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Abstract

This paper identifies the maximal domain of transferable utility games on which population monotonicity (no player is worse off when additional players enter the game) and egalitarian core selection (no other core allocation can be obtained by a transfer from a richer to a poorer player) are compatible, which is the class of games with an egalitarian population monotonic allocation scheme. On this domain, which strictly includes the class of convex games, population monotonicity and egalitarian core selection together characterize the Dutta-Ray solution. We relate the class of games with an egalitarian population monotonic allocation scheme to several other classes of games.

Keywords: population monotonicity, egalitarian core, Dutta-Ray solution
JEL classification: C71

1 Introduction

In a *transferable utility game*, players may form coalitions and obtain joint revenues by cooperation. A characteristic function assigns to each possible coalition its *worth* reflecting the economic opportunities when this coalition would form. A *solution* assigns to a game an *allocation* for the grand coalition consisting of all players together. A solution satisfies *population monotonicity* if no player is worse off when additional players enter the game. This elementary solidarity property was originally introduced and studied in the context of bargaining problems by Thomson (1983a) and Thomson (1983b), but is straightforwardly reformulated on the domain of transferable utility games.

Rosenthal (1990) showed that the famous *Shapley value* (cf. Shapley 1953), which assigns to each player a specific weighted average of its marginal contributions such that the full worth of the grand coalition is allocated, satisfies population monotonicity on the specific class of *convex games* (cf. Shapley 1971). On this class, the Shapley value selects from the *core*, i.e. subcoalitions are not better off by allocating their worth among their members.

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If a solution satisfies population monotonicity and core selection, then it induces for each game a *population monotonic allocation scheme* (cf. Sprumont 1990). Such a scheme, describing for each coalition how to fully allocate the worth among its members such that each player's payoff is nondecreasing as coalitions grow, is obtained by applying the corresponding solution to each subgame. Moulin (1990) introduced the population monotonic core, i.e. the set of all population monotonic allocation schemes of a game. Sprumont (1990) described games with a population monotonic allocation scheme as positive linear combinations of monotonic simple games with veto control. A dual description was provided by Norde and Reijnierse (2002). The class of games with a population monotonic allocation scheme contains the class of convex games, where the Shapley value satisfies population monotonicity and core selection, and consequently induces for each convex game a population monotonic allocation scheme. Getán et al. (2014) characterized convex games by means of population monotonic allocation schemes. Existence of a population monotonic allocation scheme, i.e. a nonempty population monotonic core, is a necessary condition for a game to allow for a solution that satisfies population monotonicity and core selection. Remarkably however, we show that population monotonicity and core selection are incompatible on the class of games with a population monotonic allocation scheme.

The economic doctrine of egalitarianism is characterized by the belief that all humans are fundamentally equal in terms of value, status, and rights. A common evaluation for distributional egalitarianism is the Lorenz criterion. An allocation *Lorenz dominates* another allocation if it assigns to each subgroup of the poorest at least what the other does. The Lorenz dominating allocation of a single amount of money would simply be equal division. This is considered extreme in the context of cooperative games, where it ignores any of the potential roles that players may take in other coalitions. A well-known compromise between plain egalitarianism and coalitional externalities in transferable utility games is the Dutta-Ray solution (cf. Dutta and Ray 1989). Based on a framework where individuals believe in equality as a desirable social goal, although private preferences dictate selfish behavior, this solution assigns to each game the Lorenz undominated allocation in the Lorenz core. Although such allocation is unique whenever it exists, only sufficient conditions for existence are known, the central one being convexity of the underlying game.

For convex games, the Dutta-Ray solution prescribes the Lorenz dominating core allocation. Hougaard et al. (2001) studied the set of all Lorenz undominated core allocations for general games with a nonempty core, to which we refer as the *strong egalitarian core*. A further extension is the *egalitarian core* introduced by Arin and Iñarra (2001), which consists of all core allocations from which no other core allocation can be obtained by a transfer from a richer to a poorer player. On the class of convex games, egalitarian core selection characterizes the Dutta-Ray solution. Other axiomatic characterizations of the Dutta-Ray solution for convex games were provided by Dutta (1990), Klijn et al. (2000), Arin et al. (2003), and Calleja et al. (2021a).

Dutta (1990) showed that the Dutta-Ray solution for convex games satisfies population monotonicity. Several generalizations of the Dutta-Ray solution on the class of convex games have been proposed in the literature. The sequential Dutta-Ray solutions (cf. Hokari 2000), the monotone-path Dutta-Ray solutions (cf. Hokari 2002), and the generalized Lorenz solutions (cf. Hougaard et al. 2005) all satisfy population monotonicity.

Llerena and Mauri (2017) introduced the larger class of *exact partition games* and shows that the Dutta-Ray solution for exact partition games behaves as for convex games. In line with this statement, Dietzenbacher and Yanovskaya (2020) and Dietzenbacher and Yanovskaya (2021) showed that the Dutta-Ray solution for exact partition games inherits many properties, structures, and axiomatizations from the class of convex games. Recently, Dietzenbacher and Yanovskaya (2023) generalized this to exact partition games with non-transferable utility. However, we show that the Dutta-Ray solution for exact partition games does not satisfy population monotonicity.

This paper studies the compatibility of population monotonicity and egalitarian core selection. Dietzenbacher (2021) identified the maximal domain of transferable utility games on which aggregate monotonicity and egalitarian core selection are compatible. In the same spirit, the current paper identifies the maximal domain of transferable utility games on which population monotonicity and egalitarian core selection are compatible. This turns out to be the class of games with an *egalitarian population monotonic allocation scheme*. In other words, existence of such a scheme is a necessary and sufficient condition for a game to allow for a solution that satisfies population monotonicity and egalitarian core selection. The class of games with an egalitarian population monotonic allocation scheme contains the class of convex games and is contained in the class of exact partition games. Interestingly, on this class, population monotonicity and egalitarian core selection together characterize the Dutta-Ray solution. This means that if an egalitarian population monotonic allocation scheme exists, it is unique and can be obtained by applying the Dutta-Ray solution to each subgame.

This paper is organized as follows. Section 2 provides some preliminary notions and notations for transferable utility games. Section 3 studies the compatibility of population monotonicity and egalitarian core selection. Section 4 relates this to aggregate monotonicity. Section 5 formulates some concluding remarks and suggestions for future research.

2 Preliminaries

Let N be a nonempty and finite set. Denote $2^N = \{S \mid S \subseteq N\}$. An *allocation* $x \in \mathbb{R}^N$ describes a payoff $x_i \in \mathbb{R}$ for each $i \in N$. An *allocation scheme* $(x^S)_{S \in 2^N \setminus \{\emptyset\}}$ describes an allocation $x^S \in \mathbb{R}^S$ for each $S \in 2^N \setminus \{\emptyset\}$. For each allocation $x \in \mathbb{R}^N$, define $R_0^x = \emptyset$ and $R_k^x = \{i \in N \mid \forall_{j \in N \setminus R_{k-1}^x} : x_j \leq x_i\}$ for all $k \in \mathbb{N}$. Then $R_{k-1}^x \subseteq R_k^x$ for all $k \in \mathbb{N}$, and $R_k^x = N$ if $k \geq |N|$. For each two allocations $x, y \in \mathbb{R}^N$ with $\sum_{i \in N} x_i = \sum_{i \in N} y_i$, x *Lorenz dominates* y, denoted by $x \succ_L y$, if $\min_{S \in 2^N : |S|=k} \sum_{i \in S} x_i \geq \min_{S \in 2^N : |S|=k} \sum_{i \in S} y_i$ for each $k \in \{1, \ldots, |N|\}$, with at least one strict inequality.

A *transferable utility game* is a pair (N, v), where N is a nonempty and finite set of *players* and $v : 2^N \to \mathbb{R}$ assigns to each *coalition* $S \in 2^N$ its *worth* $v(S) \in \mathbb{R}$ with $v(\emptyset) = 0$. The *subgame* (T, v_T) of (N, v) on $T \in 2^N \setminus \{\emptyset\}$ is defined by $v_T(S) = v(S)$ for all $S \in 2^T$. Let Γ_{all} denote the class of all games. Throughout this paper, Γ denotes a generic class of games.

Let $(N, v) \in \Gamma_{all}$. The *core* $C(N, v) \subseteq \mathbb{R}^N$ consists of all allocations of the worth of the grand coalition such that no coalition could be better off by itself, i.e.

$$C(N, v) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i \in N} x_i = v(N), \forall_{S \in 2^N} : \sum_{i \in S} x_i \geq v(S) \right\}.$$

The *egalitarian core* $EC(N, v) \subseteq \mathbb{R}^N$ (cf. Arin and Iñarra 2001) consists of all core allocations from which no other core allocation can be obtained by a transfer from a richer to a poorer player, i.e.

$$EC(N, v) = \left\{ x \in C(N, v) \,\middle|\, \forall_{i,j \in N : x_i > x_j} \exists_{S \in 2^N : i \in S, j \notin S} : \sum_{h \in S} x_h = v(S) \right\}.$$

The *strong egalitarian core* $SEC(N, v) \subseteq \mathbb{R}^N$ (cf. Hougaard et al. 2001) consists of all core allocations from which no other core allocation can be obtained by a sequence of transfers from a richer to a poorer player, i.e.

$$SEC(N, v) = \left\{ x \in C(N, v) \,\middle|\, \forall_{y \in C(N,v)} : y \not\succ_L x \right\}.$$

Note that $SEC(N, v) \subseteq EC(N, v) \subseteq C(N, v)$. Moreover, $SEC(N, v) \neq \emptyset$ if $C(N, v) \neq \emptyset$.

Let $(N, v) \in \Gamma_{all}$. The game (N, v) is *balanced* if $C(N, v) \neq \emptyset$, and *totally balanced* if $C(T, v_T) \neq \emptyset$ for all $T \in 2^N \setminus \{\emptyset\}$. The game (N, v) is an *exact partition game* (cf. Llerena and Mauri 2017) if there exists $x \in C(N, v)$ such that $\sum_{i \in R_k^x} x_i = v(R_k^x)$ for all $k \in \mathbb{N}$. The game (N, v) is *convex* (cf. Shapley 1971) if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \in 2^N$. Let Γ_{bal}, Γ_{totbal}, Γ_{exp}, and Γ_{conv} denote the classes of balanced games, totally balanced games, exact partition games, and convex games, respectively. Then $\Gamma_{conv} \subset \Gamma_{totbal} \subset \Gamma_{bal} \subset \Gamma_{all}$ and $\Gamma_{conv} \subset \Gamma_{exp} \subset \Gamma_{bal} \subset \Gamma_{all}$.

A *solution* f on Γ assigns to each game $(N,v) \in \Gamma$ an allocation $f(N,v) \in \mathbb{R}^N$. Throughout this paper, f denotes a generic solution.

A solution satisfies *population monotonicity* if no player is worse off when additional players enter the game.

Population monotonicity
for all $(N,v) \in \Gamma$ and all $T \in 2^N \setminus \{\emptyset\}$, we have $(T, v_T) \in \Gamma$ and

$$f_i(T, v_T) \leq f_i(N,v) \text{ for all } i \in T.$$

A solution satisfies *core selection* if it assigns an element of the core, *egalitarian core selection* if it assigns an element of the egalitarian core, and *strong egalitarian core selection* if it assigns an element of the strong egalitarian core.

Core selection
for all $(N,v) \in \Gamma$, we have $f(N,v) \in C(N,v)$.

Egalitarian core selection
for all $(N,v) \in \Gamma$, we have $f(N,v) \in EC(N,v)$.

Strong egalitarian core selection
for all $(N,v) \in \Gamma$, we have $f(N,v) \in SEC(N,v)$.

Note that strong egalitarian core selection implies egalitarian core selection, and egalitarian core selection implies core selection. If a solution satisfies *efficiency*, i.e. $\sum_{i \in N} f_i(N,v) = v(N)$ for all $(N,v) \in \Gamma$, then population monotonicity implies core selection. We have deliberately excluded efficiency from the definition of a solution to separate these two properties.

The Dutta-Ray solution DR on Γ_{exp} (cf. Dutta and Ray 1989) assigns to each exact partition game $(N,v) \in \Gamma_{exp}$ the unique allocation[1]

$$DR(N,v) \in \left\{ x \in C(N,v) \;\middle|\; \forall_{k \in \mathbb{N}} : \sum_{i \in R_k^x} x_i = v(R_k^x) \right\}.$$

Llerena and Mauri (2017) showed that the Dutta-Ray solution for exact partition games is characterized by strong egalitarian core selection. As the following example shows, it is not characterized by egalitarian core selection.

[1] Llerena and Mauri (2017) showed that this definition is equivalent to the original definition of Dutta and Ray (1989).

Example 1
Let $N = \{1,2,3,4\}$ and let $(N,v) \in \Gamma_{totbal} \cap \Gamma_{exp}$ be given by

$$v(S) = \begin{cases} 4 & \text{if } S = \{1,2,3,4\}; \\ 2 & \text{if } S \in \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\} \text{ or } |S| = 3; \\ 0 & \text{otherwise.} \end{cases}$$

Then $EC(N,v) = \text{conv}(\{(2,2,0,0),(0,0,2,2)\})$ and $SEC(N,v) = \{(1,1,1,1)\}$. This means that, in contrast to strong egalitarian core selection, egalitarian core selection does not characterize a unique solution on the class of exact partition games. $\triangle$

3 Population monotonicity and egalitarianism

This section studies the compatibility of population monotonicity and egalitarian core selection for solutions for transferable utility games. If a solution satisfies population monotonicity and egalitarian core selection, then it satisfies population monotonicity and core selection. If a solution satisfies population monotonicity and core selection, then it induces for each game a *population monotonic allocation scheme* (cf. Sprumont 1990).

Population monotonic allocation scheme
A game $(N,v) \in \Gamma_{all}$ is a game with a *population monotonic allocation scheme* if there exists $(x^S)_{S \in 2^N \setminus \{\emptyset\}}$ with $x^S \in C(S, v_S)$ for all $S \in 2^N \setminus \{\emptyset\}$ such that for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$, we have

$$x_i^S \leq x_i^T \text{ for all } i \in S.$$

Let Γ_{pmas} denote the class of games with a population monotonic allocation scheme. Then $\Gamma_{conv} \subset \Gamma_{pmas} \subset \Gamma_{totbal} \subset \Gamma_{bal} \subset \Gamma_{all}$. As in Example 1, totally balanced games and exact partition games are not necessarily games with a population monotonic allocation scheme. Moreover, as the following example shows, population monotonic allocation schemes are not necessarily unique.

Example 2
Let $N = \{1,2,3\}$ and let $(N,v) \in \Gamma_{pmas}$ with two of its population monotonic allocation schemes be given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	2	2	0	3
x^S	$(0,\cdot,\cdot)$	$(\cdot,0,\cdot)$	$(\cdot,\cdot,0)$	$(1,1,\cdot)$	$(1,\cdot,1)$	$(\cdot,0,0)$	$(1,1,1)$
x^S	$(0,\cdot,\cdot)$	$(\cdot,0,\cdot)$	$(\cdot,\cdot,0)$	$(2,0,\cdot)$	$(2,\cdot,0)$	$(\cdot,0,0)$	$(3,0,0)$

$\triangle$

If a solution satisfies population monotonicity and core selection, then it induces for each game a population monotonic allocation scheme. Remarkably, as the following example shows, population monotonicity and core selection are not compatible on the class of games with a population monotonic allocation scheme.

Example 3
Let $N = \{1, 2, 3\}$ and let $(N, v), (N, v') \in \Gamma_{pmas}$ with their unique population monotonic allocation schemes be given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	12	12	0	12
x^S	$(0,\cdot,\cdot)$	$(\cdot,0,\cdot)$	$(\cdot,\cdot,0)$	$(12,0,\cdot)$	$(12,\cdot,0)$	$(\cdot,0,0)$	$(12,0,0)$

and

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v'(S)$	0	0	0	12	0	12	12
x^S	$(0,\cdot,\cdot)$	$(\cdot,0,\cdot)$	$(\cdot,\cdot,0)$	$(0,12,\cdot)$	$(0,\cdot,0)$	$(\cdot,12,0)$	$(0,12,0)$

Let f be a solution on Γ_{pmas} satisfying population monotonicity and core selection. By core selection, $f(N, v) = (12, 0, 0)$ and $f(N, v') = (0, 12, 0)$. By population monotonicity and core selection, $f(\{1, 2\}, v_{\{1,2\}}) = (12, 0, \cdot)$ and $f(\{1, 2\}, v'_{\{1,2\}}) = (0, 12, \cdot)$. However, $v_{\{1,2\}} = v'_{\{1,2\}}$ so this is not possible. This means that population monotonicity and core selection are incompatible on the class of games with a population monotonic allocation scheme. △

Similarly, if a solution satisfies population monotonicity and egalitarian core selection, then it induces for each game an *egalitarian population monotonic allocation scheme.*

Egalitarian population monotonic allocation scheme
A game $(N, v) \in \Gamma_{all}$ is a game with an *egalitarian population monotonic allocation scheme* if there exists $(x^S)_{S \in 2^N \setminus \{\emptyset\}}$ with $x^S \in EC(S, v_S)$ for all $S \in 2^N \setminus \{\emptyset\}$ such that for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$, we have

$$x_i^S \leq x_i^T \text{ for all } i \in S.$$

Let Γ_{epmas} denote the class of games with an egalitarian population monotonic allocation scheme. Clearly, all games with an egalitarian population monotonic allocation scheme are games with a population monotonic allocation scheme. However, as in Example 3, games with a population monotonic allocation scheme are not necessarily games with an egalitarian population monotonic allocation scheme.

Example 3 (continued)
Let $(x^S)_{S \in 2^N \setminus \{\emptyset\}}$ with $x^S \in EC(S, v_S)$ for all $S \in 2^N \setminus \{\emptyset\}$. Then $(x^S)_{S \in 2^N \setminus \{\emptyset\}}$ is given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	12	12	0	12
x^S	$(0,\cdot,\cdot)$	$(\cdot,0,\cdot)$	$(\cdot,\cdot,0)$	$(6,6,\cdot)$	$(6,\cdot,6)$	$(\cdot,0,0)$	$(12,0,0)$

Then $x_2^{\{1,2\}} > x_2^{\{1,2,3\}}$ and $x_3^{\{1,3\}} > x_3^{\{1,2,3\}}$. This means that (N, v) is not a game with an egalitarian population monotonic allocation scheme, i.e. $(N, v) \notin \Gamma_{epmas}$. $\triangle$

On the class of convex games, the Dutta-Ray solution satisfies population monotonicity and egalitarian core selection. This means that the Dutta-Ray solution induces for each convex game an egalitarian population monotonic allocation scheme. However, as in Example 2, games with an egalitarian population monotonic allocation scheme are not necessarily convex.

Example 2 (continued)
The unique egalitarian population monotonic allocation scheme is given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	2	2	0	3
x^S	$(0,\cdot,\cdot)$	$(\cdot,0,\cdot)$	$(\cdot,\cdot,0)$	$(1,1,\cdot)$	$(1,\cdot,1)$	$(\cdot,0,0)$	$(1,1,1)$

We have $v(\{1,2\}) + v(\{1,3\}) > v(\{1,2,3\}) + v(\{1\})$. This means that $(N, v) \in \Gamma_{epmas}$ but (N, v) is not convex, i.e. $(N, v) \notin \Gamma_{conv}$. $\triangle$

We have $\Gamma_{conv} \subset \Gamma_{epmas} \subset \Gamma_{pmas} \subset \Gamma_{totbal} \subset \Gamma_{bal} \subset \Gamma_{all}$. Population monotonicity cannot be satisfied on the class of exact partition games since this class is not closed under subgames, i.e. subgames of exact partition games are not necessarily exact partition games. Nevertheless, we could take the closure with respect to subgames and define the class of *total exact partition games.*

Total exact partition games
A game $(N, v) \in \Gamma_{all}$ is a *total exact partition game* if $(S, v_S) \in \Gamma_{exp}$ for all $S \in 2^N \setminus \{\emptyset\}$.

Let Γ_{totexp} denote the class of total exact partition games. It turns out that the class of games with an egalitarian population monotonic allocation scheme is also contained in the class of total exact partition games.

Lemma 3.1
All games with an egalitarian population monotonic allocation scheme are total exact partition games.

Proof. Let $(N,v) \in \Gamma_{epmas}$. Let $(x^S)_{S \in 2^N \setminus \{\emptyset\}}$ with $x^S \in EC(S, v_S)$ for all $S \in 2^N \setminus \{\emptyset\}$ be such that for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$, we have $x_i^S \leq x_i^T$ for all $i \in S$. For the sake of a proof by induction, assume that for all $S \in 2^N \setminus \{\emptyset, N\}$, we have $\sum_{i \in R_k^{x^S}} x_i^S = v(R_k^{x^S})$ for all $k \in \mathbb{N}$, so $(S, v_S) \in \Gamma_{exp}$. For the sake of a proof by induction, let $k \in \mathbb{N}$ and assume that $\sum_{i \in R_{k-1}^{x^N}} x_i^N = v(R_{k-1}^{x^N})$. If $R_k^{x^N} = N$, then $\sum_{i \in R_k^{x^N}} x_i^N = v(R_k^{x^N})$. Suppose that $R_k^{x^N} \neq N$. Let $i \in R_k^{x^N}$ and let $j \in N \setminus R_k^{x^N}$. Then $x_i^N > x_j^N$ and there exists $S \in 2^N$ with $i \in S$ and $j \notin S$ such that $\sum_{h \in S} x_h^N = v(S)$. We have

$$v(S) = \sum_{h \in S} x_h^S \leq \sum_{h \in S} x_h^N = v(S),$$

so $x_h^S = x_h^N$ for all $h \in S$. Then there exists $R \subseteq S$ with $i \in R$ and $R \subseteq R_k^{x^N}$ such that $\sum_{h \in R} x_h^S = v(R)$. We have

$$v(R) = \sum_{h \in R} x_h^R \leq \sum_{h \in R} x_h^S = v(R),$$

so $x_h^R = x_h^S$ for all $h \in R$. This means that $x_i^{R_k^{x^N}} \geq x_i^R = x_i^S = x_i^N$. In general, $x_i^{R_k^{x^N}} \geq x_i^N$ for all $i \in R_k^{x^N}$. We have

$$v(R_k^{x^N}) = \sum_{i \in R_k^{x^N}} x_i^{R_k^{x^N}} \geq \sum_{i \in R_k^{x^N}} x_i^N \geq v(R_k^{x^N}).$$

Hence, $\sum_{i \in R_k^{x^N}} x_i^N = v(R_k^{x^N})$ for all $k \in \mathbb{N}$, $(N,v) \in \Gamma_{exp}$, and $(N,v) \in \Gamma_{totexp}$. □

However, as the following example shows, total exact partition games are not necessarily games with an egalitarian population monotonic allocation scheme. In fact, population monotonicity and egalitarian core selection are incompatible on the class of total exact partition games.

Example 4

Let $N = \{1,2,3\}$ and let $(N,v) \in \Gamma_{totexp}$ be given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	2	6	0	6

Let f be a solution on Γ_{totexp} satisfying population monotonicity and egalitarian core selection. By egalitarian core selection, $f(\{1,2\}, v_{\{1,2\}}) = (1,1,\cdot)$ and $f(\{1,3\}, v_{\{1,3\}}) = (3,\cdot,3)$. Then $f(N,v) \geq (3,1,3)$ by population monotonicity. This contradicts egalitarian core selection, which means that population monotonicity and egalitarian core selection are incompatible on the class of total exact partition games. △

We have $\Gamma_{conv} \subset \Gamma_{epmas} \subset \Gamma_{totexp} \subset \Gamma_{totbal} \subset \Gamma_{bal} \subset \Gamma_{all}$ and $\Gamma_{conv} \subset \Gamma_{epmas} \subset \Gamma_{totexp} \subset \Gamma_{exp} \subset \Gamma_{bal} \subset \Gamma_{all}$. In Example 2, population monotonic allocation schemes are not unique but there is a unique egalitarian population monotonic allocation scheme. The latter holds for all games with an egalitarian population monotonic allocation scheme.

Lemma 3.2

A game has at most one egalitarian population monotonic allocation scheme.

Proof. Let $(N, v) \in \Gamma_{epmas}$. Let $(x^S)_{S \in 2^N \setminus \{\emptyset\}}$ with $x^S \in EC(S, v_S)$ for all $S \in 2^N \setminus \{\emptyset\}$ be such that for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$, we have $x_i^S \leq x_i^T$ for all $i \in S$. For the sake of a proof by induction, assume that x^S is uniquely defined for all $S \in 2^N \setminus \{\emptyset, N\}$. For all $i \in N$,

$$x_i^N \geq \max_{S \subset N : i \in S} x_i^S.$$

If $v(N) = \sum_{i \in N} \max_{S \subset N : i \in S} x_i^S$, then $x_i^N = \max_{S \subset N : i \in S} x_i^S$ for all $i \in N$. Suppose that $v(N) > \sum_{i \in N} \max_{S \subset N : i \in S} x_i^S$. Then $x_i^N > \max_{S \subset N : i \in S} x_i^S$ for some $i \in N$. For all $i \in N$ with $x_i^N > \max_{S \subset N : i \in S} x_i^S$ and all $S \subset N$ with $i \in S$, we have

$$\sum_{j \in S} x_j^N > \sum_{j \in S} x_j^S = v_S(S) = v(S).$$

Then $x^N \in EC(N, v)$ implies that $x_i^N \leq x_j^N$ for all $i, j \in N$ with $x_i^N > \max_{S \subset N : i \in S} x_i^S$. This means that for all $i \in N$, we have

$$x_i^N = \max\left\{\max_{S \subset N : i \in S} x_i^S, \lambda\right\},$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} x_i^N = v(N)$.[2] Hence, x^N is uniquely defined. □

If a solution satisfies population monotonicity and egalitarian core selection, then it induces for each game an egalitarian population monotonic allocation scheme. This means that each domain of games on which population monotonicity and egalitarian core selection are compatible is necessarily contained in the class of games with an egalitarian population monotonic allocation scheme. Lemma 3.2 and its constructive proof essentially show that each egalitarian population monotonic allocation scheme recursively defines a solution satisfying population monotonicity and egalitarian core selection. Lemma 3.1 and its proof show that this is the Dutta-Ray solution. This means that, in contrast to the class of games with a population monotonic allocation scheme where population monotonicity and core selection are incompatible, population monotonicity and egalitarian core selection are compatible on the class of games with an egalitarian population monotonic allocation scheme and together characterize the Dutta-Ray solution. Hence, the main theorem follows directly.

[2]In other words, we are applying *constrained welfare egalitarianism* (cf. Calleja et al. 2021b). A similar process was used by Doğan and Esmerok (2024) in the context of minimum cost spanning tree problems.

Theorem 3.1

(i) The maximal domain on which population monotonicity and egalitarian core selection are compatible is the class of games with an egalitarian population monotonic allocation scheme.

(ii) The Dutta-Ray solution is the unique solution satisfying population monotonicity and egalitarian core selection on the class of games with an egalitarian population monotonic allocation scheme.

Unfortunately, we could neither prove nor disprove that population monotonicity and egalitarian core selection are independent on the class of games with an egalitarian population monotonic allocation scheme. In particular, it is an open question whether or not egalitarian core selection implies population monotonicity on the class of games with an egalitarian population monotonic allocation scheme, which could be shown by proving that the egalitarian core is actually single-valued on that domain.

4 Relation with aggregate monotonicity

This section relates the maximal domain on which population monotonicity and egalitarian core selection are compatible to the maximal domain on which aggregate monotonicity and egalitarian core selection are compatible. By Theorem 3.1, the maximal domain on which population monotonicity and egalitarian core selection are compatible is the class of games with an egalitarian population monotonic allocation scheme. Dietzenbacher (2021) showed that the maximal domain on which aggregate monotonicity and egalitarian core selection are compatible is the class of *PES stable games*, i.e. games where the *procedural egalitarian solution* (cf. Dietzenbacher et al. 2017) selects from the core. This solution is based on an egalitarian procedure where players iteratively fix their payoffs.

Let $(N, v) \in \Gamma_{all}$. Before the procedure, $P^{v,0} = \emptyset$ since no player has acquired a fixed payoff yet. Let $k \in \mathbb{N}$ be an iteration. The function $\chi^{v,k}$ assigns in each coalition $S \in 2^N \setminus \{\emptyset\}$ the fixed payoffs to the corresponding members, and divides the remaining worth equally among the other members,

$$\chi_i^{v,k}(S) = \begin{cases} \gamma_i^{v,k-1} & \text{if } i \in S \cap P^{v,k-1}; \\ \frac{v(S) - \sum_{j \in S \cap P^{v,k-1}} \gamma_j^{v,k-1}}{|S \setminus P^{v,k-1}|} & \text{if } i \in S \setminus P^{v,k-1}. \end{cases}$$

The collection $\mathcal{A}^{v,k} \subseteq 2^N \setminus \{\emptyset\}$ consists of all coalitions of which no member is allocated a higher payoff in any other coalition,

$$\mathcal{A}^{v,k} = \left\{ S \in 2^N \setminus \{\emptyset\} \;\middle|\; \sum_{i \in S} \chi_i^{v,k}(S) = v(S), \forall_{i \in S} \forall_{T \in 2^N : i \in T} : \chi_i^{v,k}(T) \leq \chi_i^{v,k}(S) \right\}.$$

The set $P^{v,k} \in 2^N \setminus \{\emptyset\}$ consists of all members of these coalitions $P^{v,k} = \bigcup_{S \in \mathcal{A}^{v,k}} S$, and $\gamma^{v,k} \in \mathbb{R}^{P^{v,k}}$ describes their corresponding fixed payoffs $\gamma_i^{v,k} = \chi_i^{v,k}(S)$ for each $i \in P^{v,k}$, where $S \in \mathcal{A}^{v,k}$ and $i \in S$. Let $n^v \in \{1, \ldots, |N|\}$ be the first iteration where all players have acquired a fixed payoff $n^v = \min\{k \in \mathbb{N} \mid P^{v,k} = N\}$. The *egalitarian claims* $\widehat{\gamma}^v \in \mathbb{R}^N$ are the fixed payoffs $\widehat{\gamma}^v = \gamma^{v,n^v}$. The *egalitarian admissible coalitions* $\widehat{\mathcal{A}}^v \subseteq 2^N \setminus \{\emptyset\}$ are the coalitions where the egalitarian claims of all members are attainable $\widehat{\mathcal{A}}^v = \mathcal{A}^{v,n^v} = \{S \in 2^N \setminus \{\emptyset\} \mid \sum_{i \in S} \widehat{\gamma}_i^v = v(S)\}$. The *strong egalitarian claimants* $D^v \in 2^N$ are the players which are member of all the inclusion-wise maximal egalitarian admissible coalitions $D^v = \bigcap \{S \in \widehat{\mathcal{A}}^v \mid \forall_{T \in \widehat{\mathcal{A}}^v} : S \not\subset T\}$. The *procedural egalitarian solution* $PES(N,v) \in \mathbb{R}^N$ assigns to all strong claimants their egalitarian claims and divides the remaining worth of the grand coalition as equally as possible among the other players, provided that they do not get more than their egalitarian claims,

$$PES(N,v) = \left((\widehat{\gamma}_i^v)_{i \in D^v}, (\min\{\widehat{\gamma}_i^v, \lambda\})_{i \in N \setminus D^v} \right),$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} PES_i(N,v) = v(N)$. The maximal domain on which the procedural egalitarian solution satisfies core selection is the class of PES stable games (cf. Dietzenbacher et al. 2017).[3]

PES stability
A game $(N,v) \in \Gamma_{all}$ is *PES stable* if $N \in \widehat{\mathcal{A}}^v$.

Let Γ_{PES} denote the class of PES stable games. Dietzenbacher (2021) showed that the class of PES stable games is the maximal domain on which aggregate monotonicity and egalitarian core selection are compatible. It turns out that the class of PES stable games not only contains all convex games, but also all exact partition games. For the proof, we use that the vector of egalitarian claims is an *aspiration* (cf. Bennett 1983).

Lemma 4.1 (cf. Dietzenbacher 2021)
Let $(N,v) \in \Gamma_{all}$. Then $\sum_{j \in S} \widehat{\gamma}_j^v \geq v(S)$ for all $S \in 2^N$ and for each $i \in N$ there exists $S \in 2^N$ with $i \in S$ such that $\sum_{j \in S} \widehat{\gamma}_j^v = v(S)$ and $\widehat{\gamma}_i^v \leq \widehat{\gamma}_j^v$ for all $j \in S$.

Lemma 4.2
All exact partition games are PES stable.

Proof. Let $(N,v) \in \Gamma_{exp}$. Let $x \in C(N,v)$ be such that $\sum_{i \in R_k^x} x_i = v(R_k^x)$ for all $k \in \mathbb{N}$. For the sake of a proof by induction, let $k \in \mathbb{N}$ with $R_{k-1}^x \neq N$ and assume that $\widehat{\gamma}_i^v = x_i$ for all $i \in R_{k-1}^x$. Suppose for the sake of contradiction that there exists $i \in R_k^x$ with $\widehat{\gamma}_i^v > x_i$. Let $S \in 2^N$ with $i \in S$ be such that $\sum_{j \in S} \widehat{\gamma}_j^v = v(S)$ and $\widehat{\gamma}_i^v \leq \widehat{\gamma}_j^v$ for all $j \in S$, which exists by Lemma 4.1. Then $\widehat{\gamma}_j^v = x_j$ for all $j \in S \cap R_{k-1}^x$, and $\widehat{\gamma}_j^v \geq \widehat{\gamma}_i^v > x_i \geq x_j$ for all $j \in S \setminus R_{k-1}^x$.

[3] Dietzenbacher et al. (2017) called such games *egalitarian stable*.

This means that

$$v(S)=\sum_{j\in S}\widehat{\gamma}^v_j>\sum_{j\in S}x_j\geq v(S).$$

This is a contradiction, so $\widehat{\gamma}^v_i\leq x_i$ for all $i\in R^x_k$. Then

$$v(R^x_k)\leq\sum_{i\in R^x_k}\widehat{\gamma}^v_i\leq\sum_{i\in R^x_k}x_i=v(R^x_k).$$

This means that $\widehat{\gamma}^v_i=x_i$ for all $i\in R^x_k$. Hence, $\widehat{\gamma}^v=x$, $N\in\widehat{\mathcal{A}}^v$, and $(N,v)\in\Gamma_{PES}$. □

As the following example shows, PES stable games are not necessarily exact partition games.

Example 5
Let $N=\{1,2,3\}$ and let $(N,v)\in\Gamma_{PES}$ with the corresponding egalitarian procedure be given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	3	3	0	0	0	0	6
$\chi^{v,1}(S)$	$(\underline{3},\cdot,\cdot)$	$(\cdot,\underline{3},\cdot)$	$(\cdot,\cdot,0)$	$(0,0,\cdot)$	$(0,\cdot,0)$	$(\cdot,0,0)$	$(2,2,2)$
$\chi^{v,2}(S)$	$(\mathbf{3},\cdot,\cdot)$	$(\cdot,\mathbf{3},\cdot)$	$(\cdot,\cdot,\underline{0})$	$(\mathbf{3},\mathbf{3},\cdot)$	$(\mathbf{3},\cdot,-3)$	$(\cdot,\mathbf{3},-3)$	$(\mathbf{3},\mathbf{3},\underline{0})$
$\chi^{v,3}(S)$	$(\mathbf{3},\cdot,\cdot)$	$(\cdot,\mathbf{3},\cdot)$	$(\cdot,\cdot,\mathbf{0})$	$(\mathbf{3},\mathbf{3},\cdot)$	$(\mathbf{3},\cdot,\mathbf{0})$	$(\cdot,\mathbf{3},\mathbf{0})$	$(\mathbf{3},\mathbf{3},\mathbf{0})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

In the first iteration, the worth of each coalition is equally divided among its members. The payoff of 3 to players 1 and 2 is fixed since no member of coalitions $\{1\}$ and $\{2\}$ is allocated a higher payoff in any other coalition. This means that $\mathcal{A}^{v,1}=\{\{1\},\{2\}\}$, $P^{v,1}=\{1,2\}$, and $\gamma^{v,1}=(3,3,\cdot)$. In the second iteration, players 1 and 2 are allocated their fixed payoff of 3 in each coalition they belong to, and the remaining worth is equally divided among the other members. The payoff of 3 to players 1 and 2 and the payoff of 0 to player 3 is fixed since no member of coalitions $\{1\}$, $\{2\}$, $\{3\}$, and $\{1,2,3\}$ is allocated a higher payoff in any other coalition. This means that $\mathcal{A}^{v,2}=\{\{1\},\{2\},\{3\},\{1,2,3\}\}$, $P^{v,2}=\{1,2,3\}$, and $\gamma^{v,2}=(3,3,0)$. Moreover, $n^v=2$, the egalitarian claims are $\widehat{\gamma}^v=(3,3,0)$, the egalitarian admissible coalitions are $\widehat{\mathcal{A}}^v=\{\{1\},\{2\},\{3\},\{1,2,3\}\}$, the strong egalitarian claimants are $D^v=\{1,2,3\}$, and the procedural egalitarian solution is given by $PES(N,v)=(3,3,0)$. In fact, the core is given by $C(N,v)=\{(3,3,0)\}$. This means that (N,v) is not an exact partition game, i.e. $(N,v)\notin\Gamma_{exp}$. △

We have $\Gamma_{conv}\subset\Gamma_{exp}\subset\Gamma_{PES}\subset\Gamma_{bal}\subset\Gamma_{all}$. Dietzenbacher (2021) showed that the procedural egalitarian solution is one of several solutions satisfying strong egalitarian core selection on the class of PES stable games. This means that the procedural egalitarian solution coincides with the Dutta-Ray solution on the class of exact partition games. Dietzenbacher et al. (2017) showed that this is not the case on the full class of PES stable games.

The class of PES stable games is not closed under subgames. We define the closure as the class of *totally PES stable games.*

Total PES stability

A game $(N, v) \in \Gamma_{all}$ is *totally PES stable* if $(S, v_S) \in \Gamma_{PES}$ for all $S \in 2^N \setminus \{\emptyset\}$.

Let Γ_{totPES} denote the class of totally PES stable games. Clearly, all totally PES stable games are totally balanced. However, as in Example 3, totally balanced games are not necessarily totally PES stable.

Example 3 (continued)

The egalitarian procedure is given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	12	12	0	12
$\chi^{v,1}(S)$	$(0,\cdot,\cdot)$	$(\cdot,0,\cdot)$	$(\cdot,\cdot,0)$	$(\underline{6},\underline{6},\cdot)$	$(\underline{6},\cdot,\underline{6})$	$(\cdot,0,0)$	$(4,4,4)$
$\chi^{v,2}(S)$	$(\mathbf{6},\cdot,\cdot)$	$(\cdot,\mathbf{6},\cdot)$	$(\cdot,\cdot,\mathbf{6})$	$(\mathbf{6},\mathbf{6},\cdot)$	$(\mathbf{6},\cdot,\mathbf{6})$	$(\cdot,\mathbf{6},\mathbf{6})$	$(\mathbf{6},\mathbf{6},\mathbf{6})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

We have $\mathcal{A}^{v,1} = \{\{1,2\},\{1,3\}\}$, $P^{v,1} = \{1,2,3\}$, and $\gamma^{v,1} = (6,6,6)$. Moreover, $n^v = 1$, $\widehat{\gamma}^v = (6,6,6)$, $\widehat{\mathcal{A}}^v = \{\{1,2\},\{1,3\}\}$, $D^v = \{1\}$, and $PES(N,v) = (6,3,3)$, so $N \notin \widehat{\mathcal{A}}^v$ and consequently $PES(N,v) \notin C(N,v)$. This means that (N,v) is not PES stable, i.e. $(N,v) \notin \Gamma_{PES}$. $\triangle$

By Lemma 4.2, all total exact partition games are totally PES stable. In fact, the class of total exact partition games coincides with the class of totally PES stable games.

Theorem 4.1

A game is a total exact partition game if and only if it is a totally PES stable game.

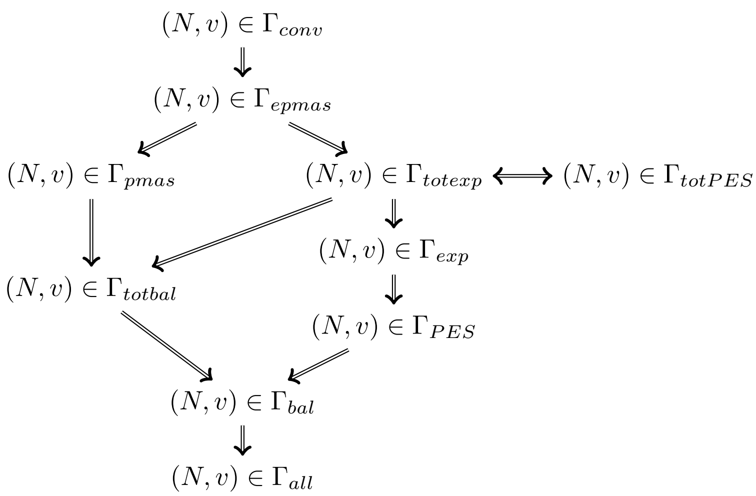
Proof. Let $(N,v) \in \Gamma_{totexp}$. Then $(S, v_S) \in \Gamma_{exp}$ for all $S \in 2^N \setminus \{\emptyset\}$. Then Lemma 4.2 implies that $(S, v_S) \in \Gamma_{PES}$ for all $S \in 2^N \setminus \{\emptyset\}$. Hence, $(N,v) \in \Gamma_{totPES}$.

Let $(N,v) \in \Gamma_{totPES}$. For the sake of a proof by induction, assume that $P^{v_S,k} \in \mathcal{A}^{v_S,k}$ for all $S \in 2^N \setminus \{\emptyset, N\}$ and all $k \in \mathbb{N}$. For the sake of a proof by induction, let $k \in \mathbb{N}$ and assume that $P^{v,k-1} = P^{v_S,k-1}$ and $\gamma_i^{v,k-1} = \gamma_i^{v_S,k-1}$ for all $S \in 2^N \setminus \{\emptyset\}$ with $P^{v,k-1} \subseteq S$ and all $i \in P^{v,k-1}$. If $P^{v,k} = N$, then $P^{v,k} \in \mathcal{A}^{v,k}$ because $N \in \widehat{\mathcal{A}}^v$. Suppose that $P^{v,k} \neq N$. For all $S \in 2^N$ with $P^{v,k} \subseteq S$, we have $\mathcal{A}^{v,k} \subseteq \mathcal{A}^{v_S,k}$, so $P^{v,k} \subseteq P^{v_S,k}$. For all $S \subset N$ with $P^{v,k} \subseteq S$, since $P^{v_S,k} \in \mathcal{A}^{v_S,k}$, we have $\gamma_i^{v_S,k} = \gamma_j^{v_S,k}$ for all $i,j \in P^{v_S,k} \setminus P^{v_S,k-1}$. For all $S \in 2^N$ with $P^{v,k} \subseteq S$, this means that $\mathcal{A}^{v,k} = \mathcal{A}^{v_S,k}$, so $P^{v,k} = P^{v_S,k}$. Moreover, $P^{v,k} \in \mathcal{A}^{v_S,k}$ and $\gamma_i^{v,k} = \gamma_i^{v_S,k}$ for all $S \in 2^N$ with $P^{v,k} \subseteq S$ and all $i \in P^{v,k}$. In particular, $P^{v,k} \in \mathcal{A}^{v,k}$ for all $k \in \mathbb{N}$. This means that $R_k^{\widehat{\gamma}^v} = P^{v,k}$ and $\sum_{i \in R_k^{\widehat{\gamma}^v}} \widehat{\gamma}_i^v = v(R_k^{\widehat{\gamma}^v})$ for all $k \in \mathbb{N}$. Hence, $(N,v) \in \Gamma_{totexp}$. $\square$

5 Concluding remarks

This paper studies the compatibility of population monotonicity and egalitarian core selection for solutions for transferable utility games. If a solution satisfies population monotonicity and core selection, then it induces for each game a population monotonic allocation scheme. Remarkably however, population monotonicity and core selection are not compatible on the class of games with a population monotonic allocation scheme. If a solution satisfies population monotonicity and egalitarian core selection, then it induces for each game an egalitarian population monotonic allocation scheme. The class of games with an egalitarian population monotonic allocation scheme is the maximal domain on which population monotonicity and egalitarian core selection are compatible. In fact, on this class, population monotonicity and egalitarian core selection together characterize the Dutta-Ray solution.

The class of games with an egalitarian population monotonic allocation scheme strictly contains the class of convex games and is strictly contained in the class of games with a population monotonic allocation scheme. Moreover, it is strictly contained in the class of total exact partition games. That class is equivalent to the class of total PES stable games and is strictly contained in both the class of totally balanced games and the class of exact partition games. The class of exact partition games is strictly contained in the class of PES stable games, which is in turn, like the class of totally balanced games, strictly contained in the class of balanced games. All relations between these classes of games are presented in the following diagram.



Dietzenbacher (2021) showed that all *large core games* (cf. Sharkey 1982) and all *exact games* (cf. Schmeidler 1972) with at most four players are PES stable. However, these classes are not closed under subgames. Moulin (1990) showed that all subgames are large core games if and only if the game is convex. Biswas et al. (1999) showed that all subgames are exact games if and only if the game is convex.

Oishi et al. (2016) showed that the Dutta-Ray solution is self-antidual on the class of convex games. Dietzenbacher and Yanovskaya (2020) showed that the Dutta-Ray solution is self-antidual on the class of exact partition games. Although the classes of total exact partition games and games with an egalitarian population monotonic allocation scheme strictly contain the class of convex games and are strictly contained in the class of exact partition games, the Dutta-Ray solution is not self-antidual on these classes because their antidual games do not necessarily belong to the same class. This is shown by the following example.

Example 2 (continued)
The corresponding antidual game $(N, v^*) \in \Gamma_{exp}$ is given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^*(S)$	-3	-1	-1	-3	-3	-3	-3

Then $C(\{2,3\}, v_{\{2,3\}}) \neq \emptyset$, so $(N, v) \notin \Gamma_{totbal}$. This means that the classes of total balanced games, games with a population monotonic allocation scheme, total exact partition games, and games with an egalitarian population monotonic allocation scheme are each not closed under antiduality. △

Whether other structures, properties, and axiomatizations of the Dutta-Ray solution for convex games and exact partition games are preserved for games with an egalitarian population monotonic allocation scheme is an interesting question for future research.

To conclude, we would like to point out two other open questions for future research. Doğan (2016) introduced absence-proofness as a weakening of population monotonicity. What is the compatibility of absence-proofness with egalitarian core selection? Recently, Solymosi (2024) characterized assignment games with a population monotonic allocation scheme. What characterizes assignment games with an egalitarian population monotonic allocation scheme?

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