \in \{1, \ldots, r\}$, $m^j \notin S_i^{\pi}$. Let $S'' \subseteq S$ and define a bijection $\phi : S' \to S''$ such that for every $j \in \{1, \ldots, r\}$ and $s^j \in S'$ it holds that $\phi_i\left(s_i^j\right) = m^j$ and $\phi_{-i}\left(s_{-i}^j\right) = s_{-i}^j$.

Now, for any $\omega \in \Omega$ and any $s' \in (S^{\pi} \setminus S') \cup S''$, let $\pi' \in \Pi$ be defined by:

$$\pi'(s'|\omega) = \begin{cases} \pi(\phi^{-1}(s')|\omega) & \text{if } s' \in S'', \\ \pi(s'|\omega) & \text{if } s' \in S^{\pi} \setminus S'. \end{cases}$$

That is, $S^{\pi'} = (S^{\pi} \setminus S') \cup S''$. By the definition of an anchor, for any $t \in S''$, we have $\alpha_i^{\pi',g}(t_i(g)) = x$, since *by construction* i observes the unique message only in this anchor t. Therefore, there are $k+1$ receivers voting for x in t, which implies that the center node is *no longer pivotal*. Since π' preserves all probability weights it is true that if $s \in S''$, then $s \in An(\pi')$.

Moreover, i's votes in signals that are not in S'' are unchanged, i.e. for any $t \in S^{\pi'} \setminus S''$, $\alpha_i^{\pi,g}(t_i(g)) = \alpha_i^{\pi',g}(t_i(g))$. This holds because if $s \in S'$ and $t \in A_i^{\pi}(g, s)$, then it holds that $\alpha_i^{\pi,g}(t_i(g)) = y$ by the definition of S' and the selection of i. The transformation removes the anchors in S'' from the association set of every signal $t \in S^{\pi'} \setminus S''$, so if $s \in S''$ then for every $t \in S^{\pi'} \setminus S''$ it is true that $t \notin A_i^{\pi'}(g, s)$. This makes it even less likely that i would vote for x in such signals, preserving its y votes in them between π and π'. The transformation does not affect any other receivers' votes, hence $V_k^{\pi'}(g) = V_k^{\pi}(g)$. □

By Lemma C.1, assume without loss of generality that under π, the center node is not pivotal in signals in which he votes for x.

For all nodes $i \in N$ and all $t \notin An(\pi)$, if $t_c \neq s_c$ for all $s \in An(\pi)$ then $\lambda_i^{t,g}(X) < 1/2$. So, if in a certain signal the center of a star network receives a message different from all anchors, *all receivers* would vote y in this signal.

Note that for two anchors $s, t \in An(\pi)$ with $s_c \neq t_c$, it holds that $A^{\pi}(g, s) \cap A^{\pi}(g, t) = \emptyset$. Define a bijection $\phi : S^{\pi} \to S'$ such that $\phi(s) = s'$ if $s'_c = x$ and for every $j \in N \setminus \{c\}$, $s'_j = (s_j, s_c)$. That is, in signals in S' the center node always observes x and the periphery nodes' messages are modified so that they contain the information previously provided by the center in signal s. In other words, the information that the center reveals to the periphery nodes is shifted to them while the center observes the same message x in every signal.

For every $s' \in S'$ such that $\phi(s) = s'$ and $\omega \in \Omega$, let $\pi' \in \Pi$ be defined by $\pi'(s'|\omega) = \pi(\phi^{-1}(s')|\omega)$. As the probabilities of corresponding signals are the same under π' as under π and the centre's information under π is shifted to the periphery nodes under π' (which are observed by the center), the center node's vote does not change. Moreover, the votes of the periphery nodes do not change either. To see this, note that for any $t' \in A_i^{\pi'}(g, s')$ there exists $t \in A_i^{\pi}(g, s)$ such that $\phi(t) = t'$. This, together with the definition of ϕ implies that $\sum_{t' \in A_i^{\pi'}(g,s')} \pi'(t'|\omega) = \sum_{t \in A_i^{\pi}(g,s)} \pi(t|\omega)$. Thus, each periphery node has the same posterior belief upon observing $s \in S^{\pi}$ and $\phi(s) \in S^{\pi'}$. Hence, $V_k^{\pi'}(g) = V_k^{\pi}(g)$.

As the center node always observes the same message under π', it has no effect on the voting decisions of the other receivers. Moreover, the center node is never pivotal in signals in which he votes for x. Observe that under π', it is *as if* the center is always voting for y, since *all* of his y votes are preserved in π' and *none* of his x votes have an impact on whether a signal implements x or not. Thus, the setup is equivalent to having an empty network with $n-1$ nodes. Hence, we can assume without loss of generality that there exists a communication strategy $\pi'' \in \Pi$ with $|S_i^{\pi''}| = 2$ for any $i \in N$ such that $V_k^{n-1} = V_k^{\pi''}(g) \geq V_k^{\pi'}(g)$. $\square$

Proof of Proposition 3.6. Take the following communication strategy π:

π	$\omega = X$	$\omega = Y$
$(x, x, x, x, \ldots, x, x, x, x)$	1	0
$(\underbrace{y, \ldots, y}_{a}, x, x, x, \ldots, x, x)$	0	$\frac{w_1}{n}$
$(x, \underbrace{y, \ldots, y}_{a}, x, x, \ldots, x, x)$	0	$\frac{w_1}{n}$
$\ldots$	$\ldots$	$\ldots$
$(x, x, \ldots, x, x, \underbrace{y, \ldots, y}_{a}, x)$	0	$\frac{w_1}{n}$
$(x, x, \ldots, x, x, x, \underbrace{y, \ldots, y}_{a})$	0	$\frac{w_1}{n}$
$\ldots$	$\ldots$	$\ldots$
$(\underbrace{y, \ldots, y}_{a-1}, x, x, \ldots, x, x, y)$	0	$\frac{w_1}{n}$
$(y, y, y, y, \ldots, y, y, y, y)$	0	w_2

with $w_1 = r_k^*, w_2 = r_0^*$ from Proposition 3.1 and $a = n - 2 - k$. This makes a total of $n+2$ signals. Every node observes a message y in their information neighbourhood in exactly $a+3$ signals. This leaves $n - 1 - a$ signals in which i and all neighbours of i observe x. Given $s' \in S^{\pi}$, denote the information neighbourhood $s'_i(g)$ of $i \in N$ by $\tilde{x}(i)$ if for all $j \in \bar{N}_i(g)$ it holds that $s'_j = x$. Let $i \in N$ and $s \in S^{\pi}$ be such that $s_i(g) = \tilde{x}(i)$. It holds that:

$$\lambda_i^{s,g}(X) = \frac{\sum_{t \in A_i^{\pi}(g,s)} \pi(t|X)\lambda^0(X)}{\sum_{t \in A_i^{\pi}(g,s)} \pi(t|X)\lambda^0(X) + \sum_{t \in A_i^{\pi}(g,s)} \pi(t|Y)\lambda^0(Y)} = \frac{\lambda^0(X)}{\lambda^0(X) + \frac{(n-2-a)w_1}{n}\lambda^0(Y)}.$$

Therefore,

$$\lambda_i^{s,g}(X) = \begin{cases} \frac{\lambda^0(X)}{1\lambda^0(X) + \frac{(n-2-a)}{n}\frac{\lambda^0(X)n}{\lambda^0(Y)k}\lambda^0(Y)} = \frac{1}{1+\frac{n-2-a}{k}} = 1/2 & \text{if } \lambda^0(X) < \frac{k}{k+n}, \\ \frac{\lambda^0(X)}{\lambda^0(X) + (n-2-a)\lambda^0(Y)/n} \geq 1/2 & \text{if } \lambda^0(X) \geq \frac{k}{k+n}, \end{cases}$$

as the second condition always holds for $\lambda^0(X) \geq \frac{k}{k+n}$.

In each signal $s \in S^\pi$ such that there exists $i \in N$ with $s_i = y$, there are $n-2-a$ many receivers $j \in N$ such that $s_j(g) = \tilde{x}(j)$. Therefore, all these signals persuade at least k agents, i.e. for such $s \in S^\pi$ we have $|\{i \in N : \alpha_i^{\pi,g}(s_i(g)) = x\}| \geq k$. The value is equal to the empty network value, i.e. $V_k^\pi(g) = \lambda^0(X) \cdot 1 + \lambda^0(Y) w_1 = \min\left\{\frac{n+k}{k}, 1\right\} = V_k^n$. □

Proof of Proposition 3.7. Let $S' = \{x, y\}^n$. Let T be the set of signals in S' in which exactly one receiver observes y, i.e. $T = \{s \in S' | \exists i \in N : s_i = y \text{ and } s_{-i} = x\}$. Let $\bar{x} \in S$ be such that $\bar{x}_i = x$ for all $i \in N$ and define $\bar{y}$ analogously. Define $\pi \in \Pi$:

$$\pi(s|\omega) = \begin{cases} 1 & \text{if } s = \bar{x} \text{ and } \omega = X, \\ \frac{w_1}{n} & \text{if } s \in T \text{ and } \omega = Y, \\ w_2 & \text{if } s = \bar{y} \text{ and } \omega = Y, \\ 0 & \text{otherwise,} \end{cases}$$

with $w_1 = r_k^*, w_2 = r_0^*$ from Theorem 3.1. This makes a total of $n+2$ signals. Since the network is regular, every node observes message y in their information neighbourhood in exactly $\delta + 2$ signals. This leaves $n - \delta$ signals where an agent $i \in N$ observes $\tilde{x}(i)$ (in particular, there are $n - \delta - 1$ such signals in state Y). Let $i \in N$ and $s \in S^\pi$ be such that $s_i(g) = \tilde{x}(i)$. It holds that:

$$\lambda_i^{s,g}(X) = \frac{\sum_{t \in A_i^\pi(g,s)} \pi(t|X)\lambda^0(X)}{\sum_{t \in A_i^\pi(g,s)} \pi(t|X)\lambda^0(X) + \sum_{t \in A_i^\pi(g,s)} \pi(t|Y)\lambda^0(Y)} = \frac{\lambda^0(X)}{\lambda^0(X) + \frac{(n-\delta-1)w_1}{n}\lambda^0(Y)}.$$

Therefore,

$$\lambda_i^{s,g}(X) = \begin{cases} \frac{1\lambda^0(X)}{1\lambda^0(X) + \frac{(n-\delta-1)}{n}\frac{\lambda^0(X)n}{\lambda^0(Y)k}\lambda^0(Y)} = \frac{1}{1+\frac{n-\delta-1}{k}} = \frac{1}{2} & \text{if } \lambda^0(X) < \frac{k}{k+n}, \\ \frac{\lambda^0(X)}{\lambda^0(X) + \frac{k}{n}\lambda^0(Y)} \geq 1/2 & \text{if } \lambda^0(X) \geq \frac{k}{k+n}, \end{cases}$$

as the second condition always holds for for $\lambda^0(X) \geq \frac{k}{k+n}$.

In each signal $s \in S^\pi$ such that there exists $i \in N$ with $s_i = y$, there are $n - 1 - \delta = k$ many receivers $j \in N$ such that $s_j(g) = \tilde{x}(j)$. Therefore all these signals persuade exactly k agents, i.e., for such $s \in S^\pi$ we have $|\{i \in N : \alpha_i^{\pi,g}(s_i(g)) = x\}| \geq k$ The value is equal to the empty network value, $V_k^\pi(g) = \lambda^0(X) \cdot 1 + \lambda^0(Y) w_1 = \min\left\{\frac{n+k}{k}, 1\right\} = V_k^n$. □

Proof of Proposition 3.8. In the empty network the value corresponding to $k = q\alpha, n = q\beta$ is the same as the value for $k' = \alpha, n' = \beta$, since $V_k^n(\lambda^0) = \min\left\{\frac{n+k}{k}\lambda^0(X), 1\right\}$ (Theorem 3.1) and $\frac{n+k}{k}\lambda^0(X) = \frac{q\alpha+q\beta}{q\beta}\lambda^0(X) = \frac{\alpha+\beta}{\beta}\lambda^0(X)$. Therefore, if the network allows uniform treatment of parts with the minimal necessary size (q) so that an equal number of nodes in every part has a neighbourhood with at least one y message in it, the setup becomes equivalent to the empty network and allows obtaining the optimal value with private communication, so that if $V_k^n = V_{q\alpha}^{q\beta}$, then $V_k^n = V_\alpha^\beta$. □

Proof of Claim 3.1. Assume the opposite, i.e. $V_2^{\hat{\pi}}(g_2) = 1$. Therefore:

- $\hat{\pi}$ is *optimal* on g_2 and for every $t \in S^{\hat{\pi}}, t \in Z_x^{g_2}(\hat{\pi})$, so *all* signals in $\hat{\pi}$ implement x.

- For every $t \in Z_x^{g_2}(\hat{\pi})$, there exists $s \in An(\hat{\pi})$ such that $s_2 = t_2$. If $t_2 \neq s'_2$ for all $s' \in An(\hat{\pi})$, then agents $1, 2$ and 3 will vote for y and the signal will not implement x.

- Node 4 never receives any information from any other node, so changing anything in the other nodes' messages would not change the vote of 4.

Observe that if player 2 is *never* pivotal, then any transformation of the communication strategy which preserves the other nodes' votes does *not* change the value of the strategy.

Claim C.1. *For any $\hat{\pi} \in \Pi$ with $V_2^{\hat{\pi}}(g_2) = 1$ there exists $\hat{\pi}' \in \Pi$ such that: (i) 2 is never pivotal, (ii) $|S_2^{\hat{\pi}'}|= 1$ and (iii) $V_2^{\hat{\pi}'}(g_2) = V_2^{\hat{\pi}}(g_2)$.*

If the claim holds, then node 2 never reveals or receives any consequential information (in terms of value). That is, the situation would be equivalent to an empty network with $n = 3$ and $k = 2$ where $V_2^{\hat{\pi}'}(g_2) \leq V_2^3 = 5/6 < 1$, which would contradict the initial assumption.

Proof of Claim C.1. ***(i)*** It is clear that for $s \in S^{\hat{\pi}}$ with $\alpha_2^{\hat{\pi},g_2}(s_2(g_2)) = y$, node 2 is *not* pivotal since all signals implement x.

Suppose that there is a signal $t \in S^{\hat{\pi}}$ in which node 2 votes for x. Hence, there is at least one anchor $s \in An(\hat{\pi})$ with $(s_1, s_2, s_3) = (t_1, t_2, t_3)$ and for every $r \in S^{\hat{\pi}}$ such that $(r_1, r_2, r_3) = (s_1, s_2, s_3)$, node 2 also votes for x. The possible voting patterns of nodes $1, 2$ and 3 in such signals are: *(a)* (x, x, x); *(b)* (y, x, x); *(c)* (x, x, y); *(d)* (y, x, y).

In the case (a), 2 is not pivotal. Consider case (b). It must be true that node 1 votes for y because it associates t with *more* signals than 2. In other words, in all signals $r \in S^{\hat{\pi}}$ where $(r_1, r_2) = (t_1, t_2)$ node 1 votes for y and this *includes* the signals in which 2 *does not* vote for x. (This also includes the associated anchors.) Thus, $A_2^{\pi}(g_2, t) \subsetneq A_1^{\pi}(g_2, t)$.

Notice the trivial fact that for every $s, t \in S^{\hat{\pi}}$ with $s_2 \neq t_2$ and $i \in \{1, 2, 3\}$, it holds that $A_i^{\pi}(g_2, s) \cap A_i^{\pi}(g_2, t) = \emptyset$, so that whenever node 2 receives a different message in different signals, these signals belong to *disjoint* association sets (for node 2) and the same observation holds for its neighbours, 1 and 2.

Let $S_2^{\hat{\pi}} = \{m^1, \ldots, m^{\ell}\}$ and define the set of signals in which receiver 1 votes for y and receiver 2 votes for x as $T = \left\{t \in S^{\hat{\pi}} | \, \alpha_1^{\hat{\pi},g_2}(t_1(g_2)) = y \text{ and } \alpha_2^{\hat{\pi},g_2}(t_2(g_2)) = x\right\}$.

Define a bijection such that in signals in $\hat{\pi}$ in which 1 votes for y and 2 votes for x, we change the message of 1 to a *unique* message that is specific to each distinct message of 2 and keep all other messages the same. Formally, let $T' \subsetneq S$ and define $\phi : T \to T'$ such that for any $t \in T$ it holds that $\phi(t) = t'$ if $t'_1 = (t_1, t_2) \in S'_1 \setminus S_1^{\hat{\pi}}$ and $t'_{-1} = t_{-1}$.

Now for any $\omega \in \Omega$ define a new strategy $\hat{\pi}' \in \Pi$, which transforms the signals in T according to ϕ, keeps all other signals the same while preserving the probability weights:

$$\hat{\pi}'(s'|\omega) = \begin{cases} \hat{\pi}(s'|\omega) & \text{if } s' \in S^{\hat{\pi}} \setminus T, \\ \hat{\pi}(\phi^{-1}(s')|\omega) & \text{if } s' \in T. \end{cases}$$

Let $s' \in S^{\hat{\pi}'}$ be such that $\phi(s) = s'$ for some $s \in T$. Then,

$$\lambda_1^{s',g_2}(X) = \frac{\sum_{t' \in A_1^{\hat{\pi}'}(g_2,s')} \hat{\pi}'(t'|X)\lambda^0(X)}{\sum_{\omega\in\Omega}\sum_{t' \in A_1^{\hat{\pi}'}(g_2,s')} \hat{\pi}'(t'|\omega)\lambda^0(\omega)} = \frac{\sum_{t' \in A_1^{\hat{\pi}'}(g_2,s')} \hat{\pi}(\phi^{-1}(t')|X)\lambda^0(X)}{\sum_{\omega\in\Omega}\sum_{t' \in A_1^{\hat{\pi}'}(g_2,s')} \hat{\pi}(\phi^{-1}(t')|\omega)\lambda^0(\omega)}$$
$$= \frac{\sum_{t \in A_1^{\hat{\pi}}(g_2,s)\cap A_2^{\hat{\pi}}(g_2,s)} \hat{\pi}(t|X)\lambda^0(X)}{\sum_{\omega\in\Omega}\sum_{t \in A_1^{\hat{\pi}}(g_2,s)\cap A_2^{\hat{\pi}}(g_2,s)} \hat{\pi}(t|\omega)\lambda^0(\omega)} = \frac{\sum_{t \in A_2^{\hat{\pi}}(g_2,s)\subseteq T} \hat{\pi}(t|X)\lambda^0(X)}{\sum_{\omega\in\Omega}\sum_{t \in A_2^{\hat{\pi}}(g_2,s)\subseteq T} \hat{\pi}(t|\omega)\lambda^0(\omega)} \geq \frac{1}{2},$$

where $\phi(t) = t'$ and the third equality follows from the definition of ϕ; $A_1^{\hat{\pi}}(g_2,s) \cap A_2^{\hat{\pi}}(g_2,s) = A_2^{\hat{\pi}}(g_2,s) \subseteq T$ follows from $A_2^{\pi}(g_2,t) \subsetneq A_1^{\pi}(g_2,t)$ and the inequality follows from the definition of case (b). Similarly, it holds that $\lambda_2^{s',g_2}(X) \geq 1/2$. This implies that in $\hat{\pi}'$ node 1 will vote for x whenever 2 votes for x in $\hat{\pi}'$. Additionally, node 2 will keep its vote for x in the corresponding signals in $\hat{\pi}$ and $\hat{\pi}'$. Thus, the transformation does not change the vote of 2 in any signals. It only *increases* the number of x votes in signals which already implement x (since all signals must do under $\hat{\pi}$). Observe that for $s \in A_1^{\pi}(g_2,t) \setminus A_2^{\pi}(g_2,t)$ such that $t \in T$, it holds that $\alpha_1^{\hat{\pi},g_2}(t_1(g_2)) = y$ and the transformation will not decrease the value, as in such s nodes 1 and 2 must already be voting for y. Hence, $V_2^{\hat{\pi}'}(g_2) = V_2^{\hat{\pi}}(g_2)$. A similar transformation can be applied in cases (c) and (d).

Therefore, for any communication strategy $\hat{\pi}$ with $V_k^{\hat{\pi}}(g_2) = 1$, there exists $\hat{\pi}' \in \Pi$ such that in every signal in which 2 votes for x in $\hat{\pi}$, nodes 1, 2 and 3 vote for x in $\hat{\pi}'$ such that $V_2^{\hat{\pi}'}(g_2) = V_2^{\hat{\pi}}(g_2)$. Thus, 2 is never pivotal in $\hat{\pi}'$.

(ii) Keeping the message of node 4 the same as in $\hat{\pi}$ (and in $\hat{\pi}'$), from here onwards, the transformation is the same as in the star network (see proof of Proposition 3.5). Note that for two anchors $s, t \in S^{\hat{\pi}}$ with $s_2 \neq t_2$, it holds that $A_2^{\pi}(g_2,s) \cap A_2^{\pi}(g_2,t) = \emptyset$. Let $S' \subseteq S$. Define a bijection $\tau : S^{\hat{\pi}'} \to S'$ such that $\tau(s) = s'$ if $s'_2 = x$, for $j \in \{1,3\}$, $s'_j = (s_j, s_2)$, and $s'_4 = s_4$. That is, in signals in S' node 2 always observes x and the messages of node 1 and 3 are modified so that they contain the information previously provided by node 2 in signal s. In other words, the information that node 2 reveals to node 1 and 3 is shifted to them while node 2 observes the same message in every signal.

For any $s' \in S'$ such that $\tau(s) = s'$ and $\omega \in \Omega$, let $\hat{\pi}'' \in \Pi$ be defined by $\hat{\pi}''(s'|\omega) = \hat{\pi}'(\tau^{-1}(s')|\omega)$. As the probabilities of corresponding signals are the same under $\hat{\pi}''$ as under $\hat{\pi}'$ and node 2's information under $\hat{\pi}'$ is shifted to nodes 1 and 3 under $\hat{\pi}''$ (which are observed by node 2), node 2's vote does not change. Moreover, the votes of nodes 1, 3, and 4 do not change either. To see this, note that for any $i \in \{1,3,4\}$ and $t' \in A_i^{\hat{\pi}''}(g,s')$ there exists $t \in A_i^{\hat{\pi}'}(g,s)$ such that $\tau(t) = t'$. This, together with the definition of τ implies that $\sum_{t' \in A_i^{\hat{\pi}''}(g,s')} \hat{\pi}''(t'|\omega) = \sum_{t \in A_i^{\hat{\pi}'}(g,s)} \hat{\pi}'(t|\omega)$. Thus, every node has the same posterior belief upon observing $s \in S^{\hat{\pi}'}$ and $\tau(s) \in S^{\hat{\pi}''}$.

(iii) Parts (i) and (ii) imply that the value is preserved, i.e. $V_2^{\hat{\pi}''}(g_2) = V_2^{\hat{\pi}'}(g_2) = V_2^{\hat{\pi}}(g_2)$.

Hence, for $k = 2$ a communication strategy $\hat{\pi}$ with $V_2^{\hat{\pi}}(g_2) = 1$ can be transformed into a strategy such that: node 2 is never pivotal, it always receives the same message and the strategy preserves the value of the initial strategy. This proves Claim C.1. □

As Claim C.1 holds, we have $V_2^{\hat{\pi}}(g_2) \leq 5/6$, which contradicts the initial assumption that $V_2^{\hat{\pi}}(g_2) = 1$. Therefore, $V_2^{\hat{\pi}}(g_2) < 1$, which proves Claim 3.1. □

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General Discussion

Overall, the chapters in this dissertation maintain that many (economic) investigations of human behaviour should be interpreted in the context of social networks and that there are few contexts in which a model which has a different representation would not fall short in some aspects. There are multiple valid ways to approach the subject of modelling social networks. Human beings are not fully rational and their decision-making often relies on some fast and frugal heuristics which can work well in many situations. Ignoring this fact and assuming full rationality might be a more standard approach from an economist's perspective but this is increasingly less so as more and more empirically discovered behavioural aspects find their way into theoretical models. Nevertheless, flawless rational behaviour is maybe the most important benchmark for economic analysis. Therefore, economic models would ideally have aspects of both heuristic use and some type of utility maximisation to reflect the full gamut of tools for decision-making people have at their disposal. This is not always a trivial task from the modeller's perspective. The chapters comprising this dissertation combine those two approaches to modelling to the author's best ability, the first two stressing the heuristic aspect, while the third one offering much more calculating and rational agents. Finally, it is important to recognise the role of stochasticity in all human interactions and hence chance and random matching have played a prominent part in the first two chapters.

Summary

Chapter One

The chapter studies a local interaction model in which agents are situated on a circle and play bilateral prisoners' dilemmas with their immediate neighbours. They have three possible strategies: cooperate in all interactions (a fully altruistic strategy), defect in all interactions (a fully egoistic strategy), or cooperate with one immediate neighbour with probability 1/2 (a strategy dubbed partial cooperation), which allows a middle option between the two extremes. Agents apply a naive imitation decision rule – after the first period they use the strategy which has the highest average payoff from the ones they have observed in their local neighbourhood, which consists of their two immediate neighbours. The absorbing states of the process are outlined theoretically and analysed further with extensive computer simulations. There does not exist an absorbing state in which the partially cooperative strategy coexists with any of the other strategies. The partially cooperative strategy limits the diffusion of altruistic behaviour in the population and gives the model a "rock-paper-scissors" feel, so that every strategy's propagation in the population is inhibited by one other strategy and supported by the third one, especially when interacting in big homogeneous groups. Even though clustering of altruists is generally beneficial for sustaining altruism, relatively big groups of altruists at the onset of the process actually *enable* the spread of the partially cooperative strategy, while relatively scattered altruists in the initial state benefit the propagation of *egoists*.

Chapter Two

This chapter studies a model of network formation in which agents create links following a simple heuristic - they invest their limited resources proportionally more in neighbours who have fewer links. More precisely, in an equal-degree network every node spreads its resources equally amongst its neighbours. However, if a node has two neighbours, one with degree two and the other one with degree one, the neighbour with degree one will get an investment twice as big as the other neighbour. This decision rule captures the notion that when considering *social* (rather than informational) value more connected agents are on average less beneficial as neighbours and node degree is a useful proxy when payoffs are difficult to compute (or just uncertain). The decision rule illustrates an externalities effect whereby an agent's actions also influence his neighbours' neighbours. Besides complete networks and fragmented networks

with complete components, the pairwise stable networks produced by this model include many non-standard ones with characteristics observed in real life like clustering and irregular components. Multiple stable states can develop from the same initial structure - the stable networks could have cliques linked by intermediary agents while sometimes they have a core-periphery structure. The observed pairwise stable networks have close to optimal welfare. This limited loss of welfare is due to the fact that when a link is established, this is beneficial to the linking agents, but makes them less attractive as neighbours for others, thereby partially internalising the externalities the new connection has generated.

Chapter Three

This chapter studies a multiple-receiver Bayesian persuasion model, where a sender communicates with receivers who have homogeneous preferences and vote sincerely. The sender wants to implement a proposal and commits to a communication strategy which sends correlated messages to the receivers. Receivers are connected in a commonly known network and perfectly observe not only their own messages but also their direct neighbours' messages. After updating their beliefs accordingly, they vote on the proposal. Under a standard multiple-receiver Bayesian persuasion model, the sender can improve upon public communication (i.e. treating all receivers *uniformly*) by using private messages. The chapter examines how networks of shared information affect persuasion in contrast to the empty network setting. This presents multiple difficulties because most of the assumptions that hold under standard multiple-receiver Bayesian persuasion models are no longer without loss of generality. For instance, straightforward or anonymous strategies are not generally optimal, and neither are strategies which are truth-telling in the sender's preferred state. Nevertheless, the chapter determines many interesting cases in which it is optimal for the sender to employ a private communication strategy and outlines sufficient conditions and the corresponding optimal strategies to achieve the optimal persuasion probability. In particular, this is done for the star, circle and wheel networks and for networks with sufficiently many singleton nodes. Surprisingly, in many cases the sender's gain from persuasion does not decrease due to the additional information provided by the receivers' neighbourhoods and is the same as in the empty network case. Moreover, the value of the optimal communication strategy does not go down monotonically when the network becomes *denser*.

Impact

This dissertation started with the assertion that theoretical models are not a perfect representation of reality. Their findings usually do not provide exact predictions about the world, but rather produce the spectrum of possible outcomes and factors which should be taken into account in decision-making. However, this is exactly where their main utility lies, as they provide the necessary context in which (empirical) results can be interpreted and understood.

The findings in Chapter One aim at contributing to the fundamental understanding of the nature of altruistic behaviour and the circumstances which are most favourable towards it. This is of interest to economists because they investigate systems which rely on cooperative behaviour, such as the financing of public goods. The chapter also comes close to the strand of literature related to evolutionary (biological and game theoretic) models which consider the survival of altruistic behaviour when faced with the (short term) dominant strategy to free-ride. Chapter One also seeks to expand the possible types that are considered in economic models beyond the binary representation of constant cooperators (altruists) and non-cooperators (egoists), by providing an alternative type of agent, one who cooperates only partially.

The findings of Chapter Two have a twofold purpose. Firstly, social network formation and specifically the role of heuristics in it has not been a prominent topic in the economic literature on networks. By showing that the model produces non-trivial stable networks with properties similar to real social networks, the chapter shows that heuristics are a plausible factor for network formation. Secondly, the chapter is especially interesting in the fact that starting from the same conditions, without initial differentiation between agents or employing complicated optimisation, but only following a random matching process, a diverse set of stable outcomes emerges. Specifically, networks formed by the simple heuristic can stabilise at states in which many individual agents get unequal outcomes, but the overall utilitarian welfare is close to maximum. These results can alert policy-makers to the possibility that when observing interconnected groups which are not well integrated with each other this is not *necessarily* harmful to welfare.

The research in the first two chapters also tries to strengthen the position of computer simulations in theoretical economic modelling. As some of the most interesting observations in the chapters come from their simulation parts, they show that simulations are a viable option when the questions are not fully within the reach of a purely analytical approach.

The findings in Chapter Three readily bridge the gap between theory and the real world. The increased use of social media and its growing influence as a source of information, as opposed to traditional channels and mainstream media, make an investigation into the power of persuasion of a single entity over a wide range of users a straightforward application. Moreover, as already mentioned, the model assumes that the agents *know* that they are targeted with a persuasion campaign and also *how* this is done. While this is unlikely in real life, the fact that in theory the persuasion campaign can nevertheless be quite successful raises a lot of relevant questions about the *actual* persuasion power of social media giants, especially when people are *not* always cognisant of the fact that they might be targeted. This is exacerbated by the fact that many markets related to social media have a single dominant actor. Therefore, research modelling a single sender persuading multiple receivers like the one in Chapter Three can inform policy-makers' deliberations when considering whether and/or how to regulate such markets.

Biography

Anastas Plamenov Tenev (*Анастас Пламенов Тенев*) was born on September 1st, 1990 in Haskovo, Bulgaria. After moving to the Netherlands for his Bachelor's degree in 2009 and obtaining two Master's degrees in Economics from Maastricht University in 2016, he started the odyssey called PhD the products of which are contained in this little book. During the years of his PhD he did a lot of soul searching and found many things he really did not want to do. This was not one of them.