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RESEARCH ARTICLE

The Average Tree permission value for games with a permission tree

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Abstract In the literature, various models of games with restricted cooperation can be found. In those models, instead of allowing for all subsets of the set of players to form, it is assumed that the set of feasible coalitions is a subset of the power set of the set of players. In this paper, we consider such sets of feasible coalitions that follow from a permission structure on the set of players, in which players need permission to cooperate with other players. We assume the permission structure to be an oriented tree. This means that there is one player at the top of the permission structure, and for every other player, there is a unique directed path from the top player to this player. We introduce a new solution for these games based on the idea of the Average Tree value for cycle-free communication graph games. We provide two axiomatizations for this new value and compare it with the conjunctive permission value.

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1 Introduction

A *cooperative game with transferable utility*, or simply a TU game, consists of a finite set of players and for every coalition of players a worth representing the total payoff that the coalition can obtain by cooperating. A value is a single-valued solution that assigns to every TU game a payoff vector whose components are the individual payoffs of the players. One of the most applied solutions for cooperative TU games is the *Shapley value* (Shapley 1953), which is applied to economic allocation problems in, e.g. Graham et al. (1990), Maniquet (2003), Chun (2006), Tauman and Watanabe (2007), van den Brink et al. (2007), Bergantiños and Lorenzo-Freire (2008), Ligett et al. (2009), and Faigle and Grabisch (2012).

In its classical interpretation, a TU game describes a situation in which the players of every subset of the set of players are able to cooperate to form a feasible coalition and earn its worth. In the literature, various restrictions on coalition formation are developed.[1] For example, in Myerson (1977), a coalition is feasible if it is a connected set in a given communication graph on the set of players. The *Myerson value* for such so-called *graph games* is the Shapley value of the corresponding *Myerson restricted game* in which the worth of any coalition is the sum of the worths of its maximally connected subsets.

On the class of cycle-free graph games, the *Average Tree value* has been proposed in Herings et al. (2008). Each player in a cycle-free graph game can be associated with a particular payoff vector introduced in Demange (2004), called *hierarchical outcome*. The Average Tree value assigns to any cycle-free graph game the average of all its hierarchical outcomes. Both the Myerson value and the Average Tree value are characterized by component efficiency and some kind of fairness. *Fairness* of the Myerson value states that, after deleting a link between two players, the payoffs of these two players change by the same amount, see Myerson (1977). *Component fairness* of the Average Tree value states that deleting a link between two players in a cycle-free graph game yields the same average change in payoff over the players in the two resulting components, see Herings et al. (2008). Herings et al. (2010) extend the definition of the Average Tree value to the entire class of graph games and thereby also cover situations where the underlying communication graph is not cycle-free.

In van den Brink et al. (2011) *games on union closed systems* are considered. In such games, the collection of feasible coalitions is closed under union, meaning that for any pair of feasible coalitions also their union is feasible. This class of union closed systems contains the class of *antimatroids*; games on antimatroids have been studied in Algaba et al. (2004). An example of an antimatroid is a permission structure, where players need permission from their superiors in a hierarchical structure, given by a directed

[1] For a survey we refer to Bilbao (2000).

graph, when they want to cooperate with other players. Games with a permission structure are considered in Gilles et al. (1992), van den Brink and Gilles (1996), Gilles and Owen (1994), and van der Brink (1997).[2] In the first two papers, the conjunctive approach, in which each player needs permission of all its predecessors, is investigated, while in the latter two papers, the disjunctive approach is considered, in which a player needs permission of at least one of its predecessors, if it has any. This leads to the conjunctive restricted game and the disjunctive restricted game, in which the worth of a coalition is set equal to the worth of its largest conjunctive and its largest disjunctive feasible subcoalition, respectively. The corresponding *conjunctive (disjunctive) permission value* is then the Shapley value of the induced conjunctive (disjunctive) restricted game. We restrict ourselves to games with an oriented tree as permission structure, i.e., there is a unique top player having no predecessors and for every other player there is a unique path from the top player to this player, and simply call such a situation a *permission tree game*. Since in a tree permission structure every player has at most one predecessor, in this case the conjunctive and disjunctive approaches coincide.

In this paper, we define and axiomatize a new value for games with an oriented tree as permission structure. Given a digraph we obtain the associated undirected graph by replacing every directed link from one node to another by an undirected link between the two nodes. When the digraph is an oriented tree, the associated undirected graph is cycle-free. To define the new value, we first take the induced permission tree game and then apply the Average Tree value to this permission tree game on the associated undirected graph.

We provide two axiomatizations of this new solution for permission tree games, one with and one without additivity. The first axiomatization uses axioms similar to those that characterize the conjunctive permission value in van den Brink and Gilles (1996), but adding a collusion neutrality axiom in the spirit of Haller (1994) and van den Brink (2012a). The second one imposes a fairness property related to the one in Herings et al. (2008) for cycle-free graph games.

By applying the Average Tree principle, we put some features of communication into a solution for permission tree games, whereas Demange (2004) puts hierarchical features into a solution for communication graph games by using hierarchical outcomes. In van den Brink (2012b), a comparison of sets of feasible coalitions in games with hierarchical and communication restrictions is made. Some other contributions in the economic literature that combine communication with hierarchies are Bolton and Dewatripont (1994) who describe a model where efficient (i.e., cost minimizing) information processing in a communication network implies a hierarchical structure in the sense that efficient networks take a pyramidal form, Chwe (2000) who studies directed communication networks and shows that the minimal sufficient networks for coordination can be seen as hierarchies, and Hart and Moore (2005) who consider (optimal) hierarchical structures where some agents coordinate and others specialize. Examples of asymmetric communication relations are given by Dewatripont and Tirole (2005) and Dessein (2002). The first two authors consider communication as a 'transfer of knowledge' between a sender and a receiver and they formulate a principal-agent

[2] Other models of games with a hierarchy on the set of players are, for example, Faigle and Kern (1992) and Li and Li (2011) or the more general model of Derks and Peters (1993).

model to analyze communication as a moral hazard problem between the sender and receiver. Dessein (2002) extends the model of Crawford and Sobel (1982) and studies a principal-agent model of an organization where the principal can make a trade-off between delegation (implying a loss of control) and communication (implying a loss of information).

The paper is organized as follows. Section 2 is a preliminary section on cooperative TU games, the Average Tree value for cycle-free communication graph games, and games with a permission structure. Section 3 introduces the Average Tree permission value for games with a permission tree structure and provides the first axiomatization. A characterization with a fairness property is given in Sect. 4. A comparison with the conjunctive permission value is made in Sect. 5, where we modify the two axiomatizations of the AT permission value to obtain new axiomatizations of the conjunctive permission value. Section 6 contains concluding remarks.

2 Cooperative games and restricted cooperation

2.1 Transferable utility games

A *cooperative game with transferable utility* in characteristic function form, or *TU game*, is a pair (N, v), where $N \subset \mathbb{N}$ is a finite set of $|N|$ players and $v\colon 2^N \to \mathbb{R}$ is a characteristic function, where $v(\emptyset) = 0$. A subset $S \in 2^N$ is called a coalition. For any coalition S, $v(S)$ displays the worth of coalition S, which the members of coalition S are able to divide among themselves when they decide to cooperate. For given player set N, we denote the collection of all TU games on N by $\mathcal{G}^N$.

For $T \in 2^N\backslash\{\emptyset\}$, the unanimity game (N, u^T) in $\mathcal{G}^N$ is given by the characteristic function $u^T(S) = 1$ if $T \subset S$, and $u^T(S) = 0$ otherwise. For any $(N, v) \in \mathcal{G}^N$, v can be written in a unique way as a linear combination of the characteristic functions u^T, $T \in 2^N\backslash\{\emptyset\}$, as $v = \sum_{T\in 2^N\backslash\{\emptyset\}} \Delta_v(T)u^T$, where the real numbers $\Delta_v(T)$ are the Harsanyi dividends, see Harsanyi (1959).

For arbitrary $K \subset \mathbb{N}$, we denote $\mathbb{R}^K$ as the $|K|$-dimensional Euclidean space with elements $x \in \mathbb{R}^K$ having components x_i, $i \in K$. A *payoff vector* of a game $(N, v) \in \mathcal{G}^N$ is a vector $x \in \mathbb{R}^N$ giving a payoff $x_i \in \mathbb{R}$ to player $i \in N$. A *value* for TU games is a single-valued solution f that assigns to every TU game $(N, v) \in \mathcal{G}^N$ a payoff vector $f(N, v) \in \mathbb{R}^N$. A solution f is *efficient* if $\sum_{i\in N} f_i(N, v) = v(N)$ for every $(N, v) \in \mathcal{G}^N$. The best-known efficient solution is the *Shapley value*, denoted by Sh. This solution is efficient and is originally introduced by Shapley (1953) as the solution in which each player receives its average marginal contribution to the coalitions when all orders of entrance (permutations) of the players have equal probability. In terms of Harsanyi dividends, the Shapley value is given by $\text{Sh}_i(N, v) = \sum_{\{T\in 2^N|i\in T\}} \Delta_v(T)/|T|$, $i \in N$, so the Harsanyi dividends $\Delta_v(T)$ are distributed uniformly over the players in coalition T.

2.2 TU games with graph structure

A *graph* is a pair (N, L) where N is a set of nodes and $L \subset \{\{i, j\} \in 2^N | i \neq j\}$ is a set of *unordered* pairs of distinct elements of N. In this paper, the nodes represent the players in a game (N, v) and so we refer to them as players. The elements of L are called *links* or *edges*. For $j \in N$, we denote $N^L(j) \subset N$ as the set of neighbors of j in L, so $N^L(j) = \{h \in N \mid \{j, h\} \in L\}$. The set of all graphs on N is denoted by $\mathcal{L}^N$.

For given $S \in 2^N \setminus \{\emptyset\}$ and $(N, L) \in \mathcal{L}^N$, the graph $(S, L(S))$ with $L(S) = \{\{i, j\} \in L | i, j \in S\}$ is the *subgraph* of L on S. Notice that $L(N) = L$. A sequence of k distinct players $(i_1, \ldots, i_k)$ is a *path* in $L(S)$ if $\{i_\ell, i_{\ell+1}\} \in L(S)$ for $\ell = 1, \ldots, k-1$. Two players $i, j \in N$ are connected in $(S, L(S))$ if there is a path $(i_1, \ldots, i_k)$ in $L(S)$ with $i_1 = i$ and $i_k = j$. A subgraph $(S, L(S))$ is *connected*, or shortly coalition S is connected, if every two players in S are connected in $(S, L(S))$. A coalition $K \subset S$ is a *component* of $(S, L(S))$ if K is a maximally connected subset of S, i.e., K is connected and for every $i \in S \setminus K$ the set $K \cup \{i\}$ is not connected. The set of components of $(S, L(S))$ is denoted by $\Sigma^L(S)$, with $\Sigma^L = \Sigma^L(N)$. The graph (N, L) is *cycle-free* if for every two different players i and j either i and j are not connected or there is precisely one path in L connecting i and j. When (N, L) is connected and cycle-free, then N is the unique component of (N, L) and (N, L) has precisely $|N| - 1$ links. Following Béal et al. (2010) (see also Béal et al. 2012 for multi-choice forest games), we call N as well as each of the two components in $(N, L \setminus \{\{i, j\}\})$, $\{i, j\} \in L$, a *cone* when (N, L) is connected and cycle-free. Therefore, a connected cycle-free graph (N, L) has $2(|N| - 1) + 1$ cones.

A *TU game with graph structure*, shortly *graph game*, is a triple (N, v, L) with $(N, v) \in \mathcal{G}^N$ and $(N, L) \in \mathcal{L}^N$. We denote the collection of all TU games with graph structure and player set N by $\mathcal{G}^N_{\mathcal{L}}$ and the class of all cycle-free graph games on N by $\mathcal{G}^N_{\mathcal{F}}$. A solution f on a subclass $\mathcal{G}$ of $\mathcal{G}^N_{\mathcal{L}}$ assigns a unique payoff vector $f(N, v, L) \in \mathbb{R}^N$ to every $(N, v, L) \in \mathcal{G}$.

For a graph game (N, v, L), Myerson (1977) introduced the *Myerson restricted game* $(N, v^L) \in \mathcal{G}^N$, defined by $v^L(S) = \sum_{T \in \Sigma^L(S)} v(T)$ for every $S \in 2^N$. The Myerson value, denoted by My, is defined as $\text{My}(N, v, L) = \text{Sh}(N, v^L)$, for every $(N, v, L) \in \mathcal{G}^N_{\mathcal{L}}$. The Myerson value is characterized by component efficiency and fairness, where a solution f is *component efficient* if for any $(N, v, L) \in \mathcal{G}^N_{\mathcal{L}}$ it holds that $\sum_{i \in K} f_i(N, v, L) = v(K)$ for every $K \in \Sigma^L$, and a solution f satisfies *fairness* if for any $(N, v, L) \in \mathcal{G}^N_{\mathcal{L}}$ and any link $\{i, j\} \in L$ it holds that $f_i(N, v, L) - f_i(N, v, L \setminus \{\{i, j\}\}) = f_j(N, v, L) - f_j(N, v, L \setminus \{\{i, j\}\})$.

On the class $\mathcal{G}^N_{\mathcal{F}}$ of cycle-free graph games, Herings et al. (2008) introduce the Average Tree value, denoted by AT. When the graph is connected, the AT value assigns to each graph game $(N, v, L) \in \mathcal{G}^N_{\mathcal{F}}$ the average of $|N|$ payoff vectors. Each of these payoff vectors is associated with precisely one of the players, the so-called hierarchical outcome for that player as introduced by Demange (2004). To define the hierarchical outcome for a particular player $i \in N$, for each $j \in N$ let $C^L_i(j)$ be defined as

$$C^L_i(j) = \{h \in N \mid \text{the path in } L \text{ from } h \text{ to } i \text{ contains } j\}.$$

Notice that $C^L_i(i) = N$, and for $j \neq i$, $C^L_i(j)$ is the cone containing j that results from deleting the first link of the unique path in L from j to i. The hierarchical outcome associated with player i is the vector $t^i(N, v, L) \in \mathbb{R}^N$ defined as

$$t^i_j(N, v, L) = v\left(C^L_i(j)\right) - \sum_{h \in C^L_i(j) \cap N^L(j)} v\left(C^L_i(h)\right), \quad j \in N. \tag{2.1}$$

The payoff to player j in this vector is equal to the worth of the cone $C^L_i(j)$ minus the worths of the cones $C^L_i(h)$ for the neighbors h of j in $C^L_i(j)$. Since $t^i_i(N, v, L) = v(N) - \sum_{h \in N^L(i)} v(C^L_i(h))$, the hierarchical outcome $t^i(N, v, L)$ is efficient.

On the class of connected cycle-free graph games, the AT value is then defined as

$$\mathrm{AT}(N, v, L) = \frac{1}{|N|} \sum_{i \in N} t^i(N, v, L).$$

If (N, L) is connected, the AT value depends only on the worths of the $2(|N| - 1) + 1$ cones. When (N, L) is not connected, the AT value is applied to each of the components in Σ^L, i.e., on each component K the AT value is the average of $|K|$ hierarchical outcomes of length K associated with each of the players in K. This construction defines the AT value on the class of cycle-free graph games $\mathcal{G}^N_{\mathcal{F}}$.

On the class $\mathcal{G}^N_{\mathcal{F}}$, the AT value is characterized by component efficiency and component fairness. For $(N, v, L) \in \mathcal{G}^N_{\mathcal{F}}$, take $K \in \Sigma^L$ and link $\{i, j\} \in L(K)$. Then K consists of two components in the graph $(N, L \backslash \{\{i, j\}\})$, obtained from (N, L) by deleting the link $\{i, j\}$. Let K^{ij}_h, $h = i, j$, denote the component of K that contains player h after deleting the link $\{i, j\}$.[3] Component fairness requires that, when deleting link $\{i, j\}$ in $L(K)$, the resulting average change in payoff to the players in K^{ij}_i is equal to the average change in payoff to the players in K^{ij}_j.

Axiom 2.1 (*Component Fairness*)

A solution f on the class $\mathcal{G}^N_{\mathcal{F}}$ of cycle-free graph games satisfies *component fairness* if, for every $(N, v, L) \in \mathcal{G}^N_{\mathcal{F}}$ and for any link $\{i, j\} \in L$, it holds that

$$\frac{\sum_{h \in K^{ij}_i} [f_h(N, v, L) - f_h(N, v, L \backslash \{\{i, j\}\})]}{|K^{ij}_i|} = \frac{\sum_{h \in K^{ij}_j} [f_h(N, v, L) - f_h(N, v, L \backslash \{\{i, j\}\})]}{|K^{ij}_j|}.$$

Theorem 2.2 (Herings et al. 2008)

On the class $\mathcal{G}^N_{\mathcal{F}}$ of cycle-free graph games, the AT value is the unique solution that satisfies component efficiency and component fairness.

Both the Myerson value and the AT value satisfy efficiency on the class of connected cycle-free graph games.

[3] When (N, L) is connected, then K^{ij}_h, $h = i, j$, are the two cones in L that result from deleting $\{i, j\}$.

2.3 TU games with permission structure

A *permission structure* on the set of players of a TU game describes a situation where some players need permission from other players to cooperate within a coalition. A permission structure is assumed to be described by a directed graph, shortly digraph, (N, D) with the finite set of players of the game N as the set of nodes and with set of arcs $D \subset \{(i, j) \in N \times N | i \neq j\}$ a collection of ordered pairs of players in N. For a digraph (N, D), the undirected graph (N, L_D) on N associated with D is defined by $L_D = \{\{i, j\}| (i, j) \in D\}$. The digraph (N, D) is *connected* if (N, L_D) is connected, and a coalition $K \in 2^N \backslash \{\emptyset\}$ is a component of (N, D) if it is a component of (N, L_D).

For a given digraph (N, D), node i is a *predecessor* of j and j is a *successor* of i if $(i, j) \in D$. A *directed path* in (N, D) from i to j is a sequence of distinct nodes $(i_1, \ldots, i_m)$ such that $i_1 = i$, $i_m = j$, and $(i_k, i_{k+1}) \in D$ for $k = 1, \ldots, m - 1$. If there is a directed path in (N, D) from node i to a different node j, then i is a *superior* of j and j is a *subordinate* of i. A directed path from i to j is a *cycle* in (N, D) if $(j, i) \in D$. The digraph (N, D) is *acyclic* if it does not contain cycles. An acyclic digraph on a finite set has at least one *top node*, being a node that has no predecessors. A digraph (N, D) is an *oriented tree* if it has only one top node and from the top node to any other node there is precisely one directed path in the digraph. The collection of all oriented trees on N is denoted by $\mathcal{D}_{\mathcal{T}}^N$. The associated undirected graph (N, L_D) of an oriented tree $(N, D) \in \mathcal{D}_{\mathcal{T}}^N$ is both connected and cycle-free.

For an oriented tree $(N, D) \in \mathcal{D}_{\mathcal{T}}^N$ every node $j \in N$, except the top node, has a unique predecessor, denoted by $p_D(j)$. For $i \in N$, $S_D(i) = \{j \in N \mid (i, j) \in D\}$ denotes the set of successors, $\widehat{S}_D(i)$ denotes the set of subordinates, and $\widehat{P}_D(i)$ denotes the set of superiors of node i. Notice that in an oriented tree, the top node is a superior of any other node and that any other node is a subordinate of the top node. Finally, for $T \in 2^N$, we denote $\widehat{S}_D(T) = \cup_{i \in T} \widehat{S}_D(i)$ and $\widehat{P}_D(T) = \cup_{i \in T} \widehat{P}_D(i)$.

A *TU game with permission structure* is a triple (N, v, D) with player set N, TU game $(N, v) \in \mathcal{G}^N$, and digraph (N, D) on the set of players. A solution f on a class of games with permission structure assigns a unique payoff vector $f(N, v, D) \in \mathbb{R}^N$ to every (N, v, D) in the class. In a game with permission structure, it is assumed that players need permission of their predecessors to cooperate with other players. In the *conjunctive approach* as introduced in Gilles et al. (1992) and van den Brink and Gilles (1996), it is assumed that a player needs permission from all its predecessors, while in the disjunctive approach as considered in Gilles and Owen (1994) and van der Brink (1997), it is assumed that a player needs permission of at least one of its predecessors if it has any.

In this paper, we consider the class of games with permission tree structure, shortly *permission tree games* and denote this collection of games by $\mathcal{G}_{\mathcal{T}}^N$. Let $(N, v, D) \in \mathcal{G}_{\mathcal{T}}^N$ and let i_0 be the top node of (N, D). Since every player in a permission tree has at most one predecessor, on the class of permission tree games the conjunctive and disjunctive approaches coincide and a coalition $S \in 2^N$ is feasible if for every player $j \in S \backslash \{i_0\}$, its predecessor is a member of S. It follows that all its superiors, including the top node player i_0, are members of S. The smallest feasible coalition containing S is equal to $F_D(S) = S \cup \widehat{P}_D(S)$. The set of feasible coalitions is given by

$$\Omega_D = \left\{ S \in 2^N \,|\forall i \in S\backslash\{i_0\},\ p_D(i) \in S \right\} = \{S \in 2^N | F_D(S) = S\}.$$

As shown by Algaba et al. (2004), the collection Ω_D is an antimatroid and is therefore union closed,[4] i.e., for every two sets $S,\ T \in \Omega_D$ it holds that $S \cup T \in \Omega_D$. Therefore, for any $S \in 2^N$, the largest feasible subset of S is uniquely defined and is equal to $\sigma_D(S) = \cup_{\{T \in \Omega_D | T \subset S\}} T$. The induced *permission restricted game* of a permission tree game $(N, v, D) \in \mathcal{G}^N_{\mathcal{T}}$ is the game $(N, v_D) \in \mathcal{G}^N$ given by

$$v_D(S) = v(\sigma_D(S)), \quad S \in 2^N, \tag{2.2}$$

i.e., the permission restricted game assigns to each coalition $S \in 2^N$ the worth of the largest feasible subset of S in the game (N, v). Note that $(u^T)_D = u^{T \cup \widehat{P}_D(T)}$ for all $T \in 2^N \backslash \{\emptyset\}$ and $D \in \mathcal{D}^N_{\mathcal{T}}$. The *conjunctive permission value* φ is the solution that assigns to every game $(N, v, D) \in \mathcal{G}^N_{\mathcal{T}}$ the Shapley value of the permission restricted game, $\varphi(N, v, D) = \mathrm{Sh}(N, v_D)$.

Since the conjunctive and disjunctive permission values coincide for permission tree games, we refer to this solution simply as the *permission value*.

3 The Average Tree permission value

In this section, we introduce a new value on the class of permission tree games and characterize it by a set of six independent axioms. This new solution for permission tree games has similarities with the idea behind the AT value as defined for graph games. For permission tree game (N, v, D) we first take the permission restricted game (N, v_D) and then apply the Average Tree value to this game with (N, L_D) as the underlying graph. The solution we obtain in this way is called the AT permission value. Thus, the AT permission value is obtained as the average of $|N|$ marginal vectors of v_D.

Definition 3.1 (*AT permission value*) On the class $\mathcal{G}^N_{\mathcal{T}}$ of permission tree games, the *Average Tree (AT) permission value* is the function ψ given by

$$\psi(N, v, D) = \mathrm{AT}(N, v_D, L_D), \quad (N, v, D) \in \mathcal{G}^N_{\mathcal{T}}.$$

We remark that in the definition instead of the permission restricted game v_D we could also take the Myerson restriction of v_D on graph L_D or the permission restricted game of the Myerson restriction v^{L_D} on L_D, since all these games are the same as the next proposition shows.

Proposition 3.2 *For every $(N, v, D) \in \mathcal{G}^N_{\mathcal{T}}$ it holds that $(v_D)^{L_D} = (v^{L_D})_D = v_D$.*

[4] A collection of feasible coalitions $\mathcal{A} \subset 2^N$ is an antimatroid if, besides being union closed, it contains the empty set and it satisfies *accessibility* meaning that $S \in \mathcal{A}$ implies that there is a player $i \in S$ such that $S\backslash\{i\} \in \mathcal{A}$, see Dilworth (1940) and Edelman and Jamison (1985).

Proof Take any $S \subset N$. Then $(v_D)^{L_D}(S) = \sum_{T \in \Sigma^{L_D}(S)} v_D(T) = v(\sigma_D(S))$, where the last equality follows since (i) $\sigma_D(T) = \emptyset$ if $i_0 \notin T$, (ii) there can be at most one $T \in \Sigma^{L_D}(S)$ with $i_0 \in T$, and (iii) $\sigma_D(S) = T$ if $i_0 \in T$ and $T \in \Sigma^{L_D}(S)$. On the other hand, $(v^{L_D})_D(S) = v^{L_D}(\sigma_D(S)) = \sum_{T \in \Sigma^{L_D}(\sigma_D(S))} v(T) = v(\sigma_D(S))$, where the last equality follows since $\sigma_D(S)$ is connected in L_D and thus $\Sigma^{L_D}(\sigma_D(S)) = \{\sigma_D(S)\}$. □

Note that Proposition 3.2 also implies that $\text{My}(N, v_D, L_D) = \text{Sh}(N, (v_D)^{L_D}) = \text{Sh}(N, v_D) = \varphi(N, v, D)$. Another consequence of Proposition 3.2 is that the AT permission value is core stable with respect to the permission restricted game. It always assigns a payoff vector in $\text{core}(N, v_D) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v_D(N)$ and $\sum_{i \in S} x_i \geq v_D(S)$ for all $S \subset N\}$ if the game (N, v) is monotone,[5] as shown in the next proposition.

Proposition 3.3 *For every* $(N, v, D) \in \mathcal{G}_T^N$ *with* v *monotone, it holds that* $\psi(N, v, D) \in \text{core}(N, v_D)$.

Proof All hierarchical outcomes of a superadditive[6] game with cycle-free graph structure belong to the core of its Myerson restricted game, see Demange (2004). Since the core is convex, in case of a superadditive game also the AT value assigns a payoff vector in this core, see Herings et al. (2008). By Theorem 4.6.(iv) of Gilles et al. (1992), it follows that for permission tree games (N, v, D) with v a monotone game, the permission restricted game v_D is superadditive. Hence, $\text{AT}(N, v_D, L_D)$ belongs to $\text{core}(N, v_D)$ if v is monotone. □

The permission value, being the Shapley value of v_D, need not be core stable, even when the underlying game is superadditive. For example, for $N = \{1, 2, 3\}$, $v = u^{\{1,2\}} + u^{\{1,3\}} - u^{\{1,2,3\}}$ and $D = \{(1, 2), (1, 3)\}$, we have $\psi(N, v, D) = (1, 0, 0)$, $\varphi(N, v, D) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$, and $\text{core}(N, v_D) = \{(1, 0, 0)\}$.[7]

Next, we give a first characterization of the AT permission value by means of six independent axioms, some axioms being similar to axioms characterizing the conjunctive permission value, and some axioms taken from the literature on the AT value for games with a graph structure. The first three axioms are also used in van den Brink and Gilles (1996) to characterize the conjunctive permission value. Efficiency states that the total sum of payoffs equals the worth of the grand coalition.

Axiom 3.4 (*Efficiency*) For every $(N, v, D) \in \mathcal{G}_T^N$ it holds that $\sum_{i \in N} f_i(N, v, D) = v(N)$.

Linearity is a straightforward generalization of the linearity axiom for TU games.

Axiom 3.5 (*Linearity*) For every pair $(N, v, D), (N, w, D) \in \mathcal{G}_T^N$ and real numbers α and β it holds that $f(N, \alpha v + \beta w, D) = \alpha f(N, v, D) + \beta f(N, w, D)$.

[5] A TU game (N, v) is monotone if $v(S) \leq v(T)$ whenever $S \subset T$.

[6] A game (N, v) is superadditive if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subset N$ with $S \cap T = \emptyset$.

[7] Notice that also the Myerson value, being the Shapley value of the Myerson restricted game v^L, need not to be in the core of v^L, even when v is superadditive and (N, L) is cycle-free.

Player $i \in N$ is a *null player* in $(N, v) \in \mathcal{G}^N$ if for all $T \subset N\backslash\{i\}$ it holds that $v(T \cup \{i\}) - v(T) = 0$. Player $i \in N$ is an *inessential player* in the permission tree game $(N, v, D) \in \mathcal{G}_T^N$ if both i and all its subordinates in (N, D) are null players in (N, v). The inessential player property states that inessential players earn zero payoff.[8]

Axiom 3.6 (*Inessential player property*) For every $(N, v, D) \in \mathcal{G}_T^N$ it holds that if player $i \in N$ is an inessential player in (N, v, D), then $f_i(N, v, D) = 0$.

Player $i \in N$ is a *necessary player* in game (N, v) if $v(S) = 0$ for all $S \subset N\backslash\{i\}$.[9] Necessary players whose neighbors in L^D also are all necessary can be considered equally important in the permission tree game (N, v, D) since, although they are needed to generate any nonzero contribution, they need all their neighbors to be able to generate this contribution. Therefore, we require equal payoffs for necessary players whose neighbors are all necessary in the game. We denote the set of all necessary players in v by $\mathrm{Nec}(N, v) = \{i \in N \mid v(S) = 0 \text{ for all } S \subset N\backslash\{i\}\}$.

Axiom 3.7 (*Interior necessary player symmetry*) For every $(N, v, D) \in \mathcal{G}_T^N$ and necessary players $i, j \in \mathrm{Nec}(N, v)$ satisfying $N^{L_D}(i) \subset \mathrm{Nec}(N, v)$ and $N^{L_D}(j) \subset \mathrm{Nec}(N, v)$ it holds that $f_i(N, v, D) = f_j(N, v, D)$.

The next axiom reflects domination of predecessors and states that the payoff distribution does not change if a predecessor i becomes necessary for its successor j in the sense that the marginal contribution of player j to every coalition that does not contain player i becomes zero. For game (N, v) and players $i, j \in N$ we define the game (N, v_j^i) by $v_j^i(S) = v(S\backslash\{j\})$ for all $S \subset N\backslash\{i\}$, and $v_j^i(S) = v(S)$ otherwise.

Axiom 3.8 (*Predecessor necessity*) For every $(N, v, D) \in \mathcal{G}_T^N$ and $i, j \in N$ such that $(i, j) \in D$, it holds that $f(N, v, D) = f(N, v_j^i, D)$.

Interior necessary player symmetry and predecessor necessity are also satisfied by the permission value. The next axiom is not satisfied by the permission value. For TU games, Haller (1994) considers collusion neutrality properties, one of them stating that when two players act together in the sense that either both players are together in a coalition or both stay out of a coalition, then the sum of the payoffs of the two players does not change. For a similar collusion neutrality property [introduced by Malawaski (2002), but equivalent to those of Haller (1994) as shown by Casajus (2013)], van den Brink (2012a) shows that there is no solution for TU games that satisfies efficiency, collusion neutrality, and the null player property, while on the class of communication graph games, all hierarchical outcomes and their convex combinations, and thus also the AT value, do satisfy these three properties when only collusion is allowed among neighbors. Here we restrict the axiom to any two players that are neighbors in the permission structure. For a game $(N, v) \in \mathcal{G}^N$ and two players $i, j \in N$, the game in which players i and j act together is defined as the game $(N, v^{ij}) \in \mathcal{G}^N$ given by $v^{ij}(T) = v(T\backslash\{i, j\})$ if $\{i, j\} \not\subset T$, and $v^{ij}(T) = v(T)$ otherwise.

[8] It weakens the null player property, which states that a null player earns zero payoff.

[9] In voting games necessary players are usually called veto players.

Axiom 3.9 (*Collusion neutrality*) For every $(N, v, D) \in \mathcal{G}_{\mathcal{T}}^{N}$ and $i, j \in N$ with $j \in N^{L_D}(i)$ it holds that $f_i(N, v^{ij}, D) + f_j(N, v^{ij}, D) = f_i(N, v, D) + f_j(N, v, D)$.

The AT permission value is characterized by Axioms 3.4–3.9.

Theorem 3.10 *On the class $\mathcal{G}_{\mathcal{T}}^{N}$ of permission tree games, the AT permission value is the unique solution that satisfies efficiency, linearity, the inessential player property, interior necessary player symmetry, predecessor necessity, and collusion neutrality.*

Proof For notational convenience, in the proof we denote the cone $C_i^{L_D}(j)$ of player j in the associated undirected graph (N, L_D) of (N, D) with respect to i by $C_i^D(j)$ and the set of neighbors $N^{L_D}(j)$ of player j by $N^D(j)$.

We first verify that the AT permission value satisfies all six axioms. Take any $(N, v, D) \in \mathcal{G}_{\mathcal{T}}^{N}$.

1. Since $(N, D) \in \mathcal{D}_{\mathcal{T}}^{N}$, the associated undirected graph (N, L_D) is connected and so every vector $t^i(N, v_D, L_D)$, $i \in N$, is efficient with respect to v_D, thus $\sum_{k \in N} t_k^i(N, v_D, L_D) = v_D(N)$. Also, since $(N, D) \in \mathcal{D}_{\mathcal{T}}^{N}$, it holds that $N \in \Omega_D$, and thus $v_D(N) = v(N)$. It follows that $\psi(N, v, D)$ is efficient.
2. Consider (N, v) and (N, w) in $\mathcal{G}^N$, real numbers α, β, and define $z = \alpha v + \beta w$. Since $z_D(S) = z(\sigma_D(S)) = \alpha v(\sigma_D(S)) + \beta w(\sigma_D(S)) = \alpha v_D(S) + \beta w_D(S)$ for every $S \in 2^N$, and, for every $i \in N$, t^i is linear in its second argument, it follows that $t^i(N, z_D, L_D) = t^i(N, \alpha v_D + \beta w_D, L_D) = \alpha t^i(N, v_D, L_D) + \beta t^i(N, w_D, L_D)$. Since the AT permission value is the average over all vectors t^i, $i \in N$, it follows that ψ is linear.
3. Let j be an inessential player in (N, v, D). We distinguish two cases.

   Case 1. When j is the unique top node in D, then all players are null players and $v(T) = 0$ for all $T \in 2^N$. It follows that $v_D(T) = 0$ for every $T \in 2^N$ and thus $t_j^i(N, v_D, L_D) = 0$ for all $i \in N$. Taking the average over all $i \in N$ yields $\psi_j(N, v, D) = 0$.
   Case 2. Next we consider the case that j is not the top node of (N, D). Take an arbitrary player $i \in N$ and consider the vector $t^i(N, v_D, L_D)$. We show that $t_j^i(N, v_D, L_D) = 0$.

   First, when i is a subordinate of j, then $C_i^D(j) \cap N^D(j)$ contains the unique predecessor of j, say h, while all other players in this set are successors of j. So,

$$t_j^i(N, v_D, L_D) = v_D\left(C_i^D(j)\right) - v_D\left(C_i^D(h)\right) - \sum_{k \in C_i^D(j) \cap S_D(j)} v_D\left(C_i^D(k)\right).$$

   For every $k \in C_i^D(j) \cap S_D(j)$, the set $C_i^D(k)$ is a set of subordinates of j and so $\sigma_D(C_i^D(k)) = \emptyset$. Further, since D is an oriented tree, both $C_i^D(j)$ and $C_i^D(h)$ are feasible in (N, v, D). Hence,

$$t_j^i(N, v_D, L_D) = v\left(C_i^D(j)\right) - v\left(C_i^D(h)\right).$$

Since $C_i^D(h) \subset C_i^D(j)$, $C_i^D(j)\backslash C_i^D(h) \subset \widehat{S}_D(j) \cup \{j\}$ and j is inessential, it follows that $v(C_i^D(j)) = v(C_i^D(h))$, and thus $t_j^i(N, v_D, L_D) = 0$.
Second, when i is not a subordinate of j and $i \neq j$, then $C_i^D(j) = \widehat{S}_D(j) \cup \{j\}$ and the neighbors of j within this set are his successors. So $C_i^D(j) \cap N_i^D(j) = S_D(j)$, and thus

$$t_j^i(N, v_D, L_D) = v_D\left(C_i^D(j)\right) - \sum_{h \in S_D(j)} v_D\left(C_i^D(h)\right) = 0,$$

where the last equality follows from the fact that j is not the top node in (N, D), and therefore $\sigma_D(C_i^D(k)) = \emptyset$ for every $k \in S_D(j) \cup \{j\}$.[10]
Third, we consider $i = j$. Since j is not the top node in D, j has precisely one predecessor, say player k. From $C_j^D(k) = N\backslash(\widehat{S}_D(j) \cup \{j\})$ it follows that

$$\begin{aligned} t_j^j(N, v_D, L_D) &= v_D(N) - v_D(N\backslash(\widehat{S}_D(j) \cup \{j\})) - \sum_{h \in S_D(j)} v_D\left(C_j^D(h)\right) \\ &= v(N) - v(N\backslash(\widehat{S}_D(j) \cup \{j\})) - \sum_{h \in S_D(j)} v(\emptyset) \\ &= 0, \end{aligned}$$

where the last equality follows from the fact that j is inessential, so $v(N\backslash(\widehat{S}_D(j) \cup \{j\})) = v(N)$. So, ψ satisfies the inessential player property.

4. Suppose that player $j \in \text{Nec}(N, v)$ is a necessary player in (N, v) whose neighbors in L_D are all necessary players, i.e. $N^D(j) \subset \text{Nec}(N, v)$. Take an arbitrary player $i \neq j$. Then

$$t_j^i(N, v_D, L_D) = v_D(C_i^D(j)) - \sum_{h \in C_i^D(j) \cap N^D(j)} v_D\left(C_i^D(h)\right).$$

Since $j \notin C_i^D(h)$ for all $h \in C_i^D(j) \cap N^D(j)$ and the neighbor of j on the path from j to i is not in $C_i^D(j)$, whereas j and all its neighbors are necessary, the worth of all these coalitions is equal to zero. Hence $t_j^i(N, v_D, L_D) = 0$ for all $i \neq j$.
Taking j itself as root of the tree, we obtain

$$t_j^j(N, v_D, L_D) = v_D(N) - v_D(N\backslash(\widehat{S}_D(j) \cup \{j\})) - \sum_{h \in S_D(j)} v_D\left(C_j^D(h)\right),$$

where $v_D(N\backslash(\widehat{S}_D(j) \cup \{j\})) = 0$ and also $v_D(C_j^D(h)) = 0$ for all $h \in S_D(j)$, since j itself is a necessary player who neither belongs to $N\backslash(\widehat{S}_D(j) \cup \{j\})$ nor to any of the sets $C_j^D(h)$, $h \in S_D(j)$. So, $t_j^j(N, v_D, L_D) = v_D(N) = v(N)$. Taking the average over all $i \in N$, it follows that $\psi_j(N, v, D) = \frac{v(N)}{|N|}$. Since this holds

[10] Note that the last equality also follows because all players in $\widehat{S}_D(j) \cup \{j\}$ are null players in (N, v_D).

for every $j \in \mathrm{Nec}(N, v)$ such that $N^D(j) \subset \mathrm{Nec}(N, v)$, it follows that ψ satisfies interior necessary player symmetry.

5. Consider $T \subset N$ and $i, j \in N$ with $(i, j) \in D$. Then
   (i) $(u^T)^i_j = u^{T \cup \{i\}}$ if $j \in T$,
   (ii) $(u^T)^i_j = u^{T \cup \{i\}} = u^T$ if $j \notin T$ and $i \in T$, and
   (iii) $(u^T)^i_j = u^T$ if $\{i, j\} \cap T = \emptyset$.

   In all cases $((u^T)^i_j)_D = (u^T)_D = u^{T \cup \widehat{P}_D(T)}$, and thus $\psi(N, u^T, D) = \mathrm{AT}(N, (u^T)_D, L_D) = \mathrm{AT}(N, ((u^T)^i_j)_D, L_D) = \psi(N, (u^T)^i_j, D)$. By linearity of ψ, it follows that this holds for any permission tree game, and thus, ψ satisfies predecessor necessity.
6. Let $j \in N^D(i)$ and without loss of generality assume that $i = p_D(j)$. Let $S \in 2^N$ be a set that contains both i and j. It holds that $j \in \sigma_D(S)$ if and only if $i \in \sigma_D(S)$. Now, for some $k \in N$, consider $t^k(N, v_D, L_D)$.

Suppose that $j \in C^D_k(i)$. We define $O_i = (C^D_k(i) \cap N^D(i)) \backslash \{j\}$ and $O_j = C^D_k(j) \cap N^D(j)$. Notice that $i \notin O_j$ because $j \in C^D_k(i)$. It holds that

$$t^k_i(N, v_D, L_D) = v_D\left(C^D_k(i)\right) - v_D\left(C^D_k(j)\right) - \sum_{h \in O_i} v_D\left(C^D_k(h)\right),$$

$$t^k_j(N, v_D, L_D) = v_D\left(C^D_k(j)\right) - \sum_{h \in O_j} v_D\left(C^D_k(h)\right).$$

We obtain that

$$\begin{aligned} t^k_i(N, v_D, L_D) + t^k_j(N, v_D, L_D) &= v_D\left(C^D_k(i)\right) - \sum_{h \in O_i \cup O_j} v_D\left(C^D_k(h)\right) \\ &= v(\sigma_D\left(C^D_k(i))\right) - \sum_{h \in O_i \cup O_j} v\left(\sigma_D\left(C^D_k(h)\right)\right) \\ &= v^{ij}\left(\sigma_D\left(C^D_k(i)\right)\right) - \sum_{h \in O_i \cup O_j} v^{ij}\left(\sigma_D\left(C^D_k(h)\right)\right) \\ &= t^k_i\left(N, (v^{ij})_D, L_D\right) + t^k_j\left(N, (v^{ij})_D, L_D\right), \end{aligned}$$

where the third equality follows because both i and j are in $C^D_k(i)$ and thus either both are in $\sigma_D(C^D_k(i))$ or both are not, and i and j are both not in $C^D_k(h)$ for every $h \in O_i \cup O_j$ and so also not in $\sigma_D(C^D_k(h))$. Similarly, the same equality holds when $i \in C^D_k(j)$. By taking the average over all $k \in N$, it follows that ψ satisfies collusion neutrality.

Next we prove that the six axioms determine a unique solution f. First, for the unanimity game (N, u^N, D) it holds by efficiency and interior necessary player symmetry that $f_i(N, u^N, D) = 1/|N|$, $i \in N$. Next we determine by induction the payoffs of the unanimity games of all feasible sets.

Take any t, $1 \leq t < |N|$, and assume that $f(N, u^T, D)$ is uniquely determined for all $T \in \Omega_D$ with $|T| > t$. Take any $T \in \Omega_D$ with $|T| = t$. Since T is feasible, for every $i \notin T$ it holds that all subordinates of i are not in T and so i is inessential in (N, u^T, D). Thus, for any $i \notin T$, $f_i(N, u^T, D) = 0$ by the inessential player property. To determine the payoffs of the players in T, for a player $i \in T$ such that $S_D(i) \backslash T \neq \emptyset$, take a player $j \in S_D(i) \backslash T$. Since T is feasible, also $T' = T \cup \{j\}$ is feasible and $|T'| = t + 1$. Since $j \notin T$, $f_j(N, u^T, D) = 0$. Applying collusion neutrality to $v = u^T$ and observing that $(u^T)^{ij} = u^{T'}$ it follows that

$$\begin{aligned} f_i(N, u^T, D) &= f_i(N, u^T, D) + f_j(N, u^T, D) \\ &= f_i(N, (u^T)^{ij}, D) + f_j(N, (u^T)^{ij}, D) \\ &= f_i(N, u^{T'}, D) + f_j(N, u^{T'}, D). \end{aligned}$$

By the induction hypothesis, $f_i(N, u^{T'}, D)$ and $f_j(N, u^{T'}, D)$ are uniquely determined, and therefore $f_i(N, u^T, D)$ is uniquely determined. So, we are left to determine the payoffs of the players in the set $\widehat{T} = \{h \in T | S_D(h) \backslash T = \emptyset\}$. For every $i \in \widehat{T}$ it holds that $N^D(i) \subset T$, because T is feasible and $S_D(i) \backslash T = \emptyset$. From interior necessary player symmetry, it follows that all players in $\widehat{T}$ have equal payoff. These payoffs then follow from efficiency. By induction, it is shown that $f(N, u^T, D)$ is uniquely determined for every feasible $T \in \Omega_D$.

Take any $T \notin \Omega_D$. Predecessor necessity implies $f(N, u^T, D) = f(N, u^{T \cup \{p_D(j)\}}, D)$ for every $j \in T$. Adding subsequently all players in $\widehat{P}_D(T) \backslash T$, we obtain that $f(N, u^T, D) = f(N, u^{F_D(T)}, D)$ and so the payoffs for every unanimity game are uniquely determined.

Finally, for any $(N, v, D) \in \mathcal{G}_{\mathcal{T}}^N$, it holds that $f(N, v, D)$ is uniquely determined by linearity. □

Note that collusion between two neighbors in the permission restricted game is not the same as taking the restricted game after two neighbors colluded.[11] Consider for example the game with permission structure $(N, v, D) \in \mathcal{G}_{\mathcal{T}}^N$ with $N = \{1, 2, 3, 4\}$, $v = u^{\{1,3\}}$, and $D = \{(1, 2), (2, 3), (2, 4)\}$. Taking $S = \{1, 2, 3\}$, we see that $(v_D)^{24}(S) = v_D(\{1, 3\}) = v(\{1\}) = 0$, while $(v^{24})_D(S) = v^{24}(\{1, 2, 3\}) = v(\{1, 3\}) = 1$.

Next, we show the logical independence of the six axioms of Theorem 3.10.

1. The permission value satisfies efficiency, linearity, the inessential player property, interior necessary player symmetry, and predecessor necessity. It does not satisfy collusion neutrality.
2. The solution $f(N, v, D) = \mathrm{AT}(N, v, L_D)$ that applies the Average Tree value to the original game v on the associated undirected graph L_D satisfies efficiency, linearity, the inessential player property, interior necessary player symmetry, and collusion neutrality. It does not satisfy predecessor necessity.

[11] Otherwise, collusion neutrality would follow immediately from van den Brink (2012a).

3. The solution that assigns all worth $v(N)$ to the top node and zero to all other players satisfies efficiency, linearity, the inessential player property, predecessor necessity, and collusion neutrality. It does not satisfy interior necessary player symmetry.
4. The equal division solution given by $f_i(N, v, D) = v(N)/|N|$ for all $i \in N$ satisfies efficiency, linearity, interior necessary player symmetry, predecessor necessity, and collusion neutrality. It does not satisfy the inessential player property.
5. For $(N, v, D) \in \mathcal{G}_{\mathcal{T}}^N$, let $(N, \overline{v}) \in \mathcal{G}^N$ be given by $\overline{v} = v(N)u^{N \setminus I(N,v,D)}$, where $I(N, v, D)$ is the set of inessential players in (N, v, D). The solution $\overline{f}(N, v, D) = \psi(N, \overline{v}, D)$ satisfies efficiency, the inessential player property, interior necessary player symmetry, predecessor necessity, and collusion neutrality. It does not satisfy linearity.
6. The zero solution given by $f_i(N, v, D) = 0$ for all $i \in N$ satisfies linearity, the inessential player property, interior necessary player symmetry, predecessor necessity, and collusion neutrality. It does not satisfy efficiency.

4 An axiomatization using fairness

In this section, we characterize the AT permission value by modifying the component fairness Axiom 2.1 to the framework of permission tree games.

We say that in a permission tree game $(N, v, D) \in \mathcal{G}_{\mathcal{T}}^N$, some player $i \in N$ is enforcing power over a player $j \in S_D(i)$ when i vetoes any coalition that contains j or any of its subordinates but does not contain player i. Since i is a predecessor of j, and thus is a superior of every player in cone K_j^{ij}, any $h \in K_j^{ij}$ has j as one of its superiors and thus also needs permission of i. So, the players in K_j^{ij} cannot cooperate without permission of player i. It follows that $\sigma_D(K_j^{ij}) = \emptyset$, i.e., in the permission structure (N, D) the players in K_j^{ij} earn worth zero without permission from player i. On the other hand, neither player i nor any of the predecessors of player i can force the players in K_j^{ij} to cooperate. The corresponding game in which the players in $\{j\} \cup \widehat{S}_D(j)$ are not cooperating is the game $(N, v^{-ij}) \in \mathcal{G}^N$ given by $v^{-ij}(T) = v(T \setminus (\{j\} \cup \widehat{S}_D(j)))$ for all $T \in 2^N$.

Applying a similar idea as component fairness, but now with respect to the enforcement of permission power, we obtain the following axiom.

Axiom 4.1 (*Permission Component Fairness*) A solution f on the class $\mathcal{G}_{\mathcal{T}}^N$ of permission tree games satisfies *permission component fairness* if, for every $(N, v, D) \in \mathcal{G}_{\mathcal{T}}^N$ and for any link $(i, j) \in D$, it holds that

$$\frac{\sum_{h \in K_i^{ij}} \left[f_h(N, v, D) - f_h\left(N, v^{-ij}, D\right)\right]}{|K_i^{ij}|} = \frac{\sum_{h \in K_j^{ij}} \left[f_h(N, v, D) - f_h\left(N, v^{-ij}, D\right)\right]}{|K_j^{ij}|}.$$

The following theorem characterizes the AT permission value by efficiency, the inessential player property, and permission component fairness.

Theorem 4.2 *On the class $\mathcal{G}_{\mathcal{T}}^N$ of permission tree games, the AT permission value is the unique solution that satisfies efficiency, the inessential player property, and permission component fairness.*

Proof It follows from Theorem 3.10 that the AT permission value satisfies efficiency and the inessential player property. To prove that the AT permission value satisfies permission component fairness, take any arc $(i, j) \in D$. Then all players in K_j^{ij} are inessential in (N, v^{-ij}, D), so

$$\sum_{h \in K_j^{ij}} \psi_h \left(N, v^{-ij}, D\right) = 0.$$

Since the AT permission value is efficient, we have that

$$\sum_{h \in K_i^{ij}} \psi_h \left(N, v^{-ij}, D\right) = v^{-ij}(N) - \sum_{h \in K_j^{ij}} \psi_h \left(N, v^{-ij}, D\right) = v\left(K_i^{ij}\right).$$

For permission component fairness to hold, we therefore have to show that

$$\frac{\sum_{h \in K_j^{ij}} \psi_h(N, v, D)}{|K_j^{ij}|} = \frac{\sum_{h \in K_i^{ij}} \psi_h(N, v, D) - v\left(K_i^{ij}\right)}{|K_i^{ij}|}.$$

Recall that the AT permission value ψ is defined as the AT value applied to the cycle-free graph game (N, v_D, L_D), so as the average of the $|N|$ hierarchical outcomes of the game (N, v_D) on the graph (N, L_D), each one is associated with precisely one of the players. The hierarchical outcome associated with player $k \in N$ gives to player $\ell \in N$ payoff

$$t_\ell^k(N, v_D, L_D) = v_D\left(C_k^{L_D}(\ell)\right) - \sum_{h \in C_k^{L_D}(\ell) \cap N^{L_D}(\ell)} v_D\left(C_k^{L_D}(h)\right), \quad \ell \in N.$$

Consider any arc $(i, j) \in D$. When $k \in K_i^{ij}$, then $C_k^{L_D}(j) = K_j^{ij}$ and so the total payoff at vector $t^k(N, v_D, L_D)$ to the players in K_j^{ij} is equal to $v_D(K_j^{ij})$. Since $v_D(K_j^{ij}) = 0$, it follows that the players in K_j^{ij} get total payoff equal to zero in $|K_i^{ij}|$ of the $|N|$ hierarchical outcomes. When $k \in K_j^{ij}$, then $C_k^{L_D}(i) = K_i^{ij}$ and so the total payoff at vector $t^k(N, v_D, L_D)$ to the players in K_i^{ij} is equal to $v_D(K_i^{ij}) = v(K_i^{ij})$. From efficiency, it follows that the players in K_j^{ij} get total payoff equal to $v(N) - v(K_i^{ij})$. This occurs in $|K_j^{ij}|$ of the $|N|$ hierarchical outcomes. It follows that

$$\sum_{h\in K_j^{ij}} \psi_h(N, v, D) = \frac{\left|K_j^{ij}\right|\left(v\left(N\right) - v\left(K_i^{ij}\right)\right)}{|N|}. \tag{4.1}$$

From the reasoning above, it also follows that the players in K_i^{ij} get total payoff equal to $v(K_i^{ij})$ in $|K_j^{ij}|$ of the $|N|$ hierarchical outcomes where $k \in K_j^{ij}$, and they get total payoff $v(N)$ in the $|K_i^{ij}|$ of the $|N|$ hierarchical outcomes where $k \in K_i^{ij}$. So,

$$\sum_{h\in K_i^{ij}} \psi_h(N, v, D) = \frac{\left|K_j^{ij}\right| v\left(K_i^{ij}\right) + \left|K_i^{ij}\right| v\left(N\right)}{|N|}.$$

Substituting $|K_j^{ij}| = |N| - |K_i^{ij}|$ in the latter equation yields

$$\sum_{h\in K_i^{ij}} \psi_h\left(N, v, D\right) - v\left(K_i^{ij}\right) = \frac{\left|K_i^{ij}\right|\left(v\left(N\right) - v\left(K_i^{ij}\right)\right)}{|N|}. \tag{4.2}$$

From Eqs. (4.1) and (4.2), it follows that ψ satisfies permission component fairness.

It remains to show that the three axioms characterize a unique solution. Let f be a solution satisfying the three axioms. Then, efficiency requires that

$$\sum_{h\in N} f_h(N, v, D) = v(N) \tag{4.3}$$

and permission component fairness requires

$$\frac{\sum_{h\in K_i^{ij}} \left[f_h(N, v, D) - f_h\left(N, v^{-ij}, D\right)\right]}{|K_i^{ij}|} = \frac{\sum_{h\in K_j^{ij}} \left[f_h(N, v, D) - f_h\left(N, v^{-ij}, D\right)\right]}{|K_j^{ij}|}, \quad (i, j) \in D.$$

All players in K_j^{ij} are inessential in (N, v^{-ij}, D), so by the inessential player property

$$\sum_{h\in K_j^{ij}} f_h\left(N, v^{-ij}, D\right) = 0, \quad (i, j) \in D.$$

Since f is efficient, we have that

$$\sum_{h\in K_i^{ij}} f_h\left(N, v^{-ij}, D\right) = v^{-ij}(N) - \sum_{h\in K_j^{ij}} f_h\left(N, v^{-ij}, D\right) = v\left(K_i^{ij}\right), \quad (i, j) \in D.$$

We find that

$$\frac{\sum_{h\in K_j^{ij}} f_h(N,v,D)}{|K_j^{ij}|} = \frac{\sum_{h\in K_i^{ij}} f_h(N,v,D) - v\left(K_i^{ij}\right)}{|K_i^{ij}|}, \quad (i,j)\in D. \quad (4.4)$$

Since $(N, D) \in \mathcal{D}_T^N$, the number of arcs in D is equal to $|N| - 1$. So, the total number of equations in (4.3) and (4.4) is $|N|$. Since all $|N|$ equations are linearly independent, the system in (4.3) and (4.4) has a unique solution in the $|N|$ variables $f_h(N, v, D)$, $h \in N$. □

The three axioms of Theorem 4.2 are logically independent. The equal division solution satisfies efficiency and permission component fairness, but not the inessential player property. The permission value satisfies efficiency and the inessential player property, but not permission component fairness. Finally, the zero solution satisfies the inessential player property and permission component fairness, but not efficiency.

5 Comparison with the permission value

In this section, we compare the Average Tree permission value with the permission value defined in Sect. 2.3 As mentioned in the first case under logical independence in Sect. 3, the permission value satisfies all axioms of Theorem 3.10 except collusion neutrality. The permission value satisfies a stronger version of interior necessary player symmetry, namely that all necessary players earn the same payoff.

Axiom 5.1 (*Necessary player symmetry*) For every $(N, v, D) \in \mathcal{G}_T^N$ and necessary players $i, j \in \text{Nec}(N, v)$, it holds that $f_i(N, v, D) = f_j(N, v, D)$.

It turns out that strengthening interior necessary player symmetry in this way, we can delete collusion neutrality to obtain an axiomatization of the permission value.

Theorem 5.2 *On the class $\mathcal{G}_T^N$ of permission tree games the permission value is the unique solution that satisfies efficiency, linearity, the inessential player property, necessary player symmetry, and predecessor necessity.*

Proof It is known from van den Brink and Gilles (1996) that the permission value satisfies efficiency, linearity, and the inessential player property, while necessary player symmetry follows from linearity and the necessary player property.[12] Predecessor necessity follows similar as in the proof of Theorem 3.10 since there it is shown that $((u^T)^i_j)_D = (u^T)_D$ for all $T \subset N$ and $i, j \in N$ with $(i, j) \in D$, and thus by linearity also φ satisfies predecessor necessity.

We are left to show uniqueness. Suppose that f satisfies the five axioms and consider the unanimity game (N, u^T, D), $T \in \Omega_D$. Similar as in the proof of Theorem

[12] The necessary player property states that for monotone games, every necessary player earns at least as much as any other player. This implies equal payoffs for the necessary players in monotone games. Efficiency and the necessary player property imply that all players earn zero in the null game v_0 on D (given by $v_0(S) = 0$ for all $S \subset N$). With linearity it follows that necessary player symmetry is satisfied for all games since $\varphi(N, u^T, D) + \varphi(N, -u^T, D) = \varphi(N, v_0, D)$ and u^T is monotone.

3.10, since T is feasible, for every $i \notin T$ it holds that also all subordinates of i are not in T and so i is inessential in (N, u^T, D). Thus, $f_i(N, u^T, D) = 0$ by the inessential player property. From necessary player symmetry, it follows that all players in T have equal payoff. These payoffs then follow from efficiency. If $T \notin \Omega_D$, then, similar as in the proof of Theorem 3.10, by predecessor necessity we have that $f(N, u^T, D) = f(N, u^{F_D(T)}, D)$. Hence, for every unanimity game, the payoffs are uniquely determined by efficiency, the inessential player property, necessary player symmetry, and predecessor necessity. Finally, $f(N, v, D)$ is uniquely determined by linearity for every $(N, v, D) \in \mathcal{G}_T^N$. □

Theorems 3.10 and 5.2 show an important difference between the permission value and AT permission value. By necessary player symmetry, in a unanimity permission tree game (N, u^T, D) the permission value treats all the players in T the same. Similar to the AT value for cycle-free communication graph games, in the AT permission value, the players in T who have neighbors outside T have some 'responsibility' or 'representability' for these players. Therefore, the interior players in T are treated equally, but the other players in T, i.e., the 'boundary' players, earn a payoff that depends on the substructure where they are the top player and that contains them and their subordinates. This is taken care for by collusion neutrality which, at each step, assigns some 'joint payoff' to a player and one of its successors when they collude.

This can also be seen when we express the AT permission value for unanimity games on a permission tree, see Herings et al. (2008, Theorem 5.1) for an analogous result for the AT value for cycle-free communication graph games.

Proposition 5.3 *For $T \in 2^N \backslash \{\emptyset\}$ and $(N, D) \in \mathcal{D}_T^N$ it holds that*

$$\psi_i(N, u^T, D) = \frac{1 + p_{F_D(T)}^{L_D}(i)}{|F_D(T)| + \sum_{j \in F_D(T)} p_{F_D(T)}^{L_D}(j)} \quad \text{if } i \in F_D(T),$$

and $\psi_i(N, u^T, D) = 0$ otherwise, where $p_S^L(j) = \sum_{\{h \in N \backslash S | \{j,h\} \in L\}} |K_h^{hj}|$ for all $(N, L) \in \mathcal{L}_{\mathcal{F}}^N$, $S \subset N$ and $j \in S$.

Proof Defining $\mathcal{C}^{L_D}$ as the set of connected coalitions in (N, L_D), by applying Theorem 5.1 of Herings et al. (2008) and the fact that $(u^T)_D = u^{F_D(T)} = (u^{F_D(T)})^{L_D}$, it follows for every $i \in F_D(T)$ that

$$\begin{aligned}
\psi_i(N, u^T, D) &= \mathrm{AT}_i(N, (u^T)_D, L_D) = \mathrm{AT}_i(N, u^{F_D(T)}, L_D) \\
&= \sum_{\{S \in \mathcal{C}^{L_D} | i \in S\}} \frac{1 + p_S^{L_D}(i)}{|S| + \sum_{j \in S} p_S^{L_D}(j)} \Delta_{u^{F_D(T)}}(S) \\
&= \sum_{\{S \in \Omega_D | i \in S\}} \frac{1 + p_S^{L_D}(i)}{|S| + \sum_{j \in S} p_S^{L_D}(j)} \Delta_{u^{F_D(T)}}(S) \\
&= \frac{1 + p_{F_D(T)}^{L_D}(i)}{|F_D(T)| + \sum_{j \in F_D(T)} p_{F_D(T)}^{L_D}(j)},
\end{aligned}$$

where the fourth equality follows since $\Omega_D \subset \mathcal{C}^{L_D}$ and $\Delta_{u^{F_D(T)}}(S) = 0$ when $S \in \mathcal{C}^{L_D} \backslash \Omega_D$. Furthermore, $\psi_i(N, u^T, D) = 0$ if $i \in N \backslash F_D(T)$. □

For the permission value it holds that $\varphi_i(N, u^T, D) = \frac{1}{|F_D(T)|}$ for all $i \in F_D(T)$, and $\psi_i(N, u^T, D) = 0$ otherwise. This also makes clear another important difference between the AT permission value and the permission value, namely that the permission value satisfies structural monotonicity (see van den Brink and Gilles 1996) while the AT permission value does not. Structural monotonicity states that in a monotone game, every player that has successors earns at least as much as its successors, i.e., if v is monotone and $(i, j) \in D$, then $f_i(N, v, D) \geq f_j(N, v, D)$. Focussing on unanimity games, the permission value treats players in the unanimity coalition and their superiors equally. This follows from the necessary player property (which requires that all players in the unanimity coalition earn at least as much as any other player) and structural monotonicity (which implies that their superiors also earn this same payoff). From Proposition 5.3, it is clear that, according to the AT permission value, the payoff for a player depends on the number of players it 'represents' or is responsible for. Therefore, such a player might get more payoff than some of its predecessors.

We illustrate some of the differences between the permission value and the AT permission value by the following example.

Example 5.4 Consider the game (N, v, D) with player set $N = \{1, 2, 3, 4, 5\}$, characteristic function $v = u^{\{2,4\}}$ and permission structure $D = \{(1, 2), (1, 3), (3, 4), (3, 5)\}$. The permission value assigns payoff vector $\varphi(N, v, D) = (1/4, 1/4, 1/4, 1/4, 0)$ and the AT permission value $\psi(N, v, D) = (1/5, 1/5, 2/5, 1/5, 0)$. By predecessor necessity, $\psi(N, u^{\{2,4\}}, D) = \psi(N, u^{\{1,2,3,4\}}, D)$ and $\varphi(N, u^{\{2,4\}}, D) = \varphi(N, u^{\{1,2,3,4\}}, D)$. By the inessential player property, in both payoff vectors player 5 has payoff zero. By necessary player symmetry, in the permission value all other players earn $1/4$. This is not the case for the AT permission value. By interior necessary player symmetry, the players 1, 2, and 4 earn the same. By collusion neutrality, players 3 and 5 together earn the same as in $\psi(N, u^N, D) = (1/5, 1/5, 1/5, 1/5, 1/5)$. Since $\psi_5(N, v, D) = 0$, we obtain $\psi_3(N, v, D) = 2/5$ and so $\psi_1(N, v, D) = \psi_2(N, v, D) = \psi_4(N, v, D) = 1/5$. Also we see that the AT permission value does not satisfy structural monotonicity since player 3 earns more than its predecessor player 1. This is because player 3 also represents player 5, a feature that is taken into account in the AT permission value but not in the permission value. In fact, by the necessary player property of the permission value, player 3 earns at least as much as player 1, while according to structural monotonicity player 1 earns at least as much as player 3, yielding equality of the payoffs assigned to these two players by the permission value.

The next five solutions show that the five axioms of Theorem 5.2 are logically independent. For each solution, we state the four axioms satisfied by the solution; consequently, it does not satisfy the fifth axiom. The solution $f_i(N, v, D) = \text{My}(N, v, L_D)$ satisfies efficiency, linearity, the inessential player property, and necessary player symmetry. The AT permission value satisfies efficiency, linearity, the inessential player property, and predecessor necessity. The equal division solution satisfies efficiency, linearity, necessary player symmetry, and predecessor necessity. The solution $\overline{f}$ defined at the end of Sect. 3 (as fifth alternative solution showing logical independence of the axioms in Theorem 3.10) satisfies efficiency, the inessential player property, necessary player symmetry, and predecessor necessity. Finally, the zero solution given by $f_i(N, v, D) = 0$ for all $i \in N$ satisfies linearity, the inessential player property, necessary player symmetry, and predecessor necessity.

Next, we modify permission component fairness to get an axiomatization of the permission value. Similar as in Sect. 4, consider an arc $(i, j) \in D$ and suppose that player i is enforcing its power over j in the sense that it does not allow player j and all its subordinates to act. Then, player j and all its subordinates, i.e., the players in K_j^{ij}, become null players. On the other hand, the players in K_j^{ij} can refuse to cooperate with the players in K_i^{ij}. Applying a similar idea as Myerson's fairness, but now with respect to the enforcement of permission power, we obtain the following axiom.

Axiom 5.5 (*Permission Fairness*) A solution f on the class $\mathcal{G}_\mathcal{T}^N$ of permission tree games satisfies *permission fairness* if, for every $(N, v, D) \in \mathcal{G}_\mathcal{T}^N$ and for any pair $i, j \in N$ with $(i, j) \in D$, it holds that

$$f_i(N, v, D) - f_i\left(N, v^{-ij}, D\right) = f_j(N, v, D) - f_j\left(N, v^{-ij}, D\right).$$

Replacing in Theorem 4.2 permission component fairness by this permission fairness characterizes the permission value.

Theorem 5.6 *On the class $\mathcal{G}_\mathcal{T}^N$ of permission tree games, the permission value is the unique solution that satisfies efficiency, the inessential player property, and permission fairness.*

Proof It is known that the permission value φ satisfies efficiency and the inessential player property. To show permission fairness, consider any $(N, v, D) \in \mathcal{G}_\mathcal{T}^N$ and $(i, j) \in D$. According to Proposition 2.3 in van den Brink and Gilles (1996), it follows by applying the dividend formula of the Shapley value that

$$\varphi_i(N, v, D) = \sum_{\{T \in 2^N | i \in F_D(T)\}} \Delta_v(T)/|F_D(T)|, \quad i \in N.$$

We therefore have that

$$\begin{aligned}
&\varphi_i(N, v, D) - \varphi_i(N, v^{-ij}, D)\\
&\quad = \sum_{\{T \in 2^N | i \in F_D(T)\}} \frac{\Delta_v(T)}{|F_D(T)|} - \sum_{\{T \in 2^N | i \in F_D(T)\}} \frac{\Delta_{v^{-ij}}(T)}{|F_D(T)|}\\
&\quad = \sum_{\{T \in 2^N | i \in F_D(T)\}} \frac{\Delta_v(T)}{|F_D(T)|} - \sum_{\{T \in 2^N | i \in F_D(T), j \notin F_D(T)\}} \frac{\Delta_v(T)}{|F_D(T)|}\\
&\quad = \sum_{\{T \in 2^N | \{i,j\} \subset F_D(T)\}} \frac{\Delta_v(T)}{|F_D(T)|} = \sum_{\{T \in 2^N | j \in F_D(T)\}} \frac{\Delta_v(T)}{|F_D(T)|} = \varphi_j(N, v, D),
\end{aligned}$$

where the second equality follows since $\Delta_{v^{-ij}}(T) = \Delta_v(T)$ if $j \notin F_D(T)$, and $\Delta_{v^{-ij}}(T) = 0$ otherwise, and the fourth equality follows since $j \in F_D(T)$ implies that $i \in F_D(T)$ for all $T \in 2^N$. Since $\varphi_j(N, v^{-ij}, D) = 0$ by the inessential player property, it follows that φ satisfies permission fairness.

We show uniqueness by induction on the cardinality of the set $I(N, v, D)$ of inessential players. Let $(N, v, D) \in \mathcal{G}_{\mathcal{T}}^N$ and $i_0 \in N$ be the top player in (N, D). If $|I(N, v, D)| = |N|$, i.e., all players are inessential, then the inessential player property implies that $f_i(N, v, D) = 0$ for all $i \in N$. If $|I(N, v, D)| = |N| - 1$ then all players in $N \backslash \{i_0\}$ are inessential players, and thus the inessential player property implies that $f_i(N, v, D) = 0$ for all $i \in N \backslash \{i_0\}$. Efficiency then determines that $f_{i_0}(N, v, D) = v(N)$.

Proceeding by induction, assume that $f(N, v', D)$ is determined when $|I(N, v', D)| = k$, $1 \leq k \leq |N| - 1$, and suppose that $|I(N, v, D)| = k - 1$. The inessential player property implies that $f_i(N, v, D) = 0$ for all $i \in I(N, v, D)$. For every $j \in N \backslash (I(N, v, D) \cup \{i_0\})$ and $i = p_D(j)$, permission fairness requires that

$$f_i(N, v, D) - f_i(N, v^{-ij}, D) = f_j(N, v, D) - f_j(N, v^{-ij}, D). \tag{5.1}$$

Since the payoffs $f_i(N, v^{-ij}, D)$ and $f_j(N, v^{-ij}, D)$ are determined by the induction hypothesis and (N, D) is an oriented tree, this yields $|N| - |I(N, v, D)| - 1$ linear equations. Together with the efficiency condition

$$\sum_{h \in N} f_h(N, v, D) = v(N), \tag{5.2}$$

the total number of equations is $|N| - |I(N, v, D)|$. Since these equations are linearly independent, the system of $|N| - |I(N, v, D)|$ equations (5.1) and (5.2) has a unique solution in the $|N| - |I(N, v, D)|$ variables $f_h(N, v, D)$, $h \in N \backslash I(N, v, D)$. □

The three axioms of Theorem 5.6 are logically independent. The equal division solution satisfies efficiency and permission fairness, but not the inessential player property. The AT permission value satisfies efficiency and the inessential player property, but not permission fairness. Finally, the zero solution satisfies the inessential player property and permission fairness, but not efficiency.

6 Concluding remarks

In this paper, we have studied games with an oriented tree as permission structure. Since in such games players can only generate a surplus if they get permission to collaborate from all their superiors, only coalitions containing the superiors of all players involved in the coalition can form. In van den Brink and Gilles (1996), the conjunctive permission value for this class of games is axiomatized.

We show that the axioms of efficiency, linearity, the inessential player property, predecessor necessity, interior necessary player symmetry, and collusion neutrality characterize a unique value, the AT permission value. The AT permission value can also be axiomatized by efficiency, the inessential player property, and permission component fairness. The AT permission value can be easily computed as the average of n hierarchical outcomes, where n is the number of players. Each hierarchical outcome is an n-dimensional payoff vector, whose values can be found by the evaluation of n simple and explicit linear expressions.

We also evaluate the connection between the AT permission value and the permission value for permission tree games. If we strengthen the interior necessary player symmetry axiom to necessary player symmetry and drop the collusion neutrality axiom, we obtain a new axiomatization for the permission value. When we replace permission component fairness by permission fairness, we obtain another new axiomatization of the permission value.

Several games in the economic literature turn out to be the conjunctive restricted game on an appropriate digraph, such as the auction games of Graham et al. (1990), the so-called DR-polluted river game of Ni and Wang (2007) and its generalization in Dong et al. (2012), and the dual of the airport game of Littlechild and Owen (1973), see also Brânzei et al. (2002). These papers study the Shapley value of the corresponding game which is a special case of the conjunctive permission value. As an alternative, the AT permission value can be studied for these applications.

As an example, we consider the *polluted river problem*, as introduced by Ni and Wang (2007). Such a problem is given by a pair (N, c), where $N = \{1, \ldots n\}$ is a finite set of agents located along a river, numbered successively from upstream to downstream, and $c = (c_1, \ldots, c_n) \in \mathbb{R}^N_+$ is a pollution cost vector with c_i the cost incurred by agent $i \in N$ for cleaning the river. A *cost allocation* of a polluted river problem (N, c) is a vector $x \in \mathbb{R}^N_+$, where x_i is the cost to be paid by agent $i \in N$ in the total joint cleaning cost $\sum_{j=1}^n c_j$. Two solutions proposed in Ni and Wang (2007) are the *Local Responsibility Sharing method* given by $x^{\mathrm{LRS}}(c) = c$, i.e., each agent pays his own cost to clean its territory, and the *Upstream Equal Sharing method* given by $x_i^{\mathrm{UES}}(c) = \sum_{j=i}^n c_j/j,\ i = 1, \ldots, n$, i.e., for any agent its costs are distributed equally among itself and all its upstream agents. The LRS outcome reflects that each agent is responsible for its own cleaning costs, while the UES outcome reflects that upstream agents are held responsible for downstream pollution and thus that every agent upstream of some agent should share in the pollution costs of this agent.

The associated *LR polluted river game* (N, v) introduced by Ni and Wang (2007) is given by $v(S) = \sum_{i \in S} c_i,\ S \subset N$. It holds that $x^{\mathrm{LRS}}(c) = \mathrm{Sh}(N, v)$. Taking the directed tree $D = \{(i, i+1) \mid i \in \{1, \ldots, n-1\}\}$, where agents are ordered from upstream to downstream, it follows that $x^{\mathrm{UES}}(c) = \mathrm{Sh}(N, v_D) = \varphi(N, v, D)$, i.e., the permission value on D yields the UES outcome. Applying the AT permission value yields the cost allocation

$$\psi_i(N, v, D) = \frac{(n - i + 1)c_i}{n} + \sum_{j=i+1}^{n} \frac{c_j}{n}, \quad i = 1, \ldots, n.$$

So, according to this value, each agent pays a fraction in its own cost that is proportional to the number of its downstream agents (including itself), while it pays a share equal to $\frac{1}{n}$ in the cost of its downstream agents. Similar to the LRS method, every agent is the highest contributor in its own cost, but not as extreme as in the LRS method where an agent fully pays its own cost. Similar to the UES method, also the upstream agents contribute an equal share in the cost of an agent, but this uniform share is determined by the number of agents along the river instead of the number of upstream agents. In all three methods, the cost of any agent is shared between this agent and its upstream agents, with the AT permission value being a mixture of the other two extreme values.

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