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The Myopic Stable Set for Social Environments*

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Abstract

We introduce a new solution concept for models of coalition formation, called the myopic stable set (MSS). The MSS is defined for a general class of social environments and allows for an infinite state space. An MSS exists and, under minor continuity assumptions, it is also unique.

The MSS generalizes and unifies various results from more specific applications. It coincides with the coalition structure core in coalition function form games when this set is non-empty; with the set of stable matchings in the Gale-Shapley matching model; with the set of Pareto optimal allocations in the Shapley-Scarf housing matching model; with the set of pairwise stable networks and closed cycles in models of network formation; with the set of pure strategy Nash equilibria in pseudo-potential games and finite supermodular games; and with the set of mixed strategy Nash equilibria in several classes of two-player games.

KEYWORDS: *Social environments, group formation, stability, Nash equilibrium.*
JEL-CLASSIFICATION: *C70,C71.*

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1 Introduction

Models of coalition formation study a widespread and important pattern of human interaction: agents tend to form groups of equally interested individuals, but these groups behave in a non-cooperative way towards outsiders. For example, individuals in a community join forces to provide a local public good, voters create parties to attain their political goals, and firms set up lobby groups to influence policy-makers.

The literature studies coalition formation in many distinct settings, like networks, coalition function form games, and matching models. In this paper, we focus on a general class of social environments that covers all of these settings and many more. More precisely, we define a social environment on the basis of four components (Chwe, 1994): a finite collection of agents, a set of social states, for every agent preferences over the set of states, and an effectivity correspondence that models the feasible transitions from one state to another. We only require that the set of social states is a non-empty, compact metric space. As such, in contrast to most settings in the literature, we allow the state space to be infinite.

For such social environments, we define a new solution concept called the *myopic stable set*, abbreviated as MSS. The MSS extends the idea of level-1 farsighted stability by Herings, Mauleon, and Vannetelbosch (2009, 2014) from finite networks to the general class of social environments. The MSS is defined by three conditions, *deterrence of external deviations*, *asymptotic external stability*, and *minimality*. Deterrence of external deviations requires that no coalition benefits by deviating from a state inside the MSS to a state outside the MSS. Asymptotic external stability makes sure that from any state outside the MSS it is possible to get arbitrarily close to a state inside by a sequence of coalitional deviations. The final condition, minimality, requires that the MSS is minimal with respect to set inclusion.

Our notion of dominance is myopic as agents or coalitions do not predict how their decision to change the current state will lead to further changes by other agents or coalitions. Such a notion is natural in complex social environments where the number of possible states is large and agents have little information about the possible actions other agents may take or the incentives of other agents. The myopic stable set thereby distinguishes our approach from the ones in the literature that focus on farsightedness (see among others, Chwe, 1994; Xue, 1998; Herings, Mauleon, and Vannetelbosch, 2004, 2009, 2014; Dutta, Ghosal, and Ray, 2005; Page, Wooders, and Kamat, 2005; Page and Wooders, 2009; Ray and Vohra, 2015). Our approach is more in line with myopic concepts like the core and the von Neumann-Morgenstern stable set. As we will see in the application to normal-form games, it is also intimately connected to the notion of Nash equilibrium.

Our first main result (Theorem 3.1) shows that every social environment contains at least one non-empty MSS. Moreover, under minor continuity assumptions, we establish uniqueness of the MSS (Theorem 3.6). The existence and uniqueness results differ from many other popular solution concepts in the literature. For instance, the core and the coalition structure core for coalition function form games can be empty (Bondareva, 1963; Scarf, 1967; Shapley, 1967); the von Neumann-Morgenstern stable set may fail to exist or to be unique (Lucas, 1968, 1992), and the set of pure strategy Nash equilibria may be empty.

We provide several additional results that provide more insights about the structure of an MSS. For finite state spaces, we fully characterize the MSS as the union of all closed cycles (Theorem 3.13), i.e., subsets which are closed under coalitional better replies. This result also provides a connection to stochastic processes of coalition and network formation as in Jackson and Watts (2002) and Sawa (2014) and suggests possibilities for refinements in the spirit of Kandori, Mailath and Rob (1993) and Young (1993). For infinite spaces, the union of all closed cycles is found to be a subset of the MSS. This result is helpful in applications and in the comparison to other solution concepts. For instance, any state in the core is a closed cycle and is therefore included in the MSS. As a special case of this result, it follows that the MSS contains the set of pure strategy Nash equilibria in a normal-form game. Next we define a generalization of the weak improvement property (Friedman and Mezzetti, 2001) to social environments and we show that, under weak continuity conditions, the weak improvement property characterizes the collection of social environments for which the MSS coincides with the core. We also show that if the von Neumann-Morgenstern stable set exists, then it has a non-empty intersection with any MSS.

We demonstrate the versatility of our results by analyzing the relationship between the MSS and other solution concepts in specific social environments. In particular, we show that the MSS coincides with the coalition structure core for coalition function form games (Kóczy and Lauwers, 2004) whenever the coalition structure core is non-empty; with the set of stable matchings in the one-to-one matching model by Gale and Shapley (1962); with the set of Pareto efficient house allocations in the housing matching model of Shapley and Scarf (1974); with the set of pairwise stable networks and closed cycles in models of network formation (Jackson and Watts, 2002); and with the set of pure strategy Nash equilibria in pseudo-potential games (Dubey, Haimanko, and Zapechelnyuk, 2006) and finite supermodular games (Topkis, 1979 and Milgrom and Roberts, 1990). For the mixed extension of the game, the MSS coincides with the set of mixed strategy Nash equilibria in two-player zero-sum games and in two-player games in which one player has two actions.

The structure of the paper is as follows. Section 2 provides the primitives of our general framework of social environments and introduces and motivates the MSS. Section 3 establishes existence, non-emptiness, and uniqueness results. Section 4 analyzes our solution concept for various settings and relates it to other stability concepts from the literature. Section 5 is a conclusion. All proofs can be found in the appendix.

2 The Myopic Stable Set

In this section, we first introduce the concept of a social environment. Next, we introduce the notions of dominance and asymptotic dominance which are used to define our solution concept, the myopic stable set.

Let N be a non-empty finite set of individuals. A coalition S is a subset of N. The set of non-empty subsets of N is denoted by $\mathcal{N}$. Let (X, d) be a metric space, where X denotes our non-empty state space and d is a metric on X.[1] Let some state $x \in X$ be given and let $\varepsilon \in \mathbb{R}_{++}$. We define

$$B_\varepsilon(x) = \{y \in X | d(x, y) < \varepsilon\}$$

as the open ball around x with radius ε. The set $B_\varepsilon(x)$ contains all states in X that are in an ε-neighborhood of x. For a sequence $(x^k)_{k\in\mathbb{N}}$ in X, we write $x^k \to x$ if for all $\varepsilon > 0$, there is a number $N \in \mathbb{N}$ such that for all $k \geq N$, $x^k \in B_\varepsilon(x)$, i.e. the sequence $(x^k)_{k\in\mathbb{N}}$ converges to x.

An *effectivity correspondence* E associates with each pair of states $(x, y) \in X \times X$ a, possibly empty, collection of coalitions $E(x, y) \subseteq \mathcal{N}$. If $S \in E(x, y)$, we say that coalition S can move from state x to state y. If $E(x, y) = \emptyset$, then no coalition can move from x to y.

Each individual $i \in N$ has a complete and transitive preference relation $\succeq_i$ over the state space X. The profile $(\succeq_i)_{i\in N}$ then lists the preferences of all individuals in N. We denote the asymmetric part of $\succeq_i$ by $\succ_i$, i.e., $x \succ_i y$ if and only if $x \succeq_i y$ and not $y \succeq_i x$.

A social environment is now defined as follows.

Definition 2.1 (Social Environment)**.** A *social environment* is a tuple

$$\Gamma = (N, (X, d), E, (\succeq_i)_{i\in N})$$

consisting of a non-empty, finite set of agents N, a non-empty, compact metric space (X, d) of states, an effectivity correspondence E on X, and a profile of preference relations $(\succeq_i)_{i\in N}$ over X.

In Section 4, we specify different social environments which correspond to applications such as coalition function form games, one-to-one matching, housing matching, network formation, and non-cooperative normal-form games.

For a given social environment $\Gamma = (N, (X, d), E, (\succeq_i)_{i\in N})$, we say that a state $y \in X$ dominates another state $x \in X$ if there is a coalition such that (i) it can move from x to y and (ii) each of its members strictly prefers y over x.

[1] A metric is a function $d : X \times X \to \mathbb{R}_+$ such that (i) for every $x, y \in X$: $d(x, y) = 0$ if and only if $x = y$, (ii) for every $x, y \in X$: $d(x, y) = d(y, x)$, and (iii) for every $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

Definition 2.2 (Dominance)**.** A state $y \in X$ *dominates* the state $x \in X$ under E if there exists a coalition $S \in E(x, y)$ such that for every $i \in S$ it holds that $y \succ_i x$.

An alternative notion is the one of weak dominance. A state y *weakly dominates* x if there exists a coalition $S \in E(x, y)$ such that for all $i \in S$, $y \succeq_i x$ and there is at least one $j \in S$ such that $y \succ_j x$. The notion of weak dominance only requires that no member of S loses while at least one strictly gains. Definition 2.2 on the other hand requires that everybody in the coalition S strictly gains. As it turns out, when we restrict ourselves to settings with a finite state space X, all theoretical results from this section remain valid when we replace dominance by weak dominance. For settings where X is infinite, most results remain valid with the exception of Theorem 3.9 and Corollary 3.10 below which provide sufficient conditions for uniqueness of the MSS. We will come back to this issue when we present these two results.

Let some state $x \in X$ be given. The subset of X consisting of all states that dominate x together with the state x itself is denoted by $f(x)$, so

$$f(x) = \{x\} \cup \{y \in X | y \text{ dominates } x \text{ under } E\}.$$

We refer to f as the *dominance correspondence.* We define the two-fold composition of f by

$$f^2(x) = \{z \in X | \exists y \in X : y \in f(x) \text{ and } z \in f(y)\}.$$

By induction, we can define the k-fold iteration $f^k(x)$ as the subset of X that contains all states obtained by a composition of dominance correspondences of length $k \in \mathbb{N}$, i.e., $y \in f^k(x)$ if there is a $z \in X$ such that $y \in f(z)$ and $z \in f^{k-1}(x)$. Since by definition $x \in f(x)$, it holds that, for all $k, \ell \in \mathbb{N}$, if $k \leq \ell$, then $f^k(x) \subseteq f^\ell(x)$. We define the set of all states that can be reached from x by a finite number of dominations by $f^{\mathbb{N}}(x)$, so

$$f^{\mathbb{N}}(x) = \textstyle\bigcup_{k \in \mathbb{N}} f^k(x).$$

A state y is said to asymptotically dominate the state x if starting from x, it is possible to get arbitrarily close to y in a finite number of dominations.

Definition 2.3 (Asymptotic Dominance)**.** A state $y \in X$ *asymptotically dominates* the state $x \in X$ under E if for all $\varepsilon > 0$ there exists $k \in \mathbb{N}$ and a state $z \in f^k(x)$ such that $z \in B_\varepsilon(y)$.

We denote by $f^\infty(x)$ the set of all states in X that asymptotically dominate x. Formally, we have

$$f^\infty(x) = \{y \in X | \forall \varepsilon > 0,\ \exists k \in \mathbb{N},\ \exists z \in f^k(x) \text{ such that } z \in B_\varepsilon(y)\}.$$

It is easy to see that the set $f^\infty(x)$ coincides with the closure of the set $f^{\mathbb{N}}(x)$.

We are now ready to define our solution concept, the myopic stable set, abbreviated as MSS.

Definition 2.4 (Myopic Stable Set)**.** Let $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ be a social environment. The set $M \subseteq X$ is a *myopic stable set* if it is closed and satisfies the following three conditions:

1. **Deterrence of external deviations:** For all $x \in M$, $f(x) \subseteq M$.

2. **Asymptotic external stability:** For all $x \notin M, f^\infty(x) \cap M \neq \emptyset$.

3. **Minimality:** There is no closed set $M' \subsetneq M$ that satisfies Conditions 1 and 2.

Let M be an MSS. Deterrence of external deviations requires that no coalition can profitably deviate to a state outside M. Next, asymptotic external stability requires that any state outside M is asymptotically dominated by a state in M. Hence, from any state outside M it is possible to get arbitrary close to a state in M by a finite number of myopic deviations. Observe that the empty set would necessarily violate asymptotic external stability, so any MSS is non-empty.

Although the property of asymptotic external stability resembles a notion of farsightedness, there is an important distinction. In models with farsighted behavior, coalitions deviate from the current state because they expect to profit from a move in some future period, i.e., after possible subsequent moves by other coalitions. Our definition of asymptotic external stability, however, is myopic in the sense that coalitions deviate only because they see an immediate gain, without anticipating potential future deviations.

Finally, minimality imposes that there is no smaller closed set of states that satisfies deterrence of external deviations and asymptotic external stability.

For finite state spaces, the restriction imposed by asymptotic external stability remains unchanged if f^∞ is replaced by $f^{\mathbb{N}}$. We refer to the property that for all states $x \notin M$, $f^{\mathbb{N}} \cap M \neq \emptyset$ as *iterated external stability.* For infinite state spaces, the two concepts differ. In particular, if one uses iterated external stability instead of asymptotic external stability, an MSS might fail to exist. This is illustrated in the following example.

Example 2.5. Consider the social environment $\Gamma = (\{1\}, (X, d), E, \succeq_1)$, where the state space is given by $X = \{1/k | k \in \mathbb{N}\} \cup \{0\}$ and $d(x, y) = |x - y|$. Note that X is compact. Preferences $\succeq_1$ are defined by $x \succeq_1 y$ if and only if $x = y$ or $x < y$. The effectivity correspondence E is such that $\{1\} \in E(1/k, 1/(k+1))$ for every $k \in \mathbb{N}$ and $E(x, y) = \emptyset$ otherwise. It follows that

$$f(\tfrac{1}{k}) = \left\{\tfrac{1}{k}, \tfrac{1}{k+1}\right\}.$$

Observe that $0 \in f^\infty(x)$ for every $x \in X$ and that $f(0) = \{0\}$. It now follows easily that $\{0\}$ is an MSS.

Suppose we replace the requirement of asymptotic external stability by the stronger requirement of iterated external stability. We show that there is no closed set satisfying iterated external stability together with deterrence of external deviations and minimality. Towards a contradiction, suppose that the closed set $M \subseteq X$ satisfies these properties. Since, for every $k \in \mathbb{N}$, $0 \notin f^{\mathbb{N}}(1/k)$, the set $\{0\}$ does not satisfy iterated external stability. Given that $M \neq \{0\}$ and M is non-empty, there is $k \in \mathbb{N}$ such that $1/k \in M$. Let k be the smallest such number. From deterrence of external deviations, we have that also $1/(k+1) \in M$. Based on the corresponding properties of M, it is easy to verify that the closed, non-empty set $M' = M \setminus \{1/k\}$ satisfies deterrence of external deviations and iterated external stability. Since M' is a proper subset of M, M violates the minimality property.

An MSS is defined as a minimal set satisfying deterrence of external deviations and asymptotic external stability. Dropping the minimality requirement, we can define the concept of a quasi myopic stable set (QMSS) which is useful in the proofs.

Definition 2.6 (Quasi Myopic Stable Set)**.** Let $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ be a social environment. The set $M \subseteq X$ is a *quasi myopic stable set* if it is closed and satisfies deterrence of external deviations and asymptotic external stability.

3 General Properties

This section establishes existence of the myopic stable set in general and, under weak additional assumptions, its uniqueness. We also derive some additional structural properties of myopic stable sets that are used in the next section. We provide a brief discussion of the relationship between the MSS and the von Neumann-Morgenstern stable set. Finally, we relate our approach to dynamic models that rely on stochastic processes.

3.1 Existence and Uniqueness

The following main result shows existence of the myopic stable set.

Theorem 3.1 (Existence)**.** Let Γ be a social environment. Then an MSS exists.

The proof of Theorem 3.1 starts from the observation that the set of all states is a QMSS. The collection of all sets of states that are a QMSS is partially ordered by inclusion. We verify that the partially ordered set satisfies the conditions for Zorn's lemma and apply it to conclude that there is a minimal QMSS, i.e., an MSS.

Having established existence of an MSS, we now turn to the cardinality of such sets. The next theorem derives a property that is very useful in the proofs and applications to come.

Theorem 3.2. Let Γ be a social environment and let M be an MSS of Γ. If $x \in M$ and $y \in f^{\infty}(x)$, then $y \in M$.

Theorem 3.2 states that if a state belongs to an MSS, then every state that asymptotically dominates it belongs to the same MSS.

The following theorem shows that two myopic stable sets cannot be disjoint.

Theorem 3.3. Let Γ be a social environment and let M_1 and M_2 be two myopic stable sets of Γ. Then $M_1 \cap M_2 \neq \emptyset$.

The intuition behind the proof of Theorem 3.3 is as follows. If a state in M_1 does not belong to M_2, then it is asymptotically dominated by a state in M_2 since M_2 satisfies asymptotic external stability. By Theorem 3.2, the asymptotically dominating state belongs to M_1 as well, so we have found a state in the intersection of M_1 and M_2.

The following example shows that uniqueness of an MSS cannot be demonstrated without any additional assumptions.

Example 3.4. Consider the social environment $\Gamma = (\{1\}, (X, d), E, \succeq_1)$, where

$$X = \{0, \tfrac{1}{2}, 1\} \cup \left\{\tfrac{1}{k} \mid k \in \mathbb{N} \setminus \{1,2\}\right\} \cup \left\{1 - \tfrac{1}{k} \mid k \in \mathbb{N} \setminus \{1,2\}\right\}$$

and the metric is given by $d(x, y) = |x - y|$.

The effectivity correspondence is such that the individual can move from both states 0 and 1 to state 1/2 and, for every $k \in \mathbb{N} \setminus \{1,2\}$, from state $1-1/k$ to state $1/k$ and from state $1/k$ to state $1-1/(k+1)$. The individual cannot make any other moves. The preferences of the individual are such that

$$\tfrac{2}{3} \prec_1 \tfrac{1}{3} \prec_1 \tfrac{3}{4} \prec_1 \tfrac{1}{4} \prec_1 \tfrac{4}{5} \prec_1 \tfrac{1}{5} \prec_1 \cdots \prec_1 1 \prec_1 0 \prec_1 \tfrac{1}{2}.$$

We claim that both $\{0, 1/2\}$ and $\{1/2, 1\}$ are myopic stable sets. Since the effectivity correspondence admits no move outside the respective sets, both $\{0, 1/2\}$ and $\{1/2, 1\}$ satisfy deterrence of external deviations. For asymptotic external stability, observe that for every $k \in \mathbb{N} \setminus \{1,2\}$ it holds that $\{0,1\} \subset f^{\infty}(1/k)$ and $\{0,1\} \subset f^{\infty}(1-1/k)$. Moreover, we have $1/2 \in f(0) = f^{\infty}(0)$ and $1/2 \in f(1) = f^{\infty}(1)$. For minimality, the sets $\{0\}$ and $\{1\}$ violate deterrence of external deviations since $1/2 \in f(0)$ and $1/2 \in f(1)$. The set $\{1/2\}$ violates asymptotic external stability as $1/2 \notin f^{\infty}(x)$ for any $x \in X$ different from 0, 1/2 and 1.

We can restore uniqueness by imposing the following mild continuity assumption on the dominance correspondence f.

Definition 3.5 (Lower Hemi-continuity of f)**.** The dominance correspondence $f : X \to X$ is *lower hemi-continuous* if for every sequence $(x^k)_{k \in \mathbb{N}}$ in X such that $x^k \to x$ and for every $y \in f(x)$ there is a sequence $(y^k)_{k \in \mathbb{N}}$ in X such that for all k, $y^k \in f(x^k)$ and $y^k \to y$.

This continuity assumption allows us to state the following uniqueness result.

Theorem 3.6. Let Γ be a social environment such that the corresponding dominance correspondence f is lower hemi-continuous. Then Γ has a unique MSS.

The continuity condition of Theorem 3.6 is trivially satisfied when the state space X is finite. As such, for all applications with a finite state space, we have uniqueness of the MSS.

The dominance correspondence f is defined in terms of the individual preference relations $(\succeq_i)_{i \in N}$ and the effectivity correspondence E. To ease the verification of lower hemi-continuity of f, we provide sufficient conditions on the primitives of a social environment.

As a first condition, we impose lower hemi-continuity of the effectivity correspondence E. Towards this end, consider, for every $S \in \mathcal{N}$, the correspondence $G_S : X \to X$ defined by

$$G_S(x) = \{x\} \cup \{y \in X \mid S \in E(x, y)\}, \quad x \in X,$$

which associates to every state $x \in X$ the union of $\{x\}$ and the set of states coalition S can move to from x.

Definition 3.7 (Lower Hemi-continuity of E)**.** The effectivity correspondence E is *lower hemi-continuous* if for every coalition $S \in \mathcal{N}$ the correspondence $G_S : X \to X$ is lower hemi-continuous, i.e., for every sequence $(x^k)_{k \in \mathbb{N}}$ in X such that $x^k \to x$ and for every $y \in G_S(x)$ there is a sequence $(y^k)_{k \in \mathbb{N}}$ such that $y^k \in G_S(x^k)$ and $y^k \to y$.

Our second condition is continuity of the preferences.

Definition 3.8 (Continuity of Preferences)**.** The preference relation $\succeq_i$ of individual $i \in N$ is *continuous* if for any two sequences $(x^k)_{k \in \mathbb{N}}$ and $(y^k)_{k \in \mathbb{N}}$ in X with $x^k \to x$ and $y^k \to y$ and, for every $k \in \mathbb{N}$, $x^k \succeq_i y^k$, it holds that $x \succeq_i y$.

Theorem 3.9 shows that lower hemi-continuity of E and continuity of preferences is sufficient for the dominance correspondence f to be lower hemi-continuous.

Theorem 3.9. Let $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ be a social environment such that the effectivity correspondence E is lower hemi-continuous and the preferences $(\succeq_i)_{i \in N}$ are continuous. Then the dominance correspondence f is lower hemi-continuous.

Combining Theorems 3.6 and 3.9 directly yields the following corollary which gives a sufficient condition on the primitives of the model to obtain a unique MSS.

Corollary 3.10. Let $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ be a social environment such that the effectivity correspondence E is lower hemi-continuous and the preferences $(\succeq_i)_{i \in N}$ are continuous. Then Γ has a unique MSS.

Recall the alternative notion of weak dominance, where a state y weakly dominates a state x when there is a coalition $S \in E(x,y)$ such that for all $i \in S$, $y \succeq_i x$ and there is at least one $j \in S$ such that $y \succ_j x$. When we replace dominance in the definition of the MSS by weak dominance, all results in this section remain valid except for Theorem 3.9 and Corollary 3.10. The following example presents a social environment in which the effectivity correspondence is lower hemi-continuous and the preferences are continuous, but using weak dominance in the definition of the MSS does not yield a unique MSS.

Example 3.11. Consider the social environment $\Gamma = (\{1,2\},(X,d),E,(\succeq_1,\succeq_2))$, where

$$X = \{(0,0),(1,0),(2,0)\} \cup \{(0,\tfrac{2}{k}),(1,\tfrac{1}{k}),(2,\tfrac{1}{k}) | k \in \mathbb{N}\}$$

and d is the Euclidean metric on X, so $d(x,y) = \|x-y\|_2$. It clearly holds that X is compact.

Individual 1 only cares about the first component of the state while individual 2 only cares about the second component. Both individuals prefer states where the component they care about is lower over states where it is higher. Note that these preferences are continuous.

The effectivity correspondence is as follows. For every $k \in \mathbb{N}$, the singleton $\{1\}$ can move from state $(2,1/k)$ to state $(1,1/k)$ and the singleton $\{2\}$ can move from state $(1,1/k)$ to state $(2,1/(k+1))$. Moreover, for every $k \in \mathbb{N}$, the singleton $\{2\}$ can move from state $(0,2/k)$ to state $(1,1/k)$. Coalition $\{1,2\}$ can move from states $(1,0)$ and $(2,0)$ to state $(0,0)$ and, for every $k \in \mathbb{N}$, from states $(1,1/k)$ and $(2,1/k)$ to state $(0,2/k)$. No other moves are possible.

To see that the effectivity correspondence is lower hemi-continuous, let the sequence $(x^k)_{k\in\mathbb{N}}$ in X be such that $x^k \to x$. There are only three relevant sequences of states in X: the sequence $((0,2/k))_{k\in\mathbb{N}}$, the sequence $((1,1/k))_{k\in\mathbb{N}}$, and the sequence $((2,1/k))_{k\in\mathbb{N}}$. The first converges to $(0,0)$, the second to $(1,0)$, and the third to $(2,0)$.

Let some $x \in \{(0,0),(1,0),(2,0)\}$ be given. Since $G_{\{1\}}(x) = \{x\}$ and $G_{\{2\}}(x) = \{x\}$, it is immediate that $G_{\{1\}}$ and $G_{\{2\}}$ are lower hemi-continuous.

For $G_{\{1,2\}}$, the only non-trivial cases are $x = (1,0)$ and $x = (2,0)$. We give the argument for state $x = (1,0)$ explicitly. The argument for state $(2,0)$ follows by symmetry. For every $y \in G_{\{1,2\}}(1,0)$ we have to find a sequence $(y^k)_{k\in\mathbb{N}}$ such that $y^k \in G_{\{1,2\}}(1,1/k)$ and $y^k \to y$. If $y = (0,0)$, we take the sequence $((0,2/k))_{k\in\mathbb{N}}$. If $y = (1,0)$, we take the sequence $((1,1/k))_{k\in\mathbb{N}}$.

Since $f^\infty(0,0) = \{(0,0)\}$, $f^\infty(1,0) = \{(1,0)\}$, and $f^\infty(2,0) = \{(2,0)\}$, it follows from asymptotic external stability that $\{(0,0),(1,0),(2,0)\}$ is a subset of any MSS. Since this set satisfies deterrence of external deviations and asymptotic external stability, it follows from minimality that the unique MSS is equal to $\{(0,0),(1,0),(2,0)\}$.

On the other hand, when we replace dominance in the definition of the MSS by weak dominance, we obtain both the sets $\{(0,0),(1,0)\}$ and $\{(0,0),(2,0)\}$ as solutions. Indeed, from both $(1,0)$ and $(2,0)$, the coalition $\{1,2\}$ can deviate to $(0,0)$ if only weak dominance is imposed. To satisfy asymptotic external stability, it is sufficient that on top of the state $(0,0)$, either the state $(1,0)$ or the state $(2,0)$ should be present. By minimality, it follows that only one of these states is included.

3.2 Closed Cycles and the Core

In this subsection, we give two general results about the structure of an MSS. The first result relates the MSS to the union of all closed cycles. The second result characterizes the social environments for which the MSS is equal to the core.

Definition 3.12 (Closed Cycle)**.** A *closed cycle* of a social environment Γ is a set $C \subseteq X$ such that for every $x \in C$ it holds that $f^{\infty}(x) = C$.

Thus, a closed cycle is a subset of X which is closed under the asymptotic dominance correspondence f^{∞}. We denote the union of all closed cycles by CC, so CC contains all states that are part of some closed cycle. The following result characterizes the MSS for social environments with a finite state space as the union of all closed cycles and shows that this union is a subset of the MSS for social environments with an infinite state space.

Theorem 3.13. Let $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ be a social environment and let M be an MSS of Γ. It holds that $\mathrm{CC} \subseteq M$. If the state space X is finite, we have $\mathrm{CC} = M$.

A *sink* is a closed cycle which consists of only one state, i.e., $f(x) = \{x\}$. The union of all sinks is called the core.

Definition 3.14 (Core)**.** Let $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ be a social environment. The *core* CO of Γ is given by

$$\mathrm{CO} = \{x \in X \mid f(x) = \{x\}\}.$$

It is well-known that the core may be empty for some social environments. However, if it is not empty, then it is always contained in the myopic stable set by virtue of Theorem 3.13 and the observation that a sink is a closed cycle which consists of one state.

Corollary 3.15. Let Γ be a social environment and let M be an MSS of Γ. Then we have $\mathrm{CO} \subseteq M$.

The next definition is inspired by the finite analogue for normal-form games as presented in Friedman and Mezzetti (2001).

Definition 3.16 (Weak (Finite) Improvement Property)**.** A social environment Γ satisfies the *weak finite improvement property* if for each state $x \in X$, $f^{\mathbb{N}}(x)$ contains a sink and the *weak improvement property* if for each state $x \in X$, $f^{\infty}(x)$ contains a sink.

The following theorem provides a characterization for the MSS in social environments with the weak improvement property.

Theorem 3.17. Let Γ be a social environment with a lower hemi-continuous dominance correspondence f. An MSS of Γ is equal to the core if and only if Γ satisfies the weak improvement property.

It follows easily from the proof of Theorem 3.17 that the requirement of lower hemi-continuity of f in Theorem 3.17 can be weakened to the requirement that CO is closed.

3.3 The von Neumann-Morgenstern Stable Set

The von Neumann-Morgenstern (vNM) stable set provides a solution concept for an environment consisting of a set of states X and a dominance relation on this set X (von Neumann and Morgenstern, 1944).

Definition 3.18 (vNM Stable Set)**.** Let $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ be a social environment. The set $V \subseteq X$ is a *vNM stable set* if it satisfies the following two conditions:

1. **Internal stability:** For all $x, y \in V$ such that $x \neq y$ it holds that $y \notin f(x)$.

2. **External stability:** For all $x \notin V, f(x) \cap V \neq \emptyset$.

Internal stability requires that no state in the set is dominated by another state in the set. External stability requires that every state outside the set should be dominated by a state in the set.

Our notion of asymptotic external stability has a similar flavor as the vNM notion of external stability. However, the vNM stable set looks at one-step dominations while our notion of asymptotic external stability uses asymptotic dominance, which can be seen as an infinite iteration of one-step dominations. In fact, extending the definition of the vNM stable set by allowing for a finite iteration of one-step dominations, i.e., replacing f by $f^{\mathbb{N}}$ in Definition 3.18, has also been advocated by several authors, see Harsanyi (1974), van Deemen (1991), Page and Wooders (2009), and Herings, Mauleon, and Vannetelbosch (2017).

On the other hand, our notion of deterrence of external deviations is quite different from the vNM notion of internal stability. While we allow that a state in the MSS is dominated by another state in the MSS, this is prohibited in the vNM stable set. Moreover, unlike our concept, in the vNM stable set it is allowed that a state in the set is dominated by a state outside the set.

In terms of predictions, first note that an MSS always exists and is unique under weak continuity assumptions, whereas the vNM stable set may not exist and if it exists may fail to be unique. If the vNM stable set exists, there are a few connections between the MSS and the vNM stable set. First of all, both sets contain the core. Second, the intersection between the vNM stable set and the MSS is non-empty as is stated in the next result.

Theorem 3.19. Let $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ be a social environment for which a vNM stable set V exists. If M is an MSS of Γ, then $M \cap V \neq \emptyset$.

By Theorem 3.13, the MSS contains the union of all closed cycles. If the vNM stable set exists, one can show that it contains at least one state from every closed cycle.

It is easily verified that Theorem 3.19 remains true if the dominance correspondences $f^{\mathbb{N}}$ or f^{∞} are used in Definition 3.18.

3.4 Dynamic Stochastic Processes

Stochastic approaches have been frequently used in non-cooperative settings like normal-form games. Sawa (2014) presents a general framework which extends such a stochastic analysis to cooperative settings. In each period, one of the coalitions that can make a move is randomly selected and chooses one of its moves at random. The move is carried out with probability 1 if all members of the coalition would be strictly better off. If no member is worse off, but at least one agent is indifferent, the move is carried out with a probability strictly between 0 and 1. Otherwise, the move is not carried out. The resulting dynamic process can be thought of as a better-response dynamic. To establish a connection to the MSS, we consider a variation of the process in Sawa (2014) in which a coalition only moves with positive probability if all coalition members are strictly better off.

Consider a social environment $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ such that X is finite. For states $x, y \in X$, let $Q(x, y)$ denote the transition probability from state x to state y and let Q be the matrix of transition probabilities. We say that Q is *consistent* with f if for every $y \in f(x) \setminus \{x\}$ it holds that $Q(x, y) > 0$ and for every $y \notin f(x)$ it holds that $Q(x, y) = 0$. In particular, the state x need not change even if the set $f(x) \setminus \{x\}$ is non-empty. An interesting special case is the one of uniform transition probabilities, which is obtained by setting, for every $x \in X$ and $y \in f(x)$, $Q(x, y) = 1/|f(x)|$, where $|f(x)|$ denotes the cardinality of the set $f(x)$.

The next result presents an equivalence between the MSS and the set of recurrent states of the Markov chain (X, Q).

Theorem 3.20. Let $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ be a social environment with finite state space X, let f be the corresponding dominance correspondence, and let (X, Q) be a Markov chain such that Q is consistent with f. Then the MSS of Γ is equal to the set of recurrent states of (X, Q).

For social environments with a finite number of states, Theorem 3.20 gives an equivalence between the set of recurrent states of a dynamic process that selects all better responses with positive probability and the MSS. This suggests the possibility of refinements based on notions of stochastic stability that have been used in non-cooperative game theory by Kandori, Mailath, and Rob (1993) and Young (1993), in network formation by Jackson and Watts (2002), and in more general coalitional settings by Sawa (2014).

The above results do not readily extend to settings where X is infinite. The next example illustrates some of the complications that arise with an infinite state space.

Example 3.21. Consider the social environment $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$, where $N = \{1, 2\}$, $X = [0, 1] \times [0, 1]$, and the metric is $d(x, y) = \|x - y\|_1 = |x_1 - y_1| + |x_2 - y_2|$. The effectivity correspondence is such that individual 1 can change the first component of the state and individual 2 the second component, so $\{1\} \in E(x, y)$ if and only if $x_2 = y_2$ and $\{2\} \in E(x, y)$ if and only if $x_1 = y_1$. The coalition $\{1, 2\}$ is never effective. The preferences of the individuals are such that

$$\begin{aligned} x \succeq_1 y \quad &\text{if and only if} \quad 2x_1x_2 - x_1 - x_2 \geq 2y_1y_2 - y_1 - y_2, \\ x \succeq_2 y \quad &\text{if and only if} \quad 2x_1x_2 - x_1 - x_2 \leq 2y_1y_2 - y_1 - y_2. \end{aligned}$$

It is not hard to see that this social environment corresponds to the normal-form game of matching pennies, where x_1 is the probability of the row player choosing "up" and x_2 is the probability of the column player choosing "left". The unique Nash equilibrium of this game is equal to $x^* = (1/2, 1/2)$.

For every $x \in X$, we define

$$\begin{aligned} f_1(x) &= \begin{cases} \{y \in X \mid y_1 \leq x_1 \text{ and } y_2 = x_2\} & \text{if } x_2 < \frac{1}{2}, \\ \{x\} & \text{if } x_2 = \frac{1}{2}, \\ \{y \in X \mid y_1 \geq x_1 \text{ and } y_2 = x_2\} & \text{if } x_2 > \frac{1}{2}, \end{cases} \\ f_2(x) &= \begin{cases} \{y \in X \mid y_1 = x_1 \text{ and } y_2 \geq x_2\} & \text{if } x_1 < \frac{1}{2}, \\ \{x\} & \text{if } x_1 = \frac{1}{2}, \\ \{y \in X \mid y_1 = x_1 \text{ and } y_2 \leq x_2\} & \text{if } x_1 > \frac{1}{2}, \end{cases} \end{aligned}$$

so we can express the dominance correspondence as

$$f(x) = f_1(x) \cup f_2(x).$$

We consider the better-response dynamics where each element of $f(x)$ is selected with equal probability. To do so, we define $\rho_1 : X \to [0, 1]$ and $\rho_2 : X \to [0, 1]$ as the functions that project x on its first and second coordinate, respectively. We use λ to denote the Lebesgue measure. Let $\mathcal{B}(X)$ denote the Borel σ-algebra on X. The transition probability kernel resulting from better-response dynamics is obtained by defining, for every $x \in X$, and for every $A \in \mathcal{B}(X)$,

$$Q(x, A) = \begin{cases} 0 & \text{if } x = (\frac{1}{2}, \frac{1}{2}) \notin A, \\ 1 & \text{if } x = (\frac{1}{2}, \frac{1}{2}) \in A, \\ 2\lambda(\rho_1(A \cap f_1(x))), & \text{if } x_1 = \frac{1}{2}, x_2 \neq \frac{1}{2}, \\ 2\lambda(\rho_2(A \cap f_2(x))), & \text{if } x_1 \neq \frac{1}{2}, x_2 = \frac{1}{2}, \\ \frac{\lambda(\rho_1(A \cap f_1(x))) + \lambda(\rho_2(A \cap f_2(x)))}{\lambda(\rho_1(f_1(x))) + \lambda(\rho_2(f_2(x)))}, & \text{if } x_1 \neq \frac{1}{2}, x_2 \neq \frac{1}{2}. \end{cases}$$

The first and second equality above show that the better-response dynamics never leaves the Nash equilibrium once reached. The third equality concerns the case where only player 1 likes to move. Observe that if $x_1 = 1/2$ and $x_2 \neq 1/2$, then $\lambda(\rho_1(f_1(x))) = 1/2$, which explains the multiplication by 2. A similar remark applies to the fourth equality above. For the last equality, notice that $x_1 \neq 1/2$ and $x_2 \neq 1/2$ implies that $\lambda(\rho_1(f_1(x))) > 0$ or $\lambda(\rho_2(f_2(x))) > 0$, so there is no division by zero.

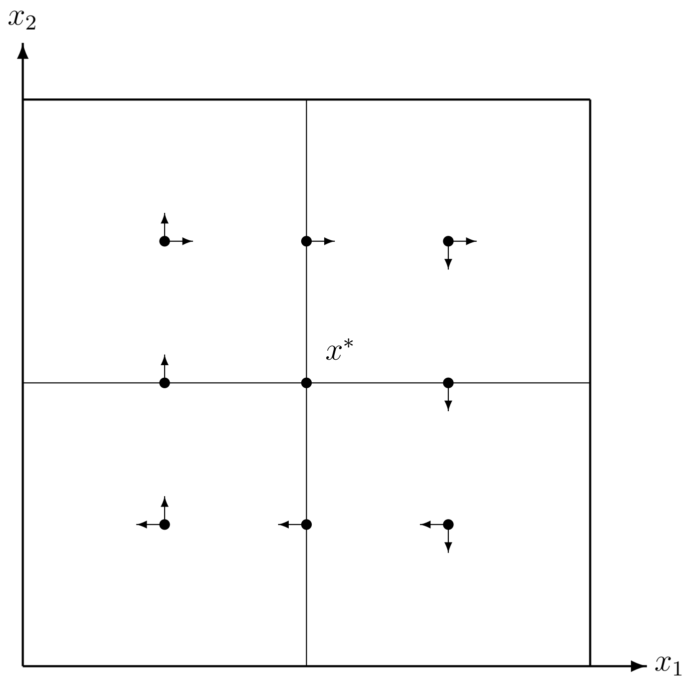

Figure 1: Better-response dynamics for the game of matching pennies.

The Markov process is illustrated in Figure 1. The arrows indicate the direction in which a state changes. A typical state can change in two directions, either west or east and either north or south, thereby generating two line segments on which the next state lies.

For every $A \in \mathcal{B}(X)$, $Q(\cdot, A)$ is a measurable function on X, but it is in general not continuous. For instance, if $A = \{x^*\}$, then $Q(x, A) = 1$ if $x = x^*$ and $Q(x, A) = 0$, otherwise. Indeed, the state x^* does not belong to $f(x)$ unless $x = x^*$ and in that case $f(x^*) = \{x^*\}$.

In this setting and other settings with an infinite state space, the Markov chain returns to a given state with probability zero, so the concept of a recurrent state is of less use and importance. Instead, for infinite settings, the property of irreducibility is often studied, which expresses that all parts of the state space can be reached by the Markov chain, no matter what the starting point is. Given a state $x \in X$ and a set A in the Borel σ-algebra $\mathcal{B}(X)$ on X, let $L(x, A)$ denote the probability that the Markov chain has a realization belonging to A at some point in the future when starting from x. Let φ be the measure on X that assigns to each set in $\mathcal{B}(X)$ its Lebesgue measure. A Markov process (X, Q) is called *φ-irreducible* if for every $A \in \mathcal{B}(X)$ such that $\varphi(A) > 0$ it holds that $L(x, A) > 0$ for every $x \in X$.

The Markov process (X, Q) in Example 3.21 is such that X can be decomposed in two parts, namely $\{x^*\}$ and $X \setminus \{x^*\}$. There is no transition between these two sets of states and the restriction of the Markov process to each set is irreducible. This is obvious for $\{x^*\}$. The next result shows this for $X \setminus \{x^*\}$.

Theorem 3.22. The restriction of the Markov process (X, Q) in Example 3.21 to $X \setminus \{x^*\}$ is φ-irreducible.

Example 3.21 shows that for the social environment corresponding to the normal-form game of matching pennies, none of the strategy profiles is singled out by the stochastic better-response dynamics. In contrast, we will show in Subsection 4.4 that the unique MSS consists of the Nash equilibrium x^*.

4 Applications

In this section, we illustrate the generality of our setting and the useful common structure of our results by means of four specific models that have been studied extensively in the literature: coalition function form games, one-to-one matching models, models of network formation, and normal-form games. For each of these settings, we first specify the social environment, i.e., the set of individuals N, the state space (X, d), the effectivity correspondence E, and the preferences $(\succeq_i)_{i \in N}$. Subsequently, we discuss how the results from the previous section can be applied.

4.1 Coalition Function Form Games

A coalition function form game is defined by a tuple (N, v), where N is the set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function that assigns to each coalition $S \subseteq N$ a number

$v(S) \in \mathbb{R}$, called the coalitional value of S, with the usual convention that $v(\emptyset) = 0$. A coalition structure is a partition π of N. It describes how the grand coalition is divided into various sub-coalitions. The collection of all coalition structures, i.e., the collection of partitions of N, is denoted by Π.

For coalition function form games, we define X as the set of coalition structures Π together with all individually rational payoff vectors that can be obtained by allocating the coalitional values among the members of the respective coalitions:

$$X = \left\{ (\pi, u) \in \Pi \times \mathbb{R}^N \middle| \forall i \in N : u_i \geq v(\{i\}) \text{ and } \forall S \in \pi : \sum_{i \in S} u_i = v(S) \right\}.$$

Given a state $x \in X$, we denote by $\pi(x)$ the projection to its first component, i.e., the coalition structure, and by $u(x)$ the projection to its second component, i.e., the payoff vector, so we can write $x = (\pi(x), u(x))$. The restriction of the payoff vector $u(x)$ to the members in coalition S is denoted by $u_S(x)$. The set X is non-empty since it always contains the state where N is partitioned into singletons and each player $i \in N$ receives the payoff $v(\{i\})$.

We define the metric d on X by setting for all $x, y \in X$,

$$d(x, y) = \mathbb{1}_{\{\pi(x) \neq \pi(y)\}} + \|u(x) - u(y)\|_\infty,$$

where $\mathbb{1}$ is the indicator function and $\|.\|_\infty$ is the infinity norm. It is easily seen that (X, d) is compact.

We define preferences $\succeq_i$ over the state space X by setting $x \succeq_i y$ if and only if $u_i(x) \geq u_i(y)$, i.e., individual i's payoff in state x is at least as high as in state y.

For each ordered pair of states x, y, the effectivity correspondence $E(x, y)$ specifies which coalitions can change state x into state y. As an example that imposes some reasonable structure on the effectivity correspondence, we provide a brief outline of the notion of *coalitional sovereignty* (Konishi and Ray, 2003; Kóczy and Lauwers, 2004; Ray and Vohra, 2014, 2015; Herings, Mauleon, and Vannetelbosch, 2017).

When a coalition of players S—called the leaving players—decides to leave their old coalitions to create a new group, the state changes to a new state y characterized by a new coalition structure $\pi(y)$ and a new payoff vector $u(y)$. The collection of coalitions of $\pi(x)$ that are unaffected by this change is denoted by $\mathcal{U}(x, S)$ and the set of all players in this group by $U(x, S)$. Formally, we have

$$\begin{aligned} \mathcal{U}(x, S) &= \{T \in \pi(x) | S \cap T = \emptyset\}, \\ U(x, S) &= \cup_{T \in \mathcal{U}(x,S)} T. \end{aligned}$$

This notation helps us in defining coalitional sovereignty.

Definition 4.1 (Coalitional sovereignty)**.** An effectivity correspondence E satisfies *coalitional sovereignty* if the following two conditions hold:

(1) *Non-interference:* For every $x, y \in X$, if $S \in E(x, y)$ and $T \in \mathcal{U}(x, S)$, then $S \in \pi(y)$, $T \in \pi(y)$, and $u_T(x) = u_T(y)$.

(2) *Full support:* For every $x \in X$, for every $S \in \mathcal{N}$, and for every $u \in \mathbb{R}^S$ such that for all $i \in S : u_i \geq v(\{i\})$ and $\sum_{i \in S} u_i = v(S)$, there is a state $y \in X$ such that $S \in E(x, y)$ and $u_S(y) = u$.

Intuitively, non-interference requires that if a coalition S induces a change from a state x to a state y, then the unaffected coalitions in $\mathcal{U}(x, S)$ are still part of the new coalition structure $\pi(y)$ and every unaffected player $i \in U(x, S)$ keeps his old payoff, i.e., $u_i(x) = u_i(y)$. Full support requires that every coalition S has the opportunity to move to a new state where it has the freedom to redistribute its worth $v(S)$ at will.

Coalitional sovereignty does not fully specify the effectivity correspondence E. In particular, it leaves unspecified the payoffs and coalition structure of players that are neither part of the leaving coalition S nor part of the unaffected players $U(x, S)$, i.e., players in the set $N \setminus (S \cup U(x, S))$. We call these players residual players. Indeed, one of the more controversial issues is to what extent the leaving players have the power to influence the coalition structure and payoffs of these residual players; see Shubik (1962), Hart and Kurz (1983), Konishi and Ray (2003), and Ray and Vohra (2014) for related discussions and alternative viewpoints. One frequently used specification is the γ-model (Hart and Kurz, 1983). The γ-model prescribes that the residual players are divided into singletons. This assumption is justified by the viewpoint that a coalition is only maintained if there is unanimous agreement among its members. As such, the departure of one individual implies the collapse of the entire coalition. In our setting, the γ-model imposes the following restriction on the effectivity correspondence.

Definition 4.2 (γ-model)**.** The effectivity correspondence E is induced by the γ*-model* if it satisfies coalitional sovereignty and

(3) For every $x, y \in X$, for every $S \in E(x, y)$, if $i \in N \setminus (S \cup U(x, S))$, then $\{i\} \in \pi(y)$.

The γ-model associates a unique social environment to each coalition function form game. We know, by Theorem 3.1, that there exists at least one non-empty MSS. Let us first show that for coalition function form games, the MSS is also unique. Towards this end, we first show that the effectivity correspondence E is lower hemi-continuous and that the preference relations $\succeq_i$ are continuous.

Theorem 4.3. Let (N, v) be a coalition function form game and $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ be the social environment induced by the γ-model. Then the effectivity correspondence E is lower hemi-continuous and the preferences $(\succeq_i)_{i \in N}$ are continuous.

Theorem 4.3 together with Corollary 3.10 shows uniqueness of the MSS.

Corollary 4.4. Let (N, v) be a coalition function form game and Γ be the social environment induced by the γ-model. Then Γ has a unique MSS.

In fact, most alternatives of the γ-model will also lead to lower hemi-continuity of E, so will also have a unique MSS. However, establishing the lower hemi-continuity of E must be done case by case.

One of the most prominent set-valued solution concepts for coalition function form games is the coalition structure core.

Definition 4.5 (Coalition Structure Core)**.** Let (N, v) be a coalition function form game and Γ be the social environment induced by the γ-model. The *coalition structure core* of (N, v) is the set of states $x \in X$ such that for every coalition $S \in \mathcal{N}$

$$\sum_{i \in S} u_i(x) \geq v(S).$$

In words, the coalition structure core gives to the members of each coalition at least the payoff they can obtain by forming that coalition.

Theorem 4.6. Let (N, v) be a coalition function form game and $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ be the social environment induced by the γ-model. The coalition structure core of (N, v) is equal to the core of Γ.

Kóczy and Lauwers (2004) define the coalition structure core to be accessible if from any initial state there is a finite sequence of states ending with an element of the coalition structure core and each element in that sequence outsider independently dominates the previous element. Accessibility of the coalition structure core thus corresponds to iterated external stability of the coalition structure core with respect to outsider independent domination. The notion of outsider independent domination differs from our notion of a myopic improvement in the γ-model in two ways. First, residual players are not required to become singletons after a move has taken place. Second, improvements for the members of the coalition that moves are not necessarily strict improvements. The following example illustrates that under the requirement of strict improvements of all members involved in a move, as in our dominance correspondence f, the coalition structure core does not satisfy iterated external stability, i.e., it is not the case that for all states $x \in X$, there is a state y in the coalitional structure core such that $y \in f^{\mathbb{N}}(x)$.

Example 4.7. Let (N, v) be a coalition function form game such that $N = \{1, 2, 3\}$, $v(\{1, 2\}) = 1$, and $v(\{2, 3\}) = 1$. All other coalitions have a coalitional value of 0. Thus, player 2 can choose to form a coalition with either player 1 or player 3 to form a two-person coalition generating a surplus equal to one. The coalition structure core therefore consists of only two states, y and y', with equal payoffs, $u(y) = u(y') = (0, 1, 0)$, and coalitional structures $\pi(y) = \{\{1, 2\}, \{3\}\}$ and $\pi(y') = \{\{1\}, \{2, 3\}\}$.

Consider an initial state $x^0 \in X$ such that $\pi(x^0) = \{\{1\}, \{2\}, \{3\}\}$ and $u(x^0) = (0, 0, 0)$. Under our notion of a myopic improvement, where all players involved in a move have to gain strictly, a state $x^1 \neq x^0$ belongs to $f(x^0)$ if and only if either $\pi(x^1) = \{\{1, 2\}, \{3\}\}$ and $u(x^1) = (\varepsilon, 1 - \varepsilon, 0)$ for some $\varepsilon \in (0, 1)$ or $\pi(x^1) = \{\{1\}, \{2, 3\}\}$ and $u(x^1) = (0, 1 - \varepsilon, \varepsilon)$ for some $\varepsilon \in (0, 1)$. It follows that x^1 is a state where either player 1 or player 3 receives a payoff of zero and the other two players receive a strictly positive payoff summing up to 1.

Now consider any state x^k such that either player 1 or player 3 receives 0 and the other two players receive a strictly positive payoff summing up to 1. We claim that any state $x^{k+1} \in f(x^k)$ has the same properties. Without loss of generality, assume that $u_3(x^k) = 0$. Let x^{k+1} be an element of $f(x^k)$ different from x^k. Since $u_1(x^k) + u_2(x^k) = 1$, the moving coalition is $\{2, 3\}$ and it holds that $\pi(x^{k+1}) = \{\{1\}, \{2, 3\}\}$. Moreover, it must also hold that $u_2(x^{k+1}) > u_2(x^k) > 0$ and $u_3(x^{k+1}) > u_3(x^k) = 0$, which proves the claim. Thus, for every $k \in \mathbb{N}$, if $x^k \in f^k(x^0) \setminus \{x^0\}$, then x^k is such that there are two players with a strictly positive payoff. It follows that there is no $k \in \mathbb{N}$ such that x^k belongs to the coalition structure core.

The definition of an MSS uses asymptotic external stability rather than iterated external stability. Theorem 4.8 shows that the MSS coincides with the coalition structure core whenever the coalition structure core is non-empty.

Theorem 4.8. Let (N, v) be a coalition function form game, Γ the social environment induced by the γ-model, and Y the coalition structure core of (N, v). If Y is non-empty, then the unique MSS of Γ is equal to Y.

4.2 Matching

As a second application, we study a number of matching models. In particular, we consider the one-to-one matching model of Gale and Shapley (1962), a different one-to-one matching model of Knuth (1976), and the housing matching model of Shapley and Scarf (1974).

Gale and Shapley (1962) introduced a one-to-one matching model for a finite set N of individuals, partitioned in the two exhaustive subgroups, men M and women W. The model can be described by a tuple $(M, W, (P_m)_{m \in M}, (P_w)_{w \in W})$ of individuals and their preference relations. A matching is a function $\mu : M \cup W \to M \cup W$ satisfying the following properties:

1. For every man $m \in M$, $\mu(m) \in W \cup \{m\}$.

2. For every women $w \in W$, $\mu(w) \in M \cup \{w\}$.

3. For all men $m \in M$ and women $w \in W$, $\mu(m) = w$ if and only if $\mu(w) = m$.

In this setting, our state space X consists of all possible matchings μ. Since X is finite, we can endow it with the discrete metric

$$d(\mu, \mu') = \mathbb{1}_{\{\mu \neq \mu'\}}.$$

Each man $m \in M$ has a complete and transitive strict preference relation P_m over the set $W \cup \{m\}$ and each woman $w \in W$ has a complete and transitive strict preference relation P_w over the set $M \cup \{w\}$. The preferences of the individuals $(\succeq_i)_{i \in M \cup W}$ over the set X are induced by their preferences over their match, i.e., for all $m \in M$ it holds that $\mu \succ_m \mu'$ if and only if $\mu(m) P_m \mu'(m)$ and for all $w \in W$ it holds that $\mu \succ_w \mu'$ if and only if $\mu(w) P_w \mu'(w)$.

The formulation of the effectivity correspondence allows us to study the consequences of different hypothesis on the matching process. We introduce two common assumptions from the literature on matching. First, every man or woman is allowed to break the link with the current partner, in which case this man or woman and the former partner become single.

(1) For all $i \in N$ and $\mu \in X$ with $\mu(i) \neq i$, we have $\{i\} \in E(\mu, \mu')$ where $\mu' \in X$ is such that

   (i) $\mu'(i) = i$,
   (ii) $\mu'(\mu(i)) = \mu(i)$,
   (iii) for every $j \in N \setminus \{i, \mu(i)\}$, $\mu'(j) = \mu(j)$.

The second assumption is that any man and woman that are currently not matched to each other can deviate by creating a link and thereby leaving their former partners single.

(2) For all $m' \in M$, $w' \in W$, and $\mu \in X$ with $\mu(m') \neq w'$, we have that $\{m', w'\} \in E(\mu, \mu')$, where $\mu' \in X$ is such that

   (i) $\mu'(m') = w'$,
   (ii) $\mu(m') \in W$ implies $\mu'(\mu(m')) = \mu(m')$,
   (iii) $\mu(w') \in M$ implies $\mu'(\mu(w')) = \mu(w')$,
   (iv) for every $j \in N \setminus \{m', w', \mu(m'), \mu(w')\}$, $\mu'(j) = \mu(j)$.

Observe that these two conditions are in line with the γ-model of coalitional sovereignty. This completes the description of the effectivity correspondence and thereby of the social environment of the model by Gale and Shapley (1962).

A matching μ is said to be stable in the matching problem $(M, W, (P_m)_{m\in M}, (P_w)_{w\in W})$ if for every $i \in M \cup W$ it does not hold that $iP_i\mu(i)$ and if for every pair $(m, w) \in M \times W$ it does not hold that $wP_m\mu(m)$ and $mP_w\mu(w)$. It can easily be shown that a matching is stable if and only if it is in the core of the social environment Γ.

In a seminal contribution to the literature, Gale and Shapley (1962) showed the existence of a stable matching. The following result of Roth and Vande Vate (1990) will be helpful in determining the relation between the set of stable matchings and the MSS.

Theorem 4.9. (Roth and Vande Vate, 1990) For every matching $\mu \in X$ there is a stable matching μ' such that $\mu' \in f^{\mathbb{N}}(\mu)$.

Since the set of states is finite in this application, it holds that $f^{\mathbb{N}}(\mu) = f^{\infty}(\mu)$. Recalling Definition 3.16, the result of Roth and Vande Vate (1990) means that Γ satisfies the weak improvement property. For finite settings, f is always lower hemi-continuous. Thus, by Theorem 3.17, the MSS of the social environment induced by the one-to-one matching model coincides with the set of stable matchings, which is the statement of the following corollary.

Corollary 4.10. Let $(M, W, (P_m)_{m\in M}, (P_w)_{w\in W})$ be a matching problem and let Γ be the induced social environment. The unique MSS of Γ is equal to the set of stable matchings.

An alternative one-to-one matching model is due to Knuth (1976). This model differs from the model of Gale and Shapley (1962) in that no individual is allowed to be single. Therefore, it requires the number of men to be equal to the number of women. If a blocking pair forms, the deserted partners are matched together. The primitives of the matching model are given by a tuple $(M, W, (P_m)_{m\in M}, (P_w)_{w\in W})$ with $|M| = |W|$. A matching is a function $\mu : M \cup W \to M \cup W$ satisfying the following properties:

1. For every man $m \in M$, $\mu(m) \in W$.

2. For every women $w \in W$, $\mu(w) \in M$.

3. For all men $m \in M$ and women $w \in W$, $\mu(m) = w$ if and only if $\mu(w) = m$.

The state space X consists of all matchings μ satisfying the above three properties and is endowed with the discrete metric $d(\mu, \mu') = \mathbb{1}_{\{\mu\neq\mu'\}}$. The preferences of the individuals $(\succeq_i)_{i\in M\cup W}$ over the set X are induced by their preferences over their match, i.e., for all $m \in M$ it holds that $\mu \succ_m \mu'$ if and only if $\mu(m)P_m\mu'(m)$ and for all $w \in W$ it holds that $\mu \succ_w \mu'$ if and only if $\mu(w)P_w\mu'(w)$. If a man and woman create a new link, the effectivity correspondence also requires a link between their deserted partners. Formally,

1. For all $m' \in M$, $w' \in W$, and $\mu \in X$ with $\mu(m') \neq w'$, we have that $\{m', w'\} \in E(\mu, \mu')$, where $\mu' \in X$ is such that

(i) $\mu'(m') = w'$,

(ii) $\mu'(\mu(w')) = \mu(m')$,

(iii) for every $j \in N \setminus \{m', w', \mu(m'), \mu(w')\}$, $\mu'(j) = \mu(j)$.

This completes the definition of the effectivity correspondence and thereby of the social environment.

For this social environment, the core is non-empty. Moreover, as demonstrated by Tamura (1993), when there are at least four women, there are preferences and a matching $\mu \in X$ such that $f^{\mathbb{N}}(\mu) = f^{\infty}(\mu)$ does not contain a stable matching. In these cases, the MSS contains matchings outside the core and can thus be rather large.

To obtain an intuition which states outside the core are part of an MSS, recall that, by Theorem 3.13, the MSS coincides with the union of all closed cycles. Thus, if the MSS contains states outside the core, these states are part of a closed cycle with more than one element. In such a cycle, agents myopically form new matches and eventually come back to the initial match. These additional states are included in the MSS due to two restrictions on the agents. First, agents are myopic and thus only consider deviations which result in an immediate gain. Second, agents are additionally restricted by the effectivity correspondence which only allows for pairwise deviations.

Another prominent matching model is the housing matching model of Shapley and Scarf (1974). This model can be represented by a tuple $(N, H, (P_i)_{i \in N})$, where N is a finite set of individuals, H is a finite set of houses with the same cardinality as the set of individuals, and each individual $i \in N$ has a strict preference relation P_i over H. The aim is to allocate the set of houses such that each house is owned by exactly one individual and no coalition can benefit by exchanging houses among themselves. An allocation is represented by a permutation matrix A with rows indexed by elements of N and columns indexed by elements of H. All entries of A are 0 or 1 and both rows and columns of A sum up to 1. If for some $h \in H$, for some $i \in N$, entry $A_{ih} = 1$, then house h has been assigned to individual i. Row $i \in N$ of the matrix A is denoted by A_i.

In this setting, it is convenient to define the state space X as the set of all permutation matrices A. Since X is finite, we can endow it with the discrete metric

$$d(A, A') = \mathbb{1}_{\{A \neq A'\}}.$$

The preferences of the individuals $(\succeq_i)_{i \in N}$ over the set X are induced by their preferences over houses. Let some individual $i \in N$ be given as well as $A, A' \in X$. Let $h, h' \in H$ be such that $A_{ih} = A'_{ih'} = 1$. Notice that h and h' are uniquely determined. It holds that $A \succ_i A'$ if and only if $h P_i h'$.

A coalition $S \in \mathcal{N}$ can redistribute its houses within the coalition, but it cannot interfere with the allocation of houses outside the coalition. More formally, the effectivity correspondence satisfies the following condition: For every $S \in \mathcal{N}$, for every $A, A' \in X$, it holds that $S \in E(A, A')$ if and only if for every $i \in N \setminus S$, $A_i = A'_i$. This completes the description of the social environment.

We define the subset Y of X as the Pareto efficient house allocations, so

$$Y = \{A^* \in X \mid \forall A \in X \setminus \{A^*\},\ \exists i \in N \text{ such that } A^* \succ_i A\}.$$

Theorem 4.11. Let $(N, H, (P_i)_{i \in N})$ be a housing matching problem and let Γ be the induced social environment. The unique MSS of Γ is equal to the set of Pareto efficient house allocations.

The proof of Theorem 4.11 consists of two steps. The first step is to show that the core of the social environment Γ is equal to the set of Pareto efficient house allocations. The second step consists of applying the top trading cycle algorithm of Shapley and Scarf (1974) to show that, starting from any initial allocation, it is possible to reach an allocation in the core of Γ in a finite number of steps. By Theorem 3.17, the MSS is then equal to the set of Pareto efficient house allocations.

4.3 Network Formation

As a third application, we look at the model of network formation by Jackson and Wolinsky (1996). A network is given by a tuple $g = (N, \mathcal{E})$, where the nodes N are the players of the network and $\mathcal{E}$ is the set of undirected edges of the network. An undirected edge is represented as a set of two distinct players. Two players $i, j \in N$ are linked in g if and only if $\{i, j\} \in \mathcal{E}$. We abuse notation and write $ij \in g$ if i and j are linked in the network g. The set of all networks with node set N is denoted by $\mathbb{G}$. A value function for player i is a function $v_i : \mathbb{G} \to \mathbb{R}$ that associates payoffs for player i for each network in $\mathbb{G}$. A network problem is thus given by $(N, \mathbb{G}, (v_i)_{i \in N})$.

We identify X with the set $\mathbb{G}$ of all possible networks on N and endow it with the discrete metric

$$d(g, g') = \mathbb{1}_{\{g \neq g'\}}.$$

Every player $i \in N$ has a preference relation $\succeq_i$ over the set X of all possible networks defined by $g \succeq_i g'$ if and only if $v_i(g) \geq v_i(g')$. Let $g + ij$ be the network obtained from network g by adding the link ij to g and let $g - ij$ be the network obtained by deleting the link ij from g.

We follow Jackson and Wolinsky (1996) by considering deviations by coalitions of size one or two and by assuming link-deletion to be one-sided and link addition to be two-sided. One-sided link deletion allows every player to delete one of his links.

(1) For all players $i \in N$, all networks $g \in X$, and all links $ij \in g$, $\{i\} \in E(g, g - ij)$.

Two-sided link addition allows any two players that are currently not linked to change the network by forming a link between themselves.

(2) For all players $i, j \in N$, all networks $g \in X$ with $ij \notin g$, we have $\{i, j\} \in E(g, g + ij)$.

This completes the description of our social environment for the network formation model. It is straightforward to adjust the effectivity correspondence to incorporate models of network formation where more than one link at a time can be changed by coalitions of arbitrary size (Dutta and Mutuswami, 1997; Jackson and van den Nouweland, 2005) or where link formation is one-sided (Bala and Goyal, 2000) into our framework. We refer to Page and Wooders (2009) for a more extensive discussion of alternative rules of network formation.

A network g is said to be pairwise stable (Jackson and Wolinsky, 1996) if for every $ij \in g$ it holds that $v_i(g - ij) \leq v_i(g)$ and $v_j(g - ij) \leq v_j(g)$ and for every $ij \notin g$ it holds that $v_i(g+ij) > v_i(g)$ implies $v_j(g+ij) \leq v_j(g)$.[2] It is not hard to show that a network is pairwise stable if and only if it is in the core of the social environment Γ as defined in Definition 3.14.

Corollary 3.15 shows that any pairwise stable network is in the myopic stable set. However, it is not necessarily the case that the MSS only contains the pairwise stable networks.

Consider the binary relation R on X defined by gRg' if $g \in f^{\mathbb{N}}(g')$, i.e., g can be reached from g' by a finite number of dominations. Let I be the symmetric part of R, i.e., gIg' if and only if gRg' and $g'Rg$. Consider the set of equivalence classes $\mathbb{E}$ induced by I. Let us denote the equivalence class of network g by $[g]$, i.e., $g' \in [g]$ if and only if $g'Ig$. For two distinct equivalence classes $[g]$ and $[g']$ write $[g]P[g']$ if gRg'. It is easy to see that $[g]P[g']$ if and only if gRg' and not gRg'.

Let V be the collection of maximal elements of $(\mathbb{E}, P)$, i.e., $[g] \in V$ if there is no $[g'] \in \mathbb{E}$ such that $[g']P[g]$. Since an element of V simply represents a closed cycle as defined in Definition 3.12, the following result follows from Theorem 3.13.

Corollary 4.12. Let $(N, \mathbb{G}, (v_i)_{i \in N})$ be a network problem and let Γ be the induced social environment. A network g belongs to the unique MSS M if and only if the equivalence class $[g]$ belongs to V, i.e., $M = \{g \in X | [g] \in V\}$.

[2]Pairwise stability as defined in Jackson and Wolinsky (1996) is somewhat stronger and also requires that there is no $ij \notin g$ such that $v_i(g+ij) > v_i(g)$ and $v_j(g+ij) = v_j(g)$. The weaker notion used here is discussed as an alternative in Section 5 of Jackson and Wolinsky (1996) and is also widely used in the literature. For generic network problems, there are no indifferences, so the two definitions are equivalent.

Herings, Mauleon, and Vannetelbosch (2009) define the pairwise myopically stable sets for network problems using the weaker notion of dominance corresponding to pairwise stability as defined in Jackson and Wolinsky (1996). It is not hard to see that the MSS for social environments Γ coincides with the pairwise myopically stable set for generic network problems. For such network problems, Corollary 4.12 is therefore equivalent to Theorem 1 of Herings, Mauleon, and Vannetelbosch (2009) that characterizes the pairwise myopically stable set as the union of closed cycles. In their paper, a closed cycle is defined in the sense of Jackson and Watts (2002) for network problems. The notion of closed cycle of Definition 3.12 is the appropriate generalization to social environments.

4.4 Normal-Form Games

As a final application, we consider normal-form games. We split our analysis into pure and mixed environments. They differ in whether players are allowed to use mixed strategies in the underlying game.

Pure Environments A normal-form game $G = (N, ((\Sigma_i, d_i), u_i)_{i \in N})$ consists of a set of players, N, and for each player a non-empty and compact metric space (Σ_i, d_i) of pure strategies and a utility function $u_i : \Sigma \to \mathbb{R}$ over the set of strategy profiles $\Sigma = \prod_{i \in N} \Sigma_i$. A typical element of Σ is denoted by s.

For the corresponding social environment $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$, we equate the state space X with the set of strategy profiles Σ and endow it with the product metric

$$d(s, s') = \sum_{i \in N} d_i(s_i, s'_i).$$

The preferences $(\succeq_i)_{i \in N}$ are such that $s \succeq_i s'$ if and only if $u_i(s) \geq u_i(s')$.

We write (s_S, s_{-S}) for the strategy profile where s_S is the list of strategies of players in coalition $S \in \mathcal{N}$ and s_{-S} is the list of strategies of all other players, i.e., $s_{-S} = (s_j)_{j \in N \setminus S}$. With a slight abuse of notation, we write s_i and $s_{-i} = (s_j)_{j \in N \setminus \{i\}}$ for single-player coalitions $S = \{i\}$.

It remains to specify the effectivity correspondence E of the social environment Γ. By allowing for all coalitional deviations, we describe coalitional normal-form games. Formally, for a coalition $S \in \mathcal{N}$, the effectivity correspondence is such that $S \in E(s, s')$ if and only if $s_{-S} = s'_{-S}$. For this effectivity correspondence, a strategy profile $s \in \Sigma$ is a strong Nash equilibrium (Aumann, 1959) if and only if it is in the core of the social environment $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$. It follows that any strong Nash equilibrium must also be in the MSS of the associated social environment.

Instead of the coalitional approach, we will henceforth restrict attention to the more frequently analyzed case of a non-cooperative game. In this case, the effectivity correspondence E only allows singletons to move. Formally, we have that $S \in E(s, s')$ if and only if $s_{-S} = s'_{-S}$ and $|S| = 1$. A non-cooperative normal-form game $G = (N, ((\Sigma_i, d_i), u_i)_{i \in N})$ then induces a social environment $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ that is identical to the social environment defined for normal-form games with coalitional moves, except that the effectivity correspondence E only allows coalitions of size one.

A strategy profile $s \in \Sigma$ is said to be a pure strategy Nash equilibrium of the game G if for every $i \in N$ and for every $s'_i \in \Sigma_i$ it holds that $u_i(s) \geq u_i(s'_i, s_{-i})$. Note that a strategy profile is a pure strategy Nash equilibrium if and only if it is in the core of the social environment Γ. Corollary 3.15 then shows that every pure strategy Nash equilibrium belongs to every MSS.

It is easy to see that the effectivity correspondence E is lower hemi-continuous. Moreover, continuity of $(\succeq_i)_{i \in N}$ is identical to continuity of the utility functions $(u_i)_{i \in N}$. As such, Theorems 3.9 and 3.17 imply the following result.

Corollary 4.13. Let $G = (N, ((\Sigma_i, d_i), u_i)_{i \in N})$ be a normal-form game and let Γ be the induced social environment. If the utility functions $(u_i)_{i \in N}$ are continuous, then the MSS of Γ is equal to the set of pure strategy Nash equilibria of G if and only if Γ satisfies the weak improvement property.

In the next step, we define pseudo-potential games (Dubey, Haimanko, and Zapechelnyuk, 2006) and prove that the weak improvement property holds for this class of games.

Definition 4.14 (Pseudo-Potential Game)**.** The game $G = (N, ((\Sigma_i, d_i), u_i)_{i \in N})$ is a pseudo-potential game if there exists a continuous function $P : \Sigma \to \mathbb{R}$ such that for all $i \in N$ and all $s \in \Sigma$,

$$\arg\max_{s_i \in \Sigma_i} u_i(s_i, s_{-i}) \supseteq \arg\max_{s_i \in \Sigma_i} P(s_i, s_{-i}).$$

Pseudo-potential games generalize ordinal potential games (Monderer and Shapley, 1996) and best-response potential games (Voorneveld, 2000). Moreover, Dubey, Haimanko, and Zapechelnyuk (2006) show that the class of pseudo-potential games contains games of strategic complements or substitutes with aggregation such as Cournot oligopoly games. Jensen (2010) extends this result to generalized quasi-aggregative games.[3]

Theorem 4.15. Let $G = (N, ((\Sigma_i, d_i), u_i)_{i \in N})$ be a pseudo-potential game and let Γ be the induced social environment. If the utility functions $(u_i)_{i \in N}$ are continuous, then the MSS of Γ coincides with the set of pure strategy Nash equilibria of G.

[3]Generalized quasi-aggregative games include aggregative games (Selten, 1970). For a subclass of aggregative games, the equivalence result in Theorem 4.15 could be obtained from Dindoš and Mezzetti (2006).

For finite supermodular games, Friedman and Mezzetti (2001) establish the weak finite improvement property which implies the weak improvement property. Thus, the equivalence between the set of pure strategy Nash equilibria and the MSS also extends to this class of games.

Mixed Environments Let $G = (N, ((\Sigma_i, d_i), u_i)_{i \in N})$ be a finite normal-form game, so for each player $i \in N$ it holds that Σ_i is finite and $d_i(s_i, s'_i) = \mathbb{1}_{\{s_i \neq s'_i\}}$.

Let us now introduce the mixed extension $\widetilde{G} = (N, ((\Delta_i, \delta_i), v_i)_{i \in N})$ of G, where Δ_i is the set of probability distributions on Σ_i. For $\sigma_i \in \Delta_i$, σ_{i,s_i} denotes the probability that player i uses pure strategy s_i. The metric δ_i on Δ_i is defined by

$$\delta_i(\sigma_i, \sigma'_i) = \max_{s_i \in \Sigma_i} |\sigma_{i,s_i} - \sigma'_{i,s_i}|.$$

We denote $\Delta = \prod_{i \in N} \Delta_i$ and endow Δ with the product metric $\delta(\sigma, \sigma') = \sum_{i \in N} \delta_i(\sigma_i, \sigma'_i)$. For a given strategy profile $\sigma \in \Delta$, we denote the probability that pure strategy profile $s \in \Sigma$ is played by $\sigma_s = \prod_{i \in N} \sigma_{i,s_i}$. Let $v_i : \Delta \to \mathbb{R}$ be the expected utility associated to strategy profiles $\sigma \in \Delta$,

$$v_i(\sigma) = \sum_{s \in \Sigma} \sigma_s u_i(s).$$

Preferences $(\succeq_i)_{i \in N}$ are such that $\sigma \succeq_i \sigma'$ if and only if $v_i(\sigma) \geq v_i(\sigma')$. The social environment $\widetilde{\Gamma} = (N, (\Delta, \delta), E, (\succeq_i)_{i \in N})$ corresponds to the game $\widetilde{G}$ where E only allows singletons to deviate and $\{i\} \in E(\sigma, \sigma')$ if and only if $\sigma_{-i} = \sigma'_{-i}$.

A strategy profile $\sigma \in \Delta$ is said to be a mixed strategy Nash equilibrium of G if it is a pure strategy Nash equilibrium of $\widetilde{G}$. The core of $\widetilde{\Gamma}$ coincides with the set of mixed strategy Nash equilibria of G. Additionally, note that the expected utility functions $(v_i)_{i \in N}$ are continuous on Δ and that E is lower hemi-continuous. As such, Theorems 3.9 and 3.17 give the following result.

Corollary 4.16. Let $\widetilde{G}$ be the mixed extension of the finite normal-form game G and let $\widetilde{\Gamma}$ be the social environment corresponding to $\widetilde{G}$. The MSS of $\widetilde{\Gamma}$ coincides with the set of mixed strategy Nash equilibria of G if and only if $\widetilde{\Gamma}$ satisfies the weak improvement property.

Clearly, the pure strategy Nash equilibria of G are also mixed strategy Nash equilibria of G, so belong to the MSS of $\widetilde{\Gamma}$. On the other hand, it is easy to find examples such that some profiles in the MSS of Γ are not in the MSS of $\widetilde{\Gamma}$.

A finite two-player game $G = (N, ((\Sigma_i, d_i), u_i)_{i \in \{1,2\}})$ is zero-sum if for all strategy profiles $s \in \Sigma$, $u_1(s) + u_2(s) = 0$. The following result shows that for such games the MSS of $\widetilde{\Gamma}$ coincides with the set of mixed strategy Nash equilibria of G.

Theorem 4.17. Let $\widetilde{G}$ be the mixed extension of a finite two-player zero-sum game G and let $\widetilde{\Gamma}$ be the social environment corresponding to $\widetilde{G}$. Then the MSS of $\widetilde{\Gamma}$ coincides with the set of mixed strategy Nash equilibria of G.

As a final result, we show the equivalence between the set of mixed strategy Nash equilibria of G and the MSS of the social environment $\widetilde{\Gamma}$ for finite two-player games where one of the two players has two pure strategies.

Theorem 4.18. Let $\widetilde{G}$ be the mixed-extension of a finite two-player game G and let $\widetilde{\Gamma}$ be the social environment corresponding to $\widetilde{G}$. Assume that one player has two pure strategies in G. Then the MSS of $\widetilde{\Gamma}$ coincides with the set of mixed strategy Nash equilibria of G.

We analyzed the game of matching pennies in Example 3.21 and concluded that better-response dynamics did not single out any strategy profile. The game of matching pennies satisfies the assumptions of both Theorems 4.17 and 4.18. The MSS of this game therefore consists of the unique mixed strategy Nash equilibrium where each pure strategy is played with probability 1/2.

5 Conclusion

The myopic stable set provides a solution concept for a wide variety of social environments. As we have shown, the setting encompasses coalition function form games, models of network formation, matching models, and non-cooperative games. These environments have been chosen based on their prominence in the literature but are by no means exhaustive. In particular, promising environments for future research on the myopic stable set include exchange processes in general equilibrium models and many-to-many matching models with transfers.

The following three features boost the appeal of the myopic stable set as a solution concept. First, the myopic stable set unifies standard solution concepts in many social environments. For instance, it coincides with the coalition structure core in coalition function form games (Kóczy and Lauwers, 2004) if the coalition structure core is non-empty, the set of stable matchings in the one-to-one matching model of Gale and Shapley (1962), the set consisting of pairwise stable networks and closed cycles of networks (Jackson and Watts, 2002), the set of pure strategy Nash equilibria in finite supermodular games (Topkis, 1979 and Milgrom and Roberts, 1990) and pseudo-potential games (Dubey, Haimanko, and Zapechelnyuk, 2006), and the set of mixed strategy Nash equilibria in two-player zero-sum-games and two-player games where one player has two actions.

Second, our solution concept exists for any social environment and—under weak continuity assumptions—provides a unique set-valued prediction. This differs from well-known concepts in the literature which fail to satisfy these properties even in social environments with more structure. In important classes of problems, the MSS gives sharp predictions. For instance, for matching markets empirical findings starting with Roth and Peranson (1999) suggest that the core is small and Ashlagi, Kanoria, and Leshno (2017) provide theoretical arguments for why this is the case. The equivalence between the core of matching problems and the MSS then implies that MSS has significant predictive power. Other examples are the equivalence between MSS and the set of pure Nash equilibria pseudo-potential games and finite supermodular games and between the MSS and the set of mixed Nash equilibria in two-player zero-sum games and in two-player normal-form games with one player having two actions.

At the same time, there are cases where the MSS may be large. For example, when the MSS contains states that do not belong to the core as in the matching model by Knuth (1976) as discussed in Section 4.2. Intuitively, the combination of myopic behavior and a restrictive effectivity correspondence may result in cycling and hence, a large MSS. This reflects the trade-off between a general solution concept for which existence and non-emptiness is guaranteed, like the MSS, and a clear prediction for every class of social environments. The investigation of refinements of the MSS in such cases is a natural direction for future research.

Appendix

Proof of Theorem 3.1: First observe that the set of states X is a QMSS. Indeed, since it is compact, it is closed and it trivially satisfies deterrence of external deviations and asymptotic external stability.

Let $\mathcal{Z}$ be the collection of all sets of states that are a QMSS. Notice that $\mathcal{Z}$ is non-empty as $X \in \mathcal{Z}$. A set $Z' \in \mathcal{Z}$ is a maximal element in the partially ordered set $(\mathcal{Z}, \supseteq)$ if for all $Z \in \mathcal{Z}$ with $Z' \supseteq Z$, we have $Z = Z'$. We will use Zorn's lemma to show the existence of a maximal element in the partially ordered set $(\mathcal{Z}, \supseteq)$.

Let $\mathcal{S}$ be a chain in $\mathcal{Z}$, i.e., $(\mathcal{S}, \supseteq)$ is a totally ordered subset of $(\mathcal{Z}, \supseteq)$. Let I be an index set for the sets in $\mathcal{S}$, i.e., $\mathcal{S} = \{Z^\alpha | \alpha \in I\}$. Let $\triangleright$ be the order on I that is induced by the order on $\mathcal{S}$, i.e., $\beta \triangleright \alpha$ if and only if $Z^\alpha \supseteq Z^\beta$. In order to apply Zorn's Lemma, we have to show that $\mathcal{S}$ has an upper bound in $\mathcal{Z}$. Let $M = \bigcap_{\alpha \in I} Z^\alpha$. Clearly, M is an upper bound of $\mathcal{S}$. We proceed by showing that $M \in \mathcal{Z}$, i.e., M is a QMSS. First of all, observe that M is closed as it is defined as the intersection of a collection of closed sets. We need to show that it satisfies deterrence of external deviations and asymptotic external stability.

Deterrence of external deviations: Let $x \in M$ and $y \notin M$ be given. Then there is $\alpha \in I$ such that $y \notin Z^\alpha$, since otherwise $y \in Z^\alpha$ for all $\alpha \in I$, which means that $y \in M$. Since

$x \in Z^\alpha$ and Z^α satisfies deterrence of external deviations, we obtain $y \notin f(x)$ as was to be shown.

Asymptotic external stability: Consider some $y \notin M$. Then there is $\alpha \in I$ such that $y \notin Z^\alpha$. As $\mathcal{S}$ is a chain, it follows that for all $\beta \rhd \alpha$ we have $y \notin Z^\beta$.

For every $\beta \rhd \alpha$, there is $x^\beta \in Z^\beta$ such that $x^\beta \in f^\infty(y)$, since Z^β satisfies asymptotic external stability. This defines a net $(x^\beta)_{\beta \rhd \alpha}$. Given that X is compact, it follows by Theorem 2.31 of Aliprantis and Border (2006) that this net has a convergent subnet, say $(x^{\beta'})_{\beta' \in I'}$, where $I' \subseteq I$ is such that for all $\beta \in I$ there is a $\beta' \in I'$ such that $\beta' \rhd \beta$. Let $\overline{x}$ be the limit of this convergent subnet. We split the remaining part of the proof in two steps. First, we show that $\overline{x} \in M$. Second, we show that $\overline{x} \in f^\infty(y)$.

Step 1: $\boldsymbol{\overline{x} \in M}$: Towards a contradiction, suppose that $\overline{x} \notin M$. Then, there exists $\gamma \in I$ such that $\overline{x} \notin Z^\gamma$. In particular, given that Z^γ is a closed set, there is $\varepsilon > 0$ such that $B_\varepsilon(\overline{x}) \cap Z^\gamma = \emptyset$. Since $\mathcal{S}$ is a chain, we have that $B_\varepsilon(\overline{x}) \cap Z^\delta = \emptyset$ for all $\delta \rhd \gamma$. Since $\overline{x}$ is the limit of the subnet $(x^{\beta'})_{\beta' \in I'}$, there is $\gamma' \in I'$ such that $\gamma' \rhd \gamma$ and $x^{\gamma'} \in B_\varepsilon(\overline{x})$. Then we have $x^{\gamma'} \in Z^{\gamma'}$, $x^{\gamma'} \in B_\varepsilon(\overline{x})$, and $B_\varepsilon(\overline{x}) \cap Z^{\gamma'} = \emptyset$, a contradiction. We conclude that $\overline{x} \in M$.

Step 2: $\boldsymbol{\overline{x} \in f^\infty(y)}$: We need to show that for every $\varepsilon > 0$ there is $k \in \mathbb{N}$ and $x \in f^k(y)$ such that $x \in B_\varepsilon(\overline{x})$.

Let some $\varepsilon > 0$ be given. The subnet $(x^{\beta'})_{\beta' \in I'}$ converges to $\overline{x}$. As such, there exists $\gamma' \in I'$ such that $x^{\gamma'} \in B_{\varepsilon/2}(\overline{x})$. In addition, $x^{\gamma'} \in f^\infty(y)$, so there is $k \in \mathbb{N}$ and $x \in f^k(y)$ such that $x \in B_{\varepsilon/2}(x^{\gamma'})$. Then, by the triangle inequality, it holds that $x \in B_\varepsilon(\overline{x})$. Together with $x \in f^k(y)$, this concludes the proof, i.e., $\overline{x} \in f^\infty(y)$. □

Proof of Theorem 3.2: Let $x \in M$ and $y \in f^\infty(x)$ and suppose, towards a contradiction, that $y \notin M$. Given that M is closed, there is $\varepsilon > 0$ such that $B_\varepsilon(y) \cap M = \emptyset$. Also, by definition, there is $k \in \mathbb{N}$ and $z \in f^k(x)$ such that $z \in B_\varepsilon(y)$, i.e. $z \notin M$. Since $z \in f^k(x)$, there is a sequence $z^0, z^1, \ldots, z^k$ of length k such that

$$z^0 = x,\ z^1 \in f(z^0),\ \ldots,\ z^k = z \in f(z^{k-1}).$$

Let $k' \in \{1, \ldots, k\}$ be such that $z^{k'}$ is the first element in this sequence with the property that $z^{k'} \notin M$. Given that $z^0 = x \in M$ and $z^k = z \notin M$, such an element exists. It holds that $z^{k'-1} \in M$, $z^{k'} \in f(z^{k'-1})$, and $z^{k'} \notin M$. This contradicts deterrence of external deviations for M. □

Proof of Theorem 3.3: Consider a state $x_1 \in M_1$. If $x_1 \in M_2$, then we are done. Otherwise, by asymptotic external stability of M_2, we know that there is $x_2 \in M_2$ such that $x_2 \in f^\infty(x_1)$. Theorem 3.2 tells us that $x_2 \in M_1$, so $x_2 \in M_1 \cap M_2$. □

The following technical lemma is helpful in proving uniqueness of an MSS, i.e. Theorem 3.6.

Lemma A.1. If the dominance correspondence $f : X \to X$ is lower hemi-continuous, then the asymptotic dominance correspondence $f^\infty : X \to X$ is transitive.

Proof. Let $x, y, z \in X$ be such that $y \in f^\infty(x)$ and $z \in f^\infty(y)$. We have to show that $z \in f^\infty(x)$, so we need to show that for every $\varepsilon > 0$, there is $k' \in \mathbb{N}$ and $z' \in f^{k'}(x)$ such that $z' \in B_\varepsilon(z)$.

By assumption, $z \in f^\infty(y)$, so there is $k \in \mathbb{N}$ and $z_1 \in f^k(y)$ such that $z_1 \in B_{\varepsilon/2}(z)$. In addition, as $y \in f^\infty(x)$, we know that for every $\ell \in \mathbb{N}$ there is $k_\ell \in \mathbb{N}$ and $y^\ell \in f^{k_\ell}(x)$ such that $y^\ell \in B_{1/\ell}(y)$. This generates a sequence $(y^\ell)_{\ell\in\mathbb{N}}$ that converges to y, i.e., $y^\ell \to y$.

Note that f^k is lower hemi-continuous, since it is a composition of k lower hemi-continuous correspondences. Given lower hemi-continuity of f^k and the fact that $z_1 \in f^k(y)$, we know that there is a sequence $(z_2^\ell)_{\ell\in\mathbb{N}}$ such that $z_2^\ell \to z_1$ and $z_2^\ell \in f^k(y^\ell)$. Now, we have that $y^\ell \in f^{k_\ell}(x)$ and $z_2^\ell \in f^k(y^\ell)$, which gives $z_2^\ell \in f^{k+k_\ell}(x)$.

We take ℓ large enough such that $z_2^\ell \in B_{\varepsilon/2}(z_1)$. Since $z_1 \in B_{\varepsilon/2}(z)$, the triangular inequality gives $z_2^\ell \in B_\varepsilon(z)$. This completes the proof. □

Proof of Theorem 3.6: Suppose not, then, by Theorems 3.1 and 3.3, there exists an MSS M_1 and an MSS M_2 such that $M_1 \neq M_2$ and their intersection $M_3 = M_1 \cap M_2$ is non-empty. Let us show that M_3 is a QMSS, contradicting the minimality of M_1 and M_2, and establishing the uniqueness of the MSS. First of all, notice that M_3, being the intersection of two closed sets, is also closed.

For deterrence of external deviations, let $x \in M_3$ and $y \in f(x)$. Then given that $x \in M_1$ and M_1 satisfies deterrence of external deviations, it must be that $y \in M_1$. Also given that $x \in M_2$ and M_2 satisfies deterrence of external deviations, it must be that $y \in M_2$. This implies that $y \in M_1 \cap M_2 = M_3$ as was to be shown.

For asymptotic external stability, take any $y \notin M_3$. There are three cases to consider.

Case 1: $\boldsymbol{y \in M_1 \setminus M_3}$: Then, by asymptotic external stability of M_2, there is $x \in M_2$ such that $x \in f^\infty(y)$. By Theorem 3.2, we have that $x \in M_1$. This means that $x \in M_1 \cap M_2 = M_3$, which is what we needed to show.

Case 2: $\boldsymbol{y \in M_2 \setminus M_3}$: The proof is symmetric to Case 1 with M_1 and M_2 interchanged.

Case 3: $\boldsymbol{y \in X \setminus (M_1 \cup M_2)}$: We know, by asymptotic external stability of M_1, that there is $x \in M_1$ such that $x \in f^\infty(y)$. If $x \in M_3$, we are done. If not, we know from Case 1 above that there is $z \in M_3$ such that $z \in f^\infty(x)$. It follows from $x \in f^\infty(y)$ and $z \in f^\infty(x)$ that $z \in f^\infty(y)$ by Lemma A.1. □

Proof of Theorem 3.9: Let $x, y \in X$ and sequences $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$ in X such that $x^k \to x$ and $y^k \to y$ be given. Let us first show that if individual $i \in N$ strictly prefers y to x, so $y \succ_i x$, then there is $\ell \in \mathbb{N}$ such that for all $k \geq \ell$, $y^k \succ_i x^k$. Suppose not, then for every $\ell \in \mathbb{N}$ we can find $k_\ell \geq \ell$ such that $x^{k_\ell} \succeq_i y^{k_\ell}$. This creates sequences $(x^{k_\ell})_{\ell\in\mathbb{N}}$, $(y^{k_\ell})_{\ell\in\mathbb{N}}$ in X with $x^{k_\ell} \to x$ and $y^{k_\ell} \to y$ such that, for every $\ell \in \mathbb{N}$, $x^{k_\ell} \succeq_i y^{k_\ell}$. By continuity of $\succeq_i$, it holds that $x \succeq_i y$, a contradiction.

Let $(x^k)_{k\in\mathbb{N}}$ be a sequence in X such that $x^k \to x$ and consider some $y \in f(x)$. Then either $y = x$ or $y \neq x$ and there is a coalition $S \in E(x, y)$ such that, for every $i \in S$, $y \succ_i x$.

If $y = x$, take the sequence $(y^k)_{k\in\mathbb{N}}$ in X defined by $y^k = x^k$. We immediately have that, for every $k \in \mathbb{N}$, $y^k \in f(x^k)$, and $y^k \to y$.

We now consider the case where $y \neq x$ and there is a coalition $S \in E(x, y)$ such that, for every $i \in S$, $y \succ_i x$. By lower hemi-continuity of the correspondence G_S, we know that there is a sequence $(y^k)_{k\in\mathbb{N}}$ such that $y^k \in G_S(x^k)$ and $y^k \to y$. By the first paragraph of the proof, we know that for every $i \in S$ there is $\ell_i \in \mathbb{N}$ such that, for every $k \geq \ell_i$, $y^k \succ_i x^k$. Let $\ell = \max_{i\in S} \ell_i$. Then, for every $k \geq \ell$, for every $i \in S$, $y^k \succ_i x^k$, and $S \in E(x^k, y^k)$, which shows that $y^k \in f(x^k)$. The sequence $(z^k)_{k\in\mathbb{N}}$ defined by $z^k = x^k$ if $k < \ell$ and $z^k = y^k$ if $k \geq \ell$ therefore has all the desired properties: for every $k \in \mathbb{N}$, $z^k \in f(x^k)$, and $z^k \to y$. $\square$

Proof of Theorem 3.13: Towards a contradiction, suppose there is a closed cycle C which is not a subset of M. Let $x \in C$ and $x \notin M$. By asymptotic external stability there is $y \in M$ such that $y \in f^\infty(x)$. By definition of a closed cycle, it follows that $y \in C$. As $x \in C$, again by definition of a closed cycle, we also have that $x \in f^\infty(y)$. By Theorem 3.2, it follows that $x \in M$, a contradiction. Since the choice of C was arbitrary, we have shown that $\text{CC} \subseteq M$.

We show next that if X is finite, then $\text{CC} = M$. Since $\text{CC} \subseteq M$, we only need to show that CC is a QMSS. The set CC satisfies deterrence of external deviations, since for all $x \in \text{CC}$, $f(x) \subseteq f^\infty(x) \subseteq \text{CC}$. It remains to verify asymptotic external stability of CC, i.e., for every state $x \notin \text{CC}$, $f^\infty(x) \cap \text{CC} \neq \emptyset$.

Let $x \notin \text{CC}$ and define $Y = f^\infty(x)$. Note that Y is non-empty since $x \in f(x)$. It also holds that Y is finite and $f^\infty(y) \subseteq Y$ for every $y \in Y$. Let us represent the set Y and the dominance correspondence f on Y by a finite directed graph D, i.e., (i) Y is the set of vertices of D and (ii) D has an arc from y to z if and only if $z \in f(y)$. By contracting each strongly connected component of D to a single vertex, we obtain a directed acyclic graph, which is called the condensation of D. As the condensation is finite and acyclic, it has a maximal element, say c. Observe that c represents a closed cycle C, so $Y \cap \text{CC} \neq \emptyset$. $\square$

Proof of Theorem 3.17: Assume that Γ satisfies the weak improvement property.

Let M be an MSS of Γ. By Corollary 3.15, we have $\mathrm{CO} \subseteq M$. We will show that CO is a QMSS. By minimality, it then follows that $\mathrm{CO} = M$.

In order to see that CO is closed, let $(x^k)_{k\in\mathbb{N}}$ be a sequence in CO, so for all $k \in \mathbb{N}$ it holds that $f(x^k) = \{x^k\}$. Now assume that $x^k \to x$ and $x \notin \mathrm{CO}$. This means that there is $y \neq x$ such that $y \in f(x)$. By lower hemi-continuity of f, there is a sequence $(y^k)_{k\in\mathbb{N}}$ such that $y^k \in f(x^k)$ and $y^k \to y$. As for all $k \in \mathbb{N}$, $x^k \in \mathrm{CO}$, we have that $y^k = x^k$, which means that $y^k \to x$, so $y = x$, a contradiction. Deterrence of external deviations is immediate for the core as it is the union of the sinks. If the social environment satisfies the weak improvement property, we have that for all $x \notin \mathrm{CO}$, $f^\infty(x) \cap \mathrm{CO} \neq \emptyset$, thus the core satisfies asymptotic external stability.

For the reverse, let M be an MSS equal to CO. Now, if $x \in M$, it is a sink, so $f^\infty(x) = \{x\} \subseteq \mathrm{CO}$. If $x \notin \mathrm{CO}$ we have by asymptotic external stability of M that $f^\infty(x) \cap M \neq \emptyset$, so $f^\infty(x)$ contains a sink, i.e., Γ satisfies the weak improvement property. □

Proof of Theorem 3.19: Suppose, towards a contradiction, that M is an MSS such that $M \cap V = \emptyset$. Recall that M is non-empty. Let $x \in M$ and $x \notin V$. Since V satisfies external stability, there is $y \in f(x)$ such that $y \in V$. Since M satisfies deterrence of external deviations, we have that $y \in M$, so $M \cap V \neq \emptyset$, a contradiction. □

Proof of Theorem 3.20: By Theorem 3.13, the MSS is unique and equal to the union of all closed cycles CC. Since Q is consistent with f, a state is recurrent if and only if it belongs to a closed cycle. □

Proof of Theorem 3.22: According to Proposition 4.2.1 of Meyn and Tweedie (1993), we have to show that for every $x \in X \setminus \{x^*\}$, for every $A \in \mathcal{B}(X \setminus \{x^*\})$ such that $\varphi(A) > 0$, there exists $k \in \mathbb{N}$ such that $Q^k(x, A) > 0$, where $Q^k(x, A)$ denotes the probability of reaching A from x in k transitions.

It is convenient to partition the set $X \setminus \{x^*\}$ in four subsets,

$$\begin{aligned}
X^1 &= \{x \in X \mid x_1 \leq \tfrac{1}{2}, x_2 > \tfrac{1}{2}\},\\
X^2 &= \{x \in X \mid x_1 > \tfrac{1}{2}, x_2 \geq \tfrac{1}{2}\},\\
X^3 &= \{x \in X \mid x_1 \geq \tfrac{1}{2}, x_2 < \tfrac{1}{2}\},\\
X^4 &= \{x \in X \mid x_1 < \tfrac{1}{2}, x_2 \leq \tfrac{1}{2}\}.
\end{aligned}$$

Let some $x \in X^4$ and some $A \in \mathcal{B}(X \setminus \{x^*\})$ such that $\varphi(A) > 0$ be given. We partition A in the four subsets $A^1 \subseteq X^1$, $A^2 \subseteq X^2$, $A^3 \subseteq X^3$, and $A^4 \subseteq X^4$. At least one of these four sets has positive Lebesgue measure. From x, the probability to reach a point in the set $Y^1 = \{y^1 \in X^1 \mid y^1_1 = x_1\}$ is at least 1/3 and the probability distribution over Y^1 is uniform.

From $y^1 \in Y^1$, the probability to reach a point in the set $Y^2(y^1) = \{y^2 \in X^2 \mid y^2_2 = y^1_2\}$ is at least $1/3$ and the probability distribution over $Y^2(y^1)$ is uniform. Thus, the probability to reach a point in X^2 after 2 transitions is at least $1/9$ and, conditional on reaching X^2, the distribution of this point is uniform on X^2. It now follows that $Q^2(x, A) \geq \varphi(A^2)/9$. Repeating this argument, we find that $Q^3(x, A) \geq \varphi(A^3)/27$, $Q^4(x, A) \geq \varphi(A^4)/81$, and $Q^5(x, A) \geq \varphi(A^1)/243$. Since at least one of A^1, A^2, A^3, and A^4 has strictly positive Lebesgue measure, we have shown that the restriction of the Markov process to $X \backslash \{x^*\}$ is φ-irreducible. An analogous argument holds for $x \in X^i$, where $i \neq 4$. □

Proof of Theorem 4.3: To show lower hemi-continuity of E, let some $S \in \mathcal{N}$, a sequence $(x^k)_{k\in\mathbb{N}}$ in X such that $x^k \to x$, and some $y \in G_S(x)$ be given. We show that there is a sequence $(y^k)_{k\in\mathbb{N}}$ such that $y^k \in G_S(x^k)$ and $y^k \to y$. If $y = x$, then the choice $y^k = x^k$ would do, so consider the case $y \neq x$.

First of all, there is $k' \in \mathbb{N}$ such that for all $k \geq k'$, $\pi(x^k) = \pi(x)$, so in particular $\mathcal{U}(x^k, S) = \mathcal{U}(x, S)$. For every $k < k'$, we define $y^k = x^k$. For every $k \geq k'$, we define $y^k \in X$ by $\pi(y^k) = \pi(y)$ and

$$u_i(y^k) = \begin{cases} u_i(y), & i \in N \setminus U(x, S), \\ u_i(x^k), & i \in U(x, S). \end{cases}$$

This completely specifies the state y^k. Consider some $k \geq k'$. Since $y \neq x$ and $y \in G_S(x)$, it holds that $S \in \pi(y)$ and, for every $i \in N \setminus (S \cup U(x, S))$, we have that i is a residual player. The properties of the γ-model imply that $\{i\} \in \pi(y)$. The same properties hold for $\pi(y^k)$. For every $i \in S$, it holds that $u_i(y^k) = u_i(y)$, so $u_i(y^k) \geq v(\{i\})$ and $\sum_{i\in S} u_i(y^k) = v(S)$. For every $i \in N \setminus (S \cup U(x, S))$, we have that $u_i(y^k) = v(\{i\}) = u_i(y)$. For every $i \in U(x, S)$ it holds that $u_i(y) = u_i(x)$ and $u_i(y^k) = u_i(x^k)$. By coalitional sovereignty, we have that $y^k \in G_S(x^k)$. Using that $x^k \to x$, it follows easily that $y^k \to y$.

Let some $i \in N$ be given. To show continuity of $\succeq_i$, let $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$ be sequences in X such that $x^k \to x$ and $y^k \to y$. Then, by continuity of the projection, we have that $u_i(x^k) \to u_i(x)$ and $u_i(y^k) \to u_i(y)$. So if $u_i(x^k) \geq u_i(y^k)$ for all $k \in \mathbb{N}$, we obtain $u_i(x) \geq u_i(y)$, which shows that $x \succeq_i y$. □

Proof of Theorem 4.6: Let Y be the coalition structure core of (N, v). Let $y \in \mathrm{CO}$ and suppose $y \notin Y$. Then there is a coalition $S \in \mathcal{N}$ such that $\sum_{i\in S} u_i(y) < v(S)$. Since $y \in X$, it holds for all $i \in S$ that $u_i(y) \geq v(\{i\})$. Now, let u_S be a vector of payoffs for the members in S such that $\sum_{i\in S} u_i = v(S)$ and for all $i \in S$, $u_i > u_i(y)$. Then, by full support, there exists a state $y' \in X$ such that $S \in E(y, y')$ and $u_S = u_S(y')$. We conclude that $y' \in f(y)$. This contradicts the fact that $y \in \mathrm{CO}$.

For the reverse, let $y \in Y$ and suppose there is $z \in f(y)$ such that $z \neq y$, i.e., $y \notin \mathrm{CO}$. Then there is $S \in E(y,z)$ such that for all $i \in S$ it holds that $u_i(z) > u_i(y)$. Also,

$$v(S) = \sum_{i \in S} u_i(z) > \sum_{i \in S} u_i(y) \geq v(S),$$

where the first equality follows from the definition of the state space and the last inequality from the definition of Y. We have obtained a contradiction. □

Proof of Theorem 4.8: From Theorem 4.3 we know that f is lower hemi-continuous. Theorem 4.6 shows that Y is equal to the core of Γ. If we can show that Γ satisfies the weak improvement property whenever $Y \neq \emptyset$, then we can use Theorem 3.17 to establish our result. Since the proof is trivial when the number of individuals n is equal to 1, we assume $n \geq 2$ throughout.

Assume that $Y \neq \emptyset$. We need to show that for all $x^0 \in X$, $f^\infty(x^0) \cap Y \neq \emptyset$. If x^0 in Y, then nothing needs to be shown, so assume that $x^0 \in X \setminus Y$. We need to show that for every $\varepsilon > 0$ there is a number $k' \in \mathbb{N}$, a state $x^{k'} \in f^{k'}(x^0)$, and a state $y \in Y$ such that $d(x^{k'}, y) < \varepsilon$.

Let some $\varepsilon > 0$ be given. Béal, Rémila, and Solal (2013) show that there exists a sequence of states $(x^0, \ldots, x^{k'})$ such that $x^{k'} \in Y$, k' is less than or equal to $(n^2+4n)/4$, and, for every $k \in \{1, \ldots, k'\}$,

1. there is $S^k \in \mathcal{N}$ such that $S^k \in E(x^{k-1}, x^k)$,
2. $u_{S^k}(x^{k-1}) < u_{S^k}(x^k)$.

Notice that the inequality in 2. only means that at least one of the players in S^k gets a strictly higher payoff, though not necessarily all of them. Let P^k be the set of partners of the players in S^k at state x^{k-1}, more formally defined as

$$P^k = \cup_{\{S \in \pi(x^{k-1}) | S \cap S^k \neq \emptyset\}} S,$$

so P^k is equal to the moving coalition S^k together with the residual players. Since $S^k \in E(x^{k-1}, x^k)$, it follows that

$$\begin{aligned} u_i(x^k) &= v(\{i\}), & i &\in P^k \setminus S^k, \\ u_i(x^k) &= u_i(x^{k-1}), & i &\in N \setminus P^k. \end{aligned}$$

We define $W^k \subset S^k$ to be the, possibly empty, proper subset of S^k consisting of players that only weakly improve when moving from state x^{k-1} to state x^k, so for every $i \in W^k$ it holds that $u_i(x^{k-1}) = u_i(x^k)$. We define

$$\begin{aligned} \delta &= \min_{k \in \{1,\ldots,k'\}} \min_{i \in S^k \setminus W^k} u_i(x^k) - u_i(x^{k-1}), \\ \varepsilon' &= \min\{\delta, \varepsilon\}, \end{aligned}$$

so δ is the smallest improvement of any of the strictly improving players involved in any move along the sequence. It holds that $\delta > 0$ and therefore that $\varepsilon' > 0$. For $k \in \{0, \ldots, k'\}$, define

$$\nu_k = \frac{n^{2k}}{n^{2k'+1}}.$$

We define $e(W^k) = 0$ if $W^k = \emptyset$ and $e(W^k) = 1$ otherwise. We use the sequence $(x^0, x^1, \ldots, x^{k'})$ of states as constructed by Béal, Rémila, and Solal (2013) to define a new sequence $(\widetilde{x}^0, \widetilde{x}^1, \ldots, \widetilde{x}^{k'})$ of states by setting $\widetilde{x}^0 = x^0$ and, for every $k \in \{1, \ldots, k'\}$,

$$\pi(\widetilde{x}^k) = \pi(x^k),$$

$$\begin{aligned}
u_i(\widetilde{x}^k) &= u_i(x^k) + \varepsilon'\nu_k \tfrac{|S^k \setminus W^k|}{|W^k|}, && i \in W^k,\\
u_i(\widetilde{x}^k) &= u_i(x^k) - \varepsilon'\nu_k e(W^k), && i \in S^k \setminus W^k,\\
u_i(\widetilde{x}^k) &= u_i(x^k) = v(\{i\}), && i \in P^k \setminus S^k,\\
u_i(\widetilde{x}^k) &= u_i(\widetilde{x}^{k-1}), && i \in N \setminus P^k.
\end{aligned}$$

Notice that the first line does not entail a division by zero, since if $i \in W^k$, then $W^k \neq \emptyset$.

Compared to the sequence $(x^0, x^1, \ldots, x^{k'})$, the sequence $(\widetilde{x}^0, \widetilde{x}^1, \ldots, \widetilde{x}^{k'})$ is such that each strictly improving player in $S^k \setminus W^k$ donates an amount $\varepsilon'\nu_k/|W^k|$ to each of the players in W^k whenever the latter set is non-empty. It is also important to observe that the fraction ν_k is an n^2 multiple of ν_{k-1} and that $\nu_{k'} = 1/n$.

We show first by induction that, for every $k \in \{0, \ldots, k'\}$, $\widetilde{x}^k \in X$. Obviously, it holds that $\widetilde{x}^0 = x^0 \in X$. Assume that, for some $k \in \{1, \ldots, k'\}$, $\widetilde{x}^{k-1} \in X$. We show that $\widetilde{x}^k \in X$. It holds that

$$\begin{aligned}
u_i(\widetilde{x}^k) &> u_i(x^k) \geq v(\{i\}), && i \in W^k,\\
u_i(\widetilde{x}^k) &\geq u_i(x^{k-1}) + \delta - \varepsilon'\nu_k > u_i(x^{k-1}) + \delta - \varepsilon' \geq u_i(x^{k-1}) \geq v(\{i\}), && i \in S^k \setminus W^k,\\
u_i(\widetilde{x}^k) &= v(\{i\}), && i \in P^k \setminus S^k,\\
u_i(\widetilde{x}^k) &= u_i(\widetilde{x}^{k-1}) \geq v(\{i\}), && i \in N \setminus P^k,
\end{aligned}$$

where the very last inequality follows from the induction hypothesis. Moreover, for every $S \in \pi(x^k)$, it holds that either $S = S^k$ and $W^k = \emptyset$, so

$$\sum_{i \in S} u_i(\widetilde{x}^k) = \sum_{i \in S^k} u_i(x^k) = v(S),$$

or $S = S^k$ and $W^k \neq \emptyset$, so

$$\sum_{i \in S} u_i(\widetilde{x}^k) = \sum_{i \in W^k} \left(u_i(x^k) + \varepsilon'\nu_k \frac{|S^k \setminus W^k|}{|W^k|} \right) + \sum_{i \in S^k \setminus W^k} \left(u_i(x^k) - \varepsilon'\nu_k \right) = \sum_{i \in S^k} u_i(x^k) = v(S),$$

or $S = \{i'\}$ with $i' \in P^k \setminus S^k$ and

$$\sum_{i \in S} u_i(\widetilde{x}^k) = u_{i'}(\widetilde{x}^k) = u_{i'}(x^k) = v(\{i'\}) = v(S),$$

or $S \subseteq N \setminus P^k$, so $S \in \pi(\widetilde{x}^{k-1})$, and

$$\sum_{i \in S} u_i(\widetilde{x}^k) = \sum_{i \in S} u_i(\widetilde{x}^{k-1}) = v(S),$$

where the last equality makes use of the induction hypothesis. We have now completed the proof of the fact that for every $k \in \{0, \ldots, k'\}$, $\widetilde{x}^k \in X$.

We show next by induction that, for every $k \in \{0, \ldots, k'\}$, and for every $i \in N$,

$$|u_i(\widetilde{x}^k) - u_i(x^k)| \leq \varepsilon' \nu_k (n-1).$$

Obviously, for every $i \in N$, it holds that $|u_i(\widetilde{x}^0) - u_i(x^0)| = 0 \leq \varepsilon' \nu_0 (n-1)$. Assume that, for some $k \in \{1, \ldots, k'\}$, for every $i \in N$, $|u_i(\widetilde{x}^{k-1}) - u_i(x^{k-1})| \leq \varepsilon' \nu_{k-1}(n-1)$. We show that, for every $i \in N$, $|u_i(\widetilde{x}^k) - u_i(x^k)| \leq \varepsilon' \nu_k (n-1)$. If $i \in W^k$, then $W^k \neq \emptyset$, and the statement follows from the observation that

$$0 \leq u_i(\widetilde{x}^k) - u_i(x^k) = \varepsilon' \nu_k \tfrac{|S^k \setminus W^k|}{|W^k|} \leq \varepsilon' \nu_k (n-1).$$

If $i \in S^k \setminus W^k$, then we have that

$$0 \geq u_i(\widetilde{x}^k) - u_i(x^k) \geq -\varepsilon' \nu_k \geq -\varepsilon' \nu_k (n-1).$$

If $i \in P^k \setminus S^k$, then we have $|u_i(\widetilde{x}^k) - u_i(x^k)| = 0$. If $i \in N \setminus P^k$, then it holds that

$$|u_i(\widetilde{x}^k) - u_i(x^k)| = |u_i(\widetilde{x}^{k-1}) - u_i(x^{k-1})| \leq \varepsilon' \nu_{k-1}(n-1) < \varepsilon' \nu_k (n-1),$$

where the first inequality makes use of the induction hypothesis and the last inequality of the fact that $\nu_{k-1} < \nu_k$.

Let some $k \in \{1, \ldots, k'\}$ and some $i \in S^k$ be given. We show that $u_i(\widetilde{x}^k) > u_i(\widetilde{x}^{k-1})$. If $i \in W^k$, then it holds that

$$\begin{aligned} u_i(\widetilde{x}^k) &= u_i(x^k) + \varepsilon' \nu_k \frac{|S^k \setminus W^k|}{|W^k|} \\ &= u_i(x^{k-1}) + \varepsilon' \nu_k \frac{|S^k \setminus W^k|}{|W^k|} \\ &\geq u_i(\tilde{x}^{k-1}) - \varepsilon' \nu_{k-1}(n-1) + \varepsilon' \nu_k \frac{1}{n-1} \\ &> u_i(\widetilde{x}^{k-1}), \end{aligned}$$

where the strict inequality uses that $\nu_k = n^2\nu_{k-1}$. If $i \in S^k \setminus W^k$, then it holds that

$$\begin{aligned} u_i(\widetilde{x}^k) &\geq u_i(x^k) - \varepsilon'\nu_k \\ &\geq u_i(x^{k-1}) + \delta - \varepsilon'\nu_k \\ &\geq u_i(\widetilde{x}^{k-1}) - \varepsilon'\nu_{k-1}(n-1) + \delta - \varepsilon' n^2 \nu_{k-1} \\ &> u_i(\widetilde{x}^{k-1}), \end{aligned}$$

where the strict inequality uses the facts that $\delta \geq \varepsilon'$ and

$$(n^2 + (n-1))\nu_{k-1} < 2n^2\nu_{k-1} \leq 2\nu_k \leq 1.$$

Combining the statements proven so far, it follows that $\widetilde{x}^{k'} \in f^{k'}(x^0)$. We complete the proof of the weak improvement property by noting that $x^{k'} \in Y$ by the result of Béal, Rémila, and Solal (2013) and by observing that $d(\widetilde{x}^{k'}, x^{k'}) < \varepsilon$ since $\pi(\widetilde{x}^{k'}) = \pi(x^{k'})$ and, for every $i \in N$,

$$|u_i(\widetilde{x}^{k'}) - u_i(x^{k'})| \leq \varepsilon'\nu_{k'}(n-1) < \varepsilon' \leq \varepsilon.$$

$\square$

PROOF OF THEOREM 4.11: To apply Theorem 3.17, we show first that the core of Γ is equal to the set of Pareto efficient house allocations Y, i.e. for every $A \in X$ it holds that $f(A) = \{A\}$ if and only if $A \in Y$.

"$\Rightarrow$" Suppose $A \notin Y$. Then there is $A' \in X$ such that, for every $i \in N$, $A' \succeq_i A$ and for some $j \in N$, $A' \succ_j A$. Let $S = \{i \in N \mid A' \succ_i A\}$ be the set of agents that strictly prefer A' to A. Since preferences are strict, for every $i \in N \setminus S$, we have $A_i = A'_i$. It therefore holds that $S \in E(A, A')$, so $A' \in f(A)$, a contradiction.

"$\Leftarrow$" Suppose $A' \in X \setminus \{A\}$ belongs to $f(A)$. Then there is $S \in \mathcal{N}$ such that $S \in E(A, A')$ and, for every $i \in S$, $A' \succ_i A$. Since for every $i \in N \setminus S$, $A_i = A'_i$, it is immediate that A' Pareto dominates A, so $A \notin Y$, a contradiction.

Since the state space X is finite, the correspondence f is lower hemi-continuous.

We complete the proof by showing that Γ satisfies the weak improvement property, so for every $A \in X$, $f^\infty(A) \cap Y \neq \emptyset$. If $A \in Y$, then evidently $f^\infty(A) \cap Y \neq \emptyset$. Consider the case $A \notin Y$. We apply the top trading cycle algorithm of Shapley and Scarf (1974) and, since preferences are assumed to be strict, we find a Pareto optimal allocation $A' \in Y$. It follows that $A' \in f^{\mathbb{N}}(A) \cap Y = f^\infty(A) \cap Y$. $\square$

PROOF OF THEOREM 4.15: Given Corollary 4.13 it suffices to show that Γ satisfies the weak improvement property.

Consider a function $b : \Sigma \to \Sigma$ such that, for every $s \in \Sigma$,

$$b(s) \in \arg\max_{\{\tau \in \Sigma \mid \exists i \in N,\ \tau_{-i} = s_{-i}\}} P(\tau),$$

and $b(s) = s$ if $P(b(s)) = P(s)$. For each strategy profile $s \in \Sigma$, there is $j \in N$ and $\tau_j^* \in \Sigma_j$ such that $b(s) = (\tau_j^*, s_{-j})$ and $b(s)$ maximizes the value of $P(\tau)$ over all strategies τ such that there is $i \in N$ with $\tau_{-i} = s_{-i}$. If the maximal value of $P(\tau)$ is equal to $P(s)$, then $b(s)$ is taken equal to s. As P is continuous and, for every $i \in N$, Σ_i is compact, the maximization problem has a solution.

Observe that $s \in \Sigma$ is a pure strategy Nash equilibrium of G if and only if

$$P(s) \geq P(\tau_i, s_{-i}), \quad i \in N,\ \tau_i \in \Sigma_i.$$

It follows that s is a pure strategy Nash equilibrium of G if and only if $b(s) = s$. Let NE be the set of all pure strategy Nash equilibria of G. The set NE is non-empty as it contains all the strategy profiles that maximize P over Σ, a non-empty set since P is continuous and Σ is compact.

Let us show that Γ satisfies the weak improvement property, i.e. for all states $s \in \Sigma$, $f^\infty(s) \cap \mathrm{NE} \neq \emptyset$.

Let $s^1 \in \Sigma \setminus \mathrm{NE}$ be given. Consider the sequence of states $(s^k)_{k\in\mathbb{N}}$ in Σ defined by

$$s^2 = b(s^1),\ s^3 = b(s^2), \ldots$$

For every $k \in \mathbb{N}$, it holds that $s^{k+1} \in f(s^k)$, so any accumulation point of the sequence $(s^k)_{k\in\mathbb{N}}$ belongs to $f^\infty(s^1)$.

Observe that, by definition, $P(s^1) \leq P(s^2) \leq P(s^3) \leq \cdots$, so the sequence $\big(P(s^k)\big)_{k\in\mathbb{N}}$ is non-decreasing. Further, as $(s^k)_{k\in\mathbb{N}}$ takes values in the compact set Σ, it has a convergent subsequence. Let us denote such a subsequence by $(x^\ell)_{\ell\in\mathbb{N}}$ and let $x^\ell \to x$. It holds that $x \in f^\infty(s^1)$. By continuity of P it holds that $P(x^\ell) \to P(x)$. It also holds that, for every $k \in \mathbb{N}$, $P(s^k) \leq P(x)$.

Since $x \in f^\infty(s_1)$, we finish the proof by showing that $x \in \mathrm{NE}$. Suppose not, then there is $j \in N$ and a best response $\tau_j' \in \Sigma_j$ such that $u_j(\tau_j', x_{-j}) > u_j(x)$. As G is a pseudo-potential game, there is a best response $\tau_j^* \in \Sigma_j$ such that

$$P(\tau_j^*, x_{-j}) > P(x).$$

Since P is continuous, there is $\varepsilon > 0$ such that for every $\tau \in B_\varepsilon((\tau_j^*, x_{-j}))$ it holds that

$$P(\tau) > P(x).$$

Since $x^\ell \to x$, there is $\ell' \in \mathbb{N}$ such that $x^{\ell'} \in B_\varepsilon(x)$, so $(\tau_j^*, x_{-j}^{\ell'}) \in B_\varepsilon((\tau_j^*, x_{-j}))$ and

$$P(\tau_j^*, x_{-j}^{\ell'}) > P(x).$$

We have that

$$P(x) < P((\tau_j^*, x_{-j}^{\ell'})) \leq P(b(s^{\ell'})) \leq P(x),$$

a contradiction. Consequently, it holds that $x \in \mathrm{NE}$. $\square$

PROOF OF THEOREM 4.17: Using Corollary 4.16, it remains to show that $\widetilde{\Gamma}$ satisfies the weak improvement property, i.e., for every strategy profile $\sigma \in \Delta$, $f^\infty(\sigma)$ contains a mixed strategy Nash equilibrium of G. Let v denote the value of the game.

Let some $\sigma \in \Delta$ be given which is not a mixed strategy Nash equilibrium of G, i.e., there is a player i such that σ_i is not a minmax strategy. We distinguish between two cases.
Case 1: σ_1 and σ_2 are not minmax strategies.

1.1 If $v_1(\sigma) \neq v$, then there exists a player i who is below his minmax payoff. Without loss of generality, let this be player 1, so $v_1(\sigma) < v$. Let (σ_1^*, σ_2^*) be a profile of minmax strategies. Note that $v_1(\sigma_1^*, \sigma_2) \geq v$. Since σ_2 is not a minmax strategy, there exists a pure strategy $s_1 \in \Delta_1$ such that $v_1(s_1, \sigma_2) > v$. Thus, for every $\varepsilon \in (0, 2]$, it holds that

$$v_1(\tfrac{\varepsilon}{2} s_1 + (1 - \tfrac{\varepsilon}{2})\sigma_1^*, \sigma_2) > v.$$

It holds that

$$v_2(\tfrac{\varepsilon}{2} s_1 + (1 - \tfrac{\varepsilon}{2})\sigma_1^*, \sigma_2^*) \geq -v,$$

so for every $\epsilon > 0$, $f^2(\sigma)$ contains a state which is in an ε-neighborhood of a mixed strategy Nash equilibrium of G, and therefore $f^\infty(\sigma)$ contains a mixed strategy Nash equilibrium of G.

1.2 Suppose $v_1(\sigma) = v$. Then, there exists a pure strategy $s_1 \in \Delta_1$ such that

$$v_1(s_1, \sigma_2) > v,$$

since otherwise σ_2 would be a minmax strategy. If s_1 is a minmax strategy, then player 2 can deviate to a minmax strategy σ_2^* to obtain $v_2(s_1, \sigma_2^*) = -v$, i.e., $f^2(\sigma)$ contains a mixed strategy Nash equilibrium of G. If s_1 is not a minmax strategy, then $(s_1, \sigma_2) \in f_1(\sigma)$ is a state as in Case 1.1, so for every $\varepsilon > 0$, $f^3(\sigma)$ contains a state which is in a ε-neighborhood of a mixed strategy Nash equilibrium of G, and therefore $f^\infty(\sigma)$ contains a mixed strategy Nash equilibrium of G.

Case 2: σ_1 is a minmax strategy and σ_2 is not, or σ_1 is not a minmax strategy and σ_2 is. Without loss of generality, assume σ_1 is a minmax strategy.

2.1 If $v_1(\sigma) > v$, then player 2 can profitably switch to a minmax strategy σ_2^* and we are done.

2.2 If $v_1(\sigma) = v$, then since σ_2 is not a minmax strategy, there exists a deviation to a pure strategy $s_1 \in \Delta_1$ such that $v_1(s_1, \sigma_2) > v$. If s_1 is a minmax strategy, then $(s_1, \sigma_2) \in f_1(\sigma)$ is a state as in Case 2.1, so $f_2(\sigma)$ contains a mixed strategy Nash equilibrium of G. If s_1 is not a minmax strategy, then $(s_1, \sigma_2) \in f_1(\sigma)$ is a state as in Case 1.1, and for every $\varepsilon > 0$ it holds that $f^3(\sigma)$ contains a state which is in an ε-neighborhood of a mixed strategy Nash equilibrium of G, so $f^\infty(\sigma)$ contains a mixed strategy Nash equilibrium of G.

□

Proof of Theorem 4.18: Assume without loss of generality that player 1 has two pure strategies. Let the set of pure strategies of player 1 be $\{U, D\}$ with generic element $A \in \{U, D\}$ and let the set of pure strategies of player 2 be given by $\{s^1, \ldots, s^\ell\}$ with generic element s^j. We also use the notation U and D for the mixed strategy that puts probability 1 on pure strategy U and D, respectively, and similarly for s^j.

Let some $\sigma \in \Delta$ be given. By Corollary 4.16, it suffices to show the weak improvement property of $\widetilde{\Gamma}$, i.e., $f^\infty(\sigma)$ contains a mixed strategy Nash equilibrium of G. We distinguish between two cases.

Case 1: G has a pure strategy Nash equilibrium, without loss of generality, (U, s^*).

If σ is a mixed strategy Nash equilibrium of G, we are done, so assume σ is not a mixed strategy Nash equilibrium of G. If player 2 has a profitable deviation from σ, then there is a pure strategy best response $s^j \in \Delta_2$ such that $(\sigma_1, s^j) \in f(\sigma)$. If (σ_1, s^j) is a mixed strategy Nash equilibrium of G, we are done. If not, then player 1 must have a pure strategy best response to (σ_1, s^j), say A. Thus, $f^2(\sigma)$ contains a mixed strategy Nash equilibrium of G or a pure strategy profile (A, s^j). The same conclusion holds if player 1 has a profitable deviation from σ. If the pure strategy profile (A, s^j) is a Nash equilibrium of G, we are done. If not, at least one player has a profitable deviation from it. We distinguish between two cases.

1.1 $A = D$.

- 1.1.a Assume player 1 can profitably deviate from (D, s^j). Then it holds that $(U, s^j) \in f(D, s^j)$. If (U, s^j) is a Nash equilibrium of G, we are done. If not, then player 2 can profitably deviate to the Nash equilibrium (U, s^*) of G and we are done.
- 1.1.b Assume player 2 can profitably deviate from (D, s^j). Let s^h be a best response for player 2, so $(D, s^h) \in f(D, s^j)$. If this is a Nash equilibrium of G, we are done. Otherwise, player 1 can profitably deviate to (D, s^h), which brings us back to Case 1.1.a.

1.2 $A = U$.

1.2.a Assume player 2 can profitably deviate form (U, s^j). It holds that the Nash equilibrium (U, s^*) of G belongs to $f(U, s^j)$, so we are done.

1.2.b Assume player 1 can profitably deviate from (U, s^j). Then it holds that $(D, s^j) \in f(U, s^j)$. If (D, s^j) is a Nash equilibrium of G, then we are done. Else, player 2 must have a profitable deviation from (D, s^j), which brings us back to Case 1.1.b.

Case 2: G has no pure strategy Nash equilibrium.

We first show that in every mixed strategy Nash equilibrium of G, player 1 plays both U and D with strictly positive probability. Towards a contradiction, suppose there is a mixed strategy Nash equilibrium (A, σ_2^*) of G such that player 1 plays a pure strategy, without loss of generality, strategy $A = U$. It holds that any pure strategy of player 2 in the support of σ_2^* is a best response against U. Since G has no pure strategy Nash equilibrium, it must hold that playing D against any pure strategy in the support of σ_2^* gives player 1 a strictly higher payoff than playing U. It follows that D is a profitable deviation for player 1 from (U, σ_2^*). This contradicts (U, σ_2^*) being a mixed strategy Nash equilibrium of G.

To finish the proof, we show that $f^\infty(\sigma)$ contains a mixed strategy Nash equilibrium of G. As in the first part of Case 1, we can show that $f^2(\sigma)$ contains a mixed strategy Nash equilibrium of G and we are done, or a pure strategy profile which is not a Nash equilibrium of G. Player 1 or player 2 has a profitable deviation from this pure strategy profile. In the latter case, player 2 can choose a pure strategy best response and in the next step, player 1 can profitably deviate to a pure strategy. In both cases it holds that there is $k \in \mathbb{N}$ such that $f^k(\sigma)$ contains a pure strategy profile (A, s^j) from which player 1 has a profitable deviation. Without loss of generality, let $A = U$.

Observe that for player 1 any completely mixed strategy is a profitable deviation from (U, s^j). Let σ^* be a mixed strategy Nash equilibrium of G and let $p \in (0, 1)$ denote the probability that σ_1^* puts on U. We distinguish 3 cases.

2.1 $v_2(D, \sigma_2^*) - v_2(U, \sigma_2^*) > v_2(D, s^j) - v_2(U, s^j)$.
For $\varepsilon \in (0, p)$, let σ_1' be the strategy where player 1 plays U with probability $p - \varepsilon/2$. Since any completely mixed strategy of player 1 is a profitable deviation from (U, s^j), it holds that $(\sigma_1', s^j) \in f(U, s^j)$. We have that

$$\begin{aligned} v_2(\sigma_1', s^j) &= v_2(\sigma_1^*, s^j) + \tfrac{\varepsilon}{2}(v_2(D, s^j) - v_2(U, s^j)) \\ &< v_2(\sigma^*) + \tfrac{\varepsilon}{2}(v_2(D, \sigma_2^*) - v_2(U, \sigma_2^*)) \\ &= v_2(\sigma_1', \sigma_2^*), \end{aligned}$$

where the strict inequality uses that σ_2^* is a best response against σ_1^* and the assumption of Case 2.1. It follows that $(\sigma_1', \sigma_2^*) \in f(\sigma_1', s^j)$. Since $\varepsilon > 0$ can be chosen arbitrarily small, this shows that $\sigma^* \in f^\infty(\sigma)$.

2.2 $v_2(D, \sigma_2^*) - v_2(U, \sigma_2^*) < v_2(D, s^j) - v_2(U, s^j)$.
For $\varepsilon \in (0, 1-p)$, let σ_1' be the strategy where player 1 plays U with probability $p+\varepsilon/2$. The proof now follows as in Case 2.1.

2.3 $v_2(D, \sigma_2^*) - v_2(U, \sigma_2^*) = v_2(D, s^j) - v_2(U, s^j)$.
It holds that $(D, s^j) \in f(U, s^j)$.

Let s^h be a best response of player 2 against D and, for $\varepsilon \in (0, 1)$, let σ_2' be the strategy that puts weight $(1-\varepsilon)$ on σ_2^* and weight ε on s^h. We have that

$$\begin{aligned} v_2(D, \sigma_2^*) &= v_2(\sigma^*) + pv_2(D, \sigma_2^*) - pv_2(U, \sigma_2^*) \\ &\geq v_2(\sigma_1^*, s^j) + pv_2(D, s^j) - pv_2(U, s^j) \\ &= v_2(D, s^j), \end{aligned} \tag{A.1}$$

where the inequality uses that σ^* is a mixed strategy Nash equilibrium of G and the assumption of Case 2.3. Since (D, s^j) is not a Nash equilibrium of G, it holds that $v_2(D, s^h) > v_2(D, s^j)$. By (A.1) and the definition of σ_2', it now follows that $v_2(D, \sigma_2') > v_2(D, s^j)$, so $(D, \sigma_2') \in f(D, s^j)$. Since (D, s^h) is not a Nash equilibrium of G and s^h is a best response against D, we have that $v_1(\sigma_1^*, s^h) > v_1(D, s^h)$. It follows that

$$v_1(\sigma_1^*, \sigma_2') = (1-\varepsilon)v_1(\sigma^*) + \varepsilon v_1(\sigma_1^*, s^h) > (1-\varepsilon)v_1(D, \sigma_2^*) + \varepsilon v_1(D, s^h) = v_1(D, \sigma_2'),$$

so $(\sigma_1^*, \sigma_2') \in f(D, \sigma_2')$. Since $\varepsilon > 0$ can be chosen arbitrarily small, we have that $\sigma^* \in f^\infty(\sigma)$, which concludes the proof.

□

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