Pricing swaptions and coupon bond options in affine term structure models

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We propose an approach to find an approximate price of a swaption in affine term structure models. Our approach is based on the derivation of approximate swap rate dynamics in which the volatility of the forward swap rate is itself an affine function of the factors. Hence, we remain in the affine framework and well-known results on transforms and transform inversion can be used to obtain swaption prices in similar fashion to zero bond options (i.e., caplets). The method can easily be generalized to price options on coupon bonds. Computational times compare favorably with other approximation methods. Numerical results on the quality of the approximation are excellent. Our results show that in affine models, analogously to the LIBOR market model, LIBOR and swap rates are driven by approximately the same type of (in this case affine) dynamics.

KEY WORDS: swaption, coupon bond option, affine term structure models, change of numéraire, swap measure, conditional characteristic function, option pricing using transform inversion

1. INTRODUCTION

Swaptions are among the most liquid derivatives being traded in the financial markets. The theoretical price of these instruments depends on the term structure model being used. From a different point of view, it is critical for term structure models to be able to fit swaption prices well. The affine term structure models (ATSMs) by Duffie and Kan (1996) are a popular framework for analyzing term structure movements and pricing interest rate derivatives. However, in these models swaption prices cannot be obtained in closed form. In this paper, we introduce a fast and accurate approximation to the price of a swaption in ATSMs. Using our method, the price of a swaption can be obtained by exactly the same techniques needed to price bond options or caplets (see Duffie, Pan, and Singleton 2000). Furthermore, the method can be easily generalized to price options on coupon bonds. Hence, throughout the paper we write swaptions when we could have written both swaptions and coupon bond options.

The price of a swaption can be written in terms of the distribution of the swap rate under the relevant swap measure. Our method is based on approximating the dynamics of the...
swap rate under this measure. We derive the dynamics of the swap rate and the underlying factors under the swap measure for a general ATSM. Then we suggest to approximate the dynamics by replacing some low-variance martingales (LVM) by their time zero values. After this approximation, we are again in the affine setting but now under the swap measure (and with time-dependent drift for the factors and time-dependent volatility for the swap rate). This approach allows us to remain in the affine setup and use results on transforms and transform inversion of ATSMs to price the swaption. The accuracy of this approximation is excellent. In a Gaussian framework, this simplifies considerably, and analytical pricing formulas can be derived.

The technique of replacing LVM by their martingale values is also used in the context of the LIBOR market model (LMM) to derive swap volatilities in this model (see Brace, Dunn, and Barton 2001; and Hull and White 2000). Our results contribute to the discussion on the existence of a central interest rate model by showing that in the affine class of term structure models, swap and LIBOR rates have (approximately) distributions of the same type. It seems that this property is not exclusive for LMM, but rather is common for every term structure model.

Other papers considering swaption pricing in ATSMs are Munk (1999), Singleton and Umantsev (2002), and Collin-Dufresne and Goldstein (2002). The paper by Singleton and Umantsev introduces a method which is based on approximation of the exercise region in the space of underlying factors by line segments. This reduces the exercise probability of the swaption to the form of that of a caplet. They then use transform inversion to calculate the required exercise probabilities. Similar to our method, they require a simplifying assumption to return to the affine framework. Our approach differs from theirs, as it doesn’t approximate the exercise region, but finds approximate affine dynamics for the swap rate. In this way, we decrease computational time and simplify implementation in general affine models.

Collin-Dufresne and Goldstein (2002) propose to approximate the price of a swaption based on an Edgeworth expansion of the density of the coupon bond price (i.e., swap rate). This requires the calculation of the moments of the coupon bond through the joint moments of the individual zero coupon bonds, which are available in closed form. Instead, our results on approximate swap rate dynamics enable us to calculate the approximate conditional characteristic function (CCF) of the swap rate directly. This allows us to use well-known transform inversion techniques and avoid the computation of joint moments of zero coupon bonds, which could become very time consuming. Therefore, contrary to their approach, our implementation easily generalizes to more general ATSMs without an increase in computational effort.

Munk (1999) shows that the price of a European option on a coupon bond (i.e., a swaption) is approximately proportional to the price of a European option on a zero coupon bond with maturity equal to the stochastic duration of the coupon bond. The stochastic duration approximation is closely related to our method, as it uses a similar approach to approximate the volatility of a coupon bond/swaption. However, our method is based on a formal derivation of the factor and swap rate dynamics under the swap measure, whereas stochastic duration is more of an ad hoc approximation.

We improve upon the existing literature in either speed, accuracy, or both. Most importantly, our methods are easier to apply to models with a higher number of factors or correlated factors than the approximations by Munk (1999), Singleton and Umantsev (2002), and Collin-Dufresne and Goldstein (2002).

The outline of the paper is the following. In Section 2, we shortly review the mechanics of swaptions and ATSMs. In Section 3, we review the alternative approaches by Munk
(1999), Collin-Dufresne and Goldstein (2002), and Singleton and Umantsev (2002). Section 4 introduces our approximation method in the context of the Vasicek model. In Section 5, we present our approximations for general ATSMs. The special case of Gaussian term structure models (TSMs) is discussed in an appendix. Section 6 discusses the quality, speed, and implementation of the approximation method in comparison with other methods in the literature. Section 7 concludes.

2. PRELIMINARIES: SWAPS AND SWAPTIONS, ATSMs

Let \( D(t, T) \) be the time \( t \) price of a zero coupon bond with maturity \( T \). A forward LIBOR rate \( L_{TS}(t) \) is the interest rate one can contract for at time \( t \) to put money in a money-market account for the time period \([T, S]\). We define

\[
L_{TS}(t) = \frac{1}{\Delta^L_{TS}} \frac{D(t, T) - D(t, S)}{D(t, S)},
\]

where \( \Delta^L_{TS} \) is the LIBOR market convention for the calculation of the daycount fraction for the period \([T, S]\). In the market, the tenor of the LIBOR rate, \( S - T \), is usually fixed at 3 or 6 months. Note that the LIBOR rate is fixed at time \( T \), but is not paid until time \( S \).

An interest rate swap is a contract in which two parties agree to exchange a set of fixed cash flows, consisting of a fixed rate \( K \) on the swap principal \( A \), for a set of floating rate payments, consisting of the LIBOR rate on the principal \( A \). In a payer swap you pay the fixed side and receive floating, in a receiver swap you receive the fixed side (and pay floating). Given a set of dates \( T_i, i = n + 1, \ldots, N \), at which swap payments are to be made, the value at time \( t \) of a (payer) swap contract starting at time \( T_n \) (paying out for the first time at \( T_{n+1} \)) and lasting until \( T_N \) with a principal of 1 and fixed payments at rate \( K \) is given by

\[
(2.1) \quad V^{pay}_{n,N}(t) = V^{flo}_{n,N}(t) - V^{fix}_{n,N}(t) = \{D(t, T_n) - D(t, T_N)\} - K \sum_{i=n+1}^{N} \Delta^Y_{i-1} D(t, T_i),
\]

where \( \Delta^Y_{i-1} \) is the market convention for the calculation of the daycount fraction for the swap payment at \( T_i \). Again given a set of payment dates \( T_i \), a forward par swap rate \( y_{n,N}(t) \) is defined by the fixed rate for which the value of the (forward starting) swap equals zero, solving (2.1) gives

\[
(2.2) \quad y_{n,N}(t) = \frac{D(t, T_n) - D(t, T_N)}{\sum_{i=n+1}^{N} \Delta^Y_{i-1} D(t, T_i)} = \frac{D(t, T_n) - D(t, T_N)}{P^{n+1,N}_{n,N}(t)}.
\]

The rate \( y_{n,N}(t) \) is the (arbitrage free) rate at which at time \( t \) a person would like to enter into a swap contract starting at time \( T_n \) (paying out for the first time at \( T_{n+1} \)) and lasting until \( T_N \). If we look carefully, we see that the 1-year forward LIBOR rate is equal to the 1-year forward par swap rate. The denominator of the swap rate, \( P^{n+1,N}_{n,N}(t) \), is called the present value of a basis point (PVBP), as it corresponds to the increase in value of the fixed side of the swap if the swap rate increases.

A swaption gives the holder the right to enter into a particular swap contract. A swaption with option maturity \( S_1 \) and swap maturity \( S_2 \) is termed a \( S_1 \times S_2 \)-swaption. The total time span associated with the swaption is then \( S_1 + S_2 \). When the strike equals
the forward par swap rate, the option is at-the-money-forward (ATMF). A payer swaption gives the holder the right to enter into a payer swap and can be seen as a call option on a swap rate. The option has a payoff at time $T_n$, the option maturity of

$$\left[ V_{n,N}^{\text{pay}}(T_n) \right]^+ = \left[ (D(T_n, T_n) - D(T_n, T_N)) - K \sum_{i=n+1}^{N} \Delta_i^{\text{ps}}(T_n, T_i) \right]^+$$

$$= [y_{n,N}(T_n)P_{n+1,N}(T_n) - K P_{n+1,N}(T_n)]^+$$

$$= P_{n+1,N}(T_n)[y_{n,N}(T_n) - K]^+,$$

where $K$ denotes the strike rate of the swaption. It can be seen from the first line of (2.3) that the price of a payer swaption can be interpreted as a European put option with strike 1 on a coupon bond with coupon $K$. The second line follows from the definition of the forward swap rate. Equivalently, a receiver swaption can be seen as a put option on a swap rate. Let $\mathcal{M}_t = \exp(\int_0^t r_s \, ds)$ be the money-market account at time $t$. Assuming absence of arbitrage, the value of a (payer) swaption with strike $K$ at time $t < T_n$, denoted by $\text{PS}_S(K)$, can be expressed by the following risk-neutral conditional expectation

$$\text{PS}_S(K) = M_t E_t^Q \left\{ \frac{P_{n+1,N}(T_n)}{M_{T_n}} \left[ y_{n,N}(T_n) - K \right]^+ \right\},$$

which can be rewritten using familiar change of numeraire techniques (see Geman, El Karoui, and Rochet 1995) as

$$\text{PS}_S(K) = P_{n+1,N}(t) E_t^Q \left\{ \left[ y_{n,N}(T_n) - K \right]^+ \right\},$$

where we let $Q^{n+1,N}$ be the swap measure corresponding to a particular PVBP, $P_{n+1,N}$, as numeraire. The price of the corresponding receiver swaption, $\text{RS}_S(K)$, is then

$$\text{RS}_S(K) = P_{n+1,N}(t) E_t^Q \left\{ \left[ K - y_{n,N}(T_n) \right]^+ \right\}.$$

Note that under this swap measure, the corresponding swap rate, $y_{n,N}$, is a martingale. The Radon–Nikodym (RN) derivative for this change of numeraire equals the ratio of numéraires, i.e.,

$$\frac{dQ^{n+1,N}}{dQ} = \frac{P_{n+1,N}(T_n) / P_{n+1,N}(t)}{M_{T_n} / M_t}.$$

The change of numéraire shows explicitly why swaptions can be viewed as options on swap rates. This particular choice of numéraire can be attributed to Jamshidian (1998).

ATSMs were introduced by Duffie and Kan (1996). Other publications include, among others, Dai and Singleton (2000) and Duffee (2002). Recently, the affine framework was extended to include jump diffusions by Duffie et al. (2000), from hereon DPS. In this paper, we follow Singleton and Umantsev (2002) in our definition of the family of ATSMs. In these models, the short rate is modeled as an affine function of some latent factors, $X_t$, that follow a diffusion process

$$r_t = \omega_0 + \omega_X X_t,$$

where $\omega_0$ is a scalar and $\omega_X$ is an $M$ vector. The $M$-dimensional factor dynamics are given by the following diffusion,

$$dX_t = \Lambda(t - X_t) \, dt + \Sigma \sqrt{V_t} \, dW_t^Q,$$

where $W_t^Q$ is an $M$-dimensional Brownian motion under the risk-neutral measure, $\Lambda$ and $\Sigma$ are $M \times M$ matrices, and $\theta$ is an $M$ vector. The matrix $V_t$ is a diagonal matrix holding the diffusion coefficients of the factors on the diagonal, i.e.,

$$V_{t,(ii)} = \alpha_i + \beta_i X_t \quad i = 1, 2, \ldots, M.$$
where the \( \beta_i \) are \( M \) vectors. Or directly in matrix notation, defining the matrix \( \beta = [\beta_1 \cdots \beta_M] \) and the vector \( \alpha = [\alpha_1 \cdots \alpha_M] \), we have

\[
V_t = \text{diag}(\alpha + \beta X_t).
\]

The instantaneous drift and variance of the factors are again affine functions of the factors. As a result, bond prices are exponentially affine in the factors, with coefficients which can be obtained by solving a system of ODEs, known as Riccati equations. Thus in ATSMs, we have \( D(t, T) = \exp(A(t, T) - B(t, T) \cdot X_t) \). Applying Itô’s lemma to \( D(t, T) \) gives

\[
dD(t, T) = r_t D(t, T) dt - B(t, T) D(t, T) \Sigma_t^1 \, dW_t^Q.
\]

Usually when setting up an affine model, one also specifies a vector of market prices of risk for the factors. Since we are concerned with option pricing, we require only knowledge of the \( \mathbb{Q} \) dynamics of the factors and hence leave the market prices of risk unspecified. Discussion on specification of these market prices is given in i.a. Duffee (2002).

Besides closed form solutions for bond prices, options on zero coupon bonds (i.e., caplets) are easily priced in affine models using transform inversion techniques (DPS, Carr and Madan 1999). To see this, let \( \mathbb{Q}^T \) denote the \( T \)-Forward measure, write for the time \( t \) price, \( C_t(K, T_0, T) \), of a call option on a zero coupon bond with strike \( K \), maturity \( T \), and option maturity \( T_0 \),

\[
C_t(K, T_0, T) = E_t^{\mathbb{Q}^T}[e^{-\int_0^{T_0} r_s ds}(D(T_0, T) - K)^+]
= D(t, T_0) E_t^{\mathbb{Q}^T}[(D(T_0, T) - K)^+].
\]

Now define the CCF of the log bond price,

\begin{equation}
\zeta(v, t, T_0) = E_t^{\mathbb{Q}^T}[\exp(iv \ln(D(T_0, T)))].
\end{equation}

Because \( D(T_0, T) \) is exponentially affine in the factors, \( \zeta \) is known in closed form (see DPS). Further define the dampened call price \( \xi(u, t, T_0) = \exp(\alpha \ln(K)) C_t(K, T_0, T) \) with dampening coefficient \( \alpha > 0 \) and the dampened call transform

\begin{equation}
\xi(u, t, T_0) = \int_{-\infty}^{\infty} \exp(iu \ln(K)) c_t(K) \, dK.
\end{equation}

Then (2.10) and (2.11) are related by

\begin{equation}
\xi(u, t, T_0) = \frac{D(t, T_0) \xi(u - i(\alpha + 1), t, T_0)}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u}
\end{equation}

(see Carr and Madan 1999). The price of the call option can now be calculated by a single integration,

\[
C_t(K, T_0, T) = \frac{\exp(-\alpha \ln(K))}{\pi} \int_0^\infty \text{Re}[\exp(-iu \ln(K))\xi(u, t, T_0)] \, du.
\]

It is important to note that if we know the CCF of \( y_{n,N} \), we could also price swaptions through transform inversion.\(^2\) This would, for instance, be the case if the dynamics of \( y_{n,N} \) were itself to be affine in the factors. In Sections 4 and 5, we propose an approximation method which is partly based on this observation.

\(^1\) If \( x \) is an \( M \)-vector, then we define \( \text{diag}(x) \) to be the \( M \times M \) diagonal matrix with the elements of \( x \) on the diagonal.

\(^2\) In the bond option case, we know the CCF of the logarithm of the underlying in closed form. When the CCF of the underlying itself is known in closed form, the procedure is similar and can be found in Lee (2004).
3. LITERATURE REVIEW

In this section, we give a short technical review of currently known approximation methods to the price of a swaption. Whereas we base our approximation method on (2.5), both Collin-Dufresne and Goldstein (2002) and Singleton and Umantsev (2002) rewrite (2.4) as

\[ PS_t(K) = D(t, T_n) P_t^{Q^T} (y_{n,N}(T_n) > K) - (1 + \Delta_{N-1}^Y K) D(t, T_N) P_t^{Q^T} (y_{n,N}(T_n) > K) - K \sum_{i=n+1}^{N-1} \Delta_{i-1}^Y D(t, T_i) P_t^{Q^T} (y_{n,N}(T_n) > K). \]

using a change of measure to several forward measures. The difficulty is that \( y_{n,N} \) is not a martingale under any of the forward measures. Now both methods are based on the approximation of the exercise probabilities under the different forward measures. The method by Collin-Dufresne and Goldstein is based on an Edgeworth expansion of the distribution of a coupon bond with coupon \( K \). As the \( D(t, T_i) \) are exponential affine functions of the factors, the moments of the coupon bond are exponential affine as well and the coefficients can be obtained by solving the familiar Riccati equations. The use of an Edgeworth expansion is validated by the low volatility of fixed income instruments and the coefficients can be obtained by solving the familiar Riccati equations. The use of an Edgeworth expansion is validated by the low volatility of fixed income instruments and the resulting approximation is tight. However, note that to determine the \( k \)th moment of \( \sum_{i=1}^{T_n} c_i D(t, T_i) \), one has to determine (and sum) \( \binom{t+k-1}{k} \) joint moments of zero coupon bonds. This would equal 11,628 for the fifth moment of a 15-year swap rate. Since these individual moments have to be determined by solving Riccati equations, this could slow the procedure down considerably when no analytical solutions are available.

Taking a different point of view, Singleton and Umantsev try to approximate the exercise probability through an approximation of the exercise region itself. They propose to linearize the exercise region, i.e., approximate it by a hyperplane, and use the approximation \( P_t^{Q^T} (y_{n,N}(T_n) > K) \approx P_t^{Q^T} (a \cdot X_T > b) \), where the vector \( a \) and the constant \( b \) are to be determined using some procedure involving least squares.

The stochastic duration approximation by Munk postulates the following approximate price for a payer swaption (i.e., a put option on a coupon bond with coupon \( K \) and strike 1),

\[ PS_t(K) = D(t, T_n) E_t^{Q^T} [1 - \xi D(T_n, t + SD(t))]^+, \]

where \( \xi = \frac{K \cdot PVBP_{n+1,N}(t) + D(t, T_N)}{D(t, t + SD(t))} \) and \( SD(t) \) is the stochastic duration of the coupon bond with payments of \( K \Delta_{i-1}^Y \), at time \( T_i = n + 1, \ldots, N \). The approach aims at approximating the coupon bond process by a more simple process with similar volatility, hence we can interpret \( CB_t = \xi D(t, t + SD(t)) \) as an approximation to \( CB_t = K \cdot PVBP_{n+1,N}(t) + D(t, T_N) \). We can now derive the price of a swaption using a change of measure from \( Q^T \) to \( Q^{CB} = Q^{CB} \) to obtain

\[ PS_t(K) = D(t, T_n) P_t^{Q^T} (\widetilde{CB}(T_n) < 1) - CB(t) P_t^{Q^{CB}} (\widetilde{CB}(T_n) < 1). \]

The previous equation implies two approximation errors. First, the volatility of the coupon bond is approximated and second, comparing (3.2) with (3.1), the exercise probabilities are taken under the wrong forward measure.
4. SWAPTITION PRICING WITHIN THE VASICEK MODEL: APPROXIMATION AND ERROR

Contrary to the methods of the previous section, we approximate the true factor and swap rate dynamics under the relevant swap measure by affine dynamics. Before we present the method in full generality, we will illustrate the approximation method within the context of the Vasicek model. The simplicity and analytical structure of the model allows us to focus more on the approximation and less on technical details. For completeness, Vasicek (1977) assumes the short rate, , follows an Ornstein–Uhlenbeck process,

\[ dr_t = a(\theta - r_t) dt + \sigma dW_t^Q, \]

where \( W_t^Q \) is one dimensional Brownian motion under the risk neutral measure. The parameter \( \theta \) is the risk neutral mean of \( r_t \) and \( a \) is the mean reversion rate. Note this is the ATSM with \( M = 1, \omega_0 = 0, \omega_X = 1, A = a, \beta = 0, \alpha = 1, \) and \( \Sigma = \sigma \). Bond prices are given by \( D(t, T) = \exp(A(t, T) - B(t, T)r_t) \), where \( B(t, T) = [1 - \exp(-a(T - t))] / a \).

4.1. Approximate Swap Rate and Factor Dynamics

In Section 2, we observed that if we know the characteristic function, or equivalently the distribution, of the swap rate under the swap measure, we can price a swaption using (2.5). We will exploit this observation and derive an approximation of the distribution of the swap rate under the swap measure by affine dynamics. Before we present the method in full generality, we will illustrate the approximation method within the context of the Vasicek model.

\[ d r_t = \theta (\theta - r_t) dt + \sigma d W_t^Q, \]

where \( W_t^Q \) is Brownian motion under the swap measure and

\[ d y_{n, N}(t) = \sigma \frac{\partial y_{n, N}(t)}{\partial r_t} d W_{t}^{Q+1, N}, \]

where \( W_{t}^{Q+1, N} \) is Brownian motion under the swap measure and

\[ \frac{\partial y_{n, N}(t)}{\partial r_t} = -B(t, T_n)D^P(t, T_n) + B(t, T_N)D^P(t, T_N) \]

\[ + y_{n, N}(t) \sum_{i=n+1}^{N} \Delta_{i-1} B(t, T_i)D^P(t, T_i), \]

where \( D^P(t, T_n) = D(t, T_n)/P_{n+1, N}(t) \), the bond price normalized by the numéraire, the PVBP.

For clarity, let us fix some notation. In the remainder of this paper, we let \( t \) denote running time and let \( t = 0 \) denote the specific point in time at which we want to value swaptions. The term structure at the time of valuation is thus given by \( D(0, T) \).

Now we consider the actual approximation. Since bond prices in this model are stochastic processes, the volatility of the swap rate is stochastic as well. However, the volatility of the swap rate is a function of asset prices normalized by the PVBP. Hence, the volatility is a function of martingales. We conjecture these martingales have sufficiently low variance to be approximated by their expectations, i.e., time zero values. This is similar to the approach taken in the literature on market models.

It is argued in Brace et al. (2001), Brace and Womersley (2000), and d’Aspremont (2003) that \( \frac{D(t, T_i)}{P_{n+1, N}(t)} \) are LVM in the context of a LIBOR market model (Miltersen, Sandmann, 1997).

3 An exact swaption price formula for this model can be found in Jamshidian (1989).

4 Assuming the absence of arbitrage, the swap rate is a martingale under the swap measure. Hence we have zero drift.
and Sonderman 1997; Jamshidian 1998; or Brace, Gatarek, and Musiela 1997). We conjecture, this is also the case in the affine class of term structure models. We will approximate the random terms $D(t, T_n) / P_{n+1, N}(0)$ by their conditional expected value under the swap measure, $D(t, T_n) / P_{n+1, N}(0)$. We also approximate the swap rate itself in the expression for the volatility of the swap rate by its martingale value. To be specific, we approximate the swap rate volatility by $\sigma \frac{\partial y_n(N(t), T_n)}{\partial r_t}$, where

$$\frac{\partial y_n(N(t), T_n)}{\partial r_t} = -B(t, T_n) D(t, T_n) + B(t, T_n) D(t, T_n) + y_n(N(t), T_n) \sum_{i=n+1}^{N} \Delta_i B(t, T_i) D(t, T_i).$$

With this approximation, swap rate volatility is deterministic. Therefore, the swap rate is a Gaussian martingale. However, we can take it a step further. If we take a closer look at the function $B(t, T)$, we can write

$$B(t, T) = \frac{1}{a} - \frac{e^{-aT}}{a} e^{at}.$$

We see that $B(t, T)$ can be split this into three separate functions of which only one is really time dependent. Furthermore, the constant $\frac{1}{a}$ cancels in (4.4). Using this in the approximate swap rate volatility, we can rewrite (4.4) as

$$\frac{\partial y_n(N(t), T_n)}{\partial r_t} = \frac{1}{a} e^{at} \left[ e^{-aT_n} D(t, T_n) - e^{-aT_n} D(t, T_n) - y_n(N(t), T_n) \sum_{i=n+1}^{N} \Delta_i e^{-aT_i} D(t, T_i) \right]
\equiv e^{at} \tilde{C}_{n, N},$$

where the first line defines $\tilde{C}_{n, N}$. Now if we define the integrated variance of $y_n(N(t), T_n)$ over the interval $[0, T_n]$ to be $\sigma_{n, N}$, we have

$$\sigma_{n, N} = \sqrt{\int_0^{T_n} \left( \frac{\sigma}{a} e^{at} \tilde{C}_{n, N} \right)^2 dt} = \sigma \tilde{C}_{n, N} \sqrt{\frac{e^{2aT_n} - 1}{2a}}.$$

All this leads to simple analytical pricing formulas for a swaption in the Vasicek model. Recall that

$$PS_0(K) = P_{n+1, N}(0) E_0^{Q_{n+1, N}} (y_n(N(T_n) - K)^+$$

and

$$RS_0(K) = P_{n+1, N}(0) E_0^{Q_{n+1, N}} (K - y_n(N(T_n))^+.$$

Now for an ATMF swaption, we have the following special result for the approximate price

$$\tilde{C}_{n, N}$$

5 A theoretical discussion of the approximation error is available upon request.
\[(4.5)\quad \text{PS}_0(y_{n,N}(0)) = \text{RS}_0(y_{n,N}(0))
\]
\[= P_{n+1,N}(0) \int_{y_{n,N}(0)}^{\infty} \frac{(x - y_{n,N}(0))}{\sigma_{n,N} \sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{x - y_{n,N}(0)}{\sigma_{n,N}} \right)^2 \right) dx
\]
\[= P_{n+1,N}(0) \frac{\sigma_{n,N}}{\sqrt{2\pi}}.
\]

For the swaption price approximation when the strike is not ATMF, we need to calculate the following integral, where \(\phi_{\mu,\sigma}(\cdot)\) is the density of a Gaussian r.v. with mean \(\mu\) and s.d. \(\sigma\), \(\Phi_{\mu,\sigma}\) is the corresponding distribution function and \(\Phi = \Phi_{0,1},\)

\[(4.6)\quad \text{PS}_0(K) = P_{n+1,N}(0) \int_{K}^{\infty} (x - K)\phi_{y_{n,N}(0),\sigma_{n,N}}(x) dx
\]
\[= P_{n+1,N}(0) \left[ (y_{n,N}(0) - K)\Phi \left( \frac{y_{n,N}(0) - K}{\sigma_{n,N}} \right) + \sigma_{n,N}\phi \left( \frac{K - y_{n,N}(0)}{\sigma_{n,N}} \right) \right]
\]

and for a receiver swaption,

\[(4.7)\quad \text{RS}_0(K) = P_{n+1,N}(0) \left[ (K - y_{n,N}(0))\Phi \left( \frac{K - y_{n,N}(0)}{\sigma_{n,N}} \right) + \sigma_{n,N}\phi \left( \frac{K - y_{n,N}(0)}{\sigma_{n,N}} \right) \right].
\]

For a general ATSM, a square root term containing the factors shows up in the swap rate volatility in (4.2). This complicates matters considerably. Approximations in this case are the subject of Section 5.

4.2. Numerical Results

We calculated approximate and exact prices at parameter values in Table 4.1. Numerical results on the quality of the approximation are given in Table 4.2. As one can see, these

| Table 4.1 Parameter Values of the Vasicek Model, the Two-Factor CIR Model and the Three-Factor Gaussian Model |
|---|---|---|
| Vasicek | Two-factor CIR | Three-factor Gaussian |
| \(a = 0.05\) | \(a_1 = 0.2\) | \(a_1 = 1.0\) |
| \(\sigma = 0.01\) | \(a_2 = 0.2\) | \(a_2 = 0.2\) |
| \(\theta = 0.05\) | \(\theta_1 = 0.03\) | \(a_3 = 0.5\) |
| \(r_0 = 0.05\) | \(\theta_2 = 0.01\) | \(\sigma_1 = 0.01\) |
| | \(\sigma_1 = 0.04\) | \(\sigma_2 = 0.005\) |
| | \(\sigma_2 = 0.02\) | \(\sigma_3 = 0.002\) |
| | \(\omega = 0.02\) | \(\rho_{12} = -0.2\) |
| | \(X_{1}(0) = 0.04\) | \(\rho_{13} = -0.1\) |
| | \(X_{2}(0) = 0.02\) | \(\rho_{23} = 0.3\) |
| | \(\omega = 0.06\) | \(X_{1}(0) = 0.01\) |
| | | \(X_{2}(0) = 0.005\) |
| | | \(X_{3}(0) = -0.02\) |

Note: The results on the quality of the approximation are produced using parameter values given here.
<table>
<thead>
<tr>
<th>Swap maturity</th>
<th>Option maturity</th>
<th>1</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ATMF</td>
<td>36.11 (0.00)</td>
<td>47.41 (0.00)</td>
<td>60.11 (0.00)</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>19.5% (0.00%)</td>
<td>19.1% (0.00%)</td>
<td>17.9% (0.00%)</td>
</tr>
<tr>
<td>2</td>
<td>68.78 (0.00)</td>
<td>90.30 (0.00)</td>
<td>114.51 (0.01)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>149.30 (0.01)</td>
<td>196.01 (0.03)</td>
<td>248.56 (0.08)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>239.86 (0.06)</td>
<td>314.89 (0.14)</td>
<td>399.31 (0.40)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ITM</td>
<td>81.08 (0.10)</td>
<td>87.67 (0.15)</td>
<td>92.85 (0.21)</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>21.2% (0.08%)</td>
<td>20.7% (0.08%)</td>
<td>19.4% (0.07%)</td>
</tr>
<tr>
<td>2</td>
<td>156.97 (0.19)</td>
<td>169.20 (0.29)</td>
<td>178.58 (0.40)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>357.27 (0.47)</td>
<td>381.62 (0.74)</td>
<td>398.84 (1.07)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>618.15 (1.05)</td>
<td>651.36 (1.76)</td>
<td>670.52 (2.71)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>OTM</td>
<td>11.49 (−0.10)</td>
<td>21.48 (−0.15)</td>
<td>35.87 (−0.20)</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>18.2% (−0.07%)</td>
<td>17.8% (−0.07%)</td>
<td>16.6% (−0.06%)</td>
</tr>
<tr>
<td>2</td>
<td>21.18 (−0.19)</td>
<td>40.04 (−0.29)</td>
<td>67.41 (−0.39)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>41.65 (−0.45)</td>
<td>81.39 (−0.69)</td>
<td>140.43 (−0.93)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>56.73 (−0.98)</td>
<td>117.32 (−1.53)</td>
<td>210.87 (−1.99)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>15.0% (−0.09%)</td>
<td>14.6% (−0.09%)</td>
<td>13.7% (−0.08%)</td>
</tr>
</tbody>
</table>

Note: To compare with stochastic duration, we obtain a relative error of 0.02% for a 1 × 10 ATMF swaption whereas the stochastic duration has an error of approximately 1% for an ATMF 6-month call on a 10-year coupon bond (Table 2 in Munk 1999). Collin-Dufresne and Goldstein report errors of 0.1% or less for a 2 × 10 swaption. The ITM results are for a payer swaption with a strike equal to 85% of the ATMF strike. The OTM results are for a payer swaption with a strike equal to 115% of the ATMF strike. For each swaption, we report on the first line the approximate price in basis points next to the error in basis points. On the second line, we report both approximate implied Black volatility in % and the error in implied Black volatility.

Results are excellent and compare very favorably with the stochastic duration approximation in Munk (1999) and all other results on swaption price approximations in Gaussian TSMs like, for example, Collin-Dufresne and Goldstein (2002). We show ATMF swaptions together with in-the-money (ITM) and out-of-the-money (OTM) swaptions.
We also show how the approximation performs over different option and swap maturities. These results essentially support our conjecture that we can approximate the swap rate dynamics under the swap measure by freezing some LVM at their martingale value. As one would expect, as we approximate swap rate volatility, the quality of the approximation declines with total volatility and hence maturity. The approximation performs well for both ATMF and ITM/OTM options. To compare with stochastic duration, we obtain a relative error of 0.0236% for a 1 × 10 ATMF swaption, whereas the stochastic duration has an error of approximately 1% for an ATMF 6-month call on a 10-year coupon bond (Table 2 in Munk 1999). Collin-Dufresne and Goldstein report errors of 0.1% or less for a 2 × 10 swaption. This number should be compared with the element in the fourth row of the second column in Table 4.2.

In the Appendix, we generalize the approach sketched above to multi-factor Gaussian TSMs. We also find an analytical formula for the approximate price in this general case. This result will speed up calibration of a multi-factor Gaussian model to ATMF swaption prices considerably. Besides for derivative pricing purposes, obtaining parameters for these models could be of interest when a Gaussian TSM is used in the pricing of stock options under stochastic interest rates, or for ALM purposes where a multi-factor Gaussian TSM is used in the pricing of insurance liabilities.

It is clear that for Gaussian TSM, our method is the fastest one available in the literature as it is the only one which gives analytical formulas for the approximate price of a swaption. But not only CPU performance is good. The quality of the approximation seems at least as good as the methods of Munk (1999), Singleton and Umantsev (2002), and Collin-Dufresne and Goldstein (2002).

5. APPROXIMATION METHOD FOR GENERAL AFFINE MODELS

In a general ATSM, we cannot infer the approximate distribution of the swap rate directly from the approximate SDE. However, it turns out that if we, again, replace some LVM by their martingale values, the swap rate and factor dynamics (under the swap measure) can be cast in the affine framework. We can then view a swaption as an option on one particular factor, namely, the swap rate itself. This enables us to use the valuation approach based on the CCF (see Carr and Madan 1999; DPS 2000; and Lee 2004). The approximation method develops according to three steps. We first derive the RN derivative for the change from the risk-neutral measure to the swap measure, \( Q_{n+1,N} \). Second, we derive the swap rate and factor dynamics under \( Q_{n+1,N} \). Third, we propose the approximate dynamics and the pricing formula.

5.1. Approximate Swap Rate and Factor Dynamics

Recall that \( M_t = \exp(\int_0^t r_s \, ds) \) denotes the money-market account. We will now derive the swap rate and factor dynamics under the martingale measure associated with the PVBP as a numéraire (“swap measure”). The RN derivative for a change from the risk-neutral to the swap measure is given by

\[
\frac{dQ_{n+1,N}}{dQ} = \frac{P_{n+1,N}(T) / P_{n+1,N}(t)}{M_T / M_t}.
\]

Results for a three-factor Gaussian model are presented in Section 6.
Applying Itô’s lemma to this expression and combining with (2.8) gives

\[
dP_{n+1,N}(t) = \frac{1}{M_t} dP_{n+1,N}(t) + P_{n+1,N}(t) d\frac{1}{M_t} + d\frac{1}{M_t} dP_{n+1,N}(t)
\]

\[
= -\sum_{i=n+1}^{N} \Delta^Y_{i-1} B(t, T_i) \frac{D(t, T_i)}{M_t} \Sigma \sqrt{V_t} dW_t^{Q^{n+1,N}}
\]

\[
= -\sum_{i=n+1}^{N} \left( \Delta^Y_{i-1} B(t, T_i) \frac{D(t, T_i)}{P_{n+1,N}(t)} \Sigma \sqrt{V_t} \right) \frac{P_{n+1,N}(t)}{M_t} dW_t^{Q^{n+1,N}},
\]

where \( W_t^{Q^{n+1,N}} \) is an \( M \)-dimensional Brownian motion under the swap measure and \( B(t, T_i) \) is an \( M \)-vector of loadings on the factor volatility to determine bond volatility.\(^8\) So the RN-kernel for a change of measure from the risk neutral measure to the swap measure, \( Q^{n+1,N} \), is given by

\[
\sqrt{V_t} \Sigma \frac{\partial \ln P_{n+1,N}(t)}{\partial X_t} = -\sqrt{V_t} \Sigma \left( \sum_{i=n+1}^{N} \Delta^Y_{i-1} B(t, T_i) \frac{D(t, T_i)}{P_{n+1,N}(t)} \right) \]

This implies for the \( Q^{n+1,N} \) dynamics of the factors,

\[
dX_t = \left[ A(\theta - X_t) + \Sigma V_t \Sigma' \left( \sum_{i=n+1}^{N} \Delta^Y_{i-1} B(t, T_i) \frac{D(t, T_i)}{P_{n+1,N}(t)} \right) \right] dt + \Sigma \sqrt{V_t} dW_t^{Q^{n+1,N}}
\]

If \( \frac{\partial \ln P_{n+1,N}(t)}{\partial X_t} \) would be deterministic (but time dependent), this implies an affine structure (with time varying coefficients). Note that the randomness in the RN-kernel is caused by \( \frac{D(t, T_i)}{P_{n+1,N}(t)} \), \( i = n + 1, \ldots, N \). However, these terms are asset prices normalized by the numeraire associated with the swap measure. Hence, they are martingales under this measure. The dynamics of the swap rate itself can be obtained by a simple application of Itô’s lemma. We use that the swap rate is a martingale under the swap measure, to obtain

\[
dy_{n+1,N}(t) = \left\{ \sum_{i=n}^{N} q^Y_i(t) B(t, T_i) \right\} \Sigma \sqrt{V_t} dW_t^{Q^{n+1,N}},
\]

where \( q^Y_i(t) = -\frac{D(t, T_i)}{P_{n+1,N}(t)} \), \( q^Y_i(t) = \Delta^Y_{i-1} y_{n+1,N}(t) \frac{D(t, T_i)}{P_{n+1,N}(t)} \) for \( i = n + 1, \ldots, N - 1 \) and \( q^Y_N(t) = [1 + \Delta^Y_{N-1} y_{n+1,N}(t)] \frac{D(t, T_N)}{P_{n+1,N}(t)} \). From (5.3), the swap rate volatility can be interpreted as a weighted bond volatility.

Following the approach outlined in Section 4 and motivated by the excellent results, we approximate the dynamics of the factors and the swap rate under the swap measure by substituting \( \frac{D(t, T_i)}{P_{n+1,N}(t)} \) for \( \frac{D(t, T_i)}{P_{n+1,N}(t)} \) in (5.2) and \( q^Y_i(t) \) for \( q^Y_i(t) \) in (5.3).\(^9\) We obtain the following approximate dynamics,

\(^7\) We leave the drift term unspecified. Since the RN derivative is a martingale it is irrelevant in our calculations.

\(^8\) Applying Itô’s lemma to \( D(t, T) = \exp(A(t, T) - B(t, T) \cdot X_t) \) gives

\[
dD(t, T) = \ldots dt - B(t, T) D(t, T) \Sigma \sqrt{V_t} dW_t^{Q^{n+1,N}}.
\]

\(^9\) The approximation of the swap rate volatility is not the crucial assumption for ATSMs since this doesn’t effect the factor dynamics and seems to work extremely well in Gaussian models.
get the following dynamics for \( \tilde{y}_{n,N}(t) \) and Kan 1996; and Dai and Singleton 2000). This drift change is small however. It will, however the drift change (which in our approximation we assume to be a deterministic function of time) influences the joint restrictions on \( A, \theta, \Sigma, \) and \( V \) see (Duffie and Kan 1996; and Dai and Singleton 2000). This drift change is small however. It will, most likely, not cause any problems in practice.

We can regard \( y_{n,N} \) as a pseudo factor. The swaption price in (2.5) can then be written as the expectation of a trivial linear combination of the “factors” minus the strike, if positive. This means that even after a change to the swap measure, we can calculate all quantities related to swaption pricing using the apparatus of the affine setup. To be precise, a swaption is just an option on the first “factor” of the affine model in (5.4) and (5.5). Hence swaptions can be priced in a similar manner as bond options. We will now elaborate on this.

5.2. Lévy Inversion

From the approximate dynamics, we can infer an approximation for the CCF of \( y_{n,N} \) under the swap measure (see Duffie et al. 2000). Then, as outlined at the end of Section 2, via a single one-dimensional Lévy inversion, we can find the price of a payer swaption. Recall that contrary to other approaches, we interpret a swaption as an option on the swap rate. It is important to note that this particular approach enables us to use the elegant inversion approach of Carr and Madan (1999). Not only is this approach faster than an approach based on “exercise” probabilities but it is also more accurate. No accumulation of errors takes place. This latter point is especially relevant for options with strikes away from the forward. See Carr and Madan (1999) and Lee (2004) for a full discussion.

Our approximation in (5.4) and (5.5) results in a degenerate affine diffusion for the swap rate \( y_{n,N} \) and the factors \( X \). Stacking those variables in a vector, defining \( w_t = [\sum_{i=n+1}^N \Delta_{i-1} Y B(t, T_i)] \) and \( k_t = [\sum_{i=n}^N q_i(t)(0) B(t, T_i)] \) (both \( M \times 1 \)-vectors), we get the following dynamics for \( \tilde{X}_t = [y_{n,N}(t)X'_t] \),

\[
\begin{align*}
\tilde{X}_t &= \left\{ \begin{bmatrix}
0 \\
A\theta - \Sigma \text{diag}(\alpha) \Sigma' w_t \\
\Delta Y B(t, T_i) / \theta_1 N(0) \\
& - AX_t - \Sigma \text{diag}(\beta X_t) \Sigma' w_t \\
\end{bmatrix} \right\} dt + \Sigma \sqrt{V_t} dW_{t \theta_{N+1}} \\
&+ \begin{bmatrix} k_t \\
\Sigma \sqrt{V_t} dW_{t \theta_{N+1}} \end{bmatrix}
\end{align*}
\]

\[
= \hat{A}(t)(\tilde{\theta}(t) - \tilde{X}_t) dt + \hat{\Sigma}(t)\sqrt{V_t} dW_{t \theta_{N+1}}.
\]

10 The classic use of transform inversion in option pricing, introduced by Heston (1993), uses two transform inversions to find “\( N(d_1) \)” and “\( N(d_2) \)” in the price formula of a call with strike \( K \) on a stock \( S, C_0 = D(0, T)[S_0 N(d_1) - KN(d_2)]. \)
We let \( \tilde{A}(t), \tilde{\theta}(t), \) and \( \tilde{\Sigma}(t) \) be defined implicitly. Furthermore, we define \( \tilde{\beta}_i = [0 \beta'_i'] \) and \( \tilde{\alpha}_1 = 0, \tilde{\alpha}_i = \alpha_{i-1} \) for \( i = 2, \ldots, M. \) Now, we let \( \psi \) denote the CCF of the swap rate, 

\[
\psi(v, t, T_n) \equiv E_{\mathcal{Q}_n+1} [\exp(iv \gamma_{n,N}(T_n))].
\]

Furthermore, let \( \phi \) be the transform of the dampened payer price, 

\[
\phi(u, t, T_n) = \int_{-\infty}^{\infty} \exp(iuK) \psi(u, t, T_n) dK.
\]

Then from results in Carr and Madan (1999) and Lee (2004), it follows that 

\[
\phi(u, t, T_n) = P_{n+1,N} \psi(u - i\alpha, t, T_n) \frac{\alpha + iu}{(\alpha + iu)^2}.
\]

Furthermore, the results in DPS state that 

\[
\psi(v, t, T_n) = \exp(\gamma t + \delta t \cdot \tilde{X}_t),
\]

where \( \delta, \gamma \) are solutions to the following system of complex valued Riccati equations, with initial conditions, 

\[
\delta_{T_n} = -ive_1, \gamma_{T_n} = 0,
\]

\[
\frac{d\delta_i}{dt} = \tilde{A}(t)'\delta_i - \frac{1}{2} \sum_{j=1}^{M+1} [\tilde{\Sigma}(t)'\delta_j]^{2}\tilde{\beta}_j,
\]

\[
\frac{d\gamma_i}{dt} = \tilde{\theta}(t)'\tilde{A}(t)'\delta_i - \frac{1}{2} \sum_{j=1}^{M+1} [\tilde{\Sigma}(t)'\delta_j]^{2}\tilde{\alpha}_i,
\]

where \( e_1 \) is the first basis vector of the \( M + 1 \) dimensional Euclidean space. Note that \( \delta \) is an \((M + 1)\)-vector and \( \gamma \) a scalar.

We can summarize the approach in the following proposition. We label the resulting approximation “TransformApprox.”

**Proposition 5.1.** Under the approximate dynamics for the swap rate and the factors in (5.4) and (5.5), the price of a payer swaption in terms of the dampened payer transform is given by 

\[
\text{PS}_0(K) = \frac{\exp(-\alpha K)}{\pi} \int_{0}^{\infty} \text{Re}[\exp(-iuK)\phi(u, 0, T_n)] du,
\]

where \( \phi(u, 0, T_n) \) is linked to the CCF of the swap rate by (5.6).

**Proof.** This follows from Duffie et al. (2000) and Lee (2004) in combination with our approximate dynamics.

\( \square \)

6. APPROXIMATION QUALITY AND COMPUTATIONAL SPEED

Our approximation method is partly based on Brace et al. (2001) and Hull and White (2000). These authors show that in the lognormal version of the LMM, we can approximate swap rates by lognormal martingales by replacing quantities similar to \( D(t, t_i)/P_{n+1,N}(t_i) \) by their time zero values. Variations around this martingale value should not be too important when small, since we aim at finding the average volatility of the swap rate over the interval from the current time to the option maturity. This approximation
yields swaption prices which are accurate up to a couple of basis points (in absolute terms) or a couple of tenths of a percent (in relative terms).

Singleton and Umantsev (2002) simplify the problem of pricing a swaption to pricing several caplets. However, several drawbacks to their method exist, which are mainly related to their algorithm to approximate the exercise region. First, to find the approximate exercise region, one requires complete knowledge of the density of the factors. This density can be computed analytically in the case they consider, that of a two-factor CIR model with uncorrelated factors. In the more general case, e.g., models which feature correlated factors and volatility components which are driven by multiple factors, this density must be computed by Lévy inversion. More precise, for each evaluation of the density one needs to perform a full transform inversion procedure. Their algorithm is based on finding the region outside which the density of the factors is negligible. To find this region, one would need at least a couple of inversions for each factor. In comparison, our method requires only a single inversion to obtain the price. Second, for each different strike they need to find a new approximation to the exercise region which can be troublesome in high dimensions. Third, they need to perform as much transform inversions as there are cash flows in the swap or coupon bond, whereas we need to perform only a single inversion. From a numerical point of view, the end result of these multiple inversions is less accurate than the inversion method used by our method which is based on a single inversion. Summarizing, we can say that our approach is more easily implemented for multi-factor models and is computationally less intensive.

Implementation for multi-dimensional models is certainly difficult for the method by Collin-Dufresne and Goldstein. Their method based on Edgeworth expansions requires calculation of the moments of the swap rate under several forward measures. These moments are a summation of moments of products of zero bond prices. However, when zero bond prices are not available analytically (but only by solving a system of ODEs), this method becomes computationally very intensive. To illustrate for the calculation of the fifth moment of a 15-year swap rate (annual payments), this requires \( \left( \frac{15+5-1}{5} \right) = 11,628 \) solutions to a system of Riccati equations. Again our approach extends much easier to the general case and is computationally less intensive as we calculate the approximate CCF of the swap rate directly.

Our procedure for calculating swaption prices nicely fits in the affine framework, so should not give to many implementation problems. To deal with time dependence in the coefficients, observe that this originates from the bond volatilities of the affine model, \( B(t, T_i) \). These bond volatilities (which can be calculated by solving the familiar Riccati equations) need to be calculated only once and can then be stored for further use. However, for the use of the affine model, the procedure to calculate bond volatilities should be in place anyway.

Next, we present results on the quality of our approximation. The quality is excellent for both the Gaussian and the CIR case. Tables 4.2, 6.1, and 6.2 show the performance of our approximation method relative to exact prices for \( n \times m \) ATMF, ITM, and OTM payer swaptions, where \( n = 1, 2, 5 \) is the option maturity and \( m = 1, 2, 5, 10 \) is the swap maturity, in the single factor Vasicek model, the two-factor CIR model, and the three-factor Gaussian TSM (see also Collin-Dufresne and Goldstein 2002). The performance is measured in absolute deviations of both price and implied Black volatility. The ITM strike is set at 85% of the ATMF and the OTM strike is set at 115% of the ATMF. For sake of completeness, we give the model equations. The two-factor CIR model,
Table 6.1
Prices, Implied Volatilities, and Errors of the Approximation Method of Subsection 5.2 Using Transform Inversion (TransformApprox), in a Two-Factor CIR Model for a Set of Swaptions at the Parameters Given in Table 4.1

<table>
<thead>
<tr>
<th>Swap maturity</th>
<th>1</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ATMF</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>25.34 (−0.01)</td>
<td>29.88 (−0.01)</td>
<td>28.88 (−0.01)</td>
</tr>
<tr>
<td></td>
<td>9.5% (0.00%)</td>
<td>8.9% (0.00%)</td>
<td>7.3% (0.00%)</td>
</tr>
<tr>
<td>2</td>
<td>44.57 (−0.01)</td>
<td>52.60 (−0.02)</td>
<td>50.94 (−0.03)</td>
</tr>
<tr>
<td></td>
<td>8.9% (0.00%)</td>
<td>8.2% (0.00%)</td>
<td>6.7% (0.00%)</td>
</tr>
<tr>
<td>5</td>
<td>78.98 (−0.02)</td>
<td>93.32 (−0.03)</td>
<td>90.61 (−0.05)</td>
</tr>
<tr>
<td></td>
<td>7.2% (0.00%)</td>
<td>6.7% (0.00%)</td>
<td>5.3% (0.00%)</td>
</tr>
<tr>
<td>10</td>
<td>99.82 (−0.04)</td>
<td>117.95 (−0.07)</td>
<td>114.55 (−0.10)</td>
</tr>
<tr>
<td></td>
<td>5.6% (0.00%)</td>
<td>5.1% (0.00%)</td>
<td>4.0% (0.00%)</td>
</tr>
<tr>
<td><strong>ITM</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>101.05 (0.03)</td>
<td>92.65 (0.08)</td>
<td>72.35 (0.08)</td>
</tr>
<tr>
<td></td>
<td>9.6% (0.06%)</td>
<td>8.9% (0.06%)</td>
<td>7.2% (0.04%)</td>
</tr>
<tr>
<td>2</td>
<td>190.75 (0.05)</td>
<td>174.51 (0.13)</td>
<td>136.24 (0.14)</td>
</tr>
<tr>
<td></td>
<td>8.8% (0.06%)</td>
<td>8.2% (0.06%)</td>
<td>6.5% (0.04%)</td>
</tr>
<tr>
<td>5</td>
<td>411.03 (0.06)</td>
<td>375.12 (0.22)</td>
<td>293.35 (0.26)</td>
</tr>
<tr>
<td></td>
<td>7.1% (0.08%)</td>
<td>6.5% (0.08%)</td>
<td>5.1% (0.05%)</td>
</tr>
<tr>
<td>10</td>
<td>675.49 (0.02)</td>
<td>618.51 (0.16)</td>
<td>487.83 (0.24)</td>
</tr>
<tr>
<td></td>
<td>5.1% (0.29%)</td>
<td>5.0% (0.16%)</td>
<td>3.6% (0.07%)</td>
</tr>
<tr>
<td><strong>OTM</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.07 (−0.05)</td>
<td>5.28 (−0.08)</td>
<td>8.43 (−0.10)</td>
</tr>
<tr>
<td></td>
<td>9.4% (−0.05%)</td>
<td>8.8% (−0.04%)</td>
<td>7.3 (−0.04%)</td>
</tr>
<tr>
<td>2</td>
<td>2.86 (−0.08)</td>
<td>7.88 (−0.14)</td>
<td>13.17 (−0.19)</td>
</tr>
<tr>
<td></td>
<td>8.8% (−0.05%)</td>
<td>8.2% (−0.04%)</td>
<td>6.7% (−0.03%)</td>
</tr>
<tr>
<td>5</td>
<td>2.27 (−0.13)</td>
<td>8.14 (−0.26)</td>
<td>15.74 (−0.41)</td>
</tr>
<tr>
<td></td>
<td>7.3% (−0.07%)</td>
<td>6.8% (−0.05%)</td>
<td>5.5% (−0.04%)</td>
</tr>
<tr>
<td>10</td>
<td>0.65 (−0.10)</td>
<td>3.86 (−0.34)</td>
<td>9.73 (−0.71)</td>
</tr>
<tr>
<td></td>
<td>5.7% (−0.10%)</td>
<td>5.3% (−0.08%)</td>
<td>4.2% (−0.07%)</td>
</tr>
</tbody>
</table>

Note: The ITM and OTM are set at the same levels as before (85% and 115%) relative to the ATMF.

\[ r_t = \omega + X_{1t} + X_{2t} \]

\[ dX_{it} = a_i(\theta_i - X_{it}) dt + \sigma_i \sqrt{X_{it}} dW_{it}, \quad X_{i0} = x_i, \quad i = 1, 2, \]

where \( W_{1t} \) and \( W_{2t} \) are independent Brownian motions under \( Q \). The three-factor Gaussian model,

\[ r_t = \omega + X_{1t} + X_{2t} + X_{3t} \]

\[ dX_{it} = -a_i X_{it} dt + \sigma_i dW_{it}, \quad X_{i0} = x_i, \quad i = 1, 2, 3, \]

where the \( W_{it} \) are correlated Brownian motions under \( Q \) with \( dW_{it} dW_{jt} = \rho_{ij} dt \).
To the ATMF.

Note: The ITM and OTM are set at the same levels as before (85% and 115%) relative to the ATMF.

For the CIR model and the three-factor Gaussian model, the true prices are calculated by simulation. Monte Carlo results are obtained using 500,000 simulations with standard antithetic variables. The parameter values at which these results are obtained are taken from Collin-Dufresne and Goldstein (2002) and are given in Table 4.1. Parameter values are close to realistic levels encountered in practice. Table 6.1 shows results on the approximation in the two-factor CIR model. Table 6.2 shows results on the approximation error in the three-factor Gaussian model. The quality of the approximation in these cases is also excellent and comparable to stochastic duration and the approximation methods by Collin-Dufresne and Goldstein (2002) and Singleton and Umantsev.
### Table 6.3
Computational Times for Options with Different Option Maturity and Swap Maturity for Different Approximation Methods

<table>
<thead>
<tr>
<th>Option char.</th>
<th>StochDur</th>
<th>SU</th>
<th>CDG</th>
<th>TransformApprox</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 × 30</td>
<td>0.06s</td>
<td>0.36s</td>
<td>NA</td>
<td>0.14s</td>
</tr>
<tr>
<td>2 × 10</td>
<td>0.02s</td>
<td>NA</td>
<td>0.01s</td>
<td>0.05s</td>
</tr>
</tbody>
</table>

Note: The numbers in the first line are scaled to represent the numbers reported in Singleton and Umantsev (2002). The numbers in the second line are scaled to represent the numbers reported in Collin-Dufresne and Goldstein (2002).

We have compared computational times for our method with the methods by Munk (1999), Collin-Dufresne and Goldstein (2002), and Singleton and Umantsev (2002) in Table 6.3. As could be expected, our approximation is slightly slower than stochastic duration but faster than the one by Singleton and Umantsev. The comparison with the numbers in Singleton and Umantsev and Collin-Dufresne and Goldstein is made by assuming the reported numbers are from 2001 (a year before publication) and further assuming computing speed doubles every 2 years. Effectively, we divide the reported numbers by four. Although the computation of the method by Collin-Dufresne and Goldstein is extremely fast, as we argued above, this is largely due to the analytical structure of the two-factor CIR model.

### 7. CONCLUSION

In this paper, we have introduced a method to obtain an accurate and fast approximation to the prices of swaptions in ATSMs. An analytic pricing formula is derived for Gaussian TSMs. In the general case, the approximation is based on approximate dynamics of the swap rate and the latent interest rate factors under the associated swap measure. Based on these approximate dynamics, our approximation uses techniques familiar to the affine setup, such as transforms and transform inversion, to calculate swaption prices. Contrary to other approaches, we write a swaption as an option on the swap rate. This enables us to use the faster and more accurate transform inversion method of Carr and Madan (1999). The resulting approximation is comparable in speed and superior in accuracy to the stochastic duration approximation of Munk (1999).

This shows that using our approximate swap rate dynamics results in swaption price formulas which improve on the existing methods in either accuracy, speed, or both. Furthermore, the implementation for general affine models is easily accomplished.

### APPENDIX: RESULTS FOR MULTI-FACTOR GAUSSIAN TSMs

In this Appendix, we will generalize the results in Section 4, on the single factor Vasicek model to multivariate Gaussian TSMs. Closed form formulas for the prices of swaptions in single (Jamshidian 1989) and two-factor models (through numerical integration of
the single factor result) are readily available. However, using the results in this section, calculations are sped up considerably while preserving the required accuracy. We can obtain the class of Gaussian TSMs from equations (2.7)–(2.9) by setting $\beta = 0$. Without loss of generality, we can assume that $\alpha X = 1$, $\alpha = 1$, and that $A$ is a diagonal matrix.

To simplify the derivation, we first perform a double change of variables. That is, we will recall that $DP$ is a zero coupon bond with maturity $t$

\[ r_t = \alpha^*_t + 1' Y_t \]

\[ dY_t = -AY_t dt + \hat{\Sigma} dZ_t, \]

where $\alpha^*_t = \omega_0 + 1' [\theta - e^{-A t}(X_0 - \theta)]$ is a deterministic function of time, $\hat{\Sigma} = \text{diag}(\Sigma \Sigma')$, and $Z_t = \Gamma^{-1} W_t$ is correlated $M$-dimensional Brownian motion under the risk neutral measure, where $\Gamma$ is the instantaneous correlation matrix of $X_t$ (i.e., $\Gamma = \hat{\Sigma}^{-1/2} \Sigma \Sigma' \hat{\Sigma}^{-1/2}$). To address the fit to the initial term structure, one could choose $\alpha^*_t$ such that this fit is perfect. Then the model becomes in fact a multi-factor Hull–White (1990) model. In the remainder of this section, we will not differentiate between these approaches, and with $D(t, T)$ just refer to the model generated price at time $t$ of a zero coupon bond with maturity $T$. Bond prices in this model are given by

\[ D(t, T) = \exp \left( A(t, T) - \sum_{i=1}^{M} B^{(i)}(t, T) Y^{(i)}_t \right), \]

where $B^{(i)}(t, T) = 1/A(0, 1 - e^{-A(0, t)})$. We leave $A(t, T)$ unspecified. This is not important as long as the term structure generated by the model at the time of valuation is known.

We again start by deriving the SDE of a swap rate under its own swap measure. From the change of numéraire theorem, we know that the swap rate is a martingale associated with the PVBP as a numéraire. Since the interest rate volatility does not contain a square root term, applying Itô's lemma to the swap rate gives

(A.1) \[ dy_{n,N}(t) = \frac{\partial y_{n,N}(t)}{\partial Y_t} \hat{\Sigma} dZ^{n+1,N}_t, \]

where $Z^{n+1,N}_t$ is $M$-dimensional correlated Brownian motion under the swap measure $Q^{n+1,N}$ corresponding to the numéraire $P_{n+1,N}$. Now for each element of the vector of derivatives, we have

(A.2) \[ \frac{\partial y_{n,N}(t)}{\partial Y^{(i)}_t} = -B^{(i)}(t, T_n) D^P(t, T_n) + B^{(i)}(t, T_N) D^P(t, T_N) \]

\[ + y_{n,N}(t) \sum_{j=n+1}^{N} \Delta Y_{j-1} B^{(j)}(t, T_j) D^P(t, T_j). \]

recall that $D^P(t, T_n) = D(t, T_n)/P_{n+1,N}(t)$, the bond price normalized by the numéraire, the PVBP.

As in Sections 4 and 5, our approximation consists of replacing the stochastic terms $D(t, T_i)/P_{n+1,N}(t)$ by their martingale values. We obtain for each partial derivative of the swap rate,
\[
\frac{\partial y_{n,N}(t)}{\partial Y^{(i)}_t} = -B^{(i)}(t, T_n) D^P(0, T_n) + B^{(i)}(t, T_N) D^P(0, T_N) \\
+ y_{n,N}(0) \sum_{j=n+1}^{N} \Delta Y_{j-1} B^{(i)}(t, T_j) D^P(0, T_j) \\
= \frac{\partial y_{n,N}(t)}{\partial Y^{(i)}_t}.
\]

This makes the swap rate volatility deterministic. Like in Section 4, we can write
\[
B^{(i)}(t, T) = \frac{1}{A^{(ii)}} - e^{-A^{(ii)}T} e^{A^{(ii)}t}.
\]

Using this in the approximate swap rate volatility gives
\[
\frac{\partial y_{n,N}(t)}{\partial Y^{(i)}_t} = \frac{1}{A^{(ii)}} e^{A^{(ii)}t} \left[ e^{-A^{(ii)}T_n} D^P(0, T_n) - e^{-A^{(ii)}T_N} D^P(0, T_N) \\
- y_{n,N}(0) \sum_{j=n+1}^{N} \Delta Y_{j-1} e^{-A^{(ii)}T_N} D^P(0, T_N) \right] \\
= e^{A^{(ii)}t} \tilde{C}^{(i)}_{n,N},
\]
where the first line defines \( \tilde{C}^{(i)}_{n,N} \). So in the approximate model, the swap rate at time \( T_n \) is given by
\[
\int_0^{T_n} d y_{n,N}(s) = \int_0^{T_n} \frac{\partial y_{n,N}(s)}{\partial Y^{(i)}_s} \tilde{\Sigma}_s d Z^{n+1,N}_s \\
\leq \int_0^{T_n} \frac{\partial y_{n,N}(s)}{\partial Y^{(i)}_s} \tilde{\Sigma}_s d Z^{n+1,N}_s \\
= \sum_{i=1}^{M} \tilde{C}^{(i)}_{n,N} \int_0^{T_n} e^{A^{(ii)}s} d W^{(i+1,N)}_s,
\]
which leads to an analytic expression for the volatility of a \( T_n \times T_N \) swaption,
\[
\sigma^{(i)}_{n,N} = \sqrt{\sum_{i=1}^{M} \tilde{\Sigma}^{2}_{(ii)}(\tilde{C}^{(i)}_{n,N})^2} \left[ \frac{e^{2A^{(ii)}T_n}}{2A^{(ii)}} - 1 \right] 2 \sum_{i=1}^{M} \sum_{j=i+1}^{M} \rho_{ij} \tilde{\Sigma}_{(ii)} \tilde{\Sigma}_{(jj)} \tilde{C}^{(i)}_{n,N} \tilde{C}^{(j)}_{n,N} \left[ \frac{e^{[A^{(ii)}+A^{(jj)}]T_n}}{A^{(ii)} + A^{(jj)}} - 1 \right].
\]

Or if we introduce
\[
e^{A^{(ii)}t} = \begin{pmatrix}
e^{A^{(11)}t} \\
\vdots \\
e^{A^{(MM)}t}
\end{pmatrix},
\]
\[
\text{diag}(\hat{C}_{n,N}) = \begin{pmatrix}
\hat{C}_{11,n} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \hat{C}_{M1,n}
\end{pmatrix},
\]

and \( \Omega = \hat{\Sigma} \Gamma \hat{\Sigma}' \), the covariance matrix of \( W_{t+1,n} \), we can write in vector notation,

\[
(A.4) \quad \sigma_{n,N} = \left( \int_0^{T_n} \left[ e^{At} \right]' \text{diag}(\hat{C}_{n,N}) \Omega \text{diag}(C_{n,N}) e^{At} ds \right)^{1/2}.
\]

We have established that, similar to the Vasicek case, in a Gaussian TSM, the swap rate is an approximately Gaussian martingale with volatility as in (A.3) or equivalently (A.4). Approximate pricing formulas in the case of multi-factor Gaussian TSMs are equivalent to those in (4.5), (4.6), and (4.7) with (4.4) replaced by (A.3).

**REFERENCES**


