

The economic order decision with continuous dynamic pricing and batch supply

Citation for published version (APA):

van den Berg, A. H. J., Herings, P. J. J., & Peters, H. J. M. (2014). *The economic order decision with continuous dynamic pricing and batch supply*. Maastricht University, Graduate School of Business and Economics. GSBE Research Memoranda No. 001 <https://doi.org/10.26481/umagsb.2014001>

Document status and date:

Published: 01/01/2014

DOI:

[10.26481/umagsb.2014001](https://doi.org/10.26481/umagsb.2014001)

Document Version:

Publisher's PDF, also known as Version of record

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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- The final published version features the final layout of the paper including the volume, issue and page numbers.

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with Continuous Dynamic
Pricing and Batch Supply**

RM/14/001

GSBE

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The Economic Order Decision with Continuous Dynamic Pricing and Batch Supply

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January 13, 2014

Abstract

We study an infinite horizon model, where a seller orders his product in batches of fixed size. A sales strategy determines both the order moments and the sales path between these moments. Under some natural conditions on the seller's revenue function, the strategy that maximizes the seller's time-discounted revenue is determined. The optimal strategy is shown to be unique and is characterized by increasing prices in between order moments. We analyze the sensitivity of this strategy to the main parameters of the model: batch size, batch cost, and discount rate. Surprisingly, increasing batch sizes may lead to lower optimal order times.

KEYWORDS: Dynamic pricing, economic order decision, inventory.

JEL CODES: C61, D92, L11.

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1 Introduction

Many industries, like retail, airlines, hotels, and utilities have adopted dynamic pricing strategies in recent years. Indeed, as argued in Fleischmann, Hall, and Pyke (2004), internet technologies have substantially reduced the costs of continuous price adjustments, making dynamic pricing very attractive to maximize revenues. In this paper, we consider the interplay between dynamic pricing and supply chain coordination. We consider dynamic pricing by a seller who acquires the product in batches of fixed size. It differs from classical inventory models like the economic lot-size model of Whitin (1955) and Wagner and Whitin (1958), where dynamic pricing considerations do not play a role. Instead, we contribute to the later literature on dynamic pricing in the presence of inventory considerations, see Bitran and Caldentey (2003) and Elmaghraby and Keskinocak (2003) for excellent overviews of the literature.

The seller offers the product to customers over time, and these customers buy the product according to a demand function that is time-independent. The seller does not necessarily offer the product in a constant quantity, or, equivalently, at a constant price, but looks for a sales strategy that maximizes his time-discounted profit, his revenue minus his cost of acquiring new batches, over an infinite time horizon. Inventory is replenished immediately, i.e., new batches arrive immediately.

The seller can be a retailer who can only acquire the good in large bulks, or a manufacturer who can only produce the good in batches of a certain size. Another example is an airline company which plans flight schedules ahead over a certain stretch of time and typically offers tickets at changing, usually increasing, prices. Or consider a webstore that has a fixed consumer base and sells catalog books or CD's, i.e., products that have been on the market for a while. Rather than using a constant price, it will usually be optimal for the seller to vary prices over time.

A strategy of the seller corresponds to a specification of order times and sales prices at each point in time. We derive the seller's optimal strategy and show that it is stationary. It can therefore be characterized by the time that elapses in between any two order moments and the optimal pricing policy, or equivalently the optimal sales path, in between two order moments. Under some natural conditions, in particular concavity of the revenue function, we find that there is a unique optimal strategy. The corresponding optimal sales path is characterized by decreasing sales, which corresponds to increasing prices in between order moments. This is due to the fact that discounted marginal revenue is constant over time at an optimal path: hence, undiscounted marginal revenue increases over time, which corresponds to decreasing sales and, hence, increasing prices due to concavity of the revenue function. As soon as the product runs out of stock, a new batch is ordered.

We show that the time between two order moments increases with batch cost, a result

that confirms intuition. In the case of linear demand, we show that the time between two order moments also increases with the discount rate, but it is not clear whether this is the case for more general demand functions. In fact, an increase in the discount rate both decreases discounted profit within each sales period, which would seem to call for higher sales speed and shorter time between order moments, and decreases the discounted price of future batches, which would seem to call for the opposite. As to the effect of batch size on the time between order moments, intuition suggests a positive effect. However, even for the case with linear demand functions, we show that it can be both positive and negative. In particular when profit margins are low, the optimal sales strategy approximates a sales strategy that maximizes the revenues coming out of a single batch, and has high prices and low turnover. An increase in batch size comes with a higher profit margin and incentives to increase turnover which can be so strong that the time in between order moments goes down.

As already mentioned, there is quite an extensive literature on the effect of dynamic pricing on the economic order decision. The work that is closest to our approach is Rajan, Steinberg, and Steinberg (1992) and Transchel and Minner (2009). Transchel and Minner (2009) consider both discrete and continuous dynamic pricing in a set-up where the seller maximizes average profit over a fixed time horizon. Rajan, Steinberg, and Steinberg (1992) restrict the analysis to continuous dynamic pricing, but add explicit costs of decay. Both papers refrain from discounting future profits as it is argued by Rajan, Steinberg, and Steinberg (1992) that “we have not included a discount rate, which would substantially complicate our model.” In this paper, we introduce discounting and maximize discounted profits over an infinite time horizon. We provide a complete picture of the comparative statics with respect to batch cost, discount rate, and batch size.

Our work is also related to the literature on production and inventory planning following the seminal contribution by Holt, Modigliani, Muth, and Simon (1960). Later papers introduced price as a decision variable and study the problem of determining the price and production schedule of a firm over a known horizon simultaneously, see Pekelman (1974) and Feichtinger and Hartl (1986), as well as extensions to multiple interacting players by Jørgensen (1986) and Jørgensen and Kort (2002). This literature also maximizes discounted profits using dynamic pricing, but deals with a fixed planning horizon and firms which operate under a convex increasing production cost function and choosing a production rate. We, on the other hand, consider the case where production or orders takes place at discrete instants of time in an infinite horizon world.

In Section 2 we introduce the model. Section 3 provides the optimal sales strategy in case a single batch is to be sold within a fixed time period. Based on this, Section 4 analyzes the infinite horizon model including the determination of the optimal order

moments, Section 5 provides the sensitivity analysis with respect to the parameters of the model, Section 6 discusses the case of linear demand functions, and Section 7 concludes.

2 The Model

We search for the profit-maximizing strategy of a single seller who sells a non-perishable good. The manufacturer delivers the good in batches of size $S > 0$ for a price $K > 0$. The price K is the total of costs for the manufacturer of placing an order, so it includes both the price to be paid to the manufacturer and his own order costs. The seller's inventory level can never become negative, that is, backlogging is not allowed. He can choose when to order new stock and how much he is willing to sell from the stock at every moment in time. Newly ordered stock is delivered instantly.¹ Time is continuous and the time horizon is infinite. Revenue streams and costs are discounted at a rate of $r > 0$. Because we consider a model with discounting, we fully accommodate inventory holding costs related to the opportunity cost of capital invested in inventories.

The non-negative quantity $q(t)$ is the amount of the good the seller decides to supply at time t . A function q is said to be *piecewise continuous* on an interval if the interval can be broken into a finite number of subintervals such that the function is continuous on each open subinterval and has a finite limit at the endpoints of each subinterval. A *removable discontinuity* is a point x at which $\lim_{t \uparrow x} q(t) = \lim_{t \downarrow x} q(t) \neq q(x)$. It is assumed that $q(t)$ is a piecewise continuous function of t on any finite interval of $[0, \infty)$ and does not have any removable discontinuities. Define Q as the set of functions that satisfy these assumptions.

The instantaneous revenue the seller receives for selling a quantity q is given by the revenue function R , defined on $[0, \infty)$. We assume that R is positive on an interval $(0, A)$, is twice-continuously differentiable except at A , has a unique maximum at q^m such that $0 < q^m < A$, and is strictly concave on $[0, A]$. Moreover, for $q = 0$ and $q \geq A$, it holds that $R(q) = 0$. Notice that the twice differentiability of R at 0 implies that $R'(0)$ is finite.

Let $X(t)$ be the inventory level of the seller at time $t \geq 0$, and let T_0, T_1, T_2, \dots with $T_0 = 0$ be the order moments. Between order moments, stock decreases with $q(t)$, and at each order moment it increases with S . A *strategy* is a tuple $\sigma = (q, T_1, T_2, \dots)$ such that $q \in Q$, $T_1, T_2, \dots \in \mathbb{R}$ with $0 = T_0 < T_1 < T_2 < \dots$, and such that $X(t) \geq 0$ for all $t \geq 0$. By Σ we denote the set of all strategies. Then the seller faces the following optimal control

¹This model can easily be extended to a model in which it takes a positive amount of time to deliver the good. This will have no major effect on the results. See Section 7 for a brief discussion on the assumptions of the model.

problem.

$$\begin{aligned} & \max_{(q, T_1, T_2, \dots) \in \Sigma} \sum_{i=0}^{\infty} \left(\int_{T_i}^{T_{i+1}} e^{-rt} R(q(t)) dt - e^{-rT_i} K \right) \\ & \text{subject to} \end{aligned} \tag{1}$$

$$X(0) = S, \quad \dot{X}(t) = -q(t); \text{ for all } i \geq 1, \quad X(T_i) = \lim_{t \uparrow T_i} X(t) + S.$$

Let w^* be the maximum discounted revenue stream that the seller can receive for selling a single batch of size S . We assume that $K < w^*$, which means that the seller can make a positive profit on each batch. In Lemma 3.6 we derive w^* explicitly.

3 Analysis of the Single Batch Fixed Time Problem

Before analyzing optimization problem (1) in detail in Section 4, we make some useful observations.

First, in an optimal strategy the seller will never run out of stock, so $X(t) > 0$ for all $t \geq 0$, since otherwise he could simply shift part of his strategy forward in time, namely to the moment where he first ran out of stock, and increase his profits due to discounting, contradicting optimality.

Second, in an optimal strategy the seller will never order before he runs out of stock, since in such a case he could increase profits by postponing reordering until he runs out of stock, thereby decreasing costs, again due to discounting.

Third, in an optimal strategy we have $q(t) \leq q^m$ for all $t \geq 0$, since otherwise decreasing the offered quantity to q^m both increases instantaneous profit and decreases the cost of ordering new stock due to discounting – since sales speed is reduced, reordering is postponed.

We summarize these observations in the following lemma.

Lemma 3.1 *Let $\sigma = (q, T_1, T_2, \dots) \in \Sigma$ be an optimal strategy for problem (1). Then for all $t \geq 0$ and $i \geq 1$ we have $X(t) > 0$, $\lim_{t \uparrow T_i} X(t) = 0$, and $q(t) \leq q^m$.*

As a step towards solving (1), we first analyze the optimal sales strategy between order moments. That is, we fix a time $T > 0$ and determine the optimal strategy for selling a batch S in the time interval $[0, T]$. In the next section we will use the result to analyze the general problem (1). Lemma 3.1 implies that the optimal amount of time before reordering is at least $T^m = S/q^m$, the time needed to sell S when $q(t) = q^m$ for all t . We therefore make the assumption that $T \geq T^m$ for the remainder of this section.

In an optimal strategy for the case without reordering, we still have $q(t) \leq q^m$ for all $t \geq 0$, since otherwise decreasing the offered quantity to q^m at each time t where $q(t) > q^m$ increases instantaneous profit.

The assumption $T \geq T^m$ also implies that it is optimal to sell the whole batch S within $[0, T]$. Indeed, when some stock would be left at time T , then profits would be increased by selling more at each time t where $q(t) < q^m$.

Let \bar{Q} be the set of non-negative piecewise continuous functions without removable discontinuities with domain $[0, T]$. The optimal control problem to solve for the case without reordering is

$$\begin{aligned} & \max_{q \in \bar{Q}} \int_0^T e^{-rt} R(q(t)) dt \\ & \text{subject to} \\ & X(0) = S, X(T) = 0, \dot{X}(t) = -q(t) \text{ for all } t \in [0, T]. \end{aligned} \tag{2}$$

Note that we can set $X(T) = 0$ since we are considering the case without reordering. The Hamiltonian associated with this problem is

$$H(t, X, q, \pi) = e^{-rt} R(q(t)) - \pi(t)q(t),$$

where $\pi(t)$ is the co-state variable, and since $q \in \bar{Q}$, hence $q(t) \geq 0$ for all $t \in [0, T]$, we consider the Lagrangian

$$L(t, X, q, \pi, \lambda) = e^{-rt} R(q(t)) - \pi(t)q(t) + \lambda(t)q(t).$$

Since the objective function in (2) is concave in (X, q) for every $t \in [0, T]$, according to the maximum principle the following conditions, for $t \in [0, T]$, are necessary and sufficient for an optimal solution of (2):

$$e^{-rt} R'(q(t)) - \pi(t) + \lambda(t) = 0, \tag{3}$$

$$\dot{\pi}(t) = -\partial L / \partial X = 0, \tag{4}$$

$$\lambda(t), q(t) \geq 0, \lambda(t)q(t) = 0, \tag{5}$$

$$X(0) = S, X(T) = 0, \dot{X}(t) = -q(t). \tag{6}$$

Combining (3) and (4), we find that there is a constant c such that

$$e^{-rt} R'(q(t)) + \lambda(t) = c \text{ for all } t \in [0, T]. \tag{7}$$

Clearly, $c \geq 0$ since $q(t) \leq q^m$, hence $R'(q(t)) \geq 0$, and $\lambda(t) \geq 0$. The following lemma says that the seller keeps on selling continuously but in decreasing amounts, until his stock is zero.

Lemma 3.2 *Let q be an optimal solution of (2). Then q is a continuous function on $[0, T]$. Moreover, either $T = T^m$ and $q(t) = q^m$ for all $t \in [0, T]$, or there is a $\hat{t} \in (T^m, T]$ such that $q(t) > 0$ for $t \in [0, \hat{t})$ and $q(t) = 0$ for $t \in (\hat{t}, T]$. In the latter case, the function q is strictly decreasing on $[0, \hat{t}]$.*

Proof. If $T = T^m$, then $q(t) = q^m$ for all $t \in [0, T]$ gives the highest possible instantaneous profit at each $t \in [0, T]$, so is clearly the unique optimal strategy.

Consider next the case $T > T^m$. By (5) and (7) we have $c = e^{-rt}R'(q(t))$ for all t such that $q(t) > 0$, and we have $c = e^{-rt}R'(q(t)) + \lambda(t)$ for all t such that $q(t) = 0$.

Suppose $q(t) = q^m$ for some $t \in [0, T]$. Then $c = 0$, so $q(t) = q^m$ for all $t \in [0, T]$, by (5) and (7). The assumption $T > T^m$ then leads to a violation of (6). Consequently, $q(t) < q^m$ for all $t \in [0, T]$. It follows that $c > 0$ and there is $t \in (T^m, T]$ such that $q(t) > 0$.

Let $q(t') > 0$ for some $t' \in (0, T]$, and let $0 \leq t < t'$. We prove that $q(t) > q(t')$. By (5) and (7) we have

$$c = e^{-rt'}R'(q(t')) = e^{-rt}R'(q(t)) + \lambda(t),$$

hence

$$e^{-rt'}R'(q(t')) \geq e^{-rt}R'(q(t)).$$

Since $e^{-rt'} < e^{-rt}$ and $R'(q(t')) > 0$, this implies $R'(q(t')) > R'(q(t))$ and hence $q(t) > q(t')$. It follows that there is a $\hat{t} \in (T^m, T]$ such that $q(t) > 0$ for $t \in [0, \hat{t})$ and $q(t) = 0$ for $t \in (\hat{t}, T]$.

To show the continuity of q , first note that on the interval $[0, \hat{t})$ we can write $q(t) = (R')^{-1}(ce^{rt})$, where $(R')^{-1}$ is the inverse of R' , which exists and is continuous by the conditions on R . Hence q is continuous on $[0, \hat{t})$, and obviously q is also continuous on $(\hat{t}, T]$. If $\hat{t} = T$ then $q(T) = \lim_{t \uparrow T} q(t)$ since q has no removable discontinuities. Now suppose $\hat{t} < T$. If $q(\hat{t}) > 0$ then for ε sufficiently small

$$c = e^{-r\hat{t}}R'(q(\hat{t})) < e^{-r(\hat{t}+\varepsilon)}R'(0) = e^{-r(\hat{t}+\varepsilon)}R'(q(\hat{t} + \varepsilon)) = c - \lambda(\hat{t} + \varepsilon) \leq c,$$

a contradiction. Hence $q(\hat{t}) = 0$. Now

$$c = \lim_{t \uparrow \hat{t}} e^{-rt}R'(q(t)) \leq \lim_{t \uparrow \hat{t}} e^{-rt}R'(0) = e^{-r\hat{t}}R'(0) \leq e^{-r\hat{t}}R'(q(\hat{t})) + \lambda(\hat{t}) = c,$$

which implies that all inequalities are equalities. Hence $\lim_{t \uparrow \hat{t}} q(t) = 0 = q(\hat{t})$. \square

If in Lemma 3.2 it holds that $\hat{t} = T$, then both $q(\hat{t}) > 0$ and $q(\hat{t}) = 0$ are possible. If $\hat{t} < T$, then it follows from continuity that $q(\hat{t}) = 0$. We show next that problem (2) has a unique solution and give its characterization.

Proposition 3.3 (a) *Problem (2) has a solution. If $q^* \in \bar{Q}$ is such a solution, then there is $c^* \geq 0$ and $t^* \in [T^m, T]$ such that*

$$R'(q^*(t)) = c^* e^{rt} \text{ for } t \in [0, t^*], \quad q^*(t) = 0 \text{ for } t \in (t^*, T] \quad (8)$$

and

$$\int_0^{t^*} q^*(t) dt = S. \quad (9)$$

(b) *The triple (q^*, c^*, t^*) in (a) is uniquely determined by (8) and (9).*

Proof. Part (a) follows from the preceding analysis, in particular the proof of Lemma 3.2. For part (b) note that uniqueness follows from Lemma 3.2 when $T = T^m$. Consider next the case $T > T^m$. For fixed t^* , both c^* and q^* are uniquely determined by (8) and (9). Now suppose that $(\tilde{q}, \tilde{c}, \tilde{t})$ is another solution, where without loss of generality $\tilde{t} < t^*$. Then, by (8) applied to $(\tilde{q}, \tilde{c}, \tilde{t})$ we have $\tilde{q}(t) \leq q^*(t)$ for all $t \in [0, \tilde{t}]$ or $\tilde{q}(t) \geq q^*(t)$ for all $t \in [0, \tilde{t}]$. In view of (9) the latter must be the case, hence $q^*(\tilde{t}) \leq \tilde{q}(\tilde{t}) = 0$. On the other hand, by (8)

$$R'(q^*(\tilde{t})) = c^* e^{r\tilde{t}} < c^* e^{rt^*} = R'(q^*(t^*)) = R'(0),$$

so that $q^*(\tilde{t}) > 0$, a contradiction. \square

Clearly, if (q^*, c^*, t^*) is the optimal solution for a given T and if either $t^* < T$, or $t^* = T$ and $q^*(t^*) = 0$, then (q^*, c^*, t^*) is also the optimal solution for any T' with $T' \geq t^*$. This follows, in particular, from part (b) of Proposition 3.3.

We now denote the optimal triple for T by (q^T, c^T, t^T) . The following lemma summarizes how q^T and c^T behave as a function of T .

Lemma 3.4 *$\partial q^T(t)/\partial T \leq 0$ and $\partial c^T/\partial T \geq 0$ for all $T \geq 0$ and $t \in [0, T]$.*

Proof. By (8) we have $R''(q^T(t))(\partial q^T(t)/\partial T) = e^{rt}(\partial c^T/\partial T)$ for all $t \in [0, T]$, so $\partial q^T(t)/\partial T$ and $\partial c^T/\partial T$ must have opposite signs for every $t \in [0, T]$. Since, by (9), $\int_0^T (\partial q^T(t)/\partial T) dt = -q^T(T)$, and $\partial c^T/\partial T$ does not depend on t , the lemma follows. \square

The next lemma together with the observation following the proof of Proposition 3.3 implies that $t^T < T$ when T is sufficiently large.

Lemma 3.5 *There is a $T > 0$ such that $t^T < T$.*

Proof. Suppose the lemma is not true. By Lemma 3.2, $c^T > 0$ for $T > T^m$, and by Lemma 3.4, c^T is non-decreasing in T . Hence, $c^T e^{rT} \rightarrow \infty$ for $T \rightarrow \infty$. On the other hand, by the observation following the proof of Proposition 3.3, for all T we have $q^T(T) > 0$ and therefore $c^T e^{rT} = R'(q^T(T)) \leq R'(0)$, a contradiction. \square

By Lemma 3.5 and the observation following the proof of Proposition 3.3, there is a $\widehat{T} > T^m$ such that $t^T = T$ for all $T \in [T^m, \widehat{T}]$ and $t^T = \widehat{T}$ for all $T \geq \widehat{T}$. In view of Lemma 3.1 we may therefore, in the next section, restrict attention to $T \in [T^m, \widehat{T}]$ as the time between two order moments in problem (1).

Recall that w^* equals the maximum discounted revenue that the seller can receive for selling a single batch of size S . Clearly, the selling time that the seller needs to achieve this maximum is at most \widehat{T} . On the other hand, it cannot be smaller than \widehat{T} since then the optimal solution of problem (2) for $T = \widehat{T}$ would not be unique, contradicting Proposition 3.3. Thus, we have the following result.

Lemma 3.6 $w^* = \int_0^{\widehat{T}} e^{-rt} R(q^{\widehat{T}}(t)) dt$.

4 Analysis of the Optimal Ordering Moment

In this section we turn to the seller's original problem (1). We first restrict our analysis to so-called stationary strategies. A strategy $\sigma = (q, T_1, T_2, \dots)$ is *stationary* if $T_i = iT_1$ for all $i \geq 2$ and $q(t) = q(t \bmod T_1)$ for all $t \geq 0$. We denote such a strategy by the pair (q, T) , where $T = T_1$. Since in the previous section we have seen that we may restrict attention to $T \in [T^m, \widehat{T}]$, we consider stationary strategies in $Q \times [T^m, \widehat{T}]$.

For $T \in [T^m, \widehat{T}]$, the revenue of the seller within a single sales period starting at time 0 is equal to

$$V(T) = \int_0^T e^{-rt} R(q^T(t)) dt.$$

Note that V is a continuous function. Therefore, also total profit

$$\Pi(T) = \sum_{i=0}^{\infty} e^{-riT} (V(T) - K) = \frac{V(T) - K}{1 - e^{-rT}}$$

is a continuous function on the interval $[T^m, \widehat{T}]$ and therefore attains a maximum, say at T^* .

Lemma 4.1 *The argument T^* maximizing $\Pi : [T^m, \widehat{T}] \rightarrow \mathbb{R}$ is unique and belongs to (T^m, \widehat{T}) . In particular, T^* is the solution of the following equation in T :*

$$(e^{rT} - 1)(e^{-rT} R(q^T(T)) - c^T q^T(T)) + r(K - \int_0^T e^{-rt} R(q^T(t)) dt) = 0. \quad (10)$$

The proof of this lemma is somewhat technical and can be found in the Appendix. We observe that equation (10) evaluated at $T = T^m$ is equal to

$$\begin{aligned} & (e^{rT^m} - 1) (e^{-rT^m} R(q^m)) + r(K - \int_0^{T^m} e^{-rt} R(q^m) dt) \\ & = R(q^m) - e^{-rT^m} R(q^m) + rK - R(q^m) + e^{-rT^m} R(q^m) = rK > 0 \end{aligned}$$

and equation (10) evaluated at $T = \hat{T}$ equals

$$r(K - \int_0^{\hat{T}} e^{-rt} R(q^{\hat{T}}(t)) dt) = r(K - w^*) < 0,$$

so existence of a zero point follows from the intermediate value theorem. It also follows that $T^* \in (T^m, \hat{T})$. Lemma 4.1 not only claims existence of a solution, but additionally gives uniqueness.

It is now clear that the unique optimal stationary strategy found in Lemma 4.1 is also the unique optimal strategy in Σ , due to the stationary character of optimization problem (1). For let $(q, T_1, T_2, \dots) \in \Sigma$ be any optimal strategy. Then, clearly, for every $i = 1, 2, \dots$ the stationary strategy of the form $(q^i, T^i - T^{i-1})$, where $q^i(t) = q(T^{i-1} + t)$ for every $t \in [0, T^i - T^{i-1}]$ must be optimal. Then by Lemma 4.1 we have $T^i - T^{i-1} = T^*$ with T^* satisfying (10). Summarizing, we have:

Theorem 4.2 *Problem (1) has a unique optimal strategy, namely the stationary strategy (q^*, T^*) with $T^* \in (T^m, \hat{T})$ satisfying (10) and $q^* = q^{T^*}$.*

5 Sensitivity Analysis of the Optimal Order Time

In this section we investigate how T^* changes if the parameters of the model change, namely the batch cost K , the discount rate r , and the batch size S . We denote first and second order partial derivatives with respect to T , K , r , and S by subscripts, and often omit arguments when no confusion is likely to arise. In the Appendix we prove the following result.

Proposition 5.1 *Let (q^*, T^*) be the optimal strategy of (1). Then*

- (a) $T_K^* > 0$.
- (b) $\text{sign}(T_r^*) = \text{sign}(AV_T - rV_r + V_{Tr}(e^{rT^*} - 1))$, where $A = T^*e^{rT^*} - \frac{1}{r}(e^{rT^*} - 1) \geq 0$, $V_T \geq 0$, and $V_r \leq 0$.
- (c) $\text{sign}(T_S^*) = \text{sign}(q^*(T^*)(e^{rT^*} - 1 + r) - q^*(0)r)$.

The optimal time between order moments, T^* , reflects the tradeoff between total discounted cost of buying new batches and total discounted sales revenue. If K increases, then total discounted cost increases, which can be offset by decreasing the number of order moments, i.e., increasing T^* . This provides the intuition for result (a).

The impact on T^* from a change in r is less clear, since an increase in r decreases both discounted sales revenue and discounted cost of buying new batches. To offset the first effect the seller might want to speed up sales and thus to decrease T^* , but this has the effect of increasing discounted buying cost. Indeed, we were not able to determine the sign of T_r^* in general. In the example in Section 6 we assume that the revenue function is based on linear demand and in that case we find that $T_r^* > 0$.

The intuition for a positive effect of S on T^* might be that a higher batch size makes it possible to take more time to sell the whole batch, thereby decreasing total discounted buying cost. This intuition, however, is misleading. For the linear example in Section 6 we find that the effect is ambiguous.

6 Linear Demand Functions

In this section we consider the case where demand functions are linear. In this case, the inverse demand function is given by $a - bq$ for some parameters $a, b > 0$. Since we are free to use units for quantities and prices, without loss of generality, we can study the case where $a = b = 1$, which results in the revenue function $R(q) = q(1 - q)$. Since the monopoly quantity q^m is equal to $\frac{1}{2}$, the time needed to sell S at this rate, T^m , is equal to $2S$. Then (8) and (9) imply, for $T \in [2S, \hat{T}]$,

$$S = \frac{1}{2} \left(T - \frac{1}{r} c^T e^{rT} + \frac{1}{r} c^T \right),$$

yielding

$$c^T = \frac{r(T - 2S)}{e^{rT} - 1} \quad \text{and} \quad q^T(t) = \frac{1}{2} \left(1 - \frac{e^{rt} r(T - 2S)}{e^{rT} - 1} \right) \quad \text{for } t \in [0, T].$$

The upper bound \hat{T} can be found by solving $q^{\hat{T}}(\hat{T}) = 0$, which implies that \hat{T} is the solution of the equation

$$1 - e^{-r\hat{T}} = r(\hat{T} - 2S),$$

so in particular $\hat{T} < 2S + 1/r$.

The optimal time between order moments, T^* , follows from (10). To evaluate the first term in (10), notice that

$$c^T q^T(T) = e^{-rT} R'(q^T(T)) q^T(T) = e^{-rT} (q^T(T) - 2q^T(T)^2),$$

so

$$e^{-rT}R(q^T(T)) - c^T q^T(T) = e^{-rT}(q^T(T) - q^T(T)^2) - c^T q^T(T) = e^{-rT}q^T(T)^2. \quad (11)$$

To evaluate the last term in (10), we write

$$e^{-rt}R(q^T(t)) = e^{-rt}(q^T(t) - q^T(t)^2),$$

where

$$\begin{aligned} e^{-rt}q^T(t) &= \frac{1}{2}e^{-rt} - \frac{1}{2}\frac{r(T-2S)}{e^{rT}-1}, \\ e^{-rt}q^T(t)^2 &= \frac{1}{4}e^{-rt} - \frac{1}{2}\frac{r(T-2S)}{e^{rT}-1} + \frac{1}{4}\frac{e^{rt}r^2(T-2S)^2}{(e^{rT}-1)^2}, \end{aligned}$$

so

$$e^{-rt}(q^T(t) - q^T(t)^2) = \frac{1}{4}e^{-rt} - \frac{1}{4}\frac{e^{rt}r^2(T-2S)^2}{(e^{rT}-1)^2}.$$

We find that

$$\int_0^T e^{-rt}R(q^T(t))dt = \frac{1}{4r}(1 - e^{-rT}) - \frac{1}{4}\frac{r(T-2S)^2}{e^{rT}-1}. \quad (12)$$

Substituting (11) and (12) in (10) and dividing by r , we find for this case of a linear demand function that (10) reduces to

$$K - \frac{1}{2}(T^* - 2S) + \frac{r e^{rT^*} + 1}{4 e^{rT^*} - 1}(T^* - 2S)^2 = 0. \quad (13)$$

The latter formula implies immediately that $T^* > 2(S + K)$.

Proposition 6.1 *Let (q^*, T^*) be the optimal strategy of (1) when the demand function is linear. Then $T_r^* > 0$.*

Proof. We make use of the implicit function theorem, from which it follows that T_r^* is equal to minus the ratio between the partial derivative of the left-hand side of equation (13) with respect to r and the partial derivative with respect to T^* . It has been shown in the next-to-last line of the proof of (a) of Proposition 5.1 that the latter derivative is negative. The partial derivative of the left-hand side of equation (13) with respect to r equals $(T^* - 2S)^2/4$ times

$$\frac{(e^{rT^*} + rT^*e^{rT^*} + 1)(e^{rT^*} - 1) - (re^{rT^*} + r)(e^{rT^*}T^*)}{(e^{rT^*} - 1)^2} = \frac{e^{2rT^*} - 2rT^*e^{rT^*} - 1}{(e^{rT^*} - 1)^2}.$$

The sign of this expression is equal to the sign of the numerator. We have that

$$\begin{aligned} e^{2rT^*} - 2rT^*e^{rT^*} - 1 &= (e^{rT^*} - rT^*)^2 - r^2T^{*2} - 1 \\ &> (1 + rT^* + \frac{1}{2}r^2T^{*2} - rT^*)^2 - r^2T^{*2} - 1 \\ &= \frac{1}{4}r^4T^{*4} > 0, \end{aligned}$$

where the inequality follows from a second-order Taylor approximation to e^{rT^*} . \square

An increase in r increases the benefits from postponing an order. For linear demand functions, this leads to an unambiguous positive effect of the discount rate on the optimal order time.

One may have the intuition that an increase in S should also lead to an increase in the optimal order time, as one may guess that it takes longer to sell a bigger stock. Surprisingly, this is not always the case. The sign of T_S^* is equal to the sign of the partial derivative of (13) with respect to S . The latter derivative is equal to

$$1 - r \frac{e^{rT^*} + 1}{e^{rT^*} - 1} (T^* - 2S).$$

The sign of this derivative is positive if and only if

$$T^* - 2S < \frac{1 - e^{-rT^*}}{r(1 + e^{-rT^*})}.$$

Since

$$\widehat{T} - 2S = \frac{1 - e^{-r\widehat{T}}}{r} > \frac{1 - e^{-r\widehat{T}}}{r(1 + e^{-r\widehat{T}})},$$

the sign of the derivative is negative when T^* is sufficiently close to \widehat{T} .

For instance, when $S = 4$, $r = 0.05$, and $K = 2$, we have that $\widehat{T} = 21.00$ and $T^* = 17.36$. When S is increased to 4.1, we find that $T^* = 17.17$, so T_S^* is approximately equal to -1.9 . In this example, the order costs are so high that the only way for the seller to make a positive profit is to choose a sales strategy that maximizes the revenues coming out of a single batch, i.e. is close to $q^{\widehat{T}}$, and the profits are barely positive. When S increases, the seller can make a relatively much higher profit per batch, and by lowering the order time, he can increase the number of batches per time-period, which explains why T_S^* can be negative in an optimal sales strategy. This case is not an artifact, but would be realistic for products with a low profit margin. Since $T^* > 2(S + K)$, it is clear that for higher values of S , the sign of T_S^* becomes positive.

In Figure 1 we consider the case where $S = 10$, $r = 0.05$, and $K = 2$ and plot the optimal sales path. In Table 1 we take $K = 2$ and display the value of T^* for S ranging from 3 to 10 and r from 0.01 to 0.09. The value ∞ is used when the order costs K exceed the maximum discounted revenue w^* , which happens for low values of S and high values of r . In these cases, the seller would not place new orders. We have shown in Proposition 6.1 that $T_r^* > 0$. The values for T^* in the columns of Table 1 are therefore increasing with r . We have argued that intuitively one would expect $T_S^* > 0$, in particular for high values of S , but that $T_S^* < 0$ may occur, in particular when profit margins are low. In Table 1 we have two instances of negative values of T_S^* when $S = 3$ and $r = 0.01$ and when $S = 4$ and $r = 0.05$. In these cases, an increase in S leads to a lower value of T^* .

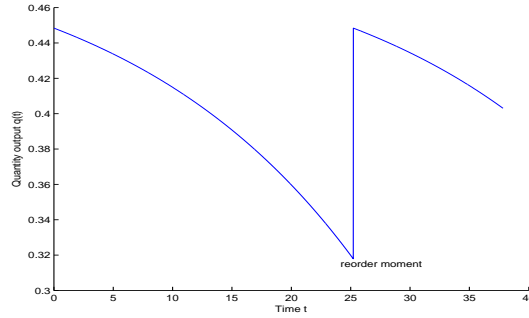


Figure 1 The optimal sales path for $R(q) = q(1 - q)$, $S = 10$, $r = 0.05$, and $K = 2$. It holds that $T^m = 20$, $\hat{T} = 36.83$, and $T^* = 25.22$.

r	Batch size S							
	3	4	5	6	7	8	9	10
0.01	18.21	16.03	16.68	18.01	19.61	21.34	23.15	25.01
0.03	∞	16.35	16.83	18.11	19.69	21.42	23.22	25.08
0.05	∞	17.36	17.19	18.35	19.88	21.58	23.37	25.22
0.07	∞	∞	18.08	18.82	20.23	21.87	23.64	25.47
0.09	∞	∞	∞	20.19	20.96	22.43	24.12	25.91

Table 1 The value of T^* for different values of S and r when $K = 2$.

7 Concluding Remarks

We have studied an infinite horizon model, where a seller orders a product in batches of fixed size. We have shown that there exists a unique strategy that maximizes the total sum of discounted profits. We characterize this strategy and show that it is stationary. It follows in particular that the seller always reorders after the same amount of time has elapsed. The seller puts the largest quantity on the market the moment it starts selling its new stock. The sales quantity continuously decreases, or equivalently the sales price continuously increases, until it is at its lowest point right before the seller runs out of its current stock. This whole cycle repeats itself when the seller buys new stock again.

As expected, increases in order costs K lead to increases in the optimal order time T^* . For linear demand functions, we show that increases in the interest rate r lead to increases in T^* as well. Surprisingly, the effect of an increase in batch size S on optimal order time T^* is ambiguous. In particular for products with low profit margins, we have the somewhat counterintuitive phenomenon that increases in S may well lead to lower values for T^* .

One possible extension of the model would be to add explicit costs of decay. To some

extent, costs of decay can be captured in the current set-up by adding the rate of decay to the interest rate when discounting the revenues from future sales. However, decay does not affect the rate at which future order costs are discounted. We therefore expect that such extensions will not influence the qualitative features of the optimal sales path, although making the holding of inventory more costly will most likely result into a more aggressive sales strategy by the seller, thereby decreasing the order time.

Obviously, there are many other extensions one can conceive of, which would lead us farther away from the present model. Instead of a single batch of size S and cost K , one could have a whole menu for choices of S , each S being associated with a particular K . We could leave the set-up with time-invariant demand functions, and study time-varying demand, which, moreover, might be stochastic, and add explicit backlog penalty costs. Instead of a monopolistically behaving seller, one might consider the strategic interaction between sellers in a duopoly or an oligopoly. This would lead us into the direction of the literature on competition under capacity constraints as in Osborne and Pitchik (1986) and van den Berg, Bos, Herings, and Peters (2012).

Appendix

Proof of Lemma 4.1 We first compute the first and second derivatives of V with respect to T , denoted by V' and V'' , respectively. Differentiating the identity $\int_0^T q^T(t)dt = S$ on both sides with respect to T , we obtain

$$q^T(T) + \int_0^T \frac{\partial q^T(t)}{\partial T} dt = 0. \quad (14)$$

Now

$$\begin{aligned} V'(T) &= e^{-rT} R(q^T(T)) + \int_0^T e^{-rt} R'(q^T(t)) \frac{\partial q^T(t)}{\partial T} dt \\ &= e^{-rT} R(q^T(T)) + \int_0^T c^T \frac{\partial q^T(t)}{\partial T} dt \\ &= e^{-rT} R(q^T(T)) - c^T q^T(T), \end{aligned} \quad (15)$$

where the second equality follows from (8) and the third equality from (14).² Differentiating (15) once again we obtain

$$\begin{aligned} V''(T) &= -re^{-rT} R(q^T(T)) + e^{-rT} R'(q^T(T)) \frac{\partial q^T(T)}{\partial T} - \frac{\partial c^T}{\partial T} q^T(T) - c^T \frac{\partial q^T(T)}{\partial T} \\ &= -re^{-rT} R(q^T(T)) - \frac{\partial c^T}{\partial T} q^T(T). \end{aligned} \quad (16)$$

Combining (15), (16), and Lemma 3.4 we obtain

$$V''(T) + rV'(T) = -\frac{\partial c^T}{\partial T} q^T(T) - rc^T q^T(T) \leq 0, \quad (17)$$

which will be useful at several places below.

The first derivative Π' of Π with respect to T is

$$\Pi'(T) = \frac{V'(T)(1 - e^{-rT}) - (V(T) - K)re^{-rT}}{(1 - e^{-rT})^2}. \quad (18)$$

If T is a stationary point of Π , i.e., $\Pi'(T) = 0$, then (18) implies

$$r(V(T) - K) = V'(T)(e^{rT} - 1). \quad (19)$$

Now (10) follows from (19) and (15).

We observe that equation (10) evaluated at $T = T^m$ is equal to

$$\begin{aligned} (e^{rT^m} - 1) (e^{-rT^m} R(q^m)) + r(K - \int_0^{T^m} e^{-rt} R(q^m) dt) \\ = R(q^m) - e^{-rT^m} R(q^m) + rK - R(q^m) + e^{-rT^m} R(q^m) = rK > 0 \end{aligned}$$

²Observe that we have reestablished the fact that $V'(T)$ is equal to the Hamiltonian evaluated at the optimal path and at the terminal time.

and equation (10) evaluated at $T = \widehat{T}$ equals

$$r(K - \int_0^{\widehat{T}} e^{-rt} R(q^{\widehat{T}}(t)) dt) = r(K - w^*) < 0.$$

It follows that a stationary point T^* belongs to the open interval (T^m, \widehat{T}) , hence $q^{T^*}(T^*) > 0$ and therefore (17) holds with strict inequality at a stationary point,

$$V''(T^*) + rV'(T^*) < 0. \quad (20)$$

Differentiating (18) once again, we obtain after some simplifications:

$$(1 - e^{-rT})^3 \Pi''(T) = V''(T)(1 - 2e^{-rT} + e^{-2rT}) + V'(T)(2re^{-2rT} - 2re^{-rT}) \quad (21)$$

$$+ V(T)(r^2e^{-rT} + r^2e^{-2rT}) + K(-r^2e^{-rT} - r^2e^{-2rT}). \quad (22)$$

At a stationary point, using (19), the sum of the terms in (22) can be written as

$$V'(T)r(1 - e^{-2rT}).$$

Combining this with (21), we obtain

$$(1 - e^{-rT})^3 \Pi''(T) = (V''(T) + rV'(T))(1 - 2e^{-rT} + e^{-2rT}). \quad (23)$$

Since $1 - 2e^{-rT} + e^{-2rT} = (1 - e^{-rT})^2 > 0$ for $T > 0$, and (17) holds with strict inequality at a stationary point, the right-hand side of (23) is negative. This implies that the maximizer T^* must be unique. \square

Proof of Proposition 5.1

(a) By (19) we have $V'(T)(e^{rT} - 1) - r(V(T) - K) = 0$ at $T = T^*$. Regarding T^* as a function of K and totally differentiating both sides with respect to K , we obtain

$$rV'(T^*)T_K^* - r = V''(T^*)T_K^*(e^{rT^*} - 1) + V'(T^*)e^{rT^*} rT_K^*,$$

which after some simplification yields

$$T_K^* = \frac{r}{(1 - e^{rT^*})(rV'(T^*) + V''(T^*))}.$$

The right-hand side is positive since $1 - e^{rT^*} < 0$, and $rV'(T^*) + V''(T^*) < 0$ by (20). Hence, $T_K^* > 0$.

(b) Since V is now a function of T and r , we use the notation V_T and V_{TT} instead of V' and V'' . Differentiating the equilibrium condition (19) with respect to r and simplifying yields

$$T_r^*[(1 - e^{rT^*})(rV_T + V_{TT})] = V_{Tr}(e^{rT^*} - 1) + T^*V_T e^{rT^*} - rV_r + K - V,$$

hence by substituting for $K - V$ using (19) again, and by (20), we have

$$\text{sign}(T_r^*) = \text{sign} \left((T^* e^{rT^*} - \frac{1}{r}(e^{rT^*} - 1))V_T - rV_r + V_{Tr}(e^{rT^*} - 1) \right). \quad (24)$$

The coefficient $A = T^* e^{rT^*} - \frac{1}{r}(e^{rT^*} - 1)$ of V_T at the right hand side of (24) is positive for $T^* > 0$ since it is zero for $T^* = 0$ and has a positive derivative with respect to T^* . From (19) it follows that $V_T \geq 0$.

By differentiating the expression $V = \int_0^T e^{-rt} R(q^T(t)) dt$ with respect to r we obtain

$$V_r = \int_0^T -te^{-rt} R(q^T(t)) dt + \int_0^T e^{-rt} R'(q^T(t)) \frac{\partial q^T(t)}{\partial r} dt.$$

The second integral can be written as $c^T \int_0^T (\partial q^T(t)/\partial r) dt$, and this is equal to zero since $\int_0^T (\partial q^T(t)/\partial r) dt = \partial(\int_0^T q^T(t) dt)/\partial r = \partial S/\partial r = 0$. Hence, $V_r = \int_0^T -te^{-rt} R(q^T(t)) dt \leq 0$.

(c) We start again from (19) but now regard V as a function of both S and T , and $T = T^*$ as a function of S . By totally differentiating (19) with respect to S and for briefness again suppressing arguments of functions we obtain

$$r(V_S + V_T T_S^*) = (V_{TS} + V_{TT} T_S^*)(e^{rT^*} - 1) + V_T e^{rT^*} r T_S^*$$

which simplifies to

$$T_S^* [(e^{rT^*} - 1)(-rV_T - V_{TT})] = (e^{rT^*} - 1)V_{TS} - rV_S. \quad (25)$$

Since $rV_T + V_{TT} \leq 0$ by (17), the sign of T_S^* is equal to the sign of the right-hand side of (25). We continue by establishing expressions for V_{TS} and V_S .

Let $q = q^{T^*}$ be the optimal path with associated $c = c^{T^*}$, i.e., $\int_0^{T^*} q(t) dt = S$ and $R'(q(t)) = ce^{rt}$ for all $0 \leq t \leq T^*$. Then $\int_0^{T^*} (\partial q(t)/\partial S) dt = 1$ and thus

$$\begin{aligned} V_S &= \frac{\partial}{\partial S} \left(\int_0^{T^*} e^{-rt} R(q(t)) dt \right) = \int_0^{T^*} e^{-rt} R'(q(t)) \frac{\partial q(t)}{\partial S} dt \\ &= \int_0^{T^*} e^{-rt} e^{rt} c \frac{\partial q(t)}{\partial S} dt = c \cdot 1 = c. \end{aligned}$$

By using (15) we obtain for the partial derivative of V_T with respect to S :

$$\begin{aligned} V_{TS} &= \frac{\partial}{\partial S} (e^{-rT^*} R(q(T^*)) - cq(T^*)) \\ &= e^{-rT^*} R'(q(T^*)) \frac{\partial q(T^*)}{\partial S} - \frac{\partial c}{\partial S} q(T^*) - c \frac{\partial q(T^*)}{\partial S} = -\frac{\partial c}{\partial S} q(T^*). \end{aligned}$$

Altogether we have

$$\text{sign}(T_S^*) = \text{sign} \left((1 - e^{rT^*}) \frac{\partial c}{\partial S} q(T^*) - rc \right). \quad (26)$$

In order to obtain an expression for $\partial c/\partial S$ we write P for $(R')^{-1}$. Then P is differentiable and we can write $\int_0^{T^*} q(t)dt = S$ as $\int_0^{T^*} P(ce^{rt})dt = S$. Differentiating both sides with respect to S (keeping T^* fixed) we obtain

$$\begin{aligned} 1 &= \int_0^{T^*} P'(ce^{rt})e^{rt} \frac{\partial c}{\partial S} dt = \frac{\partial c}{\partial S} \left(\frac{1}{c} P(ce^{rt}) \Big|_0^{T^*} \right) \\ &= \frac{\partial c}{\partial S} \frac{1}{c} (P(ce^{rT^*}) - P(ce^{r0})) = \frac{1}{c} \frac{\partial c}{\partial S} (q(T^*) - q(0)) \end{aligned}$$

hence

$$\frac{\partial c}{\partial S} = \frac{c}{q(T^*) - q(0)} .$$

Hence, with (26), $\text{sign}(T_S^*) = \text{sign}(q^*(T^*)(e^{rT^*} - 1 + r) - q^*(0)r)$. □

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