Nonparametric consumer and producer analysis

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providing details and we will investigate your claim.

1. The first step is to identify the variables of interest and the population of interest.

NONPARAMETRIC CONSUMER AND PRODUCER ANALYSIS

2. The second step is to choose a nonparametric test that is appropriate for the data and the research question.

3. The third step is to apply the test to the data and interpret the results.

4. The fourth step is to draw conclusions based on the results of the test.

5. The fifth step is to report the results of the analysis in a clear and concise manner.

6. The sixth step is to discuss the limitations of the analysis and the implications of the findings.

7. The seventh step is to provide recommendations for future research.

8. The eighth step is to conclude the analysis.

9. The ninth step is to provide a summary of the findings.

10. The tenth step is to provide a final conclusion.

11. The eleventh step is to provide a final summary.

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*TO MY FAMILY
AND PARENTS*

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CONTENTS

Frequently Used Notation	xiii
Common Notation	xiii
Specific Notation	xiv
Introduction	1
Contents	2
PART I CONSUMER DEMAND	
1 Introduction	4
1.1 Neoclassical Theory	4
1.2 Basis for a Nonparametric Consumer Theory	5
1.3 Differences with Standard Theory	6
1.4 Application of the Nonparametric Theory	7
1.5 Contents of Part I	7
2 Consumer Demand	9
2.1 Introduction	9
2.2 Consumer Demand	9
2.3 Rationalizing Preference Orders and Utility Functions	11
2.4 Data Enclosure	12
2.5 Efficiency Transformations	12
2.6 Efficiency Upper Bounds	15
2.7 Existence of a Rationalizing Preference Order	15
2.8 The Revealed Preference Axiom and Dutch Consumer Data	18
2.9 Existence of a Rationalizing Utility Function	19
3 Linearly Homogeneous Consumer Demand	24
3.1 Introduction	24
3.2 Linearly Homogeneous Utility Functions	24
3.3 Existence of a Rationalizing Linearly Homogeneous Function	26
3.4 The Existence Problem in Case of Inefficiency	28
3.5 A Linearly Homogeneous Function for Dutch Consumer Data	31
3.6 Restrictions on Linearly Homogeneous Utility Functions	31
3.7 Construction of Nonparametric Bounds	32
3.8 Nonparametric Bounds for Dutch Consumer Data	35
3.9 Approximating Nonparametric Bounds for Concave Functions	39
3.10 Rationalizing Nonparametric Bounds	40
4 Weakly Separable Consumer Demand	42
4.1 Introduction	42
4.2 Weakly Separable Utility Functions	42
4.3 Existence of a Rationalizing Aggregator Function	44
4.4 Existence of a Rationalizing Weakly Separable Function	45
4.5 The Assumption of Strictly Increasing Cost Functions	48
5 Some Nonparametric Theory	49
5.1 Introduction	49
5.2 Closed Mappings	49
5.3 Approximately Finite Data	51
5.4 Examples of Approximately Finite Data	53
5.5 Utility Maximization and Cost Minimization	54
5.6 Utility Functions and Convergence	55
5.7 Existence of a Rationalizing Concave Function	56

6 Nonparametric Analysis of Aggregate Data	58
6.1 Introduction	58
6.2 Empirical Results	58
6.3 Explanations for the Empirical Results	59
6.4 The Use of Price Index Numbers	59
6.5 Results for Aggregate Dutch Consumption Data	60
6.6 Results for Aggregate Random Data	62
6.7 Evaluation and Some More Results	63
6.8 Summary	65

PART II PRODUCER DEMAND

7 Introduction	66
7.1 Theory of Producer Behaviour	66
7.2 Producer Demand	66
7.3 Technical Inefficiency and Frontier Production Functions	66
7.4 Technical Change	67
7.5 Rationalizing Production Functions	67
7.6 Application to Empirical Data	68
7.7 Contents of Part II	68

8 Producer Demand	69
8.1 Introduction	69
8.2 Producer Demand	69
8.3 Frontier Production Functions	70
8.4 Technical Progress	72
8.5 Hicks Neutral Change	74
8.6 Neutral Change	78
8.7 Efficiency Transformations for Neutral Change	81

9 Linearly Homogeneous Producer Demand	83
9.1 Introduction	83
9.2 Conditions for Linearly Homogeneous Demand	84
9.3 Upper Bounds and Producer Demand	85
9.4 Frontier Production Functions	86
9.5 Technical Progress	89
9.6 Technical Progress for Frontier Production Functions	90
9.7 Hicks Neutral Change	91
9.8 Neutral Change	94

10 Homothetic Producer Demand	99
10.1 Introduction	99
10.2 Existence of a Homothetic Production Function	99
10.3 Restrictions on the Underlying Linearly Homogeneous Function	101

11 Weakly Separable Producer Demand	104
11.1 Introduction	104
11.2 Existence of an Aggregator Function	104
11.3 Existence of a Weakly Separable Function	105
11.4 Existence of a Linearly Homogeneous Subfunction	106
11.5 Existence of a Linearly Homogeneous Weakly Separable Function	108

PART III PROFIT MAXIMIZATION

12 Introduction	112
12.1 Textbook Theory	112
12.2 Profit Maximization	113
12.3 Contents of Part III	114

13 Profit maximization	115
13.1 Introduction	115
13.2 Conditions for Profit Maximization	115
13.3 Bounds for Profit Maximization	117
13.4 Frontier Production Functions	119
13.5 Technical Progress	120
13.6 Hicks Neutral Change	121
13.7 Neutral Change	124
14 Homothetic Profit Maximization	129
14.1 Introduction	129
14.2 Existence of a Homothetic Production Function	129
15 Weakly Separable Profit Maximization	132
15.1 Introduction	132
15.2 An Aggregator Function	132
15.2 Existence of a Weakly Separable Function	133
PART IV CAPITAL	
16 Introduction	136
16.1 Static or Dynamic Producer Behaviour	136
16.2 The Dynamic Optimization Problem	136
16.3 Survival and Retirement of Investment	136
16.4 Capital Stocks	138
16.5 Contents of Part IV	140
17 User Cost of Capital	141
17.1 Introduction	141
17.2 A General User Cost of Capital Solution	141
17.3 User Cost of Capital for Survival Functions	143
17.4 User Cost of Capital for Retirement Distributions	144
17.5 Relation with Measure Theory	145
17.6 A Generalization	147
17.7 Summary and Conclusions	148
18 Exponential Discounted Investment Price	150
18.1 Introduction	150
18.2 Solution for the Exponential Price	150
18.3 Critique on the User Cost of Capital in Empirical Models	151
18.4 Other Survival Functions	152
18.5 Summary	154
19 Generation of Capital Stocks	156
19.1 Introduction	156
19.2 Contagious Erlang Distributions	157
19.3 State-Space Realization for a Contagious Erlang Distribution	159
19.5 The Discrete-Time Model	160
Summary	162
Generalizations of the Nonparametric Theory	163
Applications of the Nonparametric Theory	163

APPENDIX

A Binary Relations	165
A.1 Binary Relations	165
A.2 Basic Properties of Binary Relations	165
A.3 Particular Binary Relations	165
A.4 Extensions of Orders	166
A.5 Extensions of Binary Relations	167
B Graph Theory	169
B.1 Introduction	169
B.2 Graphs	169
B.3 Subgraphs	170
B.4 Paths in a Graph	170
B.5 Strongly Connected Vertices	170
B.6 Absorption of a Vertex	171
B.7 Feedback Vertex Sets	171
C Path Algebra's and Labelled Graphs	173
C.1 Introduction	173
C.2 Path Algebra	173
C.3 Examples of Path Algebra's	174
C.4 Labelled Graphs	174
C.5 The Weak Closure of an Adjacency Matrix	175
C.6 Theorems for Labelled Graphs	176
D Laplace Transforms	180
E Description of the Dutch Industry Data	181
E.1 Prices and Quantities	181
E.2 Capital Stocks	182
E.3 Sensitivity to the Choice of Initial Values	183
E.4 User Cost of Capital	183
F Results of Producer Tests	186
G Overview of Nonparametric Tests	188
G.1 Consumer Demand	188
G.2 Producer Demand	189
G.3 Profit Maximization	191
References	193
Related Literature	197
Index	201
Nederlandse Samenvatting / Summary in Dutch	206
Generalisatie van de niet-parametrische theorie	207
Toepassingen van de niet-parametrische theorie	207
Curriculum Vitae	209

FREQUENTLY USED NOTATION

Common Notation

The notation of, for instance,

$$f(x) = c \quad (x \in X)$$

means $f(x) = c$ for all $x \in X$.

For preorders Herstein and Milnor's (1953) notation is used. In that case a preorder is denoted as $\succsim$, using the notations:

$$x \sim y \Leftrightarrow x \succsim y \text{ and } y \succsim x,$$

$$x \succ y \Leftrightarrow x \succsim y \text{ and not } y \succsim x,$$

$$x \prec y \Leftrightarrow y \succ x,$$

$$x \sim y \Leftrightarrow y \sim x.$$

These relations have usually the following interpretation:

$$x \succsim y: y \text{ is not preferred to } x,$$

$$x \succ y: x \text{ is preferred to } y,$$

$$x \sim y: x \text{ is indifferent to } y.$$

In the text such a preorder $\succsim$ is denoted as $\succsim, \sim$ for typographical convenience.

Next, the following common notation is used.

A, B etc. operator

A, B etc. matrix, set, binary relation

$\mathbf{0}$ vector of zero's

$\emptyset$ empty set

$\approx$ approximately equal

$\equiv$ is equal by definition

$*$ convolution:

$$(f * g)(t) \equiv \int f(t - \tau)g(\tau)d\tau$$

$\circ$ function of a function: $f \circ g(x) \equiv f(g(x))$

$\mathbb{R}$ real numbers

$\mathbb{R}_+$ nonnegative real numbers

$\mathbb{R}_{++}$ positive real numbers

$\mathbb{C}$ complex numbers

$\mathbb{N}$ positive integers

$\mathbb{Z}$ integers

P^c complement of a set P

R^* transitive closure when R is a binary relation

C^* shortest path matrix when C is a cost matrix for paths

$\mathcal{L}$ Laplace transformation:

$$\mathcal{L}[f] \equiv \int f(\tau)e^{s\tau}d\tau$$

$\tilde{g}$ Laplace transform of g : $\tilde{g} \equiv \mathcal{L}[g]$

$\check{g}$ transformation of function g such that: $\check{g}(t) \equiv g(-t)$

$|x|$ absolute value of x

$\|x\|$ Euclidean length of vector x

$\dot{x}$ derivative

$$\dot{x}(t) \equiv \frac{d}{dt} x(t)$$

$\langle x, y \rangle$ innerproduct of x and y

A^t transposed matrix A

A^t adjoint operator of operator A , which satisfies: $\langle x, Ay \rangle = \langle A^t x, y \rangle$

$\delta(\cdot)$ Delta function:

$$\int f(\tau) \delta(\tau) d\tau \equiv f(0)$$

$1(\cdot)$ step function:

$$1(t) \equiv \begin{cases} 0 & (t < 0), \\ 1 & (t \geq 0). \end{cases}$$

Specific Notation

Concerning sets the following specific notation is used:

$\mathcal{R}$ set of preference relations

$\mathcal{F}$ set of functions

$\mathcal{D}$ set of data

To denote that we consider consistency with a certain type of economic behaviour, the lower indices c , u , d , and p are used:

$\mathcal{F}_c$ the set of functions consistent with cost minimization

$\mathcal{F}_u$ the set of functions consistent with utility maximization

$\mathcal{F}_d$ the set of functions consistent with demand

$\mathcal{F}_p$ the set of functions consistent with profit maximization

Concerning $\mathcal{D}$ and $\mathcal{R}$ these indices have an analogous meaning. Furthermore, the indices are used in combination with the following upper indices:

$\mathcal{F}^o$ set of functions for data that includes observations of output

$\mathcal{F}^f$ set of frontier production functions

Special functions, that have as argument a set F of functions $f: X \rightarrow \mathbb{R}$, are:

$$U(F, x) \equiv \sup_{f \in F} f(x) \quad \text{upper bound}$$

$$U_r(F, x, x') \equiv \sup_{f \in F} f(x)/f(x') \quad \text{relative upper bound}$$

$$U_-(F, x, x') \equiv \sup_{f \in F} f(x) - f(x') \quad \text{difference upper bound}$$

$$L(F, x) \equiv \inf_{f \in F} f(x) \quad \text{lower bound}$$

$$L_r(F, x, x') \equiv \inf_{f \in F} f(x)/f(x') \quad \text{relative lower bound}$$

$$L_-(F, x, x') \equiv \inf_{f \in F} f(x) - f(x') \quad \text{difference lower bound}$$

INTRODUCTION

This book contains both a theoretical and empirical research of nonparametric consumer and producer theory. The emphasis lies on the theoretical development of a nonparametric theory, while the empirical research is meant as an example of application of the developed theory to real data.

The starting-point of the nonparametric theory is a model of optimizing behaviour that satisfies a certain symmetry. This idea has applications in different fields of science. One is the field of nonparametric statistics, where the symmetry assumption is applied to probability distributions. Another field where results are derived by using symmetry assumptions is physics. The role of symmetry is very important in classical mechanics and quantum physics. In classical mechanics the Noether theorem is used to derive conservation laws in case the dynamical system has a symmetric property. For example, the conservation of energy, momentum, and angular momentum follows from the symmetry assumption that the Lagrangian is invariant under time translation, space translation, and rotation respectively. In quantum field theory the assumption that the dynamics are invariant for Poincaré transformations is a starting-point to derive important theoretical results. It is clear that this fruitful nonparametric approach in physics may become important in economics. In this book such an approach is not applied to dynamical optimization problems, but to series of statical optimization.

The usual approach to analyse a certain specified type of economic behaviour is the specification of a production or utility function after which the corresponding economic behaviour is analysed. This book stresses the importance of a reverse approach that is especially suitable for empirical applications. This reverse approach considers data of observed behaviour and analyses properties of the corresponding set of possible utility or production functions. This approach relies on the assumption of a symmetry in the optimization model.

When we consider data that might be generated by statical optimization, the following questions are appropriate for the set of possible utility or production functions, which correspond with such an optimization model:

1. Is there such a function?
2. What may be the properties of such a function?
3. Are there restrictions on such functions?

The subject of this research is divided in consumer demand, producer demand and profit maximization. Concerning these kinds of optimization behaviour, the existence of a suitable optimized function is considered. Furthermore, several properties of these functions are considered: linear homogeneity, homotheticity and weak separability. Besides utility functions, the theory of consumer demand also treats preference orders.

A fourth subject of research is capital. Using capital in production functions is easy in textbook theory, but a real problem in empirical studies. When the nonparametric producer theory was ready to be applied to data, I discovered – after solving the problem of lacking price data – that I missed prices and quantities for a simple variable in the nonparametric theory: capital. When I tried to construct these prices and corresponding quantities, I was surprised by the usual rough approach to this problem in empirical production models. In these models investments are nearly always

exponentially decaying. This hypothesis is of course not based on empirical evidence, but on easy solutions for the capital problem. Exponential decay allows using a simple dynamical model for the capital stock. For this model the user cost of capital can easily be solved by using the maximum principle of Pontryagin (1962). Except for the lack of a general theory for the user cost of capital, using unreliable capital data may be a reason for this attitude. In case of unreliable data an error more or less does not matter. Another problem is the question whether producers are behaving as rational optimizers.

I, used to fairly exact measurements in physical experiments, was at first not aware of these problems in economics. So I found it necessary to develop a rigorous theory, suitable for reliable capital data and rational optimizing producers. The basic ideas that are used in this book to derive a user cost of capital theory are inspired by the theory of quantum physics and they may be new in the context of a capital theory.

Contents

The treatment of consumer demand in Part I is based on a definition of consumer behaviour. This definition differs in several ways from the neo-classical textbook definition of utility maximization. In the neoclassical theory one often uses restrictions on the utility function to assure that a utility maximizer is a cost minimizer. Here I assume *a priori* that a consumer is both a utility maximizer and a cost minimizer.

Another difference with the standard theory is the allowance of economical inefficiency as is also proposed by Afriat (1973). The idea of Afriat is generalized by using efficiency transformations. The idea behind such a transformation is simple. Efficiency transformations are used to transform data into new data with the aim to relax the conditions for consumer demand. Such transformations are dependent on an efficiency parameter that indicates how seriously we have deformed the data.

Furthermore, I do not use prices, but functions to describe the cost of a quantity vector. This allows for the application of the theory to noncompetitive markets. Using nonlinear budget constraints in revealed preference theory is not new. Richter (1966) used the concept of budget sets as a generalization of the usual linear budget constraint.

In Part II the idea of consumer demand is applied to producer demand, where the utility function is replaced by a production function. In contrast to consumer demand, a producer demand theory includes the observation of produced output. So, the theory of producer demand can be conceived as a consumer demand theory in which output may be observed. Moreover, a producer demand theory may consider frontier production functions, technical progress and technical inefficiency.

Part III applies the ideas that are used in Part I and Part II, like efficiency transformations and using cost functions instead of prices, to profit maximization.

Part IV treats the essential problem of capital: the dynamical relation of investments and capital and the corresponding dynamical optimization problem for a producer. The point of view in this part of the book differs from the other parts. While Part I, II and III studies mainly the properties of data, derived from consumer and producer behaviour, the main subject in Part IV, however, is the development of a behaviour model itself. This part studies how one may derive prices and quantities for capital, given a dynamic capital investment relation.

The user cost of capital is analysed, using the conventional assumption that investment costs are linear. There is a connection with the nonparametric approach, because the theory does not depart from a parametric specified capital investment relation. This relation is specified by using a few general assumptions.

Another main subject in Part IV is the generation of capital stocks. To generate these stocks a simple parametric model is proposed. This model is derived from the statistical properties for the lifetime of investment. It is argued that the parametric model is flexible enough to fit with empirical survival functions.

PART I

CONSUMER DEMAND

1 INTRODUCTION

1.1 Neoclassical Theory

The neoclassical theory postulates that a consumer maximizes his utility subject to a budget constraint. This is formalized as follows. Let $f(\cdot)$ be the consumer's utility function, representing his preferences, and p the vector of prevailing prices. The choice behaviour of a *utility maximizing* consumer is now

$$f(x') = \sup_{px \leq px'} f(x).$$

A simple example of this form of consumer behaviour, which is usually presented in text books, is illustrated in Figure 1.1.

Figure 1.1 Consumer Demand

The assumption of utility maximization, however, does not guarantee that the consumer spends his budget without a waste of money. A utility maximizing consumer may waste money when there are 'thick' indifference curves, as is illustrated in Figure 1.2. A *cost minimizing* consumer, who does not waste money, displays the behaviour

$$px' = \inf_{f(x) \geq f(x')} px.$$

To exclude 'thick' indifference curves, one often assumes that consumers utility function f is a locally nonsatiated function. For a utility maximizing consumer this assumption assures that he spends his entire budget px' , while being a cost minimizer.

Definition: Suppose $f: X \rightarrow \mathbb{R}$ and $x' \in X$. Then f is said to be *locally nonsatiated at x'* if for every open neighbourhood E of x' there is a point $x \in E$ such that $f(x) > f(x')$. If f is locally nonsatiated at every point of X , then f is said to be *locally nonsatiated*.

Figure 1.2 Thick Indifference Curve

We can generalize the above theory by replacement of the utility function by a preorder. The choice behaviour for such a preorder, called a *preference order*, is derived in the same way as for a utility function. For instance, the inequality $f(x') \geq f(x)$, for a utility function f , means that the choice x is not preferred to x' . Similarly, for a preorder $\succsim$, this is the case when one has $x' \succsim x$.

1.2 Basis for a Nonparametric Consumer Theory

This part is concerned with the nonparametric analysis of consumer behaviour. The idea of such a nonparametric analysis is as follows. Suppose that one observes data of consumer behaviour. Then one may postulate that this data is generated by a rational consumer, using a preference order or utility function. Furthermore, one may postulate the hypothesis that this preference order or utility function is member of a certain nonparametric described family of preorders or functions respectively. Examples of such nonparametric families are families of functions that satisfy a certain *symmetry*, i.e. they are *invariant* under certain transformations. For example, a widely used nonparametric family of functions is the family of linearly homogeneous functions. This family can be described as the family of functions $f(x)$ that is invariant under transformations T_λ of the form

$$(T_\lambda f)(x) = \lambda^{-1} f(\lambda x) \quad (\lambda \neq 0).$$

Another family is the family of weakly separable functions $f(x)$ of the form

$f(x_1, x_2) = g(x_1, h(x_2))$, where g is increasing in h . Such a function is invariant under the transformation T_h of the form

$$(T_h f)(x_1, x_2) = \inf_{h(x'_2) \geq h(x_2)} f(x_1, x'_2).$$

So symmetry properties can be used to define a nonparametric family of functions.

In the nonparametric consumer analysis there are now two central problems. First, there is the question which properties have to be satisfied by the given data set in order to accept a postulated hypotheses. Such a property may be a symmetry that has to be satisfied by the data. Secondly, there is the question what the given data set tells us about the underlying preference order or utility function, for instance, in the form of *restrictions* on the utility function. Such restrictions can be seen as *conserved properties* for the consumer data, if they remain valid when new consistent data are added.

1.3 Differences with Standard Theory

The starting-point in this book is a generalization of the standard neoclassical theory. This generalization differs from the standard neoclassical theory in three ways:

1. In the standard theory a consumer is a utility maximizer. In this book, however, we assume a priori that a consumer is both a utility maximizer and a cost minimizer.
2. The standard assumption of given prices is generalized. It is assumed that the consumer has a given function that describes the cost of his choices. I will call such a function a cost function.
3. Economical inefficiency is allowed.

The assumption that a consumer is both a utility maximizer and a cost minimizer is advantageous, because one can skip the functional restrictions, like nonsatiation, needed to assure that the consumer is also a cost minimizer.

The second difference with the standard theory, the use of cost functions, does not cause theoretical complications. It may even help to simplify the theory. This generalization means that the analysis applies not only to competitive consumers, where prices are fixed, but also to noncompetitive consumers, where prices are a function of consumer demand.

The nonparametric approach is not very popular in the current literature. This has to do with the limited empirical use of this approach in case one supposes consumer and producer behaviour is efficient. In empirical studies the assumption of efficient consumer behaviour is often violated. In order to apply the nonparametric methods, I will introduce efficiency measures that indicate how serious such a violation is. Hence, the allowance of economical inefficiency has an important pragmatic reason; it may be of great help in empirical applications. There is however a drawback of the inclusion of economical inefficiency in the theory. It may cause complications, which interfere with the need for a simple clear theory. So, I had the choice between the presentation of a nice clean theory, concerning efficient economical behaviour, or to accept some complications due to the inclusion of economical inefficiency. I have chosen for the latter, because such a theory is more suitable for empirical applications.

1.4 Application of the Nonparametric Theory

To illustrate the use of the nonparametric theory, developed in this book, the theory is applied to Dutch empirical consumption data. The Netherlands Central Bureau of Statistics (1981) published annual private consumption expenditure data and several types of price index numbers, covering the period 1951–1977, on various levels of aggregation. In this study the geometric price index numbers for 106 commodity groups are used (Netherlands Central Bureau of Statistics, 1981, Table 4). Implicit per capita quantity measures for each commodity group are obtained by the division of the nominal commodity group expenditures per capita by the corresponding geometric price index numbers. The data used for this computation, are the expenditure shares for the 106 commodity groups, total expenditures and population data (Netherlands Central Bureau of Statistics, 1981, Table 3, 44 and 47).

1.5 Contents of Part I

Chapter 2 treats the problem whether there is a preference relation or utility function for given consumer data. The search for a utility function which fits with given consumer demand, the 'Integrability Problem', has been traced as far back as Antonelli (1886). There is however one difference between the 'Integrability Problem' and the problem in Chapter 2. In the 'Integrability Problem' it is assumed that all possible observations of consumer demand are known. However, in this book it is assumed that only certain observations of consumer demand are available, as is the case in an empirical context. This demands a *global* analysis, based on the axiom of revealed preference in the work of Samuelson (1948, 1950), Houthakker (1950) and Uzawa (1960), whereas the 'Integrability Problem' is solved with a *local* analysis.

Chapter 3 investigates the existence of linearly homogeneous utility functions for a given data set. This problem is solved for neoclassical consumer behaviour in Afriat (1972, 1981), Diewert (1973) and Varian (1983). Their results are generalized to the more general definition of consumer demand used in this book. Another point of interest is the existence of lower and upper bounds for the linearly homogeneous utility function for given consumer demand data. This allows one to derive an idea of consumers utility function without the conventional parametric specification of this function and an estimation of the parameters.

In Chapter 4 the weak separability of utility functions is treated. This problem has not got much attention in the nonparametric approach. Diewert and Parkan (1978, 1985) and Varian (1983) consider *concave* weakly separable utility functions. In that case one can obtain results by using separating hyperplanes. This book presents new results for consumer demand with no other restriction on the utility function than weak separability.

When I started this research as mathematician between economists, I tried to get some backup from the mathematical department. When I explained the subject to a mathematician there, he said: "I can't help you. This is no mathematics!" Well, Chapter 5 is an attempt to falsify this statement. It is a mathematical exercise of a nonparametric approach to consumer demand.

In Chapter 6 it is shown that aggregation of random data may result in data that display economical behaviour. This result is obtained by the application of several nonparametric tests to aggregate random data. I have to admit that this part of the research is not a result of a sharp mind. As many results in science it is a result of a mistake. Accidentally a programming mistake did transpose the price matrix in the computer program that evaluated the

8 *Consumer Demand*

corresponding quantities. When I discovered this mistake I was surprised about the nice empirical results, which can shortly be described as "Garbage In Economics Out". Of course this may also happen, when one uses unreliable data. Let this be a warning to empirical researchers!

2 CONSUMER DEMAND

2.1 Introduction

This chapter contains basic definitions and theorems concerning the nonparametric analysis of consumer behaviour. The starting-point of these theorems is the assumption that there is one and the same utility function for all observed data. Assuming the data refer to observations at different points of time, this means in terms of symmetry that the utility function in the model is invariant under time translation. Using this assumption it will be shown that the axiom of revealed preference is a necessary and sufficient condition for the existence of a rationalizing preference order. Moreover, as was shown earlier by Afriat (1973), the axiom of revealed preference appears also to be a necessary and sufficient condition for the existence of a rationalizing utility function, provided the consumer data consist of a finite number of elements. That the axiom of revealed preference is not sufficient for infinite data sets follows from counter examples, for example in Chipman *et al.* (1971). The existence of rationalizing utility functions for data sets of an infinite number of elements is not treated in this chapter. I will consider this problem, which is mainly of theoretical interest, later on.

Important in this chapter is the introduction of efficiency transformations. A feature of an efficiency transformation is the possibility to relax the conditions belonging to efficient behaviour. The idea can be applied to every nonparametric test described in this book. Furthermore, it provides a measure that describes how seriously a nonparametric test is violated.

2.2 Consumer Demand

Let us consider an arbitrary choice space X of elements, available to a consumer to a certain cost. We may represent the cost of consumers choice x in X by using a function. I will call such a function a *cost function*. Assuming that costs are nonnegative, we can describe cost functions as follows.

Definition: Let X be a given set. A *cost function* defined on X is a function $c: X \rightarrow \mathbb{R}_+$. The set of all possible cost functions $c: X \rightarrow \mathbb{R}_+$ is denoted as $C(X)$.

The cost functions c that appear in textbook theory, are simply of the form $c(x) = px$, where x is a quantity vector and p a given price vector. When I consider this particular case, I will suppose prices positive and quantities nonnegative, i.e. $p \in \mathbb{R}_{++}^n$ and $x \in \mathbb{R}_+^n$. To refer to the function $c(x) = px$ in formulae, I will use the abuse of notation p when actually the function c is meant.

Let us assume that consumer behaviour can be derived from a utility function $f: X \rightarrow \mathbb{R}$. Then a *utility maximizer* will choose the best $x \in X$ for its money, i.e., given a cost function $c: X \rightarrow \mathbb{R}_+$, his choice behaviour can be described as

$$f(x') = \sup_{c(x) \leq c(x')} f(x).$$

Similarly, a *cost minimizer* will choose the cheapest $x \in X$ for its utility value, thus

$$c(x') = \inf_{f(x) \geq f(x')} c(x).$$

The preferences of a consumer for a given utility function can be represented as a weak order of the following form.

Definition: Let $f: X \rightarrow \mathbb{R}$ be a function. Then the weak order $\succsim$ on X induced by f is defined as

$$x \succsim y \Leftrightarrow f(x) \geq f(y) \quad (x, y \in X).$$

(Note that preorders $\succsim$ are often denoted as $\succsim$ in this book, which is not a standard notation).

For the contents of a consumer theory there is no difference in using a weak order or a utility function. We may generalize this theory by using a preorder instead of a weak order. This is a generalization, because any weak order is a preorder, but not every preorder is a weak order.

Definition: Let R be a binary relation on X . Such a relation may have the following properties. It may be:

- *transitive* if xRy and yRz implies xRz , for every $x, y, z \in X$;
- *reflexive* if xRx for all $x \in X$;
- *strongly complete* if xRy or yRx for every $x, y \in X$.

A *preorder* is a transitive reflexive binary relation. A *weak order* is a transitive strongly complete binary relation.

One may expect that preferences do order things in a consistent way. So the choice of a transitive binary relation to represent preferences is a natural one. Furthermore, something is as good as itself. Thus preferences are reflexive. So in general we may represent consumers preferences as a preorder. In the special case that a consumer can compare every pair of choices such a preorder is a weak order.

For preference orders we obtain the following definition for a utility maximizer, who chooses the best element in the budget set.

Definition: Let $\succsim$ be a preorder defined on X . Then *utility maximizing behaviour* is defined as

$$D_u(\succsim) = \{(c, x') \in C(X) \times X \mid c(x) \leq c(x') \Rightarrow x \succsim x' \quad (x \in X)\}.$$

Similarly, a cost minimizer chooses the cheapest element in the set of choices of higher or equal utility. *Cost minimizing behaviour* is defined as

$$D_c(\succsim) = \{(c, x') \in C(X) \times X \mid x \succsim x' \Rightarrow c(x) \geq c(x') \quad (x \in X)\}.$$

Combining both type of behaviour results in *consumer demand*, which is defined as $D_d(\succsim) \equiv D_u(\succsim) \cap D_c(\succsim)$.

So, if we write f when we actually mean the weak order induced by f , the set of consumer demand $D_d(f)$ contains data (c, x') such that one has both

$$\begin{aligned} f(x') &= \sup_{c(x) \leq c(x')} f(x), \\ c(x') &= \inf_{f(x) \geq f(x')} c(x). \end{aligned}$$

Consumer demand is clearly invariant under monotonic transformations of the utility function, i.e. one has $D_d(f) = D_d(m \circ f)$, where m is a strictly increasing function. In physics such a symmetry is said to be a *gauge symmetry*. Gauge symmetry means that the behaviour of physical variables can be described by several internally different models. More precisely, a physical model has a gauge symmetry when the observable physical variables are invariant under a transformation of the model equations.

Now, we have given a definition of consumer demand. What is the use of this definition? A typical empirical situation is the following. We observe a set of consumer data $D = \{(c_i, x^i)\}_{i \in I}$, where $(c_i, x^i) \in C(X) \times X$, and there is no information about a preference order that might have generated the data. A question then is whether or not there may be a consumer that has generated the data, i.e. is there a preorder $\succsim$ such that $D \subset D_d(\succsim)$.

Hence, a problem of interest is to find restrictions that are imposed on consumer data by each of the given types of consumer behaviour D_d , D_u and D_c . For empirical work it is important to know whether these restrictions are strong or weak restrictions. We may also consider the assumption that such a preorder has certain properties. This again implies more restrictions, which can be studied.

To get an idea of the meaning of this questions, I will give some trivial examples of consumer behaviour. Later on I will investigate restrictions that the assumption of consumer demand imposes on consumer data.

Example 2.1 Obviously, one has $D_u(f) = C(X) \times X$ for any constant function $f: X \rightarrow \mathbb{R}$. Similarly, one has $D_u(\succsim) = C(X) \times X$ for the trivial preorder $\succsim, \sim$ defined on X such that

$$x \succsim y \quad (x, y \in X).$$

Example 2.2 One has $D_c(\succsim) = C(X) \times X$ for the preorder $\succsim, \sim$ defined on X such that

$$x \succsim y \Leftrightarrow x = y \quad (x, y \in X).$$

So, with the above it is easy to find a cost minimizing or a utility maximizing preorder for a data set. For consumer demand, however, there are not such obvious solutions.

2.3 Rationalizing Preference Orders and Utility Functions

In the previous section I described data sets that represent consumer behaviour for a given preference order or a utility function. This is the usual view on consumer behaviour in textbook theory. In empirical research, however, the point of view is another. Then one is ignorant of consumers preference order, or utility function, and one is curious how one may represent consumers preferences for available empirical data. This urges another theoretical basis, where the point of departure is an arbitrary data set of consumer behaviour $D \subset C(X) \times X$.

The existence of a utility function for consumer data is related to an old problem which, as the *Integrability Problem*, has been traced as far back as Antonelli (1886). This is the problem to construct a utility function, which generates a given demand function. The given demand function $h(p, M)$ represents the demand of a consumer with income M when the price vector p is given. The 'Integrability Problem' now is the question whether there is a utility function f , such that the demand function can be derived from neoclassical consumer behaviour, i.e.

$$f(h(p, M)) = \sup_{p \cdot x \leq M} f(x).$$

The theory concerned with this problem can be divided into two main streams. In the early work of Antonelli a *local* analysis is made using differential equations. The last decades there has been another approach to the 'Integrability Problem'. In the pioneering work by Samuelson (1948, 1950), Houthakker (1950), and Uzawa (1960), the analysis is based on *global* characteristics with the axiom of revealed preference as a starting-point. For a review of results in revealed preference theory I refer to Chipman *et*

al. (1971) and Richter (1979). The theory that shall be presented in this book is also based on global characteristics. I will investigate whether given consumer data might be generated by a rational consumer using a preference order or a utility function. Therefore the following definition may be useful.

Definition: The family of preorders $\succ, \sim$ on X , which rationalize the data D , is defined as

$$\mathcal{R}_d(D) \equiv \{\succ \in X \times X \mid \succ \text{ is a preorder, } D \subset \mathcal{D}_d(\succ)\}.$$

So $\mathcal{R}_d(D)$ contains all the preorders that might have generated the data D . The family of utility functions $f: X \rightarrow \mathbb{R}$, which rationalize D , is analogously defined as

$$\mathcal{F}_d(D) \equiv \{f: X \rightarrow \mathbb{R} \mid D \subset \mathcal{D}_d(f)\}.$$

The definitions of $\mathcal{R}_u, \mathcal{F}_u$, and $\mathcal{R}_c, \mathcal{F}_c$ are analogous by replacing $\mathcal{D}_d$ by $\mathcal{D}_u$ and $\mathcal{D}_c$ respectively.

Clearly one has the following dual relation between the above defined families and sets of consumer data:

$$\mathcal{F}_d(D_1 \cup D_2) = \mathcal{F}_d(D_1) \cap \mathcal{F}_d(D_2),$$

where $\mathcal{F}_d$ may be replaced by any of the other families $\mathcal{F}_u, \mathcal{F}_c, \mathcal{R}_u, \mathcal{R}_c$ and $\mathcal{R}_d$.

2.4 Data Enclosure

In this section I will say something about the interesting idea of data enclosure, although it will not be used in this book. For the rationalizing set of utility functions $f: X \rightarrow \mathbb{R}$ we have

$$\mathcal{F}_d(D) = \{f: X \rightarrow \mathbb{R} \mid f \in \mathcal{F}_d(d) \text{ for all } d \in D\},$$

using the abuse of notation $\mathcal{F}_d(d) \equiv \mathcal{F}_d(\{d\})$. Using this idea, we may generalize the definition of $\mathcal{D}_d(f)$ in the following symmetrical way.

Definition: Let F be a set of utility functions $f: X \rightarrow \mathbb{R}$ (or preorders). Then we define the following set of consumer demand data

$$\mathcal{D}_d(F) \equiv \{d \in C(X) \times X \mid d \in \mathcal{D}_d(f) \text{ for all } f \in F\}.$$

Now, we may write $\mathcal{D}_d(\mathcal{F}_d(D))$, where D is a data set of consumer demand. This expression contains the *data enclosure* of D .

The data enclosure has the following meaning. If we observe data D generated by consumer demand then, using this information and assuming the utility function f unknown, the set $\mathcal{D}_d(\mathcal{F}_d(D))$ is the maximal set of data that is known to be compatible with D .

If we add the hypothesis that the utility function f satisfies $f \in F$, where F is a family of utility functions, then the data enclosure is $\mathcal{D}_d(F \cap \mathcal{F}_d(D))$. Now, the condition $f \in F$ means that we have more information available. Hence, the set $\mathcal{D}_d(F \cap \mathcal{F}_d(D))$ contains more data than $\mathcal{D}_d(\mathcal{F}_d(D))$. We have in general

$$D \subset \mathcal{D}_d(\mathcal{F}_d(D)) \subset \mathcal{D}_d(F \cap \mathcal{F}_d(D)).$$

2.5 Efficiency Transformations

This book describes several nonparametric tests of hypotheses that concern consumer and producer behaviour. The result of such tests has only two possibilities: the hypothesis is accepted or it is falsified. In empirical research it appears that the tested hypothesis is nearly always falsified by the nonparametric tests. We have to agree that falsification of the tested

hypothesis does not mean that such a hypothesis is useless. In other cases we also accept errors. For example, when we try to estimate the parameters of a production function for given data. In that case we use a statistical measure to judge how serious these errors are. To develop such a measure for nonparametric tests, I introduce the idea of *efficiency transformations*. A feature of an efficiency transformation – important for empirical research – is the possibility to relax the conditions belonging to efficient consumer behaviour. Efficiency transformations are a generalization of Afriat's idea of partial efficient consumer behaviour. Afriat (1973) introduced this idea to describe approximate utility maximization.

First, I will explain Afriat's idea. Consider the conditions for consumer demand, given a utility function $f: X \rightarrow \mathbb{R}$ and price vector p . By the definition of consumer demand in the previous section, the hypothesis that $x' \in X$ is consistent with consumer demand is equivalent to:

$$\begin{aligned} px \leq px' &\Rightarrow f(x) \leq f(x') & (x \in X), \\ px < px' &\Rightarrow f(x) < f(x') & (x \in X). \end{aligned}$$

The idea of Afriat is to change this hypothesis into the weaker hypothesis

$$\begin{aligned} px \leq epx' &\Rightarrow f(x) \leq f(x') & (x \in X), \\ px < epx' &\Rightarrow f(x) < f(x') & (x \in X), \end{aligned}$$

where $e \in [0, 1]$ is an efficiency measure. Then for $e = 1$ we obtain the original hypothesis and by lowering e this hypothesis is getting weaker.

I will generalize Afriat's idea of partial efficient behaviour by making use of efficiency transformations. The idea behind such a transformation is simple. Efficiency transformations transform data into new data with the aim to relax the conditions for consumer demand. Such transformations Φ_e are dependent on an efficiency parameter e that indicates how serious we have deformed the data.

We may relax the conditions for consumer demand by changing the cost function c to another cost function that increases the cost for all $x \neq x'$. Then a tighter budget restriction with less choice freedom will result in weaker conditions for consumer demand. The idea of Afriat can now be reformulated as follows.

Example 2.3: Consider the following transformation rule that transforms each cost function $c(x)$ into a function $c^e(x)$. This depends on a parameter e , called the *efficiency level*, and is such that

$$c^e(x) = \begin{cases} c(x) & (x = x'), \\ e^{-1}c(x) & (\text{otherwise}). \end{cases}$$

This defines a transformation rule Φ_e that transforms every data element (c, x') into (c^e, x') .

Now, let us consider the hypothesis $(c^e, x') \in \mathcal{D}_d(f)$, where $e \in [0, 1]$, in the above example. In that case the conditions concerning $(c^e, x') \in \mathcal{D}_d(f)$ are equivalent to the conditions given by Afriat

$$\begin{aligned} c(x) \leq ec(x') &\Rightarrow f(x) \leq f(x') & (x \in X), \\ c(x) < ec(x') &\Rightarrow f(x) < f(x') & (x \in X). \end{aligned}$$

If $e < 1$, clearly this hypothesis is weaker than the hypothesis that we have $(c, x') \in \mathcal{D}_d(f)$. The replacement of c by c^e allows the consumer to waste a fraction of $(1-e)c(x')$ of the expenditures. This is shown in Figure 2.1, where the consumer with cost function $c(x) = px$ can achieve the same utility as attained at x' by spending epx' instead of px' . Note that in this figure one has $(c^e, x'') \in \mathcal{D}_d(f)$ for every x'' on S .

Figure 2.1 Partial Efficient Demand

So by using a transformation rule that changes all cost functions in a data set, we may weaken the hypothesis of consumer demand. For a general description of this idea, we need the following definition. It contains a summary of the properties of the given example.

Definition: An *efficiency transformation* $\Phi_e: C(X) \times X \rightarrow C(X) \times X$, where $e \in [0, 1]$ is called the *efficiency level*, maps each data element (c, x') on a data element $\Phi_e(c, x') = (c^e, x')$, such that $c^e(x') = c(x')$ and

$$\begin{aligned} c^a \geq c^b \geq c^1 = c & \quad (0 \leq a \leq b \leq 1), \\ c^0(x) = \infty & \quad (x \in X: x \neq x'). \end{aligned}$$

For data sets D we use the notation

$$\Phi_e D \equiv \{\Phi_e d \mid d \in D\}.$$

Efficiency transformations are useful, because they provide an easy way to introduce an efficiency measure e . Using such an efficiency measure e that depends on a data transformation, has the following advantage. Every theory that can be applied to consumer data D can also be applied to the transformed consumer data $\Phi_e D$. Hence, theory concerning efficient behaviour can directly be carried over to build a theory for inefficient behaviour.

The hypothesis that we have $\Phi_e(c, x') \in D_d(f)$ is getting weaker when we lower the efficiency level e . Thus clearly the following theorem is valid.

Theorem 2.1: Suppose Φ_e is an efficiency transformation and $a, e \in [0, 1]$. Then

$$\Phi_e(c, x') \in D_d(\succ) \Rightarrow \Phi_a(c, x') \in D_d(\succ) \quad (a \leq e).$$

This is also valid if D_d is replaced by D_u or D_c .

In the applications of the theory, I will often use the simple efficiency transformation given in Example 2.3.

2.6 Efficiency Upper Bounds

A feature of an efficiency transformation Φ_e with an efficiency level e is the possibility to relax the conditions belonging to efficient consumer behaviour. Suppose a given data set D of consumer behaviour and we postulate that consumers preference order belongs to a certain family F of preference orders. Then we are interested in the set of preference orders $F \cap \mathcal{R}_d(D)$. If this set is empty then we may lower the efficiency level e to find a nonempty set $F \cap \mathcal{R}_d(\Phi_e D)$. By Theorem 2.1 the efficiency levels e , for which one has $F \cap \mathcal{R}_d(\Phi_e D) \neq \emptyset$, form always a set $[0, a]$ or $[0, a)$. We may use the upper bound a to evaluate the hypothesis that consumers preference order belongs to the family F of preference orders. So the following definition is useful.

Definition: Let $D \subset C(X) \times X$ be a data set of consumer behaviour and let F be a family of preorders on X . Then the *efficiency upper bound* $e_d(F, D)$ for efficiency transformation Φ_e is defined as

$$e_d(F, D) \equiv \sup \{e \in [0, 1] \mid F \cap \mathcal{R}_d(\Phi_e D) \neq \emptyset\}.$$

Of course, this family F of preference orders may be induced by a certain family of utility functions. Note that $e_d(F, D)$ is only well-defined when $F \cap \mathcal{R}_d(\Phi_0 D) \neq \emptyset$, because otherwise $F \cap \mathcal{R}_d(\Phi_e D) = \emptyset$ for all $e \in [0, 1]$. The following theorem shows when such a problem will not occur.

Theorem 2.2: Suppose D is an arbitrary consumer data set and Φ_e an efficiency transformation. Then $\mathcal{R}_d(\Phi_0 D)$ contains every reflexive preorder $\succ, \sim$.

Proof: Let $(c, x') \in \Phi_0 D$ and let $\succ, \sim$ be a reflexive preorder. Then the definition of an efficiency transformation implies that we have $c(x) = \infty$ for all $x \neq x'$. So, we have

$$c(x) \leq c(x') \Rightarrow x = x' \Rightarrow x \sim x' \quad (x \in X),$$

and hence $(c, x') \in \mathcal{D}_u(\sim)$. Moreover, we have $(c, x') \in \mathcal{D}_c(\succ)$, because $c(x) \geq c(x')$ for all $x \in X$. $\square$

The above theorem implies that $e_d(F, D)$ is well-defined if F contains a preference order that is induced by a utility function, because such a preorder is reflexive. Thus $e_d(F, D)$ is well defined for any nonempty family F of utility functions.

Now, suppose we have a numerical test available that tests whether one has $F \cap \mathcal{R}_d(\Phi_e D) \neq \emptyset$ for arbitrary $e \in [0, 1]$. In that case we can compute $e_d(F, D)$ within any specified tolerance $\varepsilon > 0$ by using the following simple bisection algorithm:

Step 1. Set $e_0 = 0$, $e_1 = 1$, and $e = 1/2$.

Step 2. Repeat the following $n-1$ times, where n is such that $2^{-n} \leq \varepsilon$:

If $F \cap \mathcal{R}_d(\Phi_e D) \neq \emptyset$, put $e_0 = e$, otherwise, put $e_1 = e$;

Put $e = (e_0 + e_1)/2$.

Obviously the result will satisfy $|e_d(F, D) - e| < \varepsilon$.

2.7 Existence of a Rationalizing Preference Order

To define the axiom of revealed preference I will use the following notation.

Definition: In case we consider consumer demand for the data $D = \{(c_i, x^i)\}_{i \in I}$, the binary relations $R(D)$ and $P(D)$ on $\{x^i\}_{i \in I}$ are defined as follows

$$x^i R x^j \Leftrightarrow c_i(x^j) \leq c_i(x^i) \quad (i, j \in I),$$

$$x^i P x^j \Leftrightarrow c_i(x^j) < c_i(x^i) \quad (i, j \in I).$$

Table 2.1 Revealed Preferences

	5	5	5	5	5	5	5	5	6	6	6	6	6	6	6	6	6	7	7	7	7	7	7	7	7	7	
	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7
1951	R	P	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1952	.	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1953	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1954	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1955	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1956	P	P	P	P	P	R	P	P	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1957	P	P	P	P	P	.	R	P	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1958	P	P	P	P	P	.	.	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1959	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1960	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1961	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1962	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1963	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1964	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.
1965	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.
1966	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.
1967	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.
1968	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.
1969	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.
1970	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.
1971	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.
1972	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.
1973	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.
1974	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.
1975	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R
1976	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P
1977	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P

So $x^i R x^j$ means that x^j was at period i available to the consumer. Further, $x^i P x^j$ means that there was even money left. These relations R and P have clearly the following meaning for consumer preferences:

$$x^i R x^j \Rightarrow x^j \text{ is not preferred to } x^i,$$

$$x^i P x^j \Rightarrow x^i \text{ is preferred to } x^j.$$

Note that one has clearly $P \subset R$. As an example of such relations, the relations P and R of the Dutch consumption data are given in Table 2.1. The table shows that the later years are usually preferred above the earlier years, which is due to economical progress.

The axiom of revealed preference is based on the nearly trivial observation that data of consumer demand must satisfy: x^j is not preferred to x^i implies not x^j preferred to x^i . Assuming that consumers preferences are represented by a preorder, the meaning of R holds too for the transitive closure R^* . So, consumer demand data must satisfy

$$x^i R^* x^j \Rightarrow x^j \text{ is not preferred to } x^i \Rightarrow \text{not } x^j P x^i.$$

This is the axiom of revealed preference.

Definition: A set D of consumer data is said to satisfy the *axiom of revealed preference* when

$$x^i R(D)^* x^j \Rightarrow \text{not } x^j P(D) x^i \quad (i, j \in I),$$

which is equivalent to the statement $R(D)^* \cap P(D)^c = \emptyset$. (See Appendix A for the definition of the transitive closure R^* of R , and the complement P^c of P).

In case cost functions are derived from price vectors, this axiom is reduced to the Generalized Axiom of Revealed Preference (GARP) defined in Varian (1982). The axiom of revealed preference is a necessary and sufficient condition for the existence of a rationalizing preorder. If the axiom is satisfied, we can even prove the existence of a rationalizing weak order. This result is contained in the following theorem that is even valid for data sets of an infinite number of elements.

Figure 2.2 Inconsistent Behaviour

Theorem 2.3: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a set of consumer data. Then the following conditions are equivalent:

- (i) $\mathcal{R}_d(D) \neq \emptyset$.
- (ii) There exists a weak order $\succ, \sim$ in $\mathcal{R}_d(D)$.
- (iii) D satisfies the axiom of revealed preference.

Proof: Define the binary relations $\bar{R}$ and $\bar{P}$ on X by

$$\begin{aligned} x \bar{R} y &\Leftrightarrow \exists i \in I: x^i = x \text{ and } c_i(y) \leq c_i(x^i) & (x, y \in X), \\ x \bar{P} y &\Leftrightarrow \exists i \in I: x^i = x \text{ and } c_i(y) < c_i(x^i) & (x, y \in X). \end{aligned}$$

According to the definition of $\mathcal{R}_d(D)$, one has obviously

$$\mathcal{R}_d(D) = \{ \succ \subset X \times X \mid \succ \text{ is a preorder such that } \bar{R} \subset \succ \text{ and } \bar{P} \subset \succ \}.$$

Now, from $\bar{P} \subset \bar{R}$ and Theorem A.4 in Appendix A, it follows that one has

$$(i) \Leftrightarrow (ii) \Leftrightarrow \bar{R}^* \cap \bar{P}^c = \emptyset.$$

To prove the theorem it is thus sufficient to show the equality $\bar{R}^* \cap \bar{P}^c = R^* \cap P^c$. One can easily show that the following relations are valid

$$\begin{aligned}x^i \bar{R}^* x^j &\Leftrightarrow x^i R^* x^j & (i, j \in I), \\x^i \bar{P} x^j &\Leftrightarrow x^i P x^j & (i, j \in I).\end{aligned}$$

This and the observation that we have

$$\begin{aligned}x \bar{R}^* y &\Rightarrow \exists i \in I: x = x^i & (x, y \in X), \\y \bar{P} x &\Rightarrow \exists j \in I: y = x^j & (x, y \in X),\end{aligned}$$

imply the equivalence relation

$$x \bar{R}^* y \text{ and } y \bar{P} x \Leftrightarrow x R^* y \text{ and } y P x \quad (x, y \in X),$$

we had to prove. □

Figure 2.2 displays a simple example of price and quantity data, which does not satisfy the axiom of revealed preference. It has the following revealed preference table.

Table 2.2 Revealed Preferences

	1	2
1	R	P
2	P	R

Summarizing the above in terms of symmetry and conserved properties, we have the following result. In general we can obtain consumer data from a model of consumer behaviour that uses a preorder $\succsim$. If we have data d_i , for different periods $i \in I$, then we might use in the general model for each period i another preorder $\succsim_i$. When we use the symmetry assumption that this preorder is time-independent, i.e.

$$\succsim_i = \succsim_j \quad (i, j \in I),$$

then we obtain the following conserved property. Sets of data $D \subset D'$, both generated by the same time-independent preorder model, satisfy

$$\succsim \in \mathcal{R}_d(D') \Rightarrow R(D) \subset \succsim \text{ and } P(D) \subset \succsim.$$

So the right hand restriction remains valid when new data is observed and added to D . It is a conserved property. This is in fact the basis of the axiom of revealed preference.

We can draw the following parallel for this result in classical mechanics. In classical mechanics a time-independent Lagrange function means that the energy will be conserved by the system. In economic science a time-independent utility function implies the conservation of the revealed preferences. In both cases this result is independent from the specific form of the Lagrangian or utility function respectively.

2.8 The Revealed Preference Axiom and Dutch Consumer Data

Maks (1978, 1980), Landsburg (1981) and Varian (1982) tested aggregate price and quantity consumption data on the axiom of revealed preference. They find hardly any violations of axiom of revealed preference. The Dutch consumer data appear also to satisfy the axiom of revealed preference. In Table 2.3 the annual observations from Table 2.1 are ordered in such a way that the resulting matrix is lower triangular. Such an ordering is only possible if the axiom of revealed preference is satisfied. To obtain this observation order from the natural year order, the observations for the years 1951 and 1952, and the years 1956 and 1958, are interchanged. As I will show in the following section, the resulting ordering of the years is compatible with

increasing utility values for a rationalizing utility function.

At first sight it is striking that the lower triangle in Table 2.3 is completely filled, which is in general not necessary. This means that there is only one unique observation order that generates a lower triangle matrix. Furthermore, it means that the closure R^* is equal to R , and we cannot increase — in this particular case — our information about consumer preferences by taking the closure over R .

Table 2.3 Revealed Preferences

	5	5	5	5	5	5	5	5	5	6	6	6	6	6	6	6	6	6	7	7	7	7	7	7	7	7	7
	2	1	3	4	5	8	7	6	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7
1952	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1951	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1953	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1954	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1955	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1958	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1957	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1956	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1959	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1960	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1961	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1962	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1963	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1964	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.	.
1965	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.	.
1966	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.	.
1967	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.	.
1968	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.	.
1969	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.	.
1970	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.	.
1971	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.	.
1972	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.	.
1973	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.	.
1974	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.	.
1975	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.	.
1976	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R	.
1977	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	R

The cause of the filled lower triangle is a lack of variation in the price data, which was also reported by Varian (1982) for aggregate U.S. consumption data. Figure 2.3 illustrates why one may expect to obtain a filled lower triangle, when prices are slowly changing, while budgets are increasing. Then each budget restriction contains the previous quantity choices. The Dutch consumption data satisfy these properties, as is often the case for aggregate consumption data, except for the innocent interchange of the observations of a few subsequent years.

2.9 Existence of a Rationalizing Utility Function

According to Theorem 2.3 the axiom of revealed preference is a necessary condition for the existence of a utility function, which rationalizes a given set of consumer data. In this section I prove a theorem which states that the axiom of revealed preference is also a sufficient condition, if the given

data set has a finite number of elements. Thus, since the Dutch consumer data satisfy the axiom of revealed preference, there exists a utility function for the Dutch consumer data.

Figure 2.3 Little Price Variation

To prove the existence theorem a rationalizing utility function is constructed. For this construction we need the following theorem.

Theorem 2.4: Suppose $x^i \in X$ and $z_i: X \rightarrow \mathbb{R}$ for all $i \in I$, where I is a finite index set. Let the binary relations R and P on $\{x^i\}_{i \in I}$ be defined by

$$\begin{aligned} x^i R x^j &\Leftrightarrow z_i(x^j) \leq 0 & (i, j \in I), \\ x^i P x^j &\Leftrightarrow z_i(x^j) < 0 & (i, j \in I), \end{aligned}$$

and suppose there is a preorder $\succsim$ on $\{x^i\}_{i \in I}$ such that $R \subset \succsim$, $P \subset \succ$. Then there exists a function

$$f(x) = \min_{i \in I} f_i + \lambda_i z_i(x) \quad (x \in X),$$

such that $\lambda_i > 0$ and $f(x^i) \geq f_i$ for all i . Moreover, if $\succsim$ is a weak order then the numbers f_i can be chosen in such a way that they represent this weak order.

Proof: If there is a preorder $\succsim$ satisfying the above conditions, then by Theorem A.4 in Appendix A there is also a weak order that satisfies these conditions. So, we may assume that $\succsim$ is a weak order. Since $\succsim$ is a weak order on a finite set, there is a finite sequence of elements

$$x^{i_1} \succ x^{i_2} \succ \cdots \succ x^{i_n},$$

such that I is partitioned in equivalence classes

$$E_l = \{i \in I \mid x^i \sim x^{i_l}\} \quad (l = 1, \dots, n),$$

satisfying $E_l \cap E_k = \emptyset$ if $l \neq k$, and

$$I = \bigcup_{l=1}^n E_l.$$

Now, choose f_i and λ_i as follows. Put $f_i = \lambda_i = 1$ for all $i \in E_1$, and for $l = 2, \dots, n$ do the following:

Step 1. Let

$$L_l < \min \{f_j + \lambda_j z_j(x^i), f_j\} \quad (i \in E_l, j \in \bigcup_{m < l} E_m),$$

and put $f_i = L_l$ for all $i \in E_l$.

Step 2. Then let

$$U_l \geq (f_j - f_i) / z_i(x^j) \quad (i \in E_l, j \in \bigcup_{m < l} E_m),$$

and put $\lambda_i = U_l$ for all $i \in E_l$.

Now, the condition $f_i \leq f(x^i)$ for all $i \in I$ follows clearly if we can show

$$f_i \leq f_j + \lambda_j z_j(x^i) \quad (i, j \in I).$$

This inequality holds indeed, because our choice of f_i and λ_i in conjunction with the assumptions for the weak order $\succ, \sim$, imply

$$x^j \sim x^i \Rightarrow f_j = f_i \text{ and } z_j(x^i) \geq 0 \quad (i, j \in I),$$

$$x^j \succ x^i \Rightarrow f_i < f_j + \lambda_j z_j(x^i) \quad (i, j \in I),$$

$$x^j \prec x^i \Rightarrow \lambda_j \geq (f_i - f_j) / z_j(x^i) \text{ and } z_j(x^i) > 0 \quad (i, j \in I).$$

Finally, the numbers f_i are clearly chosen in such a way that they represent the weak order $\succ, \sim$. $\square$

Now we are able to prove the main theorem which states necessary and sufficient conditions for the existence of a rationalizing utility function. In Afriat (1967) such a theorem was proved, which is improved by Varian (1982, 1983). I will generalize Varian's results by using a more general definition of consumer demand. The results improve a theorem given by Afriat (1973), concerning inefficient consumer demand for given prices, and generalize it by using cost functions.

Theorem 2.5: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a finite data set of consumer behaviour.

(a) Then the following conditions are equivalent:

(i) $\mathcal{F}_d(D) \neq \emptyset$.

(ii) D satisfies the axiom of revealed preference.

(iii) There exist numbers $\lambda_i > 0$ and f_i such that

$$f_i \leq f_j + \lambda_j [c_j(x^i) - c_j(x^j)] \quad (i, j \in I).$$

(b) One has $f \in \mathcal{F}_d(D)$ for every function of the form

$$f(x) = \min_{i \in I} f_i + \lambda_i [c_i(x) - c_i(x^i)] \quad (x \in X),$$

where f_i and $\lambda_i > 0$ satisfy the conditions in (iii).

Proof (a) (i) $\Rightarrow$ (ii): If $\mathcal{F}_d(D) \neq \emptyset$ then $\mathcal{R}_d(D) \neq \emptyset$, hence (ii) follows from Theorem 2.3.

(ii) $\Rightarrow$ (iii): Put

$$z_i(x) = c_i(x) - c_i(x^i) \quad (x \in X, i \in I).$$

From Theorem 2.3 it follows that, if the axiom of revealed preference is satisfied, there is a rationalizing weak order. Hence, we may apply Theorem 2.4 to obtain a function

$$f(x) = \min_{i \in I} f_i + \lambda_i z_i(x) \quad (x \in X),$$

where f_i and λ_i satisfy the conditions in (iii).

(iii) $\Rightarrow$ (i): From (iii) it follows that the function f , defined as above in

the proof of (ii) $\Rightarrow$ (iii), satisfies

$$\begin{aligned} z_i(x) \leq 0 &\Rightarrow f(x) \leq f_i = f(x^i) & (i \in I, x \in X), \\ z_i(x) < 0 &\Rightarrow f(x) < f_i = f(x^i) & (i \in I, x \in X). \end{aligned}$$

Hence $f \in \mathcal{F}_d(D)$.

(b) That we have $f \in \mathcal{F}_d(D)$ follows from the proof of (iii) $\Rightarrow$ (i). $\square$

The above theorem can be extended to a theorem in which the utility functions are restricted to a class of functions with one or more of the following properties: concave, strictly concave, monotonically increasing, strictly monotonically increasing, continuous, uniformly continuous. The proof of such a theorem is easy, if we suppose all cost functions c_i are element of this class of functions. In that case the constructed utility function f in Theorem 2.5 (b) will be an element of this class too. So the mentioned properties of the cost functions can be carried over to properties of the utility function.

Now, the following problem occurs if we try to prove the existence of a continuous function for inefficient behaviour. We may get into problems if we use an efficiency transformation Φ_e , because this may introduce a discontinuity in the cost functions. The transformation $\Phi_e(c, x')$ increases all values of the cost function c , except the value $c(x')$ at x' . So in general we are left with a discontinuity at x' after the transformation of a continuous cost function. This means that, for data of the form $\Phi_e D$, discontinuities will be carried over to the constructed utility function. Such discontinuities have the following property.

Theorem 2.6: Suppose $c: X \rightarrow \mathbb{R}_+$ is continuous at x' , and Φ_e is an efficiency transformation. Then $\Phi_e(c, x') = (c^e, x')$ satisfies

$$\lim_{x \rightarrow x'} c^e(x) \geq c^e(x').$$

Proof: For an efficiency transformation one has in general $c^e(x') = c(x')$ and $c^e(x) \geq c(x)$ for all $x \in X$. Hence, we have

$$\lim_{x \rightarrow x'} c^e(x) \geq \lim_{x \rightarrow x'} c(x) = c(x') = c^e(x'),$$

where we have used the continuity of c at x' . $\square$

The following theorem shows how such discontinuities may be fixed.

Theorem 2.7: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a finite data set of consumer behaviour and f_i and $\lambda_i > 0$ satisfy

$$f_i \leq f_j + \lambda_j [c_j(x^i) - c_j(x^j)] \quad (i, j \in I).$$

Then one has $f \in \mathcal{F}_d(D)$ for every function of the form

$$f(x) = \min_{i \in I} f_i + \lambda_i [\hat{c}_i(x) - c_i(x^i)] \quad (x \in X),$$

where all the functions $\hat{c}_i$ satisfy

$$\begin{aligned} \hat{c}_i(x) &\geq c_i(x) & (x = x^i), \\ \hat{c}_i(x) &= c_i(x) & (\text{otherwise}). \end{aligned}$$

Proof: The construction of f implies that we have $f(x^i) \geq f_i$ for all $i \in I$. Now, let $i \in I$ and suppose $x \neq x^i$. Then we have $c_i(x) = \hat{c}_i(x)$ and thus

$$c_i(x) \leq c_i(x^i) \Rightarrow f(x) \leq f_i \leq f(x^i), \quad c_i(x) < c_i(x^i) \Rightarrow f(x) < f_i \leq f(x^i).$$

Furthermore, in case $x = x^i$, we have

$$c_i(x) \leq c_i(x^i) \Rightarrow f(x) = f(x^i).$$

Hence $f \in \mathcal{F}_d(D)$. □

From the above theorems we can derive the following theorem in which we remove the discontinuity, caused by an efficiency transformation.

Theorem 2.8: Suppose $\{(c_i, x^i)\}_{i \in I} = \Phi_e D$, where D is a finite data set and Φ_e is an efficiency transformation. Put

$$\hat{c}_i(x) = \lim_{v \rightarrow x} c_i(v) \quad (i \in I, x \in X).$$

Suppose all functions $\hat{c}_i$ and the cost functions of D' are continuous, and the cost functions c_i have only one discontinuity at x^i . Then $\mathcal{F}_d(\Phi_e D) \neq \emptyset$ implies the existence of a continuous function $f \in \mathcal{F}_d(\Phi_e D)$.

Proof: Suppose $\mathcal{F}_d(\Phi_e D) \neq \emptyset$. Then by Theorem 2.5 there are numbers $\lambda_i > 0$ and f_i , such that $f_i \leq f_j + \lambda_j [c_j(x^i) - c_j(x^j)]$ for all $i, j \in I$. The functions $\hat{c}_i$ are continuous functions, equal to c_i except at x^i , where the discontinuity is removed. By Theorem 2.6 such functions satisfy $\hat{c}_i(x^i) \geq c_i(x^i)$ for all $i \in I$. Hence by Theorem 2.7, we have $f \in \mathcal{F}_d(D)$ for the function

$$f(x) = \min_{i \in I} f_i + \lambda_i [\hat{c}_i(x) - c_i(x^i)] \quad (x \in X).$$

This function is continuous, because all functions $\hat{c}_i$ are continuous. □

If we apply the efficiency transformation of Example 2.3 to price and quantity data, we may use the above theorem to prove the existence of a concave continuous utility function.

Theorem 2.9: Suppose D is a finite price and quantity data set and Φ_e is the efficiency transformation of Example 2.3. Then $\mathcal{F}_d(\Phi_e D) \neq \emptyset$ implies the existence of a concave continuous function $f \in \mathcal{F}_d(\Phi_e D)$.

Proof: From the proof of Theorem 2.8 it follows that properties of the cost functions $\hat{c}_i$, where the discontinuity is removed, can be carried over to the constructed utility function f . If we apply the efficiency transformation of Example 2.3 to price and quantity data then we will obtain concave continuous functions $\hat{c}_i$ in Theorem 2.8. Hence, in the proof of Theorem 2.8, we obtain the existence of a concave continuous function $f \in \mathcal{F}_d(\Phi_e D)$, assuming that we have $\mathcal{F}_d(\Phi_e D) \neq \emptyset$. □

3 LINEARLY HOMOGENEOUS CONSUMER DEMAND

3.1 Introduction

The question whether there exists a linearly homogeneous utility function for a given data set, which is compatible with efficient neoclassical consumer behaviour, is treated in Afriat (1972, 1981), Diewert (1973), and Varian (1983). In the present chapter I generalize their results, using the definition of consumer demand, given in Chapter 2, and formulate a necessary and sufficient condition which has to be satisfied by linearly homogeneous consumer data. In contrast to earlier results, concerning the existence of an arbitrary rationalizing utility function, this condition can be applied to both finite and infinite data sets. The results can also be applied to the more general case where utility functions are assumed to be homothetic, because a homothetic function is a strictly increasing transformation of a linearly homogeneous function.

Another point of interest is the existence of lower and upper bounds for the linearly homogeneous utility function for given consumer demand data. These bounds follow from the symmetry of a consumer demand model in case utility and cost functions are linearly homogeneous. Such bounds may give an impression of consumers utility function without the need of a parametric specification of the utility function and estimation of the parameters.

3.2 Linearly Homogeneous Utility Functions

I consider in this chapter linearly homogeneous functions $f: X \rightarrow \mathbb{R}$ where the choice space X is cone.

Definition: A cone is a set X such that

$$\lambda x \in X \quad (x \in X, \lambda \geq 0).$$

Definition: A function $f: X \rightarrow \mathbb{R}$ is said to be linearly homogeneous when

$$f(\lambda x) = \lambda f(x) \quad (x \in X, \lambda \in \mathbb{R}).$$

An alternative way to define linear homogeneity is the following, where the idea of symmetry is used. A linearly homogeneous function is a function that is invariant under the family of transformations T_λ defined as

$$(T_\lambda f)(x) = \lambda^{-1} f(\lambda x) \quad (\lambda \neq 0, x \in X).$$

Remark: Note that $f(0) = 0$ follows in the first definition from $f(0) = \lambda f(0)$ with $\lambda = 0$ and in the second definition from $\lambda^{-1} f(0) = f(0)$ for every $\lambda \neq 0$.

The underlying idea of the theory in this chapter is the following. Consider the optimization problem

$$\sup_{c(x) \leq c(x')} f(x).$$

Using a Lagrangian

$$L(x, x') = f(x) - \nu[c(x) - c(x')] \quad (x \in X)$$

this problem can be transformed into the unconstrained optimization problem

$$\sup_x L(x, x'),$$

where the Lagrangian is maximized. When both f and c are linearly homogeneous then this Lagrangian has a symmetric property, because one has

$$L(\lambda x, \lambda x') = \lambda L(x, x') \quad (\lambda \in \mathbb{R}, x, x' \in X).$$

The existence of such a symmetry implies that we can derive a conservation law for linearly homogeneous consumer demand.

The following theorem states that linearly homogeneous consumer demand consists of rays of consumer data.

Theorem 3.1: Suppose $f: X \rightarrow \mathbb{R}$ is linearly homogeneous and X is a cone. Then

$$(c, x') \in \mathcal{D}_d(f) \Leftrightarrow (T_\lambda c, \lambda x') \in \mathcal{D}_d(f) \quad (\lambda > 0),$$

where the transformation T_λ is defined by

$$(T_\lambda c)(x) = \lambda^{-1} c(\lambda x) \quad (x \in X).$$

Proof: The elementary proof is left to the reader. $\square$

So, by the above theorem the symmetry $T_\lambda f = f$, that holds for linearly homogeneous functions, results in the symmetry $(c, x') \sim (T_\lambda c, \lambda x')$ of the corresponding consumer demand data, where we may write $T_\lambda c = c$ in case c is linearly homogeneous.

Figure 3.1 displays an example of consumer demand for a linearly homogeneous function.

Figure 3.1 Linearly Homogeneous Function

Below I prove that linearly homogeneous utility maximization is equivalent to linearly homogeneous cost minimization, assuming that the attained utility value is positive.

Theorem 3.2: Suppose $f: X \rightarrow \mathbb{R}$ and $c: X \rightarrow \mathbb{R}_+$ are linearly homogeneous, X is a cone, $x' \in X$, $c(x') > 0$ and $f(x') > 0$. Then $(c, x') \in \mathcal{D}_u(f)$ if and only if $(c, x') \in \mathcal{D}_c(f)$.

Proof ($\Rightarrow$): Let $(c, x') \in \mathcal{D}_u(f)$ and $x \in X$. Suppose $c(x) < c(x')$. Then one has $f(x) \leq f(x')$, since $(c, x') \in \mathcal{D}_u(f)$, and we may take $\lambda > 1$ such that $\lambda c(x) < c(x')$. Now, assume that we have $f(x) = f(x')$. Then

$$c(\lambda x) = \lambda c(x) < c(x'), \quad f(\lambda x) = \lambda f(x) > f(x'), \quad \lambda x \in X.$$

This contradicts however $(c, x') \in \mathcal{D}_u(f)$. Thus $f(x) \neq f(x')$, which implies that we have $f(x) < f(x')$. This means $(c, x') \in \mathcal{D}_c(f)$ according to the definition of $\mathcal{D}_c$.

($\Leftarrow$): Let $(c, x') \in \mathcal{D}_c(f)$ and $x \in X$. Suppose $f(x) > f(x')$. Then $c(x) \geq c(x')$, since $(c, x') \in \mathcal{D}_c(f)$, and we may take $\lambda < 1$ such that $\lambda f(x) > f(x')$. Now, assume $c(x) = c(x')$. Then

$$c(\lambda x) = \lambda c(x) < c(x'), \quad f(\lambda x) = \lambda f(x) > f(x'), \quad \lambda x \in X.$$

However, this contradicts $(c, x') \in \mathcal{D}_c(f)$ and we have thus $c(x) \neq c(x')$. This means $c(x) > c(x')$, and hence $(c, x') \in \mathcal{D}_u(f)$. $\square$

The following theorem shows a property of data that is generated by a consumer using a linearly homogeneous utility function.

Theorem 3.3: Suppose $f: X \rightarrow \mathbb{R}$ is linearly homogeneous and X is a cone. Then

$$(c, x') \in \mathcal{D}_c(f) \Rightarrow c(f(x')f(x)^{-1}x) \geq c(x') \quad (x \in X: f(x) > 0).$$

Proof: Since f is linearly homogeneous, one has $f(f(x')f(x)^{-1}x) = f(x')$. Thus $(c, x') \in \mathcal{D}_c(f)$ implies $c(f(x')f(x)^{-1}x) \geq c(x')$. $\square$

The following theorem provides a condition which guarantees that $f(x') > 0$ is satisfied.

Theorem 3.4: Suppose $f: X \rightarrow \mathbb{R}$ and $c: X \rightarrow \mathbb{R}_+$ are linearly homogeneous functions, X is a cone, and $c(x') > 0$. If $(c, x') \in \mathcal{D}_c(f)$ then $f(x') > 0$.

Proof: Since X is a cone it contains the origin. Furthermore, since f and c are linearly homogeneous, one has $f(0) = c(0) = 0$. Now, $c(0) = 0 < c(x')$ and $(c, x') \in \mathcal{D}_c(f)$ implies $f(0) = 0 < f(x')$. $\square$

3.3 Existence of a Rationalizing Linearly Homogeneous Function

Using the above theorems, I am able to prove a necessary and sufficient condition for the existence of a linearly homogeneous utility function, which rationalizes consumer data. The condition refers to the idea of an *absorptive* matrix. Such a matrix can be defined as follows. Let $A = [a_{ij}]$, $i, j \in I$, be a matrix for which a_{ij} represents the cost of moving from i to j . Then A is called an absorptive matrix when the total cost of every cycle path is nonnegative. For instance, for the cycle path $i \rightarrow j \rightarrow k \rightarrow i$ this means $a_{ij} + a_{jk} + a_{ki} \geq 0$. More information concerning absorptive matrices is given in Appendix C. This appendix describes, for instance, a simple algorithm which tests whether a matrix is absorptive. At this point we shall confine ourselves to give the following definition.

Definition: A matrix $A \in \mathbb{R}^n \times \mathbb{R}^n$ is said to be *absorptive* when

$$a_{ij} + a_{jk} + \dots + a_{lm} + a_{mi} \geq 0$$

for every possible sequence $i, j, k, \dots, l, m \in I$.

This definition is used in the following theorem that considers necessary and sufficient conditions for the existence of a linearly homogeneous utility function.

Theorem 3.5: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a data set of consumer behaviour, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and $c_i(x^i) > 0$, for all $i \in I$, and X is a cone.

(a) Then the following conditions are equivalent:

- (i) There exists a linearly homogeneous function $f \in \mathcal{F}_d(D)$.
- (ii) There exists a linearly homogeneous function $f \in \mathcal{F}_c(D)$.
- (iii) There exist numbers $\lambda_i > 0$ such that

$$\lambda_i \lambda_j^{-1} c_i(x^j) \geq c_i(x^i) \quad (i, j \in I).$$

(iv) The matrix A is absorptive, where

$$a_{ij} = \ln(c_j(x^i)/c_j(x^j)) \quad (i, j \in I).$$

(b) If the above conditions are satisfied then one has $f \in \mathcal{F}_d(D)$ for every function of the form

$$f(x) = \inf_{i \in I} \lambda_i c_i(x) / c_i(x^i) \quad (x \in X),$$

such that the numbers λ_i satisfy the condition given in (iii).

Proof (a) (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (iii): Suppose $f \in \mathcal{F}_c(D)$ is a linearly homogeneous function. Then by Theorem 3.4 one has $f(x^i) > 0$ for all $i \in I$. Thus from Theorem 3.3 it follows that

$$c_i(f(x^i)f(x^j)^{-1}x^j) \geq c_i(x^i) \quad (i, j \in I).$$

Hence the numbers $\lambda_i = f(x^i)$ satisfy condition (iii), since the functions c_i are by assumption linearly homogeneous.

(iii) $\Rightarrow$ (i): Put

$$f(x) = \inf_{i \in I} \lambda_i c_i(x) / c_i(x^i) \quad (x \in X).$$

Since $\lambda_i \lambda_j^{-1} c_i(x^j) \geq c_i(x^i)$ for all $i, j \in I$, one has now

$$f(x^j) = \inf_{i \in I} \lambda_i c_i(x^j) / c_i(x^i) \geq \lambda_j \quad (j \in I).$$

To show that $f \in \mathcal{F}_d(D)$, let $x \in X$ and $i \in I$. Suppose $c_i(x) \leq c_i(x^i)$. Then clearly $f(x) \leq \lambda_i \leq f(x^i)$. Similarly, $f(x) < \lambda_i \leq f(x^i)$ if $c_i(x) < c_i(x^i)$, thus $f \in \mathcal{F}_d(D)$. Further, since all functions c_i are by assumption element of F , one has obviously $f \in F$.

(iii) $\Leftrightarrow$ (iv): Using $\phi_i = \ln \lambda_i$, one has (iii) equivalent to the existence of numbers ϕ_i such that

$$\phi_i - \phi_j \leq a_{ij} \quad (i, j \in I).$$

Theorem C.2, given in Appendix C, shows that this is equivalent to the condition that A is absorptive.

(b) That we have $f \in \mathcal{F}_d(D)$ follows from the proof of (iii) $\Rightarrow$ (i). $\square$

Figure 3.2 displays an example of price and quantity data for efficient linearly homogeneous consumer behaviour. The quantity data x^i is multiplied by λ_i^{-1} , where λ_i satisfies condition (iii) in Theorem 3.5. We may interpret these numbers as $\lambda_i = f(x^i)$ of a rationalizing linearly homogeneous function f . The figure shows how the numbers λ_i are related to the corresponding unit isoquant.

Theorem 3.5 can be extended to a theorem where the utility functions are restricted to a class of functions with one or more of the following properties: concave, monotonically increasing, upper semicontinuous. Furthermore, when the data set is finite, we may include the properties: strictly concave, strictly monotonically increasing, continuous, uniformly continuous. For a simple proof of such a theorem, we need the assumption that all cost functions c_i has such a property. Then the constructed utility function f in the above proof has this property too.

Figure 3.2 Linearly Homogeneous Behaviour

We can translate the above results for linearly homogeneous consumer demand in terms of symmetry and conserved properties as follows. The symmetry assumption is that the utility function and cost functions are invariant under the family of transformations T_λ such that

$$(T_\lambda f)(x) = \lambda^{-1} f(\lambda x) \quad (\lambda \neq 0).$$

The consumer demand conditions for which this symmetry holds are invariant under translations in the commodity space along the rays with the origin as source. So, for the data we have the symmetry

$$(c, x') \in D_d(f) \Leftrightarrow (c, \lambda x') \in D_d(f) \quad (\lambda > 0).$$

This implies that sets of data D generated by the linearly homogeneous model, satisfy

$$f \in \mathcal{F}_d(D) \text{ linearly homogeneous} \Rightarrow f(x) \leq f(x')c(x)/c(x') \text{ for } (c, x') \in D,$$

assuming that we have $c(x') > 0$. This property is the basis of Theorem 3.5. The right hand inequalities remain valid when we add more data, generated by the same model, to D . So we may consider it as a conserved property. We can draw a parallel for this result in classical mechanics, where a Lagrange function that is translation invariant in space implies the conservation of momentum.

3.4 The Existence Problem in Case of Inefficiency

If we wish to apply Theorem 3.5 to transformed data $\Phi_e D$, where Φ_e is an efficiency transformation, then the transformed cost functions have to be linearly homogeneous. So we may get into problems when using the efficiency transformation of Example 2.3 in which $\Phi_e(c, x') = (c^e, x')$ is defined by

$$c^e(x) = \begin{cases} c(x) & (x = x'), \\ e^{-1}c(x) & (\text{otherwise}). \end{cases}$$

The result c^e is in that case in general not a linearly homogeneous function,

assuming that c has this property. A slightly different efficiency transformation, that does transform linearly homogeneous cost functions into linearly homogeneous cost functions, is the following.

Example 3.1: Let the efficiency transformation $\Lambda_e(c, x') = (c^e, x')$ be such that

$$c^e(x) = \begin{cases} c(x) & (x = \lambda x', \lambda \in \mathbb{R}), \\ e^{-1}c(x) & (\text{otherwise}). \end{cases}$$

Still there remains to solve the same problem as in the previous chapter. What to do when we are searching for a continuous utility function and the efficiency transformation destroys the continuity of the cost functions? In that case we shall not derive a continuous utility function in the proof of Theorem 3.5. To solve this problem we may try to fix the discontinuities in the same way as in Theorem 2.8 in the previous chapter. This approach results in the following theorem.

Theorem 3.6: Suppose $\Phi_e D = \{(c_i, x^i)\}_{i \in I}$ is a finite data set, where Φ_e is an efficiency transformation, $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and $c_i(x^i) > 0$, for all $i \in I$, and X is a cone. Suppose every function c_i is only discontinuous at $\{\lambda x^i | \lambda \in \mathbb{R}\}$ and put

$$\hat{c}_i(x) = \lim_{v \rightarrow x} c_i(v) \quad (i \in I, x \in X),$$

where $v \notin \{\lambda x^i | \lambda \in \mathbb{R}\}$ when we take the limit $v \rightarrow x$. Suppose all functions $\hat{c}_i$ and the cost functions of D are continuous. If there is a linearly homogeneous function in $\mathcal{F}_d(\Phi_e D)$ then there exists a continuous linearly homogeneous function $f \in \mathcal{F}_d(\Phi_e D)$.

Proof: The proof is basically the same as the proof of Theorem 2.8. By replacing c_i by $\hat{c}_i$, we change the rationalizing utility function of Theorem 3.5 (b) into

$$f(x) = \min_{i \in I} \lambda_i \hat{c}_i(x) / c_i(x^i) \quad (x \in X).$$

Then, analogously as is in the proof of Theorem 2.8, we have $f \in \mathcal{F}_d(\Phi_e D)$. Furthermore, f is a continuous linearly homogeneous function, because all functions $\hat{c}_i$ are continuous and linearly homogeneous. $\square$

If we apply the efficiency transformation of Example 3.1 to price and quantity data we can even derive a concave continuous linearly homogeneous function.

Theorem 3.7: Suppose D is a set of price and quantity data and Φ_e is the efficiency transformation of Example 3.1. If there is a linearly homogeneous function in $\mathcal{F}_d(\Phi_e D)$ then there exists a concave continuous linearly homogeneous function $f \in \mathcal{F}_d(\Phi_e D)$.

Proof: In this case all functions $\hat{c}_i$ in Theorem 3.7 are concave continuous linearly homogeneous. Hence the function $f \in \mathcal{F}_d(\Phi_e D)$ in the proof of Theorem 3.7 is a concave linearly homogeneous rationalizing utility function. In that case even an infinite data set is allowed, because the infimum over an arbitrary set of concave functions is a concave function. $\square$

Let us denote the efficiency transformations in Example 2.3 and Example 3.1 as Φ_e and Λ_e respectively. If we are comparing efficiency upper bounds for both efficiency transformations, one may ask which of the following assumptions is stronger: (a) $f \in \mathcal{F}_d(\Phi_e D)$ or (b) $f \in \mathcal{F}_d(\Lambda_e D)$. In the theorem below I show that both assumptions are equivalent to each other in case f is a linearly homogeneous utility function.

Theorem 3.8: Suppose $f: X \rightarrow \mathbb{R}$ and $c: X \rightarrow \mathbb{R}_+$ are linearly homogeneous, $x' \in X$, $c(x') > 0$, and X is a cone. Then for the efficiency transformations Φ_e and Λ_e as given above, one has $\Phi_e(c, x') \in D_d(f)$ if and only if $\Lambda_e(c, x') \in D_d(f)$.

Proof: Let us use the notation $(a, x') = \Phi_e(c, x')$ and $(b, x') = \Lambda_e(c, x')$. So, we have

$$a(x) = \begin{cases} c(x) & (x = x'), \\ e^{-1}c(x) & (\text{otherwise}), \end{cases}$$

$$b(x) = \begin{cases} c(x) & (x = \lambda x', \lambda \in \mathbb{R}), \\ e^{-1}c(x) & (\text{otherwise}). \end{cases}$$

($\Leftarrow$): The conditions for consumer demand are automatically satisfied for (a, x') in the area

$$\{x \in X \mid x = \lambda x', \lambda < 0\},$$

because we have

$$a(\lambda x') = \lambda a(x') < a(x') = c(x') \quad (\lambda < 0),$$

since c is linearly homogeneous and $c(x') > 0$.

For x in the remaining area we have $b(x) \leq a(x)$, because

$$c(\lambda x') \leq e^{-1}c(\lambda x') \quad (\lambda \geq 0).$$

Hence, for x in the area where we have to show that the conditions for consumer demand are satisfied, we have $b(x) \leq a(x)$. This and $b(x') = a(x')$ imply that we have

$$(b, x) \in D_d(f) \Rightarrow (a, x') \in D_d(f).$$

($\Rightarrow$): Let $(a, x') \in D_d(f)$. Then $a(x') = c(x') > 0$ implies that we have

$$a(0) = 0 < a(x') \Rightarrow f(0) = 0 < f(x').$$

So, because both b and f are linearly homogeneous and positive at x' , we have

$$b(x) \leq b(x') \Leftrightarrow f(x) \leq f(x') \quad (x = \lambda x', \lambda \in \mathbb{R}).$$

Hence, for $x = \lambda x'$, $\lambda \in \mathbb{R}$, the conditions of consumer demand are automatically satisfied. In the remaining area, we have

$$b(x) = a(x) \quad (x \in X: x \neq \lambda x' \text{ for all } \lambda \in \mathbb{R}).$$

This, together with $b(x') = a(x')$, imply that in the remaining area there is no difference between both functions a and b with respect to the conditions of consumer demand. So, we have

$$(a, x') \in D_d(f) \Rightarrow (b, x') \in D_d(f).$$

□

Varian (1983) describes also necessary and sufficient conditions for the existence of a rationalizing linearly homogeneous utility function. A difference with the theorems in this chapter is that Varian considers economical efficient consumer demand for finite data sets of prices and quantities. The theorems in the current and the previous section generalize Varian's results in two ways. One is the extension of the results to consumer demand, which includes efficient consumer behaviour for price and quantity data as a special case. The second generalization is the obvious extension from finite to infinite data sets.

3.5 A Linearly Homogeneous Function for Dutch Consumer Data

A few researchers tested whether consumption data satisfy the nonparametric hypothesis of homotheticity of the utility function. Manser and McDonald (1988) found no violation of the homotheticity hypothesis for U.S. consumption data, and Diewert and Parkan (1978) found a slight violation of the homotheticity hypothesis for Canadian consumption data. They use a measure for the violation of the homotheticity hypothesis that is not very easy to compute. Furthermore, their measure has not a clear economical interpretation.

We may use the efficiency transformation given in Example 3.1 to define an efficiency measure for price and quantity data. Using this efficiency transformation, we can compute the efficiency upper bound for the Dutch consumption data with respect to the family of homothetic utility functions. Using condition (iv) of Theorem 3.5 and a bisection algorithm, we can compute this upper bound, as is made clear in section 2.6. The result was $e = 0.9996$ and the hypothesis of homotheticity is only slightly violated.

In addition to this, I determined the maximal subsets of observations, which are consistent with the homotheticity hypothesis at efficiency level one. To obtain consistency it was sufficient to delete only one observation, where the choice is to delete the observation for 1951 or, otherwise, the observation for 1952. By coincidence both these years are mentioned in Chapter 2, concerning the existence of a rationalizing utility functions, where we had to interchange them to obtain the lower triangular matrix in Table 2.3.

3.6 Restrictions on Linearly Homogeneous Utility Functions

The nonparametric approach allows the derivation of an impression of consumers utility function without the need of a parametric specification of the utility function and estimation of the parameters. An example of this approach is given in Manser and McDonald (1988). They used the nonparametric theory of Afriat (1972, 1981) to compute nonparametric lower and upper bounds for linearly homogeneous utility function values for observed U.S. consumer data. They could compute these bounds, because the consumer data satisfied the hypothesis of efficient homothetic utility maximization. In other empirical studies, however, the homotheticity hypothesis is rejected. Diewert and Parkan (1978) found a slight violation of this hypothesis for the Canadian consumer data they considered, and a similar result is obtained in the previous section for Dutch consumer data. So, often the homotheticity hypothesis is falsified, and one is not able to compute Afriat's nonparametric bounds.

In the remaining part of this chapter I generalize Afriat's theory, that is concerned with efficient consumer demand for price and quantity data. An important result of the generalized theory is the existence of nonparametric lower and upper bounds for any given consumer data, provided one lowers the consumer's efficiency level sufficiently. I show how these nonparametric bounds may be constructed. Furthermore, I prove – under weak conditions – that the nonparametric bounds are rationalizing utility functions for the consumer data.

To derive the restrictions on rationalizing linearly homogeneous utility functions I use the following definition.

Definition: Let F be a collection of functions $f: X \rightarrow \mathbb{R}$. Then the *relative lower bound* L_f is defined as

$$L_f(F, x, x') \equiv \inf_{f \in F} f(x)/f(x') \quad (x, x' \in X).$$

Similarly the relative upper bound U_f is

$$U_f(F, x, x') \equiv \sup_{f \in F} f(x)/f(x') \quad (x, x' \in X).$$

First I derive two elementary theorems concerning relative bounds.

Theorem 3.9: Suppose $x^i \in X$ for $i=1,2$ and F is a collection of functions $f: X \rightarrow \mathbb{R}$ such that $f(x^i) > 0$ for $i=1,2$. Then

$$L_f(F, x^1, x^2) = U_f(F, x^2, x^1)^{-1},$$

where we define $\infty^{-1} = 0$.

Proof: One has

$$\inf_{a \in A} a = (\sup_{a \in A} a)^{-1} \quad (A \subset (0, \infty)). \quad \square$$

One has the following theorem for relative upper bounds. A similar theorem, where less than or equal is replaced by larger than or equal, is valid for relative lower bounds.

Theorem 3.10: Suppose $x^i \in X$ for $i=1, \dots, n$ and F is a collection of functions $f: X \rightarrow \mathbb{R}$ such that $f(x^i) > 0$ for $i=1, \dots, n$. Then

$$U_f(F, x^0, x^n) \leq \prod_{i=1}^n U_f(F, x^{i-1}, x^i).$$

Proof: One has

$$f(x^0)/f(x^n) = \prod_{i=1}^n f(x^{i-1})/f(x^i) \leq \prod_{i=1}^n U_f(F, x^{i-1}, x^i) \quad (f \in F). \quad \square$$

3.7 Construction of Nonparametric Bounds

First I consider a simple data set which contains one element.

Theorem 3.11: Suppose $c: X \rightarrow \mathbb{R}_+$ is a linearly homogeneous function, $x' \in X$, $c(x') > 0$, and X is a cone. Put

$$F = \{f \in \mathcal{F}_d(\{(c, x')\}) \mid f \text{ linearly homogeneous}\}.$$

Then one has $U_f(F, \cdot, x') \in F$ and

$$U_f(F, x, x') = c(x)/c(x') \quad (x \in X).$$

Proof: Put

$$g(x) = c(x)/c(x') \quad (x \in X).$$

One can easily show that one has $g \in F$ and $g(x') = 1$. Thus

$$g(x) \leq U_f(F, x, x') \quad (x \in X).$$

Now, let f be an arbitrary function in F . Then $f(x') > 0$ by Theorem 3.4. Furthermore, since f and c are linearly homogeneous, Theorem 3.3 implies

$$f(x)/f(x') \leq c(x)/c(x') = g(x) \quad (x \in X).$$

Hence

$$U_f(F, x, x') \leq g(x) \quad (x \in X),$$

since f was an arbitrary function in F . So, both inequalities imply $U_f(F, x, x') = g(x)$ for all $x \in X$. $\square$

The above example is simple. The aim is now to derive a similar result that applies to consumer data sets of an arbitrary number of elements. For this

result we need the definition of the *weak closure* A^* of a square matrix A , which is defined in Appendix C. The idea of the weak closure is the following. Let the cost of a single step that moves from $i \in I$ to $j \in I$ be equal to a_{ij} . Then the weak closure a_{ij}^* , for $i, j \in I$, represents the cost of the cheapest path from i to j . We may define the weak closure as follows.

Definition: The *weak closure* A^* of a matrix $A \in \mathbb{R}^n \times \mathbb{R}^n$ defined as

$$a_{ij}^* = \inf a_{ik} + a_{kl} + \dots + a_{mn} + a_{nj},$$

where the infimum is taken over all possible sequences $k, l, \dots, m, n \in I$.

Using this definition, we formulate in the following theorem relative upper and lower bounds for linearly homogeneous consumer demand.

Theorem 3.12: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and $c_i(x^i) > 0$, for all $i \in I$, and X is a cone. Put

$$F = \{f \in \mathcal{F}_d(D) \mid f \text{ linearly homogeneous}\},$$

$$a_{ij} = \ln(c_j(x^i)/c_i(x^j)) \quad (i, j \in I).$$

If $F \neq \emptyset$ then one has $U_i(F, \cdot, x^j) \in F$ for all $j \in I$, and

$$U_i(F, x, x^j) = \inf_{i \in I} [c_i(x)/c_i(x^i)] e^{a_{ij}^*} \quad (j \in I, x \in X),$$

$$U_i(F, x^i, x^j) = e^{a_{ij}^*}, \quad U_i(F, x^i, x^j)^{-1} = L_i(F, x^j, x^i) \quad (i, j \in I).$$

Proof: Suppose $F \neq \emptyset$. Then, by Theorem 3.11 and Theorem 3.5, the matrix A has to be absorptive. This means that the weak closure A^* is well-defined and we have $a_{ij}^* \leq a_{ik} + a_{kj}$, or equivalently

$$e^{a_{ij}^*} \leq [c_k(x^i)/c_k(x^k)] e^{a_{kj}^*} \quad (k, i, j \in I).$$

Let $j \in I$ and put

$$g(x) = \inf_{i \in I} [c_i(x)/c_i(x^i)] e^{a_{ij}^*} \quad (x \in X).$$

Hence, one has

$$g(x^i) = \inf_{k \in I} [c_k(x^i)/c_k(x^k)] e^{a_{kj}^*} = e^{a_{ij}^*} \quad (i \in I).$$

To prove that one has $g \in F$, define

$$f_i(x) = c_i(x)/c_i(x^i) \quad (i \in I, x \in X),$$

and thus $f_i \in \mathcal{F}_d(\{(c_i, x^i)\})$ by Theorem 3.11. So one has

$$c_i(x) \leq c_i(x^i) \Rightarrow g(x) \leq f_i(x) e^{a_{ij}^*} \leq f_i(x^i) e^{a_{ij}^*} = g(x^i) \quad (i \in I, x \in X),$$

$$c_i(x) < c_i(x^i) \Rightarrow g(x) < f_i(x) e^{a_{ij}^*} < f_i(x^i) e^{a_{ij}^*} = g(x^i) \quad (i \in I, x \in X).$$

Hence $g \in \mathcal{F}_d(D)$, and because g is obviously a linearly homogeneous function we have thus $g \in F$.

From $a_{jj} = 0$ follows $a_{jj}^* = 0$, because A is absorptive. So one has

$$g(x^j) = e^{a_{jj}^*} = 1,$$

which in conjunction with $g \in F$ means

$$g(x) = g(x)/g_j(x^j) \leq U_i(F, x, x^j) \quad (x \in X).$$

Thus to prove $g(x) = U_i(F, x, x^j)$, it remains to show that $U_i(F, x, x^j) \leq g(x)$. Put

$$F_i = \{f \in \mathcal{F}_d(\{(c_i, x^i)\}) \mid f \text{ linearly homogeneous}\}.$$

One has clearly $F \subset F_i$, so that

$$U_f(F, x, x^i) \leq U_f(F_i, x, x^i) \quad (i \in I, x \in X).$$

Now, because Theorem 3.4 implies $f(x^i) > 0$, for all $i \in I$ and $f \in F$, one has by Theorem 3.10 that

$$\begin{aligned} U_f(F, x, x^{i_m}) &\leq U_f(F, x, x^{i_0}) \prod_{n=1}^m U_f(F, x^{i_{n-1}}, x^{i_n}) \\ &\leq U_f(F_{i_0}, x, x^{i_0}) \prod_{n=1}^m U_f(F_{i_n}, x^{i_{n-1}}, x^{i_n}). \quad (i_0, \dots, i_m \in I, x \in X). \end{aligned}$$

The latter implies that we have

$$U_f(F, x, x^j) \leq U_f(F_i, x, x^i) e^{a_{ij}^*} \quad (i \in I, x \in X).$$

Hence, since by Theorem 3.11 one has

$$U_f(F_i, x, x^i) = c_i(x)/c_i(x^i) \quad (i \in I, x \in X),$$

we obtain

$$U_f(F, x, x^j) \leq g(x) \quad (x \in X),$$

which we had to show.

Finally, from Theorem 3.9 follows $L_f(F, x^j, x^i) = U_f(F, x^i, x^j)^{-1}$ for all $i, j \in I$. $\square$

Theorem 3.12 generalizes Afriat's theory in several ways. First, I have used a more general description of consumer behaviour. Secondly, the theorem is valid for infinite data sets. Thirdly, I have derived the upper bound $U_f(F, x, x^j)$ for any $x \in X$, whereas Afriat is only concerned with the upper bounds $U_f(F, x^i, x^j)$, restricted to consumer's quantity choices.

Note that the lower bound $L_f(F, x, x^j)$ is not very interesting if $x \neq x^i$ for all $i \in I$. Then one has clearly $L_f(F, x, x^j) = -\infty$. For a finite lower bound in this case one has to restrict F to a smaller family of functions, for instance, the family of nonnegative linearly homogeneous functions.

Theorem 3.13: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and $c_i(x^i) > 0$, for all $i \in I$, and X is a cone. Put

$$F = \{f \in \mathcal{F}_d(D) \mid f \text{ linearly homogeneous}\}, \quad \bar{F} = \{f \in F \mid f \text{ nonnegative}\}.$$

Then one has:

$$L_f(\bar{F}, x, x^i) = \max \{L_f(F, x, x^i), 0\} \quad (i \in I, x \in X).$$

Proof: One can easily show that one has

$$\bar{F} = \{g \mid f \in F, g(x) = \max \{f(x), 0\}\}.$$

From this it is straightforward to derive the result. $\square$

Summarizing the above we have derived the following result. For linearly homogeneous consumer demand there is a symmetry for the cost functions and the utility function. For the resulting consumer behaviour this symmetry implies a translation symmetry in the commodity space along the rays that have the origin as source. Using this translation symmetry we have derived an expression for the relative upper bound. This bound represents a conserved property that can be described as follows. Let D be a set of data and let $\mathcal{L}$ denote the family of linearly homogeneous functions. Then we have

$$f \in \mathcal{F}_d(D) \cap \mathcal{L} \Rightarrow f(x)/f(x') \leq U_f(\mathcal{F}_d(D') \cap \mathcal{L}, x, x') \quad (f(x') > 0, D' \subset D).$$

The relative upper bound $U_f(\mathcal{F}_d(D') \cap \mathcal{L}, x, x')$ is given in Theorem 3.12. So, we have found a conserved property for linearly homogeneous consumer demand, because the right hand inequalities remain conserved when data is added to D .

Remark: The above implication is a special case of the following result. For sets F, F' of functions one has in general

$$f \in F \Rightarrow f(x)/f(x') \leq U_f(F', x, x') \quad (f(x') > 0, F \subset F').$$

Note that $D' \subset D$ implies $\mathcal{F}_d(D) \subset \mathcal{F}_d(D')$, because an increase in data information decreases the rationalizing set of functions, so that we have indeed

$$\mathcal{F}_d(D) \cap \mathcal{L} \subset \mathcal{F}_d(D') \cap \mathcal{L} \quad (D' \subset D).$$

3.8 Nonparametric Bounds for Dutch Consumer Data

In this section the above theory is applied to the Dutch consumer data. The nonparametric lower and upper bounds are computed for rationalizing linearly homogeneous utility functions at efficiency level $e=0.9996$. This efficiency level is the efficiency upper bound for linearly homogeneous behaviour, we found earlier in this chapter. The first year 1951 is used as base year. As described in Theorem 3.12, these nonparametric bounds follow from the shortest paths in a labelled graph with the adjacency matrix $A=[a_{ij}]$, such that

$$a_{ij} = \begin{cases} \lambda & (x^i = \lambda x^j, \lambda > 0), \\ e^{-1} P_{ij} & (\text{otherwise}), \end{cases}$$

where $P_{ij} = p^j x^i / p^i x^j$ is the direct Laspeyres index for current period i and base period j . Because the Dutch consumption data have the property $x^i \neq \lambda x^j$ if $i \neq j$, as one may expect of empirical data, this matrix has the form

$$a_{ij} = \begin{cases} 1 & (i = j), \\ e^{-1} P_{ij} & (\text{otherwise}). \end{cases}$$

Figure 3.3 Bounds Linearly Homogeneous Utility

I computed the nonparametric lower and upper bounds for every year i with base year $k=1951$. The resulting bounds in Figure 3.3 provide a tight

estimation for the utility values of a rationalizing linearly homogeneous utility function. By Theorem 3.12 these bounds are

$$L_j(F, x^i, x^k) = e^{-a_{ki}^*}, \quad U_j(F, x^i, x^k) = e^{a_{ik}^*},$$

where F is the family of rationalizing linearly homogeneous utility functions for data transformed using efficiency level e .

Table 3.1 Quantity Index Numbers (base year 1951)

Year	Direct		Chained		Nonparametric	
	Paasche	Laspeyres	Paasche	Laspeyres	Lower	Upper
1951	1.000	1.000	1.000	1.000	1.000	1.000
1952	0.968	0.967	0.968	0.967	0.967	0.967
1953	1.031	1.034	1.030	1.032	1.031	1.033
1954	1.104	1.113	1.104	1.109	1.104	1.110
1955	1.157	1.170	1.158	1.164	1.158	1.166
1956	1.242	1.269	1.246	1.256	1.246	1.259
1957	1.219	1.250	1.227	1.239	1.227	1.240
1958	1.192	1.217	1.202	1.215	1.202	1.214
1959	1.244	1.276	1.255	1.269	1.256	1.269
1960	1.311	1.359	1.328	1.344	1.328	1.344
1961	1.373	1.433	1.394	1.412	1.394	1.413
1962	1.432	1.507	1.463	1.484	1.462	1.485
1963	1.534	1.631	1.573	1.597	1.572	1.599
1964	1.579	1.700	1.621	1.654	1.620	1.656
1965	1.655	1.803	1.711	1.748	1.709	1.752
1966	1.668	1.822	1.730	1.770	1.730	1.773
1967	1.706	1.865	1.773	1.815	1.772	1.820
1968	1.771	1.955	1.846	1.891	1.844	1.896
1969	1.850	2.069	1.942	1.992	1.940	1.998
1970	1.953	2.226	2.062	2.117	2.061	2.124
1971	2.014	2.288	2.120	2.179	2.122	2.186
1972	2.063	2.360	2.184	2.248	2.185	2.256
1973	2.140	2.459	2.272	2.339	2.271	2.349
1974	2.183	2.517	2.313	2.388	2.315	2.399
1975	2.240	2.601	2.377	2.456	2.379	2.468
1976	2.325	2.720	2.482	2.568	2.485	2.581
1977	2.408	2.827	2.558	2.652	2.562	2.666

Table 3.1 compares the nonparametric lower and upper bounds for base year 1951 with direct and chained Paasche and Laspeyres quantity index numbers with base year 1951.

As known, and as one can derive from the theory in this chapter, the Paasche and Laspeyres index numbers are lower and upper bounds of the nonparametric lower and upper bounds respectively, provided consumer behaviour is efficient and linearly homogeneous. However, since the consumption data was only nearly efficient linearly homogeneous, the Paasche and Laspeyres index numbers do in some cases slightly violate the nonparametric bounds. It is remarkable how close the chained Paasche and Laspeyres index numbers are to the nonparametric lower and upper bounds respectively, while this is not the case for the direct Paasche and Laspeyres index numbers.

In order to analyse the results for the nonparametric bounds, I investigated the shortest paths for a_{ij}^* , from which these bounds follow. These paths are

given in Table 3.2 and Table 3.3.

It is striking that the sequences for the shortest paths are nearly all in time order except for the recession years 1956, 1957 and 1958 for the upper bound computations. Especially because the method to derive the shortest paths is independent of the order of the observations.

Table 3.2 Shortest Paths for Computation Lower Bounds

Year	Bound	Path
1951	1.000	51 51
1952	0.967	51 52
1953	1.031	51 53
1954	1.104	51 54
1955	1.158	51 54 55
1956	1.246	51 54 56
1957	1.227	51 54 56 57
1958	1.202	51 54 56 58
1959	1.256	51 54 56 59
1960	1.328	51 54 56 59 60
1961	1.394	51 54 56 59 60 61
1962	1.462	51 54 56 59 60 62
1963	1.572	51 54 56 59 60 62 63
1964	1.620	51 54 56 59 60 62 64
1965	1.709	51 54 56 59 60 62 64 65
1966	1.730	51 54 56 59 60 62 64 66
1967	1.772	51 54 56 59 60 62 64 66 67
1968	1.844	51 54 56 59 60 62 64 65 68
1969	1.940	51 54 56 59 60 62 64 65 69
1970	2.061	51 54 56 59 60 62 64 65 68 70
1971	2.122	51 54 56 59 60 62 64 65 68 71
1972	2.185	51 54 56 59 60 62 64 65 68 71 72
1973	2.271	51 54 56 59 60 62 64 65 68 71 72 73
1974	2.315	51 54 56 59 60 62 64 65 68 74
1975	2.379	51 54 56 59 60 62 64 65 68 75
1976	2.485	51 54 56 59 60 62 64 65 68 74 76
1977	2.562	51 54 56 59 60 62 64 65 68 75 77

One might think this is caused by the increase of real income per capita over time. However, this is certainly not the case, because multiplication of the quantity vectors x^i by arbitrary numbers $\gamma_i > 0$ may change the nonparametric bounds for the data set, but will not affect the years which appear in Table 3.2 and 3.3.

Remark: After multiplication of x^i with γ_i , the new adjacency matrix $A' = [a'_{ij}]$ will be of the form

$$a'_{ij} = a_{ij} + \alpha_i - \alpha_j, \quad \alpha_i = \ln \gamma_i.$$

Now, the difference of the cost of any path μ from i to j at adjacency matrix A' and adjacency matrix A , respectively, will be $\alpha_i - \alpha_j$, because all α_k terms from years (vertices) in between will cancel each other out. Hence, the same years (vertices) appear in the shortest paths for both adjacency matrices A and A' , although the cost of the shortest paths may differ. So an increase of real income is no explanation for the resulting shortest paths.

The reason why the shortest paths are in time order becomes clear by

considering the nonparametric upper bound

$$U_j(F, x, x^j) = \min_i [c_i(x)/c_i(x^i)] e^{a_{ij}^*},$$

given in Theorem 3.12, where

$$F = \{f \mid f \in \mathcal{F}_d(\{(c_i, x^i)\}_{i \in I}), f \text{ linearly homogeneous}\},$$

$$(c_i, x^i) = \Phi_e(p^i, x^i) \quad (i \in I).$$

The reader may verify that a shortest path from x^k to x^j with cost a_{kj}^* can be represented by a curve from x^k to x^j at the surface of the function $U_j(F, x, x^j)$. It visits the vertices x^i on the shortest path from x^k to x^j and is a curve along the corresponding active constraints

$$[c_i(x)/c_i(x^i)] e^{a_{ij}^*},$$

that determine $U_j(F, x, x^j)$. We may scale all points x on this curve, using a factor $\lambda(x)$ such that $\lambda(x)^{-1}x$ represents a indifference curve. The shortest paths will be in time order when the scaled quantities $\lambda(x^i)^{-1}x^i$ on such an indifference curve tend to move in the same direction when time increases.

Table 3.3 Shortest Paths for Computation Upper Bounds

Year	Bound	Path
1951	1.000	51 51
1952	0.967	52 51
1953	1.033	53 52 51
1954	1.110	54 52 51
1955	1.166	55 54 52 51
1956	1.259	56 58 55 54 52 51
1957	1.240	57 58 55 54 52 51
1958	1.214	58 55 54 52 51
1959	1.269	59 58 55 54 52 51
1960	1.344	60 58 55 54 52 51
1961	1.413	61 60 58 55 54 52 51
1962	1.485	62 60 58 55 54 52 51
1963	1.599	63 62 60 58 55 54 52 51
1964	1.656	64 63 62 60 58 55 54 52 51
1965	1.752	65 64 63 62 60 58 55 54 52 51
1966	1.773	66 64 63 62 60 58 55 54 52 51
1967	1.820	67 66 64 63 62 60 58 55 54 52 51
1968	1.896	68 67 66 64 63 62 60 58 55 54 52 51
1969	1.998	69 66 64 63 62 60 58 55 54 52 51
1970	2.124	70 69 66 64 63 62 60 58 55 54 52 51
1971	2.186	71 69 66 64 63 62 60 58 55 54 52 51
1972	2.256	72 71 69 66 64 63 62 60 58 55 54 52 51
1973	2.349	73 71 69 66 64 63 62 60 58 55 54 52 51
1974	2.399	74 73 71 69 66 64 63 62 60 58 55 54 52 51
1975	2.468	75 73 71 69 66 64 63 62 60 58 55 54 52 51
1976	2.581	76 73 71 69 66 64 63 62 60 58 55 54 52 51
1977	2.666	77 76 73 71 69 66 64 63 62 60 58 55 54 52 51

This phenomenon is illustrated in Figure 3.4 for a simple two-dimensional example of efficient consumer behaviour. When the rays, from the origin through the quantity observations, tend to move in a certain direction when time increases, then the shortest paths will tend to be in time order. This in turn, in conjunction with an efficiency level close to one, explains the

resemblance of the chained Paasche and Laspeyres index numbers to the nonparametric lower and upper bounds respectively.

Figure 3.4 Linearly Homogeneous Demand

3.9 Approximating Nonparametric Bounds for Concave Functions

Unfortunately, when we consider concave linearly homogeneous functions, the nonparametric upper bound restrictions become more complicated. There is no problem if the nonparametric upper bound for all rationalizing linearly homogeneous functions is concave. Then this bound is equal to the upper bound for the subset of *concave* linearly homogeneous functions. However, it may happen that we do not obtain such a concave upper bound. For instance, this is the case if price and quantity data is transformed, using an efficiency transformation. In that case we may try to approximate the upper bound for concave linearly homogeneous functions, using the nonparametric bounds for the case without the restriction of concavity. This approach is used in the following theorem.

Theorem 3.14: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and $c_i(x^i) > 0$, for all $i \in I$, and X is a cone. Put

$$F = \{f \in \mathcal{F}_d(D) \mid f \text{ linearly homogeneous}\}, \quad \bar{F} = \{f \in F \mid f \text{ concave}\},$$

$$\hat{c}_i(x) = \lim_{v \rightarrow x} c_i(v) \quad (i \in I, x \in X),$$

where we take the limit along $v \in \{\lambda x^i \mid \lambda \in \mathbb{R}\}$. Suppose all functions $\hat{c}_i$ are concave linearly homogeneous and

$$\hat{c}_i(x) \geq c_i(x) \quad (i \in I, x \in X).$$

Then the nonparametric bounds for $\bar{F}$ satisfy

$$[c_j(x^j)/\hat{c}_j(x^j)]U_I(F, x, x^j) \leq U_I(\bar{F}, x, x^j) \leq U_I(F, x, x^j) \quad (j \in I, x \in X),$$

$$U_I(\bar{F}, x^i, x^j)^{-1} = L_I(\bar{F}, x^j, x^i) \quad (i, j \in I).$$

Proof: Put

$$a_{ij} = \ln(c_j(x^i)/c_j(x^j)) \quad (i, j \in I).$$

Moreover, let $j \in I$ and put

$$\hat{g}(x) = \inf_{i \in I} [\hat{c}_i(x)/c_i(x^i)] e^{a_{ij}^*} \quad (x \in X),$$

$$g(x) = \inf_{i \in I} [c_i(x)/c_i(x^i)] e^{a_{ij}^*} \quad (x \in X).$$

As is shown in the proof of Theorem 3.6 and using Theorem 3.12, it follows that one has $g \in F$, $\hat{g} \in \bar{F}$, and $g \leq \hat{g}$. Furthermore, by Theorem 3.12 one has

$$g(x) = U_I(F, x, x^j) \quad (x \in X).$$

Now, because $\bar{F} \subset F$, we have

$$U_I(\bar{F}, x, x^j) \leq U_I(F, x, x^j) \quad (x \in X)$$

and thus

$$g(x)/\hat{g}(x^j) \leq \hat{g}(x)/\hat{g}(x^j) \leq U_I(\bar{F}, x, x^j) \leq g(x) \quad (x \in X),$$

where

$$g(x) = U_I(F, x, x^j) \quad (x \in X).$$

So the result follows from the inequality

$$\hat{g}(x^j) \leq [\hat{c}_j(x^j)/c_j(x^j)] e^{a_{jj}^*} = \hat{c}_j(x^j)/c_j(x^j),$$

that is obtained by using the definition of $\hat{g}$ and $a_{jj}^* = 0$. Finally, as in the proof of Theorem 3.12, it follows from Theorem 3.9 that

$$L_I(\bar{F}, x^j, x^i) = U_I(\bar{F}, x^i, x^j)^{-1} \quad (i, j \in I). \quad \square$$

In case the efficiency transformation A_e given in Example 3.1 is applied to data D with concave cost functions, then the above theorem yields the following simple estimation of the nonparametric upper bound belonging to the transformed data

$$eU_I(F, x, x^j) \leq U_I(\bar{F}, x, x^j) \leq U_I(F, x, x^j) \quad (j \in I, x \in X),$$

where

$$F = \{f \in \mathcal{F}_d(A_e D) \mid f \text{ linearly homogeneous}\}, \quad \bar{F} = \{f \in F \mid f \text{ concave}\}.$$

This follows immediately from Theorem 3.14, because we obtain in this theorem concave linearly homogeneous functions $\hat{c}_i$, such that

$$\hat{c}_i(x) \geq c_i(x) \quad (i \in I, x \in X),$$

$$c_j(x^j)/\hat{c}_j(x^j) = e \quad (j \in I).$$

3.10 Rationalizing Nonparametric Bounds

The following theorem shows, under weak conditions, that nonparametric lower and upper bounds of sets of rationalizing linearly homogeneous functions do rationalize consumer data.

Theorem 3.15: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and $c_i(x^i) > 0$, for all $i \in I$, and X is a cone. Put

$$F = \{f \in \mathcal{F}_d(D) \mid f \text{ linearly homogeneous}\},$$

where $\bar{F} \neq \emptyset$ is an arbitrary subset of F . Then one has for all $j \in I$:

(a) $L_I(\bar{F}, \cdot, x^j) \in F$, if it is a well-defined finite function.

(b) $U_I(\bar{F}, \cdot, x^j) \in F$.

Proof (a): Obviously $L_I(\bar{F}, \cdot, x^j)$ is a linearly homogeneous function as the

pointwise infimum over linearly homogeneous functions of the form

$$\{f(\cdot)/f(x^j) \mid f \in \bar{F}\}.$$

In Theorem 5.4, that is proved later on, it is shown that such a pointwise infimum over utility maximizing functions is a utility maximizing function itself, assuming that it is a well-defined finite function. So, because $f \in \mathcal{F}_u(D)$ implies clearly $f(\cdot)/f(x^j) \in \mathcal{F}_u(D)$, we have $L_j(\bar{F}, \cdot, x^j) \in \mathcal{F}_u(D)$.

Now, it remains to prove that $L_j(\bar{F}, \cdot, x^j)$ is a cost minimizing function. From Theorem 3.4 we obtain $f(x^i) > 0$ for all $f \in F$ and $i \in I$. Hence, Theorem 3.3 implies

$$c_i(x^j)f(x^i)/f(x^j) \geq c_i(x^i) \quad (f \in F, i, j \in I),$$

so that

$$L_j(\bar{F}, x^i, x^j) \geq c_i(x^i)/c_i(x^j) > 0 \quad (i, j \in I).$$

Thus, since $L_j(\bar{F}, \cdot, x^j) \in \mathcal{F}_u(D)$, it follows from Theorem 3.2 that one has $L_j(\bar{F}, \cdot, x^j) \in \mathcal{F}_c(D)$.

(b): The proof is similar to the proof of (a). We only have to show that every $U_j(\bar{F}, \cdot, x^j)$ is a well-defined finite function. From Theorem 3.3 we obtain the inequality

$$f(x)/f(x^j) \leq c_j(x)/c_j(x^j) \quad (f \in F, j \in I, x \in X).$$

Thus every $U_j(\bar{F}, \cdot, x^j)$ is a well-defined finite function. $\square$

From the above theorem, we can derive a theorem concerning the nonparametric lower bounds for concave linearly homogeneous functions.

Theorem 3.16: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and $c_i(x^i) > 0$, for all $i \in I$, and X is a cone. Put

$$F = \{f \in \mathcal{F}_d(D) \mid f \text{ linearly homogeneous}\}, \bar{F} = \{f \in F \mid f \text{ nonnegative concave}\},$$

where $\bar{F} \neq \emptyset$ is an arbitrary subset of F . Then $L_j(\bar{F}, \cdot, x^j) \in \bar{F}$ for all $j \in I$, and:

(a) If X is convex then

$$L_j(\bar{F}, x, x^j) = \sup \left\{ \sum_{i \in I} \lambda_i L_j(\bar{F}, x^i, x^j) \mid \sum_{i \in I} \lambda_i x^i \leq x, \lambda_i \geq 0 \right\} \quad (j \in I, x \in X),$$

where only a finite number of λ_i may be unequal to zero.

(b) If all functions c_i , $i \in I$, are concave then

$$L_j(\bar{F}, x^i, x^j) = U_j(F, x^j, x^i)^{-1} \quad (i, j \in I).$$

Proof: Let $j \in I$. From Theorem 3.15(a) one obtains $L_j(\bar{F}, \cdot, x^j) \in F$. Furthermore, $L_j(\bar{F}, x, x^j) \in \bar{F}$ as the pointwise infimum over functions in $\bar{F}$.

(a): Let $j \in I$ and put

$$f(x) = \sup \left\{ \sum_{i \in I} \lambda_i L_j(\bar{F}, x^i, x^j) \mid \sum_{i \in I} \lambda_i x^i \leq x, \lambda_i \geq 0 \right\} \quad (x \in X),$$

where only a finite number of λ_i may be unequal to zero. The function $L_j(\bar{F}, \cdot, x^j)$ is a nonnegative concave linearly homogeneous function on a convex cone X . Hence the definition of f implies

$$\begin{aligned} L_j(\bar{F}, x, x^j) &\geq f(x) \quad (x \in X), \\ f(x^i) &= L_j(\bar{F}, x^i, x^j) \quad (i \in I). \end{aligned}$$

Thus obviously $f \in \bar{F}$, since $L_j(\bar{F}, \cdot, x^j) \in \bar{F}$. Now, this implies

$$L_j(\bar{F}, x, x^j) \leq f(x)/f(x^j) = f(x)/L_j(\bar{F}, x^j, x^j) = f(x) \quad (x \in X),$$

so we have $L_j(\bar{F}, x, x^j) = f(x)$ for all $x \in X$.

(b): Follows immediately from Theorem 3.14. $\square$

4 WEAKLY SEPARABLE CONSUMER DEMAND

4.1 Introduction

In empirical studies there is an extensive use of weakly separable utility functions. In literature there are many examples which study the local properties of these functions. For an encyclopedic discussion of separability, see Blackorby *et al.* (1979). However, there only are few theoretical results, concerning weakly separable functions, based on a revealed preference approach. Some results in this area may be found in Diewert and Parkan (1978, 1985), and Varian (1982), who consider only concave weakly separable utility functions. This chapter considers weakly separable utility functions in general. Although it will be pointed out how one may apply the theory to data, the chapter contains no application to the Dutch consumer data.

4.2 Weakly Separable Utility Functions

In this book I will use the following definition of weak separability.

Definition: We say that a function $f: X_1 \times X_2 \rightarrow \mathbb{R}$ is *weakly separable* in X_2 if there exists a function $h: X_2 \rightarrow \mathbb{R}$ such that f can be written in the form

$$f(x_1, x_2) = g(x_1, h(x_2)) \quad (x_1 \in X_1, x_2 \in X_2).$$

The function h is called a *subfunction* and g an *aggregator* function.

Remark: In literature one assumes often that the aggregator function $g(x_1, \eta)$ is strictly increasing in its second argument.

If rational demand is generated by a weakly separable utility function $g(x_1, h(x_2))$, such that g is strictly increasing in its second argument, then its subfunction is also compatible with rational demand, as is stated in the following theorem.

Theorem 4.1: Suppose $f(x) = g(x_1, h(x_2))$ is weakly separable and $g(x_1, \eta)$ is strictly increasing in η . If $(c, x') \in \mathcal{D}_d(f)$ then $(\bar{c}, x'_2) \in \mathcal{D}_d(h)$, where $\bar{c}(\cdot) = c(x_1, \cdot)$.

Proof: Suppose $\bar{c}(x_2) < \bar{c}(x'_2)$ for arbitrary x_2 . Then $(c, x') \in \mathcal{D}_u(f)$ implies $g(x_1, h(x_2)) < g(x_1, h(x'_2))$, so that we must have $h(x_2) < h(x'_2)$. Hence $(\bar{c}, x'_2) \in \mathcal{D}_u(h)$. In the same way we obtain $(\bar{c}, x'_2) \in \mathcal{D}_c(h)$ from $(c, x') \in \mathcal{D}_c(f)$. $\square$

In terms of symmetry we have the following alternative definition of weak separability.

Theorem 4.2: A function is weakly separable of the form $f(x) = g(x_1, h(x_2))$ if and only if $T_h f = f$, where

$$(T_h f)(x_1, x_2) = \inf_{h(v)=h(x_2)} f(x_1, v).$$

Proof: This follows by using the aggregator function

$$g(x_1, \eta) = \inf_{h(v)=\eta} f(x_1, v). \quad \square$$

If the aggregator function $g(x_1, \eta)$ is increasing in the second argument then the weakly separable function satisfies the following symmetry.

Theorem 4.3: A function $f(x) = g(x_1, h(x_2))$ is weakly separable such that the

aggregator function $g(x_1, \eta)$ is increasing in η , if and only if $T_h f = f$, where

$$(T_h f)(x_1, x_2) = \inf_{h(v) \geq h(x_2)} f(x_1, v).$$

Proof: Use the aggregator function

$$g(x_1, \eta) = \inf_{h(v) \geq \eta} f(x_1, v). \quad \square$$

One may use the above transformations T_h to construct a weakly separable function $T_h f$ from an arbitrary function f . The result is a weakly separable function, because obviously the following theorem holds.

Theorem 4.4: The above transformations T_h satisfy $T_h T_h = T_h$.

Proof: Obvious. $\square$

The underlying idea of the theory in this chapter is the following. Consider the optimization problem

$$\sup_{c(x) \leq c(x')} f(x),$$

for a given x' . Using a Lagrangian, this problem can be transformed to an unconstrained optimization problem

$$\sup_x f(x) - \lambda[c(x) - c(x')],$$

where $\lambda > 0$. Now, suppose $T_h f = f$ holds, where T_h is one of the above transformations. Then we can show that

$$\sup_x f(x) - \lambda[c(x) - c(x')] = \sup_x f(x) - \lambda[(T_h c)(x) - c(x')]$$

holds. Furthermore, if we suppose that x' is a cost minimizing solution of the above optimization problem then $c(x') = (T_h c)(x')$, as is shown in the following theorem.

Theorem 4.5: Suppose T_h is one of the transformations given above, i.e.

$$(T_h f)(x_1, x_2) = \inf_{h(v) = h(x_2)} f(x_1, v) \text{ or } \inf_{h(v) \geq h(x_2)} f(x_1, v).$$

If $(c, x') \in D_c(f)$ and $T_h f = f$ then $(T_h c)(x') = c(x')$.

Proof: This follows from Theorem 4.2 and 4.3. If $T_h f = f$, we can write $f(x) = g(x_1, h(x_2))$. Now, suppose $h(v) = h(x_2)$ for arbitrary v . Then $g(x'_1, h(v)) = g(x'_1, h(x_2))$ and $(c, x') \in D_c(f)$ implies thus $c(x'_1, v) \geq c(x')$.

Next, suppose $g(x_1, \eta)$ is increasing in η and $h(v) \geq h(x_2)$. Then we have $g(x'_1, h(v)) \geq g(x'_1, h(x_2))$ and $(c, x') \in D_c(f)$ implies again $c(x'_1, v) \geq c(x')$. $\square$

The above theorem shows again that a symmetry $f = T_h f$ for the utility function can be translated in a symmetry for the corresponding consumer demand data. The symmetry $(T_h c)(x') = c(x')$ is a necessary condition for the existence of a rationalizing symmetric function $f = T_h f$.

So, concerning consumer demand for a weakly separable utility function, we can transform the original optimization problem into a problem for which the Lagrangian is of the symmetric form

$$g(x_1, h(x_2)) - \lambda[(T_h c)(x) - (T_h c)(x')].$$

This observation leads to necessary and sufficient conditions for weakly separable consumer demand.

4.3 Existence of a Rationalizing Aggregator Function

Suppose a given set of consumer data on $X_1 \times X_2$ and a given function $h: X_2 \rightarrow \mathbb{R}$. Then we may investigate whether there is an aggregator function $g(x_1, h(x_2))$, which rationalizes the consumer data. In the following theorem a dual approach is used to construct such a rationalizing aggregator function g from cost functions. Although only Theorem 4.6 (a) and (b) will be used later on, I give also Theorem 4.6 (c) in order to show when the necessary condition in (b) is a sufficient condition.

Theorem 4.6: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a finite data set, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$. Suppose $h: X_2 \rightarrow \mathbb{R}$ and let the transformation T_h of any function $f: X_1 \times X_2 \rightarrow \mathbb{R}$ be defined by

$$(T_h f)(x_1, x_2) = \inf_{h(v) \geq h(x_2)} f(x_1, v) \quad (x \in X_1 \times X_2).$$

Suppose $(T_h c_i)(x^i) = c_i(x^i)$, for all $i \in I$, and put $D' = \{(T_h c_i, x^i)\}_{i \in I}$.

(a) Then $\mathcal{F}_d(D') \subset \mathcal{F}_d(D)$.

(b) If $\mathcal{F}_d(D') \neq \emptyset$ then there exists a function $f \in \mathcal{F}_d(D)$ such that $T_h f = f$.

(c) Suppose $T_h f = f$ and suppose that the infima for each transformation $T_h c_i$ are attained, i.e. for every $i \in I$ and $x \in X$ there is a $v \in X_2$ such that

$$(T_h c_i)(x) = c_i(x_1, v), \quad h(v) \geq h(x_2).$$

Then $f \in \mathcal{F}_d(D)$ if and only if $f \in \mathcal{F}_d(D')$.

Proof (a): By definition of T_h , we have $T_h c_i \leq c_i$. Hence, using $(T_h c_i)(x^i) = c_i(x^i)$, we have

$$\begin{aligned} c_i(x) \leq c_i(x^i) &\Rightarrow (T_h c_i)(x) \leq (T_h c_i)(x^i) & (x \in X, i \in I), \\ c_i(x) < c_i(x^i) &\Rightarrow (T_h c_i)(x) < (T_h c_i)(x^i) & (x \in X, i \in I). \end{aligned}$$

So $f \in \mathcal{F}_d(D')$ implies $f \in \mathcal{F}_d(D)$, and thus $\mathcal{F}_d(D') \subset \mathcal{F}_d(D)$.

(b): If $\mathcal{F}_d(D') \neq \emptyset$ then by Theorem 2.5 we can construct a function $f \in \mathcal{F}_d(D')$ of the form

$$f(x) = \min_{i \in I} f_i + \lambda_i [(T_h c_i)(x) - (T_h c_i)(x^i)] \quad (x \in X_1 \times X_2),$$

such that $\lambda_i > 0$ and $f(x^i) = f_i$. This function satisfies $T_h f = f \in \mathcal{F}_d(D') \subset \mathcal{F}_d(D)$.

(c): Suppose $T_h f = f \in \mathcal{F}_d(D)$. Because of (a) it is sufficient to prove that one has now $f \in \mathcal{F}_d(D')$. Let $x \in X$ and $i \in I$ and suppose $(T_h c_i)(x) \leq (T_h c_i)(x^i)$. Then by assumption there is a $v \in X_2$ such that

$$(T_h c_i)(x) = c_i(x_1, v), \quad h(v) \geq h(x_2).$$

Hence, we have $f(x) \leq f(x_1, v)$ and $c_i(x_1, v) \leq c_i(x^i)$. Moreover, $f \in \mathcal{F}_d(D)$ implies $f(x_1, v) \leq f(x^i)$. So, we obtain

$$(T_h c_i)(x) \leq (T_h c_i)(x^i) \Rightarrow f(x) \leq f(x^i).$$

Now, suppose $(T_h c_i)(x) < (T_h c_i)(x^i)$. Then in general there is a $v \in X_2$ such that

$$(T_h c_i)(x) \leq c_i(x_1, v) < (T_h c_i)(x^i), \quad h(v) \geq h(x_2).$$

Thus, similarly as above, we obtain

$$(T_h c_i)(x) < (T_h c_i)(x^i) \Rightarrow f(x) < f(x^i),$$

which means that we have $f \in \mathcal{F}_d(D')$. □

So by Theorem 4.6 (c) the information concerning the possible symmetric functions $f = T_h f$ in $\mathcal{F}_d(D)$ is under weak conditions invariant under the data transformation $D \rightarrow D'$.

Note that we can also prove the above theorem, using the transformation

$$(T_h f)(x_1, x_2) = \inf_{h(v)=h(x_2)} f(x_1, v).$$

In that case we consider weakly separable functions $g(x_1, h(x_2))$ without the requirement that the aggregator function $g(x_1, \eta)$ is increasing in η .

4.4 Existence of a Rationalizing Weakly Separable Function

This section states conditions that are satisfied by weakly separable rational demand. In the previous section a rationalizing aggregator function g is constructed for a given subfunction h . This was no problem, because the family of weakly separable utility function for a given subfunction h can be described by using a symmetry. As is shown in the previous sections, there is a transformation T_h that leaves such a family invariant. Further, as we have seen above and in Chapter 3 for linearly homogeneous consumer demand, a symmetry of the Lagrange function implies a symmetry for the corresponding consumer data. Moreover, it implies the existence of conserved properties, which can be used to gain information concerning the utility function.

The case of weak separability, however, differs from the problem in which linear homogeneity is considered. For every subfunction h there is another transformation T_h that leaves the weakly separable utility function invariant. So, if h is not known then we do not know the symmetric property in advance. Furthermore, the assumption that the aggregator function $g(x_1, \eta)$ is strictly increasing in η can not be derived from a symmetry assumption. This is why there is no simple solution for the construction of a complete rationalizing weakly separable function.

With some efforts I succeeded to derive results for weakly separable consumer demand, using the theory given in the previous section. The following theorem is based on the construction of a suitable subfunction h , that is constructed in such a way that Theorem 4.6 (b) can be applied to derive a suitable aggregator function.

Theorem 4.7: Suppose $D = \{(c_i, x^i)\}_{i \in I}$, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$, is a finite data set of consumer behaviour. Consider the following conditions:

- (i) There exists a function $g(x_1, h(x_2)) \in \mathcal{F}_d(D)$, which is weakly separable in X_2 and such that $g(x_1, \eta)$ is increasing in η .
- (ii) There exists a function $g(x_1, h(x_2)) \in \mathcal{F}_d(D)$, which is weakly separable in X_2 and such that $g(x_1, \eta)$ is strictly increasing in η .
- (iii) There exist preorders $\succ, \sim$ on $\{x^i\}_{i \in I}$ and $\succ', \sim'$ on $\{x_2^i\}_{i \in I}$ such that

$$\begin{aligned} c_i(x_1^j, x_2^k) \leq c_i(x^i) &\Rightarrow (x^j \preceq x^i \text{ or } x_2^k \prec' x_2^j) \text{ and} \\ &(x^j \prec x^i \text{ or } x_2^k \preceq' x_2^j) \quad (i, j, k \in I), \\ c_i(x_1^j, x_2^k) < c_i(x^i) &\Rightarrow x^j \prec x^i \text{ or } x_2^k \prec' x_2^j \quad (i, j, k \in I). \end{aligned}$$

- (iv) There exists a weakly separable function $g(x_1, h(x_2)) \in \mathcal{F}_d(D)$, such that h and g are strictly monotonically increasing continuous functions, and furthermore h and $g(\cdot, h(x_2))$, for every fixed $x_2 \in X_2$, are concave.

Then:

(a) (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(b) Suppose every function

$$h(x_2) = \min_{i, j \in I} \alpha_{ij} + \lambda_i c_i(x_1^j, x_2) \quad (x_2 \in X_2),$$

where $\alpha_{ij} \in \mathbb{R}$ and $\lambda_i > 0$, is such that the functions

$$\bar{c}_i(x_1, \eta) = \inf_{h(x_2) \geq \eta} c_i(x_1, x_2) \quad (i \in I, x_1 \in X_1, \eta \in \mathbb{R}),$$

are strictly increasing in η . Then (iii) $\Leftrightarrow$ (ii).

(c) Suppose D is a price and quantity data set. Then (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof (a) (ii) $\Rightarrow$ (iii): Suppose $f(x) = g(x_1, h(x_2)) \in \mathcal{F}_d(D)$ is weakly separable and $g(x_1, \eta)$ is strictly increasing in η . Put $y_i = f(x^i)$ for all $i \in I$. Let $i, j, k \in I$ and suppose $c_i(x_1^j, x_2^k) \leq c_i(x^i)$. Then $f \in \mathcal{F}_d(D)$ implies

$$g(x_1^j, h(x_2^k)) \leq g(x_1^i, h(x_2^i)),$$

and g strictly increasing in h , means

$$g(x_1^j, h(x_2^j)) > g(x_1^i, h(x_2^i)) \Rightarrow g(x_1^j, h(x_2^j)) > g(x_1^i, h(x_2^k)) \Rightarrow h(x_2^j) > h(x_2^k),$$

$$g(x_1^j, h(x_2^j)) \geq g(x_1^i, h(x_2^i)) \Rightarrow g(x_1^j, h(x_2^j)) \geq g(x_1^i, h(x_2^k)) \Rightarrow h(x_2^j) \geq h(x_2^k).$$

Hence, one has then

$$(y_j \leq y_i \text{ or } h(x_2^j) < h(x_2^i)) \text{ and } (y_j < y_i \text{ or } h(x_2^j) \leq h(x_2^i)).$$

Now, suppose $c_i(x_1^j, x_2^k) < c_i(x^i)$. Then it follows in a similar way that one has

$$y_j < y_i \text{ or } h(x_2^j) < h(x_2^i).$$

So, let $\succ, \sim$ be the weak order on $\{x^i\}_{i \in I}$ induced by f , and let $\succ', \sim'$ be the weak order on $\{x_2^i\}_{i \in I}$ induced by h . Then, by the above, these weak orders do satisfy (iii).

(iii) $\Rightarrow$ (i): If there are preorders satisfying the conditions in (iii), then by Theorem A.2 given in Appendix A there are weak orders that satisfy the same conditions. Let $\succ, \sim$ and $\succ', \sim'$ be that kind of weak orders. Now, (iii) implies that we can choose γ_{ij} such that

$$0 < \gamma_{ij} < c_i(x_1^j, x_2^k) - c_i(x^i) \quad (i, j, k \in I: x_2^k \succ' x_2^j \text{ and } x^j \succ x^i),$$

$$\gamma_{ij} = 0 \quad (i, j \in I: x^j \sim x^i).$$

Now, define

$$z_j(x_2) = \min \{c_i(x_1^j, x_2) - c_i(x^i) - \gamma_{ij} \mid i \in I, x^i \sim' x^j\} \quad (j \in I, x_2 \in X_2).$$

From the construction of z_j and (iii), it follows that we have

$$z_j(x_2^k) \leq 0 \Rightarrow x_2^k \sim' x_2^j \quad (k, j \in I),$$

$$z_j(x_2^k) < 0 \Rightarrow x_2^k \prec' x_2^j \quad (k, j \in I).$$

Hence by Theorem 2.4 there is a function

$$h(x_2) = \min_{i \in I} h_i + \lambda_i z_i(x_2) \quad (x_2 \in X_2),$$

such that

$$\lambda_i > 0, h(x_2^i) \geq h_i \quad (i \in I).$$

I will show that such a function h satisfies the conditions of Theorem 4.6. Let $j \in I$ and $x_2 \in X_2$, and suppose $h(x_2) \geq h(x_2^j)$. By the construction of h we have then $h(x_2) \geq h_j$ and thus

$$c_i(x_1^j, x_2) - c_i(x^i) - \gamma_{ij} \geq 0 \quad (i \in I: x^j \succ x^i).$$

So one has

$$(T_h c_i)(x^j) \geq c_i(x^i) + \gamma_{ij} \quad (i, j \in I: x^j \succ x^i),$$

where

$$(T_h c_i)(x_1, x_2) = \inf_{h(v) \geq h(x_2)} c_i(x_1, v) \quad (i \in I, x \in X_1 \times X_2).$$

Hence, from

$$\begin{aligned} \gamma_{ij} &> 0 & (i, j \in I: x^j > x^i), \\ \gamma_{ij} &= 0 & (i, j \in I: x^j \sim x^i), \end{aligned}$$

it follows that we have $(T_h c_i)(x^i) = c_i(x^i)$ for all $i \in I$. So, the function h satisfies the conditions of Theorem 4.6. Since $>, \sim$ is a weak order, we have moreover

$$\begin{aligned} (T_h c_i)(x^j) \leq (T_h c_i)(x^i) &\Rightarrow x^j \preceq x^i & (i, j \in I), \\ (T_h c_i)(x^j) < (T_h c_i)(x^i) &\Rightarrow x^j < x^i & (i, j \in I), \end{aligned}$$

where we have used the property of a weak order $>, \sim$ that

$$\text{not } x^j > x^i \Leftrightarrow x^j \preceq x^i \quad (i, j \in I).$$

Thus $D' = \{(T_h c_i, x^i)\}_{i \in I}$ satisfies the axiom of revealed preference, so that $\mathcal{F}_d(D') \neq \emptyset$ by Theorem 2.5. Hence, there is a weakly separable function $T_h f = f \in \mathcal{F}_d(D)$ by Theorem 4.6 (b).

(b) (iii) $\Rightarrow$ (ii): Every function $T_h c_i$ in the above proof is weakly separable. Now, the assumption in (b) implies that its aggregator function $\bar{c}_i(x_1, \eta)$ is strictly increasing in η . In that case the constructed weakly separable function $g(x_1, h(x_2)) \in \mathcal{F}_d(D)$ in the proof of Theorem 4.6 (b) will also have these properties.

(c) One can show that the assumption made in (b) is satisfied by price and quantity data. Theorem 4.8, which can be found at the end of this chapter, contains a complete proof to which I would like to refer the interested reader.

To prove (iii) $\Rightarrow$ (iv) we may construct functions g and h as in the proof of (b). These functions will have the mentioned properties, because they are derived from price and quantity data. Finally, the implication (iv) $\Rightarrow$ (ii) is trivial. $\square$

Condition (iii) in the above theorem is complicated. However, one may often simplify this condition. For empirical data one has in practice

$$c_i(x_1^j, x_2^k) \neq c_i(x^i) \quad (i, j, k \in I: j \neq i \text{ or } k \neq i).$$

So in practice we may simplify condition (iii) in Theorem 4.7 into:

(iii') *There exist strict partial orders $>$ on $\{x^i\}_{i \in I}$ and $>'$ on $\{x_2^i\}_{i \in I}$ such that*

$$c_i(x_1^j, x_2^k) < c_i(x^i) \Rightarrow x^j < x^i \text{ or } x_2^k <' x_2^i \quad (i, j, k \in I).$$

(Note that for any preorder $>, \sim$ it holds that $>$ is a strict partial order. More information concerning this subject may be found in Appendix A.)

There is no simple test for the above condition. However, starting with a given strict partial order $>$, we can rewrite the implication in condition (iii') as

$$c_i(x_1^j, x_2^k) < c_i(x^i) \text{ and not } x^j < x^i \Rightarrow x_2^k <' x_2^i \quad (i, j, k \in I).$$

This, however, is a problem of the revealed preference type for which a simple test is available. Assuming that $>$ is given, one may use Theorem A.5 in Appendix A, to test whether the corresponding strict partial order $>'$ does exist. So, to test condition (iii'), we may search for a suitable strict partial order $>$, such that there is a corresponding strict partial order $>'$. I have used this approach in a computer program, which solves the problem quite fast.

Note that we cannot derive conserved properties from the conditions for weak

separability. The reason for this is the fact that weak separability is not a property that can be derived from a given symmetry. Although we do know that a weakly separable utility function satisfies a certain symmetry, this symmetry is unknown till the subfunction is h available.

The proof of the above theorem is quite complex. One may ask whether it would be simpler to specialize it to linear prices. However, this is not the case. There is only one important difference. For linear prices we can prove that the functions $\bar{c}_i(x_1, \eta)$ are strictly increasing in η . A general theorem that can be applied to prove this is given in the next section.

4.5 The Assumption of Strictly Increasing Cost Functions

The application of Theorem 4.7 (b) is limited to consumer data sets for which the functions $\bar{c}_i(x_1, \eta)$, given in this theorem, are strictly increasing in η . Clearly they are increasing in η and under certain conditions we can show that they are strictly increasing in η . In the following theorem such conditions are applied to a simplified case.

Theorem 4.8: Suppose the functions $h: X \rightarrow \mathbb{R}$ and $c: X \rightarrow \mathbb{R}$ are continuous, and the lower level sets

$$\{x \in X \mid c(x) \leq \alpha\} \quad (\alpha \in \mathbb{R})$$

compact. Let the transformation T_h be defined by

$$(T_h c)(x) = \inf_{h(v) \geq h(x)} c(v).$$

Then:

(a) For every $x \in X$ there exists a $v \in X$ such that $(T_h c)(x) = c(v)$ and $h(v) \geq h(x)$.

(b) If $-c$ is locally nonsatiated on $\{x \in X \mid h(x) > \eta_0\}$, where

$$\eta_0 = \inf_{x \in X} h(x),$$

then the function

$$\bar{c}(\eta) = \inf_{h(v) \geq \eta} c(v).$$

is strictly increasing on $h(X)$.

Proof (a): Let $x \in X$, $\varepsilon > 0$ and put

$$S = \{v \in X \mid h(v) \geq h(x)\} \cap \{v \in X \mid c(v) \leq (T_h c)(x) + \varepsilon\}.$$

Then the definition of $T_h c$ implies that one has

$$(T_h c)(x) = \inf_{v \in S} c(v).$$

Hence, because c is continuous and S compact, there is a $v \in S$ such that $(T_h c)(x) = c(v)$.

(b): Let $\eta_1, \eta_2 \in h(X)$ and suppose $\eta_2 > \eta_1$. By (a) there is an $v \in X$ such that $\bar{c}(\eta_2) = c(v)$ and $h(v) \geq \eta_2$. Now, since h is continuous and $h(v) \geq \eta_2 > \eta_1$, there is a neighbourhood V of v such that $h(x) > \eta_1$, for every $x \in V$. Furthermore, if $-c$ is locally nonsatiated at v , we may pick $x \in V$ such that $c(x) < c(v)$ and we have thus

$$\bar{c}(\eta_1) \leq c(x) < c(v) = \bar{c}(\eta_2).$$

□

5 SOME NONPARAMETRIC THEORY

5.1 Introduction

In the previous chapters I suggested an approach to consumer theory from an empirical point of view. This approach departs from a given empirical data set – instead of given preferences – and opens a promising field of research with many empirical applications. This chapter, however, contains a taste of a theoretical research, which is not directly connected with empirical applications.

The basis of the previous chapters was symmetry. The present chapter considers the following other interesting theoretical problems:

1. What are the properties of the rationalizing set of preorders or functions for a given data set?
2. What can we say about convergence of such functions?
3. Is there a way to derive a utility function from the axiom of revealed preference theorem for an infinite data set?
4. Which conditions do we need to derive rational demand from cost minimizing behaviour or from utility maximizing behaviour?

I will try to give an answer to such questions. The theory, discussed in this chapter, is suitable for application to data sets of an infinite number of elements.

5.2 Closed Mappings

This section considers transformations that map a rationalizing set of preference orders or utility functions onto itself. The next theorem contains an example that applies to preference orders.

Theorem 5.1: Suppose $R_0 \subset R_u(D)$ and let the preorder $\succsim$ be defined as the intersection of R_0 , i.e. the preorder that satisfies

$$x \succsim y \Leftrightarrow x \succsim' y \text{ for all } \succsim' \in R_0.$$

Then this preorder is element of $R_u(D)$.

Proof: Left to the reader. □

To describe the general idea we need the following definition.

Definition: Let S be a set of elements. Then S is said to be *closed* with respect to a given transformation T , if T maps all elements of S on S . Similarly, we say that S is closed with respect to T , if T is a transformation that maps all subsets of S , or all sequences in S , on S .

For example, the set $[0,1] \subset \mathbb{R}$ is closed with respect to the infimum operation, because the infimum of a subset of $[0,1]$ is element of $[0,1]$.

Now, we may translate Theorem 5.1 into a more compact version that is also valid for $R_c(D)$.

Theorem 5.2: For every data set D , the families $R_u(D)$ and $R_c(D)$ are closed with respect to the intersection of collections preorders.

Families of functions $\mathcal{F}_d(D)$ may also be closed with respect to certain transformations, and it seems fruitful to pay further attention to the implications of this property. Below are some examples of such

transformations.

Theorem 5.3: Every family $\mathcal{F}_u(D)$ and $\mathcal{F}_c(D)$ is closed with respect to the following transformations of $\{f_i\}_{i \in I}$ to a function f :

(a) A monotonic transformation $f = m \circ f_1$, where m is a strictly increasing function.

(b) $f = \min_i f_i$ and $f = \max_i f_i$.

(c) $f = \sum_{i=1}^n \alpha_i f_i$ ($\alpha_i > 0$).

(d) $f = \prod_{i=1}^n f_i^{\alpha_i}$ ($\alpha_i > 0, f_i \geq 0$).

Proof: This follows directly from Theorem 5.4 given below. □

Below follows a theorem that treats the common property of the transformations in the above theorem.

Theorem 5.4: Suppose T is a transformation, which maps collections of functions $\{f_i: X \rightarrow \mathbb{R}\}_{i \in I}$ on an image function $f: X \rightarrow \mathbb{R}$. If T has for every map of $\{f_i\}_{i \in I}$ on f the property

$$f_i(x) \leq f_i(y) \quad \forall i \in I \Rightarrow f(x) \leq f(y) \quad (x, y \in X),$$

then any family $\mathcal{F}_u(D)$ is closed with respect to T . Similarly, if T has the property

$$f_i(x) < f_i(y) \quad \forall i \in I \Rightarrow f(x) < f(y) \quad (x, y \in X),$$

then any family $\mathcal{F}_c(D)$ is closed with respect to T .

Proof: Obvious. □

For utility maximizing behaviour one has the following theorem, which has no counterpart for cost minimizing behaviour:

Theorem 5.5: Every family $\mathcal{F}_u(D)$ is closed with respect to the pointwise infimum and supremum over collections of functions, such that the limit function is a well-defined finite function.

Proof: For any set $\{a_i\}$ and $\{b_i\}$ one has

$$a_i \leq b_i \quad \forall i \Rightarrow \sup_i a_i \leq \sup_i b_i \quad \text{and} \quad \inf_i a_i \leq \inf_i b_i.$$

Thus we may apply Theorem 5.4.

A direct proof for the infimum operation is as follows. We have for $\{f_i\} \subset \mathcal{F}_u(D)$ and arbitrary $(c, x') \in D$ the implication

$$c(x) \leq c(x') \Rightarrow f_i(x) \leq f_i(x') \quad \forall i \Rightarrow \inf_i f_i(x) \leq \inf_i f_i(x').$$

Hence, for the function

$$f = \inf_i f_i$$

we have $f \in \mathcal{F}_u(D)$. A similar result holds for the supremum operation. □

There is no analogous theorem for $\mathcal{F}_c(D)$. However, there is a similar theorem valid for finite index sets I , or more general, for attained maxima and minima. This is stated in Theorem 5.3(b).

Remark: An illustration of the construction of a counter example, such that one has $\{f_i\} \subset \mathcal{F}_c(D)$ and the function

$$f = \inf_i f_i$$

is not contained in $\mathcal{F}_c(D)$, is the following. Suppose $h: X \rightarrow \mathbb{R}_{++} \in \mathcal{F}_c(D)$,

$x', x'' \in X$, $(c, x') \in D$ and $c(x'') < c(x')$. Put $f_i = h/i$ for all $i \in \mathbb{N}$. In that case one has $f_i \in \mathcal{F}_c(D)$ for all i and $f = 0$. If $f \in \mathcal{F}_c(D)$ then we must have $f(x'') < f(x')$, since $c(x'') < c(x')$. However, this is not the case, because $f = 0$. So, we have $f \notin \mathcal{F}_c(D)$.

Why is there no analogous theorem for $\mathcal{F}_c(D)$? As we have seen in the counter example in the remark above, there may occur satiation when the limit is taken. In case of utility maximization this is no problem, any constant utility function is a rationalizing function, but concerning cost minimization we have the following problem. Suppose $\{f_i\} \subset \mathcal{F}_c(D)$ and $(c, x') \in D$. Now, in order to derive an analogous theorem for cost minimization, we need the implications

$$c(x) < c(x') \Rightarrow f_i(x) < f_i(x') \quad \forall i \Rightarrow \inf_i f_i(x) < \inf_i f_i(x'),$$

However, the latter implication is not valid in general. As we have seen in the remark above, one may have $f_i(x) < f_i(x')$ for all i and

$$\inf_i f_i(x) = \inf_i f_i(x').$$

For the supremum over functions we have the same problem.

5.3 Approximately Finite Data

An important problem in the revealed preference theory is the specification of conditions, which have to be satisfied by a given consumer data set, such that there exists a rationalizing utility function. When the data set has a finite number of elements then the axiom of revealed preference is a necessary and sufficient condition. However, this is not the case for data sets that contain an infinite number of elements, for instance, a data set described by using a continuous index parameter $i \in I$. Then we need additional restrictions on the data besides the axiom of revealed preference. In this section I give an example of such restrictions.

Definition: When we have

$$S \subset \bigcup_j B_j,$$

then $\{B_j\}$ is called a *cover* of S .

Using the above definition, we may now formulate the central definition in this section.

Definition: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a data set of consumer behaviour, Φ_e is an efficiency transformation, and $e \in [0, 1]$. Let us use the notation

$$\Phi_e D = \{(c_i^\alpha, x^i)\}_{i \in I}.$$

If for every $a \in [0, e]$ there is a finite subset $J \subset I$, a cover $\{B_j\}_{j \in J}$ of $\{x^i\}_{i \in I}$, and strictly increasing monotonic transformations m_i , $i \in I$, that satisfy

$$m_k \circ c_k^a(x^i) - m_k \circ c_k(x^k) \geq c_l^e(x^j) - c_l(x^l) \quad (j, l \in J, i, k \in I: x^i \in B_j, x^k \in B_l),$$

then D is said to be *approximately finite* at efficiency level e .

The idea behind this definition is that one may apply theoretical results, concerning finite data sets, to approximately finite data sets. This is done in the following theorem, where a rationalizing utility function is derived for an infinite data set.

Theorem 5.6: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a data set of consumer behaviour, Φ_e is an efficiency transformation. If $\Phi_e D$ satisfies the axiom of revealed

preference and D is approximately finite at efficiency level e then

$$\mathcal{F}_d(\Phi_a D) \neq \emptyset \quad (a \in [0, e]).$$

Proof: Let $a \in [0, e]$ and let us use the notation

$$\Phi_a D = \{(c_i^\alpha, x^i)\}_{i \in I}.$$

Suppose D is approximately finite at efficiency level e . Then we may choose a finite subset $J \subset I$, a cover $\{B_j\}_{j \in J}$ of $\{x^i\}_{i \in I}$, and strictly increasing monotonic transformations m_i that satisfy

$$m_k \circ c_k^a(x^i) - m_k \circ c_k(x^k) \geq c_l^e(x^j) - c_l(x^l) \quad (j, l \in J, i, k \in I: x^i \in B_j, x^k \in B_l).$$

The data $\Phi_a D$ satisfy the axiom of revealed preference, hence this is also the case for any finite data subset

$$D' = \{(c_j^e, x^j)\}_{j \in J} \subset \Phi_a D.$$

Hence, as shown in the proof of Theorem 2.5, there is a function $g \in \mathcal{F}_d(D')$ of the form

$$g(x) = \min_{j \in J} g_j + \lambda_j [c_j^e(x) - c_j(x^j)] \quad (x \in X),$$

where $\lambda_j > 0$ and $g(x^j) \geq g_j$ for all $j \in J$. Now, define

$$I_j = \{i \in I \mid x^i \in B_j\},$$

$$f(x) = \min_{j \in J} g_j + \lambda_j \inf_{i \in I_j} [m_i \circ c_i^a(x) - m_i \circ c_i(x^i)] \quad (x \in X).$$

To show that $f \in \mathcal{F}_d(\Phi_a D)$, let $i \in I$ be arbitrary. Since $\{B_j\}$ is a cover of $\{x^i\}$, we may pick $j \in J$ such that $x^i \in B_j$ and thus $i \in I_j$. Then we have

$$c_i^a(x) \leq c_i(x^i) \Rightarrow m_i \circ c_i^a(x) \leq m_i \circ c_i(x^i) \Rightarrow f(x) \leq g_j \quad (x \in X),$$

$$c_i^a(x) < c_i(x^i) \Rightarrow m_i \circ c_i^a(x) < m_i \circ c_i(x^i) \Rightarrow f(x) < g_j \quad (x \in X).$$

So $f \in \mathcal{F}_d(\Phi_a D)$ provided we can show that one has $g_j \leq f(x^i)$. From

$$m_k \circ c_k^a(x^i) - m_k \circ c_k(x^k) \geq c_l^e(x^j) - c_l(x^l) \quad (j, l \in J, i, k \in I: x^i \in B_j, x^k \in B_l),$$

$$g(x^j) \geq g_j,$$

it follows that we have indeed

$$f(x^i) = \min_{l \in J} g_l + \lambda_l \inf_{k \in I_l} [m_k \circ c_k^a(x^i) - m_k \circ c_k(x^k)] \geq$$

$$g(x^j) = \min_{l \in J} g_l + \lambda_l [c_l^e(x^j) - c_l(x^l)] \geq g_j. \quad \square$$

One may also apply the proof of the above theorem to derive utility functions that are concave, monotonically increasing or upper semicontinuous. Therefore we only need the assumption that the cost functions have the same property. This result follows directly from the properties of the utility function, which is constructed in the proof of Theorem 5.6. We can derive these properties as follows. The pointwise infimum over concave functions is concave, see Rockafellar (1970, Theorem 5.6). For monotonically increasing cost functions the resulting property is obvious. Furthermore, the pointwise infimum over upper semicontinuous functions is upper semicontinuous, because the intersection of closed sets is a closed set.

Definition: A function $f: X \rightarrow \mathbb{R}$ is said to be *upper semicontinuous* if all its upper level sets

$$\{x \in X \mid f(x) \geq \alpha\} \quad (\alpha \in \mathbb{R})$$

are closed. Similarly, a function $f: X \rightarrow \mathbb{R}$ is said to be *lower semicontinuous* if all its lower level sets

$$\{x \in X \mid f(x) \leq \alpha\} \quad (\alpha \in \mathbb{R})$$

are closed.

5.4 Examples of Approximately Finite Data

For the analysis of infinite data sets it is useful to have conditions available which imply that the data set is approximately finite. The following theorem provides an example of approximately finite data.

Theorem 5.7: Let X be a metric space and $K \subset X$ compact. Suppose $P: K \times X \rightarrow \mathbb{R}_+$ is continuous on $K \times K$ and $P(x, x) = 1$ for all $x \in K$. Then for the efficiency transformation Φ_e in Example 2.3, any data set $D = \{(c_i, x^i)\}_{i \in I}$ of the form

$$(c_i, x^i) = (P(x^i, \cdot), x^i) \quad (i \in I)$$

is approximately finite at every efficiency level e .

Proof: Let us use the notation

$$\Phi_\alpha D = \{(c_i^\alpha, x^i)\}_{i \in I},$$

and let $a \in [0, e)$ and $\varepsilon = e/a - 1$. Because P is uniformly continuous on the compact set $K \times K$, we can choose $\delta > 0$ such that

$$\|x^k - x^l\| + \|x^i - x^j\| \leq \delta \Rightarrow |P(x^k, x^i) - P(x^l, x^j)| \leq \varepsilon \quad (i, j, k, l \in I).$$

Since K is compact, there is a finite cover of K consisting of open balls of diameter $\delta/2$. Hence, there is a finite subset $J \subset I$ and a cover $\{B_j\}_{j \in J}$ of $\{x^i\}_{i \in I}$, such that the diameter of B_j is less or equal to $\delta/2$ and

$$x_j \in B_j \quad (j \in J),$$

$$B_j \cap B_l = \emptyset \quad (j, l \in J: j \neq l).$$

Now, put

$$m(z) = e^{-1} a z \quad (z \in \mathbb{R}).$$

Pick arbitrary $j, l \in J$ and $i, k \in I$. If $x^k = x^i$ then it follows that we have

$$x^k \in B_l, x^i = x^k \in B_j \Rightarrow l = j \Rightarrow$$

$$m \circ c_k^a(x^i) - m \circ c_k^a(x^k) = m(1) - m(1) = 1 - 1 = c_l^e(x^j) - c_l^e(x^l).$$

At the other hand, in case $x^k \neq x^i$, one has

$$x^k \in B_l, x^i \in B_j \Rightarrow \|x^k - x^l\| + \|x^i - x^j\| \leq \delta \Rightarrow P(x^k, x^i) \geq P(x^l, x^j) - \varepsilon \Rightarrow$$

$$m \circ c_k^a(x^i) - m \circ c_k^a(x^k) = m(a^{-1} P(x^k, x^i)) - m(1) \geq$$

$$m(a^{-1} P(x^l, x^j) - \varepsilon) - m(1) = e^{-1} P(x^l, x^j) - 1 \geq c_l^e(x^j) - c_l^e(x^l).$$

Hence D is approximately finite at efficiency level e . □

Note that the continuity assumption in Theorem 5.7 can be weakened, because one may replace

$$|P(x^k, x^i) - P(x^l, x^j)| \leq \varepsilon,$$

in the proof of Theorem 5.7, by the weaker inequality

$$P(x^k, x^i) - P(x^l, x^j) \leq \varepsilon$$

without altering the result of the proof. This means that, by a more specific choice of $\{B_j\}_{j \in J}$, one may also prove a theorem which allows for certain discontinuities in P .

Now let us consider price and quantity data described by $(p(x), x)$, where p maps quantity data x on price data. The following example tells us when we may scale the prices p such that the resulting price and quantity data is

approximately finite.

Theorem 5.8: Suppose $K \subset \mathbb{R}_+^n$ is a closed bounded set and $p: K \rightarrow \mathbb{R}_{++}^n$. Suppose the Laspeyres price index $P(x', x) = p(x')x/p(x')x'$ is continuous on $K \times K$. If consumers choices are restricted to K then any price and quantity data set $D = \{(p^i, x^i)\}_{i \in I}$, where

$$p^i = [p(x^i)x^i]^{-1}p(x^i) \quad (i \in I),$$

is approximately finite at every efficiency level e .

Proof: We may apply Theorem 5.7, because any closed bounded subset of $\mathbb{R}^n$ is compact. $\square$

5.5 Utility Maximization and Cost Minimization

The question under which conditions is a utility maximizing consumer a cost minimizer and vice versa is an interesting one. In this section I will give two theorems that are concerned with such conditions. The first theorem states that a utility maximizing consumer with a locally nonsatiated utility function is a cost minimizer for upper semicontinuous cost functions.

Theorem 5.9: Let $(c, x') \in D_u(f)$. Suppose $f: X \rightarrow \mathbb{R}$ is locally nonsatiated on X' and c is upper semicontinuous on X' , where

$$X' = \{x \in X \mid f(x) = f(x')\} \cap \{x \in X \mid c(x) < c(x')\}.$$

Then $(c, x') \in D_c(f)$.

Proof: Suppose $c(x) < c(x')$ for an arbitrary $x \in X$. Then $(c, x') \in D_u(f)$ implies $f(x') \geq f(x)$. We have to show that $f(x) = f(x')$ is impossible. Therefore suppose $f(x) = f(x')$. Since c is upper semicontinuous at x by assumption, there is a neighbourhood $V \subset X$ of x such that $c(v) < c(x')$ for all $v \in V$. Because f is locally nonsatiated at x , there is a $v \in V$ such that $f(v) > f(x)$. Furthermore, $(c, x') \in D_u(f)$ and $c(v) < c(x')$ implies $f(x') \geq f(v)$. Thus $f(x') > f(x)$, which contradicts $f(x) = f(x')$. $\square$

The dual of the above theorem shows conditions under which a cost minimizer is a utility maximizer.

Theorem 5.10: Let $(c, x') \in D_c(f)$. Suppose $f: X \rightarrow \mathbb{R}$ is lower semicontinuous on X' , and $-c$ is locally nonsatiated on X' , where

$$X' = \{x \in X \mid f(x) > f(x')\} \cap \{x \in X \mid c(x) = c(x')\}.$$

Then $(c, x') \in D_u(f)$.

Proof: Similar to the proof of Theorem 5.9. $\square$

One may apply the above theorems as follows to price and quantity data.

Theorem 5.11: Suppose $f: \mathbb{R}_+^n \rightarrow \mathbb{R}$ is a locally nonsatiated function and let price and quantity $(p, x') \in D_u(f)$. Then $(p, x') \in D_c(f)$.

Proof: The function $c(x) = px$ is upper semicontinuous, hence we can apply Theorem 5.9. $\square$

Theorem 5.12: Suppose $f: \mathbb{R}_+^n \rightarrow \mathbb{R}$ is lower semicontinuous and let price and quantity $(p, x') \in D_c(f)$. Then $(p, x') \in D_u(f)$.

Proof: The function $-c(x) = -px$ with domain $\mathbb{R}_+^n$ is locally nonsatiated at every $x \neq 0$ in $\mathbb{R}_+^n$. Furthermore 0 is not contained in

$$X' = \{x \in X \mid f(x) > f(x')\} \cap \{x \in X \mid c(x) = c(x')\},$$

because $f(0) > f(x')$ implies $x' \neq 0$, thus $p0 \neq px'$. Hence, the result follows from Theorem 5.10. $\square$

5.6 Utility Functions and Convergence

When is a sequence of functions $f_n \in \mathcal{F}_d(D)$, $n \in \mathbb{N}$, converging pointwise to a limit function in $\mathcal{F}_d(D)$? Also interesting is the following question for an efficiency transformation Φ_e . When is a sequence of functions

$$f_n \in \mathcal{F}_d(\Phi_{e_n} D), \quad e_n \rightarrow e,$$

pointwise converging to a limit function in $\mathcal{F}_d(\Phi_e D)$?

The following theorem shows that the efficiency levels, which are consistent with cost minimizing behaviour, form a closed set.

Theorem 5.13: Suppose D is a data set, Φ_e is an efficiency transformation, and $e \in [0, 1]$. Suppose for all

$$(c^a, x') \in \Phi_a D \quad (a \in [0, e])$$

that $c^a(x)$ is continuous with respect to a , for every fixed $x \in X$. Then

$$f \in \mathcal{F}_c(\Phi_a D) \text{ for all } a \in [0, e] \Leftrightarrow f \in \mathcal{F}_c(\Phi_e D).$$

Proof: ($\Leftarrow$) This follows from Theorem 2.1.

($\Rightarrow$): Suppose $f \in \mathcal{F}_c(\Phi_e D)$ for all $a \in [0, e]$. Let $(c^e, x') \in \Phi_e D$, $x \in X$, and suppose $f(x) \geq f(x')$. Then one has

$$c^a(x) \geq c^a(x') = c^e(x') \quad (a \in [0, e]),$$

and the continuity assumption implies thus

$$c^e(x) = \lim_{a \rightarrow e} c^a(x) \geq c^e(x').$$

Hence, $f \in \mathcal{F}_c(\Phi_e D)$. □

The above theorem has no counterpart for utility maximizing behaviour, and the efficiency levels, which are consistent with utility maximizing behaviour, may form an open set. For utility maximizing behaviour one has in turn the following theorem, which has no counterpart for cost minimizing behaviour.

Theorem 5.14: Every family $\mathcal{F}_u(D)$ is closed with respect to the pointwise $\liminf f_n$ and $\limsup f_n$ for sequences of functions f_n , $n \in \mathbb{N}$, such that the limit function is a well-defined finite function.

Proof: We may apply Theorem 5.4 as in the proof for Theorem 5.5, concerning the pointwise infimum or supremum over rationalizing functions. □

Using the above theorem we can derive a theorem for converging efficiency levels.

Theorem 5.15: Suppose D is a data set and Φ_e is an efficiency transformation. Suppose there is a sequence of functions

$$f_n \in \mathcal{F}_u(\Phi_{e_n} D) \quad (n \in \mathbb{N}),$$

where $e_n \rightarrow e$, such that $\liminf f_n$, or $\limsup f_n$, converges pointwise to $f: X \rightarrow \mathbb{R}$. Then one has

$$f \in \mathcal{F}_u(\Phi_e D) \quad (e \in [0, 1]).$$

Proof: We prove the theorem for $\liminf f_n$. The proof for $\limsup f_n$ is analogous. From Theorem 2.1 follows

$$f_n \in \mathcal{F}_u(\Phi_{e_n} D) \quad (n \geq m, m \in \mathbb{N}),$$

since $f_n \in \mathcal{F}_u(\Phi_{e_n} D)$ and e_n is increasing. Hence from Theorem 5.14 it follows that

$$\lim_{n \rightarrow \infty} \inf f_n \in \mathcal{F}_u(\Phi_{e_m} D) \quad (m \in \mathbb{N}).$$

Thus the result for $a \in [0, e]$ follows from Theorem 2.1. $\square$

The following theorem shows when a sequence of functions

$$f_n \in \mathcal{F}_u(\Phi_{e_n} D), \quad e_n \uparrow e,$$

converges pointwise to a limit function in $\mathcal{F}_d(\Phi_e D)$.

Theorem 5.16: Suppose D is a data set and Φ_e is an efficiency transformation. Suppose there is a sequence of functions

$$f_n \in \mathcal{F}_u(\Phi_{e_n} D) \quad (n \in \mathbb{N}),$$

where $e_n \uparrow e$, such that $\liminf f_n$, or $\limsup f_n$, converges pointwise to $f: X \rightarrow \mathbb{R}$. Suppose for all

$$(c^a, x') \in \Phi_a D \quad (a \in [0, e])$$

that $c^a(x)$ is continuous with respect to a , for every fixed $x \in X$. Then one has $f \in \mathcal{F}_d(\Phi_e D)$, if the following conditions are satisfied:

(i) For every $(c, x') \in \Phi_a D$, $a \in [0, e]$, one has f locally nonsatiated, and c upper semicontinuous on

$$\{x \in X \mid f(x) = f(x')\} \cap \{x \in X \mid c(x) < c(x')\}$$

(ii) For every $(c, x') \in \Phi_e D$, one has f lower semicontinuous, and $-c$ locally nonsatiated, on

$$\{x \in X \mid f(x) > f(x')\} \cap \{x \in X \mid c(x) = c(x')\}.$$

Proof: Since $f_n \in \mathcal{F}_u(\Phi_{e_n} D)$ converges to the limit function f and $e_n \uparrow e$, one has

$$f \in \mathcal{F}_u(\Phi_a D) \quad (a \in [0, e]),$$

by Theorem 5.15. Thus by Theorem 5.9 and condition (i) one has

$$f \in \mathcal{F}_c(\Phi_a D) \quad (a \in [0, e]).$$

Hence $f \in \mathcal{F}_c(\Phi_e D)$ by Theorem 5.13. Finally, Theorem 5.10 together with condition (ii), imply $f \in \mathcal{F}_d(\Phi_e D)$. $\square$

5.7 Existence of a Rationalizing Concave Function

Using the nonparametric theory we can give conditions for price and quantity data, such that the existence of a rationalizing concave function for every efficiency level $a \in [0, e]$ implies the existence of a rationalizing concave function for efficiency level e .

Theorem 5.17: Suppose D is a set of price and quantity data and Φ_e is the efficiency transformation given in Example 2.3. Suppose there is a sequence of concave functions $f_n: \mathbb{R}_+^k \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, such that

$$f_n|_{\mathbb{R}_+^k} \in \mathcal{F}_d(\Phi_{e_n} D), \quad e_n \uparrow e.$$

Then there exists a continuous locally nonsatiated concave function $f: \mathbb{R}_+^k \rightarrow \mathbb{R}$ in $\mathcal{F}_d(\Phi_e D)$ if the following conditions are satisfied:

(i) The sequence f_n , $n \in \mathbb{N}$, is pointwise bounded.

(ii) There are $x^0 \in \mathbb{R}_+^k$ and $K > 0$ such that

$$f_n(\lambda x^0) \geq \lambda K \quad (\lambda > 0, n \in \mathbb{N}).$$

Proof: Because $\mathbb{R}_+^k$ is a closed subset of $\mathbb{R}^k$, there is a subsequence of $f_n|_{\mathbb{R}_+^k}$, $n \in \mathbb{N}$, converging to a continuous concave function $f: \mathbb{R}_+^k \rightarrow \mathbb{R}$ (See Rockafellar, 1970, Theorem 10.9). Now,

$$f_n(\lambda x^0) \geq \lambda K \quad (n \in \mathbb{N}),$$

implies $f(\lambda x^0) \geq \lambda K$. Thus for every $x \in \mathbb{R}_+^k$, there is $\lambda > 0$ such that $f(\lambda x^0) > f(x)$. Hence, since f is concave, it is easy to show that f is locally nonsatiated. Furthermore, f is lower semicontinuous, since f is continuous. So f is a locally nonsatiated lower semicontinuous limit function, thus $f \in \mathcal{F}_d(\Phi_e D)$ by Theorem 5.15. Both conditions (i) and (ii) in this theorem are satisfied for price and quantity data D and the given efficiency transformation Φ_e : condition (i), because the cost functions in $\Phi_a D$, $a \in [0, e]$, are upper semicontinuous; condition (ii), because for the cost functions c in $\Phi_e D$ one has $-c$ locally nonsatiated. $\square$

The above theorem may be useful in further research of the following idea. Let us start with the assumption that the consumer data $\Phi_e D$ satisfy the axiom of revealed preference. Our goal is the derivation of a concave utility function $f \in \mathcal{F}(\Phi_e D)$. A way to derive such a function might be as follows.

Suppose D is approximately finite at efficiency level e . Then by Theorem 5.6 there is a sequence of functions

$$f_n \in \mathcal{F}_d(\Phi_{e_n} D), \quad e_n \uparrow e.$$

Moreover, if we can show that the functions $m_i c_i^{e_n}$, $i \in I$, in the proof of Theorem 5.6 can be extended to finite concave functions on $\mathbb{R}^k$, there is a sequence of concave functions $f_n: \mathbb{R}^k \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, such that

$$f_n|_{\mathbb{R}_+^k} \in \mathcal{F}_d(\Phi_{e_n} D), \quad e_n \uparrow e.$$

Then, finally, we may use Theorem 5.17 to derive the existence of a concave utility function $f \in \mathcal{F}(\Phi_e D)$, *provided* we can show that the derived sequence f_n , $n \in \mathbb{N}$, satisfies conditions (i) and (ii) in this theorem.

Thus we end this chapter with an open problem: it remains to specify neat conditions that allow us to elaborate the above outline of a proof.

6 NONPARAMETRIC ANALYSIS OF AGGREGATE DATA

6.1 Introduction

Empirical studies of consumer data, which use aggregate data, often confirm the hypothesis of utility maximizing behaviour subject to a budget constraint. It is remarkable that studies, which investigate the axiom of revealed preference, show that macro consumption data satisfy the utility maximization hypothesis in the same way, or even better, as micro consumption data does. This in spite of results, given in Gorman (1953, 1959) and Muellbauer (1976), which show that the requirements for the existence of a representative consumer after aggregation over goods and individuals are stringent.

This chapter investigates the construction of aggregate consumption data by using price index numbers. The main subject is the influence of aggregation on the existence of a rationalizing utility function, or even a homothetic rationalizing utility function. Of course such an analysis has also a meaning for aggregate production data. In that case the subject is the existence of a production function, which satisfies the hypothesis of cost minimization, assuming that production levels are unknown.

The problem is sufficiently complicated to leave the theoretical approach and attack it with data experiments. This approach is suggested in Fisher (1971), and Fisher, Solow and Kearn (1977), where aggregate Cobb-Douglas and CES production functions were investigated. They used input-output data of simple fictitious economies, consisting of several cost minimizing producers. They estimated production functions for the aggregate data, using the hypothesis of cost minimization. Although the necessary conditions for aggregation were violated, the estimated production functions fitted well to the aggregate data and the wage prediction was good. The investigation in this chapter differs from Fisher's aggregation problem, in the sense that aggregation over goods is investigated, instead of aggregation over individuals. Furthermore, the analysis is based on nonparametric methods instead of parametric methods.

This chapter has the following contents. First a summary of related empirical research is given. Then a short description of several commonly used aggregation methods. These aggregation methods are used to aggregate Dutch consumption data and random data. Nonparametric tests are applied to check whether the aggregate data are compatible with the existence of aggregate rationalizing utility functions.

6.2 Empirical Results

The previous chapters contain empirical results for Dutch consumption data. In this section I give an overview of several other empirical studies in which the axiom of revealed preference, and the existence of homothetic utility functions, is tested.

Koo (1963, 1971), and Koo and Hasenkamp (1972) use consumer food panel data of 250 households with which they have empirically tested the axiom of revealed preference. They conclude that nearly every family made a relatively small number of inconsistent choices. Mossin (1972) tests the axiom of revealed preference with data based on consumers' reports about weekly purchases of everyday commodities. He compares the individual demand functions with the mean demand function and concludes that the mean demand

function agrees better with the axiom of revealed preference. Maks (1978, 1980, 1982, 1984), Landsburg (1981) and Varian (1982) investigate aggregate consumption data for the Netherlands and Germany, the United Kingdom, and the United States respectively. They find hardly any violations of the axiom of revealed preference.

Other empirical studies, concerning the existence of homothetic utility functions, are presented in Diewert and Parkan (1978), and Manser and McDonald (1988). In accordance with the results in Chapter 3, these studies show that annual aggregate consumption data are often nearly consistent with homothetic utility maximization.

6.3 Explanations for the Empirical Results

In literature several explanations are given for the question why aggregate consumption data may agree better with the hypothesis of utility maximization than micro consumption data do. Of course one has to distinguish several kinds of aggregation. There is aggregation from individual quantity choices to quantity choices of a representative consumer. Secondly, there is aggregation of the data from shorter periods to longer periods. In the third place, there is the joining of the commodities in commodity groups, and aggregation of prices and quantities of these groups into aggregated group prices and quantities. Each of these three types of aggregation may be a cause for better agreement of macro consumption data with the hypothesis of utility maximization.

The usual argument to explain why macro consumption data are consistent with the axiom of revealed preference is the presence of an upward trend in income. As argued in Chapter 2, such a trend tends to hide inconsistent choices. However, there is no upward trend in income present in the pre-war data used in Maks (1978, 1980), which appeared to be consistent with the assumption of utility maximization. Also, as is shown in Chapter 3, an upward trend in income is no valid argument to explain why consumption data are nearly, or even completely, consistent with homothetic utility maximizing behaviour. Some other arguments shall be considered in the following.

To explain that more inconsistent choices occur in weakly consumption data than in yearly data, Koo (1963) mentions seasonal influences, which may cause a taste change. In a later article Koo and Schmidt (1974) argue that inconsistent choices may be caused by the inability of consumers to rank very dissimilar alternatives in a consistent way. Mossin (1968) remarks that there may be stochastic elements in the behaviour of the consumer, which are not as strongly present in the mean demand function of the total population as in individual demand functions. One may also apply this argument to explain the occurrence of less inconsistencies in consumption data, which are aggregated over longer periods.

The above arguments are concerned with aggregation over time and individuals. This chapter investigates influence of the third aggregation type, price and quantity aggregation, on the existence of a rationalizing utility function. Before starting this investigation I will specify the price and quantity aggregation methods which are investigated.

6.4 The Use of Price Index Numbers

Often one wishes to represent data for a group of commodities as data for one single aggregate commodity. This means that the observations (p^i, q^i) , $i=1, \dots, m$, representing the price and quantity vectors for the commodity group, will be represented by a series (P_i, Q_i) , $i=1, \dots, m$, representing the

prices and quantities for the aggregate commodity. This is usually done by the choice of a price index, which maps the series (p^i, q^i) , $i=1, \dots, m$, on a series price index numbers P_i , $i=1, \dots, m$. Then, using the assumption that $p^i q^i = P_i Q_i$, the quantities Q_i for the aggregate commodity are defined by

$$Q_i = p^i q^i / P_i \quad (i=1, \dots, m).$$

In Table 6.1 definitions are given of commonly used price indices. In this table the price index P_{bc} measures the change of price vector p^c , for *current* period c , relative to price vector p^b , in *base* period b .

Table 6.1 Price Index Numbers

Laspeyres	$L_{bc} = p^c q^b / p^b q^b$
Paasche	$1/L_{cb} = p^c q^c / p^b q^c$
Fisher	$(L_{bc}/L_{cb})^{1/2}$
Arithmetic	$A_{bc}^w = \sum_n \beta_n^w p_n^c / p_n^b$, where $\beta_n^w = p_n^w q_n^w / p^w q^w$
Geometric	$G_{bc}^w = \prod_n (p_n^c / p_n^b)^{\beta_n^w}$, where $\beta_n^w = p_n^w q_n^w / p^w q^w$
Törnqvist	$(G_{bc}^b / G_{cb}^c)^{1/2}$

In the arithmetic and geometric index a *weight* period w is used, for which the price and quantity vector (p^w, q^w) supplies the weights $\beta_n^w = p_n^w q_n^w / p^w q^w$. Often the base period is chosen as weight period. In that case the arithmetic index is equal to the Laspeyres index $A_{bc}^b = L_{bc}$.

To derive a price index number for all periods, one may choose the first period as a fixed base period. Then P_{1i} provides a price index for all periods i . Such an index is called a *direct* index. Another kind of index is the *chained* index. This means that the index C_i , for a time series $(p^1, q^1), (p^2, q^2), \dots$, is computed as

$$C_i = D_{12} D_{23} \cdots D_{i-2, i-1} D_{i-1, i},$$

using a chain of direct index numbers D_{ij} . In general one may use a subset of base periods $i_1, i_2, \dots$, and compute the index numbers in the following way

$$C_i = D_{1i_1} D_{i_1 i_2} \cdots D_{i_i i} \quad (1 < i_1 < i_2 < \cdots < i_i < i).$$

Note that using the geometric or arithmetic price index number has an advantage, because computation of these indices is possible without having all quantity information available. Concerning these indices it is sufficient that we know the quantity vector q^w and nominal expenditures $p^w q^w$ for the weight periods.

For the geometric index we have $G_{12}^w G_{23}^w = G_{13}^w$ and – using a fixed weight period – there is no difference between the chained and direct geometric index.

6.5 Results for Aggregate Dutch Consumption Data

To investigate the effect of price and quantity aggregation for consumption data, the Dutch consumption data are aggregated to several aggregation levels, using the various price indices mentioned in Table 6.1. A detailed description of the 4, 12 and 26 aggregation level is given in the publication

of the Netherlands Central Bureau of Statistics (1981), which supplied the data set. The chained arithmetic index is computed with the first observation period as weight period; both the chained and direct geometric index are computed with the base period as weight period.

Each resulting data set is tested for the existence of a rationalizing utility function. It appeared that the method of aggregation had no effect on the results. I obtained in all cases exactly the same results as described in Chapter 2 for the original data set of 106 commodities. After this, for each data set the smallest subset of n observations is computed, which have to be deleted in order to be consistent with the existence of a rationalizing homothetic utility function. Furthermore, the maximal homothetic efficiency level e was computed for each data set, using the efficiency transformation given in Example 2.3.

Table 6.2 Homotheticity Results

Goods	Price index	n	e
106	Geometric CBS	1	0.9996
26	Fisher direct	5	0.9992
"	Geometric "	3	0.9993
"	Laspeyres "	6	0.9991
"	Paasche "	4	0.9992
"	Törnqvist "	4	0.9992
"	Arithmetic chained	3	0.9992
"	Fisher "	5	0.9992
"	Geometric "	4	0.9992
"	Laspeyres "	5	0.9992
"	Paasche "	5	0.9992
"	Törnqvist "	5	0.9992
12	Fisher direct	7	0.9990
"	Geometric "	8	0.9990
"	Laspeyres "	7	0.9990
"	Paasche "	5	0.9990
"	Törnqvist "	6	0.9990
"	Arithmetic chained	8	0.9990
"	Fisher "	7	0.9990
"	Geometric "	7	0.9990
"	Laspeyres "	7	0.9990
"	Paasche "	7	0.9990
"	Törnqvist "	7	0.9990
4	Fisher direct	14	0.9952
"	Geometric "	16	0.9933
"	Laspeyres "	16	0.9935
"	Paasche "	14	0.9980
"	Törnqvist "	14	0.9955
"	Arithmetic chained	16	0.9935
"	Fisher "	15	0.9950
"	Geometric "	15	0.9949
"	Laspeyres "	15	0.9949
"	Paasche "	15	0.9952
"	Törnqvist "	15	0.9950

The results, presented in Table 6.2, have three characteristic properties. First, it appears that the homotheticity hypothesis is increasingly violated for higher aggregation levels. Secondly, although the efficiency levels are

lower for higher aggregation levels, the efficiency levels remain rather close to one. In the third place, the choice of price index has no substantial influence on the results. Especially the chained indices appear to generate similar results. Before evaluating these findings, first other results, derived from random data, are presented.

6.6 Results for Aggregate Random Data

This section considers aggregate random data instead of actual consumption data, and we shall investigate what effect price and quantity aggregation may have. The aggregate Dutch consumption data in the previous section were obtained from a publication of the Netherlands Central Bureau of Statistics (1981). These data were aggregated by the Central Bureau of Statistics, using consumption data for more than a thousand commodities. In this section their aggregation scheme will be applied to random data.

To clarify the way the aggregate random data sets are constructed, the aggregation scheme of the Netherlands Central Bureau of Statistics is shortly described. To aggregate the Dutch consumption data at a 106 commodity level, the chained geometric price index numbers are computed, using several weight years. For the computation of these index numbers, a budget survey is needed for each weight year (see Section 6.4). In Table 6.3 the weight years w are given at the left and at the right side appear the years i in which the weight year w is used to compute the direct geometric index G_{i-1i}^w . Actually the 1960 budget survey covered the period 1959/1960. To keep things simple, we assume here that this budget survey represents non-aggregate data in the year 1960.

Table 6.3 Weight Periods

Year of budget survey	Period of use
1951	1951–1961
1960	1962–1964
1964	1965–1969
1969	1970–1975
1975	1976–1977

To construct the aggregate random data, random prices and quantities are generated for 1060 commodities at each year in the period 1951–1977. Prices were chosen from a uniform distribution on $[1,3]$ and quantities from a uniform distribution on $[0,1]$. The random data were divided in 106 commodity groups of 10 commodities each. These groups were aggregated by several aggregation methods. I computed chained arithmetic and geometric price index numbers, using the aggregation method of the Netherlands Central Bureau of Statistics, which is described above. In addition Laspeyres, Paasche, Fisher, and Törnqvist price index numbers were computed, using a similar aggregation method. Now the index numbers are computed as a chain of direct index numbers, where the years of budget survey serve as the base periods in the chain. Thus we have for these indices

$$C_i = D_{1w_1} D_{w_1 w_2} \dots D_{w_i i} \quad (1 < w_1 < w_2 \dots < w_i < i),$$

where w_j represent the years of the budget survey (see also Section 6.4). Furthermore, series of direct and chained price index numbers are computed, as was done in the previous section. The three aggregation methods will be referred to as CBS, direct and chained.

Table 6.4 displays the results for the aggregate random data. From this table the following is apparent. First, the aggregate random data are in nearly all cases consistent with the axiom of revealed preference. Even more surprising is the fact that the resulting aggregate data is in some cases consistent with the existence of a rationalizing *homothetic* function. It is striking that the arithmetic and geometric price index, used by the Netherlands Central Bureau of Statistics, is one of these examples. Secondly, all efficiency levels are close to one. The Fisher and Törnqvist price indices preserve the random behaviour in the aggregate data sets in a better way, but there is hardly any difference in the homothetic efficiency level. In the third place, in contrast to earlier results for the Dutch consumption data, the choice of price index has substantial influence on the results. Even the chained indices generate different results.

Table 6.4 Results for Aggregate Random Data

Goods	Price index	Any function		Homothetic	
		<i>n</i>	<i>e</i>	<i>n</i>	<i>e</i>
1060	Random	9	0.9908	21	0.9858
106	Fisher direct	0	1	17	0.9954
"	Geometric "	0	1	4	0.9974
"	Laspeyres "	0	1	10	0.9968
"	Paasche "	2	0.9993	11	0.9961
"	Törnqvist "	1	0.9999	16	0.9954
"	Arithmetic chained	0	1	0	1
"	Fisher "	2	0.9996	11	0.9974
"	Geometric "	0	1	2	0.9999
"	Laspeyres "	0	1	2	0.9992
"	Paasche "	0	1	6	0.9986
"	Törnqvist "	2	0.9991	10	0.9975
"	Arithmetic CBS	0	1	0	1
"	Fisher "	2	0.9991	15	0.9963
"	Geometric "	0	1	0	1
"	Laspeyres "	0	1	3	0.9989
"	Paasche "	0	1	4	0.9972
"	Törnqvist "	3	0.9990	14	0.9966

6.7 Evaluation and Some More Results

Bronars (1987) shows that nonparametric tests of preference maximization using per capita data are quite powerful against the alternative of random behaviour. However, from the results above it appears that the hypothesis of rational demand leads not to stringent restrictions on consumption data that are aggregated over commodities. Moreover, even the restriction to homotheticity is not stringent in case the consumption data contain many commodities. This conclusion is supported by the empirical results in Diewert and Parkan (1978), and Manser and McDonald (1988). An important question we have to answer in this respect is the question why aggregation of random data may generate rational, and even homothetic rational, consumer behaviour. The obvious reason for this finding is the way the aggregate quantity measures were computed. Each aggregate quantity measure was computed as the division of the group budget by the group price index. Hence, the movement of the aggregate quantity measures will tend to be opposite to the movement of the price indices. So, commodity aggregation of random data introduces a tendency to economical behaviour.

In order to investigate if the superposition principle is a sufficient explanation for our results, the following test is made. First, two independent random data sets are generated. From these two data sets aggregate data are constructed, using price index numbers derived from one random data set and expenditures derived from the second random data set. The resulting data were tested for rational demand. A result was derived, which confirms our conjecture but in a stronger way than expected. The constructed aggregate data were consistent with the hypothesis of efficient rational homothetic demand for any type of used price index number. This result is disturbing, because it implies that unreliable independent sources of aggregate price index numbers and expenditures, may cause aggregate data to satisfy the hypothesis of rational, and even homothetic rational, demand.

Figure 6.1 Inefficient Consumer Demand

A second point is the increasing violation of the homotheticity hypothesis at higher aggregation levels. This may be because the degree of freedom to choose a rationalizing linearly homogeneous utility function diminishes if there are less commodities in the data set. An example of this is a set of consumption data for one commodity. In that case there is no freedom of choice at all to choose the relative function values of a rationalizing linearly homogeneous utility function. To check the hypothesis of a diminishing degree of freedom, the 1060 commodity random data from Table 6.4 are aggregated to higher aggregation levels. The results obtained for the resulting data are similar to the results concerning the aggregate Dutch consumption data, thus it appears that there is indeed a diminishing degree of freedom. However, there was one difference with the aggregate Dutch consumption data. The homothetic efficiency levels were closer to one at a higher aggregation level for all aggregate random data sets. An explanation for this may be a relative lack of variation in the data at a higher aggregation level, caused by the fact that all random prices and random quantities are chosen from the same probability distribution.

The next point is the occurrence of high efficiency levels, even when many observations are inconsistent with the hypothesis of efficient rational demand. Figure 6.1 shows an area S of quantities which are consistent with a

rationalizing utility function of a certain efficiency level. This area increases fast when the efficiency level is lowered, assuming the indifference curve is not sharply curved. So we have to be careful when data are biased in such a way that it is easy to construct a smoothly curved utility function, approximately consistent with these data. For example in the case where price data are approximately linearly dependent. Then the existence of a rationalizing utility function for a high efficiency level does not exclude the possibility that many observations are inconsistent with the hypothesis of efficient rational demand.

It remains to explain the fact that all price index numbers for the Dutch aggregate data generate similar results, while the results for the random data depend substantially on the chosen price index. Especially the chained indices appear to generate similar results for the Dutch consumption data. An explanation for this result may be found in Vartia (1978) and Diewert (1978), who show that when prices and quantities do not differ too much between subsequent periods then both the Fisher and Törnqvist chained price indices yield approximately the same value. It seems that this argument applies to the other chained index numbers as well, and less to the direct indices. Clearly the random data do not consist of smoothly varying data, which explains the dependence of the random data results on the choice of price index. The fact that the Fisher and Törnqvist price indices preserve the random behaviour in the aggregate random data sets in a better way than the other indices is probably because these indices are not as strongly biased as the other indices. Both indices are of the form $(P_{bc}/P_{cb})^{1/2}$, for which the upward, or downward, bias of price index P_{bc} will be neutralized by a bias in opposite direction of $1/P_{cb}$.

6.8 Summary

The issue in this chapter was the investigation whether or not aggregation may introduce economic behaviour in the resulting data sets. It is evident from the outcomes of the nonparametric tests for consistency, homotheticity and efficiency, that this is indeed the case. An explanation for these results is the application of the superposition principle, which implies that division of the budget by the price index results in the corresponding quantity measure. This introduces a tendency for the aggregate group quantity measures and the aggregate price index numbers to move in opposite directions. Lower prices are accompanied by higher quantities, which is just what one would expect from a rational consumer.

The results show that one has to be careful in drawing conclusions from aggregate data. The assumption that an obvious rule for rational demand is satisfied, price times quantity equals expenditures, has as implication that aggregate random data may tend to rational demand. The belief in rational demand may be a self-fulfilling prophecy. Summarizing the above, one might state that the phrase *What does it matter which index you use, what matters is that it works* – which I have noticed in a book about indices – appears to contain unexpected truth. Aggregate consumption data may represent economic behaviour, even if the original data do not reflect economic behaviour at all. Moreover, the worst price index number, by accident computed from an other independent set of consumption data, may work very well. For random data I found that such price index numbers may generate aggregate data, which are fully consistent with the hypothesis of efficient rational homothetic consumer demand.

PART II

PRODUCER DEMAND

7 INTRODUCTION

7.1 Theory of Producer Behaviour

Producer theory and consumer theory overlap each other. In this book we confine ourselves to producer behaviour for a single-output production function. The mathematical problems for a single-output production function in the producer theory and the utility function in the consumer theory are often analogous. Differences between textbook producer and consumer theory are the following. First, produced output may be observed, while consumer utility is not observed. Another difference is that, while in standard consumer theory the utility function is not subject to change, the production function may change, because of technical progress. Furthermore, one may assume that the production function is a frontier production function, which represents a limiting frontier of the production output. This assumption allows the occurrence of technically inefficient production. A last point of difference will be treated later on: besides that a producer is a cost minimizer, a producer may also be a profit maximizer.

7.2 Producer Demand

The used definition of producer demand will be equivalent to consumer demand. However, in contrast to consumer behaviour, where utility is not observed, we can observe produced output. Therefore, I will often assume that producer data include the produced output $f(x)$. In that case we will observe $(c, x', f(x'))$, instead of (c, x') , and we will obtain the following slightly different version of the consumer demand definition.

Definition: For a function $f: X \rightarrow \mathbb{R}$, the set of *producer demand with observed output*, is defined as

$$D_d^o(f) \equiv \{(c, x', y) \mid (c, x') \in D_d(f), y = f(x')\}.$$

Furthermore, $D_u^o(f)$ etc. are defined in the same way. One has for example:

$$D_u^o(f) \equiv \{(c, x', y) \mid (c, x') \in D_u(f), y = f(x')\}.$$

7.3 Technical Inefficiency and Frontier Production Functions

Given a production function $f: X \rightarrow \mathbb{R}$, we assumed until now that the relation between input x and output y is simply described by $y = f(x)$. A more general description of producer behaviour allows for *technical inefficiency*. In that case the relation is described by

$$\{(x, y) \mid x \in X, y \leq f(x)\}.$$

Thus, due to technical inefficiency, produced output may also be less than the maximal possible output. The production function represents now the best-practice frontier of production, and is called a *frontier production*

function. We define producer behaviour with observed output in the following way for frontier production functions.

Definition: For a frontier production function $f: X \rightarrow \mathbb{R}$, the set of producer demand with observed output, is defined as

$$\mathcal{D}_d^f(f) \equiv \{(c, x', y) \mid (c, x') \in \bar{D}_d(f), y \leq f(x')\}.$$

Furthermore, $\mathcal{D}_u^f(f)$ etc. are defined analogously. For example

$$\mathcal{D}_u^f(f) \equiv \{(c, x', y) \mid (c, x') \in \bar{D}_u(f), y \leq f(x')\}.$$

7.4 Technical Change

The method of production may change over time. New technical methods are introduced, which may increase production. Therefore one often supposes that the production function is changing over time. I will call such a change of the production function *technical change*. In the most general description of such a change one uses several production functions: one for each observation period. A more specific definition of technical change is given by Hicks.

Definition: Hicks neutral change, for a series $\{t_i \in \mathbb{R}_+\}_{i \in I}$ and a production function $f: X \rightarrow \mathbb{R}$, results in a series of production functions $\{f_i: X \rightarrow \mathbb{R}\}_{i \in I}$ given as

$$f_i(x) = t_i f(x) \quad (i \in I, x \in X).$$

Thus for each period the output is simply multiplied by a positive factor. In a similar way we may multiply inputs of the production function, which leads to the following definition.

Definition: Given a production function $f: X_1 \times X_2 \rightarrow \mathbb{R}$. Neutral change of input X_2 for a series $\{t_i \in \mathbb{R}_+\}_{i \in I}$ results in the series of production functions $\{f_i: X_1 \times X_2 \rightarrow \mathbb{R}\}_{i \in I}$ given as

$$f_i(x_1, x_2) = f(x_1, t_i x_2) \quad (i \in I, x_1 \in X_1, x_2 \in X_2).$$

There are two special cases of neutral change. When X_2 represents labour one has *Harrod neutral change*, and when X_2 represents capital one has *Solow neutral change*.

An obvious hypothesis concerning changing production functions is the assumption that production is improving over time. In that case the production functions are increasing over time. In that case I replace the term change by the term *progress* in the above definitions.

Definition: Technical progress for a series production functions $\{f_i\}_{i \in I}$ means that we have $f_j \leq f_i$ if $j \leq i$. In a similar way we define (Hicks) neutral progress.

7.5 Rationalizing Production Functions

In this chapter I will consider the following kinds of data:

1. Data where output is not observed, like $\mathcal{D}_d$.
2. Data where output is observed, like $\mathcal{D}_d^o$.
3. Data where output is observed and limited by a frontier production function, like $\mathcal{D}_d^f$.

For all these data definitions we may define the set of functions which rationalizes producer behaviour.

Definition: Let D be a data set of producer behaviour. The family of production functions $\mathcal{F}_d^o$, which rationalizes D for producer demand with

observed output, is defined as

$$\mathcal{F}_d^o(D) \equiv \{f: X \rightarrow \mathbb{R} \mid D \subset \mathcal{D}_d^o(f)\}.$$

The definition for $\mathcal{F}_d^f$ etc. are analogous.

Just as in case of $\mathcal{F}_d$ for consumer demand one has

$$\mathcal{F}_d^o(D_1 \cup D_2) = \mathcal{F}_d^o(D_1) \cap \mathcal{F}_d^o(D_2).$$

This is in general valid for any kind of behaviour, like $\mathcal{F}_d^f$, which is defined in the same way as $\mathcal{F}_d$.

7.6 Application to Empirical Data

In the following chapters of Part II, Producer Demand, the above definitions are applied to develop a theory in the same way as is done in Part I for consumer demand. By way of illustration the theory is applied to a data set of Dutch producer behaviour. The nominal values concerning this data set are derived from input-output tables of the Dutch economy, covering the years 1969–1983, as published by the Netherlands Central Bureau of Statistics. For some years additional information was obtained from this bureau to get a consistent array of tables. To obtain prices for the nominal data, various sources were used. The large tables were aggregated to a less detailed sector format used by the Netherlands Central Planning Bureau. This made it possible to use less accurate sector price data, supplied by this bureau, for the missing prices. The way capital is treated will be discussed in Part IV, Capital. Here it is sufficient to know that there is a price and quantity for each good, including capital. For a more detailed description of the Dutch industry data I refer the reader to Appendix E.

7.7 Contents of Part II

Chapter 8 is concerned with the existence of a production function for given producer data. This chapter treats all variations of the producer demand problem that are mentioned above.

Chapter 9 is concerned with linearly homogeneous production functions for given producer data. Results are derived by searching for the upper bound of all rational linearly homogeneous production functions for the given data set. The application of the resulting theory to the Dutch industry data yields interesting graphs of, for example, technical efficiency upper bounds, bounds for the frontier production function and Hicks neutral progress.

Chapter 10 is a short chapter containing some results for homothetic production functions, i.e. a monotonic transformation of a linearly homogeneous function. This chapter is concerned with the existence of such a function for given producer data with observed output. Moreover, it treats restrictions on the underlying linearly homogeneous function.

Chapter 11 treats weakly separable production functions. This chapter is concerned with the problem whether given producer demand data, including observed output, may be generated by a weakly separable production function. These results are extended to linearly homogeneous weakly separable functions and weakly separable functions with a linearly homogeneous subfunction. The problem of technically progressing weakly separable production functions is not considered, because the results for weakly separable productions functions – without the addition of technical progress – are already complicated.

8 PRODUCER DEMAND

8.1 Introduction

This chapter considers the question whether there exists a production function for given producer data without any a priori restriction on the production function. This problem is considered for producer demand, frontier production functions and technical change. Furthermore, I will derive nonparametric restrictions on the technical efficiency coefficients and the scale of neutral progress. It turns out that for the Dutch industry data there only are slight violations of these nonparametric restrictions. This result is not very surprising, because there is a large set of production functions from which we may choose. The results are derived without using any restriction on the production function. Later on in the following chapters, when we consider restrictions on the production function, we shall encounter nonparametric conditions that are severely violated.

8.2 Producer Demand

There is no difference between producer and consumer demand when produced output is not observed. This means that in that case the axiom of revealed preference can be used to test the hypothesis of producer demand. This follows directly from Theorem 2.3, which states that the existence of a utility function for consumer demand can be tested with the axiom of revealed preference. The Dutch industry data without produced outputs accept the existence of a production function.

Since the produced outputs of the Dutch industry are available a second test can be made, which checks whether these outputs are consistent with the axiom of revealed preference. For such a test the following theorem can be used.

Theorem 8.1: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set, where $c_i: X \rightarrow \mathbb{R}_+$. Then the following conditions are equivalent:

- (i) $\mathcal{F}_d^0(D) \neq \emptyset$.
- (ii) One has

$$\begin{aligned} c_i(x^j) \leq c_i(x^i) &\Rightarrow y_j \leq y_i & (i, j \in I), \\ c_i(x^j) < c_i(x^i) &\Rightarrow y_j < y_i & (i, j \in I). \end{aligned}$$

Proof (i) $\Rightarrow$ (ii): Obvious, use the definition of $\mathcal{F}_d^0$.

(ii) $\Rightarrow$ (i): By Theorem 2.4 there exists a function

$$f(x) = \min_{i \in I} f_i + \lambda_i [c_i(x) - c_i(x^i)] \quad (x \in X),$$

such that $\lambda_i > 0$ and $f(x^i) \geq f_i$ for all $i \in I$, and

$$f_i \leq f_j \Leftrightarrow y_i \leq y_j \quad (i, j \in I).$$

So, because clearly $f(x^i) \leq f_i$, one has $f(x^i) = f_i$ for all $i \in I$, and it is not difficult to show that we have $(c_i, x^i) \in D_d(f)$ for all $i \in I$. Thus, for any strictly increasing function $m: \mathbb{R} \rightarrow \mathbb{R}$, such that $m(f_i) = y_i$ for all $i \in I$, one has $m \circ f \in \mathcal{F}_d^0(D)$. That such a function m exists follows from

$$f_i \leq f_j \Leftrightarrow y_i \leq y_j \quad (i, j \in I). \quad \square$$

When the above theorem is applied to the Dutch industry data, to test the hypothesis of producer demand, it appears that two observations have to be

eliminated.

Using the above theorem and the efficiency transformation given in Example 2.3, we can determine the economical efficiency upper bound for the complete data set. I computed this efficiency upper bound for the Dutch industry data, using bisection, and the result was $e = 99.5\%$.

8.3 Frontier Production Functions

Let $D = \{(c_i, x^i, y_i)\}_{i \in I}$ be a set of production data. For these data a frontier production function $f \in \mathcal{F}_d^f(D)$ satisfies by definition $f \in \mathcal{F}_d(D)$ and $f(x^i) \geq y_i$ for all $i \in I$. The corresponding values of technical efficiency are

$$t_i = y_i / f(x^i) \quad (i \in I).$$

So the technical efficiency values are maximal, for given production data, when the corresponding frontier production function output is minimal. Hence the following definition may be useful.

Definition: The lower bound L is equal to

$$L(F, x) \equiv \inf_{f \in F} f(x) \quad (x \in X).$$

Using this definition we may consider the maximal technical efficiency values for given production data.

Definition: For given data $D = \{(c_i, x^i, y_i)\}_{i \in I}$, the technical efficiency upper bounds t_i are defined as

$$t_i = y_i / u_i, \quad u_i = L(\mathcal{F}_d^f(D), x^i) \quad (i \in I).$$

Of course every technical efficiency upper bound will be less than or equal to one. The technical efficiency upper bounds provide a bound for the possible waste of output for producer demand. I will argue below how the problem may be solved.

The information needed to solve the problem, is provided by the axiom of revealed preference, which is described in Chapter 2. In order that any solution u_i exists we have to assume $\mathcal{F}_d(D) \neq \emptyset$. When this assumption is satisfied, we can apply the axiom of revealed preference. Application of this axiom reveals information about the ordering of the values $f(x^i)$ for frontier production functions $f \in \mathcal{F}_d^f(D)$. This information is represented by the relations

$$\begin{aligned} c_i(x^j) \leq c_i(x^i) &\Leftrightarrow x_i R x_j & (i, j \in I), \\ c_i(x^j) < c_i(x^i) &\Leftrightarrow x_i P x_j & (i, j \in I). \end{aligned}$$

These revealed preference relations R and P and all the observed values y_i contain together all the information, necessary to determine the technical efficiency upper bounds. The following simple problem determines the minimal frontier production values u_i from the revealed preference relations R and P , and all outputs y_i :

$$\inf u_i \quad (i \in I)$$

subject to

$$\begin{aligned} u_i &\geq y_i & (i \in I), \\ u_i &\leq u_j & (i, j \in I: x_i R x_j), \\ u_i &< u_j & (i, j \in I: x_i P x_j), \end{aligned}$$

Because we have $P \subset R$, the relation P plays no part. In fact we may drop the last condition and obtain for finite data sets the following solution

$$u_i = \max \{y_k \mid k \in I: x^k R^* x^i\} \quad (i \in I).$$

Remark: The question is whether we can actually minimize all values u_i together. This is indeed the case for finite data sets, because $\mathcal{F}_d^f(D)$ is closed with respect to the pointwise minimum operation. Thus if $f_i \in \mathcal{F}_d^f(D)$ for all $i \in I$ then $f \in \mathcal{F}_d^f(D)$, where

$$f = \min_i f_i$$

is the pointwise minimum over all functions f_i . This means that we can always find a function $f \in \mathcal{F}_d^f(D)$ that minimizes all values $u_i = f(x^i)$ together.

The next theorem is based on the above outline and uses this solution to determine the technical efficiency upper bounds.

Theorem 8.2: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set and $\mathcal{F}_d(D) \neq \emptyset$. Let R be the corresponding revealed preference relation, determined by

$$x^j R x^i \Leftrightarrow c_i(x^j) \leq c_i(x^i) \quad (i, j \in I).$$

Then

$$L(\mathcal{F}_d^f(D), x^i) = \max \{y_k \mid k \in I: x^k R^* x^i\} \quad (i \in I).$$

Proof: Put

$$u_i = \max \{y_k \mid k \in I: x^k R^* x^i\} \quad (i \in I).$$

First we show that one has

$$u_i \leq L(\mathcal{F}_d^f(D), x^i) \quad (i \in I).$$

Let $f \in \mathcal{F}_d^f(D)$. To prove the inequality, it is sufficient to show that we have

$$y_k \leq f(x^i) \quad (i, k \in I: x^k R^* x^i).$$

From $f \in \mathcal{F}_d(D)$ and the definition of consumer demand $\mathcal{F}_d$ and R , we obtain

$$f(x^k) \leq f(x^i) \quad (i, k \in I: x^k R x^i),$$

hence one has also $f(x^k) \leq f(x^i)$ in case $x^k R^* x^i$. So, because $f \in \mathcal{F}_d^f(D)$ and the definition of $\mathcal{F}_d^f$ implies $y_k \leq f(x^k)$, we have indeed

$$y_k \leq f(x^k) \leq f(x^i) \quad (i, k \in I: x^k R^* x^i).$$

The proof is complete if we can show that we have

$$u_i \geq L(\mathcal{F}_d^f(D), x^i) \quad (i \in I).$$

As one may verify, the numbers

$$u_i = \max \{y_k \mid k \in I: x^k R^* x^i\} \quad (i \in I)$$

are a solution of the problem

$$\inf u_i \quad (i \in I)$$

subject to

$$\begin{aligned} u_i &\geq y_i & (i \in I), \\ u_i &\leq u_j & (i, j \in I: x_i R x_j). \end{aligned}$$

Now, since we assumed $\mathcal{F}_d(D) \neq \emptyset$, there is a solution $f \in \mathcal{F}_d(D)$. For this function we may assume $f(x^i) > 0$ for all $i \in I$, because every monotonic transformation of f is element of $\mathcal{F}_d(D)$. Now we may verify that

$$v_i = u_i + \delta f(x^i) \quad (i \in I),$$

for arbitrary $\delta > 0$, satisfies the conditions

$$\begin{aligned} v_i &\geq y_i & (i \in I), \\ v_i &\leq v_j & (i, j \in I: x_i R x_j), \\ v_i &< v_j & (i, j \in I: x_i P x_j). \end{aligned}$$

By condition (ii) in Theorem 8.1, this implies

$$\mathcal{F}_d^0(\{(c_i, x^i, y_i)\}_{i \in I}) \neq \emptyset.$$

Furthermore, we have thus

$$v_i \geq L(\mathcal{F}_d^f(D), x^i) \quad (i \in I).$$

Hence, by letting δ approach to zero, we obtain

$$u_i \geq L(\mathcal{F}_d^f(D), x^i) \quad (i \in I). \quad \square$$

Applying the above theorem to the Dutch industry data resulted in two technically inefficient years. These years were 1975 with technical efficiency upper bound 98.4% and 1978 with 97.3%.

8.4 Technical Progress

First I consider technical progress with observed output. An approach which I applied successfully in nonparametric problems concerning technical progress is the following. Consider a given data set $\{d_i\}_{i \in I}$ for the periods $i \in I$. Suppose we wish to extract nonparametric information in order to derive an upper bound for the possible values of the production function at a certain period $i \in I$. Then a part of the nonparametric information, derived for the similar case without technical progress, can be used. Because technical progress means that the production function is increasing, all upper bound information concerning the production functions in the past periods $j \leq i$ is relevant. This approach is used in the following theorem.

Theorem 8.3: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set, where $c_i: X \rightarrow \mathbb{R}_+$. Then the following conditions are equivalent:

- (i) There exists a series of production functions $\{f_i\}_{i \in I}$ such that $f_j \leq f_i$ if $j \leq i$, and $(c_i, x^i, y_i) \in D_d^0(f_i)$ for all $i \in I$.
 (ii) One has

$$\begin{aligned} c_i(x^j) \leq c_i(x^i) &\Rightarrow y_j \leq y_i & (i, j \in I: j \leq i), \\ c_i(x^j) < c_i(x^i) &\Rightarrow y_j < y_i & (i, j \in I: j \leq i). \end{aligned}$$

Proof (i) $\Rightarrow$ (ii): Follows immediately from the definition of D_d^0 .

(ii) $\Rightarrow$ (i): Suppose (ii) is satisfied. Then by Theorem 2.4, we may choose numbers $\lambda_i > 0$ and ϕ_i , such that

$$\begin{aligned} \phi_i + \lambda_i[c_i(x^j) - c_i(x^i)] &\geq \phi_j & (i, j \in I: j \leq i), \\ \phi_j \leq \phi_i &\Leftrightarrow y_j \leq y_i & (i, j \in I). \end{aligned}$$

Let m be a monotonic transformation such that $m(\phi_i) = y_i$ for all $i \in I$, and put

$$f_j(x) = m(\min_{i \geq j} \phi_i + \lambda_i[c_i(x) - c_i(x^i)]) \quad (j \in I, x \in X).$$

For these functions one can easily verify that condition (i) is satisfied. We have $f_j \leq f_i$ if $j \leq i$, because in general

$$\min_{k \geq j} \phi_k + \lambda_k[c_k(x) - c_k(x^k)] \leq \min_{k \geq i} \phi_k + \lambda_k[c_k(x) - c_k(x^k)] \quad (x \in X, i, j \in I: j \leq i).$$

Furthermore, for every $i \in I$, we have $(c_i, x^i, y_i) \in D_d^0(f_i)$, since $f_i(x^i) = m(\phi_i) = y_i$ and moreover $(c_i, x^i) \in D_d(f_i)$, because

$$\begin{aligned} c_i(x) \leq c_i(x^i) &\Rightarrow f_i(x) \leq m(\phi_i + \lambda_i[c_i(x) - c_i(x^i)]) \leq m(\phi_i) = f_i(x^i) & (i \in I), \\ c_i(x) < c_i(x^i) &\Rightarrow f_i(x) \leq m(\phi_i + \lambda_i[c_i(x) - c_i(x^i)]) < m(\phi_i) = f_i(x^i) & (i \in I). \quad \square \end{aligned}$$

To satisfy the hypothesis of producer demand with technical progress, one had to eliminate one observation from the Dutch industry data. For the whole data set I derived as upper bound of economical efficiency $e = 99.8\%$, using the efficiency transformation of Example 2.3.

Now, we may consider the case where output is not observed. Then, as stated in the theorem given below, the hypothesis of technical progress can never be falsified. To test the hypothesis we have to check the existence of numbers y_i satisfying condition (ii) in Theorem 8.3 above. I will show that such numbers always exist. This will be shown while proving the following theorem.

Theorem 8.4: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a finite data set. Then there exists a series of production functions $\{f_i\}_{i \in I}$ such that $f_j \leq f_i$ if $j \leq i$, and $(c_i, x^i) \in D_d(f_i)$ for all $i \in I$.

Proof: The proof is based on the axiom of revealed preference, which is modified for the case of technical progress. As described in Theorem 8.3, we have the following information when there is technical progress

$$\begin{aligned} j \text{ is not preferred to } i & \text{ if } c_i(x^j) \leq c_i(x^i) \quad (i, j \in I: j \leq i), \\ i \text{ is preferred to } j & \text{ if } c_i(x^j) < c_i(x^i) \quad (i, j \in I: j \leq i). \end{aligned}$$

So we can define the revealed preference relations R and P for technical progress as

$$\begin{aligned} x^j R x^i & \Leftrightarrow c_i(x^j) \leq c_i(x^i) \text{ and } j \leq i \quad (i, j \in I), \\ x^j P x^i & \Leftrightarrow c_i(x^j) < c_i(x^i) \text{ and } j \leq i \quad (i, j \in I). \end{aligned}$$

By Theorem 8.3 there is a series of technically progressing functions for the data if we can find numbers y_i , such that one has

$$\begin{aligned} x^j R x^i & \Rightarrow y_j \leq y_i \quad (i, j \in I), \\ x^j P x^i & \Rightarrow y_j < y_i \quad (i, j \in I). \end{aligned}$$

As described in Chapter 2, such numbers do exist when the axiom of revealed preference $R^* \cap P^c = \emptyset$ is satisfied. This, however, is always the case, because we have

$$\begin{aligned} x^j R^* x^i & \Rightarrow j \leq i \quad (i, j \in I), \\ x^j P^c x^i & \Rightarrow j > i \quad (i, j \in I). \end{aligned} \quad \square$$

From the revealed preference relations, described in the above proof, we can determine the technical efficiency upper bounds. This is possible by a simple modification of Theorem 8.2 in order to allow for technical progress.

Theorem 8.5: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set. Let R be the corresponding revealed preference relation for technical progress, determined by

$$x^j R x^i \Leftrightarrow c_i(x^j) \leq c_i(x^i) \text{ and } j \leq i \quad (i, j \in I).$$

Then the technical efficiency upper bounds t_i for producer demand with technical progress are given as

$$t_i = y_i/u_i, \quad u_i = \max \{y_k \mid k \in I: x^k R^* x^i\} \quad (i \in I).$$

Proof: By Theorem 8.3 and the definition of demand D_d^f for frontier production functions, the following conditions are equivalent:

- (i) There exist a series of production functions $\{f_i\}_{i \in I}$ such that $f_j \leq f_i$ if $j \leq i$, and $f_i(x^i) = u_i$, $(c_i, x^i, y_i) \in D_d^f(f_i)$, for all $i \in I$.
- (ii) There exist outputs u_i such that

$$\begin{aligned} u_i & \geq y_i \quad (i \in I), \\ c_i(x^j) \leq c_i(x^i) & \Rightarrow u_j \leq u_i \quad (i, j \in I: j \leq i), \\ c_i(x^j) < c_i(x^i) & \Rightarrow u_j < u_i \quad (i, j \in I: j \leq i). \end{aligned}$$

So, to maximize the technical efficiency t_i , we have to minimize u_i in (ii). From Theorem 8.4 it follows that there exists always such a solution, satisfying above conditions in (i) and (ii). Now, using the approach in the proof of Theorem 8.2, we can show that such a solution can be found by

solving the following problem:

$$\inf u_i \quad (i \in I)$$

subject to

$$\begin{aligned} u_i &\geq y_i & (i \in I), \\ u_i &\leq u_j & (i, j \in I: x_i R x_j). \end{aligned}$$

This problem has indeed the given solution

$$u_i = \max \{y_k \mid k \in I: x^k R^* x^i\} \quad (i \in I). \quad \square$$

Application of the above theorem to the Dutch data had the following result. There is only one year for which the technical efficiency upper bound is less than one. For this year, the year $i = 1975$, we obtain $t_i = 98.4\%$.

8.5 Hicks Neutral Change

The popularity of Hicks neutral change is due to the fact that a simple transformation may convert data of producer demand with Hicks neutral change to data of producer demand behaviour without Hicks neutral change. This is the subject of the next theorem.

Theorem 8.6: Let $c: X \rightarrow \mathbb{R}_+$ and $f: X \rightarrow \mathbb{R}$. Then one has

$$(c, x, y) \in \mathcal{D}_d^0(tf) \Leftrightarrow (c, x, t^{-1}y) \in \mathcal{D}_d^0(f) \quad (t > 0).$$

Proof: Obvious. □

Choosing the scale factor t for Hicks neutral change means in fact that we may choose the resulting Hicks neutral changed output $t^{-1}y$. Hence, if we wish to test producer demand with Hicks neutral change, we may as well test producer demand without observed output. This is stated in the following theorem.

Theorem 8.7: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$, and suppose $y_i > 0$, for all $i \in I$. Then the following conditions are equivalent:

(i) There exist a function $f: X \rightarrow \mathbb{R}$ and numbers $t_i > 0$, such that

$$(c_i, x^i, y_i) \in \mathcal{D}_d^0(t_i f) \quad (i \in I).$$

(ii) $\mathcal{F}_d(D) \neq \emptyset$.

Proof (i) $\Rightarrow$ (ii): By Theorem 8.6 one has

$$(c_i, x^i, y_i) \in \mathcal{D}_d^0(t_i f) \Leftrightarrow (c_i, x^i, t_i^{-1}y_i) \in \mathcal{D}_d^0(f) \quad (i \in I),$$

and we have thus $f \in \mathcal{F}_d(D)$.

(ii) $\Rightarrow$ (i): Let $f \in \mathcal{F}_d(D)$. We may suppose $f(x^i) > 0$ for all $i \in I$. Otherwise we may apply a suitable monotonic transformation, because $m \circ f \in \mathcal{F}_d(D)$ for any monotonic transformation $m: \mathbb{R} \rightarrow \mathbb{R}$. Now, for

$$t_i = y_i / f(x^i) \quad (i \in I)$$

one has

$$(c_i, x^i, y_i) \in \mathcal{D}_d^0(t_i f) \quad (i \in I). \quad \square$$

By the above theorem the assumption of Hicks neutral progress with observed output is equivalent to the assumption of producer demand without observed output. We may test the latter using of a revealed preference test. It appeared that the Dutch industry data without observed output, were compatible with the hypothesis of producer demand. Hence, by Theorem 8.7 the whole data set – including observed output – is compatible with the hypothesis of Hicks neutral changing producer demand.

Another approach to look at the above problem is to solve the restrictions on the factors t_i for Hicks neutral change. As is done in the following theorem.

Theorem 8.8: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set, where $y_i > 0$ and $c_i: X \rightarrow \mathbb{R}_+$, for all $i \in I$. Then the following conditions are equivalent:

(i) There exist a function $f: X \rightarrow \mathbb{R}$ and numbers $t_i > 0$ such that

$$(c_i, x^i, y_i) \in D_a^0(t_i f) \quad (i \in I).$$

(ii) There exist values $\phi_i = \ln t_i$ such that

$$\phi_i - \phi_j \leq a_{ij}, \quad \phi_i - \phi_j < b_{ij} \quad (i, j \in I),$$

where

$$a_{ij} = \begin{cases} \ln(y_i/y_j) & (i, j \in I: c_i(x^j) \leq c_i(x^i)), \\ \infty & (\text{otherwise}), \end{cases}$$

$$b_{ij} = \begin{cases} \ln(y_i/y_j) & (i, j \in I: c_i(x^j) < c_i(x^i)), \\ \infty & (\text{otherwise}). \end{cases}$$

Proof: By Theorem 8.6 one has

$$(c_i, x^i, y_i) \in D_a^0(t_i f) \Leftrightarrow (c_i, x^i, t_i^{-1} y_i) \in D_a^0(f) \quad (i \in I).$$

By Theorem 8.1 the latter is equivalent to

$$c_i(x^j) \leq c_i(x^i) \Rightarrow t_j^{-1} y_j \leq t_i^{-1} y_i \quad (i, j \in I),$$

$$c_i(x^j) < c_i(x^i) \Rightarrow t_j^{-1} y_j < t_i^{-1} y_i \quad (i, j \in I).$$

Hence, we obtain that (i) is equivalent to the existence of numbers t_i such that

$$t_i/t_j < y_i/y_j \quad (i, j \in I: c_i(x^j) < c_i(x^i)),$$

$$t_i/t_j \leq y_i/y_j \quad (i, j \in I: c_i(x^j) \leq c_i(x^i)).$$

By taking the logarithm on both sides of the above inequalities, this is equivalent to (ii). $\square$

Using the above theorem, we may compute lower and upper bounds for the levels t_i of Hicks neutral change. These bounds follow from the inequalities

$$\phi_i - \phi_j \leq a_{ij} \quad (i, j \in I)$$

in Theorem 8.8, using $\phi_i = \ln t_i$. The solution follows from the shortest path matrix A^* of $A = [a_{ij}]$ and is given by

$$\phi_i - \phi_j \leq a_{ij}^* \quad (i, j \in I).$$

The definition of a shortest path matrix is given in Appendix C. The idea of this matrix is as follows. When a_{ij} , $i, j \in I$, represents the cost of moving from i to j then a_{ij}^* represents the cost of the cheapest path from i to j .

Figure 8.1 displays the lower and upper bounds

$$e^{-a_{1i}^*} \leq t_i/t_1 \leq e^{a_{i1}^*} \quad (i, j \in I),$$

with the first year as base year. In general one may derive these bounds from the output values y_i and the revealed preference relation R for unobserved output. One may prove that one has

$$e^{a_{ij}^*} = \begin{cases} y_i/y_j & (i, j \in I: i R^* j), \\ \infty & (\text{otherwise}). \end{cases}$$

For example, the Dutch industry data satisfies

$$c_i(x^1) < c_i(x^i) \quad (i \in I: i \neq 1),$$

which means

$$i R^* 1, \text{ not } 1 R^* i \quad (i \in I: i \neq 1).$$

Hence, the lower and upper bounds that are displayed in Figure 8.1, are in this particular case equal to

$$0 = e^{-a_{1i}} \leq t_i/t_1 \leq e^{a_{i1}} = y_i/y_1 \quad (i \in I: i \neq 1).$$

Figure 8.1 Bounds Hicks Neutral Change

Usually we assume that the change of the production function is due to progress. For Hicks neutral progress the following theorem is available.

Theorem 8.9: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set, where $y_i > 0$ and $c_i: X \rightarrow \mathbb{R}_+$, for all $i \in I$. Then the following conditions are equivalent:

(i) There exist a function $f: X \rightarrow \mathbb{R}$ and numbers $t_i > 0$, such that $t_j \leq t_i$ if $j \leq i$, and

$$(c_i, x^i, y_i) \in D_d^0(t_i f) \quad (i \in I).$$

(ii) There exist values $\phi_i = \ln t_i$ such that

$$\begin{aligned} \phi_i - \phi_j &\leq \min \{a_{ij}, c_{ij}\} \quad (i, j \in I), \\ \phi_i - \phi_j &< b_{ij} \quad (i, j \in I), \end{aligned}$$

where

$$\begin{aligned} a_{ij} &= \begin{cases} \ln(y_i/y_j) & (i, j \in I: c_i(x^j) \leq c_i(x^i)), \\ \infty & (\text{otherwise}), \end{cases} \\ b_{ij} &= \begin{cases} \ln(y_i/y_j) & (i, j \in I: c_i(x^j) < c_i(x^i)), \\ \infty & (\text{otherwise}), \end{cases} \end{aligned}$$

$$c_{ij} = \begin{cases} 0 & (i, j \in I: i \leq j), \\ \infty & (\text{otherwise}). \end{cases}$$

Proof: Analogous to the proof of Theorem 8.8. We only have to include condition $t_j \leq t_i$ if $j \leq i$, which is equivalent to $t_i/t_j \leq 1$ if $i \leq j$. Taking the logarithm on both sides of the above inequality $t_i/t_j \leq 1$ yields the additional restrictions of c_{ij} in (ii). $\square$

Condition (ii) in the above theorem is a shortest path problem that can be solved with Theorem C.3, given in Appendix C. When the theorem is applied to the Dutch industry data, it appears that one observation should be removed in order to obtain a data set, which is consistent with producer demand. Using bisection and Theorem C.3, we may determine the economical efficiency upper bound for the whole data set. The result, where again the efficiency transformation given in Example 2.3 is used, is $e = 99.8\%$.

For this efficiency level e , we may compute lower and upper bounds for the levels t_i of Hicks neutral progress that belong to the transformed data $\Phi_e D$. These bounds follow from the inequalities

$$\phi_i - \phi_j \leq d_{ij} = \min \{a_{ij}, c_{ij}\} \quad (i, j \in I)$$

in Theorem 8.9, where

$$\phi_i = \ln t_i \quad (i \in I).$$

The solution follows from the shortest path matrix D^* of $D = [d_{ij}]$ and is given by

$$\phi_i - \phi_j \leq d_{ij}^* \quad (i, j \in I).$$

Figure 8.2 Bounds Hicks Neutral Progress

Figure 8.2 displays the lower and upper bounds

$$e^{-d_{1i}^*} \leq t_i/t_1 \leq e^{d_{i1}^*} \quad (i, j \in I),$$

with the first year as base year. Since there is progress, the lower bound is larger or equal to one. It appears that the lower bound is equal to one,

except for the last year. For the upper bound, however, we get nontrivial information.

8.6 Neutral Change

Neutral change is similar to Hicks neutral change, but now an input vector is scaled instead of the produced output. The following theorem is similar to Theorem 8.6, which considered Hicks neutral change.

Theorem 8.10: Suppose $c: X_1 \times X_2 \rightarrow \mathbb{R}_+$ and $f: X_1 \times X_2 \rightarrow \mathbb{R}$. Then for every $t > 0$ with $[x'_1, tx'_2] \in X_1 \times X_2$, one has

$$(c, x', y) \in D_d^0(f_t) \Leftrightarrow (c_t, [x'_1, tx'_2], y) \in D_d^0(f),$$

where

$$f_t(x_1, x_2) = f(x_1, tx_2), \quad c_t(x_1, x_2) = c(x_1, t^{-1}x_2) \quad (x_1 \in X_1, x_2 \in X_2).$$

Proof: Using the transformation

$$v_1 = x_1, \quad v_2 = tx_2 \quad (x_1 \in X_1, x_2 \in X_2),$$

we obtain

$$\begin{aligned} c(x) \leq c(x') &\Leftrightarrow c_t(v) \leq c_t(x^t), \\ f_t(x) \leq f_t(x') &\Leftrightarrow f(v) \leq f(x^t). \end{aligned}$$

Hence the result follows directly from the definition of D_d^0 . $\square$

The idea to solve the restrictions on producer demand for neutral change is simple. With the above theorem we can convert the data to data, to which the usual nonparametric restrictions for producer demand can be applied. This approach is used in the following theorem.

Theorem 8.11: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set in which all functions $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$ are of the form

$$c_i(x) = v_i(x_1) + w_i(x_2) \quad (i \in I, x \in X_1 \times X_2),$$

where $w_i: X_2 \rightarrow \mathbb{R}_+$ is linearly homogeneous. Then the following conditions are equivalent:

(i) There exist a function $f: X_1 \times X_2 \rightarrow \mathbb{R}$ and numbers $t_i > 0$, such that for

$$f_i(x) = f(x_1, t_i x_2) \quad (i \in I, x \in X_1 \times X_2)$$

one has

$$(c_i, x^i, y_i) \in D_d^0(f_i) \quad (i \in I).$$

(ii) There exist values $\phi_i = \ln t_i$ such that

$$\phi_i - \phi_j \leq a_{ij}, \quad \phi_i - \phi_j < b_{ij} \quad (i, j \in I),$$

where

$$\begin{aligned} a_{ij} &= \begin{cases} \ln(w_i(x_2^j)[c_i(x^i) - v_i(x_1^j)]^{-1}) & (i, j \in I: y_j \geq y_i, c_i(x^i) > v_i(x_1^j)), \\ \infty & (\text{otherwise}), \end{cases} \\ b_{ij} &= \begin{cases} \ln(w_i(x_2^j)[c_i(x^i) - v_i(x_1^j)]^{-1}) & (i, j \in I: y_j > y_i, c_i(x^i) > v_i(x_1^j)), \\ \infty & (\text{otherwise}). \end{cases} \end{aligned}$$

Proof: From Theorem 8.1 and Theorem 8.10 we obtain that condition (i) is equivalent to the existence of numbers t_i such that

$$c_i(x_1^j, t_i^{-1} t_j x_2^j) > c_i(x^i) \quad (i, j \in I: y_j > y_i),$$

$$c_i(x_1^j, t_i^{-1} t_j x_2^j) \geq c_i(x^i) \quad (i, j \in I: y_j \geq y_i).$$

Because we have assumed that we can write

$$c_i(x_1^j, t_i^{-1} t_j x_2^j) = v_i(x_1^j) + t_i^{-1} t_j w_i(x_2^j) \quad (i, j \in I),$$

this condition can be rewritten as

$$t_i^{-1} t_j w_i(x_2^j) > c_i(x^j) - v_i(x_1^j) \quad (i, j \in I: y_j > y_i),$$

$$t_i^{-1} t_j w_i(x_2^j) \geq c_i(x^j) - v_i(x_1^j) \quad (i, j \in I: y_j \geq y_i).$$

Because the left side of the above inequalities is nonnegative, we may drop all inequalities that are not positive at the right side. Finally, we obtain (ii) by taking the logarithm at both sides of the remaining inequalities. $\square$

We may test condition (ii) using Theorem C.3, given in Appendix C. Note that the special additive form $c_i = v_i + w_i$ of the cost functions is necessary in the above theorem. It is used to derive the restrictions on $t_i^{-1} t_j$.

In case of neutral progress there is the additional restriction that the coefficients t_i for neutral change are increasing. This additional restriction leads to the following theorem, which is a simple variation of the previous theorem.

Theorem 8.12: *The following conditions are equivalent:*

(i) *There exist $t_i > 0$ satisfying Theorem 8.11 (i), such that $t_j \leq t_i$ if $j \leq i$.*

(ii) *There exist values $\phi_i = \ln t_i$ such that*

$$\phi_i - \phi_j \leq \min \{a_{ij}, c_{ij}\} \quad (i, j \in I),$$

$$\phi_i - \phi_j < b_{ij} \quad (i, j \in I),$$

where a_{ij} and b_{ij} are given in Theorem 8.11 and

$$c_{ij} = \begin{cases} 0 & (i, j \in I: i \leq j), \\ \infty & (\text{otherwise}). \end{cases}$$

Proof: Add the restriction $t_j \leq t_i$ if $j \leq i$, to the result described in Theorem 8.11. $\square$

A test for the existence of numbers ϕ_i satisfying condition (ii) in the above theorem is given in Appendix C in Theorem C.3. I applied condition (ii) of Theorem 8.12 to two different inputs of the Dutch industry data: labour and capital. Using this theorem, we may determine the largest possible data set that is consistent with neutral progressing producer demand. The result was as follows. For both neutral progress of labour and capital one observation has to be deleted to get a consistent data set.

Another application of Theorem 8.12, without removing observations from the data, is the computation of an economical efficiency upper bound. Such a bound represents a violation measure for the hypothesis of neutral progressing producer demand. To compute an efficiency upper bound, there are certain restrictions that the used efficiency transformation has to satisfy. In Theorem 8.11 and 8.12, it is assumed that we may write the cost functions in the form

$$c(x) = v(x_1) + w(x_2) \quad (x \in X_1 \times X_2),$$

where w is a linearly homogeneous function. This assumption may cause problems. If data satisfy this assumption, this is not necessarily the case for the transformed data, for instance, when we use the efficiency transformations Φ_e or Λ_e given in Example 2.3 and 3.1 respectively. However, we may prove both theorems, using the relaxed assumption that we may write

$$c_i(x_1^j, \lambda x_2^j) = v_i(x_1^j) + w_i(\lambda x_2^j) \quad (\lambda > 0, i, j \in I: i \neq j),$$

in which $w_i(\lambda x_2^j)$ is nonnegative and linearly homogeneous in λ . For empirical

data we may expect that this property remains satisfied, after transformation by the efficiency transformation Φ_e or Λ_e . So in practice there is no problem at all. However, if we wish to apply the above theorems to any possible data, we might choose the following suitable efficiency transformation, which leaves the desired property intact.

Example 8.1 Suppose (c, x') is a data observation such that

$$c(x) = v(x_1) + w(x_2) \quad (x \in X = X_1 \times X_2),$$

where w is linearly homogeneous. Using the efficiency transformation Φ_e and Λ_e , given in Example 2.3 and Example 3.1 respectively, we define the functions v^e and w^e as follows

$$(v^e, x'_1) = \Phi_e(v, x'_1), \quad (w^e, x'_2) = \Lambda_e(w, x'_2).$$

Then $\hat{\Phi}_e(c, x') = (\hat{c}^e, x')$, where

$$\hat{c}^e(x) = v^e(x_1) + w^e(x_2) \quad (x \in X_1 \times X_2),$$

defines an efficiency transformation, where a linearly homogeneous function w is transformed into a linearly homogeneous function w^e .

In the case of the Dutch industry data it was not necessary to use the above efficiency transformation. It was allowed to use the efficiency transformation Φ_e , given in Example 2.3. This resulted in both cases, neutral progress of labour and capital, in the efficiency upper bound $\epsilon = 99.8\%$.

Using this efficiency level, I computed the lower and upper bounds for the scale factors of neutral progress that belong to the transformed data $\Phi_e D$. The resulting bounds are

$$e^{-d_{ji}^*} \leq t_i/t_j \leq e^{d_{ij}^*} \quad (i, j \in I),$$

where we obtain

$$d_{ij} = \min \{a_{ij}, c_{ij}\} \quad (i, j \in I)$$

from Theorem 8.12. I computed these bounds for t_i/t_j for a fixed base year j , the first observation year. For both cases, capital and labour, the result was similar and not very informative. The lower bounds were equal to one, except for a value close to one for the last observation year, and the upper bounds equal to infinity. It seems that the specification of the production function is too wide for the derivation of narrow nonparametric restrictions. Later, when we suppose that the production function is linearly homogeneous, the results will be more promising. The reason why the results are a bit disappointing in this case is the following. We do not obtain any information, when the data have the following properties:

1. the data are consistent with the assumption of producer demand;
2. output is increasing over time.

This result is stated in the theorem below.

Theorem 8.13: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set in which all functions $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$ are of the form

$$c_i(x) = v_i(x_1) + w_i(x_2) \quad (i \in I, x \in X_1 \times X_2),$$

where $w_i: X_2 \rightarrow \mathbb{R}_+$ is linearly homogeneous. Suppose $\mathcal{F}_d^0(D) \neq \emptyset$ and output strictly increasing, i.e. $y_j > y_i$ if $j > i$. Let $t_i > 0$, $i \in I$, be an increasing series of numbers. Then there exists a function $f: X_1 \times X_2 \rightarrow \mathbb{R}$ such that every function

$$f_i(x) = f(x_1, t_i x_2) \quad (i \in I, x \in X_1 \times X_2)$$

satisfies $(c_i, x^i, y_i) \in \mathcal{D}_d^0(f_i)$.

Proof: Let $f \in \mathcal{F}_d^0(D)$. Then by Theorem 8.1, we have

$$\begin{aligned} y_j > y_i &\Rightarrow c_i(x^j) > c_i(x^i) & (i, j \in I), \\ y_j \geq y_i &\Rightarrow c_i(x^j) \geq c_i(x^i) & (i, j \in I). \end{aligned}$$

Now, for an increasing series t_i , $i \in I$, we have

$$c_i(x_1^j, t_j t_i^{-1} x_2^j) \geq c_i(x_1^j, x_2^j) \quad (i, j \in I: j \geq i).$$

Hence, the assumptions imply

$$\begin{aligned} y_j > y_i &\Rightarrow c_i(x_1^j, t_i^{-1} t_j x_2^j) \geq c_i(x^j) > c_i(x^i) & (i, j \in I), \\ y_j \geq y_i &\Rightarrow c_i(x_1^j, t_i^{-1} t_j x_2^j) \geq c_i(x^j) \geq c_i(x^i) & (i, j \in I). \end{aligned}$$

Thus by Theorem 8.1 there exists a function f such that

$$(c_i^t, [x_1^i, t_i x_2^i], y_i) \in D_d^0(f) \quad (i \in I),$$

where

$$c_i^t(x_1, x_2) = c_i(x_1, t_i^{-1} x_2) \quad (x_1 \in X_1, x_2 \in X_2, i \in I),$$

which is by Theorem 8.10 equivalent to $(c_i, x^i, y_i) \in D_d^0(f_i)$ for all $i \in I$. $\square$

So we may conclude that data with increasing output, which are nearly consistent with producer demand, tend to hide information about neutral technical progress. It is a pity that empirical production data often have these properties.

8.7 Efficiency Transformations for Neutral Change

To finish this chapter I will consider the efficiency transformation that is mentioned in the previous section. It still remains to answer how we can compare the results for efficiency transformation $\hat{\Phi}_e$, given in the previous section, with results for the standard efficiency transformation Φ_e , given in Example 2.3. At the end of this section I will show that both efficiency transformations may generate the same results.

First let us look at some properties of both efficiency transformations. Write

$$\Phi_e(c, x') = (c^e, x'), \quad \hat{\Phi}_e(c, x') = (\hat{c}^e, x'),$$

and assume $c = v + w$ with w linearly homogeneous. By the definition of both efficiency transformations, we have $\hat{c}^e \leq c^e$. So we can derive immediately the following general result.

Theorem 8.14: Suppose D is a data set. Let $\hat{\Phi}_e$ and Φ_e be the efficiency transformations given in Example 8.1 and 2.3 respectively. Then:

- (a) $\mathcal{F}_d(\hat{\Phi}_e D) \subset \mathcal{F}_d(\Phi_e D)$.
- (b) $\hat{e}_d(F, D) \leq e_d(F, D)$, where $\hat{e}_d$ and e_d are the efficiency upper bounds for the efficiency transformations $\hat{\Phi}_e$ and Φ_e respectively, and F is an arbitrary given set of functions.

Furthermore, the above is also valid if output is observed and in case frontier production functions are considered.

Proof: (a) Use the definition of $\mathcal{F}_d$ and the inequality $\hat{c}^e \leq c^e$.

(b) Follows immediately from (a). $\square$

It may happen that both efficiency measures lead to the same results. The following theorem presents an example of such a case.

Theorem 8.15: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a data set. Let $\hat{\Phi}_e$ and Φ_e be the efficiency transformations given in Example 8.1 and 2.3 respectively. Suppose

$$x^j \neq (x_1^i, \lambda x_2^i) \quad (i, j \in I, \lambda \in \mathbb{R}: \lambda \neq 1).$$

Then:

(a) The revealed preference conditions for the data sets $\hat{\Phi}_e D$ and $\Phi_e D$ are equal to each other.

(b) Let F be the set of all functions $f: X \rightarrow \mathbb{R}$. Then $\hat{e}_d(F, D) = e_d(F, D)$, where $\hat{e}_d$ and e_d are the efficiency upper bounds for the efficiency transformations $\hat{\Phi}_e$ and Φ_e respectively.

Furthermore, the above is also valid if output is observed or frontier production functions are considered.

Proof: (a) Using the same notation as usual, we obtain for the transformed cost functions the equality $\hat{c}_i^e(x^j) = c_i^e(x^j)$ for all $i, j \in I$. Hence the revealed preference conditions, which depend only on these values, are for both data sets equal to each other.

(b) There exists a rationalizing function if and only if the revealed preference conditions for the data are satisfied. So, (a) implies that both $\mathcal{F}_d(\hat{\Phi}_e D)$ and $\mathcal{F}_d(\Phi_e D)$ are together empty or nonempty. Hence, the result follows from the definition of e_d as the supremum over efficiency levels e for which a rationalizing function exists.

Now, let us consider the necessary and sufficient conditions for the existence of a production function in case output is observed or frontier production functions are considered. In that case only the revealed preference conditions and the observed outputs are of importance. Thus the necessary and sufficient conditions are the same for both efficiency transformations. $\square$

For empirical data the condition in the above theorem are usually satisfied. So, it is in practice not important, which one of the efficiency measures is chosen.

9 LINEARLY HOMOGENEOUS PRODUCER DEMAND

9.1 Introduction

In this chapter we shall consider producer demand when both the production function and cost functions are linearly homogeneous. In this case there is no need to restrict the theory to finite data sets. The following approach may simplify the search for conditions considering linearly homogeneous producer demand. This approach relies on the invariance of the hypothesis of linearly homogeneous producer demand under the scaling of observations, as is described in Theorem 3.1. Assuming output positive, one may replace the observations $\{(c_i, x_i^1, y_i)\}_{i \in I}$ by the observations $\{(c_i, y_i^{-1} x_i^1, 1)\}_{i \in I}$, scaled on the unit isoquant, which contain the same information as the original data. The advantage of using data that are scaled to an isoquant, is that we may weaken our hypothesis. As will be shown below, such data are not able to distinguish between the hypothesis of rational linearly homogeneous demand and the hypothesis of rational demand.

So, if we wish to test the hypothesis of rational linearly homogeneous demand, we may as well scale the data to an isoquant and test the resulting data without the condition that the production function is linearly homogeneous. Why is this the case? If there exists a rationalizing function for the scaled data, then – under very weak conditions – we may construct a linearly homogeneous function that shares the observed isoquant and which is compatible with the nonscaled data. The construction of such a function is as follows.

Suppose that there is a production function $f: X \rightarrow \mathbb{R}$, that passes the producer demand test for the scaled data. In that case we may use the surface

$$S = \{x \in X \mid f(x) = 1\}$$

to construct a linearly homogeneous function $g: X \rightarrow \mathbb{R}$, which passes also the producer demand test. This construction is obviously the following

$$g(\lambda s) = \lambda \quad (s \in S, \lambda \geq 0).$$

Under weak conditions both functions f and g will display the same economical behaviour on the shared unit-isoquant. So, for the scaled data, the existence of a rational production function f in general, is in such a case equivalent to the existence of a linearly homogeneous rational production function g .

Remark: There are a few small theoretical problems in connection with the above approach, which are related to the mentioned weak conditions that have to be satisfied. They only are treated to complete the story. These minor problems, which we shall not encounter later on, are the following. First, we have to prove that both f and g have the same economical behaviour on the shared unit-isoquant S . This is the case when the weak condition

$$f(x) < 1 \Leftrightarrow g(x) < 1 \quad (x \in X)$$

is satisfied. Secondly, to obtain a well-defined linearly homogeneous function g , every ray from the origin through S ,

$$\{\lambda s \mid \lambda \geq 0\} \quad (s \in S),$$

has to cut the surface S only once. Third, when these rays through S do not cover X , i.e.

$$X \neq \{\lambda s \mid \lambda \geq 0, s \in S\},$$

we have to extend the definition of g on X , for example, by defining g equal to zero for the remaining undefined values.

What can we conclude from the above in relation to the derivation of a theory concerning linearly homogeneous producer behaviour? First, scale the data to the unit-isoquant. Then apply the theory that is derived for the case where there is no restriction on the production function. This yields necessary conditions for the existence of a linearly homogeneous production function for given producer demand data. As argued above, such a necessary condition will often turn out to be a sufficient condition for linearly homogeneous producer demand.

To prove the existence of a linearly homogeneous production function, which may generate the given data, we can often derive such a function as the least upper bound over all possible linearly homogeneous production functions for the data set. Later on in this chapter, I will use such an approach to derive results. First, I derive conditions for linearly homogeneous demand using the theory given in Chapter 8.

9.2 Conditions for Linearly Homogeneous Demand

In this section we scale the data $\{(c_i, x^i, y_i)\}_{i \in I}$ to the unit-isoquant in order to derive necessary and sufficient conditions for the hypothesis of linearly homogeneous demand. We shall apply to these data the theory given in Chapter 8, concerning the case where there is no restriction on the production function. By Theorem 8.1, the scaled data $\{(c_i, y_i^{-1} x^i, 1)\}_{i \in I}$ satisfy the hypothesis of producer demand, if one has

$$c_i(y_j^{-1} x^j) \leq c_i(y_i^{-1} x^i) \Rightarrow 1 \leq 1 \quad (i, j \in I),$$

$$c_i(y_j^{-1} x^j) < c_i(y_i^{-1} x^i) \Rightarrow 1 < 1 \quad (i, j \in I).$$

The first condition has no meaning, because it is always true. The second condition can be rewritten as

$$c_i(y_j^{-1} x^j) \geq c_i(y_i^{-1} x^i) \quad (i, j \in I).$$

This condition is indeed a necessary and sufficient condition for the hypothesis of linearly homogeneous producer demand as is shown in the proof of the following theorem. A comparable theorem is stated in Hanoch and Rothschildt (1972) and Varian (1984).

Theorem 9.1: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous, $c_i(x^i) > 0$ and $y_i > 0$, for all $i \in I$, and X is a cone. Put

$$F = \{f \in \mathcal{F}_d^0(D) \mid f \text{ linearly homogeneous}\}.$$

Then the following conditions are equivalent:

- (i) $F \neq \emptyset$.
- (ii) $y_j \leq y_i c_i(x^j) / c_i(x^i) \quad (i, j \in I)$.

Proof: (i) $\Rightarrow$ (ii): Suppose $F \neq \emptyset$. Then we obtain, as is described above this theorem, the inequalities

$$c_i(y_j^{-1} x^j) \geq c_i(y_i^{-1} x^i) \quad (i, j \in I).$$

Hence, since all cost functions c_i are linearly homogeneous, we have

$$y_j \leq y_i c_i(x^j) / c_i(x^i) \quad (i, j \in I).$$

(ii) $\Rightarrow$ (i): Put

$$f(x) = \inf_{i \in I} y_i c_i(x) / c_i(x^i) \quad (x \in X).$$

Then (ii) implies that we have $f(x^i) = y_i$ for all $i \in I$. This means $f \in F$, because f is clearly linearly homogeneous as the infimum over linearly homogeneous functions. $\square$

I tested whether the Dutch industry data satisfied condition (ii) of the above theorem. It turned out that 13 observations have to be deleted. So, a subset of only two observations is consistent with the hypothesis of linearly homogeneous producer demand.

Using the above theorem, we can determine the economical efficiency upper bound for the efficiency transformation Λ_e given in Example 3.1. Application of Theorem 9.1 to the Dutch industry data, and using bisection, resulted in the efficiency level upper bound $e = 89.6\%$. The result is not very high, as we could have expected from the results in the previous section. This low efficiency level may be due to the fact that there is technical progress. As we shall see later on in this chapter, the allowance of technical progress results in a less restrictive test with higher efficiency levels.

9.3 Upper Bounds and Producer Demand

In the previous section we have derived a necessary and sufficient condition for the existence of a linearly homogeneous production function. To prove the existence of such a function, I used the following function

$$f(x) = \inf_{i \in I} y_i c_i(x) / c_i(x^i) \quad (x \in X).$$

In this section I will show that the choice of this function was not an arbitrary choice. The chosen function is the least upper bound of all rationalizing linearly homogeneous functions. That this bound is used to prove the existence of a rationalizing linearly homogeneous function is not a coincidence. Theorem 3.13 states that a least upper bound over rationalizing linearly homogeneous functions is a rationalizing linearly homogeneous function itself.

Let us recall the upper bound we have used for linearly homogeneous consumer behaviour. For a collection F of functions $f: X \rightarrow \mathbb{R}$, the *relative upper bound* U_f is equal to

$$U_f(F, x, x') \equiv \sup_{f \in F} f(x) / f(x') \quad (x, x' \in X).$$

In this chapter the following definition will be useful.

Definition: The *upper bound* U is equal to

$$U(F, x) \equiv \sup_{f \in F} f(x) \quad (x \in X).$$

First I consider the upper bound for the simple case of one data element

Theorem 9.2: Suppose $c: X \rightarrow \mathbb{R}_+$ is linearly homogeneous, $y > 0$, $x' \in X$, $c(x') > 0$, and X is a cone. Put

$$F = \{f: X \rightarrow \mathbb{R} \mid f \in \mathcal{F}_d(\{(c, x', y)\}) \text{ is linearly homogeneous}\}.$$

Then $F \neq \emptyset$ and:

- (a) $U(F, \cdot) \in F$.
- (b) One has

$$U_f(F, x, x') = c(x) / c(x') = y^{-1} U(F, x) \quad (x \in X).$$

Proof: (a) Put

$$f(x) = yc(x) / c(x') \quad (x \in X).$$

Then clearly $f \in F$ and thus we have only to prove (b).

(b): For $f \in F$, as defined above in the proof of (a), we have

$$c(x)/c(x') = f(x)/f(x') \leq U_f(F, x, x') \quad (x \in X).$$

Moreover, by Theorem 3.10 one has

$$f(x)/f(x') \leq c(x)/c(x') \quad (x \in X, f \in F).$$

Hence $U_f(F, x, x') \leq c(x)/c(x')$, which implies the desired equality

$$U_f(F, x, x') = c(x)/c(x') \quad (x \in X).$$

Moreover, by scaling the data we obtain $f(y^{-1}x') = y^{-1}f(x') = 1$ for any $f \in F$, so that we have also

$$U(F, x) = U_f(F, x, y^{-1}x') = c(x)/c(y^{-1}x') = yc(x)/c(x') \quad (x \in X). \quad \square$$

The following theorem shows that the least upper bound of the family of linearly homogeneous production functions for a given data set is a member of this family. The theorem extends the results of the above theorem, which was concerned with only one data element, and the upper bound is now constructed for a general data set.

Theorem 9.3: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous, $c_i(x^i) > 0$ and $y_i > 0$, for all $i \in I$, and X is a cone. Put

$$F = \{f \in \mathcal{F}_d^0(D) \mid f \text{ linearly homogeneous}\},$$

If $F \neq \emptyset$ then $U(F, \cdot) \in F$ and

$$U(F, x) = \inf_{i \in I} y_i c_i(x) / c_i(x^i) \quad (x \in X).$$

Proof: Put

$$f(x) = \inf_{i \in I} y_i c_i(x) / c_i(x^i) \quad (x \in X).$$

From the proof of Theorem 9.1, we obtain $f \in F$. Hence, we have $f(x) \leq U(F, x)$, and it remains to prove the reverse inequality.

Theorem 9.2 states that we have for

$$F_i = \{g \in \mathcal{F}_d^0(\{(c_i, x^i, y_i)\}) \mid g \text{ linearly homogeneous}\} \quad (i \in I),$$

the equality

$$U(F_i, x) = y_i c_i(x) / c_i(x^i) \quad (i \in I, x \in X).$$

Because clearly $F \subset F_i$, we obtain thus

$$U(F, x) \leq \inf_{i \in I} U(F_i, x) = \inf_{i \in I} y_i c_i(x) / c_i(x^i) = f(x) \quad (x \in X). \quad \square$$

9.4 Frontier Production Functions

The solution I will present for linearly homogeneous frontier production functions, has a resemblance to the solution for the case where there was no restriction on the production function. This case was treated in Section 8.3, where the infimum over rationalizing frontier production functions is determined. This infimum was not always attainable. In this section we shall find a minimal rationalizing frontier production function that attains the infimum. The following theorem states that such a production function always exists, assuming that there is at least one rationalizing production function for the data without observed outputs.

Theorem 9.4: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous, $c_i(x^i) > 0$ and $y_i > 0$, for all $i \in I$, and X is a cone. Suppose $F \neq \emptyset$ and put

$$F = \{f \in \mathcal{F}_d(D) \mid f \text{ linearly homogeneous}\},$$

$$F^f = \{f \in \mathcal{F}_d^f(D) \mid f \text{ linearly homogeneous}\},$$

$$l_i = \max_{k \in I} y_k L_j(F, x^i, x^k) \quad (i \in I),$$

$$l(x) = \min_{i \in I} l_i c_i(x) / c_i(x^i) \quad (x \in X).$$

Then $l \in F^f$ and $L(F^f, x^i) = l(x^i) = l_i$ for all $i \in I$.

Proof: Theorem 3.4 and $c_i(x^i) > 0$ imply that we have $f(x^i) > 0$ if $f \in F$. By definition of F^f and F we have

$$f(x^k) \geq y_k \quad (f \in F^f, k \in I),$$

$$f(x^i) / f(x^k) \geq L_j(F, x^i, x^k) \quad (i, k \in I, f \in F).$$

Using $F^f \subset F$ we obtain thus

$$f(x^i) \geq y_k f(x^i) / f(x^k) \geq y_k L_j(F, x^i, x^k) \quad (f \in F^f, i, k \in I).$$

So, the definition of L implies $L(F^f, x^i) \geq l_i$ for all $i \in I$. To complete the proof, we have to show that $l \in F^f$ and $l(x^i) = l_i$ holds.

First, I will show that we have $f(x^i) = l_i$. This means that we may apply Theorem 3.9 and Theorem 3.10 to obtain

$$L_j(F, x^i, x^k) \leq L_j(F, x^j, x^k) U_j(F, x^i, x^j) \quad (i, j, k \in I).$$

Moreover, using Theorem 3.12 we obtain

$$U_j(F, x^i, x^j) \leq c_j(x^i) / c_j(x^j) \quad (i, j \in I).$$

Hence

$$\begin{aligned} l_i &= \max_{k \in I} y_k L_j(F, x^i, x^k) \leq \max_{k \in I} y_k L_j(F, x^j, x^k) c_j(x^i) / c_j(x^j) \\ &= l_j c_j(x^i) / c_j(x^j) \quad (i, j \in I), \end{aligned}$$

from which we obtain $f(x^i) = l_i$ for all $i \in I$.

Further, we have $f \in F^f$, because clearly $f \in F$ and

$$f(x^i) = l_i \geq y_i L_j(F, x^i, x^i) = y_i \quad (i \in I). \quad \square$$

Remark: It is obvious that there is a rationalizing linearly homogeneous frontier production function $f \in F^f$ that attains only one lower bound, i.e. $f(x^i) = L(F^f, x^i)$ for a certain $i \in I$. It is not obvious, however, that there exists a function that attains all the bounds $L(F^f, x^i)$ at once, as is the case for l in Theorem 9.4. As is remarked in Section 8.3, there exists such a function, because $\mathcal{F}^f$ is closed with respect to the pointwise minimum operation.

I applied Theorem 9.4 to the Dutch industry data and obtained nice results. Before application of the theorem I determined the economical efficiency upper bound for the data without produced output, using the efficiency transformation A_e given in Example 3.1. The efficiency upper bound for the assumption of linearly homogeneous producer demand was $e = 99.9\%$, which is much higher than the value of 89.6%, which I obtained when output was included. There is a simple explanation for this. When output is included, one has more information available. This implies that one has in general $\mathcal{F}_d^0(D) \subset \mathcal{F}_d(D)$. Hence, the efficiency upper bound for any hypothesis is lower when output is observed.

Now, Theorem 9.4 was applied to the transformed data $A_e D$ for the obtained economical efficiency bound $e = 99.9\%$. The computed minimal linearly homogeneous frontier production function values $f(x^i)$ are given in

Figure 9.1. Below these values the actual produced output y_i is shown.

Figure 9.1 Minimal Frontier Production Function

Figure 9.2 Technical Efficiency Upper Bound

From Figure 9.1 we can derive the technical efficiency upper bounds

$$t_i = y_i / L(F^f, x^i) \quad (i \in I)$$

as the ratio of the production output y_i and the minimal frontier production output $L(F^f, x^i)$. These bounds are given in Figure 9.2. If they are not random, but steadily increasing, this may be an indication that there is technical progress. Thus the above theorem is an excellent theorem to test

whether there is technical progress, using the assumption of linearly homogeneous producer demand. Figure 9.2 contains a logarithmic regression line fit for the values of t_i , which is clearly increasing.

Note that there is always at least one observation period i , for which the output y_i equals the minimal frontier function, i.e. $y_i = L(F^f, x^i)$. At this point one has $t_i = 1$ and the minimal frontier production function touches the restrictions that follow from the observations. In case of increasing technical efficiency, this means that one has $t_i = 1$ for the last observation, as is the case in Figure 9.2.

By Theorem 9.4, the technical efficiency upper bounds $t_i = y_i/L(F^f, x^i)$ may be interpreted as Hicks neutral change. From the theorem we obtain

$$(c_i, x^i, y_i) \in D_d^0(t_i f) \quad (i \in I),$$

for the function

$$f(x) = \min_{i \in I} L(F^f, x^i) c_i(x) / c_i(x^i) \quad (x \in X).$$

If we assume that in each observation period $i = 0, \dots, n$, there is a constant fraction c of Hicks neutral technical progress c , then we have

$$t_i = t_0(1+c)^i \quad (i = 0, \dots, n).$$

This fraction c may be determined with logarithmic regression, using the relation

$$\ln t_i = \ln t_0 + i \ln(1+c) = \alpha + i\beta \quad (i = 0, \dots, n),$$

for the observed technical efficiency upper bounds t_i . The estimation result for the Dutch industry data was $c = 1.0\%$ ($R^2 = 0.89$), an increase in technical efficiency of approximately one percent per year.

The subject of Hicks neutral progress in the context of linearly homogeneous producer demand will be elaborated later on in this chapter. First I will consider the properties of technical progress.

9.5 Technical Progress

Concerning the assumption of linear homogeneity, the most general assumption of technical progress is a new linearly homogeneous production function for each period, with an increase of output. A necessary and sufficient condition for this simple type of technical progress is given in the following theorem.

Theorem 9.5: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous, $c_i(x^i) > 0$ and $y_i > 0$, for all $i \in I$, and X is a cone. Then the following conditions are equivalent:

(i) There exists a series of linearly homogeneous production functions f_i , $i \in I$, such that $f_j \leq f_i$ if $j \leq i$, and $(c_i, x^i, y_i) \in D_d^0(f_i)$ for all $i \in I$.

(ii) One has

$$y_j \leq y_i c_i(x^j) / c_i(x^i) \quad (i, j \in I: j \leq i).$$

Proof (i) $\Rightarrow$ (ii): From (i) and Theorem 9.2, we obtain

$$y_j = f_j(x^j) \leq f_i(x^j) \leq y_i c_i(x^j) / c_i(x^i) \quad (i, j \in I: j \leq i).$$

(ii) $\Rightarrow$ (i): Put

$$f_j(x) = \inf_{i \geq j} y_i c_i(x) / c_i(x^i) \quad (j \in I, x \in X).$$

For these functions one can easily verify that one has $f_j \leq f_i$ if $j \leq i$, and the condition in (ii) implies that we have $f_j(x^j) = y_j$ for all $j \in I$. Further, we have clearly $(c_i, x^i, y_i) \in D_d^0(f_i)$ for all $i \in I$. $\square$

Condition (ii) of Theorem 9.5 is used to test the hypothesis of technically progressing linearly homogeneous producer demand for the Dutch industry data. The result was that one has to remove two periods from the data set in order to get consistency with this hypothesis. Furthermore, we can use Theorem 9.5 to compute an economical efficiency upper bound for the hypothesis technically progressing linearly homogeneous producer demand. The result is $e = 97.5\%$, where the efficiency transformation A_e of Example 3.1 is used.

Now, suppose we have data $D = \{(c_i, x^i)\}_{i \in I}$ available which do not include the observed outputs y_i . Then we may ask whether the hypothesis of technically progressing linearly homogeneous producer demand can be falsified. To answer this question we may search for output values y_i that obey condition (ii) in Theorem 9.5 above. If such values do exist then there is consistency with the hypothesis of technically progressing linearly homogeneous demand. Using this idea the following theorem can be derived. It shows that we cannot falsify the hypothesis of technically progressing linearly homogeneous demand when the data do not contain observed outputs.

Theorem 9.6: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a countable data set, where $I = \{1, 2, \dots\}$, $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and $c_i(x^j) > 0$ if $j \leq i$, for all $i, j \in I$, and X is a cone. Then there exists a series of linearly homogeneous production functions f_i , $i \in I$, such that $f_j \leq f_i$ if $j \leq i$, and $(c_i, x^i) \in D_d(f_i)$ for all $i \in I$.

Proof: By Theorem 9.5 there exist technically progressing linearly homogeneous production functions with outputs y_i for the data, if condition (ii) in this theorem is satisfied. To construct such numbers y_i , let $y_1 > 0$ be an arbitrary number, and put iteratively

$$y_i = \max \{y_j c_i(x^j) / c_i(x^j) \mid i, j \in I: j < i\} \quad (i = 2, \dots).$$

These numbers satisfy obviously condition (ii) in Theorem 9.5, which is

$$y_j \leq y_i c_i(x^j) / c_i(x^j) \quad (i, j \in I: j \leq i). \quad \square$$

Thus by the above proof we can always find output values for the data, such that there is consistency with the hypothesis of technically progressing linearly homogeneous demand. So, it is impossible to falsify the hypothesis. Hence, the specification of linearly homogeneous cost minimization with technical progress does not impose any restriction on the data.

As we have seen above, it may happen in the nonparametric approach that a hypothesis is always satisfied by the data. For example, this was the case in Theorem 8.4. Furthermore, Varian (1988) derived such a result for consumer behaviour. He shows that the axiom of revealed preference places essentially no restriction on behaviour of a subset of goods.

9.6 Technical Progress for Frontier Production Functions

When the output data are observed, but we use the notion of a frontier production function, then the approach in the proof of Theorem 9.6 yields values of the technical efficiency upper bounds. This is pointed out in the following theorem.

Theorem 9.7: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $I = \{1, 2, \dots\}$, $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous, $c_i(x^j) > 0$ if $j \leq i$, and $y_i > 0$, for all $i \in I$, and X is a cone. Let each family F_i contain all technically progressing linearly homogeneous frontier production functions for producer demand at period $i \in I$. Thus if there is a series functions f_i , $i \in I$, such that $f_j \leq f_i$ if $j \leq i$, and $(c_i, x^i, y_i) \in D_d^f(f_i)$ for all $i \in I$, then $f_i \in F_i$ for every $i \in I$. Now, put $u_1 = y_1$ and iteratively

$$u_i = \max \{u_j c_i(x^i)/c_i(x^j) \mid j \in I: j < i\} \cup \{y_i\} \quad (i = 2, \dots).$$

Then $L(F_i^f, x^i) = u_i$ for all $i \in I$.

Proof: The definition of frontier producer demand D_d^f and Theorem 9.5, imply together the following. In order to find $u_i = L(F_i^f, x^i)$, we have to search for minimal numbers u_i such that

$$\begin{aligned} u_i &\geq y_i \quad (i \in I), \\ u_i &\geq u_j c_i(x^i)/c_i(x^j) \quad (i, j \in I: j \leq i). \end{aligned}$$

The numbers u_i given in this theorem, are the solution to this problem. $\square$

So, the technical upper bounds for linearly homogeneous frontier producer demand with technical progress are $t_i = u_i/y_i$, $i \in I$, in which the values u_i are given in the above theorem. From this theorem it follows that the first technical efficiency upper bound t_1 is always equal to one.

I applied Theorem 9.7 to obtain the technical upper bounds for the Dutch industry data. The result contained only two inefficient years with $t_i < 1$ for the hypothesis of technically progressing linearly homogeneous producer demand: 1975 and 1979 with technical efficiency upper bounds of 97.6% and 99.5% respectively.

9.7 Hicks Neutral Change

A test of the hypothesis of Hicks neutral change can be considered as a case where output may be chosen freely. This is in essence the content of the following theorem.

Theorem 9.8: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous, $c_i(x^i) > 0$ and $y_i > 0$, for all $i \in I$, and X is a cone. Then the following conditions are equivalent:

- (i) There exist a linearly homogeneous function $f: X \rightarrow \mathbb{R}$ and numbers $t_i > 0$, such that $(c_i, x^i, y_i) \in D_d^0(t_i f)$ for all $i \in I$.
- (ii) There exists a linearly homogeneous function in $\mathcal{F}_d(D)$, where the observed outputs are not used.

Proof (i) $\Rightarrow$ (ii): Suppose the function f and the numbers t_i satisfy condition (i). By Theorem 8.6 one has

$$(c_i, x^i, y_i) \in D_d^0(t_i f) \Leftrightarrow (c_i, x^i, t_i^{-1} y_i) \in D_d^0(f) \quad (i \in I),$$

and we have obviously

$$(c_i, x^i, t_i^{-1} y_i) \in D_d^0(f) \Rightarrow (c_i, x^i) \in D_d(f) \quad (i \in I).$$

So

$$(c_i, x^i, y_i) \in D_d^0(t_i f) \Rightarrow (c_i, x^i) \in D_d(f) \quad (i \in I),$$

and we have thus $f \in \mathcal{F}_d(D)$.

(ii) $\Rightarrow$ (i): Suppose $f \in \mathcal{F}_d(D)$ is linearly homogeneous. Then Theorem 3.4 and $c_i(x^i) > 0$ implies $f(x^i) > 0$. Now, put $t_i = y_i/f(x^i)$ for all $i \in I$. Then one has in general

$$(c_i, x^i) \in D_d(f) \Leftrightarrow (c_i, x^i, f(x^i)) \in D_d^0(f) \quad (i \in I).$$

Furthermore, since f is linearly homogeneous, the latter is equivalent to

$$(c_i, x^i, t_i f(x^i)) \in D_d^0(t_i f) \quad (i \in I).$$

Finally, we have by assumption

$$t_i f(x^i) = (y_i/f(x^i))f(x^i) = y_i \quad (i \in I). \quad \square$$

We can obtain the nonparametric restrictions on the scale of Hicks neutral

change from the following theorem.

Theorem 9.9: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ be a finite data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous, $c_i(x^i) > 0$ and $y_i > 0$, for all $i \in I$, and X is a cone. Let $t_i > 0$ for all $i \in I$. Then the following conditions are equivalent:

(i) There exists a linearly homogeneous function $f: X \rightarrow \mathbb{R}$ such that

$$(c_i, x^i, y_i) \in D_d^0(t_i f) \quad (i \in I).$$

(ii) One has $c_i(x^j) > 0$ for all $i, j \in I$, and the values $\phi_i = \ln t_i$ satisfy

$$\phi_i - \phi_j \leq a_{ij}, \quad a_{ij} = \ln(y_j c_i(x^j) / [y_i c_i(x^i)]) \quad (i, j \in I).$$

Proof: One has by Theorem 8.6

$$(c_i, x^i, y_i) \in D_d^0(t_i f) \Leftrightarrow (c_i, x^i, t_i^{-1} y_i) \in D_d^0(f) \quad (i \in I).$$

So, by Theorem 9.1, we have (i) equivalent to

$$t_j^{-1} y_j \leq t_i^{-1} y_i c_i(x^j) / c_i(x^i) \quad (i, j \in I).$$

This is equivalent to

$$t_i / t_j \leq y_i c_i(x^j) / [y_j c_i(x^i)] \quad (i, j \in I),$$

from which we obtain the inequalities in (ii), by taking the logarithm at both sides. Furthermore, because all t_i and y_i are positive, we obtain from (i) also $c_i(x^j) > 0$ for all $i, j \in I$. $\square$

Figure 9.3 Bounds Hicks Neutral Change

Figure 9.3 displays lower and upper bounds for the scale of Hicks neutral change. The bounds follow from condition (ii) in Theorem 9.9 and they are obtained from the transformed Dutch industry data $\Lambda_e D$, where $e = 99.9\%$. The latter is the efficiency upper bound e for the hypothesis of linearly homogeneous Hicks neutral changing producer demand. As is stated in Theorem 9.8, this efficiency upper bound can be obtained by testing the hypothesis of linearly homogeneous producer demand without using the observed output.

Theorem 9.10: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous, $c_i(x^i) > 0$ and $y_i > 0$, for all $i \in I$, and X is a cone. Let $t_i > 0$ for all $i \in I$. Then the following conditions are equivalent:

(i) One has $t_i \leq t_j$ if $i \leq j$, and there exists a linearly homogeneous function $f: X \rightarrow \mathbb{R}$ such that $(c_i, x^i, y_i) \in D_d^0(t_i f)$ for all $i \in I$.

(ii) One has $c_i(x^j) > 0$ for all $i, j \in I$, and the values $\phi_i = \ln t_i$ satisfy

$$\phi_i - \phi_j \leq \min \{a_{ij}, b_{ij}\} \quad (i, j \in I),$$

where

$$a_{ij} = \ln (y_j c_i(x^i) / [y_i c_i(x^j)]) \quad (i, j \in I),$$

$$b_{ij} = \begin{cases} 0 & (i, j \in I: i \leq j), \\ \infty & (\text{otherwise}). \end{cases}$$

Proof: This is similar to the proof of Theorem 9.9 with the additional condition that one has $t_i \leq t_j$ if $i \leq j$, which is equivalent to

$$t_i/t_j \leq 1 \quad (i, j \in I: i \leq j).$$

From this we obtain the additional inequalities for b_{ij} in (ii), by taking the logarithm at both sides. $\square$

Figure 9.4 Bounds Hicks Neutral Progress

We can use condition (ii) of the above theorem to test the hypothesis of Hicks neutral progress. Concerning the Dutch industry data the result was as follows. Four observation years have to be removed to obtain consistency with the hypothesis of Hicks neutral progressing linearly homogeneous demand. The economical efficiency upper bound for this hypothesis was $e = 97.5\%$, where again the efficiency transformation Λ_e from Example 3.1 is used.

Using condition (ii) in Theorem 9.10, we may obtain lower and upper bounds for the scale of Hicks neutral progress. Figure 9.4 displays these lower and upper bounds for the transformed Dutch industry data $\Lambda_e D$, where e is equal to the efficiency upper bound $e = 97.5\%$.

9.8 Neutral Change

The assumption of neutral change has in case of linearly homogeneous producer demand a relatively simple solution when output is observed. The solution becomes more difficult when output is not observed. Then both the scale of progress and the output may be chosen freely, which doubles the number of free variables. In the following theorem it is assumed that output is observed.

Theorem 9.11: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$ are linearly homogeneous, $c_i(x^i) > 0$ and $y_i > 0$, for all $i \in I$, and X_1 and X_2 are cones. Let $t_i > 0$ for all $i \in I$. Then the following conditions are equivalent:

(i) There exists a linearly homogeneous function $f: X_1 \times X_2 \rightarrow \mathbb{R}$ such that all functions

$$f_i(x) = f(x_1, t_i x_2) \quad (i \in I, x \in X_1 \times X_2)$$

satisfy $(c_i, x^i, y_i) \in \mathcal{D}_d^0(f_i)$.

(ii) One has

$$y_j \leq y_i c_i(x_1^j, t_i^{-1} t_j x_2^j) / c_i(x^i) \quad (i, j \in I).$$

Proof: By Theorem 8.10 we have (i) equivalent to

$$(c_i^t, [x_1^i, t_i x_2^i], y_i) \in \mathcal{D}_d^0(f) \quad (i \in I),$$

where

$$c_i^t(x_1, x_2) = c_i(x_1, t_i^{-1} x_2) \quad (i \in I, x_1 \in X_1, x_2 \in X_2).$$

By Theorem 9.1 this is equivalent to

$$y_j \leq y_i c_i^t(x_1^j, t_j x_2^j) / c_i^t(x_1^i, t_i x_2^i) \quad (i, j \in I).$$

This, finally, is equivalent to (ii). □

If we consider price and quantity data then there is a simple test for neutral change. This test is given in the following theorem, which follows directly from the above theorem.

Theorem 9.12: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$ are linearly homogeneous, $c_i(x^i) > 0$ and $y_i > 0$, for all $i \in I$, and X_1 and X_2 are cones. Suppose all functions c_i are of the form

$$c_i(x) = v_i(x_1) + w_i(x_2) \quad (i \in I, x \in X_1 \times X_2),$$

where $v_i: X_1 \rightarrow \mathbb{R}_+$ and $w_i: X_2 \rightarrow \mathbb{R}_+$ are linearly homogeneous. Let $t_i > 0$ for all $i \in I$. Then the following conditions are equivalent:

(i) There exists a linearly homogeneous function $f: X_1 \times X_2 \rightarrow \mathbb{R}$, such that all functions

$$f_i(x) = f(x_1, t_i x_2) \quad (i \in I, x \in X_1 \times X_2)$$

satisfy $(c_i, x^i, y_i) \in \mathcal{D}_d^0(f_i)$.

(ii) The values $\phi_i = \ln t_i$ satisfy

$$\phi_i - \phi_j \leq a_{ij} \quad (i, j \in I),$$

where

$$a_{ij} = \begin{cases} \ln(w_i(x_2^j) / [c_i(x^i) y_j / y_i - v_i(x_1^i)]) & (i, j \in I: c_i(x^i) y_j / y_i > v_i(x_1^i)), \\ \infty & (\text{otherwise}). \end{cases}$$

Proof: Condition (i) is by Theorem 9.11 equivalent to

$$y_j \leq y_i c_i(x_1^j, t_i^{-1} t_j x_2^j) / c_i(x^i) \quad (i, j \in I).$$

Hence, condition (i) is equivalent to

$$y_j \leq y_i [v_i(x_1^j) + t_i^{-1} t_j w_i(x_2^j)] / c_i(x^i) \quad (i, j \in I),$$

and these inequalities are in turn equivalent to

$$\ln t_i - \ln t_j \leq a_{ij} \quad (i, j \in I). \quad \square$$

Condition (ii) of Theorem 9.12 is used to test neutral change of labour for the Dutch industry data. The result was that four observations have to be removed in order to get consistency with the hypothesis of neutral changing linearly homogeneous producer demand.

We may also use Theorem 9.12 to derive an economical efficiency upper bound for the hypothesis of neutral changing linearly homogeneous producer demand. There is, however, a small problem with the choice of an efficiency transformation. In Theorem 9.12 it is assumed that we may write the cost functions in the form

$$c_i(x) = v_i(x_1) + w_i(x_2) \quad (x \in X_1 \times X_2),$$

where v_i and w_i are linearly homogeneous functions. Thus to apply this theorem, we have to choose a suitable efficiency transformation, such that the resulting cost functions satisfy this condition. An example is the following efficiency transformation.

Example 9.1 Suppose (c, x') is a data observation such that

$$c(x) = v(x_1) + w(x_2) \quad (x \in X_1 \times X_2),$$

where v and w are linearly homogeneous functions. Define, using the efficiency transformation Λ_e given in Example 3.1, the functions v^e and w^e as follows

$$(v^e, x_1') = \Lambda_e(v, x_1'), \quad (w^e, x_2') = \Lambda_e(w, x_2').$$

Then we may define the efficiency transformation $\hat{\Lambda}_e$ as follows

$$\hat{\Lambda}_e(v + w, x') = (v^e + w^e, x').$$

Such an efficiency transformation transforms linearly homogeneous functions v and w into linearly homogeneous functions v^e and w^e respectively.

The above efficiency transformation $\hat{\Lambda}_e$ resulted for neutral change of labour in an efficiency upper bound of $e = 99.8\%$, and for neutral change of capital in $e = 99.9\%$.

Using the transformed data $\hat{\Lambda}_e D$ for the computed economical efficiency e , I computed nonparametric restrictions on the scale of neutral change for labour. It is assumed that this scale is equal to one in the first observation year. The restrictions follow from the weak closure A^* of $A = [a_{ij}]$, with A defined as in Theorem 9.12. As is shown in appendix C.5, the inequalities

$$\phi_i - \phi_j \leq a_{ij} \quad (i, j \in I)$$

in Theorem 9.12 imply the inequalities

$$-a_{ji}^* \leq \phi_i - \phi_j \leq a_{ji}^* \quad (i, j \in I).$$

Since we had $\phi_i = \ln t_i$, where t_i is the scale of neutral change, this yields the following restrictions on the scale of neutral change

$$e^{-a_{ji}^*} \leq t_i / t_j \leq e^{a_{ji}^*} \quad (i, j \in I).$$

Figure 9.5 displays these bounds for $j = 1$, the index of the first observation. The figure shows that the resulting bounds for neutral changing labour are very narrow.

Figure 9.5 Bounds Neutral Change Labour

Figure 9.6 Bounds Neutral Change Capital

Concerning neutral change of capital there was only one outlying observation, which has to be removed in order to get consistency with the hypothesis of neutral changing linearly homogeneous producer demand. The economical efficiency upper bound was $\epsilon = 99.9\%$.

I computed – in the same way as for labour – the nonparametric bounds for the scale of neutral change of capital. In this case the results were less informative. As shown in Figure 9.6, the upper bound increased very fast to values which exceed any informative restriction. What is the interpretation

of the occurrence of an infinite upper bound? An infinite upper bound means that we have

$$t_i/t_j < e^{a_{ij}^*} = \infty,$$

which is equivalent to $a_{ij}^* = \infty$. In the following I will consider properties that may be a cause for the occurrence of such infinite upper bounds in case $j=1$, the index of the first observation.

Suppose the data are consistent with the hypothesis of technically progressing linearly homogeneous producer demand. Then we have by Theorem 9.5 the relation

$$y_j \leq y_i c_i(x^j)/c_i(x^i) \quad (i, j \in I: j \leq i).$$

This means that the inequalities, given in Theorem 9.5, imply that we have

$$c_i(x^i)y_j/y_i - c_i(x^j) \leq 0 \quad (i, j \in I: j \leq i).$$

This makes it likely that we have

$$c_i(x^i)y_j/y_i - c_i(x^j) + w_i(x_2^j) = c_i(x^i)y_j/y_i - v_i(x_1^j) \leq 0 \quad (i, j \in I: j \leq i),$$

especially when the added cost $w_i(x_2^j)$ is relatively small. The latter applies to capital and not to labour, because capital and labour costs are approximately 3% and 25% of the total costs $c_i(x^i)$ respectively. Now, because we have the equivalence

$$c_i(x^i)y_j/y_i - v_i(x_1^j) \leq 0 \Leftrightarrow a_{ij} = \infty \quad (i, j \in I),$$

it is likely that we obtain $a_{ij} = \infty$ for $j < i$. In that case every path from j to i with $j < i$ always encounters an infinite value $a_{kl} = \infty$, with $k < l$. So, we have then $a_{ij} = \infty$ for $j < i$. This in turn means that we find infinite upper bounds

$$t_i/t_1 < e^{a_{i1}^*} = \infty \quad (i > 1).$$

Summarizing the above. When the data are approximately consistent with technically progressing linearly homogeneous producer demand and when the neutral changing costs $w_i(x_2^j)$ are relatively small, then we may expect the occurrence of infinite upper bounds for t_i/t_1 .

We observed this phenomenon for neutral changing capital in Figure 9.6. It does not appear in Figure 9.5, for neutral changing labour, because labour costs are substantially higher than capital costs.

We may apply the theory derived for neutral change, to the case of neutral progress. Then we have only to add the assumption that the scale is increasing. This leads to the following theorem.

Theorem 9.13: *Concerning Theorem 9.12, the following conditions are equivalent:*

- (i) *Theorem 9.12(i) is satisfied and one has $t_j \leq t_i$ if $j \leq i$.*
- (ii) *The values $\phi_i = \ln t_i$ satisfy*

$$\phi_i - \phi_j \leq \min \{a_{ij}, b_{ij}\} \quad (i, j \in I),$$

where a_{ij} is given in Theorem 9.12(ii) and

$$b_{ij} = \begin{cases} 0 & (i, j \in I: i \leq j), \\ \infty & (\text{otherwise}). \end{cases}$$

Proof: This is a version of Theorem 9.12 with the additional assumption that $t_j \leq t_i$ if $j \leq i$. □

I applied condition (ii) of the above theorem to determine the scale of neutral progress of labour and capital for the Dutch industry data. In order

to accept the hypothesis of neutral progress of labour and capital there are five and three observations respectively, that have to be removed. The economical efficiency upper bounds were both equal to 97.5%.

Figure 9.8 Bounds Neutral Progress Capital

Figure 9.7 Bounds Neutral Progress Labour

The nonparametric restrictions on the scale of neutral progress are computed in the same way as they were computed for neutral change. The results are displayed in Figure 9.7 and 9.8. It appears that they are similar to the earlier obtained results of neutral change.

10 HOMOTHETIC PRODUCER DEMAND

10.1 Introduction

In this chapter we consider the existence of homothetic production functions for producer data. This case weakens the assumption of linear homogeneity, treated in the previous chapter, because a homothetic function is a monotonic transformation of a linearly homogeneous function. In case of unobserved output one cannot distinguish between demand behaviour concerning linearly homogeneous production functions or demand behaviour concerning homothetic functions. So, since linearly homogeneous demand behaviour is already treated in Chapter 9, I will assume in the present chapter that output is observed. The two main questions that are answered are:

1. When does a homothetic production function exist for producer demand data?
2. What are the restrictions on the corresponding set of underlying linearly homogeneous functions?

10.2 Existence of a Homothetic Production Function

Homotheticity is a generalization of linear homogeneity. The definition that is given by Varian (1983, 1984), and used in this book, is the following.

Definition: A function h is said to be *homothetic* if it can be written as $h = m \circ f$, where $m: \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing function and f linearly homogeneous.

Remark: There is another common used definition of homotheticity. Shephard (1953, 1970) defines a function h to be homothetic if there exists an increasing function $v: \mathbb{R} \rightarrow \mathbb{R}$ such that $f = v \circ h$ is linearly homogeneous. A function of the form $h = m \circ f$ satisfies the homotheticity definition given by Shephard, if $m: \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing and f is linearly homogeneous. Then the inverse function m^{-1} is a well-defined increasing function. Furthermore, for $v = m^{-1}$ one has $v \circ h = f$ linearly homogeneous.

A homothetic function $h = m \circ f$ rationalizes producer demand, when the following conditions are satisfied:

1. the data without output is rationalized by the linearly homogeneous function f ;
2. the linearly homogeneous output values are mapped on the output observations by the monotonic transformation m .

This in combination with the idea behind Theorem 9.3 leads to the following existence theorem for a rationalizing homothetic production function.

Theorem 10.1: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set of producer behaviour, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and $c_i(x^i) > 0$, for all $i \in I$, and X is a cone. Then the following conditions are equivalent:

- (i) There exists a homothetic function $h \in \mathcal{F}_D^o(D)$.
- (ii) There exists a homothetic function $h \in \mathcal{F}_C^o(D)$.
- (iii) There exist numbers ϕ_i such that

$$\phi_i - \phi_j \leq \min \{a_{ij}, b_{ij}\}, \quad \phi_i - \phi_j < c_{ij} \quad (i, j \in I),$$

where

$$a_{ij} = \ln(c_j(x^i)/c_j(x^j)) \quad (i, j \in I),$$

$$b_{ij} = \begin{cases} 0 & (i, j \in I: y_i \leq y_j), \\ \infty & (\text{otherwise}), \end{cases}$$

$$c_{ij} = \begin{cases} 0 & (i, j \in I: y_i < y_j), \\ \infty & (\text{otherwise}). \end{cases}$$

Proof (i) $\Rightarrow$ (ii) Trivial.

(ii) $\Rightarrow$ (iii): Suppose $h = m \circ f \in \mathcal{F}_c^o(D)$ is a homothetic function, where m is strictly increasing and f is linearly homogeneous. Then $f(x^i) > 0$ for all $i \in I$ by Theorem 3.4, and by Theorem 3.11 we have

$$f(x^i)/f(x^j) \leq c_j(x^i)/c_j(x^j) \quad (i, j \in I).$$

Moreover, m strictly increasing means that we have

$$y_j = h(x^j) \geq h(x^i) = y_i \Leftrightarrow f(x^j) \geq f(x^i) \quad (i, j \in I).$$

So (iii) follows using $\phi_i = \ln f(x^i)$.

(iii) $\Rightarrow$ (i): Suppose that the numbers $\phi_i = \ln f_i$ satisfies (iii). Put

$$f(x) = \min_{i \in I} f_i c_i(x)/c_i(x^i),$$

and let $m: \mathbb{R} \rightarrow \mathbb{R}$ be a strictly increasing function such that $m(f_i) = y_i$ for all $i \in I$. Such a function m exists, because (iii) implies that we have

$$f_i \geq f_j \Leftrightarrow y_i \geq y_j \quad (i, j \in I).$$

Now, $m \circ f$ is clearly a homothetic function. Further, (iii) implies that we have

$$f(x^i) = \min_{j \in I} f_j c_j(x^i)/c_j(x^j) = f_i \quad (i \in I).$$

Hence, $f \in \mathcal{F}_d(D)$ and also $m \circ f \in \mathcal{F}_d^o(D)$. □

A theorem that states also necessary and sufficient conditions for homothetic producer demand is given by Varian (1984). His theorem is inspired by a theorem given in Hanoch and Rothschildt (1972). However, both these theorems contain a small error in the given conditions. Varian does not require

$$\phi_i - \phi_j \leq b_{ij} \quad (i, j \in I),$$

and Hanoch and Rothschildt do not require the condition

$$\phi_i - \phi_j < c_{ij} \quad (i, j \in I).$$

Varian proves that his condition is sufficient using the inverse of an increasing function m . However, such an inverse function is only well-defined if m is strictly increasing.

Theorem 10.1 is an improvement of both earlier given theorems. Besides the correction of this error, cost functions may be used instead of prices. Moreover, condition (iii) of Theorem 10.1 does not have to be tested with a time consuming linear program as in the case of the condition given in Hanoch and Rothschildt (1972), and later in Varian (1984). To test condition (iii) of Theorem 10.1 one may use a simple shortest path test as is shown in Theorem C.3, given in Appendix C.

We may extend Theorem 10.1 to the case where the data set D contains an infinite number of elements. In that case we only can prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). The proof (iii) $\Rightarrow$ (i) is not valid for infinite data sets: it does not result necessarily in a strictly increasing function m , but

we may derive an increasing function m .

We may use condition (iii) of Theorem 10.1 to test the hypothesis of homothetic producer demand. This resulted for the Dutch industry data in an economical efficiency upper bound of $e=99.5\%$, using the efficiency transformation given in Example 3.1. This result is equal to the result given in Chapter 8, where the hypothesis of producer demand without any restriction is considered. To accept the hypothesis of homothetic producer demand, we have to remove four observations, while it only were two observations in Chapter 8. Still it seems that the restriction of homotheticity is not a very strong restriction.

10.3 Restrictions on the Underlying Linearly Homogeneous Function

In this section I will construct the relative upper bound for the set of underlying linearly homogeneous functions. The theory is closely related to Chapter 3, which treats restrictions on linearly homogeneous consumer demand. A difficulty in this case is that the relative upper bounds do not necessarily yield a rationalizing function, as was the case for linearly homogeneous consumer demand with unobserved produced output. However, we can use the relative upper bounds for the construction of a rationalizing function, assuming the data set is finite.

Theorem 10.2: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set of producer behaviour, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and $c_i(x^i) > 0$, for all $i \in I$, and X is a cone. Put

$$F = \{f: X \rightarrow \mathbb{R} \mid f \text{ linearly homogeneous,} \\ \text{there is a monotonic function } m \text{ such that } m \circ f \in \mathcal{F}_d^0(D)\}.$$

If $F \neq \emptyset$ then all functions of the form

$$g(x) = \sum_{i \in I} \alpha_i U_i(F, x, x^i) \quad (x \in X),$$

where $\alpha_i > 0$ for all $i \in I$, are element of F .

Proof: Put

$$F' = \{f \in \mathcal{F}_d(D) \mid f \text{ linearly homogeneous}\},$$

where the observed outputs are not used. Now, by Theorem 3.12 one has $U_i(F, \cdot, x^i) \in F'$ for all $i \in I$, and by Theorem 5.3(c) we have in that case $g \in F'$. Furthermore, the definition of F and U_i implies that one has

$$U_i(F, x^k, x^i) \geq U_i(F, x^j, x^i) \quad (j, k \in I: y_k \geq y_j),$$

$$U_i(F, x^k, x^j) > 1 = U_i(F, x^j, x^j) \quad (j, k \in I: y_k > y_j),$$

which means

$$y_k \geq y_j \Rightarrow \forall i: U_i(F, x^k, x^i) \geq U_i(F, x^j, x^i) \Rightarrow g(x^k) \geq g(x^j),$$

$$y_k > y_j \Rightarrow \forall i: U_i(F, x^k, x^i) \geq U_i(F, x^j, x^i) \text{ and } U_i(F, x^k, x^j) > U_i(F, x^j, x^j) \\ \Rightarrow g(x^k) > g(x^j),$$

for arbitrary $k, j \in I$. So, we have

$$y_k \geq y_j \Leftrightarrow g(x^k) \geq g(x^j) \quad (k, j \in I).$$

Hence, there exists a monotonic transformation m such that $m \circ g(x^i) = y_i$ for all $i \in I$, and, since $g \in F'$, one has thus $g \in F$. $\square$

Note that we have not necessarily $U_i(F, \cdot, x^i) \in F$ in the above theorem, because the ordering of both series values $U_i(F, x^j, x^i)$ and y_j , for all $j \in I$, may be different. The reason is that the relation

$$y_k > y_j \Rightarrow U_f(F, x^k, x^i) > U_f(F, x^j, x^i)$$

is not necessarily valid. We can only prove that we have

$$y_k \geq y_j \Rightarrow U_f(F, x^k, x^i) \geq U_f(F, x^j, x^i).$$

Finally I give a theorem, which determines the relative upper bound of the underlying linearly homogeneous functions that rationalize homothetic producer demand. The theorem is closely related to Theorem 3.12.

Theorem 10.3: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set of producer behaviour, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and $c_i(x^i) > 0$, for all $i \in I$, and X is a cone. Put

$$F = \{f: X \rightarrow \mathbb{R} \mid f \text{ linearly homogeneous,} \\ \text{there is a monotonic function } m \text{ such that } m \circ f \in \mathcal{F}_d^0(D)\},$$

$$a_{ij} = \ln(c_j(x^i)/c_j(x^j)) \quad (i, j \in I),$$

$$b_{ij} = \begin{cases} 0 & (i, j \in I: y_i \leq y_j), \\ \infty & (\text{otherwise}), \end{cases}$$

$$d_{ij} = \min \{a_{ij}, b_{ij}\} \quad (i, j \in I).$$

If $F \neq \emptyset$ then one has

$$U_f(F, x, x^j) = \inf_{i \in I} [c_i(x)/c_i(x^i)] e^{d_{ij}^*} \quad (j \in I, x \in X),$$

$$U_f(F, x^i, x^j) = e^{d_{ij}^*}, \quad U_f(F, x^i, x^j)^{-1} = L_f(F, x^j, x^i) \quad (i, j \in I).$$

Proof: This is a similar statement as Theorem 3.12 with this difference, the additional output information

$$U_f(F, x^i, x^j) \leq 1 \quad (i, j \in I: y_i \leq y_j)$$

is available. An outline of the proof is as follows. Put

$$f_j(x) = \inf_{i \in I} [c_i(x)/c_i(x^i)] e^{d_{ij}^*} \quad (j \in I, x \in X),$$

and let $k \in I$. Then Theorem 10.2 and $F \neq \emptyset$ implies that one has

$$f_k + \alpha \sum_{j \neq k} f_j \in F \quad (\alpha > 0).$$

Now, letting $\alpha \rightarrow 0$, and noting that $f_k(x^k) = 1$, we obtain

$$U_f(F, x, x^k) \geq f_k(x) \quad (x \in X).$$

Furthermore, we have

$$U_f(F, x, x^k) \leq f_k(x) \quad (x \in X),$$

which we can show in a similar way as in the proof of Theorem 3.12, and thus $U_f(F, x, x^k) = f_k(x)$. The remaining part of the proof follows in a similar way as in the proof of Theorem 3.12. $\square$

The above theorem is useful to obtain an impression of the monotonic transformation m of a homothetic function $m \circ f$, where f is linearly homogeneous. When we observe the output values of the function $m \circ f$, we may estimate the values of f using the above theorem. Figure 10.1 shows results of application of the theorem to the Dutch industry data. Because the Dutch industry data were not compatible with the hypothesis of homotheticity, an efficiency transformation is applied to the data. The transformation given in Example 3.1 is used with $\epsilon = 99.5\%$. The latter is the economical efficiency upper bound that was found in the previous section. Figure 10.1 displays the lower and upper bounds

$$e^{-d_{1i}^*} \leq f(x^i)/f(x^1) \leq e^{d_{1i}^*} \quad (i, j \in I),$$

with the first year as base year. In this figure we use indices such that $y_1 = 1$ and $f(x^1) = 1$. For every observed output $y_i = m \circ f(x^i)$, on the vertical axis, the figure shows two points, representing the corresponding left and right bound of $f(x^i)$, on the horizontal axis. From this figure it appears that the left part of the monotonic function is not concave. This will have implications concerning the profit maximization hypothesis, considered in Part III. Concerning price and quantity data we can only observe the concave part of a profit maximizing production function. Later on we shall discover that the hypothesis of homothetic profit maximization is indeed seriously violated.

Figure 10.1 Bounds Monotonic Function

11 WEAKLY SEPARABLE PRODUCER DEMAND

11.1 Introduction

Conditions for the existence of a weakly separable production function in case of observed output can be derived easily from the results for unobserved output. The latter is treated in Chapter 4 that considers weakly separable consumer demand. At the end of this chapter the theory about weak separability is extended in order to apply it to linearly homogeneous weakly separable functions and to weakly separable functions with a linearly homogeneous subfunction.

11.2 Existence of an Aggregator Function

Suppose a given set of producer data on $X_1 \times X_2$ and a given function $h: X_2 \rightarrow \mathbb{R}$. Then we may investigate whether there exists a corresponding aggregator function g , so that $g(x_1, h(x_2))$ rationalizes the data. In the following theorem it is done in a similar way as in Chapter 4 for weakly separable consumer behaviour.

Theorem 11.1: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$. Suppose $h: X_2 \rightarrow \mathbb{R}$ and let the transformation T_h of any function $f: X_1 \times X_2 \rightarrow \mathbb{R}$ be defined by

$$(T_h f)(x_1, x_2) = \inf_{h(v) \geq h(x_2)} f(x_1, v) \quad (x \in X_1 \times X_2).$$

Suppose $(T_h c_i)(x^i) = c_i(x^i)$, for all $i \in I$, and put $D' = \{(T_h c_i, x^i, y_i)\}_{i \in I}$.

(a) Then one has $\mathcal{F}_d^0(D') \subset \mathcal{F}_d^0(D)$.

(b) If $\mathcal{F}_d^0(D') \neq \emptyset$ then there exists a function $f \in \mathcal{F}_d^0(D)$ such that $T_h f = f$.

(c) Suppose $T_h f = f$ and suppose that the infima for each transformation $T_h c_i$ are attained, i.e. for every $i \in I$ and $x \in X$ there is a $v \in X_2$ such that

$$(T_h c_i)(x) = c_i(x_1, v), \quad h(v) \geq h(x_2).$$

Then $f \in \mathcal{F}_d^0(D)$ if and only if $f \in \mathcal{F}_d^0(D')$.

Proof (a): By Theorem 4.6 (a) one has $\mathcal{F}_d(D') \subset \mathcal{F}_d(D)$. Then we have obviously also $\mathcal{F}_d^0(D') \subset \mathcal{F}_d^0(D)$.

(b): Suppose $\mathcal{F}_d^0(D') \neq \emptyset$. Then in the proof of Theorem 8.1 it is shown that there exists a monotonic transformation $m \circ f \in \mathcal{F}_d^0(D')$ of a function of the form

$$f(x) = \min_{i \in I} f_i + \lambda_i [(T_h c_i)(x) - (T_h c_i)(x^i)] \quad (x \in X_1 \times X_2),$$

where $\lambda_i > 0$ and $f(x^i) = f_i$ for all $i \in I$. This function satisfies obviously $T_h f = f$, so that also $T_h(m \circ f) = m \circ f$, because m is monotonically increasing. Furthermore, by (a) one has $\mathcal{F}_d^0(D') \subset \mathcal{F}_d^0(D)$ and thus $m \circ f \in \mathcal{F}_d^0(D)$.

(c): By Theorem 4.6 (c) we have $f \in \mathcal{F}_d(D)$ if and only if $f \in \mathcal{F}_d(D')$. Hence, this is obviously also the case when output y_i is observed. $\square$

As it was the case for Theorem 4.6, the above theorem is also valid if T_h is defined as

$$(T_h f)(x_1, x_2) = \inf_{h(v) = h(x_2)} f(x_1, v)$$

11.3 Existence of a Weakly Separable Function

The existence of a weakly separable production function for producer demand can be easily tested. All we have to test is the existence of a preorder given in the following theorem.

Theorem 11.2: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set of producer behaviour, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$. Consider the following conditions:

- (i) There exists a function $g(x_1, h(x_2)) \in \mathcal{F}_d^0(D)$, which is weakly separable in X_2 and such that $g(x_1, \eta)$ is increasing in η .
- (ii) There exists a function $g(x_1, h(x_2)) \in \mathcal{F}_d^0(D)$, which is weakly separable in X_2 and such that $g(x_1, \eta)$ is strictly increasing in η .
- (iii) There exists a preorder $\succ, \sim$ on $\{x_2^i\}_{i \in I}$ such that

$$c_i(x_1^j, x_2^k) \leq c_i(x_1^i), y_j > y_i \Rightarrow x_2^k \prec x_2^j \quad (i, j, k \in I),$$

$$c_i(x_1^j, x_2^k) < c_i(x_1^i), y_j \geq y_i \Rightarrow x_2^k \prec x_2^j \quad (i, j, k \in I),$$

$$c_i(x_1^j, x_2^k) \leq c_i(x_1^i), y_j \geq y_i \Rightarrow x_2^k \precsim x_2^j \quad (i, j, k \in I).$$

- (iv) There exists a weakly separable function $g(x_1, h(x_2)) \in \mathcal{F}_d^0(D)$, such that h and g are strictly monotonically increasing continuous functions, and furthermore h is concave and $g(\cdot, h(x_2))$, for every fixed $x_2 \in X_2$, quasi-concave.

Then:

- (a) (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

- (b) Suppose every function

$$h(x_2) = \min_{i, j \in I} \alpha_{ij} + \lambda_i c_i(x_1^j, x_2) \quad (x_2 \in X_2),$$

where $\alpha_{ij} \in \mathbb{R}$ and $\lambda_i > 0$, is such that the functions

$$\bar{c}_i(x_1, \eta) = \inf_{h(x_2) \geq \eta} c_i(x_1, x_2) \quad (i \in I, x_1 \in X_1, \eta \in \mathbb{R}),$$

are strictly increasing in η . Then (iii) $\Leftrightarrow$ (ii).

- (c) Suppose D is a price and quantity data set. Then (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof (a) (ii) $\Rightarrow$ (iii): Suppose $f(x) = g(x_1, h(x_2)) \in \mathcal{F}_d^0(D)$ is weakly separable and $g(x_1, \eta)$ is strictly increasing in η . Let $\succ, \sim$ be the weak order induced by the subfunction h on $\{x_2^i\}_{i \in I}$ and let $i, j, k \in I$. Then one has

$$\begin{aligned} c_i(x_1^j, x_2^k) \leq c_i(x_1^i), y_j > y_i &\Rightarrow g(x_1^j, h(x_2^k)) \leq y_i < y_j = g(x_1^j, h(x_2^j)) \\ &\Rightarrow h(x_2^k) < h(x_2^j) \\ &\Rightarrow x_2^k \prec x_2^j. \end{aligned}$$

The other implications can be proved in a similar way.

- (iii) $\Rightarrow$ (i): If we rewrite (iii) using

$$y_i \geq y_j \Rightarrow x_1^i \succsim x_1^j \quad (i, j \in I),$$

$$x_2^i \succsim x_2^j \Rightarrow x_2^i \succsim x_2^j \quad (i, j \in I),$$

then the preorders $\succ, \sim$ and $\succ', \sim'$ satisfy the condition given in Theorem 4.7 (iii). Thus, as is shown in the proof of that theorem, we can construct a function h such that we have $(T_h c_i)(x^i) = c_i(x^i)$, for all $i \in I$, where

$$(T_h c_i)(x_1, x_2) = \inf_{h(v) \geq h(x_2)} c_i(x_1, v) \quad (i \in I, x \in X_1 \times X_2).$$

Note that the function h in the proof of Theorem 4.7 satisfies

$$(T_h c_i)(x^j) \leq (T_h c_i)(x^i) \Rightarrow x^j \preceq x^i \Rightarrow y_j \leq y_i \quad (i, j \in I),$$

$$(T_h c_i)(x^j) < (T_h c_i)(x^i) \Rightarrow x^j \prec x^i \Rightarrow y_j < y_i \quad (i, j \in I).$$

Hence, by Theorem 8.1 we have $\mathcal{F}_d^o(D') \neq \emptyset$ for $D' = \{(T_h c_i, x^i, y_i)\}_{i \in I}$. Thus by Theorem 11.1 (b) there exists a function $T_h f = f \in \mathcal{F}_d^o(D)$.

(b) (iii) $\Rightarrow$ (ii): Every function $T_h c_i$ in the above proof is weakly separable. Now, the assumption in (b) implies that its aggregator function $\bar{c}_i(x_1, \eta)$ is strictly increasing in η . In that case the constructed weakly separable function $g(x_1, h(x_2)) \in \mathcal{F}_d^o(D)$ in the proof of Theorem 11.1 (b) will also have these properties.

(c) One can prove that the assumption made in (b) is satisfied by price and quantity data. Theorem 4.8 contains a complete proof for the interested reader.

To prove (iii) $\Rightarrow$ (iv) we may construct functions g and h as in the proof of (b). When we use a continuous monotonic transformation m , these functions will have the mentioned properties, because they are derived from price and quantity data. Finally, the implication (iv) $\Rightarrow$ (ii) is trivial. $\square$

I used condition (iii) of the above theorem to test whether goods and services of the Dutch industry data are weakly separable, assuming the aggregator function $g(x_1, \eta)$ is strictly increasing in η . Only two observation years had to be removed in order to accept this hypothesis. The theorem does provide an estimation of the economical efficiency upper bound for the efficiency transformation of Example 2.3. By Theorem 11.2 (a), condition (iii) is a necessary condition for the transformed data to be weakly separable. So we may use this condition in order to compute an upper bound of the economical efficiency. The result of this computation was $e = 99.5\%$.

11.4 Existence of a Linearly Homogeneous Subfunction

The nonparametric restrictions for weakly separable functions with a linearly homogeneous subfunction h follow from the observation that we have

$$h(h(x_2^i)h(x_2^j)^{-1}x_2^j) = h(x_2^i)h(x_2^j)^{-1}h(x_2^j) = h(x_2^i) \quad (i, j \in I).$$

This observation is the basis of the following theorem, where the numbers h_i represent the values $h(x_2^i)$.

Theorem 11.3: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set of producer behaviour, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$ and X_2 is a cone. Consider the following conditions:

(i) There exists a weakly separable function $g(x_1, h(x_2)) \in \mathcal{F}_d^o(D)$ such that the subfunction $h: X_2 \rightarrow \mathbb{R}$ is linearly homogeneous and the aggregator function $g(x_1, \eta)$ is increasing in η .

(ii) There exist numbers $h_i > 0$ such that

$$c_i(x_1^i, h_i h_j^{-1} x_2^j) \geq c_i(x^i) \quad (i, j \in I),$$

$$y_j \geq y_i \Rightarrow c_i(x_1^j, h_j h_i^{-1} x_2^i) \geq c_i(x^i) \quad (i, j \in I),$$

$$y_j > y_i \Rightarrow c_i(x_1^j, h_j h_i^{-1} x_2^i) > c_i(x^i) \quad (i, j \in I).$$

Then:

(a) Suppose $c_i(x_1^i, 0) < c_i(x^i)$ for all $i \in I$. Then (i) $\Rightarrow$ (ii).

(b) Suppose all cost functions

$$c_i(x) = \alpha_i(x_1, \beta_i(x_2)) \quad (x \in X_1 \times X_2, i \in I)$$

are weakly separable in X_2 , where $\beta_i: X_2 \rightarrow \mathbb{R}_+$ are linearly homogeneous, $\beta_i(x_2^i) > 0$, and $\alpha_i(x_1, \beta)$ increasing in β , for all $i \in I$. Then (ii) $\Rightarrow$ (i).

Proof (a): Put $h_i = h(x_2^i)$ for all $i \in I$. These values h_i are positive, because one has

$$c_i(x_1^i, 0) < c_i(x^i) \Rightarrow g(x_1^i, h(0)) < g(x_1^i, h(x_2^i)) \Rightarrow 0 = h(0) < h(x_2^i) \quad (i \in I).$$

Now, the result follows from the definition of producer demand, because h linearly homogeneous implies that we have

$$g(x_1^i, h(h_j^{-1} x_2^j)) = g(x_1^i, h(x_2^i)), \quad g(x_1^j, h(h_i^{-1} x_2^i)) = g(x_1^j, h(x_2^j)) \quad (i, j \in I).$$

(b): Put

$$h(x_2) = \inf_{i \in I} h_i \beta_i(x_2) / \beta_i(x_2^i) \quad (x_2 \in X_2).$$

I will show that we can apply Theorem 11.1(b) to this function. From the assumption

$$c_i(x_1^i, h_i h_j^{-1} x_2^j) \geq c_i(x^i) \quad (i, j \in I)$$

and the form of the cost functions, we obtain

$$h_i h_j^{-1} \beta_i(x_2^j) \geq \beta_i(x_2^i) \quad (i, j \in I).$$

Hence, we have $h(x_2^i) = h_i$ and thus

$$h(v) \geq h(x_2^i) \Rightarrow \beta_i(v) \geq \beta_i(x_2^i) \Rightarrow c_i(x_1^i, v) \geq c_i(x^i) \quad (i \in I, v \in X_2).$$

The latter implies $(T_h c_i)(x^i) = c_i(x^i)$, for all $i \in I$, in Theorem 11.1. Furthermore, from the assumption

$$y_j \geq y_i \Rightarrow c_i(x_1^j, h_j h_i^{-1} x_2^i) \geq c_i(x^i) \quad (i, j \in I),$$

$$y_j > y_i \Rightarrow c_i(x_1^j, h_j h_i^{-1} x_2^i) > c_i(x^i) \quad (i, j \in I),$$

and Theorem 8.1 we obtain $\mathcal{F}_d^0(D') \neq \emptyset$ for $D' = \{(T_h c_i, x^i, y_i)\}_{i \in I}$. Hence the result follows from Theorem 11.1 (b). $\square$

The following theorem provides conditions, which may be tested in a computational simple way.

Theorem 11.4: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set of producer behaviour, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$ and $c_i(x^i) > 0$, for all $i \in I$, and X_2 is a cone. Suppose all functions c_i are of the form

$$c_i(x) = \alpha_i(x_1) + \beta_i(x_2) \quad (i \in I, x \in X_1 \times X_2),$$

where $\beta_i: X_2 \rightarrow \mathbb{R}_+$ is linearly homogeneous and $\beta_i(x_2^i) > 0$, for all $i \in I$. Then the following conditions are equivalent:

- (i) There exists a weakly separable function $g(x_1, h(x_2)) \in \mathcal{F}_d^0(D)$ such that the subfunction $h: X_2 \rightarrow \mathbb{R}$ is linearly homogeneous and $g(x_1, \eta)$ is increasing in η .
- (ii) There exist values $\phi_i = \ln h(x_2^i)$ such that

$$\phi_i - \phi_j \leq \min \{a_{ij}, b_{ij}\}, \quad \phi_i - \phi_j < c_{ij} \quad (i, j \in I),$$

where

$$a_{ij} = \ln(\beta_j(x_2^i) / \beta_j(x_2^j)) \quad (i, j \in I),$$

$$b_{ij} = \begin{cases} \ln(\beta_i(x_2^i) / [c_i(x^i) - \alpha_i(x_1^i)]) & (i, j \in I: y_j \geq y_i, c_i(x^i) > \alpha_i(x_1^i)), \\ \infty & (\text{otherwise}), \end{cases}$$

$$c_{ij} = \begin{cases} \ln(\beta_i(x_2^i) / [c_i(x^i) - \alpha_i(x_1^j)]) & (i, j \in I: y_j > y_i, c_i(x^i) > \alpha_i(x_1^j)), \\ \infty & (\text{otherwise}). \end{cases}$$

Proof: This follows by applying Theorem 11.3. Note that the assumptions in Theorem 11.3 (a) and (b) are satisfied, and we have thus (i) $\Leftrightarrow$ (ii) in that theorem. Further, if we write out the conditions on the numbers h_i in

Theorem 11.3 (ii), we obtain as necessary and sufficient conditions

$$\ln h_i - \ln h_j \leq \min \{a_{ij}, b_{ij}\}, \quad \ln h_i - \ln h_j < c_{ij} \quad (i, j \in I). \quad \square$$

Note that the weakly separable function in the above theorem is a generalization of a homothetic function, which was treated in the previous chapter. A homothetic function can be conceived as a weakly separable function $g(h(x))$ with a linearly homogeneous subfunction $h(x)$. In that case one has $X_1 = \emptyset$ and $x = x_2$, and we may take $\alpha_i = 0$ in the above theorem. Condition (ii) reduces then to condition (iii) of Theorem 10.1, which is a necessary and sufficient condition for the hypothesis of homothetic producer demand.

Application of Theorem 11.3, in order to test the hypothesis of linearly homogeneous weak separability of goods and services in the Dutch industry data, resulted in the elimination of seven observations.

11.5 Existence of a Linearly Homogeneous Weakly Separable Function

The above theory can be extended to test the hypothesis that the weakly separable function is linearly homogeneous. Then we need as an additional assumption that the cost functions are linearly homogeneous. For the derivation of the results I refer the reader to the beginning of Chapter 9, where the transformation

$$(c_i, x^i, y_i) \rightarrow (c_i, y_i^{-1} x^i, 1) \quad (i \in I)$$

is proposed. As argued in Chapter 9, this scaling of the data makes the assumption that the production function is linearly homogeneous in the nonparametric test superfluous. This is also the case when linearly homogeneous weak separability is considered. The following results include data sets of an infinite number of observations, which was not the case in the earlier results concerning separability.

Theorem 11.5: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set of producer behaviour, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$ are linearly homogeneous, $c_i(x^i) > 0$ and $y_i > 0$, for all $i \in I$, and X_1 and X_2 are cones. Put

$$v_i = y_i^{-1} x^i \quad (i \in I).$$

Consider the following conditions:

(i) There exists a linearly homogeneous weakly separable function $g(x_1, h(x_2)) \in \mathcal{F}_D^0(D)$ such that the subfunction $h: X_2 \rightarrow \mathbb{R}$ is linearly homogeneous and $g(x_1, \eta)$ is increasing in η .

(ii) There exist numbers $h_i > 0$ such that

$$c_i(v_i^i, h_i h_j^{-1} v_2^j) \geq c_i(v^i), \quad c_i(v_1^j, h_j h_i^{-1} v_2^i) \geq c_i(v^i) \quad (i, j \in I).$$

Then:

(a) Suppose $c_i(v_1^i, 0) < c_i(v^i)$ for all $i \in I$. Then (i) $\Rightarrow$ (ii).

(b) Suppose all cost functions

$$c_i(x) = \alpha_i(x_1, \beta_i(x_2)) \quad (x \in X_1 \times X_2, i \in I)$$

are weakly separable, where the functions $\beta_i: X_2 \rightarrow \mathbb{R}_+$ are linearly homogeneous, $\beta_i(v_2^i) > 0$, and $\alpha_i(x_1, \beta)$ is increasing in β , for all $i \in I$. Then (ii) $\Rightarrow$ (i).

Proof (a): In case of finite data sets, we may apply Theorem 11.3(a) to the scaled data $\{(c_i, v^i, 1)\}_{i \in I}$, because the hypothesis of linearly homogeneous producer demand is invariant under scaling. One can check that the proof of Theorem 11.3(a) remains valid in case infinite data sets are used.

(b): Put

$$h(x_2) = \inf_{i \in I} h_i \beta_i(x_2) / \beta_i(v_2^i) \quad (x_2 \in X_2),$$

$$f(x) = g(x_1, h(x_2)) = \inf_{i \in I} c_i(x_1, h_i^{-1} h(x_2) \beta_i(v_2^i)) / c_i(v^i) \quad (x \in X).$$

Because the hypothesis of linearly homogeneous producer demand is invariant under scaling, it is sufficient to show that we have $f \in \mathcal{F}_d^0(\{(c_i, v^i, 1)\}_{i \in I})$. First, I will show that all the output values $f(v^i)$ are equal to one. From

$$c_i(v_1^i, h_i h_j^{-1} v_2^j) \geq c_i(v^i) \quad (i, j \in I),$$

in (ii), and the assumptions about the form of the cost functions we obtain

$$h_i h_j^{-1} \beta_i(v_2^j) \geq \beta_i(v_2^i) \quad (i, j \in I),$$

which in turn means

$$h(v_2^j) = \inf_{i \in I} h_i \beta_i(v_2^j) / \beta_i(v_2^i) = h_j \quad (j \in I).$$

This and

$$c_i(v_1^j, h_j h_i^{-1} v_2^i) \geq c_i(v^i) \quad (i, j \in I)$$

in (ii), imply

$$g(v_1^j, h(v_2^j)) = \inf_{i \in I} c_i(x_1, h_i^{-1} h_j \beta_i(v_2^i)) / c_i(v^i) = 1.$$

So the output values are correct. It remains to show that f may generate the scaled producer demand data. From

$$h(x_2) \leq h_i \beta_i(x_2) / \beta_i(v_2^i) \quad (x_2 \in X_2, i \in I),$$

and the weak separability of the cost functions, we obtain

$$c_i(x) \leq c_i(x_1, h_i^{-1} h(x_2) \beta_i(v_2^i)) \quad (x \in X, i \in I).$$

So the construction of f implies

$$f(x) \leq c_i(x_1, h_i^{-1} h(x_2) \beta_i(v_2^i)) / c_i(v^i) \leq c_i(x) / c_i(v^i) \quad (x \in X, i \in I).$$

Hence

$$c_i(x) \leq c_i(v^i) \Rightarrow f(x) \leq 1, \quad c_i(x) < c_i(v^i) \Rightarrow f(x) < 1 \quad (x \in X, i \in I),$$

thus f is a rationalizing linearly homogeneous weakly separable function. $\square$

Remark: I have to make a remark about the form of the constructed function f in the above proof of Theorem 11.5(b), which is not an arbitrary one. In the theory concerning linearly homogeneous functions it is often possible to construct a suitable rationalizing function by taking the least upper bound over a set of rationalizing linearly homogeneous functions. This applies to this case as well. In the proof of Theorem 11.5(b) we have $f(x) = U(F, x)$, where F is the subset of all linearly homogeneous functions in

$$\{g(x_1, h(x_2)) \in \mathcal{F}_d^0(D) \mid h \text{ linearly homogeneous, } h(x_2) = h_i \text{ for all } i \in I\}.$$

In this particular case this function is of the form

$$U(F, x) = \inf_{i \in I} \alpha_i(x_1, U(H, x_2) h_i^{-1} \beta_i(v_2^i)) / c_i(v^i) \quad (x \in X),$$

where

$$H = \{h \in \mathcal{F}_d^0(\{(\beta_i, v_2^i, h_i)\}_{i \in I}) \mid h \text{ linearly homogeneous}\},$$

$$U(H, x_2) = \inf_{i \in I} h_i \beta_i(x_2) / \beta_i(v_2^i) \quad (x \in X).$$

It is left to the reader to verify these statements.

Just like in Theorem 11.4, which provides computational simple conditions for the test in Theorem 11.3, we can derive such a theorem for the test in the above theorem.

Theorem 11.6: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set of producer behaviour, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}$ and $y_i > 0$, for all $i \in I$, and X_1 and X_2 are cones. Suppose all functions c_i are of the form

$$c_i(x) = \alpha_i(x_1) + \beta_i(x_2) \quad (x \in X_1 \times X_2, i \in I),$$

where both $\alpha_i: X_1 \rightarrow \mathbb{R}_+$ and $\beta_i: X_2 \rightarrow \mathbb{R}_+$ are linearly homogeneous and $\beta_i(x_2^i) > 0$. Put $v_i = y_i^{-1} x^i$ for all $i \in I$. Then the following conditions are equivalent:

- (i) There exists a linearly homogeneous weakly separable function $g(x_1, h(x_2)) \in \mathcal{F}_d^0(D)$ such that the subfunction $h: X_2 \rightarrow \mathbb{R}$ is linearly homogeneous and $g(x_1, \eta)$ is increasing in η .
 (ii) There exist values $\phi_i = \ln h(x_2^i)$ such that

$$\phi_i - \phi_j \leq \min \{a_{ij}, b_{ij}\} \quad (i, j \in I),$$

where

$$a_{ij} = \ln(\beta_j(v_2^i) / \beta_j(v_2^j)) \quad (i, j \in I),$$

$$b_{ij} = \begin{cases} \ln(\beta_i(v_2^i) / [c_i(v^i) - \alpha_i(v_1^j)]) & (i, j \in I: c_i(v^i) - \alpha_i(v_1^j) > 0), \\ \infty & (\text{otherwise}). \end{cases}$$

Proof: The result follows by applying Theorem 11.5. Note that the assumptions in Theorem 11.5 (a) and (b) are satisfied. By writing out the conditions on the numbers h_i in Theorem 11.5 (ii), we obtain (ii) as a necessary and sufficient condition for (i). $\square$

Remark: The matrix $d_{ij} = \min \{a_{ij}, b_{ij}\}$ in Theorem 11.6 has an interpretation which concerns the relative upper bounds of subfunction h . As the reader may verify, one has

$$U_{\rho}(H, x_1^i, x_2^j) = e^{d_{ij}^*} \quad (i, j \in I),$$

where H contains all linearly homogeneous functions $h: X_2 \rightarrow \mathbb{R}$ such that there exists a linearly homogeneous function $g(x_1, h(x_2)) \in \mathcal{F}_d^0(D)$.

I applied condition (ii) of Theorem 11.6 to test the hypothesis of linearly homogeneous weak separability of goods and services for the Dutch industry data. The results were the same as obtained earlier, where no restriction on the aggregator function was supposed. A subset of only two observations is found to be consistent with this hypothesis.

Now, let us consider the existence of a linearly homogeneous weakly separable function $g(x_1, h(x_2))$ for which the subfunction h is not necessarily linearly homogeneous. In that case there is no simple way to construct a rationalizing production function. However, by scaling the data, we can obtain the following result from Theorem 11.2.

Theorem 11.7: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set of producer behaviour, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$ are linearly homogeneous, $c_i(x^i) > 0$, and $y_i > 0$, for all $i \in I$, and $X_1 \times X_2$ is a cone. Consider the following condition.

- (i) There exists a preorder $\succ, \sim$ on $\{v_2^i\}_{i \in I}$ such that

$$c_i(v_1^j, v_2^k) < c_i(v^i) \Rightarrow v_2^k \prec v_2^j \quad (i, j, k \in I),$$

$$c_i(v_1^j, v_2^k) \leq c_i(v^i) \Rightarrow v_2^k \preceq v_2^j \quad (i, j, k \in I),$$

where $v^i = y_i^{-1} x^i$ for all $i \in I$.

Put $D' = \{(c_i, v^i, 1)\}_{i \in I}$. Then:

- (a) If there is a linearly homogeneous function $g(x_1, h(x_2)) \in \mathcal{F}_d^0(D)$, which is weakly separable in X_2 and such that $g(x_1, \eta)$ is strictly increasing in η , then (i) is satisfied.

- (b) If (i) is satisfied then there exists a function $g(x_1, h(x_2)) \in \mathcal{F}_d^0(D')$,

which is weakly separable on X_2 .

Proof (a): Suppose the linearly homogeneous function $g(x_1, h(x_2)) \in \mathcal{F}_d^0(D)$ has the properties mentioned in (a). Then this function is also compatible with the scaled data D' . Hence, D' must satisfy condition (iii) in Theorem 11.2, from which we obtain condition (i).

(b): Suppose (i) is satisfied. Then condition (iii) in Theorem 11.2 holds for the data D' . Hence, this theorem implies the existence of the desired weakly separable function. $\square$

The problem in this case is the following. If we do not suppose that h is a linearly homogeneous function then the transformed cost functions $T_h c_i$ in Theorem 11.1 are not necessarily linearly homogeneous, even if we suppose all c_i are linearly homogeneous. Hence, the functions $T_h c_i$ are not suitable to construct a linearly homogeneous weakly separable function.

PART III

PROFIT MAXIMIZATION

12 INTRODUCTION

12.1 Textbook Theory

In the neoclassical theory the behaviour of a profit maximizing producer is described in the following way. Let $f(\cdot)$ be a production function, v the price of output, and p the price vector of inputs. Profit maximization means that the producer chooses a quantity vector x' such that

$$vf(x') - px' = \sup_x vf(x) - px.$$

This is simple unconditioned optimization without a budget restriction as is illustrated in Figure 12.1.

Figure 12.1 Profit Maximization

Profit maximization implies cost minimization, i.e.

$$px' = \inf_{f(x) \geq f(x')} px.$$

Furthermore, profit maximization implies output maximization, i.e.

$$f(x') = \sup_{px \leq px'} f(x).$$

Thus producer demand behaviour is a necessary condition for profit maximizing

behaviour.

12.2 Profit Maximization

I will use a more general definition of profit maximizing behaviour. The textbook definition is generalized in a similar way as consumer and producer demand in Part I and II. As in Part II, a production function is assumed to be a function $f: X \rightarrow \mathbb{R}$, where X is an arbitrary space. Now, suppose the cost of input $x \in X$ is described by a function $c: X \rightarrow \mathbb{R}_+$ and suppose that $v \in \mathbb{R}_+$ is the price of output. Then an input quantity vector x' , chosen by a profit maximizing producer, will be such that he maximizes his profit

$$vf(x') - c(x') = \max_x vf(x) - c(x).$$

The market for output $f(x)$ is thus a competitive market with a fixed price v , while the market for inputs is not necessarily a competitive market. It may also be a monopolistic market.

I assume henceforth $v=1$ without loss of generality, because we can always rewrite this behaviour as

$$f(x') - v^{-1}c(x') = \max_x f(x) - v^{-1}c(x).$$

In a similar way as in the definition of producer demand, we obtain the following definition.

Definition: For a function $f: X \rightarrow \mathbb{R}$, the set of *profit maximizing behaviour* is defined as

$$\mathcal{D}_p(f) \equiv \{(c, x') \in C(X) \times X \mid f(x') - c(x') \geq f(x) - c(x) \text{ for all } x \in X\}.$$

Another way to define profit maximizing behaviour is

$$(c, x') \in \mathcal{D}_p(f) \Leftrightarrow f(x') - c(x') = \sup_x f(x) - c(x).$$

Figure 12.2 Partial Efficient Profit Maximization

The condition concerning profit maximizing behaviour can also be relaxed by using an efficiency transformation. In fact we may use the same efficiency transformations as we have used for consumer and producer demand. In Figure 12.2 the efficiency transformation Φ_e of Example 2.3 is used in order to relax the profit maximization conditions on price and quantity data. One has in this figure $\Phi_e(c, x') \in D_p(f)$, where c is of the form $c(x) = px$. In this figure one has also $\Phi_e(c, x'') \in D_p(f)$ for any $x'' \in S$.

Producer demand is a necessary condition for profit maximization, as is stated in the following theorem.

Theorem 12.1: One has $F_p(D) \subset F_d(D)$ for any data set D .

Proof: If one has

$$f(x') - c(x') \geq f(x) - c(x) \quad (x \in X),$$

then one has also

$$c(x) \leq c(x') \Rightarrow f(x) \leq f(x'), \quad c(x) < c(x') \Rightarrow f(x) < f(x') \quad (x \in X). \quad \square$$

When the observed produced output $y = f(x')$ is contained in the data, we obtain the following slightly different version of the above definition of producer behaviour.

Definition: For a function $f: X \rightarrow \mathbb{R}$, the set of *profit maximizing behaviour with observed output* is defined as

$$D_p^o(f) \equiv \{(c, x', y) \mid (c, x') \in D_p(f), y = f(x')\}.$$

Profit maximization for frontier production functions is defined in the same way as producer demand in Chapter 7.

Definition: For a *frontier production function* $f: X \rightarrow \mathbb{R}$, the set of *profit maximizing behaviour with observed output* is defined as

$$D_p^f(f) \equiv \{(c, x', y) \mid (c, x') \in D_p(f), y \leq f(x')\}.$$

12.3 Contents of Part III

Chapter 13 is concerned with the existence of a production function for given producer data. Many variations on this problem, like the observation of produced output, frontier productions functions and technical progress, are considered.

Chapter 14 treats homothetic production functions, which are a monotonic transformation m of a linearly homogeneous function f . It is shown that profit maximization can be considered as a two stage process for homothetic functions $m \circ f(x)$. The linearly homogeneous function f behaves as a production function that is consistent with cost minimization, while the monotonic transformation m acts as a profit maximizing production function.

The two stage approach, that is applied to homothetic production functions in Chapter 14, is applied to weakly separable production functions in Chapter 15. Concerning weakly separable profit maximizing functions $g(x_1, h(x_2))$, it is shown that the subfunction h behaves as a cost minimizing production function and the aggregator function g behaves as a profit maximizing production function.

13 PROFIT MAXIMIZATION

13.1 Introduction

There are analogies between profit maximization and linearly homogeneous demand. Let f be a production function and c a linearly homogeneous cost function. Then a necessary and sufficient condition for x' to be a solution for linearly homogeneous demand is

$$\sup_{x \in X} f(x)/c(x) = f(x')/c(x').$$

A proof can be found in Färe and Primont (1995, Lemma 3.A.5). The above equality can be written in a profit maximizing form as

$$\sup_{x \in X} f_0(x) - c_0(x) = f_0(x') - c_0(x'),$$

where $f_0 = \ln f$ and $c_0 = \ln c$. Hence linearly homogeneous demand behaviour can be interpreted as a special case of profit maximization behaviour. The proofs of the earlier derived theorems, concerning linearly homogeneous demand, are however quite general. In order to derive results concerning profit maximization, many of these results can be carried over.

There is still another analogy. A necessary and sufficient condition for x' to be a solution for linearly homogeneous demand is also

$$\sup_{x \in X} f(x) - f(x')c(x)/c(x') = 0,$$

where $f(x')/c(x')$ is equal to the Lagrange multiplier λ in the Lagrangian $f(x) - \lambda c(x)$. The above equality can be written in a profit maximizing form as

$$\sup_{x \in X} f(x) - c_0(x) = f(x') - c_0(x'),$$

where $c_0(x) = f(x')c(x)/c(x')$.

So, the theories concerning both linearly homogeneous demand and profit maximizing behaviour, are related to each other, because linearly homogeneous behaviour appears to be a special case of profit maximizing behaviour. Note that the profit maximizing solutions for the above analogies of linearly homogeneous demand are degenerated solutions. In these cases it is not only a point x' that maximizes the profit, but the whole line $\lambda x'$, $\lambda > 0$, maximizes the profit.

13.2 Conditions for Profit Maximization

The conditions concerning profit maximization are similar to the conditions for linearly homogeneous producer demand.

Theorem 13.1: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$. Then the following conditions are equivalent:

- (i) $\mathcal{F}_p^0(D) \neq \emptyset$
- (ii) $y_j \leq y_i + c_i(x^j) - c_i(x^i) \quad (i, j \in I).$

Proof: Similar to the proof of Theorem 9.1. To prove the existence of a function $f \in \mathcal{F}_p^0(D)$, we may use the function

$$f(x) = \inf_{i \in I} y_i + c_i(x) - c_i(x^i) \quad (x \in X). \quad \square$$

Figure 13.1 shows an example of the construction of a profit maximizing

production function, using four observations of price and quantity data.

Figure 13.1 Construction Production Function

Application of condition (ii) in Theorem 13.1 to the Dutch industry data resulted in a total rejection of the profit maximization hypothesis. We have to remove all observations, except one, to obtain consistency with economical efficient profit maximization. Note that one observation is always consistent with profit maximization. If we transform the whole data set using the efficiency transformation given in Example 2.3, we obtain as efficiency upper bound $\epsilon = 81.5\%$

As is argued above in the introduction of this chapter, there is an interesting analogy between profit maximization and linearly homogeneous cost minimization. The following theorem shows an example of this analogy.

Theorem 13.2: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where all functions $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and X is a cone. Put

$$D' = \{(y_i c_i(\cdot) / c_i(x^i), x^i, y_i)\}_{i \in I}, \quad D'' = \{(\ln c_i, x^i, \ln y_i)\}_{i \in I}.$$

Then the following conditions are equivalent:

- (i) There exists a linearly homogeneous function in $\mathcal{F}_d^o(D)$.
- (ii) There exists a linearly homogeneous function in $\mathcal{F}_p^o(D')$.
- (iii) $\mathcal{F}_p^o(D') \neq \emptyset$.
- (iv) $\mathcal{F}_p^o(D'') \neq \emptyset$.

Proof: The theorem follows from Theorem 13.1 and Theorem 9.1, using

$$\begin{aligned} y_j &\leq y_i c_i(x^j) / c_i(x^i) \Leftrightarrow \\ y_j &\leq y_i + y_i c_i(x^j) / c_i(x^i) - y_i c_i(x^i) / c_i(x^i) \Leftrightarrow \\ \ln y_j &\leq \ln y_i + \ln c_i(x^j) - \ln c_i(x^i) \quad (i, j \in I). \end{aligned}$$

□

Although only the above theorem is derived from the analogy of linearly homogeneous cost minimization and profit maximization, there is such an analogy for nearly all the theorems in this chapter. The theoretical

background of Theorem 13.2 is the following. Linearly homogeneous functions $f \in \mathcal{F}_d(D)$ in this theorem satisfy

$$\sup_{c_i(x) \leq c_i(x^i)} f(x) = f(x^i) \Leftrightarrow$$

$$\sup_{x \in X} f(x) - f(x^i) c_i(x) / c_i(x^i) = f(x^i) - f(x^i) c_i(x^i) / c_i(x^i) = 0 \Leftrightarrow$$

$$\sup_{x \in X} f(x) / c_i(x) = f(x^i) / c_i(x^i) \quad (i \in I).$$

13.3 Bounds for Profit Maximization

In the theory concerning linearly homogeneous producer demand the relative upper bound was important. Now the following upper bound plays an important role.

Definition: Suppose F is a collection of functions $f: X \rightarrow \mathbb{R}$. Then the *difference upper bound* U_- is defined as

$$U_-(F, x, x') \equiv \sup_{f \in F} f(x) - f(x') \quad (x, x' \in X).$$

Similarly, the *difference lower bound* L_- is defined as

$$L_-(F, x, x') \equiv \inf_{f \in F} f(x) - f(x') \quad (x, x' \in X).$$

In relation to profit maximization with unobserved output, the theory in Chapter 3, concerning linearly homogeneous consumer demand, is of importance. The chapter started with a restriction on the utility function, using only one observation. On the basis of this a restriction was derived that applies to multiple observations. Concerning profit maximization, I will also derive restrictions from the case where only one observation is available.

Theorem 13.3: Suppose $c: X \rightarrow \mathbb{R}_+$, $x' \in X$, and put $F = \mathcal{F}_p(\{(c, x')\})$. Then

$$U_-(F, x, x') = c(x) - c(x') \quad (x \in X).$$

Proof: Put

$$f(x) = c(x) - c(x') \quad (x \in X).$$

Then one has clearly

$$f(x) - c(x) = -c(x') = f(x') - c(x') \quad (x \in X).$$

Hence $f \in F$. The definition of U_- implies now the inequality

$$c(x) - c(x') = f(x) - f(x') \leq U_-(F, x, x') \quad (x \in X).$$

To prove the reverse inequality, let $f \in F$. By the definition of profit maximization this is equivalent to

$$f(x) - f(x') \leq c(x) - c(x') \quad (x \in X).$$

Since $f \in F$ was arbitrary, the latter and the definition of U_- means that we have

$$U_-(F, x, x') \leq c(x) - c(x') \quad (x \in X). \quad \square$$

The derived upper bound can be generalized to the case of several observations. This generalization is valid for even an infinite number of observations.

Remark: Although it is in empirical work impossible to have an infinite number of observations, this may be the case in theoretical problems. For example when a data set is described using a continuous index parameter i .

As is explained in the introduction of this chapter, there is an analogy

between linearly homogeneous producer demand and profit maximization. We may obtain profit maximization from linearly homogeneous producer demand by taking the logarithm of the production function and cost function. This analogy implies that we may derive the following theorem, which is similar to Theorem 3.12 in Chapter 3.

Theorem 13.4: Suppose $D = \{(c_i, x^i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$. Put $F = \mathcal{F}_p(D)$, and

$$a_{ij} = c_i(x^j) - c_i(x^i) \quad (i, j \in I).$$

Then:

(a) If $F \neq \emptyset$ then

$$U_-(F, \cdot, x^j) \in \mathcal{F}_p(D) \quad (j \in I),$$

$$U_-(F, x, x^j) = \inf_{i \in I} c_i(x) - c_i(x^i) + a_{ij}^* \quad (x \in X, j \in I),$$

$$U_-(F, x^i, x^j) = a_{ij}^* \quad (i, j \in I),$$

$$-U_-(F, x^i, x^j) = L_-(F, x^j, x^i) \quad (i, j \in I).$$

(b) $F \neq \emptyset$ if and only if A is absorptive.

Proof (a): This theorem is an analogy of Theorem 3.12, concerning linearly homogeneous consumer demand. While Theorem 3.12 is based on restrictions of the form

$$\ln U_j(F, x^i, x^j) \leq a_{ij}^* \quad (i, j \in I),$$

we have in this case analogous restrictions of the form

$$U_-(F, x^i, x^j) \leq a_{ij}^* \quad (i, j \in I).$$

The proof is completely analogous. The only difference is that we use the function U_- instead of $\ln U_j$. So, for example, instead of

$$U_j(F, x^i, x^j) = e^{a_{ij}^*}$$

in Theorem 3.12, we write now $U_-(F, x^i, x^j) = a_{ij}^*$.

Note that one has $\ln(ab) = \ln a + \ln b$, hence every multiplication of U_j terms in Theorem 3.12 has to be replaced by an addition of U_- terms.

(b): The proof is similar to the proof of Theorem 3.5, in which we have to replace every multiplication by an addition in the same way as in the proof of (a). $\square$

I used Theorem 13.4 (b) to test whether there exists a profit maximizing production function for the Dutch industry data. This condition can be tested using the algorithm given in Appendix C, and which is based on Theorem C.4. It appears that eight observations have to be removed in order to get consistency with the hypothesis of profit maximization. The whole data set, transformed using the efficiency transformation given in Example 2.3, has the economical efficiency upper bound $e = 99.2\%$.

The difference with previous test results of profit maximization in this section, leading to much stronger violations, is that the outputs are not available. So we may conclude that we do not observe outputs that belong to one and the same production function for all periods. This finding may be an indication for technical inefficiency or technical progress. These properties will be treated later on in this chapter.

When we observe output, the theory of linearly homogeneous producer demand in Chapter 9 is useful. The following theorem corresponds with Theorem 9.2, which treats an upper bound for one observation.

Theorem 13.5: Suppose $c: X \rightarrow \mathbb{R}_+$, $x' \in X$, $y \in \mathbb{R}$. Put $F = \mathcal{F}_p^0(\{(c, x', y)\})$. Then $U(F, \cdot) \in F$ and

$$U(F, x) = y + c(x) - c(x') \quad (x \in X).$$

Proof: The arguments are similar as in Theorem 9.2. $\square$

Theorem 13.6: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$. Put $F = \mathcal{F}_p^0(D)$. If $F \neq \emptyset$ then $U(F, \cdot) \in F$ and

$$U(F, x) = \inf_{i \in I} y_i + c_i(x) - c_i(x^i) \quad (x \in X).$$

Proof: Similar to the proof of Theorem 9.3. $\square$

13.4 Frontier Production Functions

A frontier production function lies above the observed outputs. The minimal possible output values for such a function are interesting. These values are given in the following theorem.

Theorem 13.7: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$, where $c_i: X \rightarrow \mathbb{R}_+$, and $\mathcal{F}_p(D) \neq \emptyset$. Then

$$L(\mathcal{F}_p^f(D), x^i) = \max_{k \in I} y_k + L_-(\mathcal{F}_p(D), x^i, x^k) \quad (i \in I).$$

Proof: This is analogous to Theorem 9.4, concerning linearly homogeneous producer demand. Note that every multiplication operation in Theorem 9.4 is now represented by an addition. There is the following connection between both theorems

$$\ln L \rightarrow L, \ln L_f \rightarrow L_-, \ln y_k \rightarrow y_k.$$

So we have in the expression of L

$$\max_{k \in I} y_k L_f(F, x^i, x^k) \rightarrow \max_{k \in I} y_k + L_-(F, x^i, x^k).$$

$\square$

Figure 13.2 Minimal Frontier Production Function

Thus the above theorem provides the minimal values for the frontier production function outputs. The hypothesis of profit maximization without

observed output has to be satisfied in order that such values do exist. To satisfy this hypothesis we may need an efficiency transformation. The previous section gives for the Dutch industry data an efficiency upper bound of $\epsilon = 99.2\%$. The minimal frontier production values for the transformed data, using this economical efficiency level, are given in Figure 13.2.

We can consider the ratio of the observed output and the minimal frontier production value as a technical efficiency coefficient. These ratios are given in Figure 13.3, which includes a logarithmic regression line fit. This figure gives insight in the existence of technical progress. It appears from the given graph that technical progress is present in the Dutch industry data. The logarithmic regression line fit resulted in a technical progress of 1.1% per year ($R^2 = 0.87$). In the following section the subject of technical progress is further elaborated.

Figure 13.3 Technical Efficiency Upper Bound

13.5 Technical Progress

With the following theorem we may test whether there exists an increasing family of profit maximizing production functions for a given data set.

Theorem 13.8: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$. Then the following conditions are equivalent:

- (i) There exists a series production functions f_i , $i \in I$, such that $f_j \leq f_i$ if $j \leq i$, and $(c_i, x^i, y_i) \in D_p^0(f_i)$ for all $i \in I$.
- (ii) One has

$$y_j \leq y_i + c_i(x^j) - c_i(x^i) \quad (i, j \in I: j \leq i).$$

Proof: As the proof of Theorem 9.5. □

I applied condition (ii) of the above theorem to the Dutch industry data. As a result I obtained that only two observations have to be removed in order to achieve consistency with technically progressing profit maximization. The economical efficiency upper bound for the whole data set is equal to $\epsilon = 97.1\%$, where the efficiency transformation of Example 2.3 is used.

In the previous section I considered frontier production functions. When we consider technical progress a similar approach is possible. The following theorem is concerned with the minimal possible output values of a frontier production function, assuming that technical progress is allowed.

Theorem 13.9: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $c_i: X \rightarrow \mathbb{R}_+$ and $I = \{1, 2, \dots\}$ is countable. Let each family F_i^f contain the technical progressing profit maximizing frontier production functions for period i . Thus functions f_i such that $f_j \leq f_i$ if $j \leq i$ and $(c_i, x^i, y_i) \in D_p^f(f_i)$ for all $i \in I$, satisfy $f_i \in F_i^f$ for every $i \in I$. Put $u_1 = y_1$ and

$$u_i = \max \{u_j + c_i(x^i) - c_i(x^j) \mid i, j \in I: j < i\} \cup \{y_i\} \quad (i = 2, \dots).$$

Then $L(F_i^f, x^i) = u_i$ for all $i \in I$.

Proof: Using Theorem 13.8 one can show that the given numbers u_i are suitable as outputs $f_i(x^i)$ of a series technically progressing frontier production functions $f_i \in F_i^f$, $i \in I$. Furthermore, using the definition of D_p^f , one can show that these numbers are minimal. See also Theorem 9.7. $\square$

Application of the above idea of technically progressing frontier production functions to the Dutch industry data leads to the following results. The minimal value of the frontier function was equal to the observed output, i.e. $L(F_i^f, x^i) = u_i = y_i$, except for two years. These two technical inefficient years are 1975 and 1979 with technical efficiency coefficients y_i/u_i equal to 97.3% and 99.7% respectively.

13.6 Hicks Neutral Change

The theory concerning Hicks neutral change is not completely analogous to linearly homogeneous producer demand. One has $\ln(tf(x)) = \ln t + \ln f(x)$, thus for a complete analogy one should add a factor instead of multiplying the output by a factor. So, as appears from the following theorem, neutral changing profit maximization cannot directly be derived from the results concerning neutral changing linearly homogeneous producer demand.

Theorem 13.9: Let $c: X \rightarrow \mathbb{R}_+$ and $f: X \rightarrow \mathbb{R}$. Then

$$(c, x', y) \in D_p^o(tf) \Leftrightarrow (t^{-1}c, x', t^{-1}y) \in D_p^o(f) \quad (t > 0).$$

Proof: One has the equality

$$\sup_{x \in X} tf(x) - c(x) = t \left[\sup_{x \in X} f(x) - t^{-1}c(x) \right] \quad (t > 0). \quad \square$$

In contrast to the above we have for linearly homogeneous producer demand the equivalence

$$(c, x', y) \in D_d^o(tf) \Leftrightarrow (c, x', t^{-1}y) \in D_d^o(f) \quad (t > 0),$$

where it is not necessary to divide the cost function by the factor t .

For Hicks neutral changing linearly homogeneous producer demand, we can choose the values $t^{-1}y$ by choosing a suitable multiplication factor t . For profit maximization, however, this means also a change of the cost functions $t^{-1}c$. Therefore the following theorem, concerning Hicks neutral changing profit maximization, is not analogous to the corresponding theorem concerning linearly homogeneous producer demand.

Theorem 13.10: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $y_i > 0$ and $c_i: X \rightarrow \mathbb{R}_+$ for all $i \in I$. Then the following conditions are equivalent:

(i) There exist a function $f: X \rightarrow \mathbb{R}$ and numbers $t_i > 0$, such that

$$(c_i, x^i, y_i) \in D_p^o(t_i f) \quad (i \in I).$$

(ii) There exist values $\phi_i = \ln t_i$ such that

$$\phi_i - \phi_j \leq a_{ij} \quad (i, j \in I),$$

where

$$a_{ij} = \ln([y_i + c_i(x^j) - c_i(x^i)]/y_j) \quad (i, j \in I).$$

Proof: Condition (i) is by Theorem 13.9 equivalent to the existence of a function f and $t_i > 0$, such that

$$(t_i^{-1}c_i, x^i, t_i^{-1}y_i) \in D_p^0(f) \quad (i \in I).$$

The latter is by Theorem 13.1 equivalent to the condition

$$t_j^{-1}y_j \leq t_i^{-1}[y_i + c_i(x^j) - c_i(x^i)] \quad (i, j \in I),$$

which is equivalent to (ii). $\square$

I used condition (ii) in the above theorem to test the hypothesis of Hicks neutral changing profit maximization for the Dutch industry data. This condition can be tested using the theory given in Appendix C. Five observations have to be removed if we assume economical efficient behaviour. For the whole data set I derived an efficiency upper bound of $\epsilon = 99.8\%$, using the efficiency transformation of Example 2.3.

Figure 13.4 Bounds Hicks Neutral Change

For the transformed data, belonging to this efficiency upper bound, I computed the lower and upper bounds for the factor of Hicks neutral change, relative to the first observation. These bounds are in terms of Theorem 13.10 equal to

$$e^{-a_{1i}^*} \leq t_i/t_1 \leq e^{a_{1i}^*} \quad (i \in I).$$

The resulting bounds, shown in Figure 13.4, are very close together.

The following theorem concerning Hicks neutral progress is analogous to Theorem 13.10. There is only one difference: we assume that the multiplying factors t_i increase for later observation periods i .

Theorem 13.11: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $y_i > 0$ and $c_i: X \rightarrow \mathbb{R}_+$ for all $i \in I$. Then the following conditions are equivalent:

(i) There exist a function $f: X \rightarrow \mathbb{R}$ and numbers $t_i > 0$, such that $t_j \leq t_i$ if $j \leq i$, and

$$(c_i, x^i, y_i) \in D_p^0(t_i f) \quad (i \in I).$$

(ii) There exist values $\phi_i = \ln t_i$ such that

$$\phi_i - \phi_j \leq \min \{a_{ij}, b_{ij}\} \quad (i, j \in I),$$

where

$$a_{ij} = \ln ([y_i + c_i(x^j) - c_i(x^i)]/y_j) \quad (i, j \in I),$$

$$b_{ij} = \begin{cases} 0 & (i, j \in I: i \leq j), \\ \infty & (\text{otherwise}). \end{cases}$$

Proof: We may apply Theorem 13.10, concerned with Hicks neutral change, by adding the condition that the numbers t_i are increasing. In that case we obtain from the conditions in Theorem 13.10, the conditions (i) and (ii) in this theorem. $\square$

When we assume that there is Hicks neutral progress for the Dutch industry data, the same number of observations have to be removed as for Hicks neutral change. The economical efficiency upper bound, however, is a bit lower than obtained for Hicks neutral change. The computed bound is $e = 97.1\%$.

Figure 13.5 Bounds Hicks Neutral Progress

Figure 13.5 displays the lower and upper bound of the Hicks neutral progress factor for the transformed data, using this economical efficiency level. The factor for Hicks neutral progress is computed relative to the first observation. The bounds are now more diverging than in the case of Hicks neutral change, because the factor of Hicks neutral progress may not decrease.

13.7 Neutral Change

Neutral change concerning both profit maximization and linearly homogeneous producer demand is completely analogous as is shown by the following theorem.

Theorem 13.12: Suppose $c: X_1 \times X_2 \rightarrow \mathbb{R}_+$, $f: X_1 \times X_2 \rightarrow \mathbb{R}$, where X_2 is a cone. Then

$$(c, x', y) \in D_p^o(f(x_1, tx_2)) \Leftrightarrow (c(x_1, t^{-1}x_2), [x_1', tx_2'], y) \in D_p^o(f) \quad (t > 0).$$

Proof: Suppose $(c, x') \in D_p(f(x_1, tx_2))$. This is by definition of profit maximization equivalent to

$$f(x_1, tx_2) - c(x_1, x_2) \leq f(x_1', tx_2') - c(x_1', x_2') \quad (x_1 \in X_1, x_2 \in X_2).$$

By putting $x_2'' = tx_2$, we obtain that this is equivalent to

$$f(x_1, x_2'') - c(x_1, t^{-1}x_2'') \leq f(x_1', tx_2') - c(x_1', t^{-1}tx_2') \quad (x_1 \in X_1, x_2'' \in X_2),$$

which is by the definition of profit maximization equivalent to

$$(c(x_1, t^{-1}x_2), [x_1', tx_2'], y) \in D_p(f).$$

Obviously, the above result concerning D_p without observed output is also valid for D_p^o with observed output. $\square$

The following theorem states necessary and sufficient conditions for the existence of a neutral changing profit maximizing production function. The theorem is an analogy of Theorem 9.11, which considers a similar case concerning linearly homogeneous producer demand.

Theorem 13.13: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a data set, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$ and X_2 is a cone. Let $t_i > 0$ for all $i \in I$. Then the following conditions are equivalent:

(i) There exists a function $f: X_1 \times X_2 \rightarrow \mathbb{R}$ such that $(c_i, x^i, y_i) \in D_p^o(f_i)$ for all $i \in I$, where

$$f_i(x) = f(x_1, t_i x_2) \quad (i \in I, x \in X_1 \times X_2).$$

(ii) One has

$$y_j \leq y_i + c_i(x_1^j, t_i^{-1}t_j x_2^j) - c_i(x^i) \quad (i, j \in I).$$

Proof: As the proof of Theorem 9.11, where Theorem 13.12 may be used. $\square$

For a data set of prices and quantities, one may use the following theorem to check the existence of a neutral changing profit maximizing production function.

Theorem 13.14: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set and all functions $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$ are of the form

$$c_i(x) = v_i(x_1) + w_i(x_2) \quad (x \in X_1 \times X_2),$$

where $w_i: X_2 \rightarrow \mathbb{R}_+$ is linearly homogeneous and X_2 is a cone. Then the following conditions are equivalent:

(i) There exist a function $f: X_1 \times X_2 \rightarrow \mathbb{R}$ and $t_i > 0$ such that for the functions

$$f_i(x) = f(x_1, t_i x_2) \quad (i \in I, x \in X_1 \times X_2)$$

one has $(c_i, x^i, y_i) \in D_p^o(f_i)$.

(ii) There exist values $\phi_i = \ln t_i$ such that

$$\phi_i - \phi_j \leq a_{ij} \quad (i, j \in I),$$

where

$$a_{ij} = \begin{cases} \ln(w_i(x_2^j)/[c_i(x^i) + y_j - y_i - v_i(x_1^j)]) & (i, j \in I: y_j - v_i(x_1^j) > y_i - c_i(x^i)), \\ \infty & (\text{otherwise}). \end{cases}$$

Proof: Condition (i) is by Theorem 13.13 equivalent to the existence of numbers $t_i > 0$ such that

$$y_j \leq y_i + c_i(x_1^j, t_i^{-1} t_j x_2^j) - c_i(x^i) \quad (i, j \in I).$$

Since we have by assumption

$$c_i(x_1^j, t_i^{-1} t_j x_2^j) = v_i(x_1^j) + t_i^{-1} t_j w_i(x_2^j) \quad (i, j \in I),$$

these inequalities can be written as

$$(t_i/t_j)[c_i(x^i) + y_j - y_i + v_i(x_1^j)] \leq w_i(x_2^j) \quad (i, j \in I),$$

which is equivalent to the given inequalities in condition (ii). $\square$

I investigated the implications of Theorem 13.14 for both labour and capital in the Dutch industry data. One might say that the hypothesis of neutral changing capital is more satisfactory than the hypothesis of neutral changing labour. The hypothesis of neutral changing labour can be satisfied by removing six observations, while for neutral changing capital only two observations have to be removed. For the whole data set the economical efficiency upper bound concerning the assumption of neutral changing labour and capital is 99.3% and 99.8% respectively, using the efficiency transformation of Example 2.3.

Figure 13.6 Bounds Neutral Change Labour

The weak closure of the matrix A in the above theorem provides lower and upper bounds for the factor of neutral change. These bounds, relative to the factor corresponding to the first observation, are

$$e^{-a_{i1}^*} \leq t_i/t_1 \leq e^{a_{i1}^*} \quad (i \in I).$$

Figure 13.6 and 13.7 display the bounds on the factor of neutral change for the transformed data, where the efficiency level for the transformation corresponds to the economical efficiency upper bound level. The results in

Figure 13.7 concerning capital are quite different compared to the results for labour in Figure 13.6. The lower bound for capital is strongly increasing and, while the bounds for labour are tight, the bounds for capital are diverging. After 1972 there is no upper bound information available for the neutral change of capital: the upper bound for these observations is equal to infinity. This phenomenon is also observed in Chapter 9, which is concerned with linearly homogeneous producer demand.

Figure 13.7 Bounds Neutral Change Capital

How can we explain this phenomenon of infinite upper bounds? The arguments are similar to the arguments, given in Chapter 9, that apply to linearly homogeneous demand. Suppose the data are consistent with technically progressing profit maximization of economical efficiency level e . Then we have in Theorem 13.8 the relation

$$y_j \leq y_i + c_i(x^j) - c_i(x^i) \quad (i, j \in I: j \leq i).$$

This makes it likely that we have moreover

$$y_j \leq y_i + c_i(x^j) - c_i(x^i) - w_i(x_2^j) = y_i + v_i(x_1^j) - c_i(x^i) \quad (i, j \in I: j < i),$$

which is equivalent to $a_{ij} = \infty$ for $j < i$. This will be especially likely when the terms $w_i(x_2^j)$ are relatively small. This is the case for capital and not for labour: capital and labour costs are approximately 3% and 25% respectively, of the total costs $c_i(x^i)$. Now, suppose we have $a_{ij} = \infty$ for $j < i$. Then every path from j to i with $i > j$ does always encounters an infinite value $a_{kl} = \infty$, with $k < l$. This implies that we have also $a_{ij}^* = \infty$ for $j < i$. This in turn means that we find infinite upper bounds

$$t_i/t_1 < e^{a_{i1}^*} = \infty \quad (i > 1).$$

In summary: when the data are approximately consistent with technically progressing profit maximization and the neutral changing costs $w_i(x_2^j)$ are relatively small, we may expect the occurrence of infinite upper bounds on t_i/t_1 . We observe this phenomenon in Figure 13.7 for neutral changing capital. It does not appear in Figure 13.6 concerning neutral changing

labour, because labour costs are substantially higher than capital costs.

When we assume that there is neutral progress, instead of neutral change, we have to include the additional restriction that the scale factors are increasing. The resulting restrictions are given in the following theorem.

Theorem 13.15: *Concerning Theorem 13.14, the following conditions are equivalent:*

- (i) *There exist $t_i > 0$ satisfying Theorem 13.14 (i), such that $t_j \leq t_i$ if $j \leq i$.*
 (ii) *There exist values $\phi_i = \ln t_i$ such that*

$$\phi_i - \phi_j \leq \min \{a_{ij}, b_{ij}\} \quad (i, j \in I),$$

where a_{ij} is given in Theorem 13.14 and

$$b_{ij} = \begin{cases} 0 & (i, j \in I: i \leq j), \\ \infty & (\text{otherwise}). \end{cases}$$

Proof: As the proof of Theorem 13.14. We only add to Theorem 13.14 (i) the condition that we have

$$t_j \leq t_i \quad (i, j \in I: j \leq i),$$

which means that we must add to Theorem 13.14 (ii) the additional condition

$$\phi_i - \phi_j \leq 0 \quad (i, j \in I: i \leq j). \quad \square$$

In order to accept the assumption of neutral change we have to remove seven and three observations for labour and capital respectively. The assumption of neutral progress is stronger than the assumption of neutral change, thus we obtain a larger number of removed observations than in the case of neutral change. Also because of the stronger assumption, we obtain a lower economical efficiency upper bound. This economical efficiency upper bound concerning the Dutch industry data was in both cases, for labour and capital, equal to 97.1%. Figure 13.8 and 13.9 display the lower and upper bounds on t_i/t_1 that correspond to the transformed data for this efficiency upper bound.

Figure 13.8 Bounds Neutral Progress Labour

Figure 13.9 Bounds Neutral Progress Capital

14 HOMOTHETIC PROFIT MAXIMIZATION

14.1 Introduction

In this chapter we shall consider the existence of homothetic production functions for producer data with observed output. It will be shown that profit maximization for a homothetic production function can be considered as a two stage process. A homothetic function is a monotonic transformation of a linearly homogeneous function. The two stages for such a function are as follows. First, the linearly homogeneous function has to satisfy cost minimization. Secondly, the monotonic transformation acts as a simplified profit maximizing production function.

14.2 Existence of a Homothetic Production Function

Suppose $m \circ f(x)$ is a homothetic function, where m is strictly increasing and f is linearly homogeneous. Now, consider the profit maximization problem

$$\sup_x m \circ f(x) - c(x).$$

The idea is to transform this problem to a problem of a symmetric form, using the same approach as is earlier used for weak separability. Let the transformation T_f be defined by

$$(T_f c)(x) = \inf_{f(v)=f(x)} c(v) \quad (x \in X).$$

Now we have

$$\sup_x m \circ f(x) - c(x) = \sup_x \sup_{f(v)=f(x)} m \circ f(v) - c(v) = \sup_x m \circ f(x) - (T_f c)(x).$$

Because $(T_f c)(x)$ can be written in the form $d(f(x))$, any optimal solution x' has to satisfy the requirement that $z = f(x')$ solves the simplified profit maximization problem

$$\sup_z m(z) - d(z).$$

Furthermore, because any optimal solution x' is a cost minimizing solution, another condition that has to be satisfied is $(c, x') \in \mathcal{D}_c(f)$. Using this approach, the proof of the existence of a profit maximizing homothetic production function is relatively easy.

Theorem 14.1: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and $c_i(x^i) > 0$, for all $i \in I$, and X is a cone. Then the following conditions are equivalent:

- (i) There exists a homothetic function $h \in \mathcal{F}_p^o(D)$.
- (ii) There exists a linearly homogeneous function $f: X \rightarrow \mathbb{R}$ and an increasing function $m: \mathbb{R} \rightarrow \mathbb{R}$ such that $f \in \mathcal{F}_c(D)$ and $m \in \mathcal{F}_p^o(\{(P_i, f_i, y_i)\}_{i \in I})$, where

$$f_i = f(x^i), \quad P_i = c_i(x^i)/f(x^i) \quad (i \in I).$$

- (iii) There exist values ϕ_i such that

$$\phi_i - \phi_j \leq \min \{a_{ij}, b_{ij}\} \quad (i, j \in I),$$

where

$$a_{ij} = \ln(c_j(x^i)/c_j(x^j)) \quad (i, j \in I),$$

$$b_{ij} = \begin{cases} \ln(c_i(x^i)/[y_j - y_i + c_i(x^i)]) & (i, j: y_j - y_i + c_i(x^i) > 0), \\ \infty & (\text{otherwise}). \end{cases}$$

Proof (i) $\Rightarrow$ (ii): Suppose $h = m \circ f \in \mathcal{F}_p^o(D)$ is a homothetic function. Then one has $h \in \mathcal{F}_d(D)$ by Theorem 12.1, and hence $f \in \mathcal{F}_c(D)$. Furthermore, since $h \in \mathcal{F}_p^o(D)$ by assumption, one has

$$m \circ f(\lambda x^i) - c_i(\lambda x^i) \leq m \circ f(x^i) - c_i(x^i) \quad (\lambda \geq 0, i \in I),$$

thus for $\lambda = \eta f(x^i)^{-1}$ and $y_i = m \circ f(x^i)$ we obtain

$$m(\eta) - P_i \eta \leq y_i - P_i f_i,$$

so that $m \in \mathcal{F}_p(\{(P_i, f_i, y_i)\}_{i \in I})$.

(ii) $\Rightarrow$ (iii): Suppose that m and f satisfy condition (ii). Since $f \in \mathcal{F}_c(D)$ is linearly homogeneous, we may apply Theorem 3.4 to obtain $f(x^i) > 0$ for all $i \in I$, and by Theorem 9.1 we have then

$$f_i/f_j \leq c_j(x^i)/c_j(x^j) \quad (i, j \in I).$$

Furthermore, $m \in \mathcal{F}_p(\{(P_i, f_i, y_i)\}_{i \in I})$ implies

$$y_j - c_i(x^i) f_i^{-1} f_j = y_j - P_i f_j \leq y_i - P_i f_i = y_i - c_i(x^i) \quad (i, j \in I).$$

Thus

$$\ln f_i - \ln f_j \leq \min \{a_{ij}, b_{ij}\} \quad (i, j \in I),$$

and (iii) follows.

(iii) $\Rightarrow$ (i): Suppose there exist $f_i > 0$ such that

$$\ln f_i - \ln f_j \leq \min \{a_{ij}, b_{ij}\} \quad (i, j \in I).$$

Now, let

$$P_i = c_i(x^i)/f_i \quad (i \in I),$$

and put

$$f(x) = \min_{i \in I} f_i c_i(x)/c_i(x^i) \quad (i \in I, x \in X),$$

$$m(\eta) = \min_{i \in I} y_i + P_i(\eta - f_i) \quad (i \in I, \eta \in \mathbb{R}).$$

From the definition of A and B we can derive $f(x^i) = f_i$ and $m(f_i) = y_i$, for all $i \in I$. Thus the choice of m and f implies

$$\begin{aligned} m(f(x)) - c_i(x) &\leq y_i + P_i[f(x) - f_i] - c_i(x) \\ &\leq y_i + P_i[f_i c_i(x)/c_i(x^i) - f_i] - c_i(x) \\ &= y_i - c_i(x^i) \quad (i \in I, x \in X). \end{aligned}$$

Hence $m \circ f \in \mathcal{F}_p^o(D)$, which is clearly a homothetic function with the desired properties. $\square$

Remark: We may extend Theorem 14.1 to the case where the data set D contains an infinite number of elements. However, in case of infinite data sets it may happen that the constructed function m in the proof of (iii) $\Rightarrow$ (i) is not a strictly increasing – but an increasing – function. For infinite data sets we can only prove the equivalence of the conditions (i), (ii) and (iii), when we drop in the definition of a homothetic function the condition that m is *strictly* increasing. Such a theorem – concerning increasing functions m – is valid, because the entire proof of Theorem 14.1, only refers to the assumption that m is increasing.

The results in the above theorem are related to the use of price indices. The values P_i are in fact price indices corresponding to the volumes f_i of the

linearly homogeneous function. The theorem makes the following two stage process clear concerning profit maximization with a homothetic function $m \circ f$: there is cost minimization on a lower level for the linearly homogeneous function $f \in \mathcal{F}_c(D)$ and there is profit maximization on a higher level for the monotonic transformation $m \in \mathcal{F}_p^o(\{(P_i, f_i, y_i)\}_{i \in I})$.

In Varian (1984, Theorem 8) one may find weaker necessary and sufficient conditions concerning homothetic profit maximization. However, these conditions are only necessary conditions. The sufficiency proof in Varian (1984) contains a small error. Varian did forget to show that his constructed production function satisfies $m(f(x^i)) = y_i$ or - in the notation used by Varian - $f(\Phi(x^i)) = y^i$.

Application of condition (iii) of Theorem 14.1 to the Dutch industry data shows a bad fit with homothetic profit maximization. To achieve consistency with homothetic profit maximization it appears that all observations, except one, have to be removed. I could, however, save myself the trouble to compute these results. Chapter 13 contains the same results concerning profit maximization without restrictions on the production function.

We may derive restrictions on the underlying linearly homogeneous function in the same way as is derived for homothetic cost minimization in Chapter 10. These restrictions are given in the following theorem.

Theorem 14.2: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set, where $c_i: X \rightarrow \mathbb{R}_+$ are linearly homogeneous and $c_i(x^i) > 0$, for all $i \in I$, and X is a cone. Put

$$F = \{f: X \rightarrow \mathbb{R} \mid f \text{ linearly homogeneous,} \\ \text{there is a monotonic function } m \text{ such that } m \circ f \in \mathcal{F}_p^o(D)\},$$

$$d_{ij} = \min \{a_{ij}, b_{ij}\} \quad (i, j \in I),$$

with a_{ij} and b_{ij} as given in Theorem 14.1.

If $F \neq \emptyset$ then $U_I(F, \cdot, x^j) \in F$ for all $j \in I$ and

$$U_I(F, x, x^j) = \inf_{i \in I} [c_i(x)/c_i(x^i)] e^{d_{ij}^*} \quad (j \in I, x \in X),$$

$$U_I(F, x^i, x^j) = e^{d_{ij}^*}, \quad U_I(F, x^i, x^j)^{-1} = L_I(F, x^j, x^i) \quad (i, j \in I).$$

Proof: An outline of the proof is as follows. Put

$$f_j(x) = \inf_{i \in I} [c_i(x)/c_i(x^i)] e^{d_{ij}^*} \quad (j \in I, x \in X).$$

Then we can show that we have $f_j \in F$ and thus $U_I(F, x, x^j) \geq f_j(x)$. Furthermore, in a similar way as in the proof of Theorem 3.12, we can show that $U_I(F, x, x^j) \leq f_j(x)$. This means that we have $U_I(F, x, x^j) = f_j(x)$. The remaining part of the proof is also similar to the proof of Theorem 3.12. $\square$

With the above theorem one may derive bounds on the monotonic function m . In Chapter 10 such bounds are given in Figure 10.1, which concerns the case of homothetic cost minimization. However, it was not possible to apply the above theorem to the Dutch industry data, because these data are not compatible at all with homothetic profit maximization. This was already apparent in Chapter 10. As is shown in Figure 10.1, there exists no concave cost minimizing homothetic function for the data. Because any profit maximizing function for price and quantity data has to be concave and cost minimizing, there exists thus no homothetic profit maximizing function for the Dutch industry data. To allow such a function to exist, we might try to include, for example, the assumption of Hicks neutral progress.

15 WEAKLY SEPARABLE PROFIT MAXIMIZATION

15.1 Introduction

A weakly separable function consists of a subfunction and an aggregator function. In the previous chapter a two stage optimizing process was used to attack the problem of profit maximization with a homothetic production function. A similar approach is also convenient in case of weakly separable production functions. In this chapter it will be shown that weakly separable profit maximization can be conceived as a cost minimization stage for the subfunction and a profit maximization stage on a higher level for the aggregator function.

15.2 Existence of an Aggregator Function

The basic idea in this chapter is the observation that weakly separable functions $f(x) = g(x_1, h(x_2))$ satisfy

$$\sup_x f(x) - c(x) = \sup_x f(x) - (T_h c)(x),$$

where the transformation T_h is defined by

$$(T_h c)(x_1, x_2) = \inf_{h(v)=h(x_2)} c(x_1, v) \quad (x \in X_1 \times X_2).$$

Moreover, in case $g(x_1, \eta)$ is increasing in η , we may use the transformation

$$(T_h c)(x_1, x_2) = \inf_{h(v) \geq h(x_2)} c(x_1, v) \quad (x \in X_1 \times X_2).$$

Now, $(T_h c)(x)$ can be written in the form $d(x_1, h(x_2))$, hence the optimal x_1 and $\eta = h(x_2')$ are a solution of

$$\sup_{x_1, \eta} g(x_1, \eta) - d(x_1, \eta).$$

Furthermore, because any optimal solution x' is a cost minimizing solution, the condition $(T_h c)(x') = c(x')$ has to be satisfied.

So, we can transform a profit maximization problem with a weakly separable production function into a profit maximization problem of a symmetric form. This observation leads to conditions, which assure the existence of an aggregator function, given a data set on $X_1 \times X_2$ and a (sub)function $h: X_2 \rightarrow \mathbb{R}$.

Theorem 15.1: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$. Suppose $h: X_2 \rightarrow \mathbb{R}$ and let the transformation T_h of any function $f: X_1 \times X_2 \rightarrow \mathbb{R}$ be defined by

$$(T_h f)(x_1, x_2) = \inf_{h(v) \geq h(x_2)} f(x_1, v) \quad (x \in X_1 \times X_2).$$

Put $D' = \{(T_h c_i, x^i, y_i)\}_{i \in I}$. Then the following conditions are equivalent:

- (i) There exists a weakly separable function $f \in \mathcal{F}_p^o(D)$ such that $T_h f = f$.
- (ii) $(T_h c_i)(x^i) = c_i(x^i)$ for all $i \in I$, and $\mathcal{F}_p^o(D') \neq \emptyset$.

Proof (i) $\Rightarrow$ (ii): Suppose $f \in \mathcal{F}_p^o(D)$ is weakly separable such that $T_h f = f$. Then $f \in \mathcal{F}_c(D)$ and by Theorem 4.5 one has $(T_h c_i)(x^i) = c_i(x^i)$ for all $i \in I$.

To complete the proof, I will show that $f \in \mathcal{F}_p^o(D')$. Let $i \in I$ and $x \in X_1 \times X_2$. Because $T_h f = f \in \mathcal{F}_p^o(D)$, we have

$$f(x_1, x_2) - c_i(x_1, v) \leq f(x_1, v) - c_i(x_1, v) \leq y_i - c_i(x^i)$$

for any $v \in X_2$ such that $h(v) \geq h(x_2)$. Thus

$$f(x) - (T_h c_i)(x) \leq y_i - c_i(x^i) = f(x^i) - (T_h c_i)(x^i) \quad (x \in X, i \in I),$$

so that $f \in \mathcal{F}_p^0(D')$.

(ii) $\Rightarrow$ (i): Suppose $c_i(x^i) = (T_h c_i)(x^i)$, for all $i \in I$, and $\mathcal{F}_p^0(D') \neq \emptyset$. Put

$$f(x) = \min_{i \in I} y_i + (T_h c_i)(x) - (T_h c_i)(x^i) \quad (x \in X).$$

Then one has $f \in \mathcal{F}_p^0(D')$ by Theorem 13.6 and moreover $T_h f = f$. Now, noting that one has $T_h c_i \leq c_i$ by definition of T_h and $(T_h c_i)(x^i) = c_i(x^i)$, for all $i \in I$, we obtain

$$f(x) - c_i(x) \leq y_i + (T_h c_i)(x) - (T_h c_i)(x^i) - c_i(x) \leq y_i - c_i(x^i) \quad (x \in X, i \in I),$$

which implies that we have $f \in \mathcal{F}_p^0(D)$. $\square$

Note that that we can also prove a similar theorem as the above, using the transformation

$$(T_h f)(x_1, x_2) = \inf_{h(v)=h(x_2)} f(x_1, v).$$

15.2 Existence of a Weakly Separable Function

We can use Theorem 15.1 to prove the existence of a weakly separable function. Using this theorem we construct a suitable weakly separable function in order to prove the following main theorem.

Theorem 15.2: Suppose $D = \{(c_i, x^i, y_i)\}_{i \in I}$ is a finite data set, where $c_i: X_1 \times X_2 \rightarrow \mathbb{R}_+$. Consider the following conditions:

- (i) There exists a function $g(x_1, h(x_2)) \in \mathcal{F}_p^0(D)$, which is weakly separable in X_2 and such that $g(x_1, \eta)$ is increasing in η .
- (ii) There exists a function $g(x_1, h(x_2)) \in \mathcal{F}_p^0(D)$, which is weakly separable in X_2 and such that $g(x_1, \eta)$ is strictly increasing in η .
- (iii) There exists a preorder $\succ, \sim$ on $\{x_2^i\}_{i \in I}$ such that

$$y_j - c_i(x_1^j, x_2^k) \geq y_i - c_i(x^i) \Rightarrow x_2^k \succsim x_2^j \quad (k, i, j \in I),$$

$$y_j - c_i(x_1^j, x_2^k) > y_i - c_i(x^i) \Rightarrow x_2^k \prec x_2^j \quad (k, i, j \in I).$$

- (iv) There exists a weakly separable function $g(x_1, h(x_2)) \in \mathcal{F}_p^0(D)$, such that h and g are strictly monotonically increasing continuous functions, and furthermore h is concave and $g(\cdot, h(x_2))$, for every fixed $x_2 \in X_2$, quasi-concave.

Then:

(a) (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(b) Suppose every function

$$h(x_2) = \min_{i, j \in I} \alpha_{ij} + \lambda_i c_i(x_1^j, x_2) \quad (x_2 \in X_2),$$

where $\alpha_{ij} \in \mathbb{R}$ and $\lambda_i > 0$, is such that the functions

$$\bar{c}_i(x_1, \eta) = \inf_{h(x_2) \geq \eta} c_i(x_1, x_2) \quad (i \in I, x_1 \in X_1, \eta \in \mathbb{R}),$$

are strictly increasing in η . Then (iii) $\Leftrightarrow$ (ii).

(c) Suppose D is a price and quantity data set. Then (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof (a) (ii) $\Rightarrow$ (iii): Suppose $f(x) = g(x_1, h(x_2)) \in \mathcal{F}_p^0(D)$ is weakly separable as in (ii). Let $i, j, k \in I$ and suppose

$$y_j - c_i(x_1^j, x_2^k) \geq y_i - c_i(x^i).$$

Since $f \in \mathcal{F}_p^0(D)$, one has moreover

$$y_i - c_i(x^i) \geq f(x_1^j, x_2^k) - c_i(x_1^j, x_2^k).$$

Hence $y_j = f(x^j) \geq f(x_1^j, x_2^k)$, which implies $h(x_2^k) \leq h(x_2^j)$, because $f(x) = g(x_1, h(x_2))$ and $g(x_1, \eta)$ is strictly increasing in η . In a similar way one can prove

$$y_j - c_i(x_1^j, x_2^k) > y_i - c_i(x^i) \Rightarrow h(x_2^k) < h(x_2^j) \quad (k, i, j \in I).$$

This yields (iii), using the weak order $<, \sim$ on $\{x_2^i\}_{i \in I}$ induced by h .

(iii) $\Rightarrow$ (i): Put

$$z_j(x_2) = \min_{i \in I} y_i - c_i(x^i) - y_j + c_i(x_1^j, x_2) \quad (x_2 \in X_2, i \in I).$$

From (iii) it follows that we may use Theorem 2.4 to construct a function

$$h(x_2) = \min_{i \in I} h_i + \lambda_i z_i(x_2) \quad (x_2 \in X_2),$$

such that $\lambda_i > 0$ and $h(x^i) = h_i$ for all $i \in I$. Now, let the transformed functions $T_h c_i$ be defined as in Theorem 15.1

$$(T_h c_i)(x_1, x_2) = \inf_{h(v) \geq h(x_2)} c_i(x_1, v) \quad (x_1 \in X_1, x_2 \in X_2, i \in I).$$

The construction of h implies

$$h(x_2) \geq h(x_2^j) \Rightarrow z_j(x_2) \geq 0 \quad (x_2 \in X_2, j \in I),$$

which means that one has

$$y_i - c_i(x^i) - y_j + (T_h c_i)(x^j) \geq 0 \quad (i, j \in I).$$

Hence $(T_h c_i)(x^i) \geq c_i(x^i)$ and thus $(T_h c_i)(x^i) = c_i(x^i)$, for all $i \in I$, because $T_h c_i \leq c_i$ by definition of T_h . Now, we have

$$y_j - (T_h c_i)(x^j) \leq y_i - (T_h c_i)(x^i) \quad (i, j \in I),$$

thus by Theorem 13.1 we have moreover $\mathcal{F}_p^0(D') \neq \emptyset$ in Theorem 15.1. This means that condition (ii) in Theorem 15.1 is satisfied, which implies the existence of a weakly separable profit maximizing function $f = T_h f \in \mathcal{F}_p^0(D)$. Note that the resulting function in the proof of Theorem 15.1 is

$$f(x) = \min_{i \in I} y_i + (T_h c_i)(x) - (T_h c_i)(x^i) \quad (x \in X).$$

(b) (iii) $\Rightarrow$ (ii): The functions $T_h c_i$ in the above proof are weakly separable with aggregator function $\bar{c}_i(x_1, \eta)$. If the assumption in (b) is satisfied, then all these aggregator functions are strictly increasing in η . In that case the above constructed weakly separable function $f \in \mathcal{F}_p^0(D)$ will have the same property.

(c): Theorem 4.8 shows that the assumption made in (b) is satisfied by price and quantity data. To prove (iii) $\Rightarrow$ (iv), we may inspect the weakly separable function f that is constructed in the proof of (a) (iii) $\Rightarrow$ (i). For price and quantity data it will have the properties mentioned in (iv).

Finally, the implication (iv) $\Rightarrow$ (ii) is trivial. $\square$

Condition (iii) in the above theorem may be used to test the hypothesis of weakly separable profit maximization. This condition is of the revealed preference type and may be easily tested. In the same way as I remarked in the previous chapter, it has no sense to use this condition to test whether the Dutch industry data are consistent with weakly separable profit maximization. The data fitted so badly with the hypothesis of profit maximization in Chapter 13, that no valuable results can be expected for weakly separable profit maximization.

Varian (1984, Theorem 9) states weaker necessary and sufficient conditions concerning weakly separable profit maximization. However, these conditions

are only necessary conditions. Varian did forget to show in the sufficiency proof that his constructed production function satisfies $g(x_1^i, h(x_2^i)) = y_i$, or - in the notation of Varian (1984) - $g(x^i, h(z^i)) = y^i$.

Varian claims that in case of price and quantity data weakly separable profit maximization is equivalent to concave weakly separable profit maximization. However, it seems that there is not such an equivalence. The results in Varian (1984) indicate that the conditions concerning concave weakly separable profit maximization for price and quantity data might be as follows.

Conjecture: Suppose $D = \{(p^i, x^i, y_i)\}_{i \in I}$ is a finite price and quantity data set of producer behaviour, where $X = X_1 \times X_2$. Then the following conditions are equivalent:

(i) There exists a concave weakly separable function $g(x_1, h(x_2)) \in \mathcal{F}_p^0(D)$ such that g and $h: X_2 \rightarrow \mathbb{R}$ are concave and $g(x_1, \eta)$ is strictly increasing in η .

(ii) There exist numbers v_i and $\mu_i > 0$ such that

$$y_i \leq y_j + p_1^j(x_1^i - x_1^j) + (v_i - v_j)/\mu_j \quad (i, j \in I),$$

$$v_i \leq v_j + \mu_j p_2^j(x_2^i - x_2^j) \quad (i, j \in I).$$

Proof (i) $\Rightarrow$ (ii): It seems to me that this proof should be based on similar arguments as given in Diewert and Parkan (1978, 1985), where conditions concerning concave weakly separable demand are considered.

(ii) $\Rightarrow$ (i): Put

$$g(x_1, v) = \min_{i \in I} y_i + p_1^i(x_1 - x_1^i) + (v - v_i)/\mu_i \quad (x_1 \in X_1, v \in h(X_2)),$$

$$h(x_2) = \min_{i \in I} v_i + \mu_i p_2^i(x_2 - x_2^i) \quad (x_2 \in X).$$

It is now easy to show that for every $i \in I$ one has

$$g(x_1^i, v_i) = y_i, \quad h(x_2^i) = v_i,$$

$$\begin{aligned} g(x_1, h(x_2)) - p^i x &\leq y_i + p_1^i(x_1 - x_1^i) + (h(x_2) - v_i)/\mu_i - p^i x \\ &\leq y_i + p_1^i(x_1 - x_1^i) + p_2^i(x_2 - x_2^i) - p^i x = y_i - p^i x^i. \end{aligned}$$

So $g(x_1, h(x_2)) \in \mathcal{F}_p^0(D)$, which is clearly a concave function such that g and h is concave. $\square$

PART IV

CAPITAL

16 INTRODUCTION

16.1 Static or Dynamic Producer Behaviour

The producer theory presented in the parts "Producer Demand" and "Profit Maximization" is concerned with static optimization behaviour: all that counts are inputs and costs at a given moment. This static theory is not directly connected with the practical situation of a producer. A producer who is only focussed on the current moment, will not be taken serious as a rival. In practice a producer has to decide about current and future investments. He has to estimate his discounted profit flow to pay off his debts. The capital stock depends dynamically on past investments, which means that the current investments do not only contribute to current profit, but also to future profit. So in practice a producer is facing a dynamic optimization problem.

Does this mean that the static theory is worthless? Not if we are able to transform the dynamic producer problem into a static problem. This transformation is the central problem in Chapter 17, which presents solutions for the user cost of capital. In Chapter 18 the particular case is considered where the discounted price of investment is exponentially decreasing. A second issue, how to generate the capital stocks from the past investments, is treated in Chapter 19. The remaining part of this chapter is an introduction to the dynamic relation of the capital stock with investments and a specification of the dynamic producer problem.

16.2 The Dynamic Optimization Problem

In practice, a profit maximizing producer is optimizing over a long period. To take into account future cost and profit, he uses *discounted* cost and profit, which depends on a discount rate. Such a producer maximizes the total *present value* of profit

$$\max \int_0^{\infty} P(K(t), I(t), t) e^{-rt} dt,$$

where P is profit, K is the capital stock, I is the investment flow, and r is the discount rate. The producer may choose his current and future investments $I(t)$, for $t \geq 0$, but he cannot change his investments $I(t)$, for $t < 0$, made in the past. Furthermore, there is a given capital investment relation, which specifies the capital stock $K(\cdot)$ that belongs to investment flow $I(\cdot)$.

16.3 Survival and Retirement of Investment

The lifetime of a new investment can be described by using a *survival function* $h(t)$, representing the fraction of a unit investment that survives after a time t . In the beginning a unit investment is totally available and in the long run there will be nothing left. So, a survival function is a

decreasing function from one to zero at $t \geq 0$, and we may write

$$h(t) = 1 - \int_0^t p(\tau) d\tau \quad (t \geq 0),$$

where the *retirement distribution* $p(t)$ is a probability distribution. This probability distribution describes the retirement flow of a unit investment after a time t . Retirement distributions for investments are usually borrowed from statistical theory. The parametric families of linear, normal, Weibull and logistic probability distributions are widely used.

Below I will describe a few examples, which are: the exponential survival function, rectangular survival functions and the family of gamma retirement distributions. These examples will be discussed in the following chapters, in which especially the gamma distributions will be investigated.

The *exponential* survival function is of the form

$$h(t) = 1 - \int_0^t \lambda e^{-\lambda \tau} d\tau = e^{-\lambda t} \quad (t \geq 0),$$

for which the retirement distribution is the exponential function $\lambda e^{-\lambda t}$.

Another survival function is the *rectangular* survival function

$$h(t) = \begin{cases} 1 & (t \in [0, T]), \\ 0 & (\text{otherwise}), \end{cases}$$

for which a new investment remains unchanged in his lifetime T . Using the function

$$1(t) \equiv \begin{cases} 0 & (t < 0), \\ 1 & (t \geq 0), \end{cases}$$

we may describe this survival function as

$$h(t) = 1 - 1(t - T) = 1 - \int_0^t \delta(\tau - T) d\tau \quad (t \geq 0).$$

The above retirement distribution $\delta(t - T)$ is defined by using the *delta distribution* $\delta(\cdot)$, which satisfies by definition

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt \equiv f(0)$$

for every function f .

The *gamma distribution* is a probability distribution, determined by two parameters α and λ and given by

$$p(t) = \lambda^\alpha t^{\alpha-1} e^{-\lambda t} / \Gamma(\alpha) \quad (t \geq 0),$$

where

$$\Gamma(\alpha) \equiv \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

This family includes the exponential retirement distribution, for which one has $\alpha = 1$. The family gamma distributions also approaches the delta distribution $\delta(t - T)$, which can be described as the limit for $\alpha \rightarrow \infty$ of gamma distributions with mean lifetime T . Examples of retirement distributions with $T = 1$ are given in Figure 16.1, while Figure 16.2 displays the corresponding survival functions.

The Netherlands Central Bureau of Statistics (1986) estimated the gamma retirement distribution for several types of capital assets in the Dutch

industry. All estimated values of α were found in the range $2 \leq \alpha \leq 3.5$. Thus one may doubt whether the widely used exponential survival function with $\alpha = 1$, and the rectangular survival function with $\alpha = \infty$, are suitable choices.

Figure 16.1 Retirement Distributions

Figure 16.2 Survival Functions

16.4 Capital Stocks

A capital stock is dynamically related to current investments and investments made in the past. In this section I will specify this relation in a decreasing level of abstraction. A general description is $K = AI$, where A is a

linear operator. Such a linear operator maps each investment flow function $I(t)$ on the corresponding capital stock function $K(t)$.

Definition: An operator A is called *linear* when it has the following properties:

$$\begin{aligned} A(I_1 + I_2) &= AI_1 + AI_2, \\ A\lambda I &= \lambda AI \quad (\lambda \in \mathbb{R}). \end{aligned}$$

Below I will consider examples of linear operators A , which describe the survival and retirement of investments.

Most empirical estimations of capital stocks are based on the *perpetual inventory method*, which determines the capital stock as a weighted sum of past gross investment flows. In a continuous time description we have then

$$K(t) = \int_{-\infty}^t h(t-\tau)I(\tau) d\tau \quad (t \in \mathbb{R}),$$

where h is a survival function for investments, described in the previous section. We may simplify the notation, by putting $h(t) = 0$ for all $t < 0$, which implies $h(t-\tau) = 0$ for all $\tau > t$. In that case we may write

$$K(t) = \int_{-\infty}^{\infty} h(t-\tau)I(\tau) d\tau \quad (t \in \mathbb{R}),$$

So, by using the *convolution* operator $*$, which is defined as

$$f * g(t) \equiv \int_{-\infty}^{\infty} f(t-\tau)g(\tau) d\tau \quad (t \in \mathbb{R}),$$

we have

$$K(t) = h * I(t) \quad (t \in \mathbb{R}).$$

The operator A described by $AI = h * I$ is clearly an example of a linear operator.

Another way to describe the behaviour of the capital stock is

$$\frac{d}{dt}K = I - BI,$$

where B is a linear operator. The term BI represents in this equation the flow of retired investments. Such a relation can be derived by using a retirement distribution p that determines a survival function h and the capital stock in the following way.

Theorem 16.1: Suppose p is a retirement distribution and $p(t) = 0$ for all $t < 0$. Then the survival function

$$h(t) = 1 - \int_0^t p(\tau) d\tau \quad (t \geq 0)$$

can be written as $h = 1(\cdot) * (\delta(\cdot) - p)$. Moreover, the capital stock $K = h * I$ is governed by the equation

$$\frac{d}{dt}K = I - p * I,$$

where $p * I$ represents the flow of retired investments.

Proof: We have

$$K = h * I = [1(\cdot) * (\delta(\cdot) - p)] * I = 1(\cdot) * [(\delta(\cdot) - p) * I] = \int_{-\infty}^t (I - p * I) d\tau,$$

where the associative equality

$$[1(\cdot) * (\delta(\cdot) - p)] * I = 1(\cdot) * [(\delta(\cdot) - p) * I]$$

follows from Fubini's theorem, assuming that

$$\int_{-\infty}^t |I| d\tau < \infty \quad (t \in \mathbb{R})$$

and using the assumption that p is a probability distribution. $\square$

Thus summarizing the above, four ways to describe the capital investment relation are

$$K = AI, \quad K = h * I, \quad \frac{d}{dt} K = I - BI, \quad \frac{d}{dt} K = I - p * I,$$

where A and B are a linear operators, h is a survival function and p is a retirement distribution.

16.5 Contents of Part IV

Chapter 17 searches for the user cost of capital and considers several types of capital investment relations. Because this book is concerned with nonparametric theory, this chapter considers nonparametric defined relations. This complicates the problem at first sight, but using a mathematical machinery, distribution theory (e.g. Zemanian, 1965), simple elegant results can be derived. To keep the book readable the developed theory is roughly described without going into details. In Chapter 18 the particular case is considered in which the discounted price of investment is exponentially decreasing. If this assumption is satisfied, one may easily derive the user cost of capital for any survival function. In that case it is not necessary to use the assumption that investments are exponentially decaying.

Another problem is the construction of the capital stocks. An easy way to construct capital stocks is considered in Chapter 19. Although this construction is based on a parametric specification, it will be shown that this specification is flexible enough to approach empirical capital investment relations.

17 USER COST OF CAPITAL

17.1 Introduction

Given the discounted price of the investments, what is the shadow price of capital? The determination of this price, the user cost of capital, is the subject of this chapter. It is of importance, because it may transform the dynamic optimization problem of a producer into a static problem.

Computation of the user cost of capital is relatively simple in case of exponential and rectangular survival functions. That is a reason why these survival functions are popular in empirical models. Applications of dynamic optimization to exponential survival functions are made by Jorgenson (1967) and others. A rigorous description of Jorgenson's theory and several extensions may be found in Takayama (1974). Takayama considers constraints on investments and the effect of adjustment costs. The maximum principle can directly be applied in case the survival function is exponential, which is a reason why the exponential survival function is popular in economic theory.

However, one may ask whether the exponential and rectangular functions are suitable as survival function for empirical investment data. The Netherlands Central Bureau of Statistics (1986) investigated the form of survival functions for several capital goods in the Dutch industry, and their results deviate clearly from the exponential and rectangular form.

In this chapter I will derive the user cost of capital for survival functions of a general form. For a complete theory there are two aspects that have to be treated: the determination of the user cost of capital and the generation of capital stocks. The latter will be treated in Chapter 19.

17.2 A General User Cost of Capital Solution

Suppose there is a linear relation between capital $K(t)$ and investment $I(t)$, of the form $K = AI$, where A is a linear operator. This operator maps each admissible investment function $I(\cdot)$ on a capital stock function $K(\cdot)$. I use the vague term 'admissible' to avoid the introduction of restrictions, necessary for a rigorous description, but not necessary to explain the essential theory.

Before I derive the solution of the user cost of capital, I have to describe the cost of investments. This is

$$\int_0^{\infty} p_I(t)I(t)dt,$$

which represents the present value of investment costs, using discounted investment price p_I . When we define the *innerproduct* $\langle v, w \rangle$ of two functions v and w as

$$\langle v, w \rangle \equiv \int_{-\infty}^{\infty} v(t)w(t)dt$$

and we put $p_I(t) = 0$ for all $t < 0$, the present value of the investment cost can be written as

$$\langle p_I, I \rangle \equiv \int_{-\infty}^{\infty} p_I(t)I(t)dt.$$

Now, the user cost of capital p_K , given a discounted investment price

function p_I , has to satisfy

$$\langle p_I, I \rangle = \langle p_K, K \rangle \quad (K = AI),$$

for every admissible investment flow I .

To derive the user cost of capital the following idea is crucial.

Definition: Given a linear operator L and an innerproduct $\langle \cdot, \cdot \rangle$, the adjoint operator $L^\dagger$ is the linear operator that satisfies

$$\langle L^\dagger v, w \rangle = \langle v, Lw \rangle,$$

for all admissible functions v and w .

Example 17.1 A simple example is the case in which L is a matrix and the vector innerproduct is $\langle v, w \rangle = v^t w$. In that case one has $L^\dagger = L^t$, because

$$\langle L^t v, w \rangle = (L^t v)^t w = v^t Lw = \langle v, Lw \rangle.$$

Example 17.2 For the linear operator $L = \frac{d}{dt}$ with innerproduct

$$\langle v, w \rangle = \int_{-\infty}^{\infty} v(t)w(t) dt$$

we have

$$\langle Lv, w \rangle + \langle v, Lw \rangle = \int_{-\infty}^{\infty} \frac{d}{dt} [vw] dt = vw \Big|_{-\infty}^{+\infty} = 0.$$

Where the last equality follows by supposing that $v(t)w(t)$ vanishes for $t \rightarrow \pm\infty$. We have then

$$\langle v, Lw \rangle = \langle -Lv, w \rangle = \langle L^\dagger v, w \rangle$$

and

$$\left[\frac{d}{dt} \right]^\dagger = -\frac{d}{dt}.$$

Now, suppose we can reconstruct the investment function $I(\cdot)$ from the capital stock function $K(\cdot)$. In other words: there is an inverse linear operator A^{-1} with the property $I = A^{-1}AI = A^{-1}K$.

Theorem 17.1: Suppose A is linear operator and there is an inverse operator A^{-1} that satisfies $I = A^{-1}AI = A^{-1}K$. Then the user cost of capital for the problem

$$\langle p_I, I \rangle = \langle p_K, K \rangle \quad (K = AI)$$

is

$$p_K = (A^{-1})^\dagger p_I.$$

Proof: For the user cost of capital we have

$$\langle p_I, I \rangle = \langle p_I, A^{-1}K \rangle = \langle (A^{-1})^\dagger p_I, K \rangle = \langle p_K, K \rangle. \quad \square$$

With the above theoretical outline we can solve the user cost of capital for the textbook assumption of exponential decay.

Example 17.3 We have for survival function $e^{-\lambda t}$ the relation

$$\frac{d}{dt} K = I - \lambda K.$$

From this we obtain

$$I = \left[\lambda + \frac{d}{dt} \right] K.$$

and the user cost of capital p_K is thus

$$p_K = \left[\lambda + \frac{d}{dt} \right]^\dagger p_I = \left[\lambda - \frac{d}{dt} \right] p_I,$$

using $\lambda^\dagger = \lambda$ and the relation

$$\left[\frac{d}{dt}\right]^\dagger = -\frac{d}{dt}$$

given in Example 17.2, which is valid when we assume that $p_I(t)I(t)$ vanishes for $t \rightarrow +\infty$. Note that $p_I(t)I(t)$ vanishes for $t \rightarrow -\infty$, since we suppose $p_I(t) = 0$ for all $t < 0$.

17.3 User Cost of Capital for Survival Functions

With distribution theory (e.g. see Zemanian, 1965) one can derive the user cost of capital when a survival function h is given. In that case capital is given by the equation

$$K = h * I = \int_{-\infty}^{\infty} h(t-\tau)I(\tau)d\tau.$$

I will solve the user cost of capital using Laplace transforms.

Definition: The Laplace transform of a function $f(\cdot)$ is defined as

$$\mathcal{L}[f](s) = \int_{-\infty}^{\infty} f(t)e^{-st} dt \quad (s \in \mathbb{C}).$$

For Laplace transforms I use the following short hand notation

$$\tilde{f} \equiv \mathcal{L}[f].$$

The Laplace transform has useful properties from which the user cost of capital can be derived.

Theorem 17.2: Suppose h is a survival function. Then the user cost of capital for the problem

$$\langle p_I, I \rangle = \langle p_K, K \rangle \quad (K = h * I)$$

is

$$p_K = \mathcal{L}^{-1}[\tilde{p}_I(s)/\tilde{h}(-s)].$$

Proof: Let L be the linear operator such that $Lf = h * f$, for every admissible function f . Now, the distribution theory derives, under weak conditions, the equalities

$$\mathcal{L}[Lf] = \tilde{h}\tilde{f}, \quad \mathcal{L}[L^{-1}f] = \tilde{f}/\tilde{h}, \quad \mathcal{L}[L^\dagger f] = \tilde{\tilde{h}}\tilde{\tilde{f}},$$

where the notation

$$\tilde{\tilde{h}}(s) = \tilde{h}(-s) \quad (s \in \mathbb{C})$$

is used. Using the above three properties, we may obtain the Laplace transform of the user cost of capital p_K . Concerning $AI = h * I$ in Theorem 17.1, these properties imply

$$\tilde{p}_K(s) = \mathcal{L}[(A^{-1})^\dagger p_I] = \tilde{p}_I(s)/\tilde{h}(-s).$$

Now, using the inversion $\mathcal{L}^{-1}$ of the Laplace transform, we derive the given user cost of capital. $\square$

Example 17.5 Let us consider the simple case of the exponential survival function

$$h(t) = 1(t)e^{-\lambda t},$$

where

$$1(t) \equiv \begin{cases} 0 & (t < 0), \\ 1 & (t \geq 0). \end{cases}$$

In that case we have $\tilde{h}(s) = (\lambda + s)^{-1}$ and we obtain

$$p_K = \mathcal{L}^{-1}[\tilde{p}_I(s)/\tilde{h}(-s)] = \mathcal{L}^{-1}[\tilde{p}_I(\lambda - s)] = (\lambda - \frac{d}{dt})p_I.$$

17.4 User Cost of Capital for Retirement Distributions

Given the retirement distribution for investment $p(t)$, the capital stock is governed by the equation

$$\frac{d}{dt}K = I - BI = (1 - B)I,$$

where $BI = p * I$. We may apply now Theorem 17.1 to derive the user cost of capital.

Theorem 17.3: Suppose B is a linear operator. Then the user cost of capital for the problem

$$\langle p_I, I \rangle = \langle p_K, K \rangle \quad \left(\frac{d}{dt}K = I - BI \right)$$

is

$$p_K = -\frac{d}{dt} \sum_{n=0}^{\infty} (B^\dagger)^n p_I,$$

where $(B^\dagger)^0 = 1$, $(B^\dagger)^2 = B^\dagger B^\dagger$, $(B^\dagger)^3 = B^\dagger B^\dagger B^\dagger$, etc.

Proof: We have

$$I = (1 - B)^{-1} \frac{d}{dt}K.$$

The inverse linear operator $(1 - B)^{-1}$ can be represented by using Neumann series

$$(1 - B)^{-1} = \sum_{n=0}^{\infty} B^n,$$

where $B^0 = 1$, $B^2 = BB$, $B^3 = BBB$, etc. Note that this solution is analogous to the solution in the case that B is a scalar. Now, we may apply Theorem 17.1 to derive

$$\begin{aligned} \langle p_K, K \rangle &= \langle p_I, (1 - B)^{-1} \frac{d}{dt}K \rangle = \langle p_I, \sum_{n=0}^{\infty} B^n \frac{d}{dt}K \rangle = \langle \frac{d}{dt} \sum_{n=0}^{\infty} (B^n)^\dagger p_I, K \rangle = \\ &= \langle -\frac{d}{dt} \sum_{n=0}^{\infty} (B^\dagger)^n p_I, K \rangle. \quad \square \end{aligned}$$

Now, it remains to solve the operator $B^\dagger$ in case B is a linear operator of the form $Bw = p * w$. Using the notation

$$\check{p}(t) \equiv p(-t) \quad (t \in \mathbb{R}),$$

the solution of this problem is $B^\dagger w = \check{p} * w$. Using this expression in Theorem 17.3 solves the user cost of capital in case a retirement distribution p is given.

Theorem 17.4: Suppose p is a retirement distribution. Then the user cost of capital for the problem

$$\langle p_I, I \rangle = \langle p_K, K \rangle \quad \left(\frac{d}{dt}K = I - p * I \right)$$

is

$$p_K = -\frac{d}{dt} \sum_{n=0}^{\infty} (\check{p})^n * p_I \quad (t \geq 0),$$

where $(\check{p})^0 = \delta(\cdot)$ and $(\check{p})^{n+1} = \check{p} * (\check{p})^n$.

Proof: Fubini's theorem implies the equality $\langle v, p * w \rangle = \langle \check{p} * v, w \rangle$, when the weak condition $\langle |v|, |p| * |w| \rangle < \infty$ is satisfied. Hence, the result follows from Theorem 17.3. $\square$

Example 17.6 As an example I will apply the above equation, which describes the user cost of capital p_K , to the rectangular survival function

$$h(t) = 1 - \int_0^t \delta(\tau - T) d\tau \quad (t \geq 0),$$

where the retirement distribution is $p(t) = \delta(t - T)$. Hence

$$(\check{p}(t))^n = \delta(t + nT) \quad (n \in \mathbb{N}),$$

and from the given equation for the user cost of capital, we derive

$$p_K(t) = -\frac{d}{dt} \sum_{n=0}^{\infty} (\check{p})^n * p_I = -\frac{d}{dt} \sum_{n=0}^{\infty} p_I(t + nT) \quad (t \geq 0).$$

17.5 Relation with Measure Theory

In the above sections the user cost of capital is derived with a direct approach. An indirect way to derive the user cost of capital is the following use of measure theory. Suppose the capital investment relation $K = AI$ is invariant under time translation. With the latter I mean

$$AT_a I = T_a AI \quad (a \in \mathbb{R}),$$

where translation operator T_a is defined as

$$(T_a f)(t) = f(t + a) \quad (t, a \in \mathbb{R}),$$

for arbitrary functions f .

Remark: In quantum mechanics there is a symmetry for a linear operator A when there is a linear operator B such that $[A, B] = AB - BA = 0$. This is the case for the translation operator $B = T_a$.

A descriptive rule to solve the user cost of capital is now the following. First solve the investment flow I_1 that generates one unity of capital of an infinite lifetime, thus $AI_1 = 1(\cdot)$. Then we may obtain the user cost of capital from the following theorem.

Theorem 17.5: Suppose A is a linear operator, invariant under time translation, and $AI_1 = 1(\cdot)$. Then the user cost of capital for the problem

$$\langle p_I, I \rangle = \langle p_K, K \rangle \quad (K = AI)$$

is

$$p_K = -\frac{d}{dt} \check{I}_1 * p_I,$$

where we use the notation

$$\check{I}_1(t) = I_1(-t) \quad (t \in \mathbb{R}).$$

Proof: The equality $AI_1 = 1(\cdot)$ and A translation invariant means that we have

$$T_t 1(\cdot) = T_t AI_1 = AT_t I_1 \quad (t \in \mathbb{R}),$$

where T_t is the translation operator. Because the user cost of capital p_K , given a function p_I , satisfies $\langle p_K, AI \rangle = \langle p_I, I \rangle$, we have thus

$$\langle p_K, AT_t I_1 \rangle = \langle p_K, T_t 1(\cdot) \rangle = \langle p_t, T_t I_1 \rangle \quad (t \in \mathbb{R}).$$

So, we have

$$\begin{aligned} p_K(t) &= \frac{d}{dt} \int_t^\infty p_K(\tau) d\tau = -\frac{d}{dt} \langle p_K, T_t 1(\cdot) \rangle \\ &= -\frac{d}{dt} \langle p_t, T_t I_1 \rangle = -\frac{d}{dt} (\check{I}_1 * p_t)(t) \quad (t \in \mathbb{R}). \end{aligned}$$

From the above equation for the user cost of capital p_K and the result in the previous section, which was

$$\frac{d}{dt} K = I - p * I \Rightarrow p_K = -\frac{d}{dt} \sum_{n=0}^{\infty} (\check{p})^n * p_t, \text{ where } (\check{p})^0 = \delta(\cdot), (\check{p})^{n+1} = \check{p} * (\check{p})^n,$$

we may expect that we have

$$\check{I}_1 = \sum_{n=0}^{\infty} (\check{p})^n.$$

This conjecture is indeed correct as is shown below.

Figure 17.1 Capital Rectangular Function

Theorem 17.6: *The equation*

$$\frac{d}{dt} 1(\cdot) = I_1 - p * I_1$$

is solved by

$$\check{I}_1 = \sum_{n=0}^{\infty} (\check{p})^n.$$

Proof: For this solution one has

$$I_1 = \sum_{n=0}^{\infty} p^n$$

and

$$\frac{d}{dt} 1(\cdot) = \delta(\cdot) = p^0 = \sum_{n=0}^{\infty} p^n - p * \sum_{n=0}^{\infty} p^n = I_1 - p * I_1. \quad \square$$

In the following example the results are applied to the rectangular survival function.

Example 17.6 To build a unit capital stock of an infinite lifetime, using a rectangular survival function, we have to repeat our investment after the lifetime T . This is displayed in Figure 17.1. Thus the investment flow I_1 consists of a series delta distributions and is of the form

$$I_1(t) = \sum_{n=0}^{\infty} \delta(t+nT) \quad (t \in \mathbb{R}).$$

I will now use the above method to clarify why future investment prices play no part in the user cost of capital in case of exponentially decaying investments. As I will show below, this is caused by the fact that new and partly decayed investments are indistinguishable in the case of exponential decay.

Example 17.7 let the capital investment relation be given as

$$\frac{d}{dt} K = I - \lambda K, \quad K(-\infty) = 0.$$

To build capital $K(t) = 1(t)$, we may use the investment flow

$$I_1(t) = \delta(t) + \lambda 1(t),$$

because we have then

$$\frac{d}{dt} K = \frac{d}{dt} 1(t) = \delta(t) = \delta(t) + \lambda 1(t) - \lambda 1(t) = I_1 - \lambda 1(t) = I_1 - \lambda K.$$

So, the given capital investment relation

$$\frac{d}{dt} K = I_1 - \lambda K, \quad K(-\infty) = 0.$$

is satisfied. The above derived user cost of capital solution is thus

$$\begin{aligned} p_K &= -\frac{d}{dt} p_I * \tilde{I}_1 = -\frac{d}{dt} \langle p_I, T_t[\delta(\cdot) + \lambda 1(\cdot)] \rangle \\ &= -\frac{d}{dt} [p_I + \int_t^{\infty} p_I(\tau) \lambda d\tau] = -\frac{d}{dt} p_I + \lambda p_I. \end{aligned}$$

The future investment price are of no importance in this example, because there is no difference between new and decayed investments in case of exponentially decaying investments. This leads to the constant weight measure λ that appears in the formula

$$\int_t^{\infty} p_I(\tau) \lambda d\tau,$$

which is differentiated in order to compute p_K . Only because the weight measure λ is constant in this formula, the future investment prices $p_I(\tau)$, $\tau > t$, have no chance to play a part in the user cost of capital $p_K(t)$.

17.6 A Generalization

One may generalize the theory in this chapter in case investment cost is nonlinear. In that case one should use a *dynamic multiplier* in a similar way as in the maximum principle of Pontryagin (1962). For general forms of survival functions we cannot use the standard theory, but we may apply the following modified approach.

The general problem is

$$\max \int g(K, I, t) dt,$$

where $K = AI$. To convert this problem to an unconstrained maximization problem, we adjoin this constraint using a dynamic Lagrange multiplier η . Now, we obtain

$$\max_{K, I} \int [g - \eta(K - AI)] dt = \max_{K, I} \int [g - \eta K + (A^\dagger \eta) I] dt.$$

The latter integral may be maximized over I and K before integration, which implies the first order conditions

$$\frac{\partial}{\partial I} g + A^\dagger \eta = 0, \quad \frac{\partial}{\partial K} g - \eta = 0.$$

So, a solution of the optimization problem has to satisfy the restriction

$$\frac{\partial}{\partial I} g + A^\dagger \left(\frac{\partial}{\partial K} g \right) = 0.$$

In this chapter the case was investigated in which g is linear in I . In that case g is of the form $g(K, I, t) = f(K, t) - p_I(t)I$. Hence, we have

$$\frac{\partial}{\partial I} g = -p_I,$$

and we can solve the dynamic Lagrange multiplier η from

$$-p_I + A^\dagger \left(\frac{\partial}{\partial K} g \right) = 0 \Rightarrow \eta = \frac{\partial}{\partial K} g = (A^\dagger)^{-1} p_I.$$

This means that we are left with the maximization problem

$$\max_K f(K, t) - \eta K, \quad \eta = (A^\dagger)^{-1} p_I,$$

where η represents the user cost of capital.

Another case of interest is the case in which g is linear in K . In that case we have $g(K, I, t) = p_K(t)K - c(I, t)$, where p_K represents the discounted marginal revenue of capital. Now, we have

$$\eta = \frac{\partial}{\partial K} g = p_K,$$

and we are left with the following maximization problem

$$\max_I (A^\dagger \eta) I - c(I, t), \quad \eta = p_K,$$

where $(A^\dagger \eta)I$ is the discounted revenue of investment and c the discounted investment cost.

17.7 Summary and Conclusions

The basis of the theory concerning the user cost of capital is the capital investment relation $K = AI$, where A is linear operator that maps investment flow $I(\cdot)$ on a capital stock $K(\cdot)$. With the idea of the user cost of capital we may eliminate the investment flow $I(\cdot)$ from a given dynamic optimization problem of the form

$$\max_I \int_0^\infty [f(K, t) - p_I(t)I] dt,$$

where $K = AI$. This elimination transforms the dynamic optimization problem into a static problem of the form

$$\max_K \int_0^\infty [f(K, t) - p_K(t)K] dt,$$

which contains only the capital stock K as a variable. Now, we may optimize before integration in contrast to the original dynamic problem. Furthermore, we may use the relation $I = A^{-1}K$ to derive the optimal investment flow from the optimal capital stock solution.

Below a short summary of the results is given. Suppose the relation of the capital stock K with investment I is described, using a linear operator A , a survival function h , and a retirement distribution p respectively. In that case we obtain a corresponding relation between the user cost of capital p_K and the discounted investment price p_I . In this chapter the following relations are derived:

$$K = AI \Rightarrow p_K = (A^{-1})^{\dagger} p_I,$$

$$K = h * I \Rightarrow p_K = \mathcal{L}^{-1}[\tilde{p}_I(s)/\tilde{h}(-s)],$$

where $\tilde{f} = \mathcal{L}(f)$ denotes the Laplace transform of f .

Using a retirement flow and a linear operator B , we have derived

$$\frac{d}{dt} K = I - BI \Rightarrow p_K = -\frac{d}{dt} \sum_{n=0}^{\infty} (B^{\dagger})^n p_I, \text{ where } (B^{\dagger})^0 = 1, (B^{\dagger})^{n+1} = B^{\dagger}(B^{\dagger})^n.$$

In case of a retirement distribution p , we have derived

$$\frac{d}{dt} K = I - p * I \Rightarrow p_K = -\frac{d}{dt} \sum_{n=0}^{\infty} (\check{p})^n * p_I, \text{ where } (\check{p})^0 = \delta(\cdot), (\check{p})^{n+1} = \check{p} * (\check{p})^n,$$

where the notation

$$\check{f}(t) \equiv f(-t) \quad (t \in \mathbb{R})$$

is used. A result that is related to the latter result is

$$K = AI, A \text{ time-independent}, AI_1 = 1(\cdot) \Rightarrow p_K = -\frac{d}{dt} \check{I}_1 * p_I.$$

What is the global lesson we can learn from the above theory? In the standard case in literature, where the investment survival function is of the exponential form, the current user cost of capital depends on the current discounted investment price and its derivative. This allows a myopic producer to be an optimizer. However, in general an optimizing producer has to look forward in the future. Concerning survival functions that are not of the exponential form, the user cost of capital depends on future values of the discounted investment price.

One may generalize the theory in this chapter in order to apply it to nonlinear investment costs. In that case one should use dynamic multipliers in a similar way as in the maximum principle of Pontryagin (1962).

Finally, I have to make a remark about the theory in this chapter. I have not considered restrictions on the investment flow. For example, it may be required that investments are positive. The case in which such a restriction becomes active is not covered by the given theory.

18 EXPONENTIAL DISCOUNTED INVESTMENT PRICE

18.1 Introduction

In the literature one often uses exponentially decreasing discounted investment prices. In this chapter we shall solve the user cost of capital for this particular case. The given solution may be applied to any investment capital relation that is derived from a survival function. I have used the results in this chapter to compute the user cost of capital for the Dutch industry data. These data are used in this book in order to show applications of the developed producer theory.

18.2 Solution for the Exponential Price

Suppose the discounted investment price p_I is of the form

$$p_I(t) = p_I(0)e^{-rt} \quad (t \geq 0),$$

given a discount rate $r > 0$. Given an investment survival function h , the previous chapter states that the user cost of capital is equal to

$$p_K = \mathcal{L}^{-1}[\tilde{p}_I/\tilde{h}].$$

This expression is the inversion of the Laplace transform $\tilde{p}_I/\tilde{h}$, where $\tilde{p}_I$ and $\tilde{h}$ are the Laplace transforms of p_I and h respectively. For example, the Laplace transform $\tilde{h}$ of h is defined as

$$\tilde{h}(r) \equiv \int_{-\infty}^{\infty} h(t)e^{-rt} dt.$$

Furthermore, the notation $\tilde{h}(s) = \tilde{h}(-s)$, for $s \in \mathbb{C}$, is used.

Given the form of the discounted investment p_I , we may derive a simple expression for the user cost of capital. This expression is given in the following theorem.

Theorem 18.1: Suppose h is a survival function and

$$p_I(t) = p_I(0)e^{-rt} \quad (t \geq 0).$$

Then the user cost of capital for the problem

$$\langle p_I, I \rangle = \langle p_K, K \rangle \quad (K = h * I),$$

in which we assume $I(t) = 0$ for all $t < 0$, is

$$p_K = p_I/\tilde{h}(r).$$

Proof: The assumption $h(t) = I(t) = 0$ for all $t < 0$, implies $K(t) = (h * I)(t) = 0$ for all $t < 0$. Hence, we have $\langle p_I, I \rangle = p_I(0)\tilde{I}(r)$ and $\langle p_I, K \rangle = p_I(0)\tilde{K}(r)$. So, because by the convolution theorem given in Appendix D, we have

$$K = h * I \Rightarrow \tilde{K}(r) = \tilde{h}(r)\tilde{I}(r),$$

we obtain

$$\begin{aligned} \langle p_I, I \rangle &= p_I(0)\tilde{I}(r) = \tilde{h}(r)^{-1}p_I(0)\tilde{h}(r)\tilde{I}(r) = \tilde{h}(r)^{-1}p_I(0)\tilde{K}(r) \\ &= \langle \tilde{h}(r)^{-1}p_I, K \rangle = \langle p_K, K \rangle. \end{aligned}$$

□

Remark: An alternative way to prove the above theorem is the use of the inversion formula for the Laplace transform

$$f(t) = (2\pi)^{-1} \int_{c-i\infty}^{c+i\infty} \tilde{f}(s) e^{st} ds.$$

So, we may compute the user cost of capital as

$$p_K(t) = (2\pi)^{-1} \int_{c-i\infty}^{c+i\infty} [\tilde{p}_I(s)/\tilde{h}(-s)] e^{st} ds,$$

where

$$\tilde{p}_I(s) = \int_{-\infty}^{\infty} p_I(t) e^{-rt} dt = p_I(0)(s+r)^{-1}.$$

In the general case we suppose that the past investment flow

$$I_-(t) = \begin{cases} I(t) & (t < 0), \\ 0 & (\text{otherwise}), \end{cases}$$

is not equal to zero everywhere. Then we have $I(t) - I_-(t) = 0$ for all $t < 0$. Hence, by the above theorem we have the equalities

$$\langle p_I, I - I_- \rangle = \langle p_K, h * (I - I_-) \rangle, \quad p_K = \tilde{h}(r)^{-1} p_I.$$

Moreover, $p_I(t) = 0$ for all $t < 0$, implies $\langle p_I, I \rangle = \langle p_I, I - I_- \rangle$. Hence, we have

$$\langle p_I, I \rangle = \langle p_I, I - I_- \rangle = \langle p_K, h * (I - I_-) \rangle = \langle p_K, K \rangle - \langle p_K, h * I_- \rangle.$$

The subtracted cost $\langle p_K, h * I_- \rangle$, due to past investments I_- , is a given constant, because p_K , h , and the past investments I_- are known. Thus in the general case we may interpret the given solution p_K as the marginal cost of capital.

18.3 Critique on the User Cost of Capital in Empirical Models

The theory in the previous section is of interest, because many economic models are based on the assumption that the discounted investment price is of the form

$$p_I(t) = p_I(0) e^{-rt} \quad (t \geq 0),$$

where r is a given discount rate. In this special case we can easily derive the user cost of capital, given an arbitrary survival function, as is shown in the previous section. This means a freedom to choose a survival function in empirical economic models, which is not found in current literature. Nearly all investment survival functions one encounters in empirical economic models are of the exponential or rectangular form. However, if we may believe the results in a publication of the Netherlands Central Bureau of Statistics (1986), a better form of the investment survival function can be derived from a gamma retirement distribution.

The gamma probability distributions are of the form

$$p(t) = \lambda^\alpha t^{\alpha-1} e^{-\lambda t} / \Gamma(\alpha) \quad (t \geq 0),$$

where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

One has for α the relation $\alpha = (\mu/\sigma)^2$, where μ is the mean and σ^2 the variance of the gamma distribution. For $\alpha = 1$, the corresponding survival function

$$h(t) = 1 - \int_0^t p(\tau) d\tau \quad (t \geq 0)$$

is exponential, while for $\alpha \rightarrow \infty$ - keeping μ constant - the limiting

corresponding survival function is rectangular. The Netherlands Central Bureau of Statistics (1986) found values of α between 2 and 3.5, when the retirement of different capital assets was investigated.

Figure 18.1 User Cost Capital

The question what the error may be, when we use the exponential or rectangular survival function instead of a more realistic survival function, is of interest. From Theorem 18.1 we derive for gamma investment retirement distributions with an exponential discounted investment price the equality

$$p_K/p_I = \tilde{h}(r)^{-1} = r[1 - (1 + r/\lambda)^{-\alpha}]^{-1}.$$

Figure 18.1 compares the above expression in case of the exponential ($\alpha=1$) and the rectangular ($\alpha \rightarrow \infty$) function to the case in which a more realistic form ($\alpha=2,4$) is chosen. The mean investment lifetime μ is in all cases equal to 15 years. It appears that the exponential survival function overestimates the value of the user cost of capital with an error of approximately 10%, while the rectangular function underestimates the user cost of capital with an error of the same magnitude. As is evident from this example, it is not only the mean investment lifetime that determines the user cost of capital, but also the shape of the retirement curve.

18.4 Other Survival Functions

The previous section criticizes the use of survival functions in empirical models and specifies a survival function, which might be a better choice. One may reply again that it is doubtful whether this specific form of the survival function is a good choice. In this section I will argue that it is not the specific parametric form of the survival function that is the most important, but the statistical properties of the corresponding retirement distribution. To make a good choice of a survival function, it is usually sufficient to specify the mean and variance of the corresponding retirement distribution. Because this specification is possible for the family of gamma distributions, we may use this family to compute the user cost of capital.

Let us consider the case in which investment cost is of the form

$$p_I(t) = p_I(0)e^{-rt} \quad (t \geq 0).$$

Then the user cost of capital is equal to $p_K = p_I/\tilde{h}(r)$, where $\tilde{h}$ is the Laplace transform of the survival function h . The following theorem yields an approximation of $\tilde{h}(r)$ in case r is small.

Theorem 18.2: For survival functions h one has

$$\tilde{h}(r) \approx r^{-1}[1 - e^{-r\mu}(1 + \sigma^2 r^2/2)] \quad (r \rightarrow 0),$$

where μ is the mean and σ^2 is the variance of the corresponding retirement distribution p .

Proof: One can expand the Laplace transform of a probability distribution p around $t = \mu$ as follows.

$$\tilde{p}(s) = e^{-s\mu} \left[1 + \sum_{n=2}^{\infty} \mu_n (-s)^n / n! \right],$$

where

$$\mu_n = \int_{-\infty}^{\infty} p(t)(t - \mu)^n dt$$

is the n -th central moment of the probability distribution p . This follows from

$$\begin{aligned} \tilde{p}(s) &= \int_{-\infty}^{\infty} p(t)e^{-st} dt = e^{-s\mu} \int_{-\infty}^{\infty} p(t)e^{-s(t-\mu)} dt \\ &= e^{-s\mu} \int_{-\infty}^{\infty} p(t) \sum_{n=1}^{\infty} [-s(t-\mu)]^n / n! dt \\ &= e^{-s\mu} \left[1 + \sum_{n=2}^{\infty} \mu_n (-s)^n / n! \right]. \end{aligned}$$

For $|s|$ small, we may approximate $\tilde{p}$ by the first two terms of this expansion, and we obtain

$$\tilde{p}(s) \approx e^{-s\mu} (1 + \sigma^2 s^2/2),$$

where $\sigma^2 = \mu_2$ is the variance of the retirement distribution. Thus the Laplace transform of the survival function can be approximated by

$$\tilde{h}(s) \approx s^{-1} [1 - e^{-s\mu} (1 + \sigma^2 s^2/2)] \quad (|s| \rightarrow 0),$$

which depends only on the mean μ and variance σ^2 of the retirement distribution. Note that in case of symmetric retirement distributions one has

$$\mu_n = 0 \quad (n \text{ odd and } n > 1),$$

and then the given approximation of $\tilde{p}(s)$ is in fact a third order approximation. $\square$

Figure 18.2 compares the approximation of the user cost of capital, given in the above theorem, to the results in Figure 18.1. There are three examples of approximation given, which appear just above the user cost that is approximated. The fourth approximation of the rectangular survival function

$$\tilde{h}(s) = s^{-1} (1 - e^{-s\mu}) \quad (s \in \mathbb{C})$$

is exact, because in that case all higher central moments are equal to zero. From Figure 18.2 it appears that the approximation works well in case of gamma distributions with parameter α larger than one. For $\alpha=1$ the approximation does not work very well, because the approximation is based on an expansion around the mean, assuming that the mean is in the neighbourhood of the mode, and the mean and the mode of the exponential retirement

distribution do not coincide at all. The approximation works good in case α is larger than one, because the distribution is more symmetric. For α in the area where the survival function has a more realistic form the fit is very good.

Figure 18.2 Approximation User Cost

We can give another simple approximation when r is high. Then the user cost of capital will mainly depend on r . When p is a probability distribution then we may show that $\tilde{p}(r)$ approaches zero, when r increases, and we have then

$$p_K/p_I = \tilde{h}(r)^{-1} = r[1 - \tilde{p}(r)]^{-1} \approx r.$$

We may summarize the above results, concerning the determination of the user cost of capital, as follows. When the discounted investment price is of the exponential form

$$p_I(t) = p_I(0)e^{-rt} \quad (t \geq 0)$$

and the investment retirement distribution is maximal in the neighbourhood of the mean investment lifetime, then the user cost of capital is approximately determined by the mean and variance of the retirement distribution.

18.5 Summary

A nice result is the simple solution of the user cost of capital p_K in case capital K can be described as $K = h * I$ and the discounted investment price

$$p_I(t) = p_I(0)e^{-rt}$$

is of the exponential form. In this special case the user cost of capital p_K is equal to $p_I/h(r)$.

Furthermore, it is shown that the parametric specification of the investment survival function h is relatively unimportant in case the discount rate r is low. Then it are mainly the statistical properties of the corresponding retirement distribution, like mean and variance of the investment lifetime, which determine the user cost of capital. It is shown that the shape of the retirement curve has a substantial influence on the user cost of capital

solution. Hence, it is not sufficient to consider only the mean lifetime of capital assets, but one should also consider the variance of the lifetime.

In this chapter the user cost of capital for a given exponential survival function is compared with the user cost of capital derived from a gamma retirement distribution, that has a more realistic shape. It appears that, when the discounted investment price is exponentially decreasing, the exponential survival function overestimates and the rectangular survival function underestimates the user cost of capital.

19 GENERATION OF CAPITAL STOCKS

19.1 Introduction

With the perpetual inventory model the capital stock K is described as

$$K = h * I = \int_{-\infty}^{\infty} h(t-\tau)I(\tau) d\tau,$$

where I is investment flow and h the survival function of investments. A drawback to the generation of capital stocks, using the above equation, is the evaluation of the integral for every t . Another drawback is the need for investment data, covering a long period in the past. These drawbacks are not present when the capital stock is generated by a linear system of the form

$$\dot{x} = Ax + bI, \quad K = cx,$$

where A is a real matrix, b a real vector and c a transposed real vector. In that case it is sufficient to know the initial value $x(0)$, to start the simulation at $t=0$. Such a linear system generates the same results as a perpetual inventory model with survival function

$$h(t) = ce^{At}b \quad (t \in \mathbb{R}),$$

where

$$e^{At} \equiv \sum_{n=0}^{\infty} (At)^n/n! \quad (t \in \mathbb{R}),$$

e.g. see Kailath (1980, page 70). The triple (A, b, c) is said to be a *state-space realization* for survival function h .

So, we may simplify the generation of capital stocks, using a state-space realization for the investment survival function. However, not every survival function has a simple state-space realization. This urges a family of survival functions with the following desired properties:

1. Every member has a simple state-space realization.
2. The shape of the members approximates the empirical shape of the investment survival function.
3. Every given positive mean and variance of the corresponding retirement distribution has a representative member in the family.

Such a family is useful for generating capital stocks, because retirement distributions with the same mean and variance tend to generate similar capital stocks. As I will point out below, the latter is especially the case for investment flows, which can be described as the superposition of a slowly varying function and fast fluctuations. Of course, the generation of capital stocks is obviously improved when the actual shape of the retirement distribution is approximated.

That the statistical properties of the retirement distribution are important follows from the results in Chapter 18. There it was shown that the Laplace transform of the corresponding survival function h may be approximated by

$$\tilde{h}(s) \approx s^{-1}[1 - e^{-s\mu}(1 + \sigma^2 s^2/2)] \quad (s \in \mathbb{C}: |s| \neq 0).$$

This means that retirement distributions with the same mean and variance tend to generate similar capital stocks when the variation in the investment flow is small. Also this is the case for fast fluctuations in the investment flow, because one has

$$\tilde{h}(s) \approx s^{-1} \quad (s \in \mathbb{C}: |s| \rightarrow \infty).$$

This follows from $\tilde{h}(s) = s^{-1}[1 - \tilde{p}(s)]$, where p is the investment retirement distribution, and because probability distributions p satisfy $\tilde{p}(s) \rightarrow 0$ if $|s| \rightarrow \infty$. Hence, investment retirement distributions with the same mean and variance tend to generate similar capital stocks in case of a superposition of slow and fast investment fluctuations. Extension of the above results to a superposition is allowed, since the capital stocks are generated by a linear system.

19.2 Contagious Erlang Distributions

In Paccoud (1983) several families of probability distributions are given, which are used as retirement distribution to generate capital stocks. One of these families is the family of gamma distributions. The Statistisches Bundesamt in Germany generates capital stocks using gamma retirement distributions. Also in Berends-Ballast (1987), where several parametric families of distributions are investigated, the best fit to empirical data was achieved by using gamma distributions.

However, not every gamma distribution has a simple state-space realization. The Laplace transform of a gamma distribution is given by

$$\tilde{p}(s) = (1 + s/\lambda)^{-\alpha} \quad (s \in \mathbb{C}),$$

and the Laplace transform of the corresponding survival function is

$$\tilde{h}(s) = s^{-1}[1 - (1 + s/\lambda)^{-\alpha}] \quad (s \in \mathbb{C}).$$

For the existence of a state-space realization, given a survival function h , it is necessary that one can write $\tilde{h}(s) = a(s)/b(s)$, where $a(s)$ and $b(s)$ are polynomials, e.g. see Kailath (1980, page 66–69). So gamma distributions, such that α is integer, may be used to generate capital stocks with a state-space realization. For $\alpha = 1$ one obtains the exponential survival function, which is widely used. Further, the case $\alpha = 2$ is used in Almon *et al.* (1974) and Hahn and Schmoranz (1984). Finally, Stahmer (1983) propagates the use of the general case, where α is integer.

Definition: The subfamily of gamma distributions of the form

$$p_k(\lambda; t) = 1(t)\lambda^k t^{k-1} e^{-\lambda t} / (k-1)! \quad (k \in \mathbb{N}, \lambda > 0, t \in \mathbb{R})$$

are called *Erlang distributions*.

The Laplace transform of such an Erlang distribution is

$$\tilde{p}_k(\lambda; s) = (1 + s/\lambda)^{-k} \quad (s \in \mathbb{C}).$$

The family of Erlang distributions does not contain a representative member for every positive mean and variance. Therefore I define a family of contagious Erlang distributions.

Definition: A *contagious Erlang distribution* is defined as a weighted mean of two Erlang distributions

$$p(t) = \beta p_{k-1}(\lambda; t) + (1 - \beta) p_k(\lambda; t) \quad (0 \leq \beta < 1, \lambda > 0, k \in \mathbb{N}, t \in \mathbb{R}).$$

The mean and variance of contagious Erlang distributions are as follows.

Theorem 19.1: Suppose

$$p(t) = \beta p_{k-1}(\lambda; t) + (1 - \beta) p_k(\lambda; t) \quad (t \in \mathbb{R}),$$

is a contagious Erlang distribution. Then

$$\mu = E[t] = (k - \beta)/\lambda, \quad \sigma^2 = E[(t - \mu)^2] = (k - \beta^2)/\lambda^2.$$

Proof: We obtain

$$E[t] = \int t[\beta p_{k-1}(t) + (1-\beta)p_k(t)] dt = \beta(k-1)/\lambda + (1-\beta)k/\lambda = (k-\beta)/\lambda.$$

$$\begin{aligned} E[(t-\mu)^2] &= E[t^2] - \mu^2 = \int t^2[\beta p_{k-1}(t) + (1-\beta)p_k(t)] dt - \mu^2 \\ &= \beta[(k-1)^2/\lambda^2 + (k-1)/\lambda^2] + (1-\beta)[k^2/\lambda^2 + k/\lambda^2] - (k-\beta)^2/\lambda^2 \\ &= (k-\beta^2)/\lambda^2. \end{aligned}$$

□

Figure 19.1 Gamma Approximation

Now I show that for every given positive mean and variance there is a unique contagious Erlang distribution.

Theorem 19.2: Suppose $\mu > 0$ and $\sigma^2 > 0$. Then there exists a unique contagious Erlang distribution with mean μ and variance σ^2 , for which the parameters k , λ and β are determined by

$$k-1 < \mu^2/\sigma^2 \leq k, \quad \lambda = \left\{ \mu + [(\mu^2 + \sigma^2)/k - \sigma^2]^{1/2} \right\} / \left\{ (\mu^2 + \sigma^2)/k \right\}, \quad \beta = k - \lambda\mu,$$

where $k \in \mathbb{N}$.

Proof: From $\mu = (k-\beta)/\lambda$ follows $\beta = k - \lambda\mu$. Hence, one has

$$\sigma^2 = (k-\beta^2)/\lambda^2 = [k - (k - \lambda\mu)^2]/\lambda^2,$$

which can be written as

$$(\mu^2 + \sigma^2)\lambda^2 - 2k\mu\lambda + k^2 - k = 0.$$

Since λ is positive, this means

$$\begin{aligned} \lambda &= \left\{ 2k\mu + [(2k\mu)^2 - 4(\mu^2 + \sigma^2)(k^2 - k)]^{1/2} \right\} / 2(\mu^2 + \sigma^2) \\ &= \left\{ \mu + [(\mu^2 + \sigma^2)/k - \sigma^2]^{1/2} \right\} / \left\{ (\mu^2 + \sigma^2)/k \right\}. \end{aligned}$$

This means that one should have $(\mu^2 + \sigma^2)/k - \sigma^2 \geq 0$ and thus $k-1 \leq \mu^2/\sigma^2$. Furthermore, since $0 \leq \beta \leq 1$, one has $\beta^2 \leq \beta$ and thus

$$\mu^2/\sigma^2 = (k-\beta)^2/(k-\beta^2) \leq k-\beta \leq k. \quad \square$$

For $\alpha \geq 2$, the contagious Erlang distributions appear to be a good approximation of gamma distributions, which have the same mean and variance. An example of such an approximation is given in Figure 19.1, where a gamma distribution is considered with $\alpha = 2.5$.

19.3 State-Space Realization for a Contagious Erlang Distribution

In the above section I described the use of contagious Erlang distributions. It is shown that, given the mean and variance of the retirement distribution of capital, there exists a corresponding contagious Erlang distribution of the form

$$p(t) = \beta p_{k-1}(\lambda; t) + (1-\beta)p_k(\lambda; t) \quad (t \in \mathbb{R}).$$

In this section I derive the form of the corresponding state-space realization, which generates the capital stock with the equations

$$\dot{x} = Ax + bI, \quad K = cx.$$

This realization is

$$A = \begin{bmatrix} -\lambda & 0 & \cdots & 0 \\ \lambda & -\lambda & & \\ 0 & \lambda & & \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & \lambda - \lambda \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \quad c = [0 \cdots 0 \beta \ 1 - \beta],$$

where A is a $k \times k$ matrix.

The form of the survival function of an Erlang retirement distribution is shown in the following theorem.

Theorem 19.3: Suppose $\lambda > 0$ and $p_n(\lambda; t)$, $n = 1, \dots, k$, is a series of Erlang distributions. Then

$$1(\cdot) * (\delta(\cdot) - p_k) = \lambda^{-1} \sum_{n=1}^k p_n.$$

Proof: One can give a direct proof by using partial integration. The following proof with Laplace transforms is however easier. One has

$$\begin{aligned} \mathcal{L}[1(\cdot) * (\delta(\cdot) - p_k)] &= s^{-1}(1 - \tilde{p}_k) = s^{-1}[1 - (1 + s/\lambda)^{-k}] \\ &= s^{-1}[(1 + s/\lambda) - 1] \sum_{n=1}^k (1 + s/\lambda)^{-n} = \lambda^{-1} \sum_{n=1}^k (1 + s/\lambda)^{-n} \\ &= \lambda^{-1} \sum_{n=1}^k \tilde{p}_n. \end{aligned} \quad \square$$

When the above theorem is combined with the following theorem then the given state-space realization can be derived.

Theorem 19.4: Suppose $\lambda > 0$ and

$$A = \begin{bmatrix} -\lambda & 0 & \cdots & 0 \\ \lambda & -\lambda & & \\ 0 & \lambda & & \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & \lambda - \lambda \end{bmatrix},$$

where A is a $k \times k$ matrix. Then

$$e^{At} = \lambda^{-1} \begin{bmatrix} p_1(\lambda, t) & 0 & \cdots & 0 \\ p_2(\lambda, t) & p_1(\lambda, t) & & \\ \vdots & \vdots & \ddots & \vdots \\ p_k(\lambda, t) & \cdots & \cdots & p_1(\lambda, t) \end{bmatrix} \quad (t \in \mathbb{R}).$$

Proof: We may write $A = \lambda(L - 1)$, where 1 is the unity matrix and L satisfies $L^n = 0$ if $n \geq k$. Because L commutes with 1 , i.e. $L1 = 1L$, we have

$$e^{At} = e^{(L-1)\lambda t} = e^{L\lambda t} e^{-\lambda t},$$

thus we obtain

$$e^{At} = \left[\sum_{n=0}^{k-1} L^n \lambda^n t^n / n! \right] e^{-\lambda t} = \lambda^{-1} \begin{bmatrix} p_1(\lambda, t) & 0 & \cdots & 0 \\ p_2(\lambda, t) & p_1(\lambda, t) & & \\ \vdots & \vdots & \ddots & \vdots \\ p_k(\lambda, t) & \cdots & \cdots & p_1(\lambda, t) \end{bmatrix}. \quad \square$$

The above theorems prove that the given state-space realization corresponds to a contagious Erlang distribution.

Theorem 19.5: Suppose

$$p(t) = \beta p_{k-1}(\lambda; t) + (1 - \beta) p_k(\lambda; t) \quad (t \in \mathbb{R})$$

is a contagious Erlang distribution and put

$$A = \begin{bmatrix} -\lambda & 0 & \cdots & 0 \\ \lambda & -\lambda & & \\ 0 & \lambda & & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & -\lambda \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \quad c = [0 \cdots \beta \ 1 - \beta],$$

where A is a $k \times k$ matrix. Then (A, b, c) is a state-space realization for survival function

$$h = 1(\cdot) * (\delta(\cdot) - p).$$

Proof: Apply Theorem 19.3 and Theorem 19.4 to

$$h(t) = ce^{At}b \quad (t \in \mathbb{R}). \quad \square$$

19.4 Initialization of the State-Space Realization

Consider a state-space realization (A, b, c) as described above. To start the simulation of

$$\dot{x} = Ax + bI, \quad K = cx,$$

we need to initialize $x(0)$. I will give two ways to approximate $x(0)$. For the first approximation I consider a constant investment flow $I(t) = I(0)$, for all $t \leq 0$, in the past. For the second approximation I consider a fixed growth rate $s \in \mathbb{R}$ of the investment flow

$$I(t) = I(0)e^{st} \quad (t \leq 0),$$

in the past, and a known current retirement flow of investments $R(0)$.

Using Theorem 19.3 and 19.4, and the equality

$$x = [e^{At}b] * I,$$

we obtain $x_n = h_n * I$, for $n = 1, \dots, k$, where h_n is the survival function corresponding to an Erlang retirement distribution $p_n(\lambda; t)$. Hence, because

the mean lifetime for h_n is equal to n/λ , we obtain in case of a constant investment flow at $t \leq 0$ the approximation

$$x_n(0) = (n/\lambda)I(0) \quad (n = 1, \dots, k),$$

A more sophisticated approximation is derived by assuming the investment flow of the form

$$I(t) = I(0)e^{st} \quad (t \leq 0),$$

with a fixed rate of growth $s \in \mathbb{R}$. Then one has

$$x_n(0) = \tilde{h}_n(s)I(0) = s^{-1}[1 - (1 + s/\lambda)^{-n}]I(0) \quad (n = 1, \dots, k),$$

Now, to determine $x_n(0)$, we need to know the rate of growth s . The following method is useful when the retirement flow of investments $R(0)$ is known.

Since the rate of growth s is constant, we have the equality $R(0) = \tilde{p}(s)I(0)$, where $\tilde{p}$ is the retirement distribution. Because the contagious Erlang distribution approximates the gamma distribution with the same mean μ and variance σ^2 , we have

$$\tilde{p}(s) \approx (1 + s/\gamma)^{-\alpha} \quad (\alpha = \mu^2/\sigma^2, \gamma = \mu/\sigma^2).$$

By inverting this equation, we can approximate the rate of growth

$$s \approx [(R(0)/I(0))^{-1/\alpha} - 1]\gamma \quad (\alpha = \mu^2/\sigma^2, \gamma = \mu/\sigma^2).$$

19.5 The Discrete-Time Model

The theory in this chapter describes a continuous-time model. The empirical data, however, are discrete. The continuous-time model simplifies the mathematics, but to apply this model to empirical data it has to be reformulated into a discrete-time model. To derive the discrete-time model we replace $\dot{x} = Ax + bI$ by

$$[x(t + \Delta t) - x(t)]/\Delta t = Ax(t) + bI(t).$$

Thus the continuous model reformulated as a discrete-time model is

$$x^{t+1} = (I + \Delta t A)x^t + \Delta t bI^t, \quad K^t = cx^t.$$

Note that the length of simulation period Δt may not be too long. Otherwise the discrete-time model will not approximate the results of the continuous model.

SUMMARY

The subject "Nonparametric Consumer and Producer Analysis" is related to the nonparametric statistical theory. This statistical theory is not based on a parametric specified probability distribution, but postulates instead that the probability distribution is member of a nonparametric family of probability functions. Such a family is defined by means of symmetry properties of the probability distribution.

The analogy of this approach in economic theory is using a nonparametric specification of utility and production functions. One may postulate that observations, concerning consumer and producer behaviour, is generated by a function that satisfies such a nonparametric specification. By developing tests to check whether such a hypothesis is satisfied, one may test the significance of assumptions concerning economical behaviour and the form of the utility or production function. These nonparametric tests may be of help in case one chooses a parametric specification of economic behaviour.

This thesis describes many of these nonparametric tests. They are concerned with the economic behaviour of utility maximization, profit maximization and cost minimization, in relation to which the following nonparametric specifications are considered: linear homogeneity, homotheticity, separability, and technical progress. The nonparametric tests are applied to Dutch consumer and industry data, showing the possibilities which the theory offers. Moreover, as an experiment the nonparametric tests of consumer behaviour are applied to aggregate random data. The random data were aggregated in the same way as the Dutch consumer data were aggregated by the Netherlands Central Bureau of Statistics.

In general the nonparametric tests do not indicate which specification *has* to be used. However, they show whether specifications are incompatible with the data. So they indicate which specifications one should certainly *not* use. A clear example are the nonparametric test results in this book concerning the Dutch industry data. These results indicate that profit maximization, not including any assumption of technical progress, cannot be used as a behaviour model for the Dutch industry data. However, one *may* use a profit maximization model that includes technical progress. Another possibility, concerning these data, which does not assume technical progress, is the choice of a cost minimization model.

The results of the nonparametric tests show that one has to be careful in drawing conclusions from aggregate data. For it appeared that aggregate random data satisfied the hypothesis of consumer demand. This is caused by using an obvious rule concerning consumer demand: price times quantity equals expenditures. But prices and corresponding quantities tend to move in opposite direction through this rule, exactly what one expects of consumer behaviour. The belief in consumer behaviour acts in this case as a self-fulfilling prophecy.

In practice nonparametric tests concerning efficient economic behaviour nearly always reject the postulated hypothesis. Two ways are described to derive a measure of violation of the hypotheses. First, one may delete as few observations as possible from the data, to such an extent that the remaining observations do satisfy the hypothesis. Secondly, one may weaken the hypothesis by considering inefficient behaviour. In that case the maximal

efficiency, for which the data still satisfies the hypothesis, indicates how seriously the hypothesis is violated.

Although the emphasis in this book is put on the development of nonparametric tests, concerning consumer and producer behaviour, ample attention is paid to the problem of the derivation of prices and quantities of capital stocks. It is shown that the conventional exponential decay of capital assumption can be improved for both the generation of capital stocks and the user cost of capital. Assuming that the retirement of capital may be described by a gamma distribution, a simple method is presented that approximates the capital stocks by using a linear dynamic model. Furthermore, it is described how one may derive the user cost of capital for an arbitrary given investment retirement distribution.

Generalizations of the Nonparametric Theory

The nonparametric tests in this book are presented in the form of theorems. These theorems are proved by constructing a suitable utility or production function for the given data. Hence, if the data satisfy certain properties, like concave budget constraints, this implies certain properties concerning the constructed function. This means that many variations are possible on the theorems, in which properties of the constructed function are implied by assumptions concerning the data. Another generalization, which can often be achieved without problems, is the generalization from the single-output production function to the case where multiple outputs are allowed.

The differentiability of utility and production functions is not considered. An excellent short article of Chiappori and Rochet (1987) contains a proof that derives an infinitely differentiable utility function as the convolution of two functions. When one is interested in the existence of a differentiable utility or production function, their method can be applied to many results in this book.

Now that we have an overview of the results, we may search for connections between several nonparametric theorems with the aim to include them in a more general theory. Such a theory can answer the question for which specifications one may derive a nonparametric theorem. At the beginning of this research this was mainly a matter of trial and error. Later it became apparent why certain specifications are a problem and others not. The global issue in such a theory is the symmetry of the optimization problem. For example, linearly homogeneous cost minimization, for which the Lagrangian shows symmetry, did not give problems. This in contrast to the results in case of weak separability, for which the Lagrangian does not display such an obvious symmetry.

Applications of the Nonparametric Theory

What are the applications of the nonparametric theory? First, it can be used to get an overview of the properties of a data set. This may save time when searching for a parametric specification of the utility or production function. The nonparametric tests may serve in that case as a guide, showing the possibilities that one has. Next, the nonparametric theory can be a tool in solving theoretical problems. For example, problems which consider the existence of utility functions or production functions, concerning a given demand relation.

One application of the nonparametric approach, that is especially useful in a producer context, is the forecast of economic behaviour. This was not discussed in this book, but the tools are presented in the form of bounds on the production function, and bounds on technical efficiency and technical

progress. These bounds can be seen as conserved properties that are due to the symmetry of the optimization model, which generates the data. In physics one finds analogical examples of conserved properties, like energy and momentum, which also proceed from the fact that the Lagrangian is symmetric. The nonparametric approach, with symmetry assumptions as a starting-point, is a standard tool in advanced physics. For example, it has been used to predict the existence of new elementary particles. We might expect that such a nonparametric approach will also be of value in economics.

APPENDIX

A BINARY RELATIONS

A.1 Binary Relations

Let X be an arbitrary set of elements. A *binary relation* R on the set X is a subset of the Cartesian product $X \times X$, that is the set of ordered pairs (x, y) such that x and y are in X . If the ordered pair (x, y) belongs to R , one denotes indifferently $(x, y) \in R$ or xRy .

Given a binary relation R on X , the *complement* R^c is defined as the binary relation on X for which one has

$$xRy \Leftrightarrow yR^c x \quad (x, y \in X).$$

The *transitive closure* R^* is defined as the binary relation on X for which one has xR^*y if there exists a finite sequence $x = x^1, x^2, \dots, x^n = y$ in X such that

$$x^m R x^{m+1} \quad (m = 1, \dots, n-1).$$

A quick way to compute this closure, given by Warshall (1962), may be found in Appendix C.5.

A.2 Basic Properties of Binary Relations

Let R be a binary relation on the set X , and let x, y and z be arbitrary elements in X . Then R is:

• reflexive	provided that xRx ;
• irreflexive	not xRx ;
• symmetric	$xRy \Rightarrow yRx$;
• antisymmetric	xRy and $yRx \Rightarrow x = y$;
• asymmetric	$xRy \Rightarrow$ not yRx ;
• complete	$x \neq y \Rightarrow xRy$ or yRx ;
• strongly complete	xRy or yRx ;
• transitive	xRy and $yRz \Rightarrow xRz$.

A.3 Particular Binary Relations

A transitive binary relation R on the set X is a:

- *preorder* or *quasi order* if R is reflexive;
- *total preorder* or *weak order* if R is strongly complete;
- *partial order* if R is reflexive and antisymmetric;
- *strict partial order* if R is asymmetric;
- *total order* if R is antisymmetric and strongly complete;
- *strict total order* or *linear order* if R is asymmetric and complete;
- *equivalence relation* if R is reflexive and symmetric.

For preorders Herstein and Milnor's (1953) notation is used. In that case a preorder is denoted as $\succsim$, using the notations:

$$\begin{aligned}
x \sim y &\Leftrightarrow x \succsim y \text{ and } y \succsim x, \\
x \succ y &\Leftrightarrow x \succsim y \text{ and not } y \succsim x, \\
x \prec y &\Leftrightarrow y \succ x, \\
x \preceq y &\Leftrightarrow y \succeq x.
\end{aligned}$$

For any preorder it holds that $\succ$ is a strict partial order and $\sim$ is an equivalence relation. We refer in the text to a preorder $\succsim$ as $\succ, \sim$ for typographical convenience.

A.4 Extensions of Orders

Szpilrajn's theorem, which states that any partial order can be extended to a total order is important.

Theorem A.1: *Let $\succ, \sim$ be a partial order on a set X . Then there exists a total order $\succ', \sim'$ on X such that $\succ \subset \succ'$ and $\succ \subset \succ'$.*

Proof: See Szpilrajn (1930). $\square$

Similarly one may extend a preorder to a weak order. A proof of this theorem, using a graph theory result, may be found in Roubens and Vincke (1985, page 49). I will give an alternative proof that is based on Szpilrajn's theorem.

Theorem A.2: *Let $\succ, \sim$ be preorder on a set X . Then there exists a weak order $\succ', \sim'$ on X such that $\succ \subset \succ'$ and $\succ \subset \succ'$.*

Proof: The projection $\Pi: X \rightarrow X/\sim$ induces a preorder $\preceq, \sim$ on $X/\sim$, defined by

$$\Pi x \preceq \Pi y \text{ if } x \succsim y \quad (x, y \in X).$$

Since $\succ, \sim$ is a preorder, it is easy to show that $\preceq, \sim$ is a partial order on $X/\sim$. Hence, by Szpilrajn's theorem, $\preceq, \sim$ can be extended to a total order $\preceq', \sim'$ on $X/\sim$, such that $\preceq \subset \preceq'$ and $\preceq \subset \preceq'$. Now, define the preorder $\succ', \sim'$ on X as

$$x \succ' y \text{ if } \Pi x \preceq' \Pi y \quad (x, y \in X),$$

which is obviously a weak order. Furthermore, since

$$\preceq \subset \preceq' \text{ and } \preceq \subset \preceq',$$

it is straightforward to show that one has $\succ \subset \succ'$ and $\succ \subset \succ'$. $\square$

In general a preorder cannot be extended to a partial order, but the following weaker theorem is available.

Theorem A.3: *Let $\succ, \sim$ be preorder on a set X . Then there exists a partial order $\succ', \sim'$ on X such that $\succ \subset \succ'$.*

Proof: Define the partial order $\succ', \sim'$ on X by

$$x \succ' y \text{ if } (x \succ y \text{ or } x = y) \quad (x, y \in X).$$

To show that $\succ \subset \succ'$, suppose $x \succ y$ for arbitrary $x, y \in X$. Then one has

$$x \succ' y,$$

by definition, and $x \neq y$. Hence $x \succ' y$, because the partial order $\succ', \sim'$ is antisymmetric. Thus

$$x \succ y \Rightarrow x \succ' y \quad (x, y \in X),$$

which means that we have $\succ \subset \succ'$. $\square$

A.5 Extensions of Binary Relations

A nonparametric analysis of consumer data may reveal a part of consumers preferences. In general the revealed part of consumers preferences can be described by two binary relations R and P on a set X with the following interpretation for consumers preferences:

$xRy \Rightarrow y$ is not preferred to x ,

$xPy \Rightarrow x$ is preferred to y .

A problem now is the question whether the relations R and P can be extended to a preorder, representing the preferences of a rational consumer. In a mathematical formulation this problem is the following:

Given two binary relations R and P on X , is there a preorder $\succsim$ on X such that $R \subset \succsim$ and $P \subset \succ$?

Necessary and sufficient conditions for the existence of such a preorder are stated in the following theorem.

Theorem A.4: Let R and P be arbitrary binary relations on a set X . Then the following conditions are equivalent:

(i) There exists a preorder $\succsim$ on X such that

$$R \subset \succsim, P \subset \succ.$$

(ii) There exists a weak order $\succsim$ on X , satisfying the above condition.

(iii) One has $(R \cup P)^* \cap P^c = \emptyset$.

Proof (i) $\Rightarrow$ (ii): Follows from Theorem A.2.

(ii) $\Rightarrow$ (i): Obvious, because any weak order is a preorder.

(i) $\Rightarrow$ (iii): For any preorder $\succsim$ one has

$$(R \cup P)^* \cap P^c \subset (\succsim \cup \succ)^* \cap \succ^c = \succ^* \cap \prec = \succ \cap \prec = \emptyset \quad (R \subset \succsim, P \subset \succ).$$

(iii) $\Rightarrow$ (i): Define the binary relation $\succsim$ on X by

$$x \succsim y \Leftrightarrow (x, y) \in (R \cup P)^* \text{ or } x = y \quad (x, y \in X).$$

This is clearly a preorder, such that

$$R \subset \succsim \text{ and } P \subset \succ.$$

To show that $P \subset \succ$, suppose xPy for arbitrary $x, y \in X$. Then $x \succsim y$, because $P \subset \succsim$. Furthermore, (iii) implies that one has

$$xPy \Rightarrow (y, x) \notin (R \cup P)^* \Rightarrow \text{not } yPx \Rightarrow x \neq y.$$

Thus $(y, x) \notin (R \cup P)^*$ and $x \neq y$, which means that one has not $y \succsim x$. Hence we obtain $x \succ y$ and not $y \succsim x$, so that $x \succ y$. Thus $P \subset \succ$. $\square$

In empirical applications of the revealed preference axiom to consumer data one has often $R = \emptyset$ in the problem given in the beginning of this section. In that case all relevant information is contained in the binary relation P , and one may use the following theorem.

Theorem A.5: Let P be a binary relation on a set X . Then the following conditions are equivalent:

(i) There exists a weak order $\succsim$ on X such that $P \subset \succ$.

(ii) There exists a preorder $\succsim$ on X such that $P \subset \succ$.

(iii) There exists a partial order $\succsim$ on X such that $P \subset \succ$.

(iv) There exists a total order $\succsim$ on X such that $P \subset \succ$.

(v) P^* is irreflexive.

Proof: (i) $\Rightarrow$ (ii): Any weak order is a preorder.

(ii) $\Rightarrow$ (iii): Use Theorem A.3.

(iii) $\Rightarrow$ (iv): Use Theorem A.1.

(iv) $\Rightarrow$ (v): If $P \subset \succ$ then, by transitivity of $\succ$, one has $P^* \subset \succ$. Hence P^* is irreflexive, because $\succ$ is irreflexive.

(v) $\Rightarrow$ (i): If P^* is irreflexive then clearly $P^* \cap P^c = \emptyset$. Hence condition (iii) in Theorem A.4 is satisfied, if we take $R = \emptyset$. So this theorem implies the existence of a preorder $\succ, \sim$ on X such that $\emptyset \subset \prec$ and $P \subset \succ$. $\square$

B GRAPH THEORY

B.1 Introduction

The graph theory in this appendix is related to a revealed preference type of problem, treated in Appendix A. This problem is the question whether there exists a preorder $\succsim$ such that $P \subset \succsim$, given a binary relation P . We can represent the relation P as a graph, a set of arrows between points. Then xPy means that there is an arrow from x to y . Now, the solution of the problem is the following. The desired preorder exists if and only if we can never return to the same point, when following arrow paths in the graph of P .

Another problem occurs when there does not exist a preorder $\succsim$ such that $P \subset \succsim$. Then it is the question how we can find a minimal set of points that can be deleted, together with its arrows, in order to find a smaller graph for which such a preorder exists. Such a set is called a minimal feedback vertex set. The problem of finding a minimal feedback vertex set is related to the deletion of a minimal number of observations in order to get consistency with the axiom of revealed preference.

B.2 Graphs

A *graph* can be depicted by a diagram in which *vertices* are represented by points in the plane, and each *arc* (x,y) is represented by an arrow drawn from the point representing x to the point representing y .

Figure B.1 A Graph

So a graph $G=(X,U)$ is a set X of vertices together with a subset U of the Cartesian product $X \times X$, whose elements are called arcs. For example, Figure B.1 represents the graph $G=(X,U)$ in which

$$X = \{x_1, x_2, x_3, x_4\},$$

$$U = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1)\}.$$

There is a close relation between graphs and binary relations. Any binary relation R on a set X defines a graph $G=(X,R)$ and *vice versa*.

Another representation of a graph $G=(X,U)$ with vertices $X=\{x_1, \dots, x_n\}$ is given by its $n \times n$ *adjacency matrix* $A=[a_{ij}]$, which is a Boolean matrix defined by

$$a_{ij} = \begin{cases} 1 & ((x_i, x_j) \in U), \\ 0 & ((x_i, x_j) \notin U). \end{cases}$$

For example, the adjacency matrix of the graph in Figure B.1 is the Boolean matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The adjacency matrix provides a convenient form to store a graph in a computer.

The terminology of graph theory is not standardized. As described above I consider graphs $G=(X,U)$ in which no multiple arcs may occur, i.e. no element of $X \times X$ appears more than one time in U . However, loops (x,x) may occur. Some authors prefer to use the terms 'linear oriented graph', 'directed graph', or 'digraph' for the above definition of a graph, and often the term 'node' is used rather than 'vertex'.

When considering a graph $G=(X,U)$ we use the following terminology. If $u=(x,y)$ is an element of U then u is called *incident from* x and *incident to* y . Furthermore, x is called a *predecessor* of y and x a *successor* of y . Any vertex x which is end point of a loop (x,x) is called an *essential* vertex.

B.3 Subgraphs

When we cut out a piece of a graph we are left with a subgraph. More precisely, if we remove from a graph $G=(X,U)$ a subset of its vertices together with all the arcs incident to or from those vertices, we are left with a graph of the form $G_Y=(Y,U_Y)$, where $Y \subset X$ and $U_Y=U \cap Y \times Y$. Such a graph is called a *subgraph* of G . We may describe G_Y more precisely as the subgraph of G *generated by* Y . In case one has $Y=X-\{x\}$ we also use the notation G_x instead of G_Y .

B.4 Paths in a Graph

When we follow the arrows in a graph then we are following a path. So a *path* on a graph $G=(X,U)$ is a finite sequence of arcs $u_1, \dots, u_r$ in U of the form

$$(x, x_1), (x_1, x_2), \dots, (x_r, y).$$

One says that this is a path from x to y , and this path *encounters* every vertex in the sequence.

If the end points x and y coincide then the path is called a *cycle*. A graph is called an *acyclic graph* if it does not contain any cycle.

B.5 Strongly Connected Vertices

Strongly connected vertices are connected to each other by a cycle path and a strongly connected component is a strongly connected set of vertices. More precisely:

Let $G=(X,U)$ be a graph. A pair of vertices $x, y \in X$ is said to be *strongly connected* if there are paths from x to y and from y to x . The graph G is said to be a *strongly connected graph* if each pair of distinct vertices $x, y \in X$ is strongly connected.

Now, let us define the binary relation $\sim$ on X by

$$x \sim y \Leftrightarrow x=y \text{ or the pair } x, y \text{ is strongly connected } (x, y \in X).$$

This relation is clearly an equivalence relation. The classes of this equivalence relation partition X into subsets which generate strongly connected subgraphs of G . These subsets are called the *strongly connected components* of G . For example, the strongly connected components of the graph

in Figure B.1 are $\{x_1, x_4\}$, $\{x_2\}$ and $\{x_3\}$.

B.6 Absorption of a Vertex

To absorb a vertex in a graph, we pull out the vertex and connect the outgoing and incoming arrows with each other. More precisely:

Let $G = (X, U)$ be a graph and let $x \in X$. Define the graph $G'_x = (X', U')$ as follows:

$$X' = X - \{x\}, \quad U' = \{(y, z) \in X' \times X' \mid (y, z) \in U \text{ or } (y, x), (x, z) \in U\}.$$

The graph G'_x is said to be obtained by the *absorption* of vertex x .

The following theorems concern the absorption of a vertex in graph. The elementary proofs are left to the reader.

Theorem B.1: Let $G = (X, U)$ be a graph. Suppose $x \in X$ is not an essential vertex. Then G is acyclic if and only if G'_x is acyclic.

Theorem B.2: Let $G = (X, U)$ be a graph. Then

$$(G'_x)_Y = (G_Y)'_x \quad (x \in Y \subset X).$$

B.7 Feedback Vertex Sets

When we cut out a feedback vertex set then the resulting graph is acyclic. More precisely, given a graph $G = (X, U)$ we say that a subset Y of X is a *feedback vertex set* of G if every cycle in G encounters at least one vertex in Y . So a feedback vertex set has the following property.

Theorem B.3: Let $G = (X, U)$ be a graph and $Y \subset X$. Then Y is a feedback vertex set of G if and only if G_{X-Y} is an acyclic graph.

Proof: Obvious. □

We denote the set of all feedback vertex sets of G as $\mathcal{F}(G)$. A feedback vertex set $Y \in \mathcal{F}(G)$ of minimal cardinality is called a *minimal feedback vertex set*.

The following theorems are valid for feedback vertex sets. The first theorem says that in case we search for all feedback sets containing a certain vertex, we may remove this vertex and search further.

Theorem B.4: Let $G = (X, U)$ be a graph and let $x \in X$. Then

$$\{Y \in \mathcal{F}(G) \mid x \in Y\} = \{Y \cup \{x\} \mid Y \in \mathcal{F}(G_x)\}.$$

Proof: Let $x \in Y \subset X$. Then one has obviously $G_{X-Y} = (G_x)_{X-Y}$. So the result follows from Theorem B.3. □

At the other hand, when we search for all feedback sets *not* containing a certain vertex, we may absorb this vertex and search further.

Theorem B.5: Let $G = (X, U)$ be a graph and $x \in X$ not an essential vertex. Then

$$\{Y \in \mathcal{F}(G) \mid x \notin Y\} = \mathcal{F}(G'_x).$$

Proof: Let $x \notin Y \subset X$. Then Theorem B.1 implies that G_{X-Y} is acyclic if $(G_{X-Y})'_x$ is acyclic. Furthermore, by Theorem B.2 one has

$$(G_{X-Y})'_x = (G'_x)_{X-Y}.$$

Thus the result follows from Theorem B.3. □

We combine both above theorems to prove the following main theorem. This theorem is used by Guardabassi (1971) in a branch-and-bound algorithm to find minimal feedback vertex sets.

Theorem B.6: Let $G = (X, U)$ be a graph and $x \in X$ not an essential vertex. Then

$$\mathcal{F}(G) = \{Y \cup \{x\} \mid Y \in \mathcal{F}(G_x)\} \cup \mathcal{F}(G'_x).$$

Proof: Follows immediately from Theorem B.4 and B.5. $\square$

Thus in case we are searching for minimal feedback sets, we may split the problem in two smaller problems using the above theorem. Furthermore, we may simplify the problem by only considering the strongly connected components.

C PATH ALGEBRA'S AND LABELLED GRAPHS

C.1 Introduction

In Carré (1979) the definition of a graph is generalized, using a *path algebra*, which allows for a more unified treatment of path problems. The reason I introduce this idea is that it clarifies the connection between the two following problems:

- (a) Given a relation P on X , is there a preorder $\succ, \sim$ such that $P \subset \succ$?
- (b) Given a square matrix A , are there numbers ϕ_i such that

$$\phi_i - \phi_j \leq a_{ij} \quad (i, j \in I).$$

Problem (a) is related to acyclic Boolean graphs and is treated in Appendix B. Problem (b) is related to shortest paths in a labelled graph. The connection of the problem (a) and (b) becomes clear when we use a path algebra. A similar connection is valid for the following two problems:

- (a') Given R and P , is there a preorder $\succ, \sim$ such that

$$R \subset \succ, P \subset \succ.$$

- (b') Given A and B , are there numbers ϕ_i such that

$$\phi_i - \phi_j \leq a_{ij}, \quad \phi_i - \phi_j < b_{ij} \quad (i, j \in I).$$

Problem (a') is solved in Appendix A, which considers binary relations, but can also be translated into a graph problem. Both problems (b) and (b') are solved in this appendix, using path algebra's. Carré's definition of a path algebra is repeated in the following section for convenience.

C.2 Path Algebra

A path algebra is a set $\mathcal{P}$ equipped with two binary operations $\vee$ and $\cdot$ which have the following properties, for all $x, y, z \in \mathcal{P}$:

1. The $\vee$ operation is idempotent, commutative, and associative:

$$\begin{aligned} x \vee x &= x, \\ x \vee y &= y \vee x, \\ (x \vee y) \vee z &= x \vee (y \vee z). \end{aligned}$$

2. The $\cdot$ operation is associative, and distributive over $\vee$:

$$\begin{aligned} (x \cdot y) \cdot z &= x \cdot (y \cdot z), \\ x \cdot (y \vee z) &= (x \vee y) \cdot z = (x \cdot y) \vee (x \cdot z). \end{aligned}$$

3. The set $\mathcal{P}$ contains a zero element $\emptyset$ such that

$$\begin{aligned} \emptyset \vee x &= x, \\ \emptyset \cdot x &= x \cdot \emptyset = \emptyset, \end{aligned}$$

and a unit element e such that

$$e \cdot x = x \cdot e = x.$$

The operation $\vee$ is called the *join operation* and the operation $\cdot$ is called *multiplication*. Matrix operations are defined as follows:

$$X \vee Y = [x_{ij} \vee y_{ij}],$$

$$X \cdot Y = [\bigvee_k x_{ik} \cdot y_{kj}].$$

The notation x^k and X^k have the meaning of a repeated multiplication as in $x \cdot x = x^2$.

The binary relation $\preceq$ on $\mathcal{P}$ is defined as

$$x \preceq y \Leftrightarrow x \vee y = y \quad (x, y \in \mathcal{P}).$$

As the reader may verify this is a reflexive and antisymmetric relation, and thus a partial order, on $\mathcal{P}$. Similarly, for matrices we define

$$X \preceq Y \Leftrightarrow X \vee Y = Y.$$

C.3 Examples of Path Algebra's

We shall consider the following path algebra's:

1. The *Boolean path algebra*, where

$$\begin{aligned} \mathcal{P} &= \{0, 1\}, \quad \emptyset = 0, \quad e = 1, \\ x \vee y &= (x \text{ or } y), \quad x \cdot y = (x \text{ and } y). \end{aligned}$$

2. The *shortest path algebra*, where

$$\begin{aligned} \mathcal{P} &= \mathbb{R} \cup \{\infty\}, \quad \emptyset = \infty, \quad e = 0, \\ x \vee y &= \min\{x, y\}, \quad x \cdot y = x + y. \end{aligned}$$

Note that the notation is a bit confusing for the shortest path algebra, because one has then $x \preceq y$ equivalent to $x \geq y$.

As the reader may verify, the Boolean path algebra can be represented as a subset of the shortest path algebra. This is done by using the bijective mapping

$$\Pi(0) = \infty, \quad \Pi(1) = 0.$$

One has for example

$$\Pi(x \text{ and } y) = \min\{\Pi(x), \Pi(y)\}, \quad \Pi(x \text{ or } y) = \Pi(x) + \Pi(y).$$

C.4 Labelled Graphs

Carré (1971) defines a *labelled graph* as a graph $G = (X, U)$ together with a *labelling* $l: U \rightarrow \mathcal{P} - \{\emptyset\}$, where $\mathcal{P}$ is a path algebra. I will extend for convenience the labelling l on $X \times X$ by assuming $l(u) = \emptyset$ for $u \notin U$, so that a labelled graph is represented as $l: X \times X \rightarrow \mathcal{P}$.

Any labelled graph l with n vertices can be described by its $n \times n$ *adjacency matrix* $A = [a_{ij}]$ with entries $a_{ij} = l((x_i, x_j))$. For example, concerning a graph labelled with the Boolean path algebra one has $l(u) = 1$, for $u \in U$. Because $\emptyset = 0$, we obtain in this case the usual Boolean adjacency matrix.

An example of a labelled graph for the shortest path algebra is given in Figure C.1.

Figure C.1 A Labelled Graph

The adjacency matrix of the labelled graph in Figure C.1 is

$$A = \begin{bmatrix} \infty & 3 & \infty & 2 \\ \infty & -1 & \infty & \infty \\ \infty & \infty & \infty & -3 \\ 4 & \infty & \infty & \infty \end{bmatrix}.$$

We can generalize the definitions, concerned with graphs, easily for application to labelled graphs. The term 'labelled graph' will often be abbreviated to 'graph' for convenience.

Let $l: X \times X \rightarrow P$ be a labelled graph. The subgraph of l generated by $Y \subset X$ is the graph $l_Y = l|_Y$. The *length* (or *cost*) of a path $\mu = u_1, u_2, \dots, u_r$ is

$$l(\mu) = l(u_1) \cdot l(u_2) \cdot \dots \cdot l(u_r).$$

A labelled graph l is said to be *absorptive* if one has

$$l(\gamma) \leq e,$$

for every cycle γ . The adjacency matrix of a graph is said to be absorptive, when the corresponding labelled graph is absorptive.

We might use the following definition for an acyclic labelled graph. A labelled graph is *acyclic* when $l(\gamma) < e$ for every cycle γ . Note that any graph labelled with the Boolean algebra is absorptive, but not necessarily acyclic.

For absorptive adjacency matrices one can prove the following theorem.

Theorem C.1: Suppose A and B are adjacency matrices and $A \leq B$. If B is absorptive then A is absorptive.

Proof: See Carré (1979, page 105). □

C.5 The Weak Closure of an Adjacency Matrix

The *weak closure* of an adjacency matrix A is defined as

$$A^* = \lim_{n \rightarrow \infty} \bigvee_{k=1}^n A^n.$$

In case of the Boolean algebra the limit results in the transitive closure of the binary relation corresponding to A . In case of the shortest path algebra the limit a_{ij}^* is equal to the shortest path length taken over all paths from vertex x_i to x_j . A quick way to compute transitive closures is introduced by Warshall (1962) in relation to Boolean matrices. This algorithm is also described by Floyd (1962), but now with the aim to determine shortest path lengths. Both algorithms are identical, when considered in path algebra terms, and as follows:

For all $i \in I$ do
 For all $j \in I$ do
 For all $k \in I$ do
 Set $a_{ij} = a_{ik} \vee a_{kj}$.

This simple algorithm transforms adjacency matrix A into the weak closure A^* .

For the shortest path algebra one has $a_{ij}^* \in [-\infty, \infty)$, because the matrices

$$\bigvee_{k=1}^n A^n \quad (n \in \mathbb{N})$$

are decreasing for $n \rightarrow \infty$. So it may happen that $a_{ij}^* = -\infty$, thus A^* is not necessarily a well-defined adjacency matrix for the shortest path algebra. Furthermore, in that case the algorithm given above is not reliable. However,

the following theorem is valid.

Theorem C.2: Let A be the adjacency matrix of a graph labelled with the shortest path algebra and with vertices $\{x_i\}_{i \in I}$. Suppose (a) or (b) is satisfied:

(a) $a_{ij} \in \mathbb{R} \quad (i, j \in I)$.

(b) A is finite-dimensional.

Then the following conditions are equivalent:

(i) A is absorptive.

(ii) A^* is a well-defined adjacency matrix.

(iii) There exist numbers ϕ_i such that

$$\phi_i - \phi_j \leq a_{ij} \quad (i, j \in I).$$

Proof (i) $\Rightarrow$ (ii): Suppose A is an absorptive matrix. To show that A^* is a well-defined adjacency matrix it is sufficient to show

$$a_{ij}^* \neq -\infty \quad (i, j \in I).$$

So let $i, j \in I$ and suppose $a_{ij}^* = -\infty$. We have to show this is impossible.

First, suppose (a) is satisfied. Since $a_{ij}^* = -\infty$, there exists a path μ from i to j such that $l(\mu) + a_{ji} < 0$. Hence $l(\gamma) < 0$, for the cycle $\gamma = \mu, (x_j, x_i)$, contradicting the assumption that A is absorptive.

Now, suppose (b) is satisfied. Since $a_{ij}^* = -\infty$, there exists a sequence of paths μ_n from x_i to x_j , such that

$$\lim_{n \rightarrow \infty} l(\mu_n) = -\infty.$$

Construct a sequence paths μ'_n without cycles by removing all cycles from the paths μ_n . The resulting sequence satisfies again

$$\lim_{n \rightarrow \infty} l(\mu'_n) = -\infty,$$

because each removed cycle is of nonnegative cost, since A is absorptive. Hence the number of arcs approaches infinity when $n \rightarrow \infty$. Now, all arcs in μ'_n are different, because otherwise there is a cycle left. So we obtain a contradiction with (b), because there are maximal m^2 different arcs in a graph with m vertices.

(ii) $\Rightarrow$ (iii): Suppose A^* is a well-defined adjacency matrix. Then one has

$$a_{ik}^* \leq a_{ij} + a_{jk}^*, \quad -\infty < a_{ij}^* \leq a_{ij} \quad (k, i, j \in I).$$

Suppose now that (a) is satisfied. Let $k \in I$ and define $\phi_i = a_{ik}^*$ for all $i \in I$. Then clearly $\phi_i \in \mathbb{R}$, for every i , and

$$\phi_i - \phi_j \leq a_{ij} \quad (i, j \in I).$$

Now, suppose (b) is satisfied. Then the above proof cannot be used, because we may have $a_{ik}^* = \infty$. To avoid this, let us define the following adjacency matrix

$$b_{ij} = \begin{cases} \alpha & (a_{ij} = \infty), \\ a_{ij} & (\text{otherwise}). \end{cases}$$

Because A^* is a well-defined adjacency matrix and finite-dimensional, we may choose α such that B^* is well-defined. Otherwise we can show the existence of a cycle on B , which is of negative length for every $\alpha \in \mathbb{R}$. (For the latter we need the assumption that A is finite-dimensional). Such a cycle will be of negative length on A , which contradicts the assumption that A^* is a well-defined adjacency matrix.

Now, because B^* is well-defined and satisfies condition (a), we may apply the previous results, derived for the case that (a) is satisfied, to obtain

$$\phi_i - \phi_j \leq b_{ij} \leq a_{ij} \quad (i, j \in I).$$

(iii) $\Rightarrow$ (i): Define the adjacency matrix B as follows

$$b_{ij} = \phi_i - \phi_j \quad (i, j \in I).$$

Then obviously $l(\gamma) = 0$ for any cycle path γ on the graph of B , thus B is absorptive. Furthermore, (iii) implies that we have $A \preceq B$. Hence A is absorptive by Theorem C.1. $\square$

Let us define an adjacency matrix A *irreflexive* when $a_{ii} \neq e$ for all i .

Theorem C.3: Let A and B be two finite-dimensional adjacency matrices of a graph labelled with the shortest path algebra and with vertices $\{x_i\}_{i \in I}$. Then the following conditions are equivalent:

(i) There exist numbers ϕ_i such that

$$\phi_i - \phi_j \leq a_{ij}, \quad \phi_i - \phi_j < b_{ij} \quad (i, j \in I).$$

(ii) $A \vee B$ is absorptive and $(A \vee B)^* \cdot B$ is irreflexive.

Proof (i) $\Rightarrow$ (ii): Put $C = A \vee B$. That C has to be absorptive follows from Theorem C.2. Further, because one has

$$\phi_i - \phi_j \leq c_{ij}^*, \quad \phi_j - \phi_i < b_{ji} \quad (i, j \in I),$$

it follows that one has $c_{ij}^* + b_{ji} > 0$, $i, j \in I$. Hence $C^* \cdot B$ is irreflexive.

(ii) $\Rightarrow$ (i): Put $C = A \vee B$ and suppose $C \in \mathbb{R}^m$. Because C is absorptive, C^* is a well-defined adjacency matrix by Theorem C.2. Now, put

$$\phi_i = m^{-1} \sum_{k \neq i} c_{ik}^* \quad (i \in I).$$

The definition of C^* implies now

$$c_{ik}^* - c_{jk}^* \leq c_{ij}, \quad c_{ij}^* \leq c_{ij}, \quad -c_{ji}^* \leq c_{ij} \quad (k, i, j \in I).$$

Thus we have

$$\phi_i - \phi_j = m^{-1} \left(\sum_{k \neq i, j} c_{ik}^* - c_{jk}^* \right) + c_{ij}^* - c_{ji}^* \leq m^{-1} m c_{ij} = c_{ij} \leq a_{ij} \quad (i, j \in I).$$

Furthermore, $C^* \cdot B$ irreflexive implies $c_{ji}^* + b_{ij} \neq 0$ for all $i, j \in I$. From this and $-c_{ji}^* \leq c_{ij} \leq b_{ij}$ we obtain that

$$-c_{ji}^* < b_{ij} \quad (i, j \in I).$$

Hence, the inequalities in condition (i) are satisfied, because we have

$$\phi_i - \phi_j \leq m^{-1} ((m-1)c_{ij} - c_{ji}^*) < m^{-1} m b_{ij} = b_{ij} \quad (i, j \in I).$$

Now suppose that we may have $c_{ij} = \infty$. In that case we can derive the result in a similar way as in the proof (ii) $\Rightarrow$ (iii) of Theorem C.2. This is done by replacement of all infinite values in A and B by a finite value α , such that condition (ii) remains valid. Then the result follows from the above proof in case of finite values. $\square$

There is an interesting connection between the above theorem and Theorem A.4. Condition (iii) in Theorem A.4 was $(R \cup P)^* \cap P^c = \emptyset$, where R and P are binary relations. When we identify these relations with their Boolean matrices, this is clearly equivalent to the condition that $(R \vee P)^* \cdot P$ is irreflexive. But this is exactly what appears as condition (ii) in Theorem C.3. Note that $R \vee P$ is absorptive as any Boolean matrix is absorptive.

We can explain this resemblance by representing the Boolean path algebra as a

shortest path algebra, using the mapping

$$\Pi(0) = \infty, \quad \pi(1) = 0.$$

Now the following clarifies why Theorem A.4 is a special case of Theorem C.3. The existence of a weak order $\succsim$ such that

$$R \subset \succsim, \quad P \subset \succ,$$

in Theorem A.4 is in fact equivalent to the existence of numbers ϕ_i , such that

$$\phi_i - \phi_j \leq a_{ij} = \Pi(x^i R x^j), \quad \phi_i - \phi_j < b_{ij} = \Pi(x^i P x^j) \quad (i, j \in I).$$

C.6 Theorems for Labelled Graphs

We may apply the approach in Theorem B.1 to B.6 to feedback sets in labelled graphs. First I give some similar definitions concerning labelled graphs as I have earlier given for normal graphs.

The labelled graph $l'_x: X' \times X' \rightarrow \mathcal{P}$, obtained from *absorption* of a vertex x in a labelled graph $l: X \times X \rightarrow \mathcal{P}$, is defined by:

$$\begin{aligned} X' &= X - \{x\}, \\ l'_x((y, z)) &= l((y, z)) \vee [l((y, x)) \cdot l((x, z))] \quad ((y, z) \in X' \times X'). \end{aligned}$$

Note that this definition is a generalization of the definition for the absorption of a vertex in a Boolean graph, given in Appendix B.

Given a labelled graph $l: X \times X \rightarrow \mathcal{P}$, a subset Y of X is said to be an *feedback vertex set* if every cycle γ , such that

$$l(\gamma) \preceq e,$$

encounters at least one vertex in Y . The set of all feedback vertex sets for l is denoted by $A(l)$. A vertex $x \in X$ is said to be an *essential vertex* when $l((x, x)) \preceq e$.

We may prove the following six theorems, which are straightforward generalizations of the theorems B.1 until B.6.

Theorem C.4: Let $l: X \times X \rightarrow \mathcal{P}$ be a labelled graph and suppose that $x \in X$ is not an essential vertex. Then l is absorptive if and only if l'_x is absorptive.

So, if we do wish to test whether a labelled graph l is absorptive, we may derive from Theorem C.4 the following algorithm:

Step 1. Set absorptive = True.

If l contains an essential vertex then put absorptive = False.

Step 2. While absorptive = True and l contains more than one vertex do the following:

Take an arbitrary vertex x of l and put $l = l'_x$.

If l contains an essential vertex then put absorptive = False.

The result absorptive will contain the answer whether l is absorptive or not.

Theorem C.5: Let $l: X \times X \rightarrow \mathcal{P}$ be a labelled graph. Then

$$(l'_x)_Y = (l_Y)'_x \quad (x \in Y \subset X).$$

Theorem C.6: Suppose $l: X \times X \rightarrow \mathcal{P}$ is labelled graph and $Y \subset X$. Then Y is an feedback vertex set of l if and only if l_{X-Y} is absorptive.

Theorem C.7: Suppose $l: X \times X \rightarrow \mathcal{P}$ is a labelled graph and $x \in X$. Then

$$\{Y \in A(l) \mid x \in Y\} = \{Y \cup \{x\} \mid Y \in A(l'_x)\}.$$

Theorem C.8: Suppose $l: X \times X \rightarrow \mathcal{P}$ is a labelled graph and $x \in X$ is not an essential vertex. Then

$$\{Y \in A(l) \mid x \notin Y\} = A(l'_x).$$

Theorem C.9: Suppose $l: X \times X \rightarrow \mathcal{P}$ is a labelled graph and $x \in X$ is not an essential vertex. Then

$$A(l) = \{Y \cup \{x\} \mid Y \in A(l'_x)\} \cup A(l'_x).$$

The above theorem can be used as a branch rule in a branch-and-bound algorithm which determines minimal feedback sets in a labelled graph.

D LAPLACE TRANSFORMS

The (bilateral) Laplace transform $\tilde{f}: \mathbb{C} \rightarrow \mathbb{C}$ of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t) e^{-st} dt.$$

The Laplace transform $\tilde{f}$ is only well-defined at $s \in \mathbb{C}$, if the above integral is well-defined. When a Laplace transform is well-defined in the strip $a \leq \operatorname{Re} s \leq b$, then the Laplace transform is analytic in this region and the inverse of the Laplace transform is given by

$$f(t) = (2\pi)^{-1} \int_{c-i\infty}^{c+i\infty} \tilde{f}(s) e^{st} ds \quad (a < c < b).$$

The notations $\tilde{f} = \mathcal{L}[f]$ and $f = \mathcal{L}^{-1}[\tilde{f}]$ are used to denote the Laplace transform and the inverse of the Laplace transform respectively. One has at a simple discontinuity t of the function f

$$\mathcal{L}^{-1}[\mathcal{L}[f]](t) = \frac{1}{2}(f(t^+) + f(t^-)).$$

The convolution theorem states the following useful property of convolutions.

Theorem D.1: Suppose the Laplace transforms of two functions $f(\cdot)$ and $g(\cdot)$ well-defined in a vertical strip in the complex plain. Then the convolution product $f * g$ is well-defined. Furthermore, for s in this vertical strip the Laplace transform of $f * g$ is equal to $\tilde{f}(s)\tilde{g}(s)$ and well-defined.

Table D.1 Examples of Laplace transforms

Function	Laplace transform
$f * g(t)$	$\tilde{f}(s)\tilde{g}(s)$
$\int_{-\infty}^t f(\tau) d\tau$	$s^{-1}\tilde{f}(s)$
$\frac{d}{dt}f(t)$	$s\tilde{f}(s)$
$e^{-at}f(t)$	$\tilde{f}(s+a)$
$f(t-a)$	$e^{-as}\tilde{f}(s)$
$\delta(t)$	1
$1(t)t^{n-1}/(n-1)!$	s^{-n}

E DESCRIPTION OF THE DUTCH INDUSTRY DATA

E.1 Prices and Quantities

The input data of the Dutch industry consist of prices and quantities of the commodities that are given in Table E.1. The sectors in this table are used by the Netherlands Central Planning Bureau. The Dutch industry output data consist of aggregate total output data for the sectors 2-13, excluding indirect taxes and including subsidies. So when I refer to 'Dutch industry' in this book I mean the aggregation of the sectors 2-13.

Table E.1 Inputs of the Dutch Industry

1	Agriculture, horticulture, forestry and fishing
2	Animal food processing industry (meat and dairy products)
3	Other food processing industry (like processing fish, fruit and vegetables, grain, flour processing, sugar and margarine)
4	Beverages and tobacco-processing industry
5	Textiles and clothing industry (including leather and footwear industry)
6	Wood and furniture industry, manufacturing of building materials, earthenware and glass products
7	Paper, printing and publishing industry
8	Chemical, rubber and plastic-processing industry
9	Basic metal industry
10	Manufacturing of metal products, instruments and machinery
11	Electrotechnical industry
12	Transport equipment and automobile industry
13	Petroleum industry
14	Mining and quarrying
15	Public utilities
16	Construction and installation
17	Trade
18	Sea and air transport
19	Other transportation and communication
20	Banking and insurance
21	Other market services (hotels, restaurants, repair of consumer goods, business services, cleaning, washing and other personal services)
22	Health and veterinary services
23	Government
24	Goods and services not specified
25	Import
26	Labour
27	Capital stock buildings
28	Capital stock equipment

The above table looks simple, but it was quite a lot of work to gather a complete data set. I have constructed complete input-output data sets concerning the twelve industry branches 2-13, including capital data. In this book I describe only the results concerning the aggregate data, generated using Paasche price indices, because it was not my aim to write an empirical study about Dutch industry branches.

The main source of nominal data were input-output tables published by the

Netherlands Central Bureau of Statistics in the *National Accounts*. These publications also contain tables of taxes and subsidies. These tables are used to derive nominal sales data, excluding indirect taxes and including subsidies. Only for recent years there were a few deflated input-output tables available, so it was necessary to gather price data for the nominal data. Furthermore, the input-output tables do not contain data on capital formation by destination, a problem that will be discussed in the following section.

The input-output tables were deflated, using price data from various sources. A main source of price data was the *Central Economic Plan* published by the Netherlands Central Planning Bureau. To obtain accurate price data for the input commodities the following publications of the Netherlands Central Bureau of Statistics are used: *Maandstatistiek van de Prijzen*, *Maandstatistiek van de Binnenlandse Handel*, *Statistical Yearbook of the Netherlands*, and *Prijsanalyse*. The output prices of the Central Planning Bureau were used in case input prices were not available. Further, labour volume data was also obtained from the Central Planning Bureau. In labour volume the self-employed are included. The nominal cost of labour, given in the input-output tables and excluding the self-employed, is proportionally increased.

E.2 Capital Stocks

In this section I describe briefly the way I have generated the capital stocks concerning the Dutch industry data, which are used as a data example in this book. To generate capital stocks there were a few problems, which had to be solved. First I needed the mean and the variance of the lifetime of investments. I obtained from the Netherlands Central Bureau of Statistics parameter values for the gamma distribution for several types of investment goods at a low aggregation level. For these parameters I computed the corresponding mean and variance. However, the investment data for the generation of the capital stock was only available at a higher aggregation level. Therefore an 'aggregate' mean and variance had to be estimated. These were obtained from an available data series of expenditures for investment goods at a low aggregation level for the years 1969-1984, given in *Statistics on Fixed Capital Formation in Industry* published by the Netherlands Central Bureau of Statistics. From these data and the available mean and the variance of the lifetime of a money unit investment at a low aggregation level, I estimated the mean and variance of the lifetime of a money unit investment at a higher aggregation level.

For the obtained mean and variance of the investment lifetime there is a corresponding contagious Erlang retirement distribution, as is described in Section 19.2. For this distribution a corresponding state-space realization is described in Section 19.3. I generated the capital stocks with the discrete-time model in Section 19.5, using this state-space realization. To generate the capital stocks, which I have used in this book for the years 1969-1983, I had investment data available from the Netherlands Central Planning Bureau for the years 1950-1983.

To start the generation of the capital stocks a second problem had to be solved. The model has to be initialized. For this initialization I estimated the rate of investment growth around 1950. To obtain this rate I used available data of the retirement of investments in 1950-1956, given in a publication of the Netherlands Central Bureau of Statistics (1957). These data and the investment data yielded a mean retirement rate of investments over the years 1950-1956.

To estimate the rate of investment growth, I assumed that the retirement probability distribution is a gamma function with the above computed mean and variance. As is described in Section 19.4, this assumption yields an estimation of the rate of investment growth. This section also described the initial values, which correspond to a given rate of investment growth.

The above procedure yielded time series of capital stocks, one for buildings and one for equipment, for 15 branches of industry. Finally, these stocks were aggregated, using chained Paasche price indices, to capital stocks concerning the entire Dutch industry.

Figure E.1 Volume Indices Capital Stocks and Output

E.3 Sensitivity to the Choice of Initial Values

The sensitivity of the capital stock data to the choice of initial values depends on the length of the initial period in which the capital stocks are computed, but not actually used as data. The capital stock generation starts in 1950 and the data are used in the period 1969–1983. Hence, the equipment stocks will not be very sensitive to the way the capital stocks are initialized. The mean lifetime of equipment is around 15 to 25 year and after 18 years there is not much left of the initialized stock. The mean lifetime of buildings is longer. Hence, the stocks of buildings may contain an error. However, the error due to the choice of initial values will be small, because investments are increasing over time. The capital stocks of buildings consists in the period 1969–1983 mainly of investments made after 1950. The contribution of the initialized stock to the capital stock is after 1969 only approximately 10% of the total stock.

E.4 User Cost of Capital

To compute the user cost of capital, I used the method described in Chapter 18. The retirement distributions were assumed to be gamma functions. These functions are determined by the mean and variance, which are derived as described in the previous section. To compute the discount factor for the

investment prices, the real interest is calculated as the long-term interest rate minus the investment price inflation rate. The investment price inflation rate is calculated as the average rate over the past four years. The source of the long-term interest is De Nederlandsche Bank (1985). The investment price data were obtained from the Central Planning Bureau. The data were corrected to allow for profit tax, investment tax credit and investment premium. The additional data, necessary for this correction, were found in Vermeend (1983), and Gelauff and Hasselman (1985).

It may be interesting for empirical researchers to know that using the conventional assumption of exponential decay would result in a negative user cost of capital for several periods. Of course, this is not a desirable result. It might be a reason why one sometimes increases the user cost of capital by adding an arbitrarily chosen risk premium to the discount factor. In the computation of the user cost of capital, using gamma retirement distributions, however, this problem did not occur. The resulting user cost of capital was positive for all periods.

A negative user cost of capital may occur when the discounted investment price is exponentially increasing, instead of decreasing. Suppose the investment price p at time t is equal to

$$p(t) = p(0)e^{at},$$

where a is the rate of price inflation. Now, let u be the profit tax rate and r the interest rate. Then the discounted investment price p_I is equal to

$$p_I(t) = p_I(0)e^{-(d-a)t}, \quad d = (1-u)r,$$

where d is the discount rate. The discount rate d was approximately 4% to 6%, while the price inflation a often was higher. When this is the case then one obtains an increasing discounted investment price.

Figure E.2 Price Indices User Cost and Output

In order to check whether the results of the nonparametric tests are strongly influenced by the used assumptions concerning the user cost of capital, the nonparametric tests were applied to several types of user cost of capital

data. Both the exponential and the gamma retirement distribution were used. Moreover, observed price inflation and profit tax rate were used, and variants in which price inflation or profit tax rate, or both, were ignored and set to zero. This procedure resulted in eight data variants for the user cost of capital. As already remarked, a negative user cost of capital was obtained for the exponential retirement distribution, using price inflation and a profit tax rate. So there remained only seven usable data types, which appeared to produce similar nonparametric test results. Especially the results of efficiency levels and the deletion of observations were similar. Concerning the nonparametric bounds, there was only some difference for the neutral change of capital. However, in general the results of the nonparametric tests were only slightly influenced by the used assumptions concerning the user cost of capital. The results described in Part II and Part III, concern the assumption of a gamma retirement distribution, using price inflation and profit tax rate.

F RESULTS OF PRODUCER TESTS

This appendix contains a summary of results, obtained in Part II and III, of the application of several nonparametric tests to the Dutch industry data. The following table shows the values of e , the upper bound of the economical efficiency level, and the number n of observations, that have to be deleted in order to get consistency with the assumed hypothesis.

Table F.1 Test Results, Observed Outputs Not Used

<i>Existence Production Function</i>		
	<i>n</i>	<i>e</i>
Cost minimization	0	100.0%
Linearly homogeneous cost minimization	3	99.9%
Profit maximization	8	99.2%

Table F.2 Test Results, Observed Outputs Used

<i>Existence Production Function</i>		
	<i>n</i>	<i>e</i>
Cost minimization	2	99.5%
Linearly homogeneous cost minimization	13	89.6%
Profit maximization	14	81.5%
<i>Technical Progress</i>		
	<i>n</i>	<i>e</i>
Cost minimization	1	99.8%
Linearly homogeneous cost minimization	2	97.5%
Profit maximization	2	97.1%
<i>Hicks Neutral Change</i>		
	<i>n</i>	<i>e</i>
Cost minimization	0	100.0%
Linearly homogeneous cost minimization	3	99.9%
Profit maximization	5	99.8%
<i>Neutral Change of Labour</i>		
	<i>n</i>	<i>e</i>
Cost minimization	0	100.0%
Linearly homogeneous cost minimization	4	99.8%
Profit maximization	6	99.3%
<i>Neutral Change of Capital</i>		
	<i>n</i>	<i>e</i>
Cost minimization	0	100.0%
Linearly homogeneous cost minimization	1	99.9%
Profit maximization	2	99.8%

Table F.2 Continued

<i>Hicks Neutral Progress</i>		
	<i>n</i>	<i>e</i>
Cost minimization	1	99.8%
Linearly homogeneous cost minimization	4	97.5%
Profit maximization	5	97.1%
<i>Neutral Progress of Labour</i>		
	<i>n</i>	<i>e</i>
Cost minimization	1	99.8%
Linearly homogeneous cost minimization	5	97.5%
Profit maximization	7	97.1%
<i>Neutral Progress of Capital</i>		
	<i>n</i>	<i>e</i>
Cost minimization	1	99.8%
Linearly homogeneous cost minimization	3	97.5%
Profit maximization	3	97.1%
<i>Weak Separability of Goods and Services¹</i>		
	<i>n</i>	<i>e</i>
Cost minimization	2	99.5%
Linearly homogeneous cost minimization	13	89.6%
Profit maximization	14	81.5%
<i>Weak Linearly Homogeneous Separability of Goods and Services²</i>		
	<i>n</i>	<i>e</i>
Cost minimization	7	—
Linearly homogeneous cost minimization	13	—
Profit maximization	14	—
<i>Homothetic Function³</i>		
	<i>n</i>	<i>e</i>
Cost minimization	4	99.5%
Profit maximization	14	—

¹ In this case *e* is an upper bound for the efficiency upper bound.

² The subfunction is assumed to be linearly homogeneous. There are no efficiency upper bounds available for these tests.

³ While testing the conditions for the existence of a homothetic production function, it appeared that even efficiency level $e=0$ did not satisfy the hypothesis of profit maximization.

G OVERVIEW OF NONPARAMETRIC TESTS

Finally, I give an overview of several nonparametric tests that are a result of the main theorems in Part I, II and III. These tests are formulated for application to a finite set of price and quantity data. Such a data description is also used in Afriat (1972, 1981), Hanoch and Rothschildt (1972), Diewert (1973) and Varian (1982, 1983).

G.1 Consumer Demand

In case of demand behaviour it is assumed that price and quantity data $\{(p^i, x^i)\}_{i \in I}$ are given, where p^i and x^i are the price and quantity vector for period i respectively. The following list of necessary and sufficient conditions is available for hypotheses concerning consumer demand. Of course, they can also be applied to producer demand with unobserved output.

Consumer Demand: There exists a function f such that

$$f(x^i) = \sup_{p^i x \leq p^i x^i} f(x), \quad p^i x^i = \min_{f(x) \leq f(x^i)} p^i x \quad (i \in I).$$

Condition: Let R and P be defined such that

$$\begin{aligned} p^i x^j \leq p^i x^i &\Leftrightarrow x^i R x^j, \\ p^i x^j < p^i x^i &\Leftrightarrow x^i P x^j. \end{aligned}$$

Then one must have

$$x^i R^* x^j \Rightarrow \text{not } x^j P x^i \quad (i, j \in I).$$

See Theorem 2.5, and also Afriat (1967) and Varian (1982, 1983).

Linear Homogeneity: There exists a linearly homogeneous function f which satisfies the demand hypothesis.

Condition: There exist numbers Φ_i such that

$$\phi_i - \phi_j \leq \ln(p^i x^j / p^j x^i) \quad (i \in I).$$

See Theorem 3.5, and also Afriat (1972, 1981), Diewert (1973) and Varian (1983).

Weak Separability: There exists a weakly separable function $f(x) = g(x_1, h(x_2))$, such that $g(x_1, \eta)$ is strictly increasing in η , which satisfies the demand hypothesis.

Condition: There exist preorders $\succsim$ on $\{x^i\}_{i \in I}$ and $\succsim'$ on $\{x_2^i\}_{i \in I}$ such that

$$\begin{aligned} p_1^i x_1^j + p_2^i x_2^k \leq p^i x^i &\Rightarrow (x^j \succsim x^i \text{ or } x_2^k \prec' x_2^i) \text{ and} \\ &\quad (x^j \prec x^i \text{ or } x_2^k \succsim' x_2^i) \quad (i, j, k \in I), \\ p_1^i x_1^j + p_2^i x_2^k < p^i x^i &\Rightarrow x^j \prec x^i \text{ or } x_2^k \prec' x_2^i \quad (i, j, k \in I). \end{aligned}$$

See Theorem 4.7.

Concave Weak Separability: There exists a concave weakly separable function $f(x) = g(x_1, h(x_2))$, such that g and h are concave and $g(x_1, \eta)$ is strictly increasing in η , which satisfies the demand hypothesis.

Condition: There exist numbers $u_i, v_i, \lambda_i > 0$ and $\mu_i > 0$ such that

$$u_i \leq u_j + \lambda_j [p_1^j(x_1^i - x_1^j) + (v_i - v_j)/\mu_j] \quad (i, j \in I),$$

$$v_i \leq v_j + \mu_j p_2^j(x_2^i - x_2^j) \quad (i, j \in I).$$

See Diewert and Parkan (1978, 1985) and Varian (1983). Note that the theorem in Varian (1983) contains an omission. He does not require that the aggregator function g and the subfunction h are concave.

G.2 Producer Demand

In case of producer demand it is assumed that the data include the produced quantities y_i for period i . Then the data is of the form $\{(p^i, x^i, y_i)\}_{i \in I}$. The list of necessary and sufficient conditions for the several hypotheses is as follows.

Producer Demand: There exists a function f which satisfies the demand hypothesis and $f(x^i) = y_i$ for all $i \in I$, given the data $\{(p^i, x^i, y_i)\}_{i \in I}$.

Condition: One has

$$p^i x^j \leq p^j x^i \Rightarrow y_j \leq y_i, \quad p^i x^j < p^j x^i \Rightarrow y_j < y_i \quad (i, j \in I).$$

See Theorem 8.1. See also Hanoch and Rothschildt (1972) and Varian (1984).

Technical Progress: There exists an increasing series of functions f_i , $i \in I$, i.e. $f_i \geq f_j$ if $i \geq j$, such that each function f_i satisfies the producer demand hypothesis for its corresponding data element (p^i, x^i, y_i) .

Condition: One has

$$p^i x^j \leq p^j x^i \Rightarrow y_j \leq y_i, \quad p^i x^j < p^j x^i \Rightarrow y_j < y_i \quad (j \leq i).$$

See Theorem 8.3.

Hicks Neutral Change: There exists a function f and numbers $t_i > 0$, such that each function $t_i f$ satisfies the producer demand hypothesis for its corresponding data element (p^i, x^i, y_i) .

Condition: There exist values $\phi_i = \ln t_i$ such that

$$\begin{aligned} \phi_i - \phi_j &\leq \ln(y_i/y_j) & (i, j \in I: p^i x^j \leq p^j x^i), \\ \phi_i - \phi_j &< \ln(y_i/y_j) & (i, j \in I: p^i x^j < p^j x^i). \end{aligned}$$

See Theorem 8.8.

Neutral Change: There exists a function f and numbers $t_i > 0$, such that each function $f_i(x) = f(x_1, t_i x_2)$ satisfies the producer demand hypothesis for its corresponding data element (p^i, x^i, y_i) .

Condition: There exist values $\phi_i = \ln t_i$ such that

$$\begin{aligned} \phi_i - \phi_j &\leq \ln(p_2^i x_2^j / [p^i x^i - p_1^i x_1^j]) & (i, j \in I: y_j \geq y_i, p^i x^i > p_1^i x_1^j), \\ \phi_i - \phi_j &< \ln(p_2^i x_2^j / [p^i x^i - p_1^i x_1^j]) & (i, j \in I: y_j > y_i, p^i x^i > p_1^i x_1^j). \end{aligned}$$

See Theorem 8.10.

Linear Homogeneity: There exists a linearly homogeneous function f which satisfies the producer demand hypothesis.

Condition: One has

$$y_j \leq y_i p^i x^j / p^j x^i \quad (i, j \in I).$$

See Theorem 9.1, and also Hanoch and Rothschildt (1972) and Afriat (1972).

Linear Homogeneity with Technical Progress: There exists an increasing series of linearly homogeneous functions f_i , $i \in I$, such that each function f_i satisfies the producer demand hypothesis for its corresponding data element

(p^i, x^i, y_i) .

Condition: One has

$$y_j \leq y_i p^i x^j / p^i x^i \quad (j \leq i).$$

See Theorem 9.5.

Linear Homogeneity with Hicks Neutral Change: There exists a linearly homogeneous function f and numbers $t_i > 0$, $i \in I$, such that each function $t_i f$ satisfies the producer demand hypothesis for its corresponding data element (p^i, x^i, y_i) .

Condition: There exist values $\phi_i = \ln t_i$ such that

$$\phi_i - \phi_j \leq \ln(y_j p^i x^i / y_i p^i x^j) \quad (i, j \in I).$$

See Theorem 9.9.

Linear Homogeneity with Neutral Change: There exists a linearly homogeneous function f and numbers $t_i > 0$, $i \in I$, such that each function $f_i(x) = f(x_1, t_i x_2)$ satisfies the producer demand hypothesis for its corresponding data element (p^i, x^i, y_i) .

Condition: There exist values $\phi_i = \ln t_i$ such that

$$\phi_i - \phi_j \leq \ln(p_2^i x_2^j / [p^i x^i y_j / y_i - p_1^i x_1^j]) \quad (i, j \in I: p^i x^i y_j / y_i > p_1^i x_1^j).$$

See Theorem 9.12.

Homotheticity: There exists a homothetic function f which satisfies the producer demand hypothesis.

Condition: There exist numbers ϕ_i such that

$$\begin{aligned} \phi_i - \phi_j &\leq \ln(p^i x^j / p^i x^i) \quad (i, j \in I), \\ \phi_i - \phi_j &\leq 0 \quad (i, j \in I: y_i \leq y_j), \\ \phi_i - \phi_j &< 0 \quad (i, j \in I: y_i < y_j). \end{aligned}$$

See Theorem 10.1, and also Hanoch and Rothschildt (1972) and Varian (1984). Note that both their theorems contain an omission. Hanoch and Rothschildt do not require

$$\phi_i - \phi_j < 0 \quad (i, j \in I: y_i < y_j),$$

as is given in the condition above, and Varian does not require

$$\phi_i - \phi_j \leq 0 \quad (i, j \in I: y_i \leq y_j).$$

Weak Separability: There exists a weakly separable function $g(x_1, h(x_2))$, such that $g(x_1, \eta)$ is strictly increasing in η , which satisfies the producer demand hypothesis.

Condition: There exists a preorder $\succsim$ on $\{x_2^i\}_{i \in I}$ such that

$$\begin{aligned} p_1^i x_1^j + p_2^i x_2^k &\leq p^i x^i, \quad y_j > y_i \Rightarrow x_2^k < x_2^j \quad (i, j, k \in I), \\ p_1^i x_1^j + p_2^i x_2^k &< p^i x^i, \quad y_j \geq y_i \Rightarrow x_2^k < x_2^j \quad (i, j, k \in I), \\ p_1^i x_1^j + p_2^i x_2^k &\leq p^i x^i, \quad y_j \geq y_i \Rightarrow x_2^k \precsim x_2^j \quad (i, j, k \in I). \end{aligned}$$

See Theorem 11.2.

Concave Weak Separability: There exists a concave weakly separable function $g(x_1, h(x_2))$, such that g and h are concave and $g(x_1, \eta)$ is strictly increasing in η , which satisfies the producer demand hypothesis.

Condition: There exist numbers $v_i, \lambda_i > 0$ and $\mu_i > 0$ such that

$$y_i \leq y_j + \lambda_j [p_1^j(x_1^i - x_1^j) + (v_i - v_j)/\mu_j] \quad (i, j \in I),$$

$$v_i \leq v_j + \mu_j p_2^j(x_2^i - x_2^j) \quad (i, j \in I).$$

This is a simple modification of the conditions of Diewert and Parkan (1978, 1985), concerning concave weakly separable consumer demand and where output y_i is not observed.

Weak Linearly Homogeneous Separability: There exists a weakly separable function $g(x_1, h(x_2))$, such that h is a linearly homogeneous and $g(x_1, \eta)$ is strictly increasing in η , which satisfies the producer demand hypothesis.

Condition: There exist values $\phi_i = \ln h(x_2^i)$ such that

$$\phi_i - \phi_j \leq \ln(p_2^j x_2^i / p_2^i x_2^j) \quad (i, j \in I),$$

$$\phi_i - \phi_j \leq \ln(p_2^j x_2^i / [p^i x^i - p_1^i x_1^j]) \quad (i, j \in I: y_j \geq y_i, p^i x^i > p_1^i x_1^j),$$

$$\phi_i - \phi_j < \ln(p_2^j x_2^i / [p^i x^i - p_1^i x_1^j]) \quad (i, j \in I: y_j > y_i, p^i x^i > p_1^i x_1^j).$$

See Theorem 11.4. Note that for price and quantity data we may assume that $g(x_1, \eta)$ is strictly increasing.

Linearly Homogeneous Weak Linearly Homogeneous Separability: There exists a linearly homogeneous weakly separable function $g(x_1, h(x_2))$, such that the subfunction h is linearly homogeneous and $g(x_1, \eta)$ is strictly increasing in η , which satisfies the producer demand hypothesis.

Condition: There exist values $\phi_i = \ln h(x_2^i)$ such that

$$\phi_i - \phi_j \leq \ln(p_2^j x_2^i y_j / p_2^i x_2^j y_i) \quad (i, j \in I),$$

$$\phi_i - \phi_j \leq \ln(p_2^j x_2^i / y_i [p^i x^i / y_i - p_1^i x_1^j / y_j]) \quad (i, j \in I: p^i x^i / y_i - p_1^i x_1^j / y_j > 0).$$

See Theorem 11.6. Note that for a finite price and quantity data set we may assume that $g(x_1, \eta)$ is strictly increasing.

Linearly Homogeneous Weak Separability: There exists a linearly homogeneous weakly separable function $g(x_1, h(x_2))$, such that $g(x_1, \eta)$ is strictly increasing in η , which satisfies the producer demand hypothesis.

Condition: There exists a preorder $\succsim$ on $\{v_2^i\}_{i \in I}$ such that

$$p_1^i v_1^j + p_2^i v_2^k < p^i v^i \Rightarrow v_2^k < v_2^j \quad (i, j, k \in I),$$

$$p_1^i v_1^j + p_2^i v_2^k \leq p^i v^i \Rightarrow v_2^k \precsim v_2^j \quad (i, j, k \in I),$$

where $v^i = y_i^{-1} x^i$ for all $i \in I$.

See Theorem 11.7. Note that this theorem proves only that the above is a necessary condition, and that for price and quantity data we may assume that $g(x_1, \eta)$ is strictly increasing.

G.3 Profit Maximization

In case of profit maximization the data is also of the form $\{(p^i, x^i, y_i)\}_{i \in I}$.

Profit Maximization: There exists a function f such that

$$f(x^i) - p^i x^i = \max_x f(x) - p^i x, \quad f(x^i) = y_i \quad (i \in I).$$

Condition: One has

$$y_j \leq y_i + p^i x^j - p^i x^i \quad (i, j \in I).$$

See Theorem 13.1, and also Hanoch and Rothschildt (1972) and Varian (1984).

Technical Progress: There exists an increasing series of functions f_i , $i \in I$, such that each function f_i satisfies the profit maximization hypothesis for

its corresponding data element (p^i, x^i, y_i) .

Condition: One has

$$y_j \leq y_i + p^i x^j - p^i x^i \quad (i, j \in I: j \leq i).$$

See Theorem 13.8.

Hicks Neutral Change: There exists a function f and numbers $t_i > 0$, such that each function $t_i f$ satisfies the profit maximization hypothesis for its corresponding data element (p^i, x^i, y_i) .

Condition: There exist values $\phi_i = \ln t_i$ such that

$$\phi_i - \phi_j \leq \ln([y_i + p^i x^j - p^i x^i]/y_j) \quad (i, j \in I).$$

See Theorem 13.10.

Neutral Change: There exists a function f and a series of numbers $t_i > 0$, such that each function $f_i(x) = f(x_1, t_i x_2)$ satisfies the profit maximization hypothesis for its corresponding data element (p^i, x^i, y_i) .

Condition: There exist values $\phi_i = \ln t_i$ such that

$$\phi_i - \phi_j \leq \ln(p_2^i x_2^j / [p^i x^i + y_j - y_i - p_1^i x_1^i]) \quad (i, j \in I: y_j - p_1^i x_1^i > y_i - p^i x^i).$$

See Theorem 13.14.

Homotheticity: There exists a homothetic function f which satisfies the profit maximization hypothesis.

Condition: There exist values ϕ_i such that

$$\begin{aligned} \phi_i - \phi_j &\leq \ln(p^j x^i / p^j x^j) \quad (i, j \in I), \\ \phi_i - \phi_j &\leq \ln(p^i x^i / [y_j - y_i + p^i x^i]) \quad (i, j: y_j - y_i + p^i x^i > 0). \end{aligned}$$

See Theorem 14.1. In Varian (1984) weaker conditions are given concerning homothetic profit maximization, but the corresponding proof is incomplete.

Weak Separability: There exists a weakly separable function $g(x_1, h(x_2))$, such that $g(x_1, \eta)$ is strictly increasing in η , which satisfies the profit maximization hypothesis.

Condition: There exists a preorder $\succsim$ on $\{x_2^i\}_{i \in I}$ such that

$$\begin{aligned} y_j - p_1^i x_1^j - p_2^i x_2^k &\geq y_i - p^i x^i \Rightarrow x_2^k \succsim x_2^j \quad (k, i, j \in I), \\ y_j - p_1^i x_1^j - p_2^i x_2^k &> y_i - p^i x^i \Rightarrow x_2^k < x_2^j \quad (k, i, j \in I). \end{aligned}$$

See Theorem 15.2.

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INDEX

Emboldened page numbers indicate an entry in a **definition**

- absorption of a vertex **171, 178**
- absorptive matrix or labelled graph **26, 27, 33, 118, 175, 176-178**
- active constraints **38, 149**
- acyclic graph **170, 171, 173, 175**
- adjacency matrix **35, 37, 169, 170, 174, 175-177**
- adjoint operator **xiv, 142**
- adjustment costs **141**
- Afriat, S.N.* **2, 7, 9, 13, 21, 24, 31, 34, 188, 189**
- aggregate data **7, 18-19, 58-65, 162, 181-183**
- aggregator function **42, 43-45, 47, 104, 106, 110, 114, 132, 134, 189**
- algorithm
 - absorptive test **178**
 - bisection **15**
 - minimal feedback sets **171, 179**
 - shortest path, transitive closure **175**
- Almon, C.* **157**
- analytic function **180**
- angular momentum **1**
- antisymmetric binary relation **165, 166, 174**
- Antonelli, G.B.* **7, 11**
- approximately finite data **51, 52-54, 57**
- approximation
 - gamma distribution **159**
 - user cost of capital **153-154**
- arcs **169, 170, 176**
- arithmetic price index **60, 61-63**
- assets **137, 152, 155**
- asymmetric relation **165**
- axiom of revealed preference **7, 9, 11, 15-16, 17, 18-21, 47, 49, 51-52, 57-59, 63, 69-70, 73, 90, 169**
- base period **35, 60, 61-62**
- belief in consumer behaviour **65, 162**
- Berends-Ballast, H.* **157**
- bias of price indices **65**
- binary relation **xiii, 10, 15, 17, 20, 165, 166-167, 169-170, 173-175, 177**
- bisection **15, 31, 70, 77, 85**
- Blackorby, C.* **42**
- Boolean matrix and graph **169-170, 173-175, 177-178**
- Boolean path algebra **174, 177**
- bounded function **54, 56**
- bounds **xiii, 7, 15, 24, 29, 31-40, 68, 70-80, 84-97, 101-103, 106, 109-110, 116-127, 131, 163-164, 185-186**
- branch-and-bound algorithm **171, 179**
- branches of industry **181, 183**
- Bronars, S.G.* **63**
- budget **2, 4-5, 10, 13, 19, 58, 62-63, 65, 112, 163**
- buildings **181, 183**
- capital **1-3, 67-68, 79-80, 95-98, 125-127**
- capital stock **2, 3, 136, 138-142, 144, 147-149, 156-157, 159, 163, 181-183**
- Carré, B.* **173-175**
- CES-function **58**
- chained index **36, 39, 60, 61-63, 65, 183**
- Chiappori, P.* **163**
- Chipman, J.S.* **9, 11**
- closed mapping **49, 50, 52, 54-56, 71, 78**
- closure of a relation or graph **xiii, 16-17, 19, 33, 95, 125, 165, 175**
- Cobb-Douglas function **58**
- coefficient of technical efficiency **120, 121**
- commodities **7, 28, 34, 58-64, 181-182**
- compact set **48-49, 53-54**
- competitive
 - consumer **6**
 - market for output **113**
- complement **xiii, 17, 165**
- complete relation **165**
- computer program **8, 47, 100**
- concave function **7, 22-23, 27, 29, 39-42, 45, 52, 56-57, 103, 105, 131, 133, 135, 163, 188-191**

- cone 24, 25–41, 84–110, 116, 124, 129, 131
- connected component 170, 172
- constraint 2, 4, 38, 58, 141, 148, 163
- conserved properties 1, 6, 18, 25, 28, 34, 45, 47, 164
- consumer demand 1–2, 4–9, 10, 11–65, 69, 71, 101, 104, 117–118, 162, 188, 191
- contagious erlang distribution 157, 160–161, 182
- continuous function 22–23, 27, 29, 45, 48, 51, 53–57, 105–106, 117, 133, 139
- continuous-time model 161
- convergence 49, 55–56
- convolution **xiii**, 139, 150, 163, 180
- cost function 2, 6, 9, 10–135
- cost minimizing behaviour 9, 10, 49–50, 55
- coefficients of
 - technical efficiency 69
 - neutral change 79
- countable data set 90, 121
- counter example 9, 50–51
- cover 51–53
- current period 35, 60
- cycle path 26, 170, 171, 175–178
- data enclosure 12
- De Nederlandsche Bank N.V.* 184
- decay of investments 2, 140, 142, 147, 163, 184,
- delta distribution **xiv**, 137, 147
- Diewert W.E.* 7, 24, 31, 42, 59, 63, 65, 135, 188–189, 191
- difference
 - upper bound **xiv**, 117
 - lower bound **xiv**, 117
- differentiability 163
- differential equations 11
- digraph 170
- direct index 60, 62
- directed graph 170
- discontinuity 22–23, 29, 53, 180
- discounted cost 136, 140–141, 148–155, 183–184
- discrete-time model 161, 182
- distribution theory 140, 143
- diverging bounds 123, 126
- dual
 - approach 44
 - relation 12
 - theorem 54
- Dutch industry data 66–135, 181, 182–187
- dynamic multiplier 147, 149
- economical inefficiency 2, 6
- efficiency
 - level 13–15, 31, 35–36, 38, 51–57, 61–65, 77, 80, 82, 85, 120, 123, 125, 126, 185–186
 - transformation 2, 9, 12, 13–14, 29–30, 79–82, 114
 - upper bound 15, 29, 31, 35, 68, 70–74, 77, 79–82, 85, 87–93, 95–96, 98, 101–102, 106, 116, 118, 120, 122–123, 125, 127
- efficient behaviour 6, 9, 13–15, 24, 27, 30–31, 36, 38, 64–65, 116, 122, 162
- elementary particles 164
- energy 1, 18, 164
- equipment 181, 183
- equivalence relation 18, 165, 166, 170
- Erlang distribution 157, 158–159, 182
- essential vertex 170, 171, 178, 179
- exponential
 - decay 2, 142, 147, 163, 184
 - discounted price 136, 140, 150–155, 184
 - survival function 137, 138, 141–143, 149, 152, 155, 157
 - retirement distribution 137, 185
- factors
 - Hicks neutral change 75
 - Hicks neutral progress 122
 - neutral progress capital and labour 80, 127
- Färe, R. 115
- feedback vertex set 169, 171, 172, 178, 179
- fictitious economies 58
- finite-dimensional matrix 176–177
- Fisher, F.M.* 58
- Fisher price index 60, 61–63, 65
- Floyd, R.W.* 175
- forecast of economic behaviour 163
- frontier production function **xiv**, 2, 66–67, 68–70, 73, 81–82, 86–87, 89–90, 114, 119, 121
- Fubini's theorem 139, 145
- gamma distribution 137, 151–153, 155, 157, 159, 161, 163, 182–185
- GARP 182–187
- gauge symmetry 10
- Gelauff, G.M.M.* 184

- geometric index 7, **60**, 61-63
 global analysis 7, 11-12
Gorman, W.M. 58
 graph **169**, 170-179
 gross investment flow 139
Guardabassi, G. 171
- Hahn, F.* 157
Hanoch, G. 84, 100, 188-191
 Harrod neutral change **67**
Hasenkamp, G. 58
Hasselman, B.H. 184
Herstein, I.N. xiii, 165
 Hicks neutral
 change **67**, 74-75, 78, 89,
 91-92, 121-123, 186,
 189-190, 192
 progress **67**, 68, 74, 76-77,
 89, 93, 122-123, 131, 187
 homothetic function 1, 24, 31, 61,
 58-59, 63-65, 68, **99**, 101-102,
 108, 114, 129-132, 162, 187,
 190, 192
Houthakker, H.S. 7, 11
 hyperplanes 7
- incident arc **170**
 inconsistent behaviour 58-59,
 64-65
 individual demand 58-59
 induced order 10, 15, 46, 105, 134
 industry data 181-182
 inefficiency 2, 6, 28, 66, 118
 inefficient behaviour 14, 21-22,
 72, 91, 121, 162
 infinitely differentiable utility
 function 163
 inflation 184-185
 innerproduct **xiv**, **141**, 142
 input-output 58, 68, 181-182
 inputs 66-67, 78-79, 112-113, 136,
 181-182
 Integrability Problem 7, **11**
 intersection 49, 52
 invariance 83
 invariant 1, 5-6, 9-10, 24, 28,
 44-45, 108-109, 145
 irreflexive **165**, 167-168, 177
 isoquant 27, 83-84
- join operation **173**
Jorgenson, D.W. 141
- Kailath, T.* 156, 157
Kearl, J.H. 58
Koo, A.Y.C. 58, 59
- labelling 174
 labelled graph 35, 173, **174**,
 175-179
 labour 67, 79, 80, 95-98, 125-127,
 182, 186-187
 Lagrange multiplier 115, 148
 Lagrangian 1, 18, 28, 24-25, 43,
 115, 163-164
Landsburg, S.E. 18, 59
 Laplace transform **xiii**, 143,
 149-150, 153, 156-157, 159, **180**
 Laspeyres price index 35-36, 39,
 54, **60**, 61-63
 length or cost of a path **175**
 lifetime of investments 3,
 136-137, 145, 147, 152,
 154-155, 161, 182-183
 linear
 homogeneity 1, 5, 7, **24**,
 25-41, 45, 64-65, 68,
 78-81, 83-97, 99-102,
 104, 106-111, 114-119,
 129-131, 162-163, 186-191
 operator **139**, 140-145,
 148-149
 order **165**
 oriented graph 170
 local analysis 7, 11, 42
 locally nonsatiated function **5**,
 48, 54, 56-57
 logistic probability distribution
 137
 long-term interest 183-184
 loop **170**
 lower bound xiv, 31-34, 37, 41,
 70, 77, 80, 87, 117, 126
 lower bound **L xiv**, **70**
 lower semicontinuous function **52**,
 54, 56-57
- macro data 58-59
Maks, J.A.H. 18, 59
Manser, E.M. 31, 59, 63
 marginal
 cost of capital 151
 revenue of capital 148
 monopolistic market 113
 maximum principle 2, 141, 147, 149
McDonald, R.J. 31, 59, 63
 measure theory 145
 mechanics 1, 18, 28, 145
 micro data 58-59
Milnor, J. xiii, 165
 minimal feedback vertex set **169**, **171**
 mode 153
 moment 153
 monotonic function 101-103, 131

- monotonic transformation 10, 50–52, 68, 71–72, 74, 99, 101–102, 104, 106, 114, 129, 131
- monotonically increasing function 22, 27, 45, 52, 104–105, 133
- Mossin, A.* 58, 59
- Muellerbauer, J.* 58
- multiple outputs 117, 163
- myopic producer 149
- narrow bounds 80, 95
- neoclassical theory 2, 4, 6–7, 11, 24, 112
- Netherlands Central Bureau of Statistics* 7, 61–63, 68, 137, 141, 151–152, 162, 182
- Netherlands Central Planning Bureau* 68, 181–182, 184
- Neumann series 144
- neutral
 - change 67, 78–79, 81, 94–98, 124–127, 185
 - progress 67, 79–80, 97–98, 127–128, 186
- node 170
- Noether theorem 1
- noncompetitive market 2, 6
- nonlinear budget constraint 2
- nonlinear investment costs 147, 149
- non-aggregate data 62
- nonsatiated function 5–6, 48, 54, 56–57
- normal probability distribution 137
- operator xiii–xiv, 139–145
- ordering 18, 70, 101
- orders 1, 10–11, 15, 46–47, 49, 166
- outlying observation 96
- Paasche price index 36, 39, 60, 61–63, 181, 183
- Paccoud, T.* 157
- panel data 58
- parametric approach 3, 5, 7, 24, 31, 58, 137, 140, 152, 154, 157, 162–163
- Parkan, C.* 7, 31, 42, 59, 63, 135, 189, 191
- partial efficient behaviour 13
- partial order 47, 165, 166–167, 174
- path algebra 173, 174–178
- paths xiii, 35–38, 169, 170, 173, 175–176
- perpetual inventory method 139, 156
- Poincaré transformation 1
- pointwise limit function 41, 50, 52, 55–56, 71, 87
- polynomials 157
- Pontryagin, L.S.* 2, 147, 149
- pre-war data 59
- predecessor 170
- preference 69–71, 73–75, 82, 90, 134
- preference order 1, 5, 6, 9–12, 15, 49
- preorder xiii, 5, 10, 11–12, 15–16, 45–47, 49, 105, 110, 133, 165, 166, 167, 169, 173, 188, 190–192
- probability 1, 64, 137, 140, 151, 153–154, 157, 183, 162
- present value of profit 136
- producer demand with observed output 66
- profit maximizing behaviour 112–113, 114–135, 191–192
- with observed output 114
- Primont, D.* 115
- quantum theory 1, 2, 145
- quasi order 165
- quasi-concave function 105, 133
- random data 7, 58, 62–64, 162
- rationalizing
 - preorder and utility function 12
 - production function 67–68
- rays 25, 28, 34, 38, 83
- recession 37
- rectangular survival function 137, 138, 141, 145, 147, 151–153, 155
- reflexive binary relation 10, 165, 174
- regression 89, 120
- relative
 - lower bound xiv, 31–32
 - upper bound xiv, 32, 34, 85, 101–102, 110, 117
- retired investments 139
- retirement distribution 3, 136, 137, 139–140, 144–145, 149, 151–157, 159–161, 163, 182–185
- Richter, M.K.* 2, 12
- risk premium 184
- Rochet, J.* 163
- Rockafellar, R.T.* 52, 56
- Rothschild, M.* 84, 100, 188–191
- Roubens, M.* 166
- Samuelson, P.A.* 7, 11
- satiation 51
- scaled data 78, 83–84, 86, 108–111

- Schmidt, P.* 59
Schmoranz, I. 157
 seasonal influences 59
 self-fulfilling prophecy 65, 162
 separating hyperplanes 7
 shadow cost of capital 141
 shape of survival function 152, 154-156
Shephard, R.W. 99
 shortest path xiii, 35-38, 75, 77, 100, 173-178
 shortest path algebra 174, 175-178
 simulation 156, 160-161
 single-output 66, 163
Solow, R.M. 58
 Solow neutral change 67
 step function xiv
 stochastic elements 59
Stahmer, C. 157
 state-space realization 156, 157, 159-160, 182
 statistical theory 156
 strict total order 165
 strongly
 complete binary relation 10, 165
 connected component 170, 172
 connected graph 170
 subfunction 42, 45, 48, 68, 104-108, 110, 114, 132, 189, 191
 subgraph 170, 175
 subsidies 181-182
 successor 170
 superposition principle 64-65
 survival function 3, 136-137, 138-160
 symmetric relation 165, 166
 symmetry 1, 5, 6, 9-10, 18, 24-25, 28, 34, 42-45, 48-49, 129, 132, 145, 153-154, 162-164
Szpilrajn, E. 166
 Szpilrajn's theorem 166

Takayama, A. 141
 taste change 59
 tax 181-182, 184-185
 technical
 progres xiv, 66, 67, 68, 72-73, 81, 85, 88-91, 114, 118, 120-121, 162, 189, 191, 186
 change 67, 69
 efficiency 68-74, 88-91, 120-121, 163
 efficiency upper bound 70, 71, 73, 68, 88-91
 textbook theory 1-2, 9, 11, 66, 112-113, 142
 'thick' indifference curve 4-5
 time-independent 18, 149
 Törnqvist price index 60, 61-63, 65
 total preorder 165
 transitive
 closure xiii, 16-17, 165, 175
 binary relation 10, 165
 triangular matrix 18-19, 31

 unit-isoquant 83-84
 unobserved output 75, 99, 101, 104, 117, 188
 unreliable data 2, 8, 64
 upper
 bound xiv, 7, 15, 24, 29-40, 68-98, 101-102, 106, 109-110, 116-127, 186
 bound U xiv, 85
 semicontinuous function 27, 52, 54, 56-57
 upward bias 59, 65
 user cost of capital 2, 3, 136, 140-144, 150-154, 163, 183-185
 utility maximizing behaviour 9-10, 49-50, 55, 58-59
Uzawa, H. 7, 11

Varian, H.R. 7, 17-19, 21, 24, 30, 42, 59, 84, 90, 99-100, 131, 134-135, 188-192
 variance 151-159, 161, 182-183
Vartia, Y.O. 65
Vermeend, W.A.T.G. 184
 vertex 37-38, 169, 170-171, 174-179
Vincke, P. 166

Warshall, S. 165, 175
 waste 4, 13, 70
 weak
 closure 33, 46, 47, 95, 105, 125, 134, 175
 order 10, 17, 20-21, 165, 166-167, 178
 separability 1, 5, 7, 42, 43-48, 68, 104-111, 114, 129, 132-135, 162-163, 187-188, 190-192
 Weibull probability distribution 137
 weight period 60, 61-62

Zemanian, A.H. 140, 143

NEDERLANDSE SAMENVATTING / SUMMARY IN DUTCH

Het onderwerp "Niet-parametrische consumenten- en producentenanalyse" is verwant aan de niet-parametrische statistiek. Deze statistiek is niet gebaseerd op parametrische specificaties van kansverdelingen, maar gaat ervan uit dat de kansverdeling lid is van een niet-parametrische familie van kansverdelingen. Zo'n familie wordt vastgelegd door symmetrie-eigenschappen voor de kansverdeling te specificeren.

De analogie van deze benadering in de economische wetenschappen is het gebruik van een niet-parametrische specificatie voor de nuts- of produktiefunctie. Voor data van economisch gedrag kan gepostuleerd worden dat deze gegenereerd is door middel van een functie, die aan een niet-parametrische specificatie voldoet. Door toetsen te ontwikkelen, om na te gaan of aan zo'n hypothese is voldaan, kan men de bruikbaarheid toetsen van aannames betreffende economisch gedrag en de vorm van de nuts- of produktiefunctie. Deze niet-parametrische toetsen kunnen hulp bieden bij het kiezen van een parametrische specificatie van economisch gedrag.

Dit proefschrift beschrijft een groot aantal van deze niet-parametrische toetsen. Het gaat hierbij om economisch gedrag van nutsmaximering, kostenminimering en winstmaximering, waarbij de volgende niet-parametrische specificaties beschouwd worden: lineaire homogeniteit, homotheticiteit, scheidbaarheid en technische vooruitgang. De toetsen zijn toegepast op prijzen en hoeveelheden, betreffende consumenten- en producentengedrag in Nederland. Zo wordt een indruk gegeven van de mogelijkheden die deze theorie biedt. Bovendien is als experiment een aantal niet-parametrische toetsen voor consumentengedrag toegepast op geaggregeerde random data. De random data werden voor dit doel geaggregeerd met aggregratie methodes, die het Centraal Bureau voor de Statistiek voor de Nederlandse consumentengegevens heeft gebruikt.

Uit de resultaten van de niet-parametrische toetsen valt in het algemeen niet op te maken welke specificatie gekozen moet worden. De toetsen geven wel aan of specificaties in strijd zijn met de data. Zo wordt duidelijk welke specificaties zeker *niet* gebruikt moeten worden. Een duidelijk voorbeeld hiervoor leveren de resultaten van de niet-parametrische toetsen voor de gegevens van de Nederlandse industrie. Deze geven aan dat winstmaximering, zonder gebruik te maken van de aanname van technische vooruitgang, *niet* in aanmerking komt als gedrags-specificatie. Winstmaximering *met* technische vooruitgang is wel een toegestane gedrags-specificatie. Een andere mogelijkheid, die niet uitgaat van technische vooruitgang, is de veronderstelling van kostenminimerend gedrag.

Uit de resultaten blijkt dat voorzichtigheid is geboden bij het trekken van conclusies over geaggregeerde consumentengegevens. Gebleken is namelijk dat geaggregeerde random data voldeden aan de hypothese van consumentengedrag. De oorzaak hiervan is het gebruik van een voor de hand liggende regel betreffende consumentengedrag: prijs maal hoeveelheid is gelijk aan de uitgaven. Prijsindices en bijbehorende hoeveelheden krijgen daardoor de neiging om zich in tegengestelde richting te bewegen, datgene wat men van het gedrag van een consument zou verwachten. Het geloof in consumentengedrag werkt hier dus als een *self-fulfilling prophecy*.

In de praktijk blijken de niet-parametrische toetsen - op de "revealed preference" toets na - de gestelde hypothesen over efficiënt economisch

gedrag te verwerpen. Er worden twee manieren aangegeven waarmee bepaald kan worden in welke mate zo'n hypothese verworpen wordt. Ten eerste kan men zo weinig mogelijk waarnemingen uit de dataverzameling verwijderen, zodanig dat de overblijvende gegevens voldoen aan de hypothese. De tweede methode is het verzwakken van de gestelde hypothese door ook inefficiënt gedrag te beschouwen. De maximale efficiëntie, waarbij de gegevens nog aan de hypothese voldoen, geeft dan een indruk van de mate waarin aan deze hypothese voldaan is.

Hoewel de nadruk ligt op de ontwikkeling van niet-parametrische toetsen, wordt ruime aandacht besteed aan het afleiden van prijzen en hoeveelheden voor kapitaalvoorraden. Voor het berekenen van de kapitaalvoorraden en de schaduwprijs van kapitaal, wordt aangetoond dat de gebruikelijke aanname van exponentieel verval van kapitaal op eenvoudige manier verbeterd kan worden. Beschreven wordt hoe kapitaalvoorraden gegenereerd kunnen worden met behulp van een lineair dynamisch model. Hierbij is aangenomen dat de afstoot van kapitaal benaderd kan worden door een gammafunctie. Bovendien wordt beschreven hoe de schaduwprijs van kapitaal afgeleid kan worden voor een willekeurig gegeven afstootfunctie van kapitaal.

Generalisatie van de niet-parametrische theorie

De niet-parametrische toetsen volgen uit theorema's, die in dit boek worden afgeleid. Deze theorema's worden bewezen door een geschikte nuts- of produktiefunctie te construeren voor de gegeven data. Als de data aan bepaalde eigenschappen voldoen, zoals concaviteit van de budgetrestricties, houdt dit in dat deze geconstrueerde functie bepaalde eigenschappen heeft. Dit betekent dat een groot aantal varianten mogelijk is op de theorema's, waarbij eigenschappen van de geconstrueerde functie volgen uit aannames over eigenschappen van de gegeven data. Een andere generalisatie, die vaak zonder problemen gemaakt kan worden, is de generalisatie van single-output naar het geval waar meerdere outputs zijn toegestaan.

Er worden geen differentieerbare nuts- en produktiefuncties beschouwd. In een artikel van Chiappori en Rochet (1987) staat echter een elegant bewijs, waarin een differentieerbare nutsfunctie is afgeleid als de convolutie van twee functies. Als men het bestaan van een differentieerbare nuts- of produktiefunctie wil bewijzen, kan deze methode op veel resultaten in dit boek worden toegepast.

Nu er een overzicht is van de resultaten, is het mogelijk om de verbanden tussen de verschillende niet-parametrische theorema's te onderzoeken. Dit met het doel ze in te passen in een algemenere theorie. Zo'n theorie kan antwoord geven op de vraag voor welke specificaties er met succes een niet-parametrisch theorema afgeleid kan worden. Bij de aanvang van dit onderzoek was dit voornamelijk een kwestie van *trial* en *error*. Later werd duidelijk waarom bepaalde specificaties een probleem waren en andere niet. De rode draad in zo'n algemene theorie is de symmetrie van het optimaliseringsprobleem. Lineair homogene kostenminimering, waarvoor de Lagrangeaan symmetrie vertoont, leverde bijvoorbeeld geen problemen op. Dit in tegenstelling tot de zwak scheidbare nutsfunctie, waarvoor de Lagrangeaan geen voor de hand liggende symmetrie vertoont.

Toepassingen van de niet-parametrische theorie

Wat zijn de toepassingen van de niet-parametrische theorie? Ten eerste kan deze theorie gebruikt worden om een overzicht te krijgen van de eigenschappen van een verzameling gegevens. Dit kan tijd besparen bij het zoeken naar een parametrische specificatie van de nuts- of produktiefunctie. De niet-

parametrise toetsen dienen dan als een leidraad, die aangeeft welke mogelijkheden er zijn. De niet-parametrise theorie kan vervolgens een hulpmiddel zijn bij het oplossen van theoretise problemen. Voorbeelden hiervoor zijn problemen waarbij voor een gegeven vraagrelatie gevraagd wordt naar het bestaan van een nuts- of produktiefunctie.

Een toepassing van de niet-parametrise benadering, die vooral bruikbaar is voor producentengedrag, is de voorspelling van economise gedrag. Dit onderwerp komt niet aan de orde in dit boek, maar het gereedschap ervoor wordt aangereikt in de vorm van grenzen voor de produktiefunctie en grenzen voor de technische efficiëntie en technische vooruitgang. Deze grenzen kunnen beschouwd worden als behouden grootheden, die volgen uit de symmetrie van het optimeringsmodel, dat de waarnemingen genereert. Voor deze behouden grootheden zijn analoge voorbeelden te vinden in de fysica, zoals energie en impulsmoment, die ook daar voortkomen uit het feit dat de Lagrangeaan symmetrie vertoont. In de theoretise fysica geldt de niet-parametrise benadering, met symmetrie-veronderstellingen als uitgangspunt, als standaard-gereedschap. Zo is bijvoorbeeld het bestaan van nieuwe elementaire deeltjes voorspeld. Te verwachten is dat deze niet-parametrise benadering ook van waarde zal blijken te zijn in de economise wetenschap.

CURRICULUM VITAE

Martijn Houtman was born on 23 June 1955 in The Hague. After finishing B.A. mathematics and physics with subsidiary subject sociology, he graduated in 1982 in mathematics with the subsidiary subjects computer science and physics. Subsequently he worked as fellow assistant at the Department of Economics of the University of Groningen. His work consisted of writing computer programs, concerning nonparametric tests of consumer behaviour. After this he was several years engaged in research at the Department of Economics of the University of Groningen. During this time he was for several months on detachment at the University of Michigan in Ann Arbor.

Besides research on nonparametric theory he studied the structure of economic models. He also made a simulation model of the Dutch economy. At the moment he is graduating in physics at the University of Leiden. His main subjects are theoretical physics and artificial intelligence. Occasionally he occupied himself with research in other areas. For example, he investigated the way foreigners are presented at the Dutch television. Since 1995 he works part-time as a researcher at the University of Limburg in Maastricht.