

# Judgment aggregation on restricted domains.

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# Judgment aggregation on restricted domains

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draft  
July 2006

## Abstract

We show that, when a group takes independent majority votes on interconnected propositions, the outcome is consistent once the profile of individual judgment sets respects appropriate structural conditions. We introduce several such conditions on profiles, based on ordering the propositions or ordering the individuals, and we clarify the relations between these conditions. By restricting the conditions to appropriate subagendas, we obtain local conditions that are less demanding but still guarantee consistent majority judgments. By applying the conditions to agendas representing preference aggregation problems, we show parallels of some conditions to existing social-choice-theoretic conditions, specifically to order restriction and intermediateness, restricted to triples of alternatives in the case of our local conditions.

## 1 Introduction

We consider a group of individuals having to decide which of different logically interrelated propositions (statements) to jointly accept (believe<sup>1</sup>). Taking an independent majority vote on each proposition is known to cause problems. For instance, suppose a group decides by separate majority votes on whether to believe  $a$ : "CO2 emissions exceed threshold X"; whether to believe  $b$ : "Global warming will continue"; and whether to believe the implication  $a \rightarrow b$  ("if  $a$  then  $b$ "). Even if each individual holds a consistent set of beliefs (her judgment set), the resulting set of group beliefs (the group's judgment set) may be logically inconsistent: the group's judgment set is  $\{a, a \rightarrow b, \neg b\}$  if the judgment sets  $\{a, a \rightarrow b, b\}$ ,  $\{a, \neg(a \rightarrow b), \neg b\}$  and  $\{\neg a, a \rightarrow b, \neg b\}$  are each held by a third of the individuals. A second example is the famous Condorcet paradox, reinterpreted as a logical inconsistency under majority voting on the propositions  $xPy$ ,  $yPz$ ,  $zPx$  in a suitable predicate logic with a binary predicate  $P$

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<sup>1</sup>Standardly in judgment aggregation, "accepting" a proposition means "believing" it, so that the goal is to reach consistent group beliefs on propositions. (If "accepting" means "desiring", the goal is to reach consistent group desires, typically on different propositions than those relevant in belief aggregation.) The motivation might lay in the need for group action based on consistent (non-probabilistic) group beliefs. Propositions can be descriptive (e.g. "Policy X leads to a budget deficit") or normative (e.g. "A budget deficit should be avoided").

for strict preference, constants  $x, y, z$  for options, and axioms for the rationality conditions like transitivity, as defined in Dietrich and List (forthcoming); see also List and Pettit (2004).

In standard social choice theory, which is concerned with aggregating preferences over alternatives, an extensive literature investigates which conditions on preference profiles guarantee acyclic majority preferences. One can broadly distinguish between conditions based on ordering the alternatives (notably Black's 1948 single-peakedness and Inada's 1964 single-cavedness), and conditions based on ordering the individuals (notably Grandmont's 1978 intermediateness and Rothstein's 1990/1991 order restriction with its special case of single-crossingness, e.g. Roberts 1977, Saporiti and Tohmé 2006, Saporiti forthcoming).<sup>2</sup> The position of an alternative/individual in the ordering could be interpreted as its/her position on some issue dimension, e.g. a political left-to-right dimension, or an economical socialist-to-libertarian dimension.

Our first goal is to show that in judgment aggregation a number of conditions on (judgment) profiles exist which guarantee the logical consistency of majority judgments. Some of the conditions are based on ordering the individuals, with analogies to known conditions in preference aggregation. Other conditions are based on ordering the propositions, without obvious analogies to existing conditions in preference aggregation. While ordering individuals could be given similar interpretations to those in preference aggregation, ordering propositions should be distinguished conceptually from ordering alternatives: propositions are (usually non-exclusive) statements about the world, whereas alternatives are exclusive descriptions of the world. From this perspective, it might appear surprising that ordered propositions lead to consistency results under majority voting.

Our second goal is to show that in judgment aggregation we can formulate *local* conditions for majority consistency: conditions of (judgment) profiles relative to certain *subagendas*. In preference aggregation, local conditions like single-peakedness on all triples of alternatives (i.e. on all subagendas of size 3) are known to guarantee acyclic majority preferences. In judgment aggregation, it is not obvious which subagendas to consider. Considering all subagendas of a certain *size* will not do: majority consistency cannot be ensured by restricting people's judgments within any subagenda of a certain size. We show how the subagendas should be selected. Again, we relate those local conditions based on ordering the individuals to local conditions in preference aggregation.

So far the only condition known to guarantee consistent majority judgments is List's (2003 and corrigendum) *unidimensional alignment*, a global condition based on ordering the individuals. The field of judgment aggregation emerged from law and political philosophy (e.g. Kornhauser and Sager 1986 and Pettit 2001), and is formalised social-choice-theoretically by List and Pettit (2002).

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<sup>2</sup>Sen's (1966) value restriction does not fall into these categories. In a private-good context, not the social alternatives but individual shares are ordered; e.g. Klaus, Peters and Storcken (1997).

The observation that majority judgments can be logically inconsistent generalises to several impossibility results (e.g. List and Pettit 2002 and 2004, Pauly and van Hees forthcoming, Dietrich 2006, Gärdenfors forthcoming, Nehring and Puppe 2006, van Hees forthcoming, Mongin 2005, Dietrich and List forthcoming, and Dokow and Holzman 2005). Other impossibility results follow from Nehring and Puppe's (2002) strategy-proofness results on property spaces. A liberal-paradox-type impossibility is derived in Dietrich and List (forthcoming). Giving up propositionwise aggregation, possibilities can be obtained by sequential rules (List forthcoming) and fusion operators (Pigozzi forthcoming). Voter manipulation is analysed in Dietrich and List (2004a). Here we use Dietrich's (forthcoming) generalised model, which allows the propositions to be expressed in rich logical languages.

## 2 Model

We consider a set of individuals  $N$  ( $2 \leq |N| < \infty$ ).

**Formal logic.** The group considers propositions expressed in a formal language  $\mathbf{L}$ .  $\mathbf{L}$  is a non-empty set of formal expressions (*propositions*), representing statements, such that if  $\mathbf{L}$  contains  $p$  then  $\mathbf{L}$  contains  $\neg p$  ("not  $p$ ").  $\mathbf{L}$  is endowed with an *entailment* relation  $\models (\subseteq \mathcal{P}(\mathbf{L}) \times \mathbf{L})$ , representing logical relations, where for all  $A \subseteq \mathbf{L}$  and  $p \in \mathbf{L}$  " $A \models p$ " means " $A$  entails  $p$ ".<sup>3</sup> For instance,  $\mathbf{L} = \{a, a \rightarrow b, \neg(c \wedge d), \dots\}$ , where  $\{a, a \rightarrow b\} \models b$ ,  $\{a, b\} \models a \wedge b$ ,  $\{a\} \not\models a \wedge b$ , ...; or (if  $\mathbf{L}$  is a predicate language)  $\mathbf{L} = \{3 + x \geq y, (\forall v_1)(v_1 \geq 0), \dots\}$ , where  $\{(\forall v_1)(v_1 \geq 0)\} \models x \geq 0$ , ... The language can be more or less expressive, depending on the decision problem at hand.<sup>4</sup> A set  $A \subseteq \mathbf{L}$  is *inconsistent* if there is a  $p \in \mathbf{L}$  such that  $A \models p$  and  $A \models \neg p$ , and *consistent* otherwise. A  $p \in \mathbf{L}$  is a *contradiction* if  $\{p\}$  is inconsistent, a *tautology* if  $\{\neg p\}$  is inconsistent, and *contingent* if  $\{p\}$  and  $\{\neg p\}$  are consistent. To ensure that  $\models$  is well-behaved, we require three conditions satisfied in most familiar logics (including most propositional, predicate or modal, and some fuzzy logics):

<sup>3</sup> $\models$  can be interpreted as semantic entailment or as syntactic derivability (often denoted by  $\vdash$ ), depending on whether we want to model a semantic or syntactic notion of rationality.

<sup>4</sup>In the simple case of classical propositional logic,  $\mathbf{L}$  is the (smallest) set such that (i)  $\mathbf{L}$  contains certain "atomic propositions"  $a, b, c, \dots$  and (ii) if  $p, q \in \mathbf{L}$  then  $\neg p \in \mathbf{L}$  and  $(p \wedge q) \in \mathbf{L}$ ; and  $A \models p$  if and only if every truth-function making all  $q \in A$  true makes  $p$  true, where a *truth-function* is a function  $v : \mathbf{L} \rightarrow \{T, F\}$ , assigning truth-values to propositions, such that, for all  $p, q \in \mathbf{L}$ ,  $v(\neg p) = T \Leftrightarrow v(p) = F$  and  $v(p \wedge q) = T \Leftrightarrow v(p) = v(q) = T$ . We may enrich the language by introducing other connectives than  $\neg$  and  $\wedge$ , e.g. a modal necessity operator  $\Box$  ( $\Box p$  means "necessarily  $p$ ", by some notion of necessity), or a moral "ought" operator  $O$  (leading to a deontic logic) or a subjunctive implication  $\rightarrow$  ( $p \rightarrow q$  means "if  $p$  then  $q$ ", in the subjunctive sense relevant, say, in our global warming example). Of course, entailment  $\models$  then has to be extended to the richer language (see Dietrich forthcoming). Our examples will either use the connectives  $\neg, \wedge, \rightarrow$  or use a predicate logic to represent preference aggregation problems. Notationally, we drop brackets when there is no ambiguity, e.g.  $a \wedge b \wedge (b \rightarrow c)$  stands for  $((a \wedge b) \wedge (b \rightarrow c))$ .

- L1** for all  $p \in \mathbf{L}$ ,  $\{p\} \models p$  (*self-entailment*)
- L2** for all  $p \in \mathbf{L}$  and  $A \subseteq B \subseteq \mathbf{L}$ , if  $A \models p$  then  $B \models p$  (*monotonicity*);
- L3**  $\emptyset$  is consistent, and each consistent set  $A \subseteq \mathbf{L}$  has a consistent superset  $B \subseteq \mathbf{L}$  containing a member of each pair  $p, \neg p \in \mathbf{L}$  (*completeness*).

**The agenda.** The *agenda* is a non-empty finite<sup>5</sup> set  $X \subseteq \mathbf{L}$  of propositions under decision, where  $X$  is a union of pairs  $\{p, \neg p\}$  (with  $p$  not itself a negated proposition). For instance, our global warming example has agenda  $X = \{a, \neg a, b, \neg b, a \rightarrow b, \neg(a \rightarrow b)\}$ . Hereafter, when we write " $\neg q$ " and  $q = \neg p \in X$  then " $\neg q$ " stands for  $p$  rather than  $\neg\neg p$  (so double-negations cancel each other out).

**Judgment sets.** A *judgment set* (held by an individual or the group) is a subset  $A \subseteq X$ , where  $p \in A$  stands for "the person/group accepts proposition  $p$ ". We will focus mainly on whether a judgment set  $A$  is *consistent*, and occasionally on whether it is *complete* (i.e. contains a member of each pair  $p, \neg p \in X$ ). We will mostly not require individuals to hold complete judgment sets: they may abstain on some or even all pairs  $p, \neg p \in X$ . It has been argued (e.g. Gärdenfors forthcoming) that individual completeness is a severe constraint, as people need opinions on all issues. As most of the literature studies *impossibility* results, the assumption of individual completeness was natural: it shrinks the domain, hence strengthens (most) impossibilities. However, we are interested in *possibilities*, which are strengthened by allowing individual incompleteness.

**Aggregation rules.** A *profile* is an  $n$ -tuple  $(A_1, \dots, A_n)$  of (individual) judgment sets  $A_i \subseteq X$ . A (*judgment*) *aggregation rule* is a function  $F$  that maps each profile  $(A_1, \dots, A_n)$  in a given non-empty domain to a (group) judgment set  $F(A_1, \dots, A_n) = A \subseteq X$ ; it is *consistent/complete* if it generates a consistent/complete judgment set for each profile in the domain. The *majority outcome on*  $(A_1, \dots, A_n)$  (a profile) is the judgment set

$$\{p \in X : \text{there are more individuals } i \text{ with } p \in A_i \text{ than with } p \notin A_i\}.$$

*Majority rule on*  $D$  (a set of profiles) is the aggregation rule with domain  $D$  generating the majority outcome on each profile. We will investigate on which domains  $D$  majority rule is consistent. Other aggregation rules (not studied here) are dictatorial rules, supermajority rules, and premise-based or conclusion-based rules.

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<sup>5</sup>The finiteness assumption is mainly for proof convenience. Without it, the results continue to hold as such or under an additional compactness assumption on the logic.

### 3 Global conditions for majority consistency

#### 3.1 Conditions based on ordering the propositions

Throughout the paper, by an *order* on a set  $S$  we mean a reflexive, transitive, complete and antisymmetric binary relation  $\leq$  on  $S$ .<sup>6</sup>

**Definition 1** Consider an order  $\leq$  on  $X$  (i.e. an order of the propositions).

- (a) A judgment set  $A$  is single-plateaued relative to  $\leq$  if  $A = \{p : p_{\text{left}} \leq p \leq p_{\text{right}}\}$  for some  $p_{\text{left}}, p_{\text{right}} \in X$ , and single-canyoned relative to  $\leq$  if  $A = X \setminus \{p : p_{\text{left}} \leq p \leq p_{\text{right}}\}$  for some  $p_{\text{left}}, p_{\text{right}} \in X$ .<sup>7</sup>
- (b) If each  $A_i$  in a profile  $(A_1, \dots, A_n)$  is single-plateaued (-canyoned) relative to  $\leq$ ,  $(A_1, \dots, A_n)$  is single-plateaued (-canyoned) relative to  $\leq$ , or simply single-plateaued (-canyoned).
- (c) In (a) and (b) we refer to  $\leq$  as a (possibly non-unique) structuring order.

For instance, for an agenda containing scientific propositions (hypotheses) related to global warming, individuals might hold single-plateaued judgment sets relative to an order of the propositions from "most pessimistic" on the climate front to "most optimistic"; the location of an individual  $i$ 's plateau reflects  $i$ 's scientific position. Or, if the agenda contains hypotheses about the effects of potential taxation or budget measures, these hypotheses might be ordered from left-most economic views to right-most economic views. Or, if the agenda contains scientific hypotheses about some animal species, the propositions might be ordered from "closest to theory X" to "closest to theory Y" (where X and Y are competing theories, say evolutionary and a creationist ones).

We now prove that on single-plateaued profiles majority voting preserves consistency; and we show that single-canyoned profiles are actually special single-plateaued profiles with the property that majority voting preserves not only consistency but also single-canyonedness. To see that a single-canyoned profile is single-plateaued, we will reorder the propositions so as to "glue together" each person's two extreme sets of propositions into a single plateau.

**Proposition 2** For any profile  $(A_1, \dots, A_n)$  of consistent judgment sets,

- (a) if  $(A_1, \dots, A_n)$  is single-plateaued, the majority outcome is consistent;
- (b) if  $(A_1, \dots, A_n)$  is single-canyoned,  $(A_1, \dots, A_n)$  is single-plateaued;
- (c) if  $(A_1, \dots, A_n)$  is single-canyoned, the majority outcome is consistent and single-canyoned (relative to the same structuring order).

*Proof.* Let each  $A_i$  be consistent. The following notation is used in this and other proofs. Let  $A$  be the majority outcome. Put  $N_p := \{i : p \in A_i\} \forall p \in X$ .

<sup>6</sup>Reflexivity:  $x \leq x \forall x \in S$ . Transitivity:  $x \leq y \& y \leq z \Rightarrow x \leq z \forall x, y, z \in S$ . Completeness:  $x \neq y \Rightarrow [x \leq y \text{ or } y \leq x] \forall x, y \in S$ . Antisymmetry:  $x \neq y \Rightarrow [x \not\leq y \text{ or } y \not\leq x] \forall x, y \in S$ .

<sup>7</sup>We do not require  $p_{\text{left}} \leq p_{\text{right}}$ , i.e.  $\{p : p_{\text{left}} \leq p \leq p_{\text{right}}\}$  may be empty.

Whenever we consider an order  $\leq$  of  $X$ , let  $[p, q] := \{r \in X : p \leq r \leq q\}$   $\forall p, q \in X$ . An order  $\leq$  is sometimes identified with the corresponding ascending list of propositions  $p_1 \dots p_{2k}$  where  $2k$  is the size of  $X$  (which is even as  $X$  is a union of pairs  $\{p, \neg p\}$ ).

(a) Assume single-plateauedness, say relative to  $\leq$ . Among all propositions in  $A$ , let  $p$  and  $q$  be the smallest resp. largest one w.r.t.  $\leq$ . So  $A \subseteq [p, q]$ . As  $N_p$  and  $N_q$  each contain a majority of the individuals,  $N_p \cap N_q \neq \emptyset$ ; so there is an  $i \in N_p \cap N_q$ . As  $A_i$  is single-plateaued and  $p, q \in A_i$ , we have  $[p, q] \subseteq A_i$ . So  $A \subseteq A_i$ . So  $A$  is consistent.

(b) Let  $(A_1, \dots, A_n)$  be single-canyoned, say relative to the order  $p_1 \dots p_{2k}$ . We consider any  $A_i$  and show that  $A_i$  is single-plateaued relative to the new order  $p_{k+1} \dots p_{2k} p_1 \dots p_k$ . By assumption, (\*)  $A_i = \{p_1, \dots, p_j\} \cup \{p_{j'}, \dots, p_{2k}\}$  for some  $0 \leq j \leq j' \leq 2k + 1$ . As  $A_i$  is consistent,  $A_i$  contains no pair  $p, \neg p \in X$ ; so  $|A_i| \leq |X|/2 = k$ , whence (\*\*)  $j \leq k$  and  $j' \geq k + 1$ . Using both (\*) and (\*\*), one can check that  $A_i$  is, as desired, an interval relative to the new order  $p_{k+1} \dots p_{2k} p_1 \dots p_k$ :  $A_i = [p_{j'}, p_j]$  if  $j \neq 0 \& j' \neq 2k + 1$ ,  $A_i = [p_1, p_j]$  if  $j \neq 0 \& j' = 2k + 1$ ,  $A_i = [p_{j'}, p_{2k}]$  if  $j = 0 \& j' \neq 2k + 1$ , and  $A_i = \emptyset$  if  $j = 0 \& j' = 2k + 1$ .

(c) Let  $(A_1, \dots, A_n)$  be single-canyoned, say relative to  $\leq$ . By (a)-(b),  $A$  is consistent. As one easily checks  $A$  is single-canyoned relative to  $\leq$  *if and only if* for all  $p \in A$  we have  $\{q : q \leq p\} \subseteq A$  or  $\{q : q \geq p\} \subseteq A$ . So it suffices to show the latter. Consider any  $p \in A$ . Check that either (i)  $|\{q : q \leq p\}| \leq k < |\{q : p \leq q\}|$  or (ii)  $|\{q : p \leq q\}| \leq k < |\{q : q \leq p\}|$ . We assume (i) and show that  $\{q : q \leq p\} \subseteq A$  (analogously, if (ii) then  $\{q : p \leq q\} \subseteq A$ ). For each  $i \in N_p$ , single-canyonedness implies that  $\{q : q \leq p\} \subseteq A_i$  or  $\{q : p \leq q\} \subseteq A_i$ . But the latter is impossible: otherwise  $|A_i| > k$  by (i), so that  $A_i$  would contain a pair  $p, \neg p$ , hence be inconsistent. So we have  $\{q : q \leq p\} \subseteq A_i$  for all  $i \in N_p$ , hence for a majority of the individuals. It follows that  $\{q : q \leq p\} \subseteq A$ , as desired. ■

## 3.2 Conditions based on ordering the individuals

For orders of the individuals we use the symbol  $\Omega$  (while  $\leq$  is used to order propositions). Moreover, for any sets of individuals  $N', N'' \subseteq N$ , we write  $N' \Omega N''$  if  $i \Omega j$  for all  $i \in N'$  and  $j \in N''$ .

**Definition 3** Consider an order  $\Omega$  on  $N$  (i.e. an order of the individuals).

- (a) A profile  $(A_1, \dots, A_n)$  such that, for all  $p \in X$ ,  $\{i : p \in A_i\} = \{i : i_{\text{left}} \Omega i \Omega i_{\text{right}}\}$  for some  $i_{\text{left}}, i_{\text{right}} \in N$ , is unidimensionally ordered relative to  $\Omega$ , or simply unidimensionally ordered.<sup>8</sup>
- (b) (List 2003) A profile  $(A_1, \dots, A_n)$  such that for all  $p \in X$   $\{i : p \in A_i\} \Omega \{i : p \notin A_i\}$  or  $\{i : p \notin A_i\} \Omega \{i : p \in A_i\}$  is unidimensionally aligned relative to  $\Omega$ , or simply unidimensionally aligned.
- (c) In (a) and (b) we refer to  $\Omega$  as a (possibly non-unique) structuring order.

<sup>8</sup>We do not require  $i_{\text{left}} \Omega i_{\text{right}}$ , i.e.  $\{i : i_{\text{left}} \Omega i \Omega i_{\text{right}}\}$  may be empty.

Unidimensional alignment relative to  $\Omega$  is in fact a special case of unidimensional orderedness relative to  $\Omega$ : that where for all  $p \in X$  at least one of  $i_{\text{left}}, i_{\text{right}}$  is "extreme", i.e. the minimum or maximum of  $\Omega$ .

**Proposition 4** *For any profile  $(A_1, \dots, A_n)$  of consistent judgment sets,*

- (a) *if  $(A_1, \dots, A_n)$  is unidimensionally ordered, the majority outcome is consistent and it is a subset of  $A_m$  (if  $n$  is even) or of  $A_{m_1} \cap A_{m_2}$  (if  $n$  is odd), where  $m$  is the middle individual (if  $n$  is odd) or  $(m_1, m_2)$  is the middle pair of individuals (if  $n$  is even) in any structuring order  $\Omega$ .*
- (b) *if  $(A_1, \dots, A_n)$  is unidimensionally aligned,  $(A_1, \dots, A_n)$  is unidimensionally ordered;*
- (c) *(List 2003) if  $(A_1, \dots, A_n)$  is unidimensionally aligned, the majority outcome is consistent and it is  $A_m$  (if  $n$  is even) or  $A_{m_1} \cap A_{m_2}$  (if  $n$  is odd), where  $m$  (if  $n$  is odd) or  $(m_1, m_2)$  (if  $n$  is even) are as in part (a).*

*Proof.* Let each  $A_i$  be consistent. We use earlier proof notation.

(a) Suppose unidimensional orderedness, say relative to  $\Omega$ . For all  $p \in A$ , by unidimensional orderedness  $N_p$  is some interval  $[i_{\text{left}}, i_{\text{right}}]$ , which by  $|N_p| > n/2$  is long enough to necessarily contain the median individual  $m$  (if  $n$  is odd) or the median pair of individuals  $m_1, m_2$  (if  $n$  is even); so that  $p \in A_m$  (if  $n$  is odd) or  $p \in A_{m_1} \cap A_{m_2}$  (if  $n$  is even). Hence, as desired,  $A \subseteq A_m$  (if  $n$  is odd) or  $A \subseteq A_{m_1} \cap A_{m_2}$  (if  $n$  is even). In particular,  $A$  is consistent.

(b) See the remark above.

(c) See List (2003), or check that in the proof of (a) the converse inclusions  $A_m \subseteq A$  (if  $n$  is odd) or  $A_{m_1} \cap A_{m_2} \subseteq A$  (if  $n$  is even) also hold under unidimensional alignment. ■

We now show that unidimensional alignment can be related to Rothstein's (1990/1991) *order restriction*, and unidimensional orderedness can be related to (the one-dimensional case of) Grandmont's (1978) *intermediateness*. For this we apply our conditions to special agendas  $X$ : so-called *preference agendas*, which represent standard (strict) preference aggregation problems. First recall that, if  $K$  is a set of alternatives with  $3 \leq |K| < \infty$ , and if  $(\succ_1, \dots, \succ_n)$  is a profile of (strict) preference relation on  $K$  (i.e. of a binary relation  $\succ_i \subseteq K \times K$ ), then<sup>9</sup>

-  $(\succ_1, \dots, \succ_n)$  is *order restricted* if there is an order  $\Omega$  of the individuals such that for all  $x, y \in X$   $\{i : x \succ_i y\} \Omega \{i : y \succ_i x\}$  or  $\{i : y \succ_i x\} \Omega \{i : x \succ_i y\}$  (Rothstein 1990/1991);

-  $(\succ_1, \dots, \succ_n)$  is *(one-dimensionally) intermediate* if there is an order  $\Omega$  of the individuals such that for all  $x, y \in X$  and all  $i, j, k \in N$  with  $i \Omega j \Omega k$ , if  $x \succ_i y \& x \succ_k y$  then  $x \succ_j y$  (Grandmont 1978).

Following Dietrich and List (forthcoming), we now define the *(strict) preference agenda for  $K$*  as  $X_K = \{xPy, \neg xPy \in \mathbf{L} : x, y \in K\}$ , formed in the

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<sup>9</sup>Rothstein and Grandmont formulate their definitions more generally for weak preferences  $\succeq_i$ .

predicate language  $\mathbf{L}$  that is given by (i) a single predicate  $P$  which is binary (and represents strict preference), (ii) a set of constants  $K$  (representing alternatives), (iii) no function symbols, and (iv) as usually, a quantifier  $\forall$  and variables  $v_1, v_2, v_3, \dots$ . Entailment on  $\mathbf{L}$  is given, for all  $A \subseteq \mathbf{L}$  and  $p \in \mathbf{L}$ , by

$$A \models p \Leftrightarrow A \cup Z \text{ entails } p \text{ in the standard sense of predicate logic,}$$

where  $Z$  is the set of rationality conditions on strict preferences.<sup>10</sup>

There is a correspondence between (strict) preference relations  $\succ$ , i.e. arbitrary binary relations on  $K$ , and judgment sets  $A \subseteq X_K$  that are decisive, i.e. contain exactly one member of each pair  $p, \neg p \in X_K$ :

- to any preference relation  $\succ$  corresponds the decisive judgment set

$$A_\succ := \{xPy : x, y \in K \& x \succ y\} \cup \{\neg xPy : x, y \in K \& x \not\succeq y\}; \quad (1)$$

- to any decisive judgment set  $A \subseteq X_K$  corresponds the preference relation

$$x \succ_A y :\Leftrightarrow xPy \in A \ (\Leftrightarrow \neg xPy \notin A) \ \forall x, y \in K.$$

Moreover, a preference relation  $\succ$  is fully rational (i.e. asymmetric, transitive and connected) if and only if  $A_\succ$  is (logically) consistent, because we have built the set of rationality conditions  $Z$  as axioms into the logic. In short, irrationalities of  $\succ$  become inconsistencies of  $A_\succ$ .

As one easily checks,  $\succ$  is order restricted if and only if  $A_\succ$  is unidimensionally aligned. But  $\succ$ 's intermediateness is not quite equivalent to  $A_\succ$ 's unidimensional orderedness: the former requires  $\{i : xPy \in A_i\}$  to be an interval for all  $x, y \in K$ , and the latter requires  $\{i : xPy \in A_i\}$  and  $\{i : \neg xPy \in A_i\}$  to be intervals for all  $x, y \in K$ . We will now give a different, also plausible, definition of  $A_\succ$ , under which  $\succ$ 's intermediateness becomes equivalent to  $A_\succ$ 's unidimensional orderedness (and  $\succ$ 's order restriction stays equivalent to  $A_\succ$ 's unidimensional alignment). But first we need to motivate the new definition.

While  $A_\succ$  is under (1) by construction decisive, a general judgment sets  $A \subseteq X_K$  need not be decisive: it can be incomplete, even empty. How should we understand this? If for a given pair  $x, y \in K$  a preference relation  $\succ$  satisfies  $x \not\succeq y$ , two subtly different interpretations are possible. We could read  $x \not\succeq y$  either as "not viewing  $x$  better than  $y$ ", or as "viewing  $x$  as not better than  $y$ ". This distinction does not appeal to whether  $y \succ x$ , but to the difference between "not believing  $p$ " and "believing  $\neg p$ ", where  $p$  is " $x$  is better than  $y$ ". The definition (1) of  $A_\succ$  implicitly assumes the second interpretation of  $x \not\succeq y$ , because  $A_\succ$  contains  $\neg xPy$  whenever  $x \not\succeq y$ . While a preference relation  $\succ \subseteq K \times K$  is ambiguous between the two interpretations, a judgment set  $A \subseteq X_K$  opts for one or the other interpretation: if  $xPy \notin A$  then  $x$  is "not viewed

<sup>10</sup> $Z$  consists of  $(\forall v_1)(\forall v_2)(v_1Pv_2 \rightarrow \neg v_2Pv_1)$  (asymmetry),  $(\forall v_1)(\forall v_2)(\forall v_3)((v_1Pv_2 \wedge v_2Pv_3) \rightarrow v_1Pv_3)$  (transitivity),  $(\forall v_1)(\forall v_2)(\neg v_1 = v_2 \rightarrow (v_1Pv_2 \vee v_2Pv_1))$  (connectedness) and, for each pair of distinct constants  $x, y \in K$ ,  $\neg x = y$  (exclusiveness of alternatives).

better" than  $y$ , and only if moreover  $\neg xPy \in A$  then  $x$  is "viewed not better" than  $y$ . For any distinct  $x, y \in K$ , a preference relation  $\succ$  can display 4 different patterns:  $x \succ y \& y \not\succeq x$ ,  $x \not\succeq y \& y \succ x$ ,  $x \not\succeq y \& y \not\succeq x$ , or  $x \succ y \& y \succ x$ ; but a judgment set  $A \subseteq X_K$  can display  $2^4 = 16$  different patterns, depending on which of  $xPy, \neg xPy, yPx, \neg yPx$  are contained in  $A$ : all four, any three, any two, any one, or none.

Under the other interpretation of  $x \not\succeq y$ , we have to define  $A_\succ$  not by (1) but as

$$A_\succ := \{xPy : x, y \in K \& x \succ y\}, \quad (2)$$

a typically incomplete judgment set because  $A_\succ$  contains none of  $xPy, \neg xPy$  if  $x \not\succeq y$ . Under this definition, a preference relation  $\succ$  is fully rational (i.e. asymmetric, transitive and connected) if and only if  $A_\succ$  is consistent *and* contains a member of each pair  $xPy, yPx \in X$  with  $x \neq y$ . As one easily checks, under the new definition (2), the (one-dimensional) intermediateness of  $\succ$  translates into the unidimensional orderedness of  $A_\succ$ .

The two parallels can be summarised as follows.

**Remark 5** Consider the preference agenda  $X_K$ .

- (a) A profile  $(\succ_1, \dots, \succ_n)$  of (strict preference) relations on  $K$  is intermediate if and only if the associated judgment profile  $(A_{\succ_1}, \dots, A_{\succ_n})$ , defined by (2), is unidimensionally ordered.
- (b) A profile  $(\succ_1, \dots, \succ_n)$  of (strict preference) relations on  $K$  is order restricted if and only if the associated judgment profile  $(A_{\succ_1}, \dots, A_{\succ_n})$ , defined by (1) or (2), is unidimensionally aligned.

### 3.3 Relations between ordering propositions and ordering individuals

A natural question to ask is whether conditions based on ordering the agenda are related to conditions based on ordering the group. To some extent they are, as we now show.<sup>11</sup>

**Proposition 6** Consider the four conditions on profiles.

- (a) Restricted to profiles of consistent judgment sets,
  - unidimensional alignment implies any of the three other conditions;
  - single-canyonedness implies single-plateauedness;
  - there are no other implications between two conditions.
- (b) Restricted to profiles of consistent and complete (or just decisive) judgment sets, the four conditions are equivalent.

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<sup>11</sup>Of course, the non-implication claims in (a) do not refer to a *fixed* agenda  $X$  and group size  $n$ , but to the *existence* of an agenda and a group size (in fact, of many ones) for which one condition holds without the other one. (For special agendas or group sizes, like  $X = \{p, \neg p\}$  or  $n = 2$ , all conditions hold trivially.)

*Proof.* We use again earlier proof notation, and abbreviations like "SP" for "single-plateaued(ness)".

(a) Under 1 we show the claimed implications, and under 2 the non-implications.

1. Already by Proposition 2, SC implies SP. By Propositions 2-4, to show that UA implies all other conditions it suffices to show that it implies SC. So let  $(A_1, \dots, A_n)$  be a profile of consistent judgment sets, and suppose UA, for simplicity relative to the order  $1, 2, \dots, n$  (i.e.  $1\Omega 2\Omega \dots \Omega n$ ). We show SC relative to  $\leq$ , where  $\leq$  is the order  $p_1 p_2 \dots p_{2k}$  that

- begins with the propositions  $p \in X$  with  $N_p = \{1, \dots, n\}$ ,
- followed by the propositions  $p \in X$  with  $N_p = \{1, \dots, n-1\}$ ,
- ...
- followed by the propositions  $p \in X$  with  $N_p = \{1\}$ ,
- followed by the propositions  $p \in X$  with  $N_p = \emptyset$ ,
- followed by the propositions  $p \in X$  with  $N_p = \{n\}$ ,
- followed by the propositions  $p \in X$  with  $N_p = \{n-1, n\}$ ,
- followed by the propositions  $p \in X$  with  $N_p = \{n-2, \dots, n\}$ ,
- ...
- ending with the propositions  $p \in X$  with  $N_p = \{2, \dots, n\}$ .

	1	2	3	4	5
$p_1$	Y	Y	Y	Y	Y
$p_2$	Y	Y	Y	Y	
$p_3$	Y				
$p_4$					
$p_5$					Y
$p_6$		Y	Y	Y	Y

Table 1: Example of the order  $p_1, \dots, p_{2k}$  for  $n = 5$  individuals and  $2k = 6$  propositions; a "Y" indicates acceptance of the row proposition by the column individual

This procedure to construct  $p_1 \dots p_{2k}$  is well-defined, as by UA each  $p \in X$  is of one of the forms considered in the procedure. In the example profile of Table 1, it is obvious that  $(A_1, \dots, A_n)$  is SC relative to  $p_1 \dots p_{2k}$ :  $A_1 = X \setminus [p_4, p_6]$ ,  $A_2 = A_3 = A_4 = X \setminus [p_3, p_5]$  and  $A_5 = A_6 = X \setminus [p_2, p_4]$ .

For the general proof, consider any  $A_h$  ( $1 \leq h \leq n$ ) and let us show that  $A_h$  is SC relative to  $\leq$ . To show this it suffices to prove that, for all  $p \in X$ , either  $[p_1, p] \subseteq A_h$  or  $[p, p_{2k}] \subseteq A_h$ . So consider any  $p \in X$ . By UA, either  $N_p = \{1, \dots, k\}$  for some  $k$ , or  $N_p = \{k, \dots, n\}$  for some  $k \geq 2$ . By construction of the order  $p_1 \dots p_{2k}$ , in the first case  $[p_1, p] \subseteq A_h$  and in the second case  $[p, p_{2k}] \subseteq A_h$ , as desired.

2. We now show all claimed non-implications by counterexamples.

$SP \not\Rightarrow SC$ . Consider an agenda  $X$  and a profile  $(A_1, \dots, A_n)$  consisting of pairwise disjoint consistent judgment sets, at least three of which are non-empty. The profile is SP, namely relative to an order starting with the propositions in

$A_1$ , followed by those in  $A_2$ , ..., ending with those in  $A_n$ . But the profile is not SC: if it were, say relative to an ordering  $\leq$ , then each non-empty  $A_i$  would contain an extreme (i.e. left- or right-most) proposition, hence there would be at least three extreme propositions (since the  $A_i$ s are disjoint and at least three  $A_i$ s are non-empty).

$SC \not\Rightarrow UO$ . Consider an agenda  $X$ , group  $N$  and profile  $(A_1, \dots, A_n)$  such that  $n = 4$ ,  $A_1 = \{p, p', q, q'\}$ ,  $A_2 = \{p, p'\}$ ,  $A_3 = \{q, q'\}$ ,  $A_4 = \{p, q\}$ , where  $p, p', q, q' \in X$  are pairwise distinct. This profile is SC: consider an order  $\leq$  such that  $p \leq p' \leq \dots \leq q' \leq q$  (where " $\dots$ " contains all remaining propositions). Suppose for a contradiction UO holds, say relative to the order  $i_1 \dots i_n$ . As  $N_{p'} = \{1, 2\}$ , individuals 1 and 2 are neighbours (in  $i_1 \dots i_n$ ) As  $N_{q'} = \{1, 3\}$ , 1 and 3 are neighbours. So 1 is "surrounded" by 2 and 3, i.e.  $i_1 \dots i_n$  contains the sublist 213 or 312; suppose it contains the sublist 213 (the proof continues analogously for the sublist 312). Further, as  $N_p = \{1, 2, 4\}$ , 4 is a neighbour of 1 or of 2; as 4 cannot be a neighbour of 1 (which is surrounded by 2 and 3), it is a neighbour of 2. So  $i_1 \dots i_n$  contains the sublist 4213. Finally, as  $N_q = \{1, 3, 4\}$ , 4 is a neighbour of 1 or 3, which is not the case since  $i_1 \dots i_n$  contains the sublist 4213.

$SC \not\Rightarrow UA$ . This follows from  $SC \not\Rightarrow UO$  by  $UO \Rightarrow UA$ .

$SP \not\Rightarrow UO$ . This follows from  $SC \not\Rightarrow UO$  by  $SC \Rightarrow SP$ .

$SP \not\Rightarrow UA$ . This follows from  $SC \not\Rightarrow UA$  by  $SC \Rightarrow SP$ .

$UO \not\Rightarrow UA$ . Consider an agenda  $X$ , group  $N$  and profile  $(A_1, \dots, A_n)$  such  $n \geq 3$  and the  $A_i$ s are pairwise disjoint and singleton. As each  $N_p, p \in X$ , is empty or singleton, the profile is UO (relative to any order of  $N$ ). It is not UA: if it were, say relative to the order  $\Omega$  of  $N$ , then each  $i \in N$  would have to be extreme, i.e. smallest or largest in  $\Omega$  (as  $i$  is the only individual accepting the proposition in  $A_i$ ), which is not possible as there are  $n \geq 3$  individuals but only two extreme positions.

$UO \not\Rightarrow SP$ . Consider a group, agenda  $X$  and profile  $(A_1, \dots, A_n)$  with  $n = 3$  and  $A_1 = \{p, p_1\}$ ,  $A_2 = \{p, p_2\}$  and  $A_3 = \{p, p_3\}$ , where  $p, p_1, p_2, p_3 \in X$  are pairwise distinct. This profile is UO, relative to any order of  $N$ . But it is not SP: if it were, say relative to an order  $p_1 \dots p_{2k}$  of  $X$ , then in this order  $p$  must be a neighbour of  $p_1$  (by  $A_1 = \{p, p_1\}$ ), one of  $p_2$  (by  $A_2 = \{p, p_2\}$ ) and one of  $p_3$  (by  $A_3 = \{p, p_3\}$ ), a contradiction.

$UO \not\Rightarrow SC$ . This follows from  $UO \not\Rightarrow SP$  by  $SC \Rightarrow SP$ .

(b) Now let  $(A_1, \dots, A_n)$  be a profile of consistent *and complete* (or just decisive) judgment sets. Then each  $A_i$  contains exactly  $k = |X|/2$  propositions. As by part (a)  $UA \Rightarrow SC$  and  $SC \Rightarrow SP$ , the equivalence of all four conditions follows from the following additional implications, which we now prove using that  $|A_i| = k$  for all  $i$ .

$SP \Rightarrow UO$ . Suppose SP, say relative to the order  $p_1 \dots p_{2k}$ . Then for all  $i$  there is (using  $|A_i| = k$ ) an index  $j(i) \in \{1, \dots, 2k\}$  such that  $A_i = [p_{j(i)}, p_{j(i)+k-1}]$ . Consider an order of the individuals  $i_1 \dots i_n$  such that  $j(i_1) \leq j(i_2) \leq \dots \leq j(i_n)$ .

To check UO relative to  $i_1\dots i_n$ , note that for all  $p = p_l \in X$  we have

$$\begin{aligned} \{i : p_l \in A_i\} &= \{i : p_l \in [p_{j(i)}, p_{j(i)+k-1}]\} = \{i : j(i) \leq l < j(i) + k\} \\ &= \{i : -l \leq -j(i) < k - l\} = \{i : l - k < j(i) \leq l\}, \end{aligned}$$

which is an interval of the order  $i_1\dots i_n$ , as desired.

$UO \Rightarrow UA$ . Let  $(A_1, \dots, A_n)$  be UO, say relative to the order  $\Omega$ . To see that  $(A_1, \dots, A_n)$  is also UA relative to the same order  $\Omega$ , consider any  $p \in X$ . As each  $A_i$  contains exactly one member of each pair  $p, \neg p \in X$ ,  $N_{\neg p} = N \setminus N_p$ . Further, by UA  $N_p$  and  $N_{\neg p}$  are  $(\Omega)$ -intervals. So  $N_p$  and  $N \setminus N_p$  are intervals. Hence  $N_p \Omega N \setminus N_p$  or  $N \setminus N_p \Omega N_p$ , as desired. ■

## 4 Local conditions for majority consistency

The four conditions discussed so far are, for many agendas, stronger than necessary. Our goal is now to introduce a move similar to that from single-peakedness to single-peakedness on triples in preference aggregation. Specifically, we will require our conditions to hold not on the entire agenda  $X$  but on appropriate subagendas. Identifying the right subagendas is not obvious. Choosing them according to their size (e.g. subagendas of size less than  $x$ ) or to the syntactic form of their member propositions (e.g. subagendas whose propositions containing only a certain type or number logical connectives) does not work. The analysis of this chapter is guided by two aims.

1. The subagendas should be chosen such that once inconsistencies (under majority rule) are excluded within these subagendas, inconsistencies are excluded in general (just like for strict preferences acyclicity on each triple implies general acyclicity);
2. If possible, there should be *few* and *small* subagendas, so as to make our local domain conditions weak.

### 4.1 General form of the local conditions

The following definitions set out the type of local conditions to be analysed, where the choice of subagendas  $Z$  is the topic of the following subsections. Definition 7 is based on ordering propositions, and Definition 8 is based on ordering individuals.

- Definition 7** (a) *A subagenda is a non-empty subset of  $X$  closed under negation (hence itself an agenda).*
- (b) *For any subagenda  $Z$ , a judgment set  $A$  is single-plateaued (-canyoned) on  $Z$  relative to  $\leq$ , an order on  $Z$ , if  $A \cap Z$ , seen as a judgment set for the agenda  $Z$ , is single-plateaued (-canyoned) relative to  $\leq$ .*
- (c) *For any subagenda  $Z$ , if each  $A_i$  in a profile  $(A_1, \dots, A_n)$  is single-plateaued (-canyoned) on  $Z$  relative to  $\leq$ , an order on  $Z$ ,  $(A_1, \dots, A_n)$  is single-*

plateaued (-canyoned) on  $Z$  relative to  $\leq$ , or simply single-plateaued (-canyoned) on  $Z$ .

- (d) For any set  $\mathcal{Z}$  of subagendas, if a profile  $(A_1, \dots, A_n)$  is for each  $Z \in \mathcal{Z}$  single-plateaued (-canyoned) on  $Z$  relative to  $\leq_Z$ , an order on  $Z$ , then  $(A_1, \dots, A_n)$  is single-plateaued (-canyoned) on  $\mathcal{Z}$  relative to  $(\leq_Z)_{Z \in \mathcal{Z}}$ , or simply single-plateaued (-canyoned) on  $\mathcal{Z}$ .
- (e) In (b) and (c)  $\leq$  is called a ( $Z$ -)structuring order, and in (d)  $(\leq_Z)_{Z \in \mathcal{Z}}$  is called a ( $\mathcal{Z}$ -)family of structuring orders.

**Definition 8** (a) For any subagenda  $Z$ , a profile  $(A_1, \dots, A_n)$  is unidimensionally ordered (aligned) on  $Z$  relative to  $\Omega$ , an order on  $N$ , or simply unidimensionally ordered (aligned) on  $Z$ , if  $(A_1 \cap Z, \dots, A_n \cap Z)$ , seen as a profile for the agenda  $Z$ , is unidimensionally ordered (aligned) relative to  $\leq$ .

- (b) For any set  $\mathcal{Z}$  of subagendas, if a profile  $(A_1, \dots, A_n)$  is for each  $Z \in \mathcal{Z}$  unidimensionally ordered (aligned) on  $Z$  relative to  $\Omega_Z$ , an order on  $N$ , then  $(A_1, \dots, A_n)$  is unidimensionally ordered (aligned) on  $\mathcal{Z}$  relative to  $(\Omega_Z)_{Z \in \mathcal{Z}}$ , or simply unidimensionally ordered (aligned) on  $\mathcal{Z}$ .
- (c) In (a)  $\Omega$  is called a ( $Z$ -)structuring order, and in (b)  $(\Omega_Z)_{Z \in \mathcal{Z}}$  is called a ( $\mathcal{Z}$ -)family of structuring orders.

Single-plateauedness on  $X$  is of course equivalent to single-plateauedness simpliciter, and single-plateauedness on  $Z (\subseteq X)$  implies single-plateauedness on any subagenda  $Z' \subseteq Z$ . Analogous remarks apply to the three other conditions.

Any implications and equivalences between our global conditions (see Proposition 6) also hold for their local versions, because the local conditions are defined by the validity on subagendas of the global conditions. So Proposition 6 has the following corollary.

**Corollary 9** Consider the four local conditions on profiles in Definitions 7-8.

- (a) Restricted to profiles of consistent judgment sets,
- unidimensional alignment on  $\mathcal{Z}$  implies any of the three other conditions;
  - single-canyonedness on  $\mathcal{Z}$  implies single-plateauedness on  $\mathcal{Z}$ .
- (b) Restricted to profiles of consistent and complete (or just decisive) judgment sets, the four conditions are equivalent.

Whether the non-implications of Propositions 6 continue to hold for the local conditions depends on the specification of  $\mathcal{Z}$ . For the two specifications analysed below the non-implications do indeed hold (e.g. single-plateauedness on  $\mathcal{Z}$  does *not* imply single-canyonedness on  $\mathcal{Z}$ ), because the counterexamples constructed in the proof of Proposition 6 work also as counterexamples for the local conditions *if* we moreover assume the agenda  $X$  to be such that each

global condition is equivalent to its local variant, an assumption that is possible without loss of generality.<sup>12</sup>

## 4.2 Conditions based on minimal inconsistent sets

In Definitions 7-8, what class of subagendas  $\mathcal{Z}$  should be chosen? This subsection takes the following approach. Note that a (collective) judgment set  $A \subseteq X$  is inconsistent if and only if it has a subset  $Y \subseteq X$  that is minimal inconsistent (i.e. is inconsistent and every proper subset of  $Y$  is consistent). So a consistent majority outcome can be ensured by requiring a local structural condition, e.g. single-plateauedness on  $\mathcal{Z}$ , where

$$\mathcal{Z} := \{\{p, \neg p : p \in Y\} : Y \text{ is a minimal inconsistent subset of } X\}. \quad (3)$$

**Proposition 10** *Let  $\mathcal{Z}$  be the class of subagendas (3), and consider a profile  $(A_1, \dots, A_n)$  of consistent judgment sets.*

- (a) *If  $(A_1, \dots, A_n)$  satisfies any of the four local structural conditions of Definitions 7-8, the majority outcome is consistent.*
- (b) *If  $(A_1, \dots, A_n)$  is unidimensionally aligned on  $\mathcal{Z}$ , the majority outcome is  $\cup_{Z \in \mathcal{Z}^*} (A_{m_Z} \cap Z)$  (if  $n$  is even) or  $\cup_{Z \in \mathcal{Z}^*} (A_{m_Z} \cap A_{m'_Z} \cap Z)$  (if  $n$  is odd), where*
  - $\mathcal{Z}^* \subseteq \mathcal{Z}$  *is any subset of subagendas with  $\cup_{Z \in \mathcal{Z}^*} Z = X$  (e.g.  $\mathcal{Z}^* = \mathcal{Z}$ );*
  - *for all  $Z \in \mathcal{Z}^*$ ,  $m_Z$  is the middle individual (if  $n$  is odd) or  $(m_Z, m'_Z)$  is the middle pair of individuals (if  $n$  is even) in any  $Z$ -structuring order.*

*Proof.* Let  $\mathcal{Z}$  and  $(A_1, \dots, A_n)$  be as specified, with majority outcome  $A$ .

(a) To prove  $A$ 's consistency, it suffices to prove that  $A$  has no minimal inconsistent subset, hence to prove that  $A \cap Z$  is consistent for all  $Z \in \mathcal{Z}$ . So consider any subagenda  $Z \in \mathcal{Z}$ . As the profile is, say, single-plateaued on  $Z$  (the proof is similar for single-canyonnedness or unidimensional orderedness/alignment),  $(A_1 \cap Z, \dots, A_n \cap Z)$  is single-plateaued for the agenda  $Z$ , hence has a consistent majority outcome by Proposition 2. But this majority outcome is  $A \cap Z$ . So  $A \cap Z$  is consistent, as desired.

(b) Let  $\mathcal{Z}^*$  and the individuals  $(m_Z)_{Z \in \mathcal{Z}^*}$  (if  $n$  is odd) or  $(m_{Z,1}, m'_{Z,1})_{Z \in \mathcal{Z}^*}$  (if  $n$  is even) be as specified. To show that  $A$  is  $\cup_{Z \in \mathcal{Z}^*} (A_{m_Z} \cap Z)$  (if  $n$  is even) or  $\cup_{Z \in \mathcal{Z}^*} (A_{m_Z} \cap A_{m'_Z} \cap Z)$  (if  $n$  is odd), it is by  $A = \cup_{Z \in \mathcal{Z}^*} (A \cap Z)$  sufficient to show that, for all  $Z \in \mathcal{Z}^*$ ,  $A \cap Z$  is  $A_{m_Z} \cap Z$  (if  $n$  is even) or  $A_{m_Z} \cap A_{m'_Z} \cap Z$  (if  $n$

<sup>12</sup>To see why the assumption is possible, check first that the assumption holds if  $\mathcal{Z}$  contains only the trivial subagendas  $X$  and  $\{p, \neg p\}$ ,  $p \in X$ . Second, it has to be checked that such a special class  $\mathcal{Z}$  can indeed arise under the specifications analysed below. (For instance, if  $\mathcal{Z}$  is given by (3),  $\mathcal{Z}$  can take the special form: let the only minimal inconsistent sets  $Y \subseteq X$  be (i) a subset  $Y \subseteq X$  containing a member of each pair  $p, \neg p \in X$ , and (ii) the binary sets  $\{p, \neg p\} \subseteq X$ .)

is odd). The latter follows from part (c) of Proposition 4 using that, for any given agenda  $Z \in \mathcal{Z}^*$ ,  $A \cap Z$  is the majority outcome on the unidimensionally aligned profile  $(A_1 \cap Z, \dots, A_n \cap Z)$ . ■

How do our local conditions look if applied to the (strict) preference agenda  $X_K := \{xPy, \neg xPx : x, y \in K\}$  discussed in Section 3.2? To answer this question, we have to identify the set  $\mathcal{Z}$  for  $X_K$ . Note that, by definition of our preference logic, for any distinct  $x, y \in K$ ,  $\neg xPy$  and  $yPx$  are equivalent, i.e. entail each other. Call two judgment sets "essentially identical" if one arises from the other by (zero, one or more) replacements of propositions by equivalent propositions. For distinct options  $x_1, \dots, x_k \in K$  ( $k \geq 1$ ), the cycle  $x_1 \succ x_2 \succ \dots \succ x_k \succ x_1$  can be represented by the set  $\{x_1Px_2, x_2Px_3, \dots, x_{k-1}Px_k, x_kPx_1\}$ ; we call such a set, and any set essentially identical to it, a *cycle*, or more precisely a *k-cycle*.

**Remark 11** Consider the preference agenda  $X_K$ .

- (a) The minimal inconsistent sets  $Y \subseteq X_K$  are the cycles.
- (b) So the class of subagendas (3) is  $\mathcal{Z} = \{\{p, \neg p : p \in Y\} : Y \subseteq X_K \text{ is a cycle}\}$ .

*Proof.* Part (b) follows from part (a). Part (a) follows by the definition of the logic  $\mathbf{L}$ . First, any cycle is obviously minimal inconsistent in  $\mathbf{L}$ . Second, suppose  $Y \subseteq X_K$  is minimal inconsistent. One may check that, by  $Y$ 's inconsistency, some subset  $Y^* \subseteq Y$  is a cycle. By minimal inconsistency, then,  $Y = Y^*$ . ■

### 4.3 Conditions based on irreducible inconsistent sets

The last subsection's class  $\mathcal{Z}$  of subagendas is often large. This is for instance reflected in the fact that, for a preference agenda  $X_K$ , all cycles  $Y \subseteq X$  (also 4-cycles, 5-cycles etc.) are minimal inconsistent, hence give rise to a subagenda in  $\mathcal{Z}$  (see part (b) of Remark 11). As a consequence, for instance unidimensional alignment on  $\mathcal{Z}$ , applied to preference agendas, is stronger than Rothstein's order restriction on triples, essentially because  $\mathcal{Z}$  contains subagendas involving more than three alternatives. Using such a rich class of subagendas  $\mathcal{Z}$  was necessary because we wanted to guarantee majority consistency even if individuals hold incomplete judgment sets (whereas order restriction on triples does not guarantees acyclic majority preferences if individual incompleteness is allowed).

We now show that, if individuals hold *complete* judgment sets, we can use a much slimmer class of subagendas  $\mathcal{Z}$ , which in the case of a preference  $X_K$  involves no more than three alternatives. We obtain the new subagendas by focussing not on *minimal* inconsistent sets but on *irreducible* inconsistent sets, i.e. by defining

$$\mathcal{Z} := \{\{p, \neg p : p \in Y\} : Y \text{ is an irreducible inconsistent subset of } X\}, \quad (4)$$

where irreducibility is the notion to be introduced and studied now.

**Definition 12** (i) For two inconsistent sets  $Y, Z \subseteq X$ ,  $Z$  is a reduction of  $Y$  (and  $Y$  is reducible to  $Z$ ) if  $|Z| < |Y|$  and each  $p \in Z \setminus Y$  is entailed by some subset  $V \subseteq Y$  with  $|Y \setminus V| \geq 2$ .

(ii) An inconsistent set  $Y \subseteq X$  is irreducible if it has no reduction.<sup>13</sup>

For instance, the inconsistent set  $\{a, a \rightarrow b, b \rightarrow c, \neg c\}$  (where  $a, b, c$  are distinct atomic propositions) is reducible to  $Z = \{b, b \rightarrow c, \neg c\}$ , as  $b$  is entailed by  $\{a, a \rightarrow b\}$ . In the definition of reducibility, the clause  $|Y \setminus V| \geq 2$  is essential. Notably, only requiring  $|Y \setminus V| \geq 1$  would render all inconsistent sets  $Y \subseteq X$  with  $|Y| \geq 3$  reducible, namely to any pair  $\{p, \neg p\}$  with  $p \in Y$ :  $\neg p$  is entailed by  $Y \setminus \{p\}$  (assuming non-paraconsistency L4).

As every *non-minimal* inconsistent set is reducible to any of its inconsistent subsets, the following holds.

**Lemma 13** Every irreducible inconsistent set  $Y \subseteq X$  is minimal inconsistent.

In particular, the class of subagendas  $\mathcal{Z}$  defined by (4) is, as desired, contained in that defined by (3).

Conversely, many minimal inconsistent sets  $Y \subseteq X$ , like the set  $\{a, a \rightarrow b, b \rightarrow c, \neg c\}$  mentioned above, are reducible. As another example, if  $X$  is the (strict) preference agenda  $X_K$  defined in Section 3.2 (for a set of options  $K$ ), then any  $k$ -cycle

$$Y = \{x_1 P x_2, x_2 P x_3, \dots, x_{k-1} P x_k, x_k P x_1\}$$

with  $k \geq 4$  is reducible, e.g. to the 3-cycle  $\{x_1 P x_2, x_2 P x_3, x_3 P x_1\}$ , as  $x_3 P x_1$  is entailed by  $\{x_3 P x_4, x_4 P x_5, \dots, x_k P x_1\}$ . The full characterisation of irreducible inconsistent sets  $Y \subseteq X_K$  is given in Remark 16 below.

In the light of the first aim stated at the beginning of Section 4, the following result is crucial.

**Proposition 14** Every complete and inconsistent judgment set  $A \subseteq X$  has an irreducible inconsistent subset.

The reason why we need individual completeness is now apparent: this guarantees that the majority outcome is complete (or close to complete if  $n$  is even), so that Proposition 14 applies.

*Proof.* Let  $A \subseteq X$  be complete and inconsistent. Among all inconsistent subsets of  $A$ , let  $B$  be one of smallest size  $|B|$ . As  $C^+$  is compact,  $|B| < \infty$ . We show that  $B$  is irreducible. Suppose for a contradiction that  $B$  is reducible to  $C \subseteq X$ . We will define an inconsistent subset of  $A$  smaller than  $B$ , in

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<sup>13</sup>The condition that  $V \vDash p$  could be weakened without affecting our results to the condition that  $V \cup \{\neg p\}$  is inconsistent. (The two conditions are equivalent if the logic is non-paraconsistent in the sense of condition L4 in Dietrich forthcoming.)

contradiction to the choice of  $B$ . By  $|C| < |B|$  and the choice of  $B$ , we have  $C \subsetneq A$ . So there is a  $p \in C \setminus A$ . Since  $A$  is complete, we have  $\neg p \in A$ . As  $C$  is a reduction of  $B$ , there is a subset  $B^* \subseteq B$  with  $|B \setminus B^*| \geq 2$  and  $B^* \models p$ . Now  $B^* \cup \{\neg p\}$  is an inconsistent subset of  $A$  smaller than  $B$ :

- $B^* \cup \{\neg p\}$  is a subset of  $A$  by  $B^* \subseteq B \subseteq A$  and  $\neg p \in A$ .
- $B^* \cup \{\neg p\}$  is inconsistent by  $B^* \models p$ ;
- $|B^* \cup \{\neg p\}| \leq |B^*| + 1 = |B| - |B \setminus B^*| + 1 \leq |B| - 2 + 1 < |B|$ . ■

We are now in a position to prove our central result: if individuals hold not only consistent but also complete judgment sets, the local conditions based on irreducible sets guarantee majority consistency.

**Proposition 15** *If a profile of complete and consistent judgment sets satisfies one (hence by Corollary 9 all) of the four local conditions in Definitions 7-8, with  $\mathcal{Z}$  given by (4), the majority outcome is consistent.*

*Proof.* We consider a profile  $(A_1, \dots, A_n)$  of the specified kind, and we use earlier proof notation.

Case 1:  $n$  is odd. Then  $A$  is complete. So, by Proposition 14, to prove  $A$ 's consistency, it suffices to prove that  $A$  has no irreducible inconsistent subset, hence to prove that  $A \cap Z$  is consistent for all  $Z \in \mathcal{Z}$ . The latter follows by an argument analogous to the one in the proof of part (a) of Proposition 10.

Case 2:  $n$  is even. Let  $A_{n+1}$  be any complete and consistent judgment set such that  $(A_1, \dots, A_{n+1})$  still satisfies the local condition, e.g. single-plateauedness on  $\mathcal{Z}$ , now for group size  $n+1$  (one might take  $A_{n+1} := A_1$ ). By case 1 the majority outcome on  $(A_1, \dots, A_{n+1})$  is a consistent judgment set  $\tilde{A}$ . Check that  $A \subseteq \tilde{A}$ . So  $A$  is consistent, as desired. ■

Finally, to illustrate that the class of subagendas  $\mathcal{Z}$  has shrunk – and our local conditions have become weaker – by defining  $\mathcal{Z}$  by (4) rather than (3), we again consider preference agendas. A profile  $(\succ_1, \dots, \succ_n)$  of strict preference relations on  $K$  is *order restricted on triples* (Rothstein 1990/1991) resp. (one-dimensionally) *intermediate on triples* (Grandmont's 1978) if, for every subagenda  $K' \subseteq K$  with  $|K'| = 3$ , the profile restricted to  $K$ ,  $(\succ_1|_{K'}, \dots, \succ_n|_{K'})$ , is order restricted resp. intermediate (as defined in Section 3.2).

**Remark 16** *Consider the preference agenda  $X_K$ .*

- (a) *The irreducible inconsistent sets  $Y \subseteq X_K$  are the 1- or 2- or 3-cycles.*
- (b) *So the class of subagendas (4) is  $\mathcal{Z} = \{\{p, \neg p : p \in Y\} : Y \subseteq X_K \text{ is a 1- or 2- or 3-cycle}\}$ .*
- (c) *A profile  $(\succ_1, \dots, \succ_n)$  of strict linear orders<sup>14</sup> on  $K$  is intermediate (equivalently, order restricted) on triples if and only if the associated judgment profile  $(A_{\succ_1}, \dots, A_{\succ_n})$ , given by (1), is unidimensionally ordered (equivalently, aligned) on  $\mathcal{Z}$ , with  $\mathcal{Z}$  given by (4).*

<sup>14</sup>i.e. of asymmetric, transitive and connected relations

*Proof.* (a) First, consider a 1- or 2- or 3-cycle  $Y$ . If  $Y$  is a 1-cycle, i.e.  $Y = \{xPx\}$  for some  $x \in K$ , or a 2-cycle, say  $Y = \{xPy, yPx\}$  with distinct  $x, y \in K$ , then  $Y$  is obviously irreducible. Now let  $Y$  be a 3-cycle, say  $Y = \{xPy, yPz, zPx\}$  for distinct  $x, y, z \in K$ . Suppose for a contradiction that  $Y$  is reducible, say to  $Z \subseteq X$ . Then  $|Z| \leq 2$ . Moreover each  $p \in Z$  is entailed by a single member of  $Y$ , i.e. by one of  $xPy, yPz, zPx$ . But the only proposition (in  $X$ ) entailed by  $xPy$  is  $xPy$  (and the logically equivalent  $\neg yPx$ ), and similarly for  $yPz$  and  $zPx$ . So each  $p \in Z$  is one of  $xPy, yPz, zPx$  (or one of  $\neg yPx, \neg zPy, \neg xPz$ ). Hence  $Z$  is (essentially identical to) a proper subset of  $Y = \{xPy, yPz, zPx\}$ . So  $Z$  is consistent, a contradiction.

Second, suppose  $Y \subseteq X_K$  is irreducible inconsistent. By Lemma 13,  $Y$  is minimal inconsistent. So, by part (a) of Remark 11,  $Y$  is a cycle, hence (essentially identical to) a set of type  $\{x_1Px_2, x_2Px_3, \dots, x_{k-1}Px_k, x_kPx_1\}$  ( $k \geq 1$ ). Now  $k \leq 3$ , as otherwise  $Y$  would be reducible to  $Z := \{x_1Px_2, x_2Px_3, x_3Px_1\}$ . So  $Y$  is a 1- or 2- or 3-cycle.

(b) follows immediately from (a).

(c): Let  $(\succ_1, \dots, \succ_n)$  be as specified. For all  $i$ , as  $\succ_i$  is a strict linear order, we have, for all  $x, y \in K$ ,  $x \succ_i y \Leftrightarrow y \not\succeq_i x$ ; so for  $(\succ_1, \dots, \succ_n)$  intermediateness on triples is indeed equivalent to order restriction on triples. Moreover, as each  $A_i$  is complete and consistent, for  $(A_1, \dots, A_n)$  unidimensional orderedness on  $\mathcal{Z}$  is indeed equivalent to unidimensional alignment on  $\mathcal{Z}$  (see Corollary 9). So it remains to show that  $(\succ_1, \dots, \succ_n)$  is intermediate on triples if and only if  $(A_1, \dots, A_n)$  is unidimensionally ordered on  $\mathcal{Z}$ .

To prove the latter, note first that  $(A_{\succ_1}, \dots, A_{\succ_n})$  is trivially unidimensional ordered on any 1-cycle  $Z = \{xPx, \neg xPx\}$ , and also on any 2-cycle  $Z = \{xPy, \neg xPy, yPx, \neg yPx\}$  (consider an order of  $N$  beginning with the individuals  $i$  with  $x \succ_i y$ , and followed by the individuals  $i$  with  $y \succ_i x$ ). For this reason, and by part (b), unidimensional orderedness on  $\mathcal{Z}$  is equivalent to unidimensional orderedness on

$$\begin{aligned} \mathcal{Z}' &:= \{ \{p, \neg p : p \in Y\} : Y \subseteq X_K \text{ is a 3-cycle} \} \\ &= \{ \{xPy, \neg xPy, yPz, \neg yPz, zPx, \neg zPx\} : x, y, z \in K \text{ all distinct} \}. \end{aligned}$$

Now unidimensional orderedness on  $\mathcal{Z}'$  is equivalent to unidimensional orderedness on

$$\mathcal{Z}'' := \{ \{xPy, yPx, yPz, zPy, zPx, xPz\} : x, y, z \in K \text{ all distinct} \}$$

(where  $\mathcal{Z}''$  arises from  $\mathcal{Z}'$  by replacing any  $\neg xPy$  in any  $Y \in \mathcal{Z}'$  by  $yPx$ ), because  $\neg xPy$  is equivalent to  $yPx$  in the logic  $\mathbf{L}$ , so that each  $A_i$  contains  $\neg xPy$  if and only if it contains  $yPx$  (as  $A_i$  is complete and consistent). One easily checks that  $(\succ_1, \dots, \succ_n)$  is intermediate on triples if and only if  $(A_1, \dots, A_n)$  is unidimensionally ordered on  $\mathcal{Z}''$ , as desired. ■

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