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Bankruptcy games with nontransferable utility

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HIGHLIGHTS

- This paper introduces a modified model for bankruptcy games with nontransferable utility.
- This paper shows that bankruptcy games are compromise stable and reasonable stable.
- This paper axiomatically characterizes the class of game theoretic bankruptcy rules by truncation invariance.

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ABSTRACT

This paper analyzes bankruptcy games with nontransferable utility as a generalization of bankruptcy games with monetary payoffs. Following the game theoretic approach to NTU-bankruptcy problems, we study some appropriate properties and the core of NTU-bankruptcy games. Generalizing the core cover and the reasonable set to the class of NTU-games, we show that NTU-bankruptcy games are compromise stable and reasonable stable. Moreover, we derive a necessary and sufficient condition for an NTU-bankruptcy rule to be game theoretic.

1. Introduction

A bankruptcy problem is an elementary allocation problem in which claimants have individual claims on an estate which cannot be satisfied together. Bankruptcy theory studies allocations of the estate among the claimants, taking into account the corresponding claims. In a bankruptcy problem with transferable utility (cf. O'Neill, 1982), the estate and claims are of a monetary nature. These problems are well-studied, both from an axiomatic perspective and a game theoretic perspective. We refer to Thomson (2003) for an extensive survey, to Thomson (2013) for recent advances, and to Thomson (2015) for an update.

Carpente et al. (2013) extended TU-bankruptcy problems by explicitly including individual but comparable utility functions on the domain of feasible monetary payoffs. Dietzenbacher et al. (2016) generalized monetary bankruptcy problems to bankruptcy problems with nontransferable utility in which individual utility is represented in incompatible measures. The estate can take a more general shape and corresponds to a set of feasible utility allocations. Dietzenbacher et al. (2016) analyzed these NTU-bankruptcy problems from an axiomatic perspective by formulating appropriate properties for bankruptcy rules and studying their implications. In particular, they focused on proportionality, equality, and duality in bankruptcy problems with nontransferable utility, which resulted in axiomatic characterizations of the proportional rule and the constrained relative equal awards rule. Dietzenbacher et al. (2017a) continued on this axiomatic approach by studying several consistency notions and formulating the relative adjustment principle.

Orshan et al. (2003) analyzed NTU-bankruptcy problems from a game theoretic perspective by introducing an associated NTU-bankruptcy game. Estévez-Fernández et al. (2014) pointed out that coalitions can attain payoff allocations outside the estate in this game, which contradicts the original idea of O'Neill (1982). They redefined NTU-bankruptcy games to stay in line with this original idea about TU-bankruptcy games, while focusing on convexity and compromise stability. However, it turns out that their NTU-bankruptcy game does not straightforwardly generalize the original TU-bankruptcy game, since the attainable payoff allocations of subcoalitions are explicitly bounded by individual claims.

This paper studies a slightly modified version of the model of Orshan et al. (2003) for NTU-bankruptcy games which both generalizes the model for TU-bankruptcy games and stays in line with the idea of O'Neill (1982). Focusing on the structure of the core, we analyze NTU-bankruptcy games along the lines of Curiel et al. (1987). They showed that TU-bankruptcy games are convex, i.e. the core equals the Weber set, and compromise stable, i.e. the core equals the core cover. We introduce the notion of reasonable stability to describe games for which the core equals the reasonable set. We show that reasonable stability is equivalent to the combination of convexity and compromise stability on the class of TU-games, which means that TU-bankruptcy games are reasonable

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stable. Generalizing the core, the core cover, and the reasonable set to the class of NTU-games, we show that NTU-bankruptcy games are compromise stable and reasonable stable as well.

Curiel et al. (1987) also showed that a TU-bankruptcy rule is game theoretic if and only if it satisfies truncation invariance. This means that there exists a solution for TU-games which coincides on the class of bankruptcy games with a certain bankruptcy rule if and only if this bankruptcy rule satisfies truncation invariance. We generalize this characterization to rules for bankruptcy problems with nontransferable utility.

This paper is organized in the following way. Section 2 provides a formal overview of notions for transferable utility games and bankruptcy problems. Section 3 generalizes some notions for transferable utility games to the class of nonnegative games with nontransferable utility. Section 4 introduces and analyzes a modified model for bankruptcy games with nontransferable utility.

2. Preliminaries

2.1. Transferable utility games

Let N be a nonempty and finite set of *players*. An *order* of N is a bijection $\sigma : \{1, \dots, |N|\} \to N$. The set of all orders of N is denoted by $\Pi(N)$ and the set of all *coalitions* is denoted by $2^N = \{S \mid S \subseteq N\}$. A *transferable utility game* is a pair (N, v) in which $v : 2^N \to \mathbb{R}$ assigns to each coalition $S \in 2^N$ its *worth* $v(S) \in \mathbb{R}$ such that $v(\emptyset) = 0$. Let TU^N denote the class of all transferable utility games with player set N. For convenience, a TU-game is denoted by $v \in \mathrm{TU}^N$.

Let $v \in \mathrm{TU}^N$. The *marginal vector* $M^\sigma(v) \in \mathbb{R}^N$ corresponding to $\sigma \in \Pi(N)$ is for all $k \in \{1, \dots, |N|\}$ given by

$$M^\sigma_{\sigma(k)}(v) = v(\{\sigma(1), \dots, \sigma(k)\}) - v(\{\sigma(1), \dots, \sigma(k-1)\}).$$

The vector $K(v) \in \mathbb{R}^N$ is for all $i \in N$ given by

$$K_i(v) = v(N) - v(N \setminus \{i\}),$$

and the vector $k(v) \in \mathbb{R}^N$ is for all $i \in N$ given by

$$k_i(v) = \max_{S \in 2^N : i \in S} \left\{ v(S) - \sum_{j \in S \setminus \{i\}} K_j(v) \right\}.$$

Let $v \in \mathrm{TU}^N$. The *core* is given by

$$\mathcal{C}(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), \forall S \in 2^N : \sum_{i \in S} x_i \geq v(S) \right\},$$

the *Weber set* (cf. Weber, 1988) is given by

$$\mathcal{W}(v) = \mathrm{Conv} \left\{ M^\sigma(v) \;\middle|\; \sigma \in \Pi(N) \right\},$$

the *core cover* (cf. Tijs and Lipperts, 1982) is given by

$$\mathcal{CC}(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), k(v) \leq x \leq K(v) \right\},$$

and the *reasonable set* (cf. Gerard-Varet and Zamir, 1987) is given by

$$\mathcal{R}(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), \forall i \in N : \min_{\sigma \in \Pi(N)} M^\sigma_i(v) \leq x_i \leq \max_{\sigma \in \Pi(N)} M^\sigma_i(v) \right\}.$$

We have $\mathcal{C}(v) \subseteq \mathcal{W}(v) \subseteq \mathcal{R}(v)$ and $\mathcal{C}(v) \subseteq \mathcal{CC}(v) \subseteq \mathcal{R}(v)$. A TU-game $v \in \mathrm{TU}^N$ is called *convex* (cf. Shapley, 1971 and Ichiishi, 1981) if $\mathcal{C}(v) = \mathcal{W}(v)$, and *compromise stable* (cf. Quant et al., 2005) if $\mathcal{C}(v) = \mathcal{CC}(v)$ and $\mathcal{CC}(v) \neq \emptyset$.

We introduce the notion of reasonable stability to describe games for which the core and the reasonable set coincide. Moreover, we show that reasonable stability is equivalent to the combination of convexity and compromise stability.

Definition 2.1 (*Reasonable Stability*). A transferable utility game $v \in \mathrm{TU}^N$ is called *reasonable stable* if $\mathcal{C}(v) = \mathcal{R}(v)$.

Theorem 2.1. *A transferable utility game is reasonable stable if and only if it is convex and compromise stable.*

Proof. Assume that $v \in \mathrm{TU}^N$ is reasonable stable. Then we have $\mathcal{C}(v) = \mathcal{R}(v)$. Since $\mathcal{C}(v) \subseteq \mathcal{W}(v) \subseteq \mathcal{R}(v)$ and $\mathcal{C}(v) \subseteq \mathcal{CC}(v) \subseteq \mathcal{R}(v)$, this means that $\mathcal{C}(v) = \mathcal{W}(v)$ and $\mathcal{C}(v) = \mathcal{CC}(v)$. Hence, $v \in \mathrm{TU}^N$ is convex and compromise stable.

Assume that $v \in \mathrm{TU}^N$ is convex and compromise stable. Since $v \in \mathrm{TU}^N$ is convex, we have $\min_{\sigma \in \Pi(N)} M^\sigma_i(v) = v(\{i\})$ and $\max_{\sigma \in \Pi(N)} M^\sigma_i(v) = v(N) - v(N \setminus \{i\})$ for all $i \in N$. Moreover, we have $k_i(v) = v(\{i\})$ for all $i \in N$. This means that $\min_{\sigma \in \Pi(N)} M^\sigma_i(v) = k_i(v)$ and $\max_{\sigma \in \Pi(N)} M^\sigma_i(v) = K_i(v)$ for all $i \in N$, so $\mathcal{CC}(v) = \mathcal{R}(v)$. Since $v \in \mathrm{TU}^N$ is compromise stable, this implies that $\mathcal{C}(v) = \mathcal{CC}(v) = \mathcal{R}(v)$. Hence, $v \in \mathrm{TU}^N$ is reasonable stable. □

2.2. Bankruptcy problems

Let N be a nonempty and finite set of *claimants*. A *bankruptcy problem with transferable utility* (cf. O'Neill, 1982) is a triple (N, M, c) in which $M \in \mathbb{R}_+$ is an *estate* and $c \in \mathbb{R}^N_+$ is a vector of *claims* of N on M for which $\sum_{i \in N} c_i \geq M$. Let TUBR^N denote the class of all bankruptcy problems with transferable utility with claimant set N. For convenience, a TU-bankruptcy problem is denoted by $(M, c) \in \mathrm{TUBR}^N$.

For any set of payoff allocations $E \subseteq \mathbb{R}^N_+$,

- the *nonnegative comprehensive hull* is given by $\mathrm{comp}(E) = \{x \in \mathbb{R}^N_+ \mid \exists y \in E : y \geq x\}$;
- the *strong Pareto set* is given by $\mathrm{SP}(E) = \{x \in E \mid \neg\exists y \in E, y \neq x : y \geq x\}$;
- the *strong upper contour set* is given by $\mathrm{SUC}(E) = \{x \in \mathbb{R}^N_+ \mid \neg\exists y \in E, y \neq x : y \geq x\}$;
- the *weak upper contour set* is given by $\mathrm{WUC}(E) = \{x \in E \mid \neg\exists y \in E : y > x\}$.

Note that $\mathrm{SP}(E) \subseteq \mathrm{SUC}(E) \subseteq \mathrm{WUC}(E)$. A set of payoff allocations $E \subseteq \mathbb{R}^N_+$ is called *(nonnegative) comprehensive* if $E = \mathrm{comp}(E)$, and called *nonleveled* if $\mathrm{SUC}(E) = \mathrm{WUC}(E)$.

A *bankruptcy problem with nontransferable utility* (cf. Dietzenbacher et al., 2016) is a triple (N, E, c) in which $E \subseteq \mathbb{R}^N_+$ is a nonempty, closed, bounded, comprehensive and nonleveled *estate* and $c \in \mathrm{SUC}(E)$ is a vector of *claims* of N on E. Let BR^N denote the class of all bankruptcy problems with nontransferable utility with claimant set N. For convenience, an NTU-bankruptcy problem is denoted by $(E, c) \in \mathrm{BR}^N$. Note that any TU-bankruptcy problem $(M, c) \in \mathrm{TUBR}^N$ gives rise to the NTU-bankruptcy problem $(E, c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i \leq M\}$.

Let $(E, c) \in \mathrm{BR}^N$. The vector of *utopia values* $u^E \in \mathbb{R}^N_+$ is for all $i \in N$ given by

$$u^E_i = \max\{x_i \mid x \in E\}.$$

The vector of *truncated claims* $\hat{c}^E \in \mathbb{R}^N_+$ is for all $i \in N$ given by

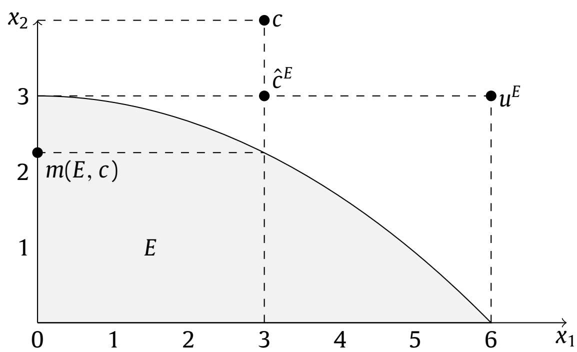
$$\hat{c}^E_i = \min\{c_i, u^E_i\}.$$

The vector of *minimal rights* $m(E, c) \in \mathbb{R}^N_+$ is for all $i \in N$ given by

$$m_i(E, c) = \begin{cases} \max\{x \mid (x, c_{N \setminus \{i\}}) \in E\} & \text{if } (0, c_{N \setminus \{i\}}) \in E; \\ 0 & \text{if } (0, c_{N \setminus \{i\}}) \notin E. \end{cases}$$

Note that $m(E, c) \in E$, $\hat{c}^E \in \mathrm{SUC}(E)$, and $m(E, c) \leq \hat{c}^E$.

Example 1. Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \text{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$. We have $\text{SP}(E) = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 = 36\}$ and $\text{SUC}(E) = \text{WUC}(E) = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \geq 36\}$. Moreover, we have $u^E = (6, 3)$, $\hat{c}^E = (3, 3)$, and $m(E, c) = (0, 2\frac{1}{4})$.



△

3. Nonnegative games with nontransferable utility

This section generalizes some notions for transferable utility games to the class of nonnegative games with nontransferable utility. Many classes of TU-games are nonnegative, e.g. cost savings games such as sequencing games and other operations research games, airport games, simple games, glove games, and bankruptcy games. This lower bound for the worth of coalitions arises naturally from the assumption that allocating nothing to a player corresponds to a payoff of zero utility, which implies for some allocation problems that negative utility payoffs do not have any interpretation. Following the same lines of reasoning, this lower bound can also be applied in the context of NTU-games.

Definition 3.1 (*Nonnegative Game with Nontransferable Utility*). A *nonnegative game with nontransferable utility* is a pair (N, V) in which N is a nonempty and finite set of *players* and V assigns to each nonempty coalition $S \in 2^N \setminus \{\emptyset\}$ a nonempty, closed, bounded and comprehensive set of payoff allocations $V(S) \subseteq \mathbb{R}^S_+$.

A nonnegative NTU-game (N, V) is called *monotonic* if $V(S) \subseteq \{x_S \mid x \in V(T)\}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$. Let NTU^N_+ denote the class of all monotonic nonnegative NTU-games with player set N. For convenience, such an NTU-game is denoted by $V \in \text{NTU}^N_+$. Note that any monotonic nonnegative TU-game $v \in \text{TU}^N$ gives rise to the NTU-game $V \in \text{NTU}^N_+$ in which $V(S) = \{x \in \mathbb{R}^S_+ \mid \sum_{i \in S} x_i \leq v(S)\}$ for all $S \in 2^N \setminus \{\emptyset\}$.

Let $V \in \text{NTU}^N_+$. Similar to Otten et al. (1998), we define the marginal vector $M^\sigma(V) \in \mathbb{R}^N$ corresponding to $\sigma \in \Pi(N)$ for all $k \in \{1, \ldots, |N|\}$ by

$$M^\sigma_{\sigma(k)}(V) = \max \left\{x \in \mathbb{R}_+ \mid (M^\sigma_{\sigma(1)}(V), \ldots, M^\sigma_{\sigma(k-1)}(V), x) \in V(\{\sigma(1), \ldots, \sigma(k)\})\right\}.$$

Note that the conditions on the game imply that this maximum exists. As in the context of TU-games, the marginal contribution of a player in a certain order can be interpreted as its maximal payoff when joining its predecessors, which have already been allocated their marginal contributions. Inspired by Borm et al. (1992),[1] we define $K(V) \in \mathbb{R}^N$ for all $i \in N$ by

$$K_i(V) = \max \left\{x_i \mid x \in V(N), x_{N \setminus \{i\}} \in \text{SUC}(V(N \setminus \{i\}))\right\},$$

and $k(V) \in \mathbb{R}^N$ for all $i \in N$ by

$$k_i(V) = \max_{S \in 2^N : i \in S} \sup \left\{x \in \mathbb{R}_+ \mid (x, K_{S \setminus \{i\}}(V)) \in V(S)\right\}.$$

Note that the conditions on the game imply that these maxima exist. As in the context of TU-games, $K_i(V)$ can be interpreted as the maximal payoff of player $i \in N$ within an allocation of $V(N)$ which is stable against a coalitional deviation of the other players together. Moreover, $k_i(V)$ can be interpreted as the minimal right of player $i \in N$, the maximal payoff which can be obtained within some coalition $S \in 2^N$, with $i \in S$, when each other member $j \in S \setminus \{i\}$ is allocated $K_j(V)$.

Using these notions, we can generalize the core, the core cover, and the reasonable set to the context of NTU-games. Let $V \in \text{NTU}^N_+$. The *(strong) core* is defined by

$$\mathcal{C}(V) = \left\{x \in V(N) \mid \forall S \in 2^N \setminus \{\emptyset\} : x_S \in \text{SUC}(V(S))\right\},$$

the *(strong) core cover* is defined by

$$\mathcal{CC}(V) = \{x \in \text{SP}(V(N)) \mid k(V) \leq x \leq K(V)\},$$

and the *reasonable set* is defined by

$$\mathcal{R}(V) = \left\{x \in \text{SP}(V(N)) \mid \forall i \in N : \min_{\sigma \in \Pi(N)} M^\sigma_i(V) \leq x_i \leq \max_{\sigma \in \Pi(N)} M^\sigma_i(V)\right\}.$$

Lemma 3.1. *Let $V \in \text{NTU}^N_+$. Then $\mathcal{C}(V) \subseteq \mathcal{CC}(V)$.*

Proof. Let $x \in \mathcal{C}(V)$. For all $i \in N$, we can write

$$\begin{aligned} x_i &\leq \max\{x_i \mid x \in \mathcal{C}(V)\} = \max\{x_i \mid x \in V(N), \\ &\quad \forall S \in 2^N \setminus \{\emptyset\} : x_S \in \text{SUC}(V(S))\} \\ &\leq \max\{x_i \mid x \in V(N), x_{N \setminus \{i\}} \in \text{SUC}(V(N \setminus \{i\}))\} = K_i(V). \end{aligned}$$

Suppose that there exists an $i \in N$ for which $x_i < k_i(V)$. Let $S \in 2^N$ with $i \in S$ be such that $(k_i(V), K_{S \setminus \{i\}}(V)) \in V(S)$. Then $x_S \leq (k_i(V), K_{S \setminus \{i\}}(V))$ and $x_S \neq (k_i(V), K_{S \setminus \{i\}}(V))$. Since $V(S)$ is comprehensive, this means that $x_S \notin \text{SUC}(V(S))$. This contradicts that $x \in \mathcal{C}(V)$, so $k(V) \leq x \leq K(V)$. Hence, $x \in \mathcal{CC}(V)$. □

Lemma 3.1 shows that the (strong) core cover indeed contains the (strong) core of an NTU-game. An NTU-game for which the core coincides with the core cover is called compromise stable.

Definition 3.2 (*Compromise Stability*). An NTU-game $V \in \text{NTU}^N_+$ is called *compromise stable* if $\mathcal{C}(V) = \mathcal{CC}(V)$ and $\mathcal{CC}(V) \neq \emptyset$.

Contrary to TU-games, the following example shows that the reasonable set does not necessarily contain the (strong) core of an NTU-game.

Example 2. Let $N = \{1, 2, 3\}$ and consider the game $V \in \text{NTU}^N_+$ which is for all $S \in 2^N \setminus \{\emptyset\}$ given by

$$V(S) = \begin{cases} \{x \in \mathbb{R}^S_+ \mid x_1^2 + x_2^2 \leq (9 - x_3)^2, x_3 \leq 9\} & \text{if } S = N; \\ \{x \in \mathbb{R}^S_+ \mid x_1 + x_2 \leq 4\} & \text{if } S = \{1, 2\}; \\ 0_S & \text{otherwise.} \end{cases}$$

All marginal vectors are presented below.

σ	$M^\sigma_1(V)$	$M^\sigma_2(V)$	$M^\sigma_3(V)$
(1, 2, 3)	0	4	5
(1, 3, 2)	0	9	0
(2, 1, 3)	4	0	5
(2, 3, 1)	9	0	0
(3, 1, 2)	0	9	0
(3, 2, 1)	9	0	0

[1] Borm et al. (1992) used certain bounds to define the weak core cover which contains the weak core. We use different bounds to define the strong core cover which contains the strong core.

This means that the reasonable set is given by

$$\mathcal{R}(V) = \{x \in SP(V(N)) \mid 0 \le x_1 \le 9, 0 \le x_2 \le 9, 0 \le x_3 \le 5\}.$$

One can verify that $(2, 2, 9 - 2\sqrt{2}) \in \mathcal{C}(V) \setminus \mathcal{R}(V)$. Hence, $\mathcal{C}(V) \not\subseteq \mathcal{R}(V)$. △

Although the reasonable set does not necessarily contain the core of an NTU-game, the minimal and maximal marginal contributions can still be considered as reasonable bounds for payoff allocations. An NTU-game for which the core coincides with the reasonable set is called reasonable stable.

Definition 3.3 (*Reasonable Stability*)**.** An NTU-game $V \in NTU_+^N$ is called *reasonable stable* if $\mathcal{C}(V) = \mathcal{R}(V)$.

Note that reasonable stability is stronger than *marginal convexity* (cf. Hendrickx et al., 2002), which requires that $M^\sigma(V) \in \mathcal{C}(V)$ for all $\sigma \in \Pi(N)$.

4. Bankruptcy games with nontransferable utility

This section introduces and analyzes a modified model for bankruptcy games with nontransferable utility. Since NTU-bankruptcy problems generalize TU-bankruptcy problems, and NTU-games generalize TU-games, NTU-bankruptcy games should generalize TU-bankruptcy games.

Let $(M, c) \in TUBR^N$. The corresponding *bankruptcy game with transferable utility* (cf. O'Neill, 1982) $v^{M,c} \in TU^N$ is given by $v^{M,c}(S) = \max\{M - \sum_{i\in N\setminus S} c_i, 0\}$ for all $S \in 2^N$. Curiel et al. (1987) showed that TU-bankruptcy games are convex and compromise stable. Quant et al. (2005) showed that any convex and compromise stable TU-game is strategically equivalent to a bankruptcy game. Using Theorem 2.1, this means that TU-bankruptcy games are reasonable stable, and any reasonable stable TU-game is strategically equivalent to a bankruptcy game.

Orshan et al. (2003) introduced a first model for NTU-bankruptcy games. For any $(E, c) \in BR^N$, their corresponding bankruptcy game $\dot{V}^{E,c}$ assigns to any $S \in 2^N \setminus \{\emptyset\}$ the set of payoff allocations $\dot{V}^{E,c}(S) = \text{comp}(\{x \in \mathbb{R}_+^N \mid (x_S, c_{N\setminus S}) \in E \text{ or } x_S = 0_S\})$.

Example 3. Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in BR^N$ in which $E = \{x \in \mathbb{R}_+^N \mid x_1^2 + 12x_2 \le 36\}$ and $c = (3, 4)$ as in Example 1. We have $\dot{V}^{E,c}(\{1\}) = \{x \in \mathbb{R}_+^N \mid x_1 = 0\}$, $\dot{V}^{E,c}(\{2\}) = \{x \in \mathbb{R}_+^N \mid x_2 \le 2\frac{1}{4}\}$, and $\dot{V}^{E,c}(N) = E$. △

Estévez-Fernández et al. (2014) pointed out that coalitions can attain payoff allocations outside the estate in this game, e.g. $(6, 2) \in \dot{V}^{E,c}(\{2\}) \setminus E$ in Example 3. This contradicts the original idea of O'Neill (1982).

Estévez-Fernández et al. (2014) redefined NTU-bankruptcy games to stay in line with the idea of O'Neill (1982). For any $(E, c) \in BR^N$, their corresponding bankruptcy game $\ddot{V}^{E,c}$ assigns E to N and assigns to any $S \in 2^N \setminus \{\emptyset, N\}$ the set of payoff allocations

$$\ddot{V}^{E,c}(S) = \begin{cases} \text{comp}_S(\{x \in SP(E) \mid x_S \le c_S, x_{N\setminus S} = c_{N\setminus S}\}) \\ \quad \text{if } (0_S, c_{N\setminus S}) \in E; \\ \text{comp}_S(\{x \in SP(E) \mid x_S = 0_S, x_{N\setminus S} \le c_{N\setminus S}\}) \\ \quad \text{if } (0_S, c_{N\setminus S}) \notin E, \end{cases}$$

where $\text{comp}_S(E) = \{x \in \mathbb{R}_+^N \mid \exists y \in E : y_S \ge x_S, y_{N\setminus S} = x_{N\setminus S}\}$.

Example 4. Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in BR^N$ in which $E = \{x \in \mathbb{R}_+^N \mid x_1^2 + 12x_2 \le 36\}$ and $c = (3, 4)$ as in Examples 1 and 3. We have $\ddot{V}^{E,c}(\{1\}) = \{x \in \mathbb{R}_+^N \mid x_1 = 0, x_2 = 3\}$, $\ddot{V}^{E,c}(\{2\}) = \{x \in \mathbb{R}_+^N \mid x_1 = 3, x_2 \le 2\frac{1}{4}\}$, and $\ddot{V}^{E,c}(N) = E$. △

However, the following example shows that their NTU-bankruptcy game actually does not straightforwardly generalize the original TU-bankruptcy game.

Example 5. Let $N = \{1, 2, 3\}$ and consider the TU-bankruptcy problem $(M, c) \in TUBR^N$ in which $M = 4$ and $c = (1, 2, 3)$. This gives rise to the NTU-bankruptcy problem $(E, c) \in BR^N$ in which $E = \{x \in \mathbb{R}_+^N \mid x_1 + x_2 + x_3 \le 4\}$ and $c = (1, 2, 3)$. The corresponding TU-bankruptcy game $v^{M,c} \in TU^N$ is presented below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v^{M,c}(S)$	0	0	1	1	2	3	4

The worth $v^{M,c}(\{2, 3\}) = 3$ gives rise to the game $V^{E,c} \in NTU_+^N$ in which $V^{E,c}(\{2, 3\}) = \{x \in \mathbb{R}_+^{\{2,3\}} \mid x_2 + x_3 \le 3\}$. However, $\ddot{V}^{E,c}(\{2, 3\}) = \{x \in \mathbb{R}_+^N \mid x_1 = 1, x_2 \le 2, x_3 \le 3, x_2 + x_3 \le 3\}$, which is essentially different due to the upper bound on the payoff of player 2. △

Next, we introduce a slightly modified version of the model of Orshan et al. (2003) for NTU-bankruptcy games, which generalizes TU-bankruptcy games and simultaneously stays in line with the original idea of O'Neill (1982).

Definition 4.1 (*Bankruptcy Game with Nontransferable Utility*)**.** Let $(E, c) \in BR^N$ be a bankruptcy problem with nontransferable utility. The corresponding *bankruptcy game with nontransferable utility* $V^{E,c} \in NTU_+^N$ is for all $S \in 2^N \setminus \{\emptyset\}$ defined by

$$V^{E,c}(S) = \begin{cases} \{x \in \mathbb{R}_+^S \mid (x, c_{N\setminus S}) \in E\} & \text{if } (0_S, c_{N\setminus S}) \in E; \\ 0_S & \text{if } (0_S, c_{N\setminus S}) \notin E. \end{cases}$$

Note that $V^{E,c}(N) = E$ and $V^{E,c}(\{i\}) = [0, m_i(E, c)]$.

Example 6. Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in BR^N$ in which $E = \{x \in \mathbb{R}_+^N \mid x_1^2 + 12x_2 \le 36\}$ and $c = (3, 4)$ as in Examples 1, 3 and 4. We have $V^{E,c}(\{1\}) = \{0\}$, $V^{E,c}(\{2\}) = [0, 2\frac{1}{4}]$, and $V^{E,c}(N) = E$. △

Contrary to the models of Orshan et al. (2003) and Estévez-Fernández et al. (2014), every subgame of this new NTU-bankruptcy game is a bankruptcy game too, as is the case for TU-bankruptcy games. For any NTU-game $V \in NTU_+^N$, the *subgame* $V_S \in NTU_+^S$ on coalition $S \in 2^N \setminus \{\emptyset\}$ is defined by $V_S(R) = V(R)$ for all $R \in 2^S \setminus \{\emptyset\}$.

Proposition 4.1. *Each subgame of a bankruptcy game is a bankruptcy game as well.*

Proof. Let $(E, c) \in BR^N$ and let $S \in 2^N \setminus \{\emptyset\}$. Then $V^{E,c}(S)$ is nonempty, closed, bounded, comprehensive and nonleveled, and $c_S \in SUC(V^{E,c}(S))$. This means that $(V^{E,c}(S), c_S) \in BR^S$. For all $R \in 2^S \setminus \{\emptyset\}$, we can write

$$\begin{aligned} V^{V^{E,c}(S),c_S}(R) &= \begin{cases} \{x \in \mathbb{R}_+^R \mid (x, c_{S\setminus R}) \in V^{E,c}(S)\} \\ \quad \text{if } (0_R, c_{S\setminus R}) \in V^{E,c}(S); \\ 0_R \quad \text{if } (0_R, c_{S\setminus R}) \notin V^{E,c}(S) \end{cases} \\ &= \begin{cases} \{x \in \mathbb{R}_+^R \mid (x, c_{S\setminus R}, c_{N\setminus S}) \in E\} \\ \quad \text{if } (0_R, c_{S\setminus R}, c_{N\setminus S}) \in E; \\ 0_R \quad \text{if } (0_R, c_{S\setminus R}, c_{N\setminus S}) \notin E \end{cases} \\ &= \begin{cases} \{x \in \mathbb{R}_+^R \mid (x, c_{N\setminus R}) \in E\} & \text{if } (0_R, c_{N\setminus R}) \in E; \\ 0_R & \text{if } (0_R, c_{N\setminus R}) \notin E \end{cases} \\ &= V^{E,c}(R) \\ &= V_S^{E,c}(R). \end{aligned}$$

Hence, $V_S^{E,c} \in NTU_+^S$ is a bankruptcy game. □

The remainder of this section studies the relationship between the core, the core cover, and the reasonable set of bankruptcy games. A useful observation for this analysis is that bankruptcy games are invariant under claim truncation.

Lemma 4.2. *Let $(E, c) \in \mathrm{BR}^N$. Then $V^{E,c} = V^{E,\hat{c}^E}$.*

Proof. Let $S \in 2^N \setminus \{\emptyset\}$. If $\hat{c}^E_{N\setminus S} = c_{N\setminus S}$, then clearly $V^{E,c}(S) = V^{E,\hat{c}^E}(S)$. Suppose that $\hat{c}^E_{N\setminus S} \neq c_{N\setminus S}$. Then there exists an $i \in N\setminus S$ for which $\hat{c}^E_i = u^E_i < c_i$. This means that $(0_S, c_{N\setminus S}) \notin E$, so $V^{E,c}(S) = 0_S$. Since E is nonleveled, we have $V^{E,\hat{c}^E}(S) = \{x \in \mathbb{R}^S_+ \mid (x, \hat{c}^E_{N\setminus S}) \in E\} = 0_S$ if $(0_S, \hat{c}^E_{N\setminus S}) \in E$. Hence, $V^{E,c}(S) = V^{E,\hat{c}^E}(S)$. □

The vector of truncated claims and the vector of minimal rights determine the upper and lower bound for the core cover of bankruptcy games, respectively.

Lemma 4.3. *Let $(E, c) \in \mathrm{BR}^N$. Then*

(i) $K(V^{E,c}) = \hat{c}^E$;
(ii) $k(V^{E,c}) = m(E, c)$.

Proof. (i) From Lemma 4.2 we know that $V^{E,c} = V^{E,\hat{c}^E}$, so $K(V^{E,c}) = K(V^{E,\hat{c}^E})$. Let $i \in N$. We have $(\hat{c}^E_i, x) \in E$ for all $x \in V^{E,\hat{c}^E}(N \setminus \{i\})$, so $(\hat{c}^E_i, x) \in V^{E,\hat{c}^E}(N)$ for all $x \in \mathrm{SP}(V^{E,\hat{c}^E}(N \setminus \{i\}))$. This implies that $K_i(V^{E,\hat{c}^E}) \geq \hat{c}^E_i$. Suppose that $K_i(V^{E,\hat{c}^E}) > \hat{c}^E_i$. Let $x \in \mathrm{SUC}(V^{E,\hat{c}^E}(N \setminus \{i\}))$ be such that $(K_i(V^{E,\hat{c}^E}), x) \in V^{E,\hat{c}^E}(N)$. Since $V^{E,\hat{c}^E}(N)$ is comprehensive, we have $(\hat{c}^E_i, x) \in V^{E,\hat{c}^E}(N)$, so $x \in V^{E,\hat{c}^E}(N \setminus \{i\})$. This means that $(\hat{c}^E_i, x) \notin \mathrm{SP}(V^{E,\hat{c}^E}(N))$ and $x \in \mathrm{SP}(V^{E,\hat{c}^E}(N \setminus \{i\}))$. Since $V^{E,\hat{c}^E}(N)$ is nonleveled, then there exists a $y \in V^{E,\hat{c}^E}(N)$ for which $y > (\hat{c}^E_i, x)$. Since $V^{E,\hat{c}^E}(N)$ is comprehensive, we have $(\hat{c}^E_i, y_{N\setminus\{i\}}) \in V^{E,\hat{c}^E}(N)$. This means that $y_{N\setminus\{i\}} \in V^{E,\hat{c}^E}(N \setminus \{i\})$, which contradicts that $x \in \mathrm{SP}(V^{E,\hat{c}^E}(N \setminus \{i\}))$. Hence, $K_i(V^{E,c}) = K_i(V^{E,\hat{c}^E}) = \hat{c}^E_i$.

(ii) Let $i \in N$. We can write

$$k_i(V^{E,c}) \geq \sup \left\{x \in \mathbb{R}_+ \mid x \in V^{E,c}(\{i\})\right\} = \max \left\{x \in V^{E,c}(\{i\})\right\} = m_i(E, c).$$

Suppose that we have $k_i(V^{E,c}) > m_i(E, c) = \max\{x \in V^{E,c}(\{i\})\}$. Let $S \in 2^N$ with $i \in S$ be such that $(k_i(V^{E,c}), K_{S\setminus\{i\}}(V^{E,c})) \in V^{E,c}(S)$. Then we know from Lemma 4.2 and (i) that $(k_i(V^{E,\hat{c}^E}), \hat{c}^E_{S\setminus\{i\}}) \in V^{E,\hat{c}^E}(S)$. This means that $(k_i(V^{E,\hat{c}^E}), \hat{c}^E_{S\setminus\{i\}}, \hat{c}^E_{N\setminus S}) \in E$, which implies that $k_i(V^{E,\hat{c}^E}) \in V^{E,\hat{c}^E}(\{i\})$. This contradicts that $k_i(V^{E,\hat{c}^E}) > m_i(E, c)$. Hence, $k_i(V^{E,c}) = k_i(V^{E,\hat{c}^E}) = m_i(E, c)$. □

From Lemma 3.1 and Example 2 we know that, contrary to TU-games, the core cover is not necessarily contained in the reasonable set of an NTU-game. Surprisingly, for the reasonable set of an NTU-bankruptcy game we find the same upper bound and lower bound as for the core cover, which means that the core cover and the reasonable set of an NTU-bankruptcy game coincide.

Lemma 4.4. *Let $(E, c) \in \mathrm{BR}^N$ and let $i \in N$. Then*

(i) $\max_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,c}) = \hat{c}^E_i$;
(ii) $\min_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,c}) = m_i(E, c)$.

Proof. (i) From Lemma 4.2 we know that $V^{E,c} = V^{E,\hat{c}^E}$, so $\max_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,c}) = \max_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,\hat{c}^E})$. Let $\hat{\sigma} \in \Pi(N)$ be such that $\hat{\sigma}(|N|) = i$. We have $(x, \hat{c}^E_i) \in E$ for all $x \in V^{E,\hat{c}^E}(N \setminus \{i\})$, so $(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(|N|-1)}(V^{E,\hat{c}^E}), \hat{c}^E_i) \in V^{E,\hat{c}^E}(N)$. This implies that $\max_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,\hat{c}^E}) \geq \hat{c}^E_i$. Suppose that $\max_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,\hat{c}^E}) > \hat{c}^E_i$. Let $\hat{\sigma} \in \Pi(N)$ be such that $M^{\hat{\sigma}}_i(V^{E,\hat{c}^E}) = \max_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,\hat{c}^E})$. Let $k \in \{2, \ldots, |N|\}$ be such that $\hat{\sigma}(k) = i$. Then we have $(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k)}(V^{E,\hat{c}^E})) \in V^{E,\hat{c}^E}(\{\hat{\sigma}(1), \ldots, \hat{\sigma}(k)\})$, which means that

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k)}(V^{E,\hat{c}^E}), \hat{c}^E_{\hat{\sigma}(k+1)}, \ldots, \hat{c}^E_{\hat{\sigma}(|N|)}\right) \in E.$$

Since E is comprehensive, we have

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k-1)}(V^{E,\hat{c}^E}), \hat{c}^E_{\hat{\sigma}(k)}, \ldots, \hat{c}^E_{\hat{\sigma}(|N|)}\right) \in E \setminus \mathrm{SP}(E).$$

Since E is nonleveled, then there exists a $y \in E$ for which

$$y > \left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k-1)}(V^{E,\hat{c}^E}), \hat{c}^E_{\hat{\sigma}(k)}, \ldots, \hat{c}^E_{\hat{\sigma}(|N|)}\right).$$

Since E is comprehensive, we have

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k-2)}(V^{E,\hat{c}^E}), y_{\hat{\sigma}(k-1)}, \hat{c}^E_{\hat{\sigma}(k)}, \ldots, \hat{c}^E_{\hat{\sigma}(|N|)}\right) \in E.$$

This means that

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k-2)}(V^{E,\hat{c}^E}), y_{\hat{\sigma}(k-1)}\right) \in V^{E,\hat{c}^E}(\{\hat{\sigma}(1), \ldots, \hat{\sigma}(k-1)\}),$$

which contradicts that $M^{\hat{\sigma}}_{\hat{\sigma}(k-1)}(V^{E,\hat{c}^E})$ equals

$$\max\left\{x \in \mathbb{R}_+ \mid (M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k-2)}(V^{E,\hat{c}^E}), x) \in V^{E,\hat{c}^E}(\{\hat{\sigma}(1), \ldots, \hat{\sigma}(k-1)\})\right\}.$$

Hence, $\max_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,c}) = \max_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,\hat{c}^E}) = \hat{c}^E_i$.

(ii) Let $\hat{\sigma} \in \Pi(N)$ be such that $\hat{\sigma}(1) = i$. We can write

$$M^{\hat{\sigma}}_i(V^{E,c}) = \max\left\{x \in \mathbb{R}_+ \mid x \in V^{E,c}(\{i\})\right\} = \max\left\{x \in V^{E,c}(\{i\})\right\} = m_i(E, c).$$

This implies that $\min_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,c}) \leq m_i(E, c)$. Suppose that $\min_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,c}) < m_i(E, c)$. From Lemma 4.2 we know that $V^{E,c} = V^{E,\hat{c}^E}$, so $\min_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,c}) = \min_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,\hat{c}^E})$. Let $\hat{\sigma} \in \Pi(N)$ be such that $M^{\hat{\sigma}}_i(V^{E,\hat{c}^E}) = \min_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,\hat{c}^E})$. Let $k \in \{2, \ldots, |N|\}$ be such that $\hat{\sigma}(k) = i$. Then we have

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k-1)}(V^{E,\hat{c}^E}), m_i(E, c)\right) \notin V^{E,\hat{c}^E}(\{\hat{\sigma}(1), \ldots, \hat{\sigma}(k)\}),$$

which means that

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k-1)}(V^{E,\hat{c}^E}), m_i(E, c), \hat{c}^E_{\hat{\sigma}(k+1)}, \ldots, \hat{c}^E_{\hat{\sigma}(|N|)}\right) \notin E.$$

Since E is comprehensive, we know from (i) that $(m_i(E, c), \hat{c}^E_{N\setminus\{i\}}) \notin E$, which contradicts that $m_i(E, c) \in V^{E,\hat{c}^E}(\{i\})$. Hence, $\min_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,c}) = \min_{\sigma\in\Pi(N)} M^\sigma_i(V^{E,\hat{c}^E}) = m_i(E, c)$. □

From Lemmas 4.3 and 4.4 we know that the core cover and the reasonable set of a bankruptcy game coincide. From Lemma 3.1 we know that the core is contained in the core cover, which means that the core of a bankruptcy game is also contained in its reasonable set. Next, we generalize the result for TU-bankruptcy games which states that the core of a bankruptcy game coincides with the core cover and the reasonable set.

Theorem 4.5. *Every bankruptcy game is compromise stable and reasonable stable.*

Proof. Let $(E, c) \in \mathrm{BR}^N$. From Lemmas 4.3 and 4.4 we know that $\mathcal{CC}(V^{E,c}) = \mathcal{R}(V^{E,c})$, so it suffices to show that $\mathcal{CC}(V^{E,c}) \neq \emptyset$ and

$\mathcal{C}(V^{E,c}) = \mathcal{CC}(V^{E,c})$. From Lemma 4.3 we know that $\mathcal{CC}(V^{E,c}) = \{x \in \text{SP}(E) \mid m(E,c) \leq x \leq \hat{c}^E\}$. Since there exists an $x \in \text{SP}(E)$ for which $m(E,c) \leq x \leq \hat{c}^E$, this means that $\mathcal{CC}(V^{E,c}) \neq \emptyset$.

Let $x \in \mathcal{CC}(V^{E,c})$. Then we have $x \leq \hat{c}^E \leq c$. Suppose that $x \notin \mathcal{C}(V^{E,c})$. Then there exists an $S \in 2^N \setminus \{\emptyset\}$ for which $x_S \notin \text{SUC}(V^{E,c}(S))$. This means that there exists a $y \in V^{E,c}(S)$ for which $y \geq x_S$ and $y \neq x_S$. Then we have $(y, c_{N\setminus S}) \in E$. Since $x \leq (y, c_{N\setminus S})$ and $x \neq (y, c_{N\setminus S})$, this means that $x \notin \text{SP}(E)$, which contradicts that $x \in \mathcal{CC}(V^{E,c})$. Hence, $x \in \mathcal{C}(V^{E,c})$ and we have $\mathcal{CC}(V^{E,c}) \subseteq \mathcal{C}(V^{E,c})$. From Lemma 3.1 we know that $\mathcal{C}(V^{E,c}) \subseteq \mathcal{CC}(V^{E,c})$, so $\mathcal{C}(V^{E,c}) = \mathcal{CC}(V^{E,c})$. □

Using Lemmas 4.3, 4.4, and Theorem 4.5, we derive a compact expression for the core of a bankruptcy game.

Corollary 4.6. *Let* $(E,c) \in \text{BR}^N$. *Then* $\mathcal{C}(V^{E,c}) = \{x \in \text{SP}(E) \mid x \leq c\}$.

A *bankruptcy rule* $f : \text{BR}^N \to \mathbb{R}^N_+$ assigns to any $(E,c) \in \text{BR}^N$ a payoff allocation $f(E,c) \in \text{SP}(E)$ for which $f(E,c) \leq c$. A bankruptcy rule $f : \text{BR}^N \to \mathbb{R}^N_+$ satisfies *truncation invariance* if $f(E,c) = f(E,\hat{c}^E)$ for all $(E,c) \in \text{BR}^N$. A *solution* for NTU-games $F : \text{NTU}^N_+ \to \mathbb{R}^N_+$ assigns to any $V \in \text{NTU}^N_+$ a payoff allocation $F(V) \in \text{SP}(V(N))$.

In other words, all bankruptcy rules assign to each bankruptcy problem a core element of the corresponding bankruptcy game. This means that a solution for NTU-games corresponds on the class of bankruptcy games to a bankruptcy rule if and only if it assigns to any bankruptcy game a core element. The other way around, the question arises under which conditions a bankruptcy rule corresponds to a solution for NTU-games on the class of bankruptcy games. Such a bankruptcy rule is called game theoretic.

Definition 4.2 (*Game Theoretic Bankruptcy Rule*)**.** A bankruptcy rule $f : \text{BR}^N \to \mathbb{R}^N_+$ is called *game theoretic* if there exists a solution $F : \text{NTU}^N_+ \to \mathbb{R}^N_+$ for which $f(E,c) = F(V^{E,c})$ for all $(E,c) \in \text{BR}^N$.

Similar to bankruptcy rules for TU-bankruptcy problems, a necessary and sufficient condition for an NTU-bankruptcy rule to be game theoretic is to satisfy truncation invariance.

Theorem 4.7. *A bankruptcy rule is game theoretic if and only if it satisfies truncation invariance.*

Proof. Let $f : \text{BR}^N \to \mathbb{R}^N_+$ be a game theoretic bankruptcy rule. Let $(E,c) \in \text{BR}^N$. From Lemma 4.2 we know that $V^{E,c} = V^{E,\hat{c}^E}$. We can write

$$f(E,c) = F(V^{E,c}) = F(V^{E,\hat{c}^E}) = f(E,\hat{c}^E).$$

Hence, f satisfies truncation invariance.

Let $f : \text{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying truncation invariance. Let $F : \text{NTU}^N_+ \to \mathbb{R}^N_+$ be a solution such that $F(V^{E,c}) = f(V^{E,c}(N), K(V^{E,c}))$ for any bankruptcy game $V^{E,c} \in \text{NTU}^N_+$. Let $(E,c) \in \text{BR}^N$. From Lemma 4.3 we know that $K(V^{E,c}) = \hat{c}^E$. We can write

$$f(E,c) = f(E,\hat{c}^E) = f(V^{E,c}(N), K(V^{E,c})) = F(V^{E,c}).$$

Hence, f is game theoretic. □

Examples of game theoretic bankruptcy rules are the constrained relative equal awards rule (cf. Dietzenbacher et al., 2016) and the truncated proportional rule (cf. Dietzenbacher et al., 2017a). Dietzenbacher et al. (2017b) showed that the constrained egalitarian solution for bankruptcy games corresponds to the constrained relative equal awards rule for the underlying bankruptcy problem. Future research could study the interpretation and axiomatic significance of other game theoretic bankruptcy rules in order to further extend the relation between NTU-bankruptcy problems and NTU-games.

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