# Strategy-proof location of public bads in a two-country model

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# Strategy-proof location of public bads in a two-country model[☆]

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## HIGHLIGHTS

- Joint decision making in two neighboring countries about the location of public bads is considered.
- The main requirement on decision rules is strategy-proofness with respect to single-dipped preference profiles.
- Admissible rules turn out to assign border locations, and in general decisions in the two countries are not independent.

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## ABSTRACT

We consider the joint decision of placing public bads in each of two neighboring countries, modeled by two adjacent line segments. Residents of the two countries have single-dipped preferences, determined by the distance of their dips to the nearer public bad (myopic preferences) or, lexicographically, by the distance to the nearer and the other public bad (lexmin preferences). A (social choice) rule takes a profile of reported preferences as input and assigns the location of the public bad in each country. For the case of myopic preferences, all rules satisfying strategy-proofness, country-wise Pareto optimality, non-corruptibility, and the far away condition are characterized. These rules pick only border locations. The same holds for lexmin preferences under strategy-proofness and country-wise Pareto optimality alone.



## 1. Introduction

We consider two neighboring countries which jointly decide on where to locate two public bads, i.e., public provisions that are beneficial for the countries but that no one likes to have in his backyard. We assume that these countries maintain their independence to a large extent: one public bad will be located in each country, but the purpose of deciding jointly is to take also the preferences of the residents of the other country into account. As an example specific for what we do in this paper, one could think of Belgium and the Netherlands each placing a windmill park: the two central governments jointly decide on where to locate the Dutch and the Belgian windmill park while taking the preferences of the residents over the locations of both parks into account.

In our analysis joint decision making will be based on voting by the residents, who report their preferences. The emphasis is on strategy-proofness, which means that the joint decision rule should provide no incentives to report insincere preferences. We assume that each preference is characterized by a single dip, representing the worst location of the public bad in the resident country. This could be the place where the resident is living, but it could also be some natural resort, or place of historical interest; so it makes sense to assume that preferences are private knowledge and, moreover, each person should be free in expressing a preference.

In our stylized model for this situation we assume that the two countries are represented by the real intervals $A = [-1, 0]$ and $B = [0, 1]$. A (social choice) *rule* then assigns a location in country $A$ and one in country $B$, based on the reported dips of the (finitely many) residents or *agents* in each country. Apart from strategy-proofness, we impose that the rule is country-wise Pareto optimal: for each country the location of the public bad should be Pareto optimal given the reported preferences and the location of the public bad in the other country. Since the agents will report truthfully, this means that the assigned locations will be country-wise Pareto optimal *ex post*. Country-wise Pareto optimality is a modification of the usual Pareto optimality condition, and reflects the assumption that countries keep their sovereignty in the decision making process.

We consider two specifications of the single-dipped preferences. A preference is *myopic* if it is completely determined by the distance between the dip and the nearer public bad (thus,

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preference increases with this distance). In this case, the location of the other public bad plays no role as long as it is farther away, and therefore myopic preferences allow for many indifferences. To break some of those indifferences we impose two further conditions on rules. Non-corruptibility (cf. Ritz, 1985) says that if an agent both before and after a change of preference is indifferent between the location pairs, then the locations themselves should not change. The 'far away' condition says that if all agents in a country, given the location in the other country, weakly prefer the non-shared border as location (i.e., $-1$ in country $A$ and 1 in country $B$) then this should be assigned. Under these two additional conditions on rules we show that only border locations can be chosen, i.e., one of the pairs $(-1, 1)$, $(-1, 0)$, $(0, 1)$, and $(0, 0)$.

The second preference specification we consider is *lexmin*: preference is determined by the distance to the nearer public bad and, in case of a tie, by the distance to the other public bad. We show that in this case, where there are much fewer indifferences, strategy-proofness and country-wise Pareto optimality are sufficient for a rule to pick only from the four border pairs.

For the case of myopic preferences we characterize all rules satisfying the four mentioned conditions by so-called decisive pairs of coalitions. Within this class of rules (which range from majority voting to almost dictatorial rules), we identify those that are strategy-proof and country-wise Pareto optimal for lexmin preferences, but also show that there are such rules outside of this class.

Our results can be seen as positive results compared to the seminal impossibility theorem of Gibbard (1973) and Satterthwaite (1975) which says that if there are three or more alternatives, then it is impossible to find a non-dictatorial social choice function which is also strategy-proof and Pareto optimal. One way out from this impossibility result is to consider restricted preference domains. A well-known example of this is the single-peaked preference domain (Moulin, 1980). Another example is the single-dipped preference domain. Peremans and Storcken (1999) have shown the equivalence between individual and group strategy proofness in subdomains of single-dipped preferences. Manjunath (2014) has characterized the class of all nondictatorial, strategy-proof and Pareto optimal social choice functions when preferences are single-dipped on an interval. Barbera et al. (2012) have characterized the class of all nondictatorial, group strategy-proof and Pareto optimal social choice functions when preferences are single-dipped on a line. The rules in the present paper bear similarities to the rules in the last two papers.

But there are impossibility results in this domain as well. Öztürk et al. (2013, 2014) have shown that there does not exist a nondictatorial social choice function that is strategy-proof and Pareto optimal when preferences are single-dipped on a disk, and on some, but not all, convex polytopes in the plane. Chatterjee et al. (2016) have extended these results to social choice functions on a sphere, when preferences are single-dipped or, equivalently in this case, single-peaked.

All these results are about strategy-proof location of one public bad. As far as we know, the present paper is the first one to consider the location of public bads in neighboring regions or countries, apart from an analysis of the one agent per country case in Öztürk (2013). There is also a literature adopting a mechanism design approach to the location of public bads, that is, including monetary side payments: e.g., recently, Lescop (2007) and Sakai (2012), but we are not aware of results in this area addressing the location in more than one region.

This paper is organized as follows. Section 2 introduces the model and some preliminary results. Section 3 shows that internal locations are excluded, and Section 4 provides the characterization of all rules satisfying our conditions. Section 5 concludes.

## 2. The two country model

### 2.1. Preliminaries

Let country $A$ be represented by the interval $[-1, 0]$ and country $B$ by $[0, 1]$. The set of possible *alternatives* is denoted by $\mathcal{A} = [-1, 0] \times [0, 1]$. The set of *agents* $N$ is partitioned into the set $N_A$ of inhabitants of country $A$ and the set $N_B$ of inhabitants of country $B$. Let the cardinalities of $N,\ N_A$ and $N_B$ be natural numbers $n$, $n_A$, $n_B$, with $n = n_A + n_B$.

Each agent $i \in N$ has a *preference* $R_{z(i)}$ over $\mathcal{A}$, characterized by its *dip* $z(i) \in [-1, 1]$. Below, we introduce the two specifications of these preferences that are considered in this paper, both based on distance. For now, it is sufficient to mention that each preference is completely determined by its dip. Therefore, preferences will often be identified with their dips and denoted by $z(i)$ instead of $R_{z(i)}$.

As usual, $P_{z(i)}$ denotes the strict or asymmetric part of $R_{z(i)}$ and $I_{z(i)}$ denotes the indifference or symmetric part.

A (preference) *profile* $z$ assigns to each agent $i \in N$ a preference $z(i)$ such that $z(i) \in [-1, 0]$ if $i \in N_A$ and $z(i) \in [0, 1]$ if $i \in N_B$. The set of all profiles is denoted by $\mathcal{R}$, where it is understood that this set will depend on the further specification of the preferences under consideration.

For a profile $z$ and a non-empty set $S \subseteq N$, let $z_S = (z(i))_{i\in S}$. For $i \in N$, profile $z'$ is an *i-deviation* of $z$ if $z_{N\setminus\{i\}} = z'_{N\setminus\{i\}}$. For $a \in A$ and $S \subseteq N_A$, $(a^S, z_{N\setminus S})$ denotes the profile where all $i \in N\setminus S$ have preference $z(i)$ and all $i \in S$ have preference $a$.

A *rule* $\varphi$ assigns to each profile $z$ an alternative $\varphi(z) = (\alpha(z), \beta(z)) \in \mathcal{A}$.[1]

For $x, y \in \mathbb{R}$, $\mu(x, y) = \frac{x+y}{2}$ denotes the midpoint of $x$ and $y$. In case there is no confusion, for a profile $z$ we write $\mu(z)$ instead of $\mu(\alpha(z), \beta(z))$ to indicate the midpoint of the interval with endpoints given by $\varphi(z)$.[2]

The main properties of a rule $\varphi$ considered in this paper are the following.

Strategy-Proofness (SP) $\varphi$ is *strategy-proof* if $\varphi(z)R_{z(i)}\varphi(z')$ for every $z \in \mathcal{R}$, every $i \in N$, and every $i$-deviation $z'$ of $z$.

Country-Wise Pareto Optimality (CPO) $\varphi$ is *Pareto optimal for country A* (resp. $B$) if for every profile $z$ there does not exist an $a \in [-1, 0]$ (resp. $b \in [0, 1]$) such that $(a, \beta(z))R_{z(i)}\varphi(z)$ for all $i \in N_A$ and $(a, \beta(z))\, P_{z(k)}\, \varphi(z)$ for at least one $k \in N_A$ (resp. $(\alpha(z), b)R_{z(i)}\varphi(z)$ for all $i \in N_B$ and $(\alpha(z), b)P_{z(i)}\varphi(z)$ for at least one $k \in N_B$). Rule $\varphi$ is *country-wise Pareto optimal* if it is both Pareto optimal for country $A$ and Pareto optimal for country $B$.

Strategy-proofness says that truth-telling is a weakly dominant strategy. Country-wise Pareto optimality is a modification of the usual Pareto optimality and reflects the fact that although the countries make a joint decision they still keep their sovereignty to some extent.

### 2.2. Myopic preferences

A preference $R_{z(i)}$ of agent $i \in N$ is a *myopic* preference if for all $(a_1, b_1), (a_2, b_2) \in \mathcal{A}$, $(a_1, b_1)$ is at least as good as $(a_2, b_2)$ at $R_{z(i)}$, with the usual notation $(a_1, b_1)R_{z(i)}(a_2, b_2)$, if

$$\min\{|a_1 - z(i)|, |b_1 - z(i)|\} \geq \min\{|a_2 - z(i)|, |b_2 - z(i)|\}.$$

[1] Of course, $\alpha(\cdot)$ and $\beta(\cdot)$ depend on $\varphi$, but this is suppressed from the notation.

[2] Observe that if $\alpha(z) = \beta(z)$ then $\alpha(z) = \beta(z) = 0$. In practice, this means that the two public bads are located on or very close to the border, on either side.

Thus, an agent with a myopic preference cares only about the position of the nearer public bad, and is indifferent as to the position of the more distant one. Myopic preferences allow for many indifferences. In order to achieve some tie-breaking we will consider the following condition on a rule $\varphi$, introduced by Ritz (1985).

Non-Corruptibility (NC) $\varphi$ is *non-corruptible* if $\varphi(z) = \varphi(z')$ for every $z \in \mathcal{R}$, $i \in N$, and $i$-deviation $z'$ of $z$ such that $\varphi(z) I_{z(i)} \varphi(z')$ and $\varphi(z) I_{z'(i)} \varphi(z')$.

At a non-corruptible rule a unilateral deviation either affects the deviator's preference somewhere or has no effect at all. The lemma below shows that strategy-proofness and non-corruptibility imply the following monotonicity condition if preferences are myopic.

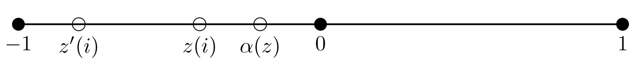
Monotonicity (MON) $\varphi$ is *monotone* if $\varphi(z) = \varphi(z')$ for all $z, z' \in \mathcal{R}$ such that for all agents $i \in N$:

- $z'(i) \leq z(i) \leq \alpha(z)$ or
- $\alpha(z) \leq z(i) \leq z'(i) \leq \mu(z)$ or
- $\mu(z) \leq z'(i) \leq z(i) \leq \beta(z)$ or
- $\beta(z) \leq z(i) \leq z'(i)$.

Monotonicity is a familiar consequence in the presence of strategy-proofness: in this case it says, roughly, that if the preference of an agent changes such that the chosen pair becomes better when evaluated according to the new preference, then it remains to be chosen. As an aside, it can be shown that this monotonicity condition is weaker than what Maskin Monotonicity would demand in this framework.

**Lemma 2.1.** *Let $\mathcal{R}$ be the set of all profiles of myopic preferences. Let $\varphi : \mathcal{R} \to \mathcal{A}$ satisfy SP and NC. Then $\varphi$ satisfies MON.*

**Proof.** It is sufficient to prove monotonicity for an $i$-deviation $z' \in \mathcal{R}$ of $z \in \mathcal{R}$ for an agent $i \in N_A$. There are three cases.
(a) $z'(i) < z(i) \leq \alpha(z)$.



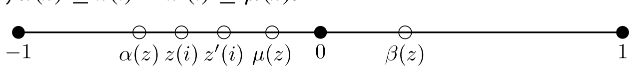

SP implies

$$|z(i) - \alpha(z)| \geq |z(i) - \alpha(z')| \tag{1}$$

and

$$|z'(i) - \alpha(z')| \geq |z'(i) - \alpha(z)|. \tag{2}$$

Let $x \in \mathbb{R}$ such that $z(i) = \mu(x, \alpha(z))$. Then by (1): $\alpha(z') \in [\max\{x, -1\}, \alpha(z)]$, hence by (2): $\alpha(z') = \alpha(z)$. By NC, $\varphi(z) = \varphi(z')$.
(b) $\alpha(z) \leq z(i) < z'(i) \leq \mu(z)$.



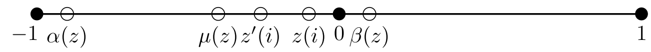

SP implies

$$|z(i) - \alpha(z)| \geq |z(i) - \alpha(z')| \quad \text{or}$$
$$|z(i) - \alpha(z)| \geq |z(i) - \beta(z')| \tag{3}$$

and

$$|z'(i) - \alpha(z')| \geq |z'(i) - \alpha(z)| \quad \text{and}$$
$$|z'(i) - \beta(z')| \geq |z'(i) - \alpha(z)|. \tag{4}$$

If the first inequality in (3) holds then the first inequality in (4) implies $\alpha(z') = \alpha(z)$. By the second inequality in (4), $\alpha(z) = \alpha(z')$ is closer to both $z(i)$ and $z'(i)$ than $\beta'(z)$ is, so by NC, $\varphi(z) = \varphi(z')$. If the second inequality in (3) holds then the second inequality in (4) is violated, a contradiction.
(c) $\alpha(z) \leq \mu(z) \leq z'(i) < z(i)$.




SP implies

$$|z(i) - \beta(z)| \geq |z(i) - \alpha(z')| \quad \text{or}$$
$$|z(i) - \beta(z)| \geq |z(i) - \beta(z')| \tag{5}$$

and

$$|z'(i) - \alpha(z')| \geq |z'(i) - \beta(z)| \quad \text{and}$$
$$|z'(i) - \beta(z')| \geq |z'(i) - \beta(z)|. \tag{6}$$

If the first inequality in (5) holds then the first inequality in (6) implies $\alpha(z') = \beta(z)$, hence $\alpha(z') = \beta(z) = 0$. Hence by NC, $\varphi(z) = \varphi(z')$. If the second inequality in (5) holds then the second inequality in (6) implies $\beta(z) = \beta(z')$. Then both at $z(i)$ and $z'(i)$ agent $i$ is indifferent between $\varphi(z)$ and $\varphi(z')$, so that by NC, $\varphi(z) = \varphi(z')$. ■

### 2.3. Lexmin preferences

A preference $R_{x(i)}$ of agent $i \in N$ is a *lexmin* preference if for all $(a_1, b_1), (a_2, b_2) \in \mathcal{A}$, $(a_1, b_1)$ is at least as good as $(a_2, b_2)$ at $R_{x(i)}$, with the usual notation $(a_1, b_1) R_{x(i)} (a_2, b_2)$, if

$$\min\{|a_1 - x(i)|, |b_1 - x(i)|\} > \min\{|a_2 - x(i)|, |b_2 - x(i)|\}, \quad \text{or}$$
$$\min\{|a_1 - x(i)|, |b_1 - x(i)|\} = \min\{|a_2 - x(i)|, |b_2 - x(i)|\} \quad \text{and}$$
$$\max\{|a_1 - x(i)|, |b_1 - x(i)|\} \geq \max\{|a_2 - x(i)|, |b_2 - x(i)|\}.$$

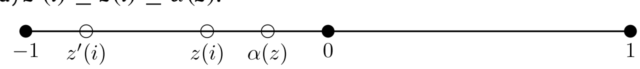
Lexmin preferences allow for fewer indifferences as myopic preferences, and it turns out that strategy-proofness alone is sufficient to obtain the following, slightly weaker version of monotonicity.

Weak Monotonicity (WMON) $\varphi$ is *weakly monotone* if $\varphi(z) = \varphi(z')$ for all $z, z' \in \mathcal{R}$ such that for all agents $i \in N$:

- $z'(i) \leq z(i) \leq \alpha(z)$, or
- $\alpha(z) \leq z(i) \leq z'(i) \leq \mu(z)$ and $\alpha(z) < 0$, or
- $\mu(z) \leq z'(i) \leq z(i) \leq \beta(z)$ and $\beta(z) > 0$, or
- $\beta(z) \leq z(i) \leq z'(i)$.

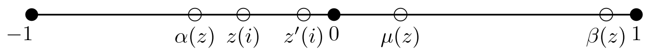
**Lemma 2.2.** *Let $\mathcal{R}$ be the set of all profiles of lexmin preferences. Let $\varphi : \mathcal{R} \to \mathcal{A}$ satisfy SP. Then $\varphi$ satisfies WMON.*

**Proof.** It is sufficient to prove weak monotonicity for an $i$-deviation from $z \in \mathcal{R}$ to $z' \in \mathcal{R}$ for an agent $i \in N$. We consider the following two cases (the remaining two cases are analogous).
(a) $z'(i) \leq z(i) \leq \alpha(z)$.



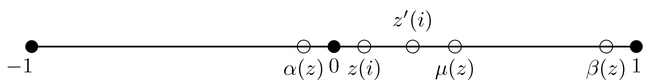

(i) If $\alpha(z) < \alpha(z') \leq 0$, then agent $i$ manipulates from $z(i)$ to $z'(i)$. (ii) Now suppose $-1 \leq \alpha(z') < \alpha(z)$. Then we must have $\alpha(z') \leq z'(i) - r$, where $r = \alpha(z) - z'(i)$, otherwise $i$ manipulates from $z'(i)$ to $z(i)$. In turn this implies $|z(i) - \alpha(z')| > |z(i) - \alpha(z)|$, so we must have $\alpha(z) = \beta(z') = 0$, otherwise $i$ manipulates from $z(i)$ to $z'(i)$. Now $\beta(z) \leq z'(i) + (z'(i) - \alpha(z'))$, otherwise $i$ manipulates from $z'(i)$ to $z(i)$; and $\beta(z) \geq z(i) + (z(i) - \alpha(z'))$, otherwise $i$ manipulates from $z(i)$ to $z'(i)$. These two inequalities combined, however, contradict the assumption that $z(i) > z'(i)$. (iii) The only remaining possibility is $\alpha(z) = \alpha(z')$, and by strategy-proofness this implies $\beta(z) = \beta(z')$.
(b) $\alpha(z) \leq z(i) \leq z'(i) \leq \mu(z)$ and $\alpha(z) < 0$. We consider two subcases.

(b1) $i \in N_A$.




(i) Suppose $\alpha(z') < \alpha(z)$. Then $\beta(z') < z(i) + (z(i) - \alpha(z))$ otherwise $i$ manipulates from $z(i)$ to $z'(i)$. In turn, this implies $\beta(z') - z'(i) < \beta(z') - z(i) < z(i) - \alpha(z) < z'(i) - \alpha(z')$ and therefore, $z'(i) - \alpha(z) \leq \beta(z') - z'(i)$, otherwise $i$ manipulates from $z'(i)$ to $z(i)$. These two inequalities imply $z(i) > z'(i)$, contradicting our assumption. (ii) Suppose $\alpha(z') > \alpha(z)$. Then $\alpha(z) < \alpha(z') \leq z(i) + (z(i) - \alpha(z))$, otherwise $i$ manipulates from $z(i)$ to $z'(i)$. Thus, $|z'(i) - \alpha(z')| < z'(i) - \alpha(z)$, hence $\beta(z) < z'(i) + |z'(i) - \alpha(z)|$, otherwise $i$ manipulates from $z'(i)$ to $z(i)$. This, however, contradicts the assumption that $z'(i) \leq \mu(z)$. (iii) The only remaining possibility is $\alpha(z) = \alpha(z')$, and by strategy-proofness this implies $\beta(z) = \beta(z')$.

(b2) $i \in N_B$.




(i) Suppose $\alpha(z') < \alpha(z)$. Then $\beta(z') < z(i) + (z(i) - \alpha(z))$ otherwise $i$ manipulates from $z(i)$ to $z'(i)$. In turn, this implies $z'(i) - \alpha(z) \leq |\beta(z') - z'(i)|$, otherwise $i$ manipulates from $z'(i)$ to $z(i)$. If $\beta(z') \geq z'(i)$ then $z(i) + (z(i) - \alpha(z)) > \beta(z') \geq z'(i) + (z'(i) - \alpha(z))$, which contradicts the assumption $z(i) < z'(i)$. If $\beta(z) < z'(i)$ then $\beta(z') \leq \alpha(z)$, which implies $\alpha(z) = \beta(z') = 0$, a contradiction since $\alpha(z) < 0$. (ii) If $\alpha(z') > \alpha(z)$ then $i$ manipulates from $z'(i)$ to $z(i)$. (iii) The only remaining possibility is $\alpha(z) = \alpha(z')$, and by strategy-proofness this implies $\beta(z) = \beta(z')$. ■

## 3. No internal locations

In this section we show that under certain assumptions internal locations for public bads are excluded. For myopic preferences it turns out that next to strategy-proofness, country-wise Pareto optimality and non-corruptibility we need an additional tie-breaking condition on a rule to achieve this. For lexmin preferences we find that strategy-proofness and country-wise Pareto optimality are sufficient. In both cases, the (weak) monotonicity condition is an important tool to derive the result.

### *3.1. Myopic preferences*

Throughout this subsection $\mathcal{R}$ is the set of all profiles of myopic preferences. We introduce the following condition for a rule $\varphi : \mathcal{R} \to \mathcal{A}$.

Far Away Condition (FA) $\varphi$ satisfies the *far away condition* if for every profile $z \in \mathcal{R}$:

- if $(\alpha(z), 1) R_{z(i)} \varphi(z)$ for all $i \in N$, then $\beta(z) = 1$, and
- if $(-1, \beta(z)) R_{z(i)} \varphi(z)$ for all $i \in N$, then $\alpha(z) = -1$.

The far away condition says that if all agents weakly prefer an extreme location then the rule should assign that. Throughout this subsection, $\varphi$ is a rule satisfying SP, CPO, NC, and FA.

**Theorem 3.1.** *$\varphi(z) \in \{(-1, 1), (0, 0), (-1, 0), (0, 1)\}$ for every $z \in \mathcal{R}$.*

The proof of this theorem uses the two lemmas below. For a profile $z \in \mathcal{R}$ we define $S(z) = \{i \in N_A : z(i) \geq \alpha(z)\}$ and $T(z) = \{i \in N_B : z(i) \leq \beta(z)\}$. By Lemma 2.1, $\varphi$ is monotone. Therefore we may assume that

(a) $z = (-1^{N_A \setminus S(z)}, 0^{S(z)}, \mu(z)^{T(z)}, 1^{N_B \setminus T(z)})$ if $\mu(z) \in [0, 1]$, and
(b) $z = (-1^{N_A \setminus S(z)}, \mu(z)^{S(z)}, 0^{T(z)}, 1^{N_B \setminus T(z)})$ if $\mu(z) \in [-1, 0]$.

The following lemma shows that if one of the two bads is located at 0, then the other one cannot be located at an interior point of its country.

**Lemma 3.2.** *Let $z \in \mathcal{R}$.*

*(a) If $\alpha(z) = 0 < \beta(z)$, then $z(i) \leq \frac{1}{2}$ for all $i \in N$, and $\beta(z) = 1$.*
*(b) If $\alpha(z) < 0 = \beta(z)$, then $z(i) \geq -\frac{1}{2}$ for all $i \in N$, and $\alpha(z) = -1$.*

**Proof.** We only prove part (a), part (b) is analogous. Let $\alpha(z) = 0 < \beta(z)$. Then $\mu(z) \in [0, 1]$, so $z = (-1^{N_A \setminus S(z)}, 0^{S(z)}, \mu(z)^{T(z)}, 1^{N_B \setminus T(z)})$. Since all agents $i \in T(z)$ are indifferent between (0, 0) and $(0, \beta(z))$ and all agents $i \in N_B \setminus T(z)$ strictly prefer (0, 0) to $(0, \beta(z))$, CPO implies that $T(z) = N_B$. From this, $\beta(z) = 1$ follows by FA, and thus $z(i) \leq \mu(z) = \frac{1}{2}$ for all $i \in N$. ■

The next lemma shows that if one of the two bads is located at the border, then the other is located at the border as well.

**Lemma 3.3.** *Let $z \in \mathcal{R}$. Then $\alpha(z) \in \{-1, 0\}$ if and only if $\beta(z) \in \{0, 1\}$.*

**Proof.** We show the if-direction, the other direction is analogous. By Lemma 3.2(b) it is sufficient to prove that $\alpha(z) \in \{-1, 0\}$ if $\beta(z) = 1$. To the contrary suppose $-1 < \alpha(z) < 0$ and $\beta(z) = 1$. Then $0 < \mu(z) < \frac{1}{2}$ and $T(z) = N_B$ by definition of $T(z)$, so that $z = (-1^{N_A \setminus S(z)}, 0^{S(z)}, \mu(z)^{N_B})$. By FA, $\beta(z') = 1$ for all profiles $z' \in \mathcal{R}$ with $z'(i) = \mu(z)$ for all $i \in N_B$. We will compare the four profiles in the following table:

| | $N_A \setminus S(z)$ | $S(z)$ | $N_B$ | $\alpha$ | $\beta$ |
|---|---|---|---|---|---|
| $z$ | $-1$ | $0$ | $\mu(z)$ | $\alpha(z)$ | $1$ |
| $z^-$ | $\frac{\alpha(z)-1}{2}$ | $0$ | $\mu(z)$ | $\alpha(z^-)$ | $1$ |
| $z^*$ | $\frac{\alpha(z)-1}{2}$ | $\frac{\alpha(z)}{2}$ | $\mu(z)$ | $\alpha(z^*)$ | $1$ |
| $z^+$ | $-1$ | $\frac{\alpha(z)}{2}$ | $\mu(z)$ | $\alpha(z^+)$ | $1.$ |

(For instance, the first line of this table means that $z = (-1^{N_A \setminus S(z)}, 0^{S(z)}, \mu(z)^{N_B})$ and that $\beta(z) = 1$. Note that to all these profiles $\beta$ assigns location 1 by FA.) Consider profiles $z$ and $z^-$. SP implies that $\alpha(z^-) \in \{\alpha(z), -1\}$, and then CPO implies that $\alpha(z^-) = -1$. Applying SP at profiles $z^-$ and $z^*$ now yields that $\alpha(z^*) = -1$. Considering SP at the profiles $z$ and $z^+$ yields that $\alpha(z^+) \in \{\alpha(z), 0\}$, and then CPO implies $\alpha(z^+) = 0$. Finally, comparing profiles $z^+$ and $z^*$ yields a contradiction with SP since $(0, 1) = \varphi(z^+)$ is better for dip $\frac{\alpha(z)-1}{2}$ than $(-1, 1) = \varphi(z^*)$. ■

**Proof of Theorem 3.1.** Let $z \in \mathcal{R}$ and suppose that $-1 < \alpha(z) < 0$ and $0 < \beta(z) < 1$. We will derive a contradiction: then the proof is complete by Lemma 3.3. We assume without loss of generality that $\mu(z) \in [0, 1]$, so that $z = (-1^{N_A \setminus S(z)}, 0^{S(z)}, \mu(z)^{T(z)}, 1^{N_B \setminus T(z)})$.

First note that $S(z) \neq \emptyset, N_A$ and $T(z) \neq \emptyset, N_B$ by CPO. Consider the following profiles, where $t \in \mathbb{N}$:

| | $N_A \setminus S(z)$ | $S(z)$ | $T(z)$ | $N_B \setminus T(z)$ |
|---|---|---|---|---|
| $z^0 = z$ | $-1$ | $0$ | $\mu(z)$ | $1$ |
| $z^t$ $(t \geq 1)$ | $-1$ | $\frac{\alpha(z^{t-1})}{2}$ | $\mu(z)$ | $1$ |
| $v^1$ | $\frac{\alpha(z^0)-1}{2}$ | $0$ | $\mu(z)$ | $1$ |
| $v^t$ $(t \geq 2)$ | $\frac{\alpha(z^{t-1})-1}{2}$ | $\frac{\alpha(z^{t-2})}{2}$ | $\mu(z)$ | $1$ |
| $w^t$ $(t \geq 1)$ | $\frac{\alpha(z^{t-1})-1}{2}$ | $\frac{\alpha(z^{t-1})}{2}$ | $\mu(z)$ | $1.$ |

The proof proceeds in a few steps.

*Step* 1. Let $t \geq 1$ and suppose that $-1 < \alpha(z^{t-1}) < 0$. Then $\varphi(v^t) = \varphi(w^t) = (-1, 1)$.

**Proof.** Comparing $z^{t-1}$ and $v^t$, SP implies $\alpha(v^t) \leq \alpha(z^{t-1})$ since otherwise $N_A \backslash S(z)$ can manipulate from $z^{t-1}$ to $v^t$, and therefore $\alpha(v^t) \in \{-1, \alpha(z^{t-1})\}$ since otherwise $N_A \backslash S(z)$ can manipulate from $v^t$ to $z^{t-1}$. By CPO, this implies $\alpha(v^t) = -1$. Therefore, by Lemma 3.3, $\beta(v^t) \in \{0, 1\}$. Now Lemma 3.2(b) implies that $\beta(v^t) \neq 0$. Thus, $\beta(v^t) = 1$ and $\varphi(v^t) = (-1, 1)$. By SP, going from $w^t$ to $v^t$, we obtain $\alpha(w^t) = -1$. By Lemma 3.3, $\beta(w^t) \in \{0, 1\}$. Since $\frac{\alpha(z^{t-1})-1}{2} < -\frac{1}{2}$, Lemma 3.2(b) then implies $\beta(w^t) = 1$. Thus, $\varphi(w^t) = (-1, 1)$.

*Step* 2. For all $t \geq 1$:

$$-1 < \alpha(z^t) < \alpha(z^{t-1}) < 0 \leq 2\mu(z) < \beta(z^t) \leq -\alpha(z^{t-1}) < 1.$$

**Proof.** By assumption we have $-1 < \alpha(z^0) = \alpha(z) < 0 \leq 2\mu(z) < \beta(z^0) = \beta(z) < 1$. We prove the statement in Step 2 by induction. Assume it is true for all $s < t$, where $t \geq 1$. By going from $z^t$ to $z^{t-1}$, SP implies $\alpha(z^t) \in [-1, \alpha(z^{t-1})]$ or $\alpha(z^t) = 0$. If $\alpha(z^t) = 0$, then $N_A \backslash S(z)$ could manipulate from $w^t$ to $z^t$ since $\varphi(w^t) = (-1, 1)$, as established in Step 1. Hence, $\alpha(z^t) \in [-1, \alpha(z^{t-1})]$. Now CPO applied to the profile $z^t$ implies $\alpha(z^t) \neq \alpha(z^{t-1})$, so $\alpha(z^t) \in [-1, \alpha(z^{t-1}))$. In turn, this and the induction hypothesis imply by SP, going from $z^{t-1}$ to $z^t$, that $\beta(z^t) \leq -\alpha(z^{t-1})$. Then $\alpha(z^t) > -1$, since otherwise by Lemma 3.3, $\beta(z^t) = 0$, contradicting Lemma 3.2(b), or $\beta(z^t) = 1$, contradicting that $-\alpha(z^{t-1}) < 1$ by the induction hypothesis. Finally, $2\mu(z) < \beta(z^t)$ follows since otherwise CPO would imply that $\beta(z^t) = 0$.

*Step* 3. $\mu(z) = 0$.

**Proof.** First suppose that for some $t > 1$, $\mu(z^t) \geq \frac{\alpha(z^{t-2})}{2}$. Then, by Step 2, for all $i \in S(z)$ we have $\alpha(z^t) < \frac{\alpha(z^{t-1})}{2} = z^t(i) < z^{t-1}(i) = \frac{\alpha(z^{t-2})}{2} \leq \mu(z^t)$, so that by MON we have $\varphi(z^{t-1}) = \varphi(z^t)$, a contradiction. Thus, $\mu(z^t) < \frac{\alpha(z^{t-2})}{2}$ for all $t > 1$. Hence, $\alpha(z^t) + \beta(z^t) < \alpha(z^{t-2})$ for all $t > 1$. By Step 2 this implies $2\mu(z) < \beta(z^t) < \alpha(z^{t-2}) - \alpha(z^t)$, which implies that $\mu(z) = 0$ since $\alpha(z^{t-2}) - \alpha(z^t)$ converges to 0 for $t$ going to infinity.

*Step* 4. If $\mu(z) = 0$ then $\mu(z^1) < 0$.

**Proof.** Follows from Step 2 by taking $t = 1$.

We can now complete the proof of the theorem. Step 3 implies that $\mu(z) = 0$ for *any* profile $z$ with $\alpha(z) \in (-1, 0)$ and $\beta(z) \in (0, 1)$—observe that this indeed does not depend on our initial assumption $\mu(z) \geq 0$. This contradicts Step 4 since $z^1$ is also such a profile. ■

### 3.2. *Lexmin preferences*

Let now $\mathcal{R}$ be the set of all profiles of lexmin preferences and let $\varphi : \mathcal{R} \to \mathcal{A}$ be a rule satisfying strategy-proofness and country-wise Pareto optimality. We now have the following result, the proof of which follows similar lines as the proof of Theorem 3.1 but differs in some details—see the Appendix to the paper.

**Theorem 3.4.** *$\varphi(z) \in \{(-1, 1), (0, 0), (-1, 0), (0, 1)\}$ for every $z \in \mathcal{R}$.*

This theorem is quite similar to Theorem 3.1 for myopic preferences, but in this case it is sufficient to require only strategy-proofness and country-wise Pareto optimality to obtain the result, due to the fact that there are only few indifferences in the case of lexmin preferences.

In fact, for all $(a, b), (a', b') \in \mathcal{A}$, if an agent prefers $(a, b)$ over $(a', b')$ at a lexmin preference, then that agent also prefers $(a, b)$ over $(a', b')$ at a myopic preference with the same dip. From this, it is easy to deduce that strategy-proofness under lexmin preference profiles implies strategy-proofness under myopic preference profiles; and that country-wise Pareto optimality under lexmin preference profiles implies both country-wise Pareto optimality and the far away condition under myopic preference profiles. Non-corruptibility under lexmin preferences is always satisfied but non-corruptibility under myopic preference profiles is not implied by strategy-proofness under lexmin preference profiles. The following example exhibits a rule that satisfies all relevant properties except non-corruptibility under myopic preference profiles.

**Example 3.5.** Let $\mathcal{R}$ be either the set of all profiles of myopic preferences or the set of all profiles of lexmin preferences. We define the rule $\varphi$ on $\mathcal{R}$ as follows. Let $z \in \mathcal{R}$ then $\varphi(z) = (\alpha(z), \beta(z))$, where

$$\alpha(z) = \begin{cases} -1 & \text{if } \left|\left\{i \in N_A : z(i) > -\frac{1}{2}\right\}\right| \\ & \geq \left|\left\{i \in N_A : z(i) < -\frac{1}{2}\right\}\right| \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta(z) = \begin{cases} 1 & \text{if } \left|\left\{i \in N_B : z(i) < \frac{1}{2}\right\}\right| \geq \left|\left\{i \in N_B : z(i) > \frac{1}{2}\right\}\right| \\ 0 & \text{otherwise.} \end{cases}$$

Note that this rule means that each country, independently of the other country, makes a majority decision.

Consider a profile $w$ such that $|\{i \in N_A : w(i) < -\frac{1}{2}\}| = |\{i \in N_A : w(i) > -\frac{1}{2}\}| + 1$, and such that $|\{i \in N_B : z(i) < \frac{1}{2}\}| < |\{i \in N_B : z(i) > \frac{1}{2}\}|$. Then $\varphi(w) = 00$. If, for some $i \in N_A$, $w(i) = -\frac{1}{2}$ and $w'$ is an $i$-deviation of $w$ with $w'(i) > -\frac{1}{2}$, then $\varphi(w') = -10$. Now at both $w(i)$ and $w'(i)$, agent $i$ is indifferent between 00 and $-10$. Hence $\varphi$ is corruptible (under myopic preferences). However, $\varphi$ is strategy-proof and country-wise Pareto optimal.

We conclude this section with an example of a rule that admits internal locations but is still strategy-proof and unanimous for myopic preferences.

**Example 3.6.** For every $z \in \mathcal{R}$ define

$$f(z) = (\alpha(z), \beta(z)) = (-1, \max\{0, 2\min\{z(i) : i \in N_A\} + 1\}).$$

Hence, this rule always assigns $-1$ in country $A$, and in $B$ either 0 or the point that has the same distance to the minimum dip in country $A$ as this minimum dip has to $-1$. This rule is strategy-proof and non-corruptible for myopic preferences. It is also unanimous, which here boils down to the requirement that $f(z) = -11$ if $z(i) = 0$ for all $i \in N$; and anonymous, which means that it is independent of a permutation of the residents of $A$ and of the residents of $B$. Clearly, it is not country-wise Pareto optimal and does not satisfy the FA condition.

## 4. The rules

In this section we completely characterize the class of rules satisfying the conditions in Theorem 3.1 for myopic preferences. For the case of lexmin preferences we do not present the complete characterization but highlight some similarities and differences with the class of rules under myopic preferences.

### 4.1. Rules under myopic preferences

Throughout this subsection we assume that $\mathcal{R}$ is the set of all profiles of myopic preferences. If $\varphi$ is a rule satisfying CPO, SP, NC, and FA, then Theorem 3.1 says that the range of $\varphi$ is $\mathcal{B} = \{-11, -10, 01, 00\}$, where $-11$ denotes $(-1, 1)$, etc. We first show that for such rules $\varphi$ only the preferences restricted to $\mathcal{B}$ matter.

**Lemma 4.1.** *Let rule $\varphi$ satisfy CPO, SP, NC, and FA, and let $z, z' \in \mathcal{R}$ such that $z$ and $z'$ coincide on $\mathcal{B}$. Then $\varphi(z) = \varphi(z')$.*

**Proof.** Without loss of generality we may assume that $z$ and $z'$ are $i$-deviations. Theorem 3.1 implies that $\varphi(z)$ and $\varphi(z')$ are both in $\mathcal{B}$. Since $z$ and $z'$ coincide on $\mathcal{B}$, agent $i$ has at $z(i)$ the same preference between $\varphi(z)$ and $\varphi(z')$ as at $z'(i)$. By SP, agent $i$ must be indifferent between $\varphi(z)$ and $\varphi(z')$ both at $z(i)$ and $z'(i)$. Hence $\varphi(z) = \varphi(z')$ by NC. ■

On $\mathcal{B}$ there are just four different single-dipped preferences. These preferences, with dip $x$ and symmetric and asymmetric parts $\sim$ and $\succ$, are the following:

- If $-1 \le x < \frac{1}{2}$, then $00 \sim 01 \succ -11 \sim -10$.
- If $x \in \{-\frac{1}{2}, \frac{1}{2}\}$, then $00 \sim 01 \sim -11 \sim -10$.
- If $-\frac{1}{2} < x < \frac{1}{2}$, then $-11 \succ -10 \sim 00 \sim 01$.
- If $\frac{1}{2} < x \le 1$, then $-10 \sim 00 \succ -11 \sim 01$.

We will show that set of rules satisfying CPO, SP, NC, and FA, consists of monotonic voting between $-11$ and 00, except for cases where $-11$ and 00 cannot be selected because of FA or CPO. These voting rules are characterized by families of decisive pairs of coalitions of agents. The first coalition of such a pair contains the agents with dip strictly between $-\frac{1}{2}$ and $\frac{1}{2}$: these agents strictly prefer outcome $-11$ over outcome 00. The second coalition contains the agents with dip either $-\frac{1}{2}$ or $\frac{1}{2}$: these agents are indifferent between $-11$ and 00. We will now make this more precise.

**Definition 4.2.** $\mathcal{W} \subseteq 2^N \times 2^N$ is a *family of decisive pairs* if

(d1) $(U, V) \in \mathcal{W}$ for all $U, V \subseteq N$ with $U \cup V = N$,
(d2) $(U', V') \in \mathcal{W}$ for all $U', V' \subseteq N$ for which there is $(U, V) \in \mathcal{W}$ with $U \subseteq U'$ and $U \cup V \subseteq U' \cup V'$.
(d3) $U \cap N_A \neq \emptyset$ or $N_A \subseteq V$ for all $(U, V) \in \mathcal{W}$.
(d4) $U \cap N_B \neq \emptyset$ or $N_B \subseteq V$ for all $(U, V) \in \mathcal{W}$.

With a rule $\varphi$ that has the properties CPO, SP, NC, and FA, we will associate a family of decisive pairs, as follows. For a profile $z \in \mathcal{R}$ let $U(z) = \{i \in N : -\frac{1}{2} < z(i) < \frac{1}{2}\}$, which is the set of agents who strictly prefer $-11$ to 00; and let $V(z) = \{i \in N : z(i) \in \{-\frac{1}{2}, \frac{1}{2}\}\}$, which is the set of agents who are indifferent between $-11$ and 00. We define

$$\mathcal{W}_\varphi = \{(U, V) \in 2^N \times 2^N : \text{there exists } z \in \mathcal{R} \text{ with } (U, V) = (U(z), V(z)) \text{ and } \varphi(z) = -11\}.$$

Observe that, by Lemma 4.1, $\varphi(z) = -11$ for *all* $z \in \mathcal{R}$ such that $(U(z), V(z)) \in \mathcal{W}_\varphi$. We now have:

**Lemma 4.3.** *Let $\varphi$ satisfy SP, CPO, NC, and FA. Then $\mathcal{W}_\varphi$ is a family of decisive pairs.*

**Proof.** For condition (d1), let $U, V \subseteq N$ with $U \cup V = N$. Take a profile $z$ with $z(i) \in (-\frac{1}{2}, \frac{1}{2})$ for all $i \in U$ and $z(i) \in \{-\frac{1}{2}, \frac{1}{2}\}$ for all $i \in V$. Then $\varphi(z) = -11$ by FA (or CPO), hence $(U, V) = (U(z), V(z)) \in \mathcal{W}_\varphi$.

For condition (d2), let $z \in \mathcal{R}$ with $\varphi(z) = -11$. Consider a $j$-deviation $z'$ of $z$ such that $U(z) \subseteq U(z')$ and $U(z) \cup V(z) \subseteq U(z') \cup V(z')$. It is sufficient to prove that $\varphi(z') = -11$. Without loss of generality assume that $j \in N_A$. Assume $U(z) \neq U(z')$ or $V(z) \neq V(z')$, otherwise we are done by Lemma 4.1. Since $U(z) \subseteq U(z')$ and $U(z) \cup V(z) \subseteq U(z') \cup V(z')$ it follows that $z(j) < z'(j) \le 0$. So by MON, $\varphi(z') = \varphi(z) = -11$.

For condition (d3), let $(U, V) \in 2^N \times 2^N$ and suppose that $U \cap N_A = \emptyset$ and $N_A \not\subseteq V$ (the other case is similar). Let $z$ be any profile with $(U(z), V(z)) = (U, V)$. Then $z(i) \le -\frac{1}{2}$ for all $i \in N_A$ and $z(i) < -\frac{1}{2}$ for some $i \in N_A$. By CPO, $\alpha(z) = 0$. Hence, $(U, V) \notin \mathcal{W}_\varphi$. ■

Conversely, for a family of decisive pairs $\mathcal{W}$ we define a rule $\varphi_\mathcal{W}$ as follows. For every $z \in \mathcal{R}$:

$$\varphi_\mathcal{W}(z) = \begin{cases} -11 & \text{if } (U(z), V(z)) \in \mathcal{W} \\ -10 & \text{if } (U(z), V(z)) \notin \mathcal{W} \text{ and } N_A \subseteq U(z) \cup V(z) \\ 01 & \text{if } (U(z), V(z)) \notin \mathcal{W} \text{ and } N_B \subseteq U(z) \cup V(z) \\ 00 & \text{otherwise.} \end{cases}$$

In words, $\varphi_\mathcal{W}$ assigns $-11$ to a profile $z$ if the pair $(U(z), V(z))$ is decisive. Otherwise, it assigns 00 unless FA demands otherwise, that is, $-10$ or 01. Next, we prove that $\varphi_\mathcal{W}$ satisfies our four conditions.

**Lemma 4.4.** *Let $\mathcal{W}$ be a family of decisive pairs. Then $\varphi_\mathcal{W}$ satisfies SP, CPO, NC, and FA.*

**Proof.** We first prove SP of $\varphi_\mathcal{W}$. Consider $z \in \mathcal{R}$ and an $i$-deviation $z'$ of $z$ for $i \in N_A$. It is sufficient to prove that $i$ weakly prefers $\varphi_\mathcal{W}(z)$ to $\varphi_\mathcal{W}(z')$. This is evidently the case if $\varphi_\mathcal{W}(z) = \varphi_\mathcal{W}(z')$ or if $z(i) = -\frac{1}{2}$. Therefore assume that $\varphi_\mathcal{W}(z) \neq \varphi_\mathcal{W}(z')$ and that $z(i) \neq -\frac{1}{2}$. We distinguish the following two cases.

- $-1 \le z(i) < -\frac{1}{2}$. Then $U(z) \subseteq U(z')$ and $U(z) \cup V(z) \subseteq U(z') \cup V(z')$ and because of $\varphi_\mathcal{W}(z) \neq \varphi_\mathcal{W}(z')$ at least one of these inclusions is strict. Hence, (d2) and $\varphi_\mathcal{W}(z) \neq \varphi_\mathcal{W}(z')$ imply $(U(z), V(z)) \notin \mathcal{W}$ and $\varphi_\mathcal{W}(z) \neq -11$. Since $N_A \not\subseteq U(z) \cup V(z)$, we have $\varphi_\mathcal{W}(z) \neq -10$. Hence, $\varphi_\mathcal{W}(z) \in \{00, 01\}$, so that $i$ at $z(i)$ weakly prefers $\varphi_\mathcal{W}(z)$ to $\varphi_\mathcal{W}(z')$.
- $-\frac{1}{2} < z(i) \le 0$. Then $U(z') \subseteq U(z)$ and $U(z') \cup V(z') \subseteq U(z) \cup V(z)$, and because of $\varphi_\mathcal{W}(z) \neq \varphi_\mathcal{W}(z')$ at least one of these inclusions is strict. If $\varphi_\mathcal{W}(z) \in \{00, 01\}$, then by (d2) of $\mathcal{W}$ and the definition of $\varphi_\mathcal{W}$ we have $\varphi_\mathcal{W}(z) = \varphi_\mathcal{W}(z')$, a contradiction. If $\varphi_\mathcal{W}(z) = -10$ and $z'(i) < -\frac{1}{2}$, then $\varphi_\mathcal{W}(z') = 00$ and in that case agent $i$ does not manipulate. Otherwise, $\varphi_\mathcal{W}(z) = -11$, which is the single best outcome at $z(i)$.

We next prove CPO of $\varphi_\mathcal{W}$. It is sufficient to prove this for country $A$. To the contrary, suppose that all agents in $N_A$ weakly prefer $(a, \beta_\mathcal{W}(z))$ to $\varphi_\mathcal{W}(z) = (\alpha_\mathcal{W}(z), \beta_\mathcal{W}(z))$ and some $j$ in $N_A$ strictly. We distinguish three cases.

- $\alpha_\mathcal{W}(z) = 0$. Then $(U(z), V(z)) \notin \mathcal{W}$ and all agents in $N_A$ have their dip equal to or greater than $\frac{a}{2} \ge -\frac{1}{2}$. Hence $N_A \subseteq U(z) \cup V(z)$, which implies $\alpha_\mathcal{W}(z) = -1$, a contradiction.
- $\beta_\mathcal{W}(z) = 0$ and $\alpha_\mathcal{W}(z) = -1$. Then $(U(z), V(z)) \notin \mathcal{W}$ and $N_A \subseteq U(z) \cup V(z)$. But then all agents in $N_A$ have their dip greater than or equal to $-\frac{1}{2}$, which contradicts the existence of agents $j$ who strictly prefer $(a, 0)$ to $-10$.

- $\beta_{\mathcal{W}}(z) = 1$ and $\alpha_{\mathcal{W}}(z) = -1$. Then $(U(z), V(z)) \in \mathcal{W}$. So, by condition (d3) of $\mathcal{W}$ there are agents $i \in N_A \cap U(z)$ or all agents in $N_A$ have their dip at $-\frac{1}{2}$. Since agents in $N_A \cap U(z)$ strictly prefer $-11$ to every other outcome $(a, 1)$, we must have that all agents in $N_A$ have their dip at $-\frac{1}{2}$. At dip $-\frac{1}{2}$, however, outcome $-11$ is weakly preferred to every outcome $(a, 1)$ for $-1 \le a \le 0$. This contradicts the existence of agents $j$ who strictly prefer $(a, 1)$ to $-11$.

Third, we next prove NC of $\varphi_{\mathcal{W}}$. Consider $z \in \mathcal{R}$ and an $i$-deviation $z'$ of $z$ for $i \in N_A$, and suppose that $i$ is indifferent between $\varphi_{\mathcal{W}}(z)$ and $\varphi_{\mathcal{W}}(z')$ both at $z(i)$ and at $z'(i)$. It is sufficient to prove that $\varphi_{\mathcal{W}}(z) = \varphi_{\mathcal{W}}(z')$. We may assume that $z(i) < z'(i)$ and the ordering at $z(i)$ of $\mathcal{B}$ is different from that of $z'(i)$. We distinguish two cases.

- $-1 \le z(i) < -\frac{1}{2}$. Then $\varphi_{\mathcal{W}}(z) \neq -10$ since $N_A \not\subseteq U(z) \cup V(z)$. If $\varphi_{\mathcal{W}}(z) = -11$ then by $z(i) < z'(i)$ and (d2) of $\mathcal{W}$ we have $\varphi_{\mathcal{W}}(z') = -11$ and we are done. Suppose $\varphi_{\mathcal{W}}(z) \in \{00, 01\}$. Then since at $z(i)$ outcomes $\varphi_{\mathcal{W}}(z)$ and $\varphi_{\mathcal{W}}(z')$ are indifferent we have $\varphi_{\mathcal{W}}(z') \in \{00, 01\}$. Then, since $N_B \subseteq U(z) \cup V(z)$ if and only if $N_B \subseteq U(z') \cup V(z')$, it follows that $\varphi_{\mathcal{W}}(z) = 01$ if and only if $\varphi_{\mathcal{W}}(z') = 01$.
- $z(i) = -\frac{1}{2}$ and $-\frac{1}{2} < z'(i) \le 0$. If $\varphi_{\mathcal{W}}(z') = -11$, then the indifference between $\varphi_{\mathcal{W}}(z)$ and $\varphi_{\mathcal{W}}(z')$ at $z'(i)$ yields that $\varphi_{\mathcal{W}}(z) = \varphi_{\mathcal{W}}(z') = -11$. Since $N_A \subseteq U(z') \cup V(z')$ if and only if $N_A \subseteq U(z) \cup V(z)$, it follows that $\varphi_{\mathcal{W}}(z) = -10$ if and only if $\varphi_{\mathcal{W}}(z') = -10$. Further, since $N_B \subseteq U(z) \cup V(z)$ if and only if $N_B \subseteq U(z') \cup V(z')$, it follows that $\varphi_{\mathcal{W}}(z) = 01$ if and only if $\varphi_{\mathcal{W}}(z') = 01$. Hence, we also have $\varphi_{\mathcal{W}}(z) = 00$ if and only if $\varphi_{\mathcal{W}}(z') = 00$ since that is the only remaining case. Thus, $\varphi_{\mathcal{W}}(z) = \varphi_{\mathcal{W}}(z')$.

Finally, we prove FA of $\varphi$. Suppose that all agents weakly prefer $(\alpha_{\mathcal{W}}(z), 1)$ to $\varphi_{\mathcal{W}}(z) = (\alpha_{\mathcal{W}}(z), \beta_{\mathcal{W}}(z))$. It is sufficient to prove that $\beta_{\mathcal{W}}(z) = 1$. To the contrary suppose $\beta_{\mathcal{W}}(z) = 0$. Then all agents in $N_B$ have their dip smaller than or equal to $\frac{1}{2}$. So, $N_B \subseteq U(z) \cup V(z)$. As $\beta_{\mathcal{W}}(z) = 0$, we have $(U(z), V(z)) \notin \mathcal{W}$. This however contradicts the definition of $\varphi_{\mathcal{W}}$ because if $(U(z), V(z)) \notin \mathcal{W}$ and $N_B \subseteq U(z) \cup V(z)$, then $\varphi_{\mathcal{W}}(z) = 01$. ■

We can now formulate the main result of this section, which is a corollary to the preceding two lemmas.

**Corollary 4.5.** *Let $\varphi$ be a rule. Then $\varphi$ satisfies SP, CPO, NC, and FA, if and only if there is a family $\mathcal{W}$ of decisive pairs such that $\varphi = \varphi_{\mathcal{W}}$.*

**Proof.** If $\mathcal{W}$ is a family of decisive pairs, then $\varphi_{\mathcal{W}}$ satisfies SP, CPO, NC, and FA by Lemma 4.4. Conversely, let $\varphi$ satisfy these four conditions. We show that $\varphi = \varphi_{\mathcal{W}_\varphi}$, which completes the proof by Lemma 4.3. Let $z \in \mathcal{R}$. Then $\varphi(z) = -11 \Leftrightarrow (U(z), V(z)) \in \mathcal{W}_\varphi \Leftrightarrow \varphi_{\mathcal{W}_\varphi}(z) = -11$. If $\varphi_{\mathcal{W}_\varphi}(z) = 01$ then (by the previous step) $\varphi(z) \neq -11$ and moreover, $N_B \subseteq U(z) \cup V(z)$, so that $\varphi(z) = 01$ by FA. Similarly, $\varphi_{\mathcal{W}_\varphi}(z) = -10$ implies $\varphi(z) = -10$. Hence, we also have $\varphi_{\mathcal{W}_\varphi}(z) = 00$ if and only if $\varphi(z) = 00$. ■

### *4.2. Rules under lexmin preferences*

Rather than characterizing the set of all rules when agents have lexmin preferences we will show that this set is neither contained in nor contains the set of rules in Corollary 4.5 for myopic preferences. We start with identifying the rules in the latter class that are strategy-proof and country-wise Pareto optimal on the class of profiles with lexmin preferences.

**Corollary 4.6.** *Let $\mathcal{R}$ be the set of all profiles of lexmin preferences and let $\mathcal{W}$ be a family of decisive pairs. Then rule $\varphi_{\mathcal{W}} : \mathcal{R} \to \mathcal{A}$ is country-wise Pareto optimal. Rule $\varphi_{\mathcal{W}}$ is strategy-proof if and only if $\mathcal{W}$ satisfies the following condition: for all $(U, V) \in \mathcal{W}$, if $i \in N_A \cap U$ and $N_B \not\subseteq U \cup V$ or if $i \in N_B \cap U$ and $N_A \not\subseteq U \cup V$, then $(U \backslash \{i\}, V \cup \{i\}) \in \mathcal{W}$.*

**Proof.** CPO of $\varphi_{\mathcal{W}}$ follows by exactly the same arguments as in the proof of Lemma 4.4.

For the only-if direction concerning strategy-proofness, suppose without loss of generality that there are $(U, V) \in \mathcal{W}$ and $i \in N_A \cap U$ such that $N_B \not\subseteq U \cup V$ and $(U\backslash\{i\}, V \cup \{i\}) \notin \mathcal{W}$. Consider a profile $z$ with $U(z) = U\backslash\{i\}$, $V(z) = V \cup \{i\}$, and $z(j) > \frac{1}{2}$ for some $j \in N_B$. Then $N_B \not\subseteq (U\backslash\{i\}) \cup V \cup \{i\}$, so that $\varphi_{\mathcal{W}}(z) \in \{-10, 00\}$. Consider an $i$-deviation $z'$ with $z'(i) > -\frac{1}{2}$. Then $(U(z'), V(z')) = (U, V) \in \mathcal{W}$, so that $\varphi_{\mathcal{W}}(z') = -11$, which implies that $i$ manipulates from $z$ to $z'$.

For the if-direction concerning strategy-proofness, again the same arguments as in the proof of Lemma 4.4 apply, with one exception, namely the case where $z(i) = -\frac{1}{2}$: this case follows by using the additional condition on $\mathcal{W}$ in the corollary. ■

See Example 3.5 for a rule which is not of the form $\varphi_{\mathcal{W}}$ for some family of decisive pairs $\mathcal{W}$, but which is nevertheless strategy-proof and country-wise Pareto optimal both for myopic and for lexmin preferences. According to that rule the countries make their decisions independently. An example of a rule where countries decide independently but corresponding to a family of decisive pairs is the following.

**Example 4.7.** Let $\mathcal{W} = \{(U, V) \in 2^N \times 2^N : U \cup V = N\}$, then $\mathcal{W}$ satisfies (d1)–d(4) and therefore is a family of decisive pairs. Then $\varphi_{\mathcal{W}}(z) = (\alpha(z), \beta(z))$, where $\alpha(z) = -1$ if $z(i) \ge -\frac{1}{2}$ for all $i \in N_A$ and $\alpha(z) = 0$ otherwise; and $\beta(z) = 1$ if $z(i) \le \frac{1}{2}$ for all $i \in N_B$ and $\beta(z) = 0$ otherwise. This rule satisfies SP, CPO, NC, and FA for both myopic and lexmin preferences.

**Remark 4.8.** By carefully considering the rules in Corollary 4.5, it is not hard to show that the rule in Example 4.7 is in fact the only rule in the corollary in which the countries decide independently, as follows. Let $\mathcal{W}$ be a family of decisive pairs and let $\varphi_{\mathcal{W}}$ be the corresponding rule. Suppose that there is a pair $(U, V) \in \mathcal{W}$ such that $U \cup V \neq N$, without loss of generality $i \notin U \cup V$ for some $i \in N_A$. Let $z$ be a profile with $z(i) = -1$, $U(z) = U$, and $V(z) = V$. Then $\varphi_{\mathcal{W}}(z) = -11$. Now consider a profile $z'$ with $z'(j) = z(j)$ for all $j \in N_A$ and $z'(j) = -1$ for all $j \in N_B$. Then, since $U(z') \cap N_B = \emptyset$ and $N_B \not\subseteq V(z')$, we have $(U(z'), V(z')) \notin \mathcal{W}$. Hence, since $N_A \not\subseteq U(z') \cup V(z')$, we obtain $\varphi(z') = 00$. Thus, $\alpha(z') = 0 \neq -1 = \alpha(z)$ although $z(j) = z'(j)$ for all $j \in N_A$. We conclude that the countries do not decide independently.

An example of a rule satisfying all conditions both for myopic and lexmin preferences, but in which the countries do not decide independently, is as follows.

**Example 4.9.** For convenience suppose that $n_A = n_B \ge 2$, and let $\mathcal{W}$ be the set of all $(U, V) \in 2^N \times 2^N$ satisfying at least one of the following three conditions: (i) $U \cup V = N$, (ii) $N_A \subseteq U \cup V$ and $N_B \cap U \neq \emptyset$, (iii) $N_B \subseteq U \cup V$ and $N_A \cap U \neq \emptyset$. Then $\mathcal{W}$ satisfies (d1)–(d4) as well as the condition in Corollary 4.6, so that $\varphi_{\mathcal{W}}$ satisfies SP, CPO, NC, and FA for myopic preferences as well as lexmin preferences. Let $z$ be a profile of preferences such that $z(i) \ge -\frac{1}{2}$ for all $i \in N_A$, $z(j) < \frac{1}{2}$ for some agent $j \in N_B$, and $z(k) > \frac{1}{2}$ for some $k \in N_B$. Then $\varphi_{\mathcal{W}}(z) = -11$. Pick an agent $i_1 \in N_A$ and consider the profile $\hat{z}$ equal to $z$ except that $\hat{z}(i_1) < -\frac{1}{2}$. Then $\varphi_{\mathcal{W}}(\hat{z}) = 00$, so that the decision in country $B$ has been altered by an agent in country $A$.

### *4.3. Further examples*

The class of rules $\varphi_{\mathcal{W}}$ for families $\mathcal{W}$ of decisive pairs contains many different rules, ranging from majority voting to almost dictatorial rules, as the following examples show.

**Example 4.10.** Let $\mathcal{W}$ consist of all pairs $(U, V) \in 2^N \times 2^N$ such that $|U| \geq |N \backslash (U \cup V)|$ and (d3) and (d4) are satisfied. Then $\mathcal{W}$ is a family of decisive pairs. The associated rule $\varphi_{\mathcal{W}}$ assigns to a preference profile $z$ the outcome $-11$ if there is a weak majority with strict preference for $-11$. If not, then it assigns $-10$ if all agents in country $A$ weakly prefer $-1$ and it assigns 01 if all agents in $B$ weakly prefer 1. In all other cases it assigns 00. By Corollary 4.5 this rule satisfies SP, CPO, NC, and FA for myopic preferences. By Corollary 4.6 it is not strategy-proof for lexmin preferences.

**Example 4.11.** Fix agents $i_A \in N_A$ and $i_B \in N_B$. Let $\mathcal{W}$ consist of all pairs $(U, V) \in 2^N \times 2^N$ such that both $i_A$ and $i_B$ are in $U$ or $N = U \cup V$. Then $\mathcal{W}$ is a family of decisive pairs. The associated rule $\varphi_{\mathcal{W}}$ assigns to a preference profile $z$ the outcome $-11$ if all agents weakly prefer this outcome or if $i_A$ and $i_B$ strictly prefer it. If not, then it assigns $-10$ if all agents in country $A$ weakly prefer $-1$ and it assigns 01 if all agents in $B$ weakly prefer 1. In all other cases it assigns 00. By Corollary 4.5 this rule satisfies SP, CPO, NC, and FA for myopic preferences. By Corollary 4.6 again it is not strategy-proof for lexmin preferences. In this rule, $i_A$ and $i_B$ exercise a kind of joint dictatorship. Observe that we cannot have dictatorial rules since these would violate CPO, both under myopic and leximin preferences.

The following example shows that profitable deviations by coalitions are not necessarily excluded by strategy-proofness.

**Example 4.12.** Consider the rule of Example 4.11, which is strategy-proof for myopic preferences. Let $z$ be a profile such that $z(i_A) = -\frac{1}{2}, z(i_B) = \frac{1}{4}, z(j) = -\frac{3}{4}$ for some $j \in N_A \backslash \{i_A\}$, and $z(k) = \frac{3}{4}$ for some $k \in N_B \backslash \{i_B\}$. Then $\varphi_{\mathcal{W}}(z) = 00$. Let $S = \{i_A, i_B\}$, and let $z'$ be the profile with $z'(i_A) = -\frac{1}{4}, z'(i_B) = \frac{1}{8}$, and $z'(j) = z(j)$ for all $j \in N \backslash \{i_A, i_B\}$. Then $\varphi_{\mathcal{W}}(z') = -11$. Since $-11 P_{z(i_B)} 00$ and $-11 I_{z(i_A)} 00$, $S$ has profitably deviated.

An example of a strategy-proof (and country-wise Pareto optimal) but not coalitionally strategy-proof rule for lexmin preferences can be found in the Appendix of the paper.

### 4.4. Independence of the axioms

We show that the main conditions imposed on rules in this paper are logically independent.

If $\mathcal{R}$ is the set of profiles with lexmin preferences, then any constant rule would be strategy-proof but not country-wise Pareto optimal. Examples 4.10 and 4.11 present rules which are country-wise Pareto optimal but not strategy-proof.

Now let $\mathcal{R}$ be the set of profiles with myopic preferences. We show that SP, CPO, NC, and FA are independent.

The constant rule that assigns $-11$ to every profile satisfies SP, NC, and FA, but not CPO. The rule in Example 3.5 satisfies SP, CPO, and FA, but not NC. For the independence of strategy-proofness consider the following example.

**Example 4.13.** Consider a family of decisive pairs $\mathcal{W}$ with associated rule $\varphi_{\mathcal{W}}$. Let $i_A, j_A$ be two different agents in $N_A$ and let $i_B, j_B$ be two different agents in $N_B$. Define the rule $\widehat{\varphi}$ equal to $\varphi_{\mathcal{W}}$ for all profiles except for profiles $q$ at which $q(i_A) = -1, q(i_B) = 1$, and $q(j_A) = q(j_B) = 0$: then let $\widehat{\varphi}(q) = (-\frac{1}{2}, \frac{1}{2})$. Rule $\widehat{\varphi}$ satisfies FA and CPO, as is not difficult to see. Non-corruptibility can be seen by considering the preferences of the agents $i_A, j_A, i_B, j_B$ at such a profile $q$, as given in the following table.

$$i_A: \quad 00 \sim 01 \succ -\frac{1}{2}\frac{1}{2} \succ -11 \sim -10$$
$$j_A, j_B: \quad -11 \succ -\frac{1}{2}\frac{1}{2} \succ 00 \sim -10 \sim 01$$
$$i_B: \quad 00 \sim -10 \succ -\frac{1}{2}\frac{1}{2} \succ -11 \sim 01.$$

Thus, these four agents are not indifferent between the outcome at $\widehat{\varphi}(q)$ and any $\varphi_{\mathcal{W}}(z) \in \{-11, -10, 00, 01\}$ for profile $z$ not of the form $q$. It is not difficult to see that $\widehat{\varphi}$ is not strategy-proof.

The final example exhibits a rule which satisfies SP, CPO, and NC, but not FA.

**Example 4.14.** Consider a family of decisive pairs $\mathcal{W}$ with associated rule $\varphi_{\mathcal{W}}$. Let rule $\widetilde{\varphi}$ be equal to $\varphi_{\mathcal{W}}$ except that $\widetilde{\varphi}((-\frac{1}{2})^{N_A}, (\frac{1}{2})^{N_B}) = -10$. Then $\widetilde{\varphi}$ violates the far away condition at profile $((-\frac{1}{2})^{N_A}, (\frac{1}{2})^{N_B})$ for country $B$. Clearly, $\widetilde{\varphi}$ satisfies CPO. For SP and NC consider a unilateral deviation $z$ from $((-\frac{1}{2})^{N_A}, (\frac{1}{2})^{N_B})$ by agent $i$. First suppose $z(i) < -\frac{1}{2}$. Then CPO implies that, with $\varphi_{\mathcal{W}}(z) = (\alpha(z), \beta(z)), \alpha(z) = 0$, so that $(U(z), V(z)) \notin \mathcal{W}$. Since $N_B \subseteq U(z) \cup V(z)$, it follows that $\widetilde{\varphi}(z) = 01$. At $((-\frac{1}{2})^{N_A}, (\frac{1}{2})^{N_B})$ agent $i$ is indifferent between $-10$ and 01, but at $z(i)$ agent $i$ strictly prefers 01 to $-10$. It follows that at these deviations the requirements of SP and NC hold.

Next consider the case that $0 \geq z(i) > -\frac{1}{2}$. Then $(N_A \cup N_B) \subseteq (U(z) \cup V(z))$ and $(U(z), V(z)) \in \mathcal{W}$, which implies that $\widetilde{\varphi}(z) = -11$. At $((-\frac{1}{2})^{N_A}, (\frac{1}{2})^{N_B})$ agent $i$ is indifferent between $-10$ and $-11$ and at $z(i)$ agent $i$ strictly prefers outcome $-11$ to $-10$. Hence, also at these deviations the requirements of SP and NC hold.

Now consider the case that $0 \leq z(i) < \frac{1}{2}$. Then $(N_A \cup N_B) \subseteq (U(z) \cup V(z))$ and $(U(z), V(z)) \in \mathcal{W}$, which implies that $\widetilde{\varphi}(z) = -11$. At $((-\frac{1}{2})^{N_A}, (\frac{1}{2})^{N_B})$ agent $i$ is indifferent between $-10$ and $-11$ and at $z(i)$ agent $i$ strictly prefers $-11$ to $-10$. Hence, at these deviations again the requirements of SP and NC hold.

Finally, consider $z(i) > \frac{1}{2}$. CPO implies $\beta(z) = 0$. Hence, $(U(z), V(z)) \notin \mathcal{W}$. Since $N_A \subseteq U(z) \cup V(z)$, it follows that $\widetilde{\varphi}(z) = -10$. Again the requirements of SP and NC hold at these deviations.

## 5. Conclusion

We have studied strategy-proof and country-wise Pareto optimal rules for the joint placement of two public bads in two neighboring countries. For myopic preferences, which are completely determined by the location of the nearer public bad, we have characterized all rules which satisfy two additional tie-breaking conditions. In particular, such rules always assign border locations—in agreement with early findings about single-dipped preferences in the one country case (Peremans and Storcken, 1999). For lexmin preferences, where also the location of the less near public bad is taken into account, we find, similarly that only border locations are assigned under the conditions of strategy-proofness and country-wise Pareto optimality, but the class of associated rules is different.

## Appendix. Remaining proofs

### A.1. Proof of Theorem 3.4

We prove this theorem with the help of the following two lemmas. Throughout, $\mathcal{R}$ is the set of all profiles of lexmin preferences and $\varphi$ is a rule satisfying SP and CPO. Let $z \in \mathcal{R}$. The first lemma shows that if one of the two bads is located at 0, then the other one cannot be located at an interior point of its country.

**Lemma A.1.** *$\alpha(z) = 0$ implies $\beta(z) \in \{0, 1\}$ and $\beta(z) = 0$ implies $\alpha(z) \in \{-1, 0\}$.*

**Proof.** We prove that $\alpha(z) = 0$ implies $\beta(z) \in \{0, 1\}$, the other part of the lemma is analogous. Suppose $\alpha(z) = 0$ but to the contrary $\beta(z) \in (0, 1)$. By Pareto optimality for country $B$, it follows that $T_1 := \{i \in N_B : z(i) \geq \beta(z)\} \neq \emptyset$ and $T_2 := \{i \in N_B : z(i) \leq \mu(z)\} \neq \emptyset$. By weak monotonicity (Lemma 2.2), we may assume that $z(i) = 1$ for all $i \in T_1$ and $z(i) = -1$ for all $i \in N_A$. Without loss of generality, assume that $T_2 := \{1, \dots, m\}$ where $m < n_B$. Consider the following profiles for all $j = 1, \dots, m$:

$$z_1 = (z_{N\setminus T_1}, \mu(\beta(z), 1)^{T_1})$$
$$z_2^j = (z_{N\setminus\{1,\dots,j\}}, \mu(z)^{\{1,\dots,j\}})$$
$$z_3^j = (z_{N\setminus(T_1\cup\{1,\dots,j\})}, \mu(\beta(z), 1)^{T_1}, \mu(z)^{\{1,\dots,j\}}).$$

By Pareto optimality for country $A$, it follows that $\alpha(z_1) = \alpha(z_2^j) = \alpha(z_3^j) = 0$ for every $j \in T_2$.

First, consider the deviation from $z$ to $z_1$. Since $\alpha(z_1) = 0$, strategy-proofness implies that $\beta(z_1) \in \{\beta(z), 1\}$. Pareto optimality for country $B$ then implies $\beta(z_1) = 1$. So $\varphi(z_1) = (0, 1)$. Now consider the deviation from $z_1$ to $z_3^1$. As $\mu(z) < \frac{1}{2}$, strategy-proofness implies that $\varphi(z_3^1) = (0, 1)$. Now suppose $\varphi(z_3^j) = (0, 1)$ for some $j < m$. Consider the deviation from $z_3^j$ to $z_3^{j+1}$. Since $\mu(z) < \frac{1}{2}$, strategy-proofness implies that $\varphi(z_3^{j+1}) = (0, 1)$. Hence, $\varphi(z_3^m) = (0, 1)$.

Next, consider the deviation from $z$ to $z_2^1$. Since $\alpha(z_2^1) = 0$, strategy-proofness implies that $\beta(z_2^1) \in \{0, \beta(z)\}$. So $\varphi(z_2^1) \in \{f(z), (0, 0)\}$. Consider the deviation from $z_2^1$ to $z_2^2$. Since $\alpha(z_2^2) = 0$, we have that $f(z_2^2) = (0, 0)$ if $\varphi(z_2^1) = (0, 0)$ and $f(z_2^2) \in \{\varphi(z), (0, 0)\}$ if $\varphi(z_2^1) = \varphi(z)$. Continuing this way, we conclude that $\varphi(z_2^m) \in \{(0, 0), \varphi(z)\}$. Then Pareto optimality for country $B$ implies that $\varphi(z_2^m) = (0, 0)$. Consider the deviation from $z_2^m$ to $z_3^m$. Since $\mu(\beta(z), 1) > \frac{1}{2}$, strategy-proofness implies that $\varphi(z_3^m) = (0, 0)$, which contradicts the fact that $\varphi(z_3^m) = (0, 1)$ and concludes the proof of the lemma. ∎

The second lemma shows that if one of the two bads is located at the noncommon border of a country, then the other one cannot be located at an interior point of the other country.

**Lemma A.2.** *$\alpha(z) = -1$ implies $\beta(z) \notin (0, 1)$ and $\beta(z) = 1$ implies $\alpha(z) \notin (-1, 0)$.*

**Proof.** We prove that $\alpha(z) = -1$ implies $\beta(z) \notin (0, 1)$, the other part of the lemma is analogous. Suppose $\alpha(z) = -1$ but to the contrary $\beta(z) \in (0, 1)$. So, in this case $-\frac{1}{2} < \mu(z) < 0$. By Pareto optimality for country $B$, it follows that $T := \{i \in N_B : z(i) \geq \beta(z)\} \neq \emptyset$ and $N_B\setminus T \neq \emptyset$. By weak monotonicity (Lemma 2.2), we may assume that $z(i) = \mu(z)$ for all $i \in N_A$, $z(i) = 1$ for all $i \in T$ and $z(i) = 0$ for all $i \in N_B\setminus T$. Consider now the following three profiles.

| | $N_A$ | $T$ | $N_B\setminus T$ |
|---|---|---|---|
| $z_1$ | $\mu(z)$ | $\mu(\beta(z), 1)$ | 0 |
| $z_2$ | $\mu(z)$ | 1 | $\mu(0, \beta(z))$ |
| $z_3$ | $\mu(z)$ | $\mu(\beta(z), 1)$ | $\mu(0, \beta(z))$. |

Since $-\frac{1}{2} < \mu(z) < 0$, Pareto optimality for country $A$ implies that $\alpha(z_1) = \alpha(z_2) = \alpha(z_3) = -1$.

First, consider the deviation from $z$ to $z_1$. Since $\alpha(z_1) = -1$, strategy-proofness implies $\beta(z_1) \in \{\beta(z), 1\}$. Pareto optimality for country $B$ implies $\beta(z_1) = 1$. So $\varphi(z_1) = (-1, 1)$. Now consider the deviation from $z_1$ to $z_3$. Since $\mu(0, \beta(z)) < \frac{1}{2}$, strategy-proofness implies that $\varphi(z_3) = (-1, 1)$.

Next, consider the deviation from $z$ to $z_2$. As $\alpha(z_2) = -1$, strategy-proofness implies $\beta(z_2) \in \{0, \beta(z)\}$. Pareto optimality for country $B$ implies that $\varphi(z_2) = (-1, 0)$. Now consider the deviation from $z_2$ to $z_3$. Since $\mu(\beta(z), 1) > \frac{1}{2}$, strategy-proofness implies that $\varphi(z_3) = (-1, 0)$, which contradicts the fact that $\varphi(z_3) = (-1, 1)$ and concludes the proof of the lemma. ∎

**Proof of Theorem 3.4.** Let $z \in \mathcal{R}$. In view of Lemmas A.1 and A.2 it is sufficient to show that $\varphi(z) \notin (-1, 0) \times (0, 1)$. Suppose, to the contrary, that $\varphi(z) \in (-1, 0) \times (0, 1)$. Without loss of generality assume that $\mu(z) \geq 0$. Because of country-wise Pareto optimality $S(z) := \{i \in N_A : z(i) \geq \alpha(z)\}$ and $T(z) := \{i \in N_B : z(i) \leq \beta(z)\}$ are nonempty strict subsets of $N_A$ and $N_B$, respectively. Consider the following profiles, where $t \in \mathbb{N}$:

| | $N_A\setminus S(z)$ | $S(z)$ | $T(z)$ | $N_B\setminus T(z)$ |
|---|---|---|---|---|
| $z^0 = z$ | $-1$ | 0 | $\mu(z)$ | 1 |
| $z^t$ $(t \geq 1)$ | $-1$ | $\frac{\alpha(z^{t-1})}{2}$ | $\mu(z)$ | 1 |
| $v^1$ | $\frac{\alpha(z^0)-1}{2}$ | 0 | $\mu(z)$ | 1 |
| $v^t$ $(t \geq 2)$ | $\frac{\alpha(z^{t-1})-1}{2}$ | $\frac{\alpha(z^{t-2})}{2}$ | $\mu(z)$ | 1 |
| $w^t$ $(t \geq 1)$ | $\frac{\alpha(z^{t-1})-1}{2}$ | $\frac{\alpha(z^{t-1})}{2}$ | $\mu(z)$ | 1. |

The proof proceeds in a few steps.

*Step* 1. Let $t \geq 1$ and suppose that $-1 < \alpha(z^{t-1}) < 0$. Then $\varphi(v^t) = \varphi(w^t) = (-1, 1)$.

**Proof.** Considering strategy-proofness at $z^{t-1}$ and $v^t$ yields that $\alpha(v^t) \in \{-1, \alpha(z^{t-1})\}$. Pareto optimality for country $A$ now implies that $\alpha(v^t) = -1$. Then strategy-proofness implies $\beta(v^t) \in [\beta(z^{t-1}), 1]$. Since $0 \notin [\beta(z^{t-1}), 1]$, Lemma A.2 implies $\beta(v^t) = 1$. Thus, $\varphi(v^t) = (-1, 1)$. Comparing $v^t$ and $w^t$ and noting that $-1 < \alpha(z^{t-1})$ and therewith $-\frac{1}{2} < \frac{\alpha(z^{t-1})}{2}$, strategy-proofness implies that $\varphi(w^t) = (-1, 1)$.

*Step* 2. For all $t \geq 1$:

$$-1 < \alpha(z^t) < \alpha(z^{t-1}) < 0 \leq 2\mu(z) < \beta(z^t) \leq -\alpha(z^{t-1}) < 1.$$

**Proof.** By assumption we have $-1 < \alpha(z^0) = \alpha(z) < 0 \leq 2\mu(z) < \beta(z^0) = \beta(z) < 1$. We prove the statement in Step 2 by induction. Assume it is true for all $s < t$, where $t \geq 1$. Consider the deviation from $z^{t-1}$ to $z^t$. Then strategy-proofness implies at least one of $\alpha(z^t)$ and $\beta(z^t)$ is in the closed interval $[\alpha(z^{t-1}), -\alpha(z^{t-1})]$. By considering the deviation from $z^t$ to $z^{t-1}$ it follows that $\alpha(z^t), \beta(z^t) \notin (\alpha(z^{t-1}), 0)$. Since $\alpha(z^{t-1})$ is not a Pareto optimal location for country $A$ at $z^t$ we have $\alpha(z^t) \neq \alpha(z^{t-1})$. Further, $\alpha(z^t) \neq 0$ otherwise by considering $w^t$ we would have a contradiction with strategy-proofness, since $(0, \beta(z^t)) = \varphi(z^t)$ is better for dip $\frac{\alpha(z^{t-1})-1}{2}$ than $(-1, 1) = \varphi(w^t)$ (by Step 1). So, $\alpha(z^t) < \alpha(z^{t-1})$ and $0 \leq \beta(z^t) \leq -\alpha(z^{t-1})$. Now suppose $\beta(z^t) = 0$. Then $\alpha(z^t) < \alpha(z^{t-1})$ and Lemma A.1 imply that $\alpha(z^t) = -1$. But $(-1, 1) = \varphi(w^t)$ (by Step 1) is better for dip $-1$ than $\varphi(z^t) = (-1, 0)$, which is a violation of strategy-proofness. So $\beta(z^t) \neq 0$. Country-wise Pareto optimality now implies $\beta(z^t) > 2\mu(z)$. Since $0 < \beta(z^t) < 1$, Lemma A.2 implies $\alpha(z^t) \neq -1$. Altogether we have $-1 < \alpha(z^t) < \alpha(z^{t-1}) < 0 \leq 2\mu(z) < \beta(z^t) \leq -\alpha(z^{t-1}) < 1$. Hence Step 2 follows by induction.

*Step* 3. $\mu(z) = 0$.

**Proof.** First suppose that for some $t > 1$, $\mu(z^t) \geq \frac{\alpha(z^{t-2})}{2}$. Then, by Step 2, for all $i \in S(z)$ we have $\alpha(z^t) < \frac{\alpha(z^{t-1})}{2} = z^t(i) < z^{t-1}(i) = \frac{\alpha(z^{t-2})}{2} \leq \mu(z^t)$, so that by monotonicity we have $\varphi(z^{t-1}) = \varphi(z^t)$, a contradiction. Thus, $\mu(z^t) < \frac{\alpha(z^{t-2})}{2}$ for all $t > 1$. Hence, $\alpha(z^t) + \beta(z^t) < \alpha(z^{t-2})$ for all $t > 1$. By Step 2 this implies $2\mu(z) < \beta(z^t) < \alpha(z^{t-2}) - \alpha(z^t)$, which implies that $\mu(z) = 0$ since $\alpha(z^{t-2}) - \alpha(z^t)$ converges to 0 for $t$ going to infinity.

*Step* 4. If $\mu(z) = 0$ then $\mu(z^1) \neq 0$.

**Proof.** Follows from Step 2 by taking $t = 1$.

We can now complete the proof. Step 3 implies that $\mu(z) = 0$ for *any* profile $z$ with $\alpha(z) \in (-1, 0)$ and $\beta(z) \in (0, 1)$. This contradicts Step 3 since $z^1$ is also such a profile. ■

*A.2. Further examples*

**Example A.3.** This is an example of a strategy-proof and country-wise Pareto optimal rule $\psi$ on the domain of lexmin preference profiles which is not coalitionally strategy-proof. Fix agents $i_A \in N_A$ and $i_B \in N_B$. We define two functions $\alpha$ and $\beta$ on preference profiles, as follows:
$\alpha(z) = -1$ if $z(i_A) > -\frac{1}{2}$, $\alpha(z) = 0$ if $z(i_A) < -\frac{1}{2}$ or if $z(i_A) = -\frac{1}{2}$ and $z(j) < -\frac{1}{2}$ for some $j \in N_A$, and $\alpha(z) = -1$ otherwise;
$\beta(z) = 1$ if $z(i_B) < \frac{1}{2}$, $\beta(z) = 0$ if $z(i_B) > \frac{1}{2}$ or if $z(i_B) = \frac{1}{2}$ and $z(j) > \frac{1}{2}$ for some $j \in N_B$, and $\beta(z) = -1$ otherwise.
We define $\psi$ by $\psi(z) = (\alpha(z), \beta(z))$. Note that this rule is decomposable: the decision about the location of the public bad of a country depends only on the preferences of the agents in that country. It can be seen that $\psi$ is strategy-proof and country-wise Pareto optimal. Now consider a profile $z$ such that $z(i_A) = -\frac{1}{2}$, $z(i_B) = \frac{1}{4}$ and $z(j) = -\frac{3}{4}$ for some $j \in N_A \backslash \{i_A\}$. Then $\psi(z) = 01$. Let $S = \{i_A, i_B\}$, and consider the profile $z'$ with $z'(i_A) = -\frac{1}{4}$, $z'(i_B) = \frac{1}{8}$, and $z'(j) = z(j)$ for all $j \in N \backslash \{i_A, i_B\}$. Then $\psi(z') = -11$. Since $-11 P_{z(i_B)} 01$ and $-11 I_{z(i_A)} 01$, $S$ has a profitable deviation.

**Example A.4.** We show that Pareto optimality and country-wise Pareto optimality are independent conditions.

(1) We define a rule $g$ as follows. For every profile $z \in \mathcal{R}$,

$$\alpha(z) = \begin{cases} -1 & \text{if } N_A \subseteq U(z) \cup V(z) \text{ and } U(z) \cap N_A \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$\beta(z) = \begin{cases} 1 & \text{if } N_B \subseteq U(z) \cup V(z) \text{ and } U(z) \cap N_B \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and $g(z) = (\alpha(z), \beta(z))$. Then $g$ is strategy-proof and country-wise Pareto optimal both for myopic and lexmin preferences. Now consider the profile $z$ with $z(i) = 0$ for all $i \in N_A$ and $z(i) = \frac{1}{2}$ for all $i \in N_B$. Then $g(z) = -10$. Since both for myopic and lexmin preferences we have that $-11 P_{z(i)} g(z)$ for all $i \in N_A$ and $-11 I_{z(j)} g(z)$ for all $j \in N_B$, this rule is Pareto optimal neither for myopic nor for lexmin preferences.

(2) We define a dictatorial rule $h_1$ for lexmin preferences, as follows. Fix an agent $i \in N_A$. Then $h_1(z) = 01$ if $z(i) < -\frac{1}{2}$ or if $z(i) = -\frac{1}{2}$ and $z(k) < -\frac{1}{2}$ for some $k \in N_A$, and $h_1(z) = -11$ otherwise. It can be seen that $h_1$ is Pareto optimal, but not country-wise Pareto optimal.

(3) We define a dictatorial rule $h_2$ for myopic preferences, as follows. Fix an agent $i \in N_A$. Then

$$h_2(z) = \begin{cases} (-1, 2z(i)+1) & \text{if } 2z(i)+1 \geq 0 \text{ and} \\ & \mu(2z(i)+1, 1) < z(k) \text{ for some } k \in N \\ (-1, 1) & \text{if } 2z(i)+1 \geq 0 \text{ and} \\ & \{k \in N : \mu(2z(i)+1, 1) < z(k)\} = \emptyset \\ (0, 0) & \text{otherwise.} \end{cases}$$

It can be seen that $h_2$ is Pareto optimal, but not country-wise Pareto optimal.

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