The strong sequential core in a dynamic exchange economy

The Strong Sequential Core in a Dynamic Exchange Economy

Arkadi Predtetchinski* Jean-Jacques Herings*
Hans Peters†

January 8, 2002

Abstract

Dynamic exchange economies with uncertainty are considered in which information is released over infinite time. The strong sequential core of such an economy consists of those consumption processes where no coalition of agents wishes to deviate at any moment for the rest of time. Comparable to the optimality principle in dynamic programming, necessary and sufficient conditions for non-emptiness of the strong sequential core in stationary economies are derived, based on non-emptiness of classical cores of certain static economies. The main result of the paper is an existence result for stationary economies based on the presence of a competitive equilibrium in an associated limit economy, given a high enough discount factor. Moreover, sufficient conditions are given under which the strong sequential core contains only time and history independent processes.

1 Introduction

One of the major difficulties in modeling cooperation in a dynamic environment is the potential abundance of opportunities for coalitions to deviate

*Department of Economics, University of Maastricht, P.O. Box 616, 6200 MD Maastricht, The Netherlands. E-mail: a.predtetchinsky@algec.unimaas.nl, p.herings@algec.unimaas.nl.

†Department of Quantitative Economics, University of Maastricht, P.O. Box 616, 6200 MD Maastricht, The Netherlands. E-mail: h.peters@ke.unimaas.nl.

from earlier agreements. As the initial positions of agents and coalitions and their preferences may change over time, a contract that was originally considered acceptable may no longer be so after some time. Thus, agents or coalitions may tend to break agreements reached earlier in the game. In the case of uncertainty, where information becomes available over time, this is even more likely to occur. What was perceived as an acceptable agreement when no information was available, may no longer be so after some information has been released.

In a dynamic economy the classical core concept as introduced in Aumann (1961) is still meaningful. A consumption bundle is an element of this core if no coalition can make its members better off by pooling their own resources and deviating at period zero for the rest of the time. The classical core, however, ignores incentives for coalitional improvement that may arise in subsequent periods. The classical core is essentially a static concept, for it does not depend on the time structure or on the process of the release of information.

These considerations motivate the search for a core concept that offers a more appropriate view on cooperation taking place in a dynamic, rather than a static environment. To the best of our knowledge, the concept of the so-called sequential core was first proposed in Gale (1978) in a model of a monetary economy. Other core concepts that take into account the dynamic structure of the environment have been proposed in Repullo (1988), Koutsougeras (1998), and Filar and Petrosjan (2000). Kranich *et al.* (2001) distinguish between a strong and a weak version of the sequential core. They study both concepts in a deterministic setting where agents face a finite sequence of transferable utility games. In Predtetchinski *et al.* (2001) the notion of the strong sequential core is applied to an economy with two periods and uncertainty where agents are given a possibility to exchange assets in the first period.

Despite the differences in the models, the three mentioned papers share a common approach to the definition of the strong sequential core. The dynamic nature of the economy or game allows for a number of sub-economies (or sub-games) to be identified. The consumption bundle (or vector of payoff streams) is then said to be a strong sequential core-element if it belongs to the classical core of each of the sub-economies (sub-games) of the original dynamic economy (game). Hence, at no particular moment of time can a coalition improve by deviating for the rest of the time.

In general, it turns out to be too much to hope for non-emptiness of the strong sequential core. In Predtetchinski *et al.* (2001) the strong sequential core is defined in such a way that, as a set, it is weakly increasing in the number of assets. In this sense, the presence of assets can potentially lead to non-emptiness of the strong sequential core. Nevertheless, a number of negative results is established even in this context. Generic emptiness is shown in the setting of a finance economy, and an example of emptiness is presented even for asset structures satisfying a very strong condition of completeness. The general conclusion is that, unless a complete set of state-contingent contracts is available for trade, non-emptiness of the strong sequential core cannot be guaranteed.

These negative results provide a motivation to augment the basic model in a different direction, namely to consider a dynamic exchange economy with infinite time horizon. A dynamic exchange economy is an economy with uncertainty where information is revealed gradually over infinite time. The process of the release of information is described by a sequence of partitions that gives rise to a so-called event tree. At each node of this event tree the utility functions and initial endowments are specified. Basically, we will restrict attention to stationary economies. A dynamic exchange economy is called stationary if the utility functions and the initial endowments do not depend on time or history but only on the current state, of which there are assumed finitely many. For stationary dynamic exchange economies the question of non-emptiness is solved in the sense that the strong sequential core is shown to exist whenever the discount factor that agents apply to future utility streams is sufficiently close to one.

In more detail, the content of the paper is as follows. In Section 2 we give the definitions of the dynamic exchange economy, the classical core and the strong sequential core. We will, however, not study the dynamic exchange economy in full generality but, in Section 3, introduce the subclass of stationary dynamic exchange economies. In these economies there is a finite number of so called current states that correspond one-to-one to given exchange economies. A transition matrix gives the probabilities of reaching each state from another one, and this then defines the event tree. We show that the strong sequential core of this stationary dynamic exchange economy is non-empty if and only if in each state the static economy comprising, roughly, the discounted expected value of all future utility streams has a non-empty core. This result should be seen as a characterization of the strong sequential core closely related to the idea of a value function in dynamic programming. In Section 4 an existence result for the strong sequential core in a stationary dynamic exchange economy is provided. More precisely, this result says that a competitive equilibrium allocation of a specific limit economy is in the strong sequential core whenever the discount factor on future utility streams is sufficiently close to one. Section 5 concludes. The mathematical proofs are collected in an appendix.

2 Dynamic Exchange Economies

A dynamic exchange economy is an exchange economy with uncertainty, where information is revealed gradually over time. The process of the release of information is modeled by an event tree. At all nodes of the event tree, the consumption sets, elementary utility functions, and initial endowments are specified.

Time and uncertainty. Let $\mathbf{T}$ be a finite or countable set of time periods and let Ω be a sample space. The sample space is interpreted to be the list of all possible paths that the economy can follow over time. We use the notation $\Omega \subseteq \Omega$ for an event, and $\omega \in \Omega$ for a typical path.

The process of the release of information is described by a sequence of partitions $\{\mathbb{F}_t\}_{t\in\mathbf{T}}$ of the sample space. The number of subsets in $\mathbb{F}_t$ is finite and the partition $\mathbb{F}_t$ is finer than $\mathbb{F}_{t-1}$, i.e. for every event $\Omega \in \mathbb{F}_t$ there is an event $\Omega' \in \mathbb{F}_{t-1}$ such that $\Omega \subseteq \Omega'$. At time zero there is no information, so that $\mathbb{F}_0 = \{\Omega\}$.

A pair $\xi = (t, \Omega)$ with $t \in \mathbf{T}$ and $\Omega \in \mathbb{F}_t$ is called a node; $t(\xi) = t$ is the date of node ξ. The set $\mathbb{D}$ consisting of all nodes is called the event tree induced by the sequence of partitions $\{\mathbb{F}_t\}_{t\in\mathbf{T}}$:

$$\mathbb{D} = \{(t, \Omega) : \ t \in \mathbf{T}, \ \Omega \in \mathbb{F}_t\}.$$

The subtree starting at node $\xi = (t, \Omega) \in \mathbb{D}$ is the set

$$\mathbb{D}(\xi) = \{(t', \Omega') \in \mathbb{D} : \ t' \geq t, \ \Omega' \subseteq \Omega\}.$$

Any function $\mathcal{X}$ defined on a tree or a subtree will be referred to as a *process*; we write $\mathcal{X} : \xi \mapsto \mathcal{X}_\xi$. A restriction of the process $\mathcal{X}$ defined on $\mathbb{D}$ to a subtree $\mathbb{D}(\xi)$ is denoted by $\mathcal{X}_{\mathbb{D}(\xi)}$.

We assume that there is a system of conditional probabilities $\rho(\eta|\xi)$, interpreted as the probability of reaching node η given that we are in node ξ. In particular, $\rho(\xi)$ denotes the probability of reaching node $\xi \in \mathbb{D}$, and

$\sum_{\xi:t(\xi)=t}\rho(\xi) = 1$ for all $t \in \mathbf{T}$. Note that $\rho(\eta|\xi)$ is defined even when $\rho(\xi) = 0$.

Endowments, consumption sets, and utility functions. Let N be a finite set of agents. There are L commodities at each node of the event tree. Agent $i \in N$ has a consumption set $X^i_\xi \subseteq \mathbb{R}^L$, and a Bernoulli (elementary, or instantaneous) utility function $u^i_\xi : X^i_\xi \to \mathbb{R}$ at every node $\xi \in \mathbb{D}$. The commodity space $\mathbf{C}$ is the space of all bounded processes $\mathcal{X} : \mathbb{D} \to \mathbb{R}^L$. The consumption set $\mathbf{X}^i \subseteq \mathbf{C}$ of agent i contains all bounded processes $\mathcal{X}^i$ on $\mathbb{D}$ with values $\mathcal{X}^i_\xi \in X^i_\xi$. Agent $i \in N$ owns the endowment process $e^i \in \mathbf{X}^i$.

We assume that agent i's preference ordering over $\mathbf{X}^i$ is represented by the expected utility discounted to the initial node:

$$\mathcal{V}^i(\mathcal{X}^i) = \sum_{\xi\in\mathbb{D}} \rho(\xi)\,\delta^{t(\xi)}\,u^i_\xi(\mathcal{X}^i_\xi), \quad \mathcal{X}^i \in \mathbf{X}^i,$$

where $0 < \delta < 1$ is a given common discount factor. Note that this sum is finite because $\mathbf{X}^i$ contains only bounded processes.

This concludes the description of a dynamic exchange economy $\mathbf{E}$.

Sub-economies. Let $\mathbf{E}_{\mathbb{D}(\xi)}$ denote the sub-economy of the economy $\mathbf{E}$ starting at node $\xi \in \mathbb{D}$. The consumption set $\mathbf{X}^i_{\mathbb{D}(\xi)}$ of agent i in this sub-economy is the set of all bounded processes $\mathcal{X}^i$ on $\mathbb{D}(\xi)$ with values $\mathcal{X}^i_\xi \in X^i_\xi$. We assume that the preference ordering over $\mathbf{X}^i_{\mathbb{D}(\xi)}$ can be represented by the discounted expected utility conditional on node ξ:

$$\mathcal{V}^i_{\mathbb{D}(\xi)}(\mathcal{X}^i) = \sum_{\xi'\in\mathbb{D}(\xi)} \rho(\xi'|\xi)\,\delta^{t(\xi')-t(\xi)}\,u^i_{\xi'}(\mathcal{X}^i_{\xi'}), \quad \mathcal{X}^i \in \mathbf{X}^i_{\mathbb{D}(\xi)}. \tag{1}$$

We continue with the definitions of the classical core for the sub-economy $\mathbf{E}_{\mathbb{D}(\xi)}$ and of the strong sequential core for the economy $\mathbf{E}$. Essentially, the former definition requires that no coalition be able to make its members better off by deviating at node ξ for the rest of the time. The latter definition requires that no coalition be able to improve by deviating at any of the nodes of the event tree. In this way, the strong sequential core takes away the incentives for coalitions to first agree to a proposed allocation but break the agreement later on.

Definition 1 Process $\mathcal{X} \in \prod_{i\in N}\mathbf{X}^i_{\mathbb{D}(\xi)}$ belongs to the *classical core* of the sub-economy $\mathbf{E}_{\mathbb{D}(\xi)}$, denoted $C(\mathbf{E}_{\mathbb{D}(\xi)})$, if

1. $\sum_{i\in N}\mathcal{X}^i_{\xi'}=\sum_{i\in N}e^i_{\xi'}$ for all $\xi'\in\mathbb{D}(\xi)$,

2. there exist no coalition $Q\subseteq N$ and no process $\mathcal{Y}\in\prod_{i\in Q}\mathbf{X}^i_{\mathbb{D}(\xi)}$ such that $\sum_{i\in Q}\mathcal{Y}^i_{\xi'}=\sum_{i\in Q}e^i_{\xi'}$ for all $\xi'\in\mathbb{D}(\xi)$ and $\mathcal{V}^i_{\mathbb{D}(\xi)}(\mathcal{Y}^i)>\mathcal{V}^i_{\mathbb{D}(\xi)}(\mathcal{X}^i)$ for all $i\in Q$.

Definition 2 Process $\mathcal{X}\in\prod_{i\in N}\mathbf{X}^i$ belongs to the *strong sequential core* of the dynamic exchange economy $\mathbf{E}$, denoted $SSC(\mathbf{E})$, if $\mathcal{X}_{\mathbb{D}(\xi)}\in C(\mathbf{E}_{\mathbb{D}(\xi)})$ for all $\xi\in\mathbb{D}$.

3 Stationary Dynamic Exchange Economies

In the rest of this paper attention is restricted to stationary dynamic economies.

3.1 Definition of a Stationary Dynamic Economy

In order to define a stationary dynamic exchange economy the following objects are needed:

a set $\mathbf{S}=\{0,1,\ldots,S\}$ of states;

probabilities $\pi(\sigma|s)$ of transition from state $s\in\mathbf{S}$ to state $\sigma\in\mathbf{S}$;

consumption sets $X^i_s\subset\mathbb{R}^L$, Bernoulli utility functions $u^i_s:X^i_s\to\mathbb{R}$ and initial endowments $e^i_s\in\mathbb{R}^L$ for all $i\in N,\ s\in\mathbf{S}$.

Below we will refer to the following assumption.

Assumption $\mathcal{A}$:

(1) π is an irreducible matrix;

(2) the consumption sets X^i_s are convex, closed and bounded from below for all $s\in\mathbf{S},\ i\in N$;

(3) the utility functions u^i_s are continuous and concave for all $s\in\mathbf{S},\ i\in N$;

(4) $e^i_s\in X^i_s$ for all $s\in\mathbf{S},\ i\in N$.

The stationary economy has an infinite horizon: the set of time periods is $\mathbf{T} = \{0, 1, \dots\}$. The sample space Ω consists of all paths that the state can follow over time, with the convention that only state zero can realize in period zero. Formally,

$$\Omega = \{\omega : \mathbf{T} \to \mathbf{S} : \ \omega_0 = 0\},$$

where we write ω_t instead of $\omega(t)$. A typical element Ω of the partition $\mathbb{F}_t$ comprises all paths in Ω that have a common history up to date $t+1$. Formally, $\Omega \in \mathbb{F}_t$ if and only if Ω can be written as

$$\Omega = \{\omega_0\} \times \{\omega_1\} \times \cdots \times \{\omega_t\} \times \mathbf{S} \times \mathbf{S} \times \mathbf{S} \cdots , \tag{2}$$

with $(\omega_0, \dots, \omega_t)$ being a common history preceding date $t+1$ of all the paths contained in Ω. A node $\xi = (t, \Omega)$, with $t \in \mathbf{T}$, $\Omega \in \mathbb{F}_t$ may be identified with the collection $(t; \omega_0, \dots, \omega_t)$. The state $\omega_t \in \mathbf{S}$ is referred to as the *current state at node* ξ. It follows from the definitions already given that zero is the current state at the initial node. The operator $s(\cdot) : \mathbb{D} \to \mathbf{S}$ assigns a current state $s(\xi)$ to any node ξ in the event tree.

For the case $\mathbf{S} = \{0, 1\}$, the event tree is partially reproduced in Figure 1. Numbers in bold type indicate the current states.

Let $\Omega \in \mathbb{F}_t$ and $\Omega' \in \mathbb{F}_{t'}$ be events with $t < t'$, $\Omega \supseteq \Omega'$. The probability of Ω' conditional on Ω (the probability that the node (t', Ω') will be reached conditional on the fact that the node (t, Ω) has been reached) is equal to $\prod_{\tau=t}^{t'-1} \pi(\omega_{\tau+1}|\omega_\tau)$, for any $\omega \in \Omega'$.

Finally, the consumption sets, Bernoulli utility functions, and endowments for agent i at node ξ are defined by

$$X^i_\xi = X^i_{s(\xi)}, \ \ u^i_\xi = u^i_{s(\xi)}, \ \ \ e^i_\xi = e^i_{s(\xi)}.$$

Definition 3 A process $\mathcal{X}$ with values in $\mathbb{R}^m$ is *time and history independent* if there exists a function $x : \mathbf{S} \to \mathbb{R}^m$ such that $\mathcal{X}_\xi = x_{s(\xi)}$ for all $\xi \in \mathbb{D}$; we write $\mathcal{X} = \{x\}$.

The endowment process is an example of a time and history independent process.

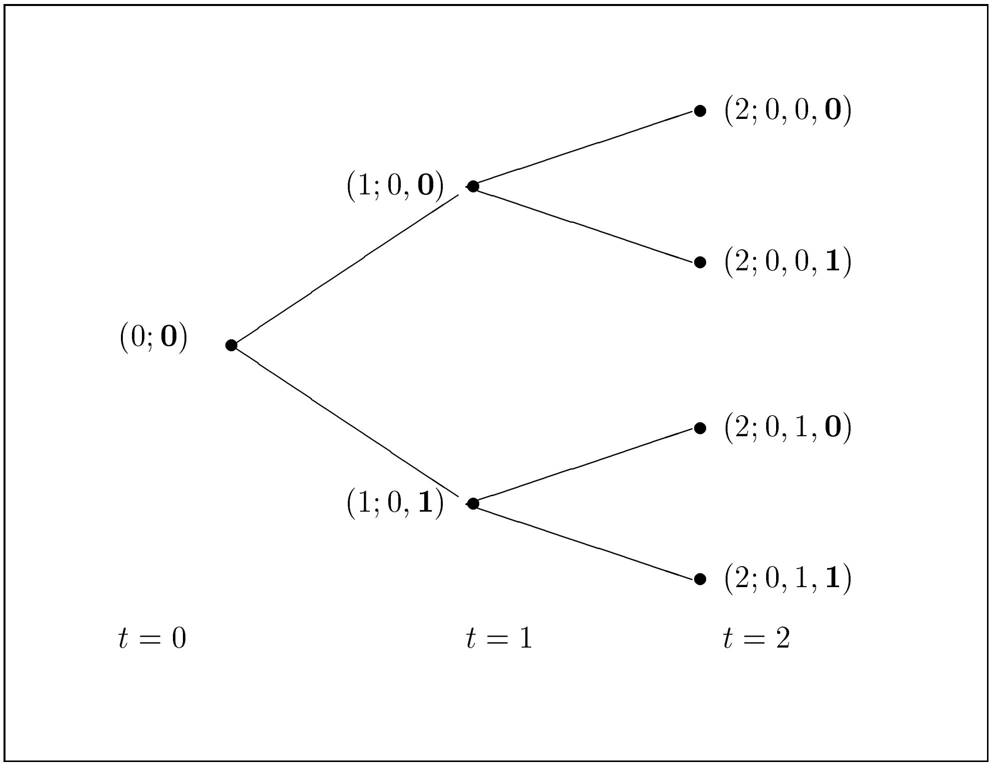

Figure 1: *Event tree of a stationary economy with* $\mathbf{S} = \{0, 1\}$.

3.2 Characterization of the Strong Sequential Core for Stationary Economies

This subsection has two objectives: (1) to provide a tractable criterion for a given time and history independent process to be an element of the strong sequential core of a stationary economy; (2) to give sufficient conditions for the strong sequential core of a stationary economy to contain *only* time and history independent processes.

Consider a time and history independent process $\mathcal{X}^i = \{x^i\}$ with values $x^i_s \in X^i_s$. The instantaneous utility of $\mathcal{X}^i$ at the node η can be written as the sum

$$u^i_\eta(\mathcal{X}^i_\eta) = \sum_{\sigma\in\mathbf{S}} \iota(s(\eta)=\sigma)\ u^i_\sigma(x^i_\sigma), \quad \text{where} \tag{3}$$

$$\iota(s(\eta)=\sigma) = \begin{cases} 1 & \text{if } s(\eta)=\sigma, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Substituting (3) and (4) into (1) gives

$$\mathcal{V}^i_{\mathbb{D}(\xi)}(\mathcal{X}^i) = \sum_{\sigma\in\mathbf{S}} \Psi(\sigma|\xi)\ u^i_\sigma(x^i_\sigma), \quad \text{where} \tag{5}$$

$$\Psi(\sigma|\xi) = \sum_{\eta\in\mathbb{D}(\xi)} \rho(\eta|\xi)\delta^{t(\eta)-t(\xi)}\ \iota\left(s(\eta)=\sigma\right). \tag{6}$$

The number $\Psi(\sigma|\xi)$ is the expected value, discounted to node ξ, of the process that promises one unit of utility at a node $\eta \in \mathbb{D}(\xi)$ with current state σ and zero utility otherwise.

Lemma 1 *$\Psi(\sigma|\cdot) : \xi \mapsto \Psi(\sigma|\xi)$ is a time and history independent process. If the matrix π is irreducible, then $\Psi(\sigma|\xi) > 0$ for all $\sigma \in \mathbf{S},\ \xi \in \mathbb{D}$.*

An analytical expression for the corresponding function $\psi(\sigma|\cdot) : s \mapsto \psi(\sigma|s)$ can be found in the appendix. One immediate implication of Lemma 1 is that the process $\mathcal{V}^i_{\mathbb{D}(\cdot)}(\mathcal{X}^i) : \xi \mapsto \mathcal{V}^i_{\mathbb{D}(\xi)}(\mathcal{X}^i)$ is a time and history independent process whenever $\mathcal{X}^i$ is.

Definition 4 The symbol $\mathcal{E}_s$ denotes a pure exchange economy in which $\mathbb{R}^{(S+1)L}$ is the commodity space, $X^i = \prod_{\sigma\in\mathbf{S}} X^i_\sigma$ is agent i's consumption set, $e^i = (e^i_\sigma)_{\sigma\in\mathbf{S}}$ is the vector of initial endowments, and $v^i_s : X^i \to \mathbb{R}$ is the utility function defined by

$$v^i_s(x^i) = \sum_{\sigma\in\mathbf{S}} \psi(\sigma|s)\ \ u^i_\sigma(x^i_\sigma), \quad x^i \in X^i. \tag{7}$$

The relationship between the functions v^i_s and $\mathcal{V}^i_{\mathbb{D}(\xi)}$ is described by the following equation:

$$v^i_{s(\xi)}(x^i) = \mathcal{V}^i_{\mathbb{D}(\xi)}\left(\{x^i\}\right) \quad \forall\ \xi \in \mathbb{D}. \tag{8}$$

That is, the discounted expected utility of the time and history independent process $\{x^i\}$ conditional on the node ξ is equal to $v^i_s(x^i)$, whenever the current state at the node ξ equals s.

The following lemma provides a criterion for a given time and history independent process to be an element of the strong sequential core of a stationary economy.

Lemma 2 *Let $\mathbf{E}$ be a stationary economy satisfying assumption $\mathcal{A}$. Let $\mathcal{X} = \{x\}$ be a time and history independent process. Then*

1. $\mathcal{X}_{\mathbb{D}(\xi)} \in C(\mathbf{E}_{\mathbb{D}(\xi)})$ *if and only if* $x \in C(\mathcal{E}_{s(\xi)})$.
2. $\mathcal{X} \in SSC(\mathbf{E})$ *if and only if* $x \in C(\mathcal{E}_s)$ *for all* $s \in \mathbf{S}$.

Lemma 3 gives sufficient conditions for the classical core of the stationary economy to contain only time and history independent processes.

Lemma 3 *Let $\mathbf{E}$ be a stationary economy that satisfies assumption $\mathcal{A}$ and in which the matrix π is positive, $X^i_\sigma = \mathbb{R}^L_+$, u^i_σ are strongly monotone*[1] *for all $i \in N$, $\sigma \in \mathbf{S}$ and strictly concave for all $i \in \{1, \ldots, n-1\}$, $\sigma \in \mathbf{S}$. Then the classical core of any sub-economy $\mathbf{E}_{\mathbb{D}(\xi)}$ contains only time and history independent processes.*

We conclude this section with a criterion for non-emptiness of the strong sequential core.

Corollary 1 *Let $\mathbf{E}$ be a stationary economy satisfying all assumptions of Lemma 3. Then*

$$SSC(\mathbf{E}) \neq \emptyset \quad \Leftrightarrow \quad \bigcap_{s \in \mathbf{S}} C(\mathcal{E}_s) \neq \emptyset.$$

4 An Existence Result for Stationary Economies

In this section we consider a family of stationary economies $\mathbf{E}_\delta$ parameterized by the discount factor $\delta \in (0,1)$. We will prove that the strong sequential core of an economy $\mathbf{E}_\delta$ is non-empty whenever δ is sufficiently close to one.

[1] The function u^i_σ is strongly monotone if the following condition is satisfied: $\left[x^i_\sigma,\ y^i_\sigma \in \mathbb{R}^L_+,\ x^i_{\sigma,l} \geq y^i_{\sigma,l}\ \forall\ l \in \{1, \ldots, L\},\ x^i_\sigma \neq y^i_\sigma\right]$ implies $\left[u^i_\sigma(x^i_\sigma) > u^i_\sigma(y^i_\sigma)\right]$.

Our approach is to compute the limits for the utility functions v_s^i as δ approaches one, and next to construct an economy using the resulting utility functions. This artificial economy will then provide us with some plausible candidates to belong to the strong sequential core of the original dynamic economy.

As is evident from equation (6), ψ is a function of the discount factor. To make this dependence explicit we will write $\psi(\sigma|s;\delta)$. The utility functions $v_s^i(\cdot)$ of the economies $\mathcal{E}_s$ depend on the discount factor through ψ; we write $v_s^i(\cdot,\delta)$ and $\mathcal{E}_{s,\delta}$.

Naturally, the value of $\psi(\sigma|s;\delta)$ goes to infinity as δ approaches one. When multiplied by $(1-\delta)$, however, it converges to a well-defined limit.

Theorem 1 *Suppose that the matrix π is irreducible. Then the limit*

$$\phi(\sigma|s) \equiv \lim_{\delta\to 1}(1-\delta)\ \psi(\sigma|s,\delta)$$

exists for all $\sigma, s \in \mathbf{S}$. Moreover,

(1) the values of $\phi(\sigma|s)$ are independent of s, i.e. $\phi(\sigma|s) = \phi(\sigma|s')$ for all $s, s' \in \mathbf{S}$; [2]

(2) $\phi(\sigma) > 0$ for all $\sigma \in \mathbf{S}$;

(3) the vector $\phi = (\phi(\sigma))_{\sigma\in\mathbf{S}}$ is the eigenvector of the matrix π, corresponding to the eigenvalue $\lambda = 1$.

Definition 5 The symbol $\mathcal{E}_\infty$ denotes a pure exchange economy, in which $\mathbb{R}^{(S+1)L}$ is the commodity space, $X^i = \prod_{\sigma\in\mathbf{S}} X_\sigma^i$ is the consumption set, $e^i = (e_\sigma^i)_{\sigma\in\mathbf{S}}$ is a vector of initial endowments, and $\vartheta^i : X^i \mapsto \mathbb{R}$ is the utility function defined by

$$\vartheta^i(x^i) = \sum_{\sigma\in\mathbf{S}} \phi(\sigma)\ u_\sigma^i(x_\sigma^i), \quad x^i \in X^i.$$

Theorem 2 *Let $\mathbf{E}_\delta$ be a family of stationary economies that satisfy assumption $\mathcal{A}$ and in which the Bernoulli utility functions u_s^i are strictly concave and locally non-satiated for all $i \in N$, $s \in \mathbf{S}$. Let $\overline{x}$ be an equilibrium allocation of the economy $\mathcal{E}_\infty$. Then there exists a $\overline{\delta} \in (0,1)$ such that $\{\overline{x}\} \in SSC(\mathbf{E}_\delta)$ for all $\delta \in (\overline{\delta}, 1)$.*

[2] In what follows we suppress the argument s of ϕ.

It remains to augment the list of assumptions of Theorem 2 by assumptions that guarantee existence of an equilibrium for the economy $\mathcal{E}_\infty$. We either require that the endowments are interior to the consumption set, or, alternatively, that the utility functions are strongly monotone.

Corollary 2 *Let $\mathbf{E}_\delta$ be a family of stationary economies satisfying all assumptions of Theorem 2 and at least one of the following conditions:*

1. $e^i_s \in intX^i_s$ *for all* $s \in \mathbf{S},\ i \in N$*;*
2. u^i_s *are strongly monotone for all* $s \in \mathbf{S},\ i \in N$.

Then there exists a $\overline{\delta} \in (0,1)$ *such that* $SSC(\mathbf{E}_\delta) \neq \emptyset$ *for all* $\delta \in (\overline{\delta}, 1)$.

Below we give an example of a family of stationary economies of which the strong sequential core is empty for all $\delta \in (0,1)$. In this example the matrix π is assumed to be positive, and the state-independent Bernoulli utility functions are continuous, concave, and strongly monotone. Such an example must necessarily violate the assumption that u^i_σ are *strictly* concave for all $\sigma \in \mathbf{S},\ i \in N$.

Example Consider a family $\mathbf{E}_\delta$ of stationary economies, in which $N = \{a, b, c\}$, $L = 1$, and the Bernoulli utility functions are given by

$$u^i_\sigma(x^i_\sigma) = \ln(x^i_\sigma), \quad x^i_\sigma \in \mathbb{R}^1_{++}, \quad i \in \{a, b\}$$

$$u^c_\sigma(x^c_\sigma) = x^c_\sigma, \quad x^c_\sigma \in \mathbb{R}^1_{+}.$$

We assume that the initial endowments satisfy the following conditions:

$$e^a_\sigma + e^b_\sigma = 1, \quad \forall\ \sigma \in \mathbf{S}$$

$$e^c_\sigma > \textstyle\sum_{i \in Q} e^i_\sigma - \left(\min_{s \in \mathbf{S}} \sum_{i \in Q} e^i_s\right), \quad \forall\ \sigma \in \mathbf{S}, \quad \forall\ Q \subseteq N :\ c \in Q.$$

The transition probabilities $\pi(\sigma|s)$ are positive for all $\sigma, s \in \mathbf{S}$.
We introduce the additional notation:

$$t^i_{s,\delta} = (1-\delta) \sum_{\sigma \in \mathbf{S}} \psi(\sigma|s;\delta) e^i_\sigma.$$

Finally, we impose a joint restriction on the transition probabilities and the endowments: for each $\delta \in (0,1)$ there exist $s, s' \in \mathbf{S}$ such that $t^a_{s,\delta} \neq t^a_{s',\delta}$. It is straightforward to show that $C(\mathcal{E}_{s,\delta})$ is a single-element set:

$$C(\mathcal{E}_{s,\delta}) = \left\{ \left(t^a_{s,\delta}\mathbf{1},\ t^b_{s,\delta}\mathbf{1},\ e^c \right) \right\}.$$

The assumption with regard to $t^a_{s,\delta}$ guarantees that $\bigcap_{s\in\mathbf{S}} C(\mathcal{E}_{s,\delta}) = \emptyset$ for all $\delta \in (0,1)$. Since all the assumptions of Lemma 3 are satisfied, the criterion for non-emptiness of the strong sequential core given in Corollary 1 is applicable. This leads to the conclusion that $SSC(\mathbf{E}_\delta) = \emptyset$ for all $\delta \in (0,1)$.

5 Concluding Remarks

The modeling of cooperation in dynamic environments has received little attention in the literature so far. Although the classical core is well-defined in this framework, it ignores incentives for coalitions to deviate at future time periods.

We study the strong sequential core for infinite horizon exchange economies with uncertainty. At each date-state in such an economy, a sub-economy can be identified. An allocation belongs to the strong sequential core if it belongs to the classical core of each sub-economy. It follows immediately that the strong sequential core is a strengthening of the core. It is therefore not surprising that the strong sequential core might be empty, even when the classical core is not.

We restrict attention to stationary economies. These are economies where at each date the economy is in one of a finite number of states. The state determines the instantaneous utility functions and endowments of all agents. Movement from one state to the next occurs according to an exogenously given Markov process. We show that the strong sequential core only contains time and history independent allocations. We are also able to characterize the strong sequential core as the intersection of the classical cores of the sub-economies corresponding to all possible initial states. This result is used to show that the strong sequential core is non-empty when all agents in the economy are sufficiently patient.

A Proofs

Proof of Lemma 1.
Let node $\xi \in \mathbb{D}$, state $\sigma \in \mathbf{S}$ and date $\tau \in \mathbf{T}$ be given. Let $\mathcal{L}(\sigma, \tau|\xi)$ be the probability that the set of nodes

$$\{\xi' \in \mathbb{D}(\xi) : \; t(\xi') = t(\xi) + \tau, \; s(\xi') = \sigma\}$$

are reached, conditional on the fact that ξ has been reached. Then equation (6) can be rewritten as

$$\Psi(\sigma|\xi) = \sum_{\tau=0}^{\infty} \delta^{\tau} \mathcal{L}(\sigma, \tau|\xi). \tag{9}$$

It is straightforward to show that $\mathcal{L}(\sigma, \tau|\cdot) : \xi \mapsto \mathcal{L}(\sigma, \tau|\xi)$ is a time and history independent process, with corresponding function $l(\sigma, \tau|\cdot) : s \mapsto l(\sigma, \tau|s)$. The value of $l(\sigma, \tau|s)$ is given by entry (σ, s) of the matrix π^{τ}. It is the probability of a transition from state s to state σ in τ periods.
Let ψ be an $(S+1) \times (S+1)$ matrix with (σ, s) entry equal to $\psi(\sigma|s)$. Then equations (9) can be written in matrix form as follows:

$$\psi = \sum_{\tau=0}^{\infty} \delta^{\tau} \pi^{\tau}.$$

Since the spectral radius of the matrix $\delta\pi$ equals $\delta \in (0, 1)$,

$$\psi = (I_{S+1} - \delta\pi)^{-1},$$

where I_{S+1} is the identity matrix of order $(S + 1)$. If the matrix π is irreducible, then the matrix ψ is positive. $\square$

Proof of Lemma 2.
$(\Rightarrow)$ Suppose that there is a deviation $y \in \prod_{i \in Q} X^i$ from x by a coalition Q in the economy $\mathcal{E}_{s(\xi)}$. It then follows from equality (8) that the regular process $\{y\}$ that starts at the node ξ is a deviation from $\{x\}$ in the sub-economy $\mathbf{E}_{\mathbb{D}(\xi)}$.
$(\Leftarrow)$ Suppose that there is a deviation $\mathcal{Y} \in \prod_{i \in Q} \mathbf{X}^i_{\mathbb{D}(\xi)}$ from $\{x\}$ by coalition Q in the sub-economy $\mathbf{E}_{\mathbb{D}(\xi)}$. Note that the process $\mathcal{Y}$ need not satisfy

Definition 3. There exists, however, a time and history independent process $\mathcal{Z} = \{z\},\ z \in \prod_{i \in Q} X^i$ that (a) is feasible for Q and (b) gives each member of the coalition Q at least as high expected discounted utility at node ξ as the process $\mathcal{Y}$ does. Such a process $\{z\}$ would then be a deviation from the process $\{x\}$ in the sub-economy $\mathbf{E}_{\mathbb{D}(\xi)}$. By equality (8), the allocation z would be a deviation from x in the economy $\mathcal{E}_{s(\xi)}$.
Define z by

$$z^i_\sigma = \frac{1}{\Psi(\sigma|\xi)} \sum_{\xi' \in \mathbb{D}(\xi)} \rho(\xi'|\xi) \delta^{t(\xi')-t(\xi)}\ \iota\left(s(\xi') = \sigma\right) \mathcal{Y}^i_{\xi'}, \tag{10}$$

for all $i \in Q$ and $\sigma \in \mathbf{S}$. Since the sets X^i_σ are assumed to be convex and closed, $z^i_\sigma \in X^i_\sigma$. It is straightforward to verify that z is feasible for Q. Continuity and concavity of the Bernoulli utility functions implies the inequalities

$$u^i_\sigma(z^i_\sigma) \geq \frac{1}{\Psi(\sigma|\xi)} \sum_{\xi' \in \mathbb{D}(\xi)} \rho(\xi'|\xi) \delta^{t(\xi')-t(\xi)}\ \iota(s(\xi') = \sigma) u^i_\sigma\left(\mathcal{Y}^i_{\xi'}\right) \tag{11}$$

for all $i \in Q,\ \sigma \in \mathbf{S}$. We substitute inequality (11) into the equality (5) to obtain:

$$\begin{aligned}
\mathcal{V}^i_{\mathbb{D}(\xi)}(\mathcal{Z}^i) &= \sum_{\sigma \in \mathbf{S}} \Psi(\sigma|\xi)\ u^i_\sigma(z^i_\sigma) \\
&\geq \sum_{\xi' \in \mathbb{D}(\xi)} \rho(\xi'|\xi) \delta^{t(\xi')-t(\xi)} \sum_{\sigma \in \mathbf{S}} \iota\left(s(\xi') = \sigma\right) u^i_\sigma(\mathcal{Y}^i_{\xi'}) \\
&= \sum_{\xi' \in \mathbb{D}(\xi)} \rho(\xi'|\xi) \delta^{t(\xi')-t(\xi)}\ u^i_{\xi'}(\mathcal{Y}^i_{\xi'}) \\
&= \mathcal{V}^i_{\mathbb{D}(\xi)}(\mathcal{Y}^i)
\end{aligned} \tag{12}$$

for all $i \in Q$. $\square$

Proof of Lemma 3.
Let the process $\mathcal{Y} \in \prod_{i \in Q} \mathbf{X}^i_{\mathbb{D}(\xi)}$ be feasible for the grand coalition in the sub-economy $\mathbf{E}_{\mathbb{D}(\xi)}$. If $\mathcal{Y}$ does not satisfy Definition 3, then there are $\eta,\ \varsigma \in \mathbb{D}(\xi),\ i_0 \in N$ such that $s(\eta) = s(\varsigma)$ and $\mathcal{Y}^{i_0}_\eta \neq \mathcal{Y}^{i_0}_\varsigma$. Since

$$\sum_{i \in N} \mathcal{Y}^i_\eta = \sum_{i \in N} \mathcal{Y}^i_\varsigma = \sum_{i \in N} e^i_{s(\eta)},$$

there must exist $i_1 \in N,\ i_1 \neq i_0$ such that $\mathcal{Y}^{i_1}_{\eta} \neq \mathcal{Y}^{i_1}_{\varsigma}$.
Let the process $\mathcal{Z} = \{z\}$ be defined as in (10) for all $\sigma \in \mathbf{S},\ i \in N$. Note that under the assumptions of Lemma 3 inequalities (11) and (12) hold true for all $\sigma \in \mathbf{S},\ i \in N$.
Since the requirement of strict concavity is violated for at most one agent, the utility function $u^i_{s(\eta)}$ must be strictly concave either for $i = i_0$ or for $i = i_1$. Let it be strictly concave for i_0. Under the assumption that all the transition probabilities are positive, $\rho(\eta|\xi)$ and $\rho(\varsigma|\xi)$ are both positive. Therefore, inequality (11) is strict for $i = i_0$ and $\sigma = s(\eta)$, and inequality (12) is strict for $i = i_0$. Thus, the process $\mathcal{Z}$ weakly dominates process $\mathcal{Y}$. Due to continuity and strong monotonicity of the utility functions, however, the equivalence between the weak and the strong notions of Pareto-optimality holds true, i.e. it is possible to construct a process $\tilde{\mathcal{Z}}$ that makes inequalities (12) strict for all $i \in N$. $\square$

Proof of Theorem 1.
Let $\lambda_1, \ldots, \lambda_n$ $(n \leq S+1)$ be the distinct eigenvalues of the matrix π.
The matrix π is a column stochastic matrix, i.e. its entries in a given column add up to 1. This implies that $\lambda_1 = 1$ is an algebraically (and hence geometrically) simple eigenvalue of the matrix π; that $\mathbf{1} = (1, \ldots, 1)'$ is the eigenvector of the matrix π' corresponding to the eigenvalue λ_1, and that the eigenvector of the matrix π corresponding to λ_1 can be set positive.
Let $\pi = T \Lambda T^{-1}$ be a Jordan decomposition of the matrix π, where Λ is a Jordan form. There is a unique Jordan block corresponding to the eigenvalue λ_1 and this block has order 1. Hence, Λ can be written as

$$\Lambda = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \Gamma \end{pmatrix}, \tag{13}$$

where Γ is a direct sum of those Jordan blocks that correspond to eigenvalues $\lambda_2, \ldots, \lambda_n$. Observe that the matrix $(I_S - \delta\Gamma)$ is non-singular, whenever $\delta \neq 1/\lambda_j,\ j = 2, \ldots, n$.
The first column of the matrix T, denoted $cn_1 T$ and the first row of the matrix T^{-1}, denoted $rw_1 T^{-1}$ are the eigenvectors of the matrices π and π', respectively. We can therefore set

$$cn_1 T >> \mathbf{0} \tag{14}$$

$$rw_1 T^{-1} = \alpha\, \mathbf{1}' \tag{15}$$

for some nonzero scalar α. Using the Jordan decomposition of the matrix π we can write

$$(I_{S+1} - \delta\pi)^{-1} = T(I_{S+1} - \delta\Lambda)^{-1}T^{-1}$$

$$(I_{S+1} - \delta\Lambda)^{-1} = \begin{pmatrix} (1-\delta)^{-1} & \mathbf{0} \\ \mathbf{0} & (I_S - \delta\Gamma)^{-1} \end{pmatrix}$$

Because $(I_S - \Gamma)$ is invertible,

$$\lim_{\delta\to 1}(1-\delta)(I_{S+1} - \delta\Lambda)^{-1} = diag\{1, \mathbf{0}\}$$

$$\begin{aligned} \lim_{\delta\to 1}(1-\delta)(I_{S+1} - \delta\pi)^{-1} &= T\, diag\{1, \mathbf{0}\}\, T^{-1} = \\ &= cn_1T \cdot rw_1T^{-1} = \alpha\, cn_1T \cdot \mathbf{1}'. \end{aligned} \tag{16}$$

The following observations complete the proof of Theorem 1:

(1) $\phi(\sigma|s) = \alpha T_{\sigma,1}$ apparently does not depend on s.

(2) Since $\phi(\sigma|s)$ cannot be negative, α must be positive. On the other hand, the vector cn_1T was set positive. Hence, $\phi(\sigma|s)$ is, in fact, positive.

(3) The vector $\phi = \alpha{\cdot}cn_1T$ is the eigenvector of the matrix π corresponding to λ_1.

□

We introduce the following additional notation:

$$X_\sigma(Q) = \left\{x_\sigma \in \textstyle\prod_{i\in Q} X^i_\sigma : \;\; \sum_{i\in Q} x^i_\sigma = \sum_{i\in Q} e^i_\sigma\right\}$$

$$X(Q) = \textstyle\prod_{\sigma\in\mathbf{S}} X_\sigma(Q).$$

Let $\mathcal{P}'_Q(\mathcal{E}_\infty)$ denote the set of weakly efficient allocations for a coalition Q in the economy $\mathcal{E}_\infty$. Recall that $\overline{x} \in \mathcal{P}'_Q(\mathcal{E}_\infty)$ if and only if $\overline{x} \in X(Q)$ and there is no $x \in X(Q)$ such that $\vartheta^i(\overline{x}^i) > \vartheta^i(x^i)$ for all $i \in Q$. The sets $\mathcal{P}'_Q(\mathcal{E}_{s,\delta})$ are similarly defined for the economies $\mathcal{E}_{s,\delta}$. To prove Theorem 2 we need the following.

Lemma 4 *Suppose that π is irreducible and that the Bernoulli utility functions u_s^i are concave. Let the coalition $Q \subseteq N$ be given. Then*

$$\mathcal{P}'_Q(\mathcal{E}_\infty) = \mathcal{P}'_Q(\mathcal{E}_{s,\delta})$$

for all $s \in \mathbf{S}$ and $\delta \in (0,1)$.

Proof of Lemma 4.
An allocation x is an element of $\mathcal{P}'_Q(\mathcal{E}_\infty)$ if and only if there exist weights $\alpha^i \geq 0$ $(i \in Q)$, $\sum_{i \in Q} \alpha^i = 1$, such that x maximizes the function

$$\sum_{i \in Q} \alpha^i \vartheta^i(x^i) = \sum_{\sigma \in \mathbf{S}} \phi(\sigma) \sum_{i \in Q} \alpha^i u_\sigma^i(x_\sigma^i)$$

over $X(Q)$. Since all components of $\phi(\cdot)$ are positive, this is equivalent to the requirement that x_σ maximizes the function

$$\sum_{i \in Q} \alpha^i u_\sigma^i(x_\sigma^i)$$

over $X_\sigma(Q)$ for all $\sigma \in \mathbf{S}$. Again, since all components of $\psi(\cdot)$ are positive, this is equivalent to the condition that x maximizes the function

$$\sum_{\sigma \in \mathbf{S}} \psi(\sigma|s;\delta) \sum_{i \in Q} \alpha^i u_\sigma^i(x_\sigma^i) = \sum_{i \in Q} \alpha^i v_s^i(x^i;\delta)$$

over $X(Q)$. This last condition is equivalent to $x \in \mathcal{P}'_Q(\mathcal{E}_{s,\delta})$. $\square$

Proof of Theorem 2.
We must show that for all values of δ sufficiently close to one and for all $s \in \mathbf{S}$ the allocation $\overline{x}$ is in the core of the economy $\mathcal{E}_{s,\delta}$ (see Lemma 2). Suppose that this is not the case. Then there exist a state $s \in \mathbf{S}$, coalition $Q \subseteq N$, sequences $\delta_m \in (0,1)$ and $x_{(m)} \in X(Q)$ such that

$$\begin{aligned} \lim \delta_m &= 1, \\ v_s^i\left(x_{(m)}^i;\delta_m\right) &> v_s^i\left(\overline{x}^i;\delta_m\right) \quad \forall\, i \in Q. \end{aligned} \tag{17}$$

Since $X(Q)$ is a compact set, there is a subsequence of $x_{(m)}$ converging to an element x of $X(Q)$. Premultiplying inequality (17) by $(1-\delta_m)$, replacing sequences by subsequences and taking the limit yields

$$\vartheta^i(x^i) \geq \vartheta^i(\overline{x}^i), \quad \forall i \in Q. \tag{18}$$

Let $\overline{p}$ be a vector of prices corresponding to equilibrium allocation $\overline{x}$. Under the assumption of local non-satiation of preferences, inequalities (18) imply that

$$\overline{p}x^i \geq \overline{p}e^i, \quad \forall i \in Q. \tag{19}$$

If there were some strict inequality in (19), then x would be not feasible for coalition Q. Hence,

$$\overline{p}x^i = \overline{p}e^i, \quad \forall i \in Q.$$

The strict concavity assumption implies that there exists a unique maximizer of the utility function in the budget set at prices $\overline{p}$, i.e.

$$x^i = \overline{x}^i, \quad \forall i \in Q.$$

Hence, $(\overline{x}^i)_{i \in Q} \in X(Q)$. This implies that allocation $(\overline{x}^i)_{i \in Q}$ is not only feasible, but also Pareto-efficient for the members of the coalition Q in the economy $\mathcal{E}_\infty$. According to Lemma 4, however, the set of weakly efficient allocations for any coalition in the economy $\mathcal{E}_\infty$ coincides with that in the economy $\mathcal{E}_{s,\delta}$ for all $s \in \mathbf{S}$ and $\delta \in (0,1)$. Therefore, $(\overline{x}^i)_{i \in Q}$ is weakly Pareto-efficient for Q in the economies $\mathcal{E}_{s,\delta}$ as well. This contradicts, however, the original assumption. □

References

Aumann, R.J. (1961), "The Core of a Cooperative Game without Side Payments," *Transactions of the American Mathematical Society, 98,* 539–552.

Filar, J.A., and L.A. Petrosjan (2000): "Dynamic Cooperative Games," *International Game Theory Review, 2*, 47–65.

Gale, D. (1978), "The Core of a Monetary Economy without Trust," *Journal of Economic Theory, 19,* 456–491.

Koutsougeras, L.C. (1998), "A Two-stage Core with Applications to Asset Market and Differential Information Economies," *Economic Theory, 11,* 563–584.

Kranich, L., A. Perea, and H. Peters (2001), "On Cores of Dynamic Cooperative Games," Mimeo, Universiteit Maastricht.

Predtetchinski, A., P.J.J. Herings, and H. Peters (2001), "The Strong Sequential Core for Two-period Economies," METEOR Discussion Paper 01/013, Universiteit Maastricht.

Repullo, R. (1988), "The Core of an Economy with Transaction Costs," *Review of Economic Studies, 55,* 447–458.