Risk management from a microeconomic perspective

Summary

To have an adequate determination of capital requirements in the banking industry and a proper performance evaluation of portfolio managers it is crucial to *measure* and *allocate* risk in an appropriate way.

A *measure of risk* assigns a real number to the probability distribution of the future value of a portfolio. It can be interpreted as the minimal amount of cash the regulated agent has to add to his portfolio, and to invest in a zero coupon bond for its risk to be acceptable to the regulator. The literature knows of numerous possible ways to measure risk; lately interest shifted to *coherent measures of risk* (Artzner, Delbaen, Eber, and Heath, 1999) satisfying the following four axioms: monotonicity, subadditivity, positive homogeneity and translation invariance. Adding two more axioms, comonotonic additivity and law invariance, one obtains a subclass of coherent measures of risk, called *spectral measures of risk*.

Most importantly, *subadditivity* requires that the risk of an aggregate portfolio should not exceed the total risk of the individual portfolios: it captures the notion of diversification. The diversification benefits should be allocated somehow, preferably in a stable way, when no collection of individual portfolios would be better off if they separate from the others.

In this thesis we analyze the aforementioned axioms using tools from general equilibrium theory and study the allocation of risk by means of cooperative game theory.

Chapter 2 of the thesis elaborates on analyzing the axioms of coherent and spectral measures of risk using an exchange economy model. By doing so we contribute to the research agenda that connects finance to general equilibrium theory. The corresponding measure of risk, the so-called *GE measure of risk* of a portfolio is the negative of its equilibrium market price in the exchange economy.

We prove that the GE measure of risk is a coherent measure of risk, thus coherent measures of risk are compatible with a natural general equilibrium approach to measure risk. However, using the insight of the Capital Asset Pricing Model (Sharpe, 1964; Lintner, 1965) that the risk of a portfolio does not only depend on the probability distribution of its payoff (law invariance), but also on how these payoffs are correlated to those of the market portfolio, we show that the GE measure of risk does not satisfy law invariance, and is therefore not a spectral measure of risk.

We model the problem of risk allocation by cooperative game theory. A *transferable utility* (TU) game consists of a finite set of players and a value function specifying the maximum attainable utility (money) for all the coalitions of players. The *core* of a TU game consists of those efficient

allocations which are robust against all coalitional deviations. One obtains a *subgame* by restricting the game to a subset of players. Games having a non-empty core in every subgame are called *totally balanced*.

In Chapter 3 we provide a set of linear programming problems to study a subclass of totally balanced games having a nice core structure, *exact* games (Schmeidler, 1972). A game is exact if for each coalition there is a core allocation such that the coalition only gets its stand-alone value. By the linear programming problems one can easily check whether a game is exact or not. Using the dual of the linear programming problems we develop two new characterizations of exact games, complementing earlier characterizations by Schmeidler (1972) and Azrieli and Lehrer (2005). The characterizations can be used to verify exactness of a game, we apply them in Chapter 4.

In Chapter 4 we come back to the question of the distribution of the risk diversification benefits. *Risk allocation games* (Denault, 2001) are transferable utility cooperative games defined to this purpose. A risk allocation game arises from a risk environment specifying a number of portfolios and a coherent measure of risk determining the risk of each portfolio. Coalitions of agents can combine their portfolios and thereby create diversification gains.

We prove that the class of risk allocation games coincides with the class of totally balanced games. This result ensures that a regulator can always allocate risk in a *stable* way. No matter how the risk environment changes, there is always an allocation of risk that no coalition of portfolios can object to.

To get exact games, we need an extra condition, since exact games are a subclass of totally balanced games. Usually the risk of the aggregate portfolio is low compared to the risk involved in the individual portfolios. As an extreme case, *no aggregate uncertainty* refers to the case when the value of the aggregate portfolio is constant over all states of nature. We prove that the class of risk allocation games with no aggregate uncertainty coincides with the class of exact games. To show that all risk allocation games with no aggregate uncertainty are exact, we use one of the characterizations of exact games developed in Chapter 3.

Thus, in the case of no aggregate uncertainty, for each coalition of portfolios, this coalition does not necessarily benefit from the diversification opportunities offered by the aggregate portfolio. As a consequence, the regulator has a high level of discretion in allocating the risk to the individual portfolios in this case.

Convex games with transferable utility introduced by Shapley (1971) provide a further refinement of exact games, since convex games are a subset of exact games. If the number of players (individual portfolios) below four, we show that one obtains the class of convex games in risk allocation games with no aggregate uncertainty.

The assumption of transferable utility can be relaxed to obtain the class of *games with non-transferable utility* (NTU games, for short). In a TU game it is assumed that utilities can be transferred. Such an assumption is justified if there is a commodity which has the same marginal utility for everyone and the utility functions are linear and separable in it. In general, this is not the case and one would like to study the more general class of NTU games. An NTU game specifies the set of attainable utility levels for the members of each coalition.

The notion of convexity can be generalized to NTU games in at least five ways. Vilkov (1977) and Sharkey (1981) have extended convexity to NTU games to define *ordinal* and *cardinal convexity*, respectively. Hendrickx, Borm, and Timmer (2002) analyze *coalition merge convexity*, *individual merge convexity*, and *marginal convexity* in an NTU setting.

The aforementioned five classes of NTU convex games do not coincide in general. The only general result is that coalition merge convexity implies individual merge convexity, and individual merge convexity implies marginal convexity. It is natural to seek a result that is analogous to convex TU games being exact. In Chapter 5 we generalize exactness to the NTU setting. In an *exact NTU game* for each coalition there is a core element on the boundary of its payoff set, meaning that this coalition does not necessarily benefit from the gains of forming the grand coalition in an allocation which is robust against all coalitional deviations. We show that each of ordinal, coalition merge, individual merge and marginal convexity can be unified under NTU exactness, which gives a sensible, common property of the core for all those games.

References

Artzner, P. F., F. Delbaen, J.-M. Eber, and D. Heath, Coherent measures of risk. *Mathematical Finance*, 9, (1999), pp. 203–228.

Azrieli, Y. and E. Lehrer, Concavification and convex games. *Working Paper*. School of Mathematical Sciences, Tel Aviv University.

Denault, M., Coherent allocation of risk capital. *Journal of Risk*, 4, no. 1, (2001), pp. 1–34.

Hendrickx, R., P. Borm, and J. Timmer, A note on NTU convexity. *International Journal of Game Theory*, 31, (2002), pp. 29–37.

Lintner, J., The valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics*, 47, (1965), pp. 13–37.

Shapley, L. S., Cores of convex games. *International Journal of Game Theory*, 1, (1971), pp. 11–26.

Sharkey, W., Convex games without side payments. *International Journal of Game Theory*, 10, (1981), pp. 101–106.

Schmeidler, D., Cores of exact games. *Journal of Mathematical Analysis and Applications*, 40, (1972), pp. 214–225.

Sharpe, W. F., Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance*, 19, (1964), pp. 425–442.

Vilkov, V., Convex games without side payments (in Russian). *Vestnik Leningradskiva Universitata*, 7, (1977), pp. 21–24.