

# Fraction auctions: the tradeoff between efficiency and running time

Citation for published version (APA):

Grigorieva, E., Herings, P. J. J., Müller, R. J., & Vermeulen, A. J. (2009). *Fraction auctions: the tradeoff between efficiency and running time*. METEOR, Maastricht University School of Business and Economics. METEOR Research Memorandum No. 045 <https://doi.org/10.26481/umamet.2009045>

## Document status and date:

Published: 01/01/2009

## DOI:

[10.26481/umamet.2009045](https://doi.org/10.26481/umamet.2009045)

## Document Version:

Publisher's PDF, also known as Version of record

## Please check the document version of this publication:

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RM/09/045

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## Fraction auctions: the tradeoff between efficiency and running time

Elena Grigorieva,<sup>1</sup> P.Jean-Jacques Herings,<sup>2</sup> Rudolf Müller,<sup>3</sup> and Dries Vermeulen<sup>4</sup>

**Abstract.** This paper studies the sales of a single indivisible object where bidders have continuous valuations. In Grigorieva et al. [13] it was shown that, in this setting, query auctions necessarily allocate inefficiently in equilibrium. In this paper we propose a new sequential auction, called the  $c$ -fraction auction. We show  $c$ -fraction auctions guarantee approximate efficiency at any desired level of accuracy, independent of the number of bidders. We discuss the running time and the efficiency in the ex-post equilibrium of the auction. We show that by changing the parameter  $c$  of the auction we can trade off efficiency against running time.

**Keywords and Phrases:** query auction, efficiency, running time

**JEL Classification Numbers:** C72, D44.

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<sup>1</sup>The author acknowledges support by the Dutch Science Foundation NWO through grant 401-01-101.

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## 1 Introduction

The English auction remains the predominant auction format used in practice, though we know since Vickrey [24] that it is strategically equivalent to the second-price, sealed bid auction. Rothkopf et al. [20] argued that third parties being able to “capture fractions of the economic rent revealed by the second price procedure” is one of the reasons why we hardly observe any Vickrey auctions in practice. Engelbrecht-Wiggans and Kahn [10] support this argument in their analysis. They study a model of a procurement auction where the winner of the auction might have to undergo a negotiation with a third party after the auction. In this negotiation, information of the third party about the winner’s cost revealed in the auction can have a negative influence on the winner’s surplus and the auctioneers revenue. Also for combinatorial auctions—settings with multiple, heterogenous goods and bidder valuations for bundles of items—ascending auctions have been the most popular approach: first in form of simultaneous ascending auctions (Cramton [7]), later in form of clock and clock-proxy auctions [1].

From a theoretical point of view an ascending auction should be implemented by a continuous, increasing price clock, where bidders drop out whenever the price exceeds what they are willing to pay. However, in practice we see almost exclusively implementations of the following two variants: (1) a discrete clock is increased by increments chosen by the auctioneer; (2) bidders submit increasing bids which exceed the current high bid plus some minimum increment, usually in terms of a percentage of the standing high bid. Cramton [7] reports that “bid increments in the 5 to 10 percent range are required to get the auction to conclude in a manageable number of rounds”. If the auction decides on the sales of radio spectrum, such increments are in the order of hundreds of millions of dollars, and can cause bidders to drop out because the next bid would exceed their valuation.

Thus, when the auctioneer sets bid increments he has to deal with a tradeoff between efficiency and running time of the auction. As bid increments define a decomposition of the continuum of valuations into intervals, a large bid increment might lead to an inefficient allocation of the item because it prohibits to distinguish between bidders whose valuations are in the same interval. Small increments decrease the chance of allocation inefficiency, but increase substantially the running time of the auction and thereby participation costs [9]. Despite its practical relevance, the tradeoff between these two goals has found very little attention in the academic literature. The furthest reaching theoretical evaluation of discrete bid levels has been given in David et al. [9], following up on [8]. They provide a recipe on how to set a finite number of discrete bid levels in order to maximize expected revenue of the auctioneer. They hereby extend the analysis of Rothkopf and Harstad [21], which was limited to either 2 bidders or 2 bid levels. However, only in a few cases the recipe of David et al. can be solved analytically. In other cases,

the auctioneer has to rely on numerical calculations. The key qualitative insight of these articles is that *decreasing* increments are preferable to constant or increasing bid levels—the common practice.

In this article we introduce a discrete query auction, called  $c$ -fraction auction, for the sale of a single item.<sup>5</sup> The auction gives the auction designer full control of the tradeoff between inefficiency and running time. Based on a prior of bidders' valuations the auctioneer can choose a single parameter, called  $c$  throughout, to regulate both running time and expected inefficiency. The practical implementation requires as only numerical tool the computation of values of the quasi-inverse of cumulative distribution functions. The auction is detail-free from bidders' perspectives by offering them an ex-post Nash equilibrium that differs only slightly from truth-telling. In other words, the equilibrium analysis does not have to make assumptions about bidders' beliefs of other bidders' valuations. Last but not least,  $c$ -fraction auctions can be applied for any single item setting with bidder's valuation being i.i.d.

Our main contribution is to discuss in detail the performance of the  $c$ -fraction auction under the bluff equilibrium. First, we investigate the running time of the auction according to two measures, namely the expected number of rounds and the expected number of queries performed in the auction.<sup>6</sup> For both measures we derive first an exact recursive formula and give then an upper bound for the function defined by this formula. We prove that for a fixed  $c$  the expected number of rounds is bounded by a function that is logarithmic in the number of players while the expected number of queries is bounded by a function that is linear in the number of players. We also give a (more) precise estimate of the number of rounds and number of queries by using computer simulation.

Second, we analyze the level of inefficiency of the auction. As measures of inefficiency we employ the probability of inefficient allocation and the expected loss of welfare. For the probability of inefficient allocation we prove that it is no more than  $c$  for any number of players, again after deriving an exact recursive formula. This is remarkable because for any choice of constant or increasing bid increments, and, say, uniform i.i.d. valuations, the probability of inefficient allocation is increasing in the number of bidders and converging to 1. Indeed, the number  $n_{max}$  of bidders with a value in the largest interval is increasing, and as the auction cannot distinguish between them, the probability of inefficient allocation is  $\frac{n_{max}-1}{n_{max}}$ . With respect to expected loss of welfare, we derive for each distribution a constant  $\gamma(c)$  which is such that the expected loss of welfare is bounded from above by  $\gamma(c)c$  for any number of players. When valuations are uniformly drawn from  $[0, 1]$  it holds that  $\gamma(c) = c$  and for the exponential distribution with parameter  $\lambda$  we have that  $\gamma(c) = -(\ln(1-c))/\lambda$ . Our results imply that by choosing the appropriate

<sup>5</sup>We prefer to use the term query auction rather than clock auction because our design allows query prices to decrease and increase.

<sup>6</sup>As a query we consider each separate question of the auctioneer to an active player. As a round we consider a sequence of queries in which each active player is asked to act exactly once.

$c$ , the minimum level of efficiency can be determined by the auctioneer *before* it is known how many players will participate in the auction. We also give a (more) precise estimate of the expected loss of welfare by using computer simulation. Furthermore we show that for a fixed number of players there is a trade-off between efficiency and running time: for increasing efficiency of the auction we have to pay by an increasing number of rounds and an increasing number of queries.

In Grigorieva et al. [13] we study the limitations of query auctions regarding the objective of economic efficiency maximization. In particular, we prove that in a setting with valuations distributed according to a continuous density function any ex-post equilibrium in an ex-post individual rational query auction that ends with positive probability after a finite number of queries, can not be fully efficient. This result implies that in the setting of continuous valuations full efficiency can only be achieved at the expense of an infinite running time of a query auction for almost all realizations of valuations. Our results on the  $c$ -fraction auction prove a counter part of this negative result: for any  $c > 0$ , the probability of inefficient allocation can be limited by  $c$  using an individually rational query auction with an ex-post equilibrium that ends after a finite number of rounds for *all* realizations of valuations. Furthermore, the number of rounds is logarithmic in the number of bidders, and independent of the range of valuations.

Our paper is closely related to David et al. [9]. They focus on revenue maximization and touch loss of efficiency only on the side by showing that revenue optimal discrete bid levels achieve better economic efficiency than equi-distant levels. Though the paper is more ambitious by trying to solve the optimization problem, a practical implementation requires complex numerical calculations and at least a prior on the number of bidders. Later, Rogers et al. [19] have provided a Bayesian machine learning model to compute, based on closing prices of previous auctions, revenue maximizing discrete bid levels when no ex-ante information on the number of bidders and their valuations distribution is available.

The earliest paper on discrete bid levels in continuous settings that we could trace is by Chwe [4]. Chwe studied the impact on revenue when discrete bids rather than continuous bids are used in sealed-bid, first price auctions. He analyzes equilibrium bidding strategies and shows that revenue with discrete bids is always lower than with continuous bids. Later Yu [25] extended his equilibrium analysis to English, Vickrey and Dutch auctions.

Another stream of literature studies iterative auctions from the viewpoint of preference elicitation. Determining one's valuation with a precision up to the last digit can be computationally demanding, see for example Larson and Sandholm [15], Parkes [18], and Sandholm [22]. In combinatorial auctions, the full revelation of agents' preferences may require a prohibitive amount of communication, see for example Nisan and Segal [17]. Such considerations lead to an interest in auctions where agents need not reveal

their information entirely but only partially. One approach is to limit communication in a sealed bid auction to a finite number of possible bits, see Blumrosen et al. [2]. Another approach is incremental elicitation of valuations in multiple rounds. It has been recognized that multi-round mechanisms can reduce revelation and associated with it computation and communication, compared to single-round mechanisms advocated by the revelation principle. See Blumrosen et al. [3] and Conitzer and Sandholm [6]. Incremental elicitation of bidder valuations has been modeled by query auctions (see e.g., Conen and Sandholm [5]). In a query auction the auctioneer sequentially queries the agents about specific aspects of their preferences. As an answer to the query an agent can choose one of a finite set of actions. Through incremental querying, the auctioneer gradually collects the information on agents' valuations. By using a query strategy in which previously revealed information guides the selection of subsequent queries, elicitation is focused on pertinent information. Incremental querying has been applied in different settings (see e.g. Conen and Sandholm [5] and Grigorieva et al. [12]) and it has been shown that only a small fraction of agents' valuation information needs to be revealed before the (approximately) optimal allocation can be determined Grigorieva et al. [11], Hudson and Sandholm [14].

When evaluating the effectiveness of elicitation we may generally care about the running time expressed in the number of queries required to determine an optimal (according to the specific objective of the designer) allocation (Sandholm and Boutilier [23]). Since information about agents' valuation becomes more refined with each query, a higher number of queries leads to a better allocation. Our proposed  $c$ -fraction auction provides a framework that lets the auctioneer explicitly tradeoff the two goals.

The paper is organized as follows. Section 2 introduces the rules of the  $c$ -fraction auction and Section 3 derives what we call the bluff equilibrium. Section 4 addresses the probabilities by which bidders make *yes* and *no* responses. In Section 5 the running time of the auction is analyzed. Section 6 is devoted to the analysis of the efficiency of the auction and remarks about the trade-off between running time and efficiency. Section 6 provides some concluding remarks.

## 2 The $c$ -fraction auction

Suppose a single indivisible object is auctioned to a set  $N = \{1, \dots, n\}$  of players. The players have quasi-linear utilities. We assume independent private valuations drawn from a common continuous probability distribution with density  $f$  and cumulative density  $F$ . The support of  $f$  belongs to  $\mathbb{R}_+$ . The minimum of this support is denoted by  $\alpha$  and the supremum by  $\beta$ , where we allow  $\beta$  to be infinite and assume that  $\beta$  strictly exceeds every possible valuation. Since  $F$  may not be strictly increasing, it may not have an inverse. By  $F^{-1} : [0, 1) \rightarrow \mathbb{R}$  we denote the function such that  $F^{-1}(y) = \max\{x \in \mathbb{R}_+ \mid F(x) = y\}$ .

Before the start of the auction there is a lottery that determines an ordering of the players. Without loss of generality we assume that this ordering is  $1 \prec 2 \prec \dots \prec n-1 \prec n$ . A player with a lower ranking is called a predecessor.

The auction runs for a number of rounds. A round  $r$  is characterized by a payment  $p_r$ , a query price  $q_r$ , an upper bound  $u_r$ , and a set of active players  $A_r$ . The payment specifies the price to be paid if an active player wins in this round. The query price is used by the auctioneer to ask the active players whether their valuation is larger than or equal to the query price. Active players are queried publicly in increasing order—player  $i$  before player  $i+1$ , such that an active player can observe the bids of his predecessors. In each round the query price  $q_r$  is chosen from the open interval  $(p_r, u_r)$ , where  $u_r$  is allowed to be infinite.

The initial set of active players is  $A_1 = N$ . The auction starts with  $p_1 = \alpha$ ,  $u_1 = \beta$ , and some  $q_1$  in  $(p_1, u_1)$ . Given the current set  $A_r$ , the payment  $p_r$ , the query price  $q_r$ , the upper bound  $u_r$ , and the bids of players in round  $r$ , the characteristics of the next round  $r+1$  are defined as follows. If all active players submit a *no* bid they all remain active, i.e.  $A_{r+1} = A_r$ , the payment remains the same, and the upper bound is set to the previous query price, i.e.  $p_{r+1} = p_r$  and  $u_{r+1} = q_r$ . If at least two active players submit a *yes* bid, all players that said *yes* remain active, the upper bound remains the same, and the payment is set equal to the previous query price, i.e.  $u_{r+1} = u_r$  and  $p_{r+1} = q_r$ . In both cases, as a function of the bounds, a new query price  $q_{r+1}$  in  $(p_{r+1}, u_{r+1})$  is determined in a way specified below. If only one active player submits a *yes* bid, the auction stops, this player wins the auction, and pays  $p_r$ . If such a moment doesn't occur, i.e. at least two players remain active forever, the winner is determined according to the order of players: among those players who remain active the player with the highest ranking wins. The price the winner pays is equal to the limit of the sequence of the payments  $(p_r)_{r \in \mathbb{N}}$  that occurred in the subsequent rounds in the auction. Since the sequence of payments is increasing, this limit is equal to the supremum of the payments, and is denoted by  $p_\infty$ .

For the  $c$ -fraction auction, where  $c \in (0, 1)$ , the query price  $q_r$  is chosen as the maximal  $q$  for which

$$\frac{F(q) - F(p_r)}{F(u_r) - F(p_r)} = c,$$

i.e.  $c$  is equal to the probability that a valuation belongs to the interval  $(p_r, q_r]$  conditional on this valuation being in the interval  $(p_r, u_r]$ . For the uniform distribution for example it holds that  $q_r = p_r + c(u_r - p_r)$ . Equivalently, we can define

$$q_r = F^{-1}((1 - c)F(p_r) + cF(u_r)).$$

Consider the following strategy for player  $i$  having valuation  $v_i$ . This player says *yes* in round  $r$  whenever

1.  $v_i \geq q_r$ , or
2.  $p_r \leq v_i < q_r$  and no active predecessor of  $i$  said *yes* in round  $r$ .

In the second part of the definition, we mean by an active predecessor of  $i$  in round  $r$  a player in  $A_r$  that is a predecessor of  $i$ . Because this part of the definition involves a certain amount of bluff, we call this strategy the bluff strategy. Formally, the auction is given by an extensive form game, the bluff strategy of player  $i$  is a function from the set of decision nodes of player  $i$  to  $\{yes, no\}$ , and is denoted by  $b_i$ . The following example illustrates how the auction proceeds when all players follow the bluff strategy.

**Example.** Suppose five players with valuations uniformly distributed on  $[0, 1)$  participate in the  $c$ -fraction auction with  $c$  equal to 0.5. Players have the following private valuations: 0.43, 0.71, 0.38, 0.79, and 0.86. The auction proceeds as follows.

| Round<br>$r$ | Payment<br>$p_r$ | Query price<br>$q_r$ | Set of active<br>players $A_r$ | Player<br>1 | Player<br>2 | Player<br>3 | Player<br>4 | Player<br>5 |
|--------------|------------------|----------------------|--------------------------------|-------------|-------------|-------------|-------------|-------------|
| 1            | 0                | 0.5                  | {1,2,3,4,5}                    | yes         | yes         | no          | yes         | yes         |
| 2            | 0.5              | 0.75                 | {1,2,4,5}                      | no          | yes         | -           | yes         | yes         |
| 3            | 0.75             | 0.875                | {2,4,5}                        | -           | no          | -           | yes         | no          |

In the first round player 1, having no predecessor and valuation larger than  $p_1$ , says *yes*. Every other player, having now an active predecessor who said *yes*, says *yes* if and only if his valuation is greater than or equal to  $q_1 = 0.5$ . All players except player 3 say *yes* and therefore remain active. The payment and the query price increase to 0.5 and 0.75, respectively. Since  $v_1 < p_2$  player 1 says *no* in the second round. Now player 2 has no active predecessor who said *yes* and since  $v_2 \geq p_2$  he says *yes*. Players 4 and 5 say *yes* since their valuations are larger than  $q_2 = 0.75$ . Again the payment and the query price increase. In the third round player 2 says *no*, player 4, having now no active predecessors who said *yes*, says *yes* and player 5 says *no*. In this round there is only one *yes* decision meaning that the auction ends. Player 4 wins the auction and pays 0.75.

Notice that the outcome in the example is not efficient - the winner is not the player with the highest valuation. Later in the paper we investigate how inefficient this auction is by analyzing the probability of an inefficient allocation and the expected loss of welfare.

### 3 Bluff equilibria

In this section we show that the profile of bluff strategies constitutes an ex-post individually rational ex-post equilibrium called the *bluff equilibrium*. An ex-post equilibrium is a strategy profile such that, given any realization of valuations, the plan of action prescribed to a bidder in the auction by his strategy

is a best response to the plans of action prescribed by the strategies of the other bidders given their valuations. A strategy is ex-post individually rational if for every realization of valuations and for any profile of actions of the player's opponents, the strategy leads to non-negative utility.

**Proposition 3.1** *The bluff strategy is ex-post individually rational.*

**Proof.** Consider an arbitrary strategy profile where player  $i$  follows his bluff strategy. If  $i$  is not the winner, then he has utility 0. When  $i$  is the winner, we have two cases. Case 1: The auction terminates after finitely many rounds, say in round  $r$ . Then all players in  $A_r$  said *no*, while player  $i$  said *yes*. Therefore,  $v_i \geq p_r$ , and since  $p_r$  is the price to be paid by  $i$ , he has a non-negative payoff. Case 2: The auction does not terminate after finitely many rounds. Suppose that  $v_i < p_\infty$ . Since  $i$  is the winner, he belongs to  $A_r$  for every value of  $r$ , and there is a player  $i'$  with the same property and such that  $i' \prec i$ . Since player  $i$  follows the bluff strategy and  $v_i < p_\infty$ , there are rounds in which player  $i$  said *no*. Let  $r$  be the first such round. Notice that all players in  $A_r$  say *no*, since otherwise player  $i$  would have ceased to be active. Since  $p_1$  is equal to the minimum of the support of  $F$  and player  $i$  uses the bluff strategy,  $r > 1$ . Player  $i$  said *yes* in round  $r - 1$ , and so did player  $i'$ , since otherwise player  $i'$  would have ceased to be active in round  $r$ . Since player  $i$  uses the bluff strategy and  $i' \prec i$ ,  $v_i \geq q_{r-1}$ . Since at least two players said *yes* in round  $r - 1$ ,  $p_r = q_{r-1} \leq v_i$ . Since all players say *no* in round  $r$ , and player  $i$  uses the bluff strategy, it follows that  $v_i < p_r$ , a contradiction to  $v_i \geq q_{r-1} = p_r$ . We conclude that  $v_i \geq p_\infty$ , so player  $i$  has a non-negative utility. ■

**Theorem 3.2** *The bluff strategy profile is an ex-post Nash equilibrium.*

**Proof.** Let  $v$  be a realization of valuations and  $f_i$  be a strategy for player  $i$ , i.e. a function from the set of decision nodes for player  $i$  to  $\{yes, no\}$ . We show that  $f_i$  is not a profitable deviation from  $b_i$  against  $b_{-i}$ . Let  $h$  be the first decision node at which player  $i$  following  $f_i$  deviates from the bluff strategy, and let  $r$  denote the corresponding round. (Obviously, if  $f_i$  coincides with  $b_i$  at all decision nodes,  $f_i$  is not a profitable deviation.) Notice that since we treat  $v$  as given and apart from  $v$  there is no imperfect information, the node  $h$  is well-defined. We consider two cases.

**Case 1.** Let  $h$  be such that at least one predecessor of  $i$  in  $A_r$  has said *yes*. If  $f_i(h) = no$  and  $b_i(h) = yes$ , the payoff of playing  $f_i$  is 0, while according to Proposition 3.1 the payoff of playing the bluff strategy is at least 0. Consider the case where  $f_i(h) = yes$  and  $b_i(h) = no$ . When player  $i$  says *yes* in  $h$ , there are at least two players who say *yes* in round  $r$ . The winning payment will be at least  $q_r$ . Further,  $v_i < q_r$  because  $b_i(h) = no$ . Hence, the payoff of playing  $f_i$  is non-positive while the payoff of playing  $b_i$  is 0.

**Case 2.** Let  $h$  be such that none of the predecessors of  $i$  in  $A_r$  has said *yes*. If  $f_i(h) = yes$  and  $b_i(h) = no$ , we know that  $v_i < p_r$ . Since the payment of the winner is at least  $p_r$ , playing  $f_i$  has non-positive payoff,

while playing according to  $b_i$  guarantees non-negative payoff. Consider the case where  $f_i(h) = no$  and  $b_i(h) = yes$ , so  $v_i \geq p_r$ . Suppose that  $v$  is such that all successors of  $i$  say  $no$  if  $i$  says  $yes$ . Then, following  $b_i$ , player  $i$  wins at a price  $p_r$  while following  $f_i$  he might win at a price at least  $p_r$ . Now suppose that  $v$  is such that there is a successor  $j$  of  $i$  that says  $yes$  if player  $i$  says  $yes$ . Since player  $j$  uses the bluff strategy, he will also say  $yes$  when player  $i$  switches to  $no$ . But then the payoff of playing  $f_i$  would be 0 while the payoff of playing  $b_i$  is non-negative. ■

**Theorem 3.3** *The allocation under the bluff equilibrium is not ex-post efficient.*

**Proof.** Consider the set  $V$  of valuations for which  $p_1 \leq v_i < q_1$  for all  $i \in N$ . For each  $v \in V$ , bidder 1 says  $yes$  in round 1 and all other bidders say  $no$ . Thus the auction ends after round 1 and bidder 1 wins the item for a price of  $p_1$ . For  $v \in V$  such that for some  $i$ ,  $v_1 < v_i$ , the allocation is not ex-post efficient. ■

We argue next that  $c$ -fraction auctions have a finite running time under the bluff equilibrium for *any* realization of valuations. Notice that this statement is stronger than just saying that we have a finite running time almost surely. We also argue that when every bidder plays according to his bluff strategy, in any round of the auction there is at least one player who says  $yes$ .

**Theorem 3.4** *The bluff equilibrium has a finite running time for every realization of valuations. Moreover, the query price increases from round to round up to the moment where the winner is found.*

**Proof.** We first claim that the winner of the auction, say player  $j$ , says  $yes$  in every round of the auction. Suppose not, then let  $r$  be the first round in which he says  $no$ . Now all players in  $A_r$  say  $no$  in round  $r$ , since otherwise  $j$  cannot be the winner of the auction. Either  $r = 1$  or in round  $r - 1$  player  $j$  said  $yes$  and so did all players in  $A_r$ . If  $r = 1$ , then  $v_j \geq p_1$ , so player  $j$  says  $yes$ , leading to a contradiction. If in round  $r - 1$  all players in  $A_r$  said  $yes$ , then so did a player  $i$  with a predecessor in  $A_r$ , implying that  $v_i \geq q_{r-1} = p_r$ . Observing a  $no$  from all his predecessors in round  $r$ , player  $i$  should say  $yes$  in round  $r$  when following the bluff strategy, a contradiction. We have shown that the winner of the auction says  $yes$  in every round of the auction. It then follows that the query price increases from round to round up to the moment where the winner is found.

Suppose the valuation  $v$  is such that the running time of the auction is infinite. Since by the previous paragraph the query price increases from round to round, we have  $p_\infty = \beta$ . Since the winner of the auction, say player  $j$ , says  $yes$  in every round  $r$ , we have for all  $r \in \mathbb{N}$ ,  $v_j \geq p_r$ , so  $v_j \geq \beta$ , a contradiction. ■

As a corollary to Theorem 3.4 we find an easy characterization of the query price in round  $r$ .

**Corollary 3.5** *In the bluff equilibrium it holds that  $q^r = F^{-1}(1 - (1 - c)^r)$ ,  $r \in \mathbb{N}$ .*

## 4 Probability distribution of player actions

In this section we evaluate the probability of saying *yes* and *no* by an active player under the bluff strategy. Knowing these probabilities enables us to derive recursive formulas for the expected number of rounds and the expected number of queries performed in the auction. Throughout this section, as well as in remaining sections, it is assumed that every player uses the bluff strategy.

Recall that in any round  $r$  of the  $c$ -fraction auction the query price  $q_r$  is determined such that, conditional on  $v_i$  belonging to  $[p_r, u_r)$ , the probability that  $v_i$  is in  $[p_r, q_r)$  is equal to  $c$ . We define  $i_r = \min\{i \mid i \in A_r\}$  - among the active players in round  $r$  the one with the lowest ranking - and  $j_r = \min\{i \mid i \in A_r, i \neq i_r\}$  - among the active players in round  $r$  the one with the second lowest ranking.

First let us observe that when player  $i_r$  says *no* for the first time, player  $j_r$  says *yes* with certainty. Indeed, in all previous rounds player  $i_r$  said *yes* and since  $j_r$  is active in round  $r$  also he said *yes* in those rounds. Both the payment and the query price increased so that  $p_r = q_{r-1}$ . Since player  $j_r$  follows the bluff strategy his previous *yes* decision implies that  $v_{j_r} \geq q_{r-1} = p_r$ . If in round  $r$  player  $i_r$  says *no*, player  $j_r$  is in the situation where he doesn't have any active predecessor with a *yes* decision and therefore says *yes* whenever his valuation is not smaller than  $p_r$ . It follows that after a round  $r$  where player  $i_r$  drops out,  $i_{r+1} = j_r$ .

Second, we need to know the probability that player  $i_r$  says *yes* in round  $r$ . Having no active predecessor, player  $i_r$  says *yes* if and only if  $v_{i_r} \geq p_r$ . Since  $p_1 = \alpha$  player  $i_1$  says *yes* with certainty in round 1. Now let us show that for any  $r > 1$  the probability that player  $i_r$  says *yes* in round  $r$  equals  $1 - c$ . Regarding the identity of player  $i_r$  there are two possibilities - either  $i_r = i_{r-1}$  if the decision of  $i_{r-1}$  was *yes*, or  $i_r = j_{r-1}$  if the decision of  $i_{r-1}$  was *no* and consequently the decision of  $j_{r-1}$  was *yes*. In both cases the decision of player  $i_r$  in round  $r - 1$  was *yes* which happens if and only if  $v_{i_r} \geq p_{r-1}$ . Thus,  $\mathbb{P}(v_{i_r} \geq p_r \mid v_{i_r} \geq p_{r-1}) = \mathbb{P}(v_{i_r} \geq q_{r-1} \mid v_{i_r} \geq p_{r-1}) = 1 - c$ . The last equality holds because  $q_{r-1}$  satisfies

$$\frac{F(q_{r-1}) - F(p_{r-1})}{F(u_{r-1}) - F(p_{r-1})} = c,$$

and Theorem 3.4 implies that  $u_{r-1} = \beta$ .

Further, we need to know the probability of saying *yes* in round  $r$  for any player  $i \in A_r \setminus \{i_r\}$ . We can distinguish two cases.

First, consider the case where player  $i_r$  says *yes* in round  $r$ . From the fact that  $i \in A_r$ , it follows that

player  $i$  said *yes* in round  $r - 1$  which is the case if and only if  $v_i \geq q_{r-1}$ . In round  $r$  he says *yes* if and only if  $v_i \geq q_r$ . Thus,  $\mathbb{P}(v_i \geq q_r \mid v_i \geq q_{r-1}) = \mathbb{P}(v_i \geq q_r \mid v_i \geq p_r) = 1 - c$ . The last equality holds because  $q_r$  satisfies

$$\frac{F(q_r) - F(p_r)}{F(u_r) - F(p_r)} = c,$$

and Theorem 3.4 implies that  $u_r = \beta$ .

Secondly, consider the case where player  $i_r$  says *no* in round  $r$ . As we described before player  $j_r$  says *yes* with certainty. For any player  $i \in A_r \setminus \{i_r, j_r\}$ , the situation is the same as in the previous case and thus such a player  $i$  says *yes* with probability  $1 - c$ . We summarize our findings in the following theorem.

**Theorem 4.1** *In the bluff equilibrium it holds that a player  $i \in A_r$  says yes in round  $r$  with probability  $1 - c$ , except when  $i = i_1$  or ( $i = j_r$  and  $i_r$  says no), in which case player  $i$  says yes with probability 1.*

Due to the query price setting rule of the  $c$ -fraction auction, Theorem 4.1 holds regardless of the distribution from which valuations are drawn. Moreover, the probability of saying *yes* or *no* by an active player does not depend on the round with the exception of round 1. This enables us to derive recursive formulas for the expected number of rounds and the expected number of queries performed in the auction.

## 5 Running time of the auction

In this section we investigate the expected running time of the  $c$ -fraction auction if the bluff strategies are played. We analyze two measures, namely the expected number of rounds and the expected number of queries performed in the auction before the winner is found. As a query we consider each separate question of the auctioneer to an active player. For both measures we derive first a recursive formula and give then an upper bound for the function defined by this formula.

### 5.1 The expected number of rounds

Let  $e_c(k)$  be the expected number of rounds of the auction with  $k$  active players, given that the decision of the active player with the lowest ranking is *yes* in the current round; and let  $e_c^*(k)$  be the expected number of rounds given that this decision is *no*. Consider round  $r$  with  $n$  active players and suppose that the decision of player  $i_r$  in the current round is *yes*. The current round contributes 1 to  $e_c(n)$ . Now let us compute the expected number of remaining rounds. If all active players apart from player  $i_r$  say *no*, the auction stops after this round. If  $k$ , where  $1 \leq k \leq n - 1$ , active players apart from player  $i_r$  say *yes*, then the auction continues with  $k + 1$  active players. Using Theorem 4.1, the probability of this

situation given the *yes* decision of player  $i_r$  is  $\binom{n-1}{k}(1-c)^k c^{n-1-k}$ . In the next round player  $i_{r+1} = i_r$  says *yes* or *no* with probability  $1-c$  and  $c$  respectively. Thus if  $k$  active players apart from player  $i_r$  say *yes* in round  $r$ , the expected number of remaining rounds is equal to  $(1-c)e_c(k+1) + ce_c^*(k+1)$ . Hence, for any  $n \geq 2$ ,

$$e_c(n) = 1 + \sum_{k=1}^{n-1} \binom{n-1}{k} (1-c)^k c^{n-1-k} \left[ (1-c)e_c(k+1) + ce_c^*(k+1) \right]. \quad (1)$$

Theorem 4.1 states that if player  $i_{r+1}$  says *no*, player  $j_{r+1}$  says *yes* with certainty, which causes player  $i_{r+1}$  to drop out of the auction. We find that  $e_c^*(2) = 1$  and  $e_c^*(k+1) = e_c(k)$  for any  $k > 1$ .

We denote  $P_k^n = \binom{n}{k}(1-c)^k c^{n-k}$  and rewrite Equation (1) as follows:

$$\begin{aligned} e_c(n) &= 1 + \sum_{k=1}^{n-1} P_k^{n-1} \left[ (1-c)e_c(k+1) + ce_c^*(k+1) \right] \\ &= 1 + (1-c) \sum_{k=1}^{n-2} P_k^{n-1} e_c(k+1) + (1-c)P_{n-1}^{n-1} e_c(n) + c \sum_{k=2}^{n-1} P_k^{n-1} e_c^*(k+1) + cP_1^{n-1} e_c^*(2) \\ &= 1 + (1-c) \sum_{k=1}^{n-2} P_k^{n-1} e_c(k+1) + (1-c)^n e_c(n) + c \sum_{k=2}^{n-1} P_k^{n-1} e_c(k) + (n-1)(1-c)c^{n-1} \\ &= 1 + (1-c)^n e_c(n) + (n-1)(1-c)c^{n-1} + (1-c) \sum_{k=2}^{n-1} P_{k-1}^{n-1} e_c(k) + c \sum_{k=2}^{n-1} P_k^{n-1} e_c(k) \\ &= 1 + (1-c)^n e_c(n) + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1} \left[ (1-c)P_{k-1}^{n-1} + cP_k^{n-1} \right] e_c(k) \\ &= 1 + (1-c)^n e_c(n) + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1} P_k^n e_c(k). \end{aligned}$$

This can be rewritten to

$$\left[ 1 - (1-c)^n \right] e_c(n) = 1 + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} e_c(k). \quad (2)$$

This formula is valid for any  $n \geq 2$ .

Now notice that since in the first round player  $i_1$  says *yes* with certainty, the expected number of rounds of the auction with  $n$  players is equal to  $e_c(n)$ . Thus using Formula 2 we can compute the expected number of rounds in the auction of  $n$  players. Plugging in  $n = 2$  yields  $e_c(2) = \frac{1+c(1-c)}{c(2-c)}$ . All other values can be determined recursively. Table 1 presents the computational results for different values of  $c$  in the auction with up to 100 players (the data is within an accuracy of 0.001). Notice that the bisection character of the  $c$ -fraction auction guarantees a remarkably low number of rounds. For instance, with 100 bidders and  $c = 1/2$ , the auction terminates in less than 7 rounds on average.

| $n \setminus c$ | 1/10   | 1/8    | 1/6    | 1/4    | 1/3    | 1/2   | 2/3   | 3/4   | 5/6   | 7/8   | 9/10  |
|-----------------|--------|--------|--------|--------|--------|-------|-------|-------|-------|-------|-------|
| 2               | 5.737  | 4.733  | 3.727  | 2.714  | 2.200  | 1.667 | 1.375 | 1.267 | 1.171 | 1.127 | 1.101 |
| 3               | 8.901  | 7.230  | 5.555  | 3.873  | 3.021  | 2.143 | 1.663 | 1.483 | 1.319 | 1.240 | 1.193 |
| 4               | 11.273 | 9.102  | 6.927  | 4.742  | 3.638  | 2.505 | 1.891 | 1.660 | 1.446 | 1.341 | 1.277 |
| 5               | 13.172 | 10.600 | 8.024  | 5.437  | 4.131  | 2.794 | 2.076 | 1.807 | 1.557 | 1.431 | 1.353 |
| 6               | 14.753 | 11.848 | 8.938  | 6.016  | 4.542  | 3.035 | 2.230 | 1.931 | 1.654 | 1.512 | 1.423 |
| 7               | 16.109 | 12.918 | 9.721  | 6.513  | 4.895  | 3.241 | 2.361 | 2.037 | 1.738 | 1.584 | 1.486 |
| 8               | 17.296 | 13.854 | 10.407 | 6.947  | 5.203  | 3.421 | 2.475 | 2.129 | 1.813 | 1.650 | 1.545 |
| 9               | 18.350 | 14.686 | 11.016 | 7.334  | 5.477  | 3.581 | 2.576 | 2.211 | 1.879 | 1.709 | 1.598 |
| 10              | 19.299 | 15.435 | 11.565 | 7.681  | 5.724  | 3.726 | 2.667 | 2.283 | 1.939 | 1.762 | 1.647 |
| 20              | 25.647 | 20.443 | 15.233 | 10.006 | 7.373  | 4.690 | 3.275 | 2.760 | 2.312 | 2.102 | 1.971 |
| 30              | 29.417 | 23.418 | 17.412 | 11.387 | 8.353  | 5.264 | 3.637 | 3.048 | 2.524 | 2.281 | 2.140 |
| 40              | 32.109 | 25.541 | 18.967 | 12.372 | 9.052  | 5.673 | 3.894 | 3.255 | 2.681 | 2.406 | 2.249 |
| 50              | 34.203 | 27.194 | 20.177 | 13.140 | 9.596  | 5.991 | 4.095 | 3.414 | 2.808 | 2.508 | 2.333 |
| 60              | 35.918 | 28.547 | 21.168 | 13.768 | 10.042 | 6.252 | 4.260 | 3.543 | 2.913 | 2.595 | 2.405 |
| 70              | 37.370 | 29.693 | 22.007 | 14.299 | 10.419 | 6.472 | 4.399 | 3.652 | 3.002 | 2.672 | 2.469 |
| 80              | 38.628 | 30.686 | 22.735 | 14.760 | 10.746 | 6.664 | 4.520 | 3.747 | 3.077 | 2.740 | 2.527 |
| 90              | 39.740 | 31.563 | 23.377 | 15.167 | 11.035 | 6.833 | 4.627 | 3.831 | 3.143 | 2.801 | 2.580 |
| 100             | 40.735 | 32.348 | 23.952 | 15.532 | 11.294 | 6.984 | 4.722 | 3.907 | 3.201 | 2.855 | 2.629 |

Table 1: The expected number of rounds  $e_c(n)$  in the  $c$ -fraction auction.

Figure 1: The expected number of rounds (a) for different fixed values of  $c$ ; (b) for different fixed numbers of players.

Figure 1(a) shows how for a fixed value of  $c$  the expected number of rounds increases in the number of players who participate in the auction. Furthermore, Figure 1(b) demonstrates how for a fixed number of players the expected number of rounds decreases as  $c$  increases.

Generally we show that the expected number of rounds of the auction is bounded from above by a function that is logarithmic in the number of players. To prove this, we introduce several notations and lemmas first.

For any  $n \geq 2$ , define  $D_n = \prod_{k=1}^n \frac{1}{1-(1-c)^k}$ . Also, for any  $n > 2$ , define  $E_2 = \frac{1+c(1-c)}{c(2-c)}$  and

$$E_n = 1 + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} E_k.$$

**Lemma 5.1** For any  $n \geq 2$ ,  $e_c(n) < E_n \cdot D_n$ .

**Proof.** The proof is by induction on  $n$ . The basis of the induction is trivial since  $e_c(2) = E_2$  and  $D_2 > 1$ . Suppose that  $e_c(k) < E_k \cdot D_k$  is true for any  $2 \leq k \leq n-1$ . Notice that  $D_n > D_{n-1} > \dots > D_2 > 1$ . Thus, using the recursive formula for  $e_c(n)$  and the induction hypothesis,

$$\begin{aligned}
\left[1 - (1 - c)^n\right] e_c(n) &= 1 + (n - 1)(1 - c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1 - c)^k c^{n-k} e_c(k) \\
&< 1 + (n - 1)(1 - c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1 - c)^k c^{n-k} E_k D_k \\
&< 1 + (n - 1)(1 - c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1 - c)^k c^{n-k} E_k D_{n-1} \\
&< D_{n-1} \left[1 + (n - 1)(1 - c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1 - c)^k c^{n-k} E_k\right] \\
&= E_n \cdot D_{n-1},
\end{aligned}$$

which completes the proof. ■

Now we find bounds on  $D_n$  and  $E_n$ .

**Lemma 5.2** *For any  $n \geq 2$ ,  $D_n < e^{\frac{1-c}{c^2}}$ .*

**Proof.** We have to show that  $\ln D_n \leq \frac{1-c}{c^2}$ . Let us define  $\lambda = \frac{1}{1-c}$ . Notice that since  $0 < c < 1$  it holds that  $\lambda > 1$ .

We have

$$\begin{aligned}
\ln D_n &= \ln \left( \prod_{k=1}^n \frac{\lambda^k}{\lambda^k - 1} \right) = \sum_{k=1}^n \left[ \ln \lambda^k - \ln(\lambda^k - 1) \right] \leq \sum_{k=1}^n \partial_x \ln x|_{x=\lambda^k-1} = \sum_{k=1}^n \frac{1}{\lambda^k - 1} \\
&< \sum_{k=1}^n \frac{1}{\lambda^k - \lambda^{k-1}} = \frac{1}{\lambda - 1} \sum_{k=1}^n \frac{1}{\lambda^{k-1}} < \frac{1}{\lambda - 1} \sum_{k=0}^{\infty} \frac{1}{\lambda^k} = \frac{\lambda}{(\lambda - 1)^2} = \frac{1 - c}{c^2}.
\end{aligned}$$

■

**Lemma 5.3** *For any  $n \geq 2$  and any  $c \leq \frac{1}{2}$ ,  $E_n < 1 + \log_a n$ , with base  $a = \frac{1}{1-c}$ .*

**Proof.** The proof is by induction on  $n$ . The basis of the induction holds since  $\frac{1+c(1-c)}{c(2-c)} < 1 + \log_a 2$  for any  $c \leq \frac{1}{2}$ . Suppose  $E_k < 1 + \log_a k$  for any  $2 \leq k \leq n - 1$ . Using the induction hypothesis,

$$\begin{aligned}
E_n &= 1 + (n - 1)(1 - c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1 - c)^k c^{n-k} E_k \\
&< 1 + (n - 1)(1 - c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1 - c)^k c^{n-k} (\log_a k + 1) \\
&= 1 + \sum_{k=2}^{n-1} \binom{n}{k} (1 - c)^k c^{n-k} \log_a k + \sum_{k=1}^{n-1} \binom{n}{k} (1 - c)^k c^{n-k} \\
&< 2 + \sum_{k=2}^{n-1} \binom{n}{k} (1 - c)^k c^{n-k} \log_a k.
\end{aligned}$$

Since the logarithm with base  $a = \frac{1}{1-c}$  is concave, we know that if  $\lambda_k \geq 0$  and  $\sum_{k=0}^n \lambda_k = 1$  then

$$\sum_{k=0}^n \lambda_k \log_a(x_k) \leq \log_a\left(\sum_{k=0}^n \lambda_k x_k\right).$$

So let us take  $\lambda_k = \binom{n}{k}(1-c)^k c^{n-k}$  for  $k = 0, \dots, n$  and take  $x_0 = x_n = 1$ ,  $x_k = k$  for any  $1 \leq k \leq n-1$ .

Then

$$\begin{aligned} E_n &< 2 + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} \log_a k \\ &= 2 + \sum_{k=0}^n \binom{n}{k} (1-c)^k c^{n-k} \log_a(x_k) \\ &\leq 2 + \log_a \left[ \sum_{k=0}^n \binom{n}{k} (1-c)^k c^{n-k} x_k \right] \\ &= 2 + \log_a \left[ \sum_{k=1}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} k + c^n + (1-c)^n \right] \\ &\leq 2 + \log_a \left[ \sum_{k=1}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} k + n(1-c)^n \right] \\ &= 2 + \log_a \left[ \sum_{k=0}^n \binom{n}{k} (1-c)^k c^{n-k} k \right] \\ &= 2 + \log_a [(1-c)n] \\ &= 1 + \log_a n. \end{aligned}$$

The last inequality holds since for any  $c \leq \frac{1}{2}$  and any  $n \geq 2$  it holds that  $c^n + (1-c)^n \leq 2(1-c)^n \leq n(1-c)^n$ . ■

A final immediate consequence of Lemmas 5.1 – 5.3 is the following theorem.

**Theorem 5.4** *For any  $c \leq \frac{1}{2}$  and any  $n \geq 2$ ,  $e_c(n) \leq e^{\frac{1-c}{c^2}} \left( \log_{\frac{1}{1-c}} n + 1 \right)$ .*

**Remark:** Since  $e_c(n) < e_{\bar{c}}(n)$  when  $c > \bar{c}$ , the upper bound for  $\bar{c} = \frac{1}{2}$  is also valid for any  $c > \frac{1}{2}$ .

We showed that the expected number of rounds of the  $c$ -fraction auction is bounded from above by a function that is logarithmic in the number of players. A comparison of the bound with the computed results suggests that this bound is not tight. It can easily be checked that for a fixed value of  $c$  the ratio between the bound and the computed result is approximately constant as a function of  $n$ , implying that the bound has approximately the correct order of magnitude.

## 5.2 The expected number of queries

Let  $b_c(k)$  be the expected number of queries of the auction with  $k$  active players, given that the decision of the active player with the lowest ranking is *yes* in the current round;  $b_c^*(k)$  be the expected number of queries given that this decision is *no*. Notice that in a round with  $k$  active players  $k$  queries are performed. Following the same argumentation as we used for determining the formula for the expected number of rounds we find that for any  $n \geq 2$

$$b_c(n) = n + \sum_{k=1}^{n-1} \binom{n-1}{k} (1-c)^k c^{n-1-k} \left[ (1-c)b_c(k+1) + cb_c^*(k+1) \right]. \quad (3)$$

Again, notice that when player  $i_{r+1} = i_r$  says *no* player  $j_{r+1}$  says *yes* with certainty, which causes player  $i_{r+1}$  to drop out of the auction. Thus,  $b_c^*(2) = 2$  and for all  $k \geq 2$  it holds that  $b_c^*(k+1) = 1 + b_c(k)$ .

Recall that  $P_k^n = \binom{n}{k} (1-c)^k c^{n-k}$ . Using the facts that  $b_c^*(2) = 2$  and  $b_c^*(k+1) = b_c(k) + 1$  for all  $k \geq 2$ , we get from (3) that

$$\begin{aligned} b_c(n) &= n + \sum_{k=1}^{n-1} P_k^{n-1} \left[ (1-c)b_c(k+1) + cb_c^*(k+1) \right] \\ &= n + (1-c) \sum_{k=1}^{n-2} P_k^{n-1} b_c(k+1) + (1-c)P_{n-1}^{n-1} b_c(n) + c \sum_{k=2}^{n-1} P_k^{n-1} b_c^*(k+1) + cP_1^{n-1} b_c^*(2) \\ &= n + (1-c) \sum_{k=1}^{n-2} P_k^{n-1} b_c(k+1) + (1-c)^n b_c(n) + c \sum_{k=2}^{n-1} P_k^{n-1} [b_c(k) + 1] + 2(n-1)(1-c)c^{n-1} \\ &= n + (1-c)^n b_c(n) + 2(n-1)(1-c)c^{n-1} + (1-c) \sum_{k=2}^{n-1} P_{k-1}^{n-1} b_c(k) + c \sum_{k=2}^{n-1} P_k^{n-1} b_c(k) + c \sum_{k=2}^{n-1} P_k^{n-1} \\ &= n + (1-c)^n b_c(n) + 2(n-1)(1-c)c^{n-1} + c - c^n - (n-1)(1-c)c^{n-1} + \\ &\quad \sum_{k=2}^{n-1} \left[ (1-c)P_{k-1}^{n-1} + cP_k^{n-1} \right] b_c(k) \\ &= n + (1-c)^n b_c(n) + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1} P_k^n b_c(k). \end{aligned}$$

Rewriting yields, for any  $n \geq 2$ ,

$$\left[ 1 - (1-c)^n \right] b_c(n) = n + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} b_c(k). \quad (4)$$

Now notice that since in the first round player  $i_1$  says *yes* with certainty, the expected number of queries in the auction of  $n$  players is equal to  $b_c(n)$ . Thus using (4) we can compute the expected number of queries performed in the auction with  $n$  players. Plugging in  $n = 2$  yields  $b_c(2) = \frac{2+2c(1-c)}{c(2-c)}$ . All other values can be determined recursively. Table 2 presents the computational results for different values of  $c$  in the auction with up to 100 players (data is within an accuracy of 0.001). Again, the  $c$ -fraction auction

| $n \setminus c$ | 1/10    | 1/8     | 1/6     | 1/4     | 1/3     | 1/2     | 2/3     | 3/4     | 5/6     | 7/8     | 9/10    |
|-----------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 2               | 11.474  | 9.467   | 7.455   | 5.429   | 4.400   | 3.333   | 2.750   | 2.533   | 2.343   | 2.254   | 2.202   |
| 3               | 21.790  | 17.779  | 13.759  | 9.718   | 7.674   | 5.571   | 4.442   | 4.029   | 3.666   | 3.496   | 3.396   |
| 4               | 32.027  | 26.013  | 19.988  | 13.935  | 10.879  | 7.752   | 6.094   | 5.495   | 4.972   | 4.727   | 4.582   |
| 5               | 42.217  | 34.200  | 26.171  | 18.109  | 14.044  | 9.897   | 7.717   | 6.938   | 6.264   | 5.949   | 5.762   |
| 6               | 52.375  | 42.356  | 32.323  | 22.254  | 17.181  | 12.017  | 9.320   | 8.365   | 7.545   | 7.162   | 6.936   |
| 7               | 62.511  | 50.490  | 38.454  | 26.378  | 20.298  | 14.120  | 10.908  | 9.778   | 8.815   | 8.368   | 8.104   |
| 8               | 72.630  | 58.607  | 44.568  | 30.487  | 23.401  | 16.211  | 12.484  | 11.180  | 10.078  | 9.568   | 9.268   |
| 9               | 82.735  | 66.711  | 50.669  | 34.583  | 26.492  | 18.291  | 14.051  | 12.575  | 11.333  | 10.763  | 10.427  |
| 10              | 92.830  | 74.804  | 56.761  | 38.670  | 29.575  | 20.363  | 15.611  | 13.962  | 12.582  | 11.952  | 11.582  |
| 20              | 193.465 | 155.430 | 117.372 | 79.251  | 60.124  | 40.845  | 31.016  | 27.653  | 24.893  | 23.679  | 22.985  |
| 30              | 293.842 | 235.802 | 177.735 | 119.597 | 90.451  | 61.132  | 46.258  | 41.203  | 37.070  | 35.264  | 34.248  |
| 40              | 394.111 | 316.068 | 237.995 | 159.843 | 120.684 | 81.336  | 61.430  | 54.691  | 49.201  | 46.802  | 45.457  |
| 50              | 494.320 | 396.274 | 298.196 | 200.035 | 150.865 | 101.495 | 76.563  | 68.144  | 61.307  | 58.319  | 56.644  |
| 60              | 594.492 | 476.443 | 358.361 | 240.192 | 181.014 | 121.626 | 91.673  | 81.574  | 73.394  | 69.825  | 67.820  |
| 70              | 694.637 | 556.587 | 418.501 | 280.325 | 211.140 | 141.736 | 106.766 | 94.989  | 85.468  | 81.320  | 78.989  |
| 80              | 794.763 | 636.711 | 478.622 | 320.440 | 241.249 | 161.832 | 121.847 | 108.394 | 97.531  | 92.809  | 90.152  |
| 90              | 894.874 | 716.820 | 538.729 | 360.542 | 271.345 | 181.916 | 136.918 | 121.790 | 109.586 | 104.290 | 101.311 |
| 100             | 994.973 | 796.918 | 598.825 | 400.633 | 301.431 | 201.992 | 151.981 | 135.180 | 121.634 | 115.766 | 112.466 |

Table 2: The expected number of queries  $b_c(n)$  in the  $c$ -fraction auction.

Figure 2: The expected number of queries (a) for different fixed values of  $c$ ; (b) for different fixed numbers of players.

needs only very few queries to allocate the object. For instance, with 100 bidders and  $c = 1/2$ , the auction needs less than 202 queries on average, surprisingly little if one realizes that 100 is the absolute minimum with 100 bidders.

Figure 2(a) demonstrates that for a fixed value of  $c$  the expected number of queries increases in the number of players participating in the auction. Figure 2(b) shows that for a fixed number of players the expected number of queries decreases as  $c$  becomes larger.

Generally we show that the expected number of queries is bounded from above by a function that is linear in the number of players. To prove this we introduce several notations and lemmas.

Define  $B_2 = \frac{2+2c(1-c)}{c(2-c)}$  and

$$B_n = n + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} B_k$$

for any  $n > 2$ .

Recall that  $D_n = \prod_{k=1}^n \frac{1}{1-(1-c)^k}$ .

**Lemma 5.5** For any  $n \geq 2$ ,  $b_c(n) \leq B_n \cdot D_n$ .

**Proof.** The proof is identical to the proof of Lemma 5.1 if we replace  $e_c(k)$  by  $b_c(k)$  and  $E_k$  by  $B_k$  for all  $2 \leq k \leq n$ . ■

From Lemma 5.2 we know that for any  $n \geq 2$ ,  $D_n \leq e^{\frac{1-c}{c^2}}$ . Now we find a bound on  $B_n$ .

**Lemma 5.6** *For any  $n \geq 2$ ,  $B_n \leq \left(\frac{2}{c} + \frac{1}{2}\right)(n+1)$ .*

**Proof.** The proof is by induction on  $n$ . The basis of the induction holds since it can be easily shown that  $B_2 < 3\left(\frac{2}{c} + \frac{1}{2}\right)$ . Now suppose that  $B_k \leq \left(\frac{2}{c} + \frac{1}{2}\right)(k+1)$  for any  $2 \leq k \leq n-1$ . Using the induction hypothesis,

$$\begin{aligned}
B_n &= n + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} B_k \\
&\leq n + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} \left(\frac{2}{c} + \frac{1}{2}\right)(k+1) \\
&\leq 2n + c + \left(\frac{2}{c} + \frac{1}{2}\right) \sum_{k=0}^n \binom{n}{k} (1-c)^k c^{n-k} k + \left(\frac{2}{c} + \frac{1}{2}\right) \sum_{k=0}^n \binom{n}{k} (1-c)^k c^{n-k} \\
&= 2n + c + \left(\frac{2}{c} + \frac{1}{2}\right) (1-c)n + \left(\frac{2}{c} + \frac{1}{2}\right) \\
&= \left(\frac{2}{c} + \frac{1}{2}\right) (n+1) + c \left(1 - \frac{n}{2}\right) \\
&\leq \left(\frac{2}{c} + \frac{1}{2}\right) (n+1).
\end{aligned}$$

The last inequality holds since  $n \geq 2$ . ■

A final immediate consequence of Lemmas 5.2, 5.5 and 5.6 is the following theorem.

**Theorem 5.7** *For any integer  $n \geq 2$ ,  $b_c(n) \leq e^{\frac{1-c}{c^2}} \left(\frac{2}{c} + \frac{1}{2}\right)(n+1)$ .*

We showed that the expected number of queries is bounded from above by a function that is linear in the number of players. Again, a comparison of the bound with the computed results suggests that this bound is not tight. It can be easily checked that for a fixed value of  $c$  the ratio between the bound and the computed result is approximately constant (as a function of  $n$ ), implying that the bound is likely to have the correct order of magnitude.

## 6 Efficiency of the auction

In this section we investigate the efficiency of the  $c$ -fraction auction when the bluff equilibrium is played. In particular, in the following two subsections, we compute the probability of inefficient allocation and the expected loss of welfare.

Figure 3: The probability of inefficient allocation.

### 6.1 The probability of inefficient allocation

We derive a recursive formula for the probability of inefficient allocation and give an upper bound for this probability using the recursive formula. We denote by  $P_c(n)$  the probability that the auction with  $n$  players terminates in an inefficient allocation.

A first important observation is that the  $c$ -fraction auction with  $k$  players having valuations drawn from  $F$  conditional on these valuations being greater than or equal to  $F^{-1}(c)$  has exactly the same structure as the original  $c$ -fraction auction with  $k$  players having valuations drawn from  $F$ . For both cases, the probability of an inefficient allocation under the bluff equilibrium is the same.

Consider the case where the valuation of all players is smaller than  $F^{-1}(c)$ . The probability of this event is  $c^n$ . In this case player  $i_1$  is the only player saying *yes*, and therefore receives the object. The auction is only efficient if the player with the lowest ranking has the highest valuation, which happens with probability  $1/n$ . Thus this case contributes  $(n-1)/nc^n$  to  $P_c(n)$ . Next consider the case where  $k \geq 1$  players have valuations larger than or equal to  $F^{-1}(c)$  and  $n-k$  players have valuations smaller than  $F^{-1}(c)$ , which happens with probability  $\binom{n}{k}c^{n-k}(1-c)^k$ . For  $k=1$  the auction is efficient, so this case adds zero to  $P_c(n)$ . Consider the case where  $k > 1$ . Either player  $i_1$  has a value below  $F^{-1}(c)$ , responds *no* in round 2, and inefficiency among the remaining  $k$  bidders takes place with probability  $P_c(k)$ . Or player  $i_1$  has a value greater than or equal to  $F^{-1}(c)$ , in which case the auction starts in round 2 with  $k$  bidders having a value greater than or equal to  $F^{-1}(c)$  and inefficiency takes place with probability  $P_c(k)$ . We find that

$$P_c(n) = \frac{n-1}{n}c^n + \sum_{k=2}^n \binom{n}{k}c^{n-k}(1-c)^kP_c(k).$$

A direct evaluation of the recursive formula yields that  $P_c(2) = \frac{1}{2} \cdot \frac{c}{2-c}$ . For  $n > 2$  rewriting leads to

$$\left[1 - (1-c)^n\right]P_c(n) = \frac{n-1}{n}c^n + \sum_{k=2}^{n-1} \binom{n}{k}c^{n-k}(1-c)^kP_c(k). \quad (5)$$

Direct computation of this expression for different combinations of  $n$  and  $c$  gives the values that are plotted in Figure 3.

The recursive formula can also be used to derive the following upper bound on  $P_c(n)$ .

**Theorem 6.1** For all  $n \in \mathbb{N}$ ,  $P_c(n) < c$ .

**Proof.** The proof is by induction on  $n$ . The basis of the induction holds since  $P_c(1) = 0$  and  $P_c(2) = \frac{1}{2} \cdot \frac{c}{2-c} < c$ . Suppose that  $n \geq 3$  and  $P_c(k) < c$  for all  $1 \leq k \leq n-1$ . Then

$$\begin{aligned}
P_c(n) &= \frac{1}{1 - (1-c)^n} \left[ \frac{n-1}{n} \cdot c^n + \sum_{k=2}^{n-1} \binom{n}{k} c^{n-k} (1-c)^k \cdot P_c(k) \right] \\
&< \frac{1}{1 - (1-c)^n} \left[ c^n + c \sum_{k=2}^{n-1} \binom{n}{k} c^{n-k} (1-c)^k \right] \\
&= \frac{1}{1 - (1-c)^n} \left[ c^n + c \left( 1 - c^n - n(1-c)c^{n-1} - (1-c)^n \right) \right] \\
&= \frac{c(1 - (1-c)^n)}{1 - (1-c)^n} + \frac{c^n - c^{n+1} - n(1-c)c^n}{1 - (1-c)^n} \\
&= c + \frac{c^n(1-c)(1-n)}{1 - (1-c)^n} \\
&< c.
\end{aligned}$$

The first inequality holds by the induction assumption and the fact that  $\frac{n-1}{n} < 1$ . The last inequality holds since  $n \geq 3$ . ■

This theorem shows in particular that by choosing an appropriate fraction  $c$  in the auction we can make the probability of inefficiency as small as we like, *independent of the number of players!* Also Figure 3 shows that the probability of inefficient allocation is quite independent from the number of players.

When we assume  $c < \frac{1}{2}$  it can be shown by the same chain of arguments that  $P_c(n) \leq \frac{1}{2}c$  for all  $n \in \mathbb{N}$ .

## 6.2 The expected loss of welfare

The welfare of an auction is equal to the valuation of the winner of the auction. The maximum welfare, given  $v$ , is  $\max\{v_i \mid i \in N\}$ . The expected loss of welfare, denoted by  $L_c(n)$ , is the expected value of the difference between the maximum welfare and the valuation of the winner. To estimate the value of  $L_c(n)$  we simulate the  $c$ -fraction auction and ran it for valuations uniformly and independently drawn from the interval  $[0, 1)$ . For each combination of value  $c$  and number of players  $n$  we ran 10000 trials. Figure 4 shows the 99% confidence interval for the expected loss of welfare. It is interesting to notice that the maximum expected loss does not occur for the minimum number of players.

For a distribution function  $F$ , we define  $\gamma(c) \in \mathbb{R}_+$  by

$$\gamma(c) = \sup_{r \in N} F^{-1}(1 - (1-c)^r) - F^{-1}(1 - (1-c)^{r-1}),$$

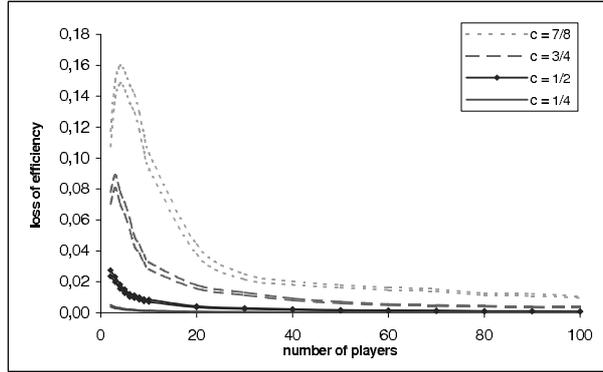


Figure 4: The expected loss of welfare, 99% confidence interval.

so  $\gamma(c)$  measures the maximal difference between the query price  $q_r$  and the payment  $p_r$  that can occur in an auction. We restrict ourselves to distributions for which  $\gamma(c)$  is finite, thereby excluding distributions with fat tails. It is easily verified that for the uniform distribution  $\gamma(c)$  is equal to  $c$  and for the exponential distribution with parameter  $\lambda$  we have  $\gamma(c) = -(\ln(1 - c))/\lambda$ .

**Theorem 6.2** For all  $n \in \mathbb{N}$ ,  $L_c(n) < \gamma(c)c$ .

**Proof.** Let  $r$  be the round in which the winner of the auction is found. The winner said *yes* in round  $r$ , so has a valuation at least equal to  $p_r$ . Since all other players say *no* in round  $r$  or before, they have a valuation strictly less than  $q_r$ . The welfare loss is therefore bounded above by  $q_r - p_r$ , and therefore by  $\gamma(c)$ . Hence,  $L_c(n) \leq \gamma(c) \cdot P_c(n)$ . Applying the result of Theorem 6.1 completes the proof. ■

Many distributions have the feature that  $\lim_{c \downarrow 0} \gamma(c) = 0$ , for instance the exponential distribution with parameter  $\lambda$ , and any distribution with compact support like for instance the uniform distribution. For such distribution, by choosing an appropriate fraction  $c$  in the auction we can limit the expected loss of welfare to an arbitrary chosen level, *independent of the number of players*.

### 6.3 Trade-off between efficiency and running time

From the analysis in this and the previous section we derive the following relation between the value of  $c$ , the level of efficiency and the running time. For a fixed number of players, a smaller fraction  $c$  leads to a lower expected loss of welfare and lower probability of inefficient allocation. But at the same time it leads to a higher expected number of rounds and queries. Thus, increasing running time is a price that we have to pay for increasing the level of welfare. Depending on the priorities of the auctioneer he may trade off welfare against running time.

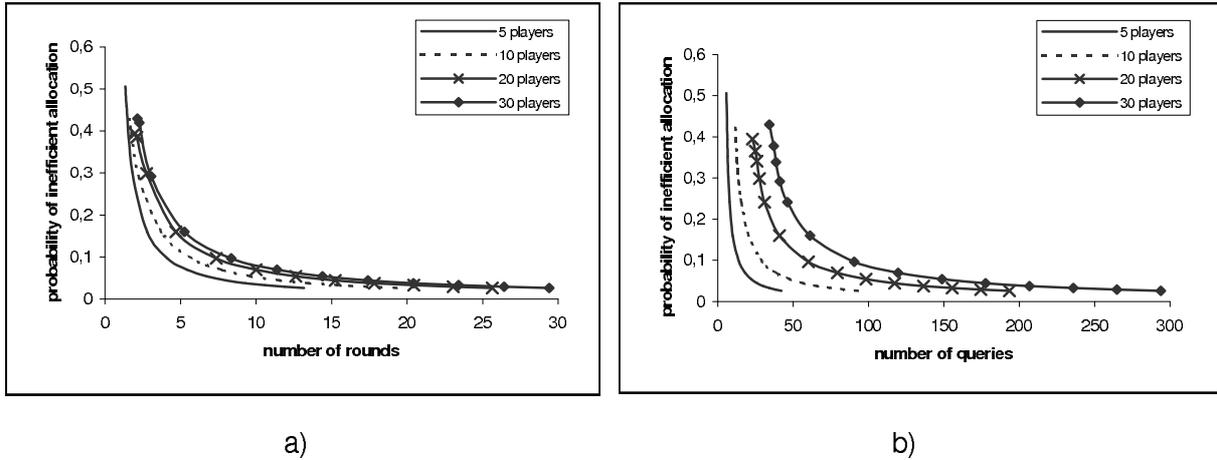


Figure 5: The trade-off between (a) the probability of inefficient allocation and the number of rounds; (b) the probability of inefficient allocation and the number of queries.

Figure 5 shows, for several fixed  $n$ , the relation between the expected running time and the probability of inefficient allocation. These relations are built on computational results based on recursive formulas 2, 4 and 5. Figure 5(a) shows for every  $n = 5, 10, 20$  and  $30$  a part of the curve  $\{(e_c(n), P_c(n) | c \in [0, 1])\}$ . In accordance with our bounds it shows that for fixed probability of inefficiency (vertical axes), we need a number of rounds (horizontal axis) that is logarithmic increasing in the number of bidders, using each time roughly the same  $c$ . Figure 5(b) shows for every  $n = 5, 10, 20$  and  $30$  a part of the curve  $\{(b_c(n), P_c(n) | c \in [0, 1])\}$ . For fixed probability of inefficiency, the number of rounds is now increasing linearly in the number of bidders.

## 7 Concluding remarks

We have shown that  $c$ -fraction auctions provide an easy way of trading-off efficiency versus running time of a single item auction. Bluff strategies form an ex-post equilibrium of these auctions. David et al. [9] have proposed a slightly different ascending price query auction. Truthful reports to the queries form an ex-post equilibrium in that auction. We expect that choosing increments in their auction in the same way as they are chosen in the  $c$ -fraction auction provides similar bounds on the number of rounds, number of queries, and efficiency losses. Setting increments dynamically according to the  $c$ -fraction rule is thus an easy to implement method that leads to auctions that dominate rules-of-thumb approaches like fixed increments or fixed-percentage increments, as for example described in McAfee et al. [16], in all relevant dimensions.

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