Club good mechanisms: from free-riders to citizen-shareholders, from impossibility to characterization

Citation for published version (APA):

Mackenzie, A., & Trudeau, C. (2018). *Club good mechanisms: from free-riders to citizen-shareholders, from impossibility to characterization*. Maastricht University, Graduate School of Business and Economics. GSBE Research Memoranda No. 012 <https://doi.org/10.26481/umagsb.2018012>

Document status and date:

Published: 08/05/2018

DOI:

[10.26481/umagsb.2018012](https://doi.org/10.26481/umagsb.2018012)

Document Version:

Publisher's PDF, also known as Version of record

Please check the document version of this publication:

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Download date: 26 Apr. 2023

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RM/18/012

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CLUB GOOD MECHANISMS: FROM FREE-RIDERS TO CITIZEN-SHAREHOLDERS, FROM IMPOSSIBILITY TO CHARACTERIZATION

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This draft: January 18, 2017

Abstract

Consider a community that shares a technology for producing a *club good* (Buchanan, 1965): any group of agents can “win” for an associated monetary cost. Who should win, and how should production be funded? To address this question, we seek *rules* (that is, direct mechanisms) where each agent participates voluntarily and is incentivized to report his valuation honestly, and where these reports are used to select winners efficiently without running a deficit.

We find that whether or not there are such rules depends on the production technology. If costs are even “somewhat concave,” then there are no such rules: the *free-rider problem* (Wicksell, 1896; Samuelson, 1954; Green and Laffont, 1979) persists even when agents who do not contribute can be excluded. If costs are symmetric and convex, however, then there are such rules that moreover satisfy *no-envy-in-trades* (Kolm, 1971; Schmeidler and Vind, 1972). We characterize this class, whose Pareto-worst member is the familiar *minimum-price Walrasian rule* (Vickrey, 1961; Clarke, 1971; Groves, 1973; Demange, 1982; Leonard, 1983); the other rules do better by treating the agents as equal shareholders in the technology and offering *social dividends* (Lange, 1936).

1 Introduction

1.1 Executive summary

According to one version of the *free-rider problem* (Wicksell, 1896; Samuelson, 1954; Green and Laffont, 1979), a community seeking to efficiently provide a public good on the basis of reported interest, with production funded by voluntary contributions, is doomed to fail. In the usual story, an agent inevitably under-reports his interest and

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[‡]We thank Yu Awaya, Eddie Dekel, Hervé Moulin, Yves Sprumont, and Rodrigo Velez; seminar participants at University of Glasgow, University of St Andrews, the 2015 Midwest Theory Conference in Rochester, and the 2016 International Conference on Game Theory in Stony Brook; and especially William Thomson.

underpays, only to later reap benefit from whatever is produced, thereby *free-riding* on the contributions of his peers. But might such a person confess his true valuation upon learning that, should he inadequately contribute, he can and will be excluded from the final product?

In this article, we consider the problem of providing a *club good* (Buchanan, 1965): a good that is (i) *nonrival*, like a public good, in that no agent's consumption impacts another's (positively or negatively), and (ii) *excludable*, like a private good, in that any agent can be prevented from consuming it. Examples include gated parks, high-speed internet cables, libraries with membership, city sewage, and the like. We find that the ability to exclude non-contributors helps to resolve the free-rider problem sometimes, but not always. In particular, when costs are even "somewhat" concave, the problem unfortunately persists. On the other hand, when costs are symmetric and convex, we identify a rich new family of mechanisms that improve on the familiar *minimum price Walrasian rule* (Vickrey, 1961; Clarke, 1971; Groves, 1973; Demange, 1982; Leonard, 1983) by treating members of the community as shareholders in the production process.

In our model, any group of *winners* can be provided the club good for an associated monetary cost, but each agent's valuation of winning is private information. Because efficient provision requires the agents' collective information, we investigate designing an institution called a *game form*, or *mechanism*, which specifies an unambiguous procedure by which the agents make choices to ultimately determine their outcome. So that these choices plausibly lead to desired outcomes, we require that no matter the agents' true preferences, a desired outcome is a dominant strategy equilibrium.

By the revelation principle (Hurwicz, 1972; Gibbard, 1973), it is without loss of generality to restrict attention to *direct mechanisms*, or *rules*, where the agents simultaneously report their valuations, then these reports determine who wins and how production is funded. Our objective is to identify rules that satisfy the following requirements:

- *no-deficit*: production is funded entirely by the contributions of the agents,
- *production efficiency*: the winning group is compatible with Pareto-efficiency (though money may be burned),¹
- *voluntarism*: no agent prefers losing without a transfer to his assignment, and
- *strategy-proofness*: honesty is a dominant strategy for each agent.

Whether or not these objectives are compatible depends on the production technology, which follows from the literature:

1. When serving nobody is free, serving everybody has some finite positive cost, and serving any other group has infinite cost, we are essentially providing a pure public good, where the Groves rules (Groves, 1973) are the only ones satisfying *production efficiency* and *strategy-proofness* (Green and Laffont, 1977; Holmström, 1979), and the *voluntary* ones necessarily run a deficit (Green and Laffont, 1979).
2. When serving up to k winners is free, while serving more winners has infinite cost, we are essentially distributing k identical indivisible objects, where it is well-known that the Vickrey rule (Vickrey, 1961) satisfies our criteria.

¹This requirement is sometimes called *allocative efficiency*, *decision efficiency*, or *partial efficiency*. Because agents have quasi-linear preferences in our setting, this axiom is equivalent to the requirement that the group of winners is selected to maximize Marshallian surplus.

Figure 1: An example of a social-dividend schedule, α_0 . Here, μ_1 is the marginal cost of producing the first object, while μ_n is the marginal cost of producing the n th. The schedule is a Lipschitz-continuous function over this interval such that the slope between any two points on the curve is between 0 and 1. Each agent faces a price in this interval, which is determined by a particular function that anonymously processes the bids of his peers. The point on the curve associated with this price is his social-dividend award, and strategically, the agent can choose whether to lose with the award or win with the award minus the price.

Our contribution is to extend both of these observations to our general club good model.

[Theorem 1](#) is our strong statement of the free-rider problem: if no agents can be served for free, and if there are weak returns to scale in the sense that some group's cost is less than the sum of costs for its members individually, then no rule satisfies our requirements. Unfortunately, this applies to all subadditive cost functions, which are the production technologies that lead naturally to monopoly ([Baumol, 1977](#)). This is disappointing, because we might have hoped that mechanisms could help improve market outcomes in these industries.

On the other hand, if costs are finite, symmetric, and convex, then there are rules that not only meet our criteria, but moreover satisfy *no-envy-in-trades* ([Kolm, 1971](#); [Schmeidler and Vind, 1972](#)), the equal-opportunity requirement that no agent prefers another's net change in resources to his own. [Theorem 2](#) characterizes the rich class of rules satisfying *production efficiency*, *strategy-proofness*, and *no-envy-in-trades*, whose rules each use the same *price function* to offer a price to each agent by processing the bids of his peers anonymously. Each rule additionally has its own *social-dividend schedule*, which offers each agent a *social-dividend award* determined solely by his price. Strategically, each agent either (i) loses with this award, which is the outcome when he bids less than his price, or (ii) wins for this award minus his price, which is the outcome when he bids more than his price. Moreover, the social-dividend schedule is a nondecreasing Lipschitz-continuous function of price, and the slope between any two points on its curve is at most 1 (see [Figure 1](#)). We refer to rules in this class as *citizen-shareholder rules*.

[Theorem 3](#) characterizes the citizen-shareholder rules that satisfy *no-deficit*, which we call *cost-constrained citizen-shareholder rules*, using a finite list of constraints on the social-dividend schedule. [Theorem 4](#) identifies the rules that are Pareto-undominated in this class, as well as the rules that are Pareto-undominated in its *voluntary* subclass.

1.2 Example: The Big Show

When costs are finite, symmetric, and convex, there is in general a large class of Pareto-undominated rules in our class, but in the special case that there are three agents, there is an essentially-unique Pareto-dominant rule. We illustrate this rule with an example.

Three fans want to go to some event, which we call the Big Show, but are disappointed to learn that it is already sold out. Undeterred, they go online to search for tickets that are being re-sold, and sure enough they find a website where there are three scalpers, each selling one ticket. Each of the fans is willing to pay \$30, while the scalpers' prices are \$10, \$20, and \$80.

This is an economy with indivisible objects and money, where each agent wants at most one object, and where each agent is endowed with at most one object, as in the classic horse market economy (Böhm-Bawerk, 1888; Shapley and Shubik, 1972); we describe our contribution to this literature in the conclusion. Here, we simply consider what happens in our Big Show economy, focusing specifically on the outcomes of the fans. Their outcomes, of course, depend on the institutional arrangements that prevail in society, of which we consider four possibilities:

- **JUNGLE:**² First, suppose that each ticket can be acquired with the click of a button at the scalper's listed price (as in eBay's Buy It Now prices), and the fans race one another to acquire tickets.

Outcome: The fastest-clicker receives a ticket and pays \$10, the second-fastest-clicker receives a ticket and pays \$20, and the slowest-clicker stays home and receives no transfer.

- **WALRASIAN MARKET** (WALRAS, 1896): Suppose now that the scalper's listed prices are reservation prices, which we might indeed see if the scalpers are trying to maximize revenue (Myerson, 1981). The fans go on to offer competing bids on each ticket. At Walrasian equilibrium, there is a Walrasian price at which supply equals demand, which must be \$30: at a lower price, three fans demand while fewer scalpers supply; at a higher price, no fans demand while some scalper supplies.

Outcome: Two fans receive a ticket and pay \$30, while the third fan stays home and receives no transfer.

- **MONOPOLY:** Suppose now that some outside party acquires ownership of the website, so that the three fans cannot directly access the scalpers, but must go through the owner. The owner views his website as a technology for producing tickets (by paying the scalpers as needed). Moreover, the owner has a monopoly on tickets, and wishes to sell to the three fans in order to maximize profit. Whether or not the owner is able to price-discriminate, two fans pay their full valuations while no ticket is sold to the third fan.

Outcome: Same as for the Walrasian market.

- **OUR RULE:** Finally, suppose that the three fans seek the help of a mediator. The mediator does not know their valuations, so he commits to using our rule and asks each fan how much he is willing to pay. It is a dominant strategy for each to report

²For lack of a better term, we borrow *jungle* from Piccione and Rubinstein (2007), though our notion differs from theirs in that we do not consider a stronger fan appropriating money from a weaker fan.

Figure 2: Logical relations between standard domains of cost functions. Theorem 1 states that there is no rule satisfying our requirements when the cost function is in the shaded region.

the truth, and in order to form our prediction we assume that the truth is indeed reported.

Outcome: Two fans receive a ticket and pay \$20, while the third fan stays home and receives \$10 as consolation.

It is straightforward to verify that each fan prefers our rule's outcome to both the Walrasian market outcome and the monopoly outcome. With respect to the jungle, our rule transfers \$10 from the fastest-clicker to the slowest-clicker, removing envy derived from clicking speed. Whether or not one prefers the jungle to our rule may depend on one's own clicking speed, or whether or not one views clicking speed as a virtue that should help determine where our institutions direct resources; we simply remark that this trait is not related to valuation or contribution.

When the scalpers are considered, our rule does not offer a Pareto-improvement over the Walrasian outcome, but rather a transfer from the scalpers to the fans. That said, it is easy to see that our analysis is legitimate if the scalpers are replaced by a technology for producing tickets, and here our rule indeed offers a Pareto-improvement over the Walrasian outcome.

For this example, our rule does not burn money: the two fans who go to the show together spend \$40, of which \$10 goes to the first scalper, \$20 goes to the second scalper, and \$10 goes to the third fan. That said, our rule sometimes burns money upon receiving reports in order to remain compatible with incentives.

1.3 Related literature

Our model is a special case of the [Green and Laffont \(1979\)](#) model: the public decisions are specifically groups of winners, and preferences are restricted to reflect this additional structure. This preference restriction does not obstruct us from a fundamental result

(Holmström, 1979) that the rules satisfying *production efficiency* and *strategy-proofness* are precisely the Groves rules (Groves, 1973).³

In turn, special cases of our model, given by particular classes of cost functions, have been previously studied (see Figure 2). In addition to the well-studied cases described earlier, there is axiomatic analysis for:

- the “almost-public” good class, where all nonempty groups share a fixed finite cost (Deb and Razzolini, 1999a; Deb and Razzolini, 1999b; Ohseto, 2000; Ohseto, 2005; Yu, 2007; Ohseto, 2009; Massó, Nicolò, Sen, Sharma, and Ülkü, 2015; Hashimoto and Saitoh, 2016),
- the monotone and concave class (Moulin and Shenker, 2001; Mutuswami, 2004), and
- the general case (Juarez, 2013; Juarez, working); these papers consider *group strategy-proofness*.

With the exception of Moulin and Shenker (2001), however, these articles do not consider *production efficiency*; they usually instead consider *budget balance*, the requirement that there is no deficit and no money is burned. Together, *production efficiency* and *budget balance* are equivalent to Pareto-efficiency, and we cannot have both: Groves rules necessarily burn money to provide incentives (Green and Laffont, 1979).

For monotone and concave cost functions, Moulin and Shenker (2001) characterize the class of rules that satisfy *production efficiency*, *voluntarism*, *strategy-proofness*, and the additional requirement that no agent receives a positive transfer, which is fittingly called *no positive transfers*. An immediate corollary is that no such rule satisfies *no-deficit*; thus for these cost functions, our objectives are incompatible with *no positive transfers*. While we do not view *no positive transfers* as a normatively appealing objective, because it is a sort of welfare *upper* bound, this axiom can alternatively be interpreted as the physical restriction that no agent can transfer the divisible good to another. This could be the case, for example, if the divisible good is *leisure* instead of *money*.

Among our citizen-shareholder rules, the zero-dividend rule is the only rule that satisfies *voluntarism* and *no positive transfers*, and this rule moreover satisfies *no deficit*. This particular rule is in fact none other than the latest incarnation of an idea, known now by many names, that regularly emerges in these kinds of models. With public goods, it is called *Clarke’s pivot rule* (Clarke, 1971; Green and Laffont, 1979; Moulin, 1986; Moulin and Shenker, 2001). When there is a social endowment of identical indivisible objects of which each agent desires at most one, it is called *Vickrey’s rule* (Vickrey, 1961; Ohseto, 2006; Chew and Serizawa, 2007; Saitoh and Serizawa, 2008; Sakai, 2008; Ashlagi and Serizawa, 2012). More generally, this is the *minimum-price Walrasian rule*.⁴ Perhaps

³Technically, the theorem of Holmström (1979) does not directly apply because it considers maximizing the sum of valuations, while we take into account the cost function as in Roberts (1979) and maximize the Marshallian surplus. That said, this is only a superficial difference: the original proof goes through with this small modification line-by-line.

⁴When there are (possibly distinct) indivisible objects of which each agent desires at most one and preferences are quasi-linear, the Walrasian price vectors form a bounded lattice (Shapley and Shubik, 1972), and the *minimum-price Walrasian rule*, which assigns an outcome associated with the minimum Walrasian price vector, is *strategy-proof* (Demange, 1982; Leonard, 1983). These facts persist when preferences are not quasi-linear (Demange and Gale, 1985), and this rule was recently characterized (Morimoto and Serizawa, 2015). In fact, even when agents may consume several objects, Walrasian

surprisingly, the zero-dividend rule is distinguished in another way among the citizen-shareholder rules satisfying *no-deficit* and *voluntarism*: it is the Pareto-worst! Thus for this class, the content of *no positive transfers* is anti-efficiency. The welfare improvements offered by positive transfers have not gone unnoticed; for example, [Guo, Markakis, Apt, and Conitzer \(2013\)](#) study the Pareto-undominated Groves rules in the general [Green and Laffont \(1979\)](#) model.

Groves rules have not only been investigated on the basis of efficiency; fairness has been focal as well. The article contributes to the large literature on fair Groves rules for economies with indivisible objects and money, where:

- *no-envy* ([Tinbergen, 1946](#); [Foley, 1967](#)) has been investigated ([Pápai, 2003](#); [Ohseto, 2006](#); [Yengin, 2012b](#); [Yengin, 2016](#)),
- *population monotonicity* ([Thomson, 1983](#)) has been investigated ([Yengin, 2013a](#)),
- *egalitarian-equivalence* ([Pazner and Schmeidler, 1978](#)) has been investigated ([Ohseto, 2004](#); [Yengin, 2012a](#); [Yengin, 2012b](#); [Yengin, 2016](#)),
- the *identical-preferences-lower-bound* ([Steinhaus, 1948](#)) has been investigated ([Yengin, 2013b](#)), and
- a parametric notion of maximizing the welfare of the worst-off agent ([Rawls, 1971](#); [Porter, Shoham, and Tennenholtz, 2004](#)) has been investigated ([Atlamaz and Yengin, 2008](#)).

We consider *no-envy-in-trades* ([Schmeidler and Vind, 1972](#)), which mathematically plays the same role as *no-envy* in the above works; we make the distinction to emphasize a conceptual point.⁵ Our contribution to this literature is allowing production. Of these works, the most closely related is [Ohseto \(2006\)](#), which characterizes the *strategy-proof* and *envy-free* rules when there is a fixed supply of identical indivisible objects that all must be consumed. In this setting, *no-envy* implies Pareto-efficiency ([Svensson, 1983](#)), so mathematically both proofs involve the same axioms. In fact, one of the cases in our proof of the [Consolation Lemma](#), which is used to prove [Theorem 2](#), uses an adaptation of an argument from [Ohseto \(2006\)](#) (though the other case in our proof uses a novel argument). That said, we do not directly generalize [Ohseto \(2006\)](#), because we cannot simply add the restriction that all objects must be consumed as an axiom; the restriction also has implications for what is feasible, and therefore for whether or not a given allocation is Pareto-dominated.

Our work is related in spirit to early contributions to the free-rider problem, which often considered a single divisible public good. One of the focal rules in this setting is

prices form a lattice under reasonable assumptions: [Gul and Stacchetti \(1999\)](#) show that Walrasian prices (for objects, not bundles) form a lattice if each agent’s preferences satisfy *single improvement*, which for monotonic preferences is equivalent to the *gross substitutes* condition of [Kelso and Crawford \(1982\)](#).

⁵In particular, *no-envy* requires that no agent prefers what another agent consumes to what he himself does, while *no-envy-in-trades* requires that no agent prefers another’s net change in resources to his own. Both axioms play the same role mathematically because (i) we suppress private endowments in our notation, and (ii) the axioms are equivalent when all agents happen to have the same endowment. Conceptually, we allow envy due to the suppressed endowments, but forbid envy due to opportunities afforded by the mechanism. In our model, when production is unavailable, a pure redistribution of money could be necessary under *no-envy*, but violates *no-envy-in-trades*; in this way our fairness axiom takes property rights into account.

the Lindahl rule (Lindahl, 1919), where one’s contribution is based on marginal benefit, and before mechanism design was formalized, Samuelson (1954) argued that this rule is manipulable. In one of the seminal contributions to mechanism design, Hurwicz (1972) showed that such manipulability is not only a problem for public goods,⁶ and an analogous result for public goods (applying to all rules, not just the Lindahl rule) was proved shortly thereafter (Ledyard and Roberts, 1975; Groves and Ledyard, 1987).

Finally, we mention that in production economies with private goods, there are *efficient* and *strategy-proof* rules only in the extraordinary circumstance that production is linear, where each agent can operate the technology on his own without impacting his peers (Maniquet and Sprumont, 1999; Leroux, 2004). One might worry that the analogous result holds for club goods: there are rules satisfying our axioms only when the cost function is additive. Fortunately, however, the situation is not quite so bleak in our model.

In Section 2, we introduce the general model, the axioms, and the two special cases we study. In Section 3, we consider “somewhat concave” cost functions, and describe and discuss Theorem 1, our impossibility result. In Section 4, we consider the production of identical indivisible objects, where we describe and discuss a series of results culminating in Theorem 2, Theorem 3, and Theorem 4. Section 5 concludes.

2 Model

2.1 General setting

We consider a community of agents $N \equiv \{1, 2, \dots, n\}$ who share a production technology. Initially there is only one good, *money*, which is infinitely-divisible and homogeneous, and which may be transferred among agents; as the initial distribution of money has no impact on our analysis, we speak of agents consuming monetary *transfers* in order to simplify notation.⁷ The technology takes money as its input in order to create a group of *winners*, whose number we often denote by w . We call an agent who does not win a *loser*.

Each agent i is *selfish*, caring only about his personal consumption of money and whether or not he wins.⁸ More precisely, i has a preference relation R_i over $\mathbb{R} \times \{0, 1\}$, where the first coordinate specifies his monetary transfer and the second coordinate indicates whether or not he wins, with 1 representing that he does. Moreover, we assume that R_i is strictly monotonic in money, quasi-linear in money, and that it satisfies the

⁶In particular, Hurwicz (1972) considered a two-agent economy with two private goods (that is, an Edgeworth box economy), and proved that every *voluntary* and *efficient* rule is manipulable. This disappointing result was generalized to the case where there are n agents and m goods (Serizawa, 2002), and in fact, the situation is even worse than that: no *efficient* and *strategy-proof* rule can guarantee that every agent receives an assignment that is bounded away from the origin (Serizawa and Weymark, 2003).

⁷Our analysis does not depend on the initial distribution of money for two reasons: (1) our assumptions imply that each agent’s willingness to pay for an object is independent of his endowment, and (2) none of our axioms make reference to endowment. In fact, our approach implicitly assumes that addressing inequality is *not* a primary objective, since both property rights and fairness—as expressed by our axioms—are violated whenever money is redistributed and nothing is produced.

⁸The selfishness assumption restricts the possible interpretations of the club good—for example, it is not a telephone network (which is more valuable as more friends consume it) or a road (which is less valuable as traffic grows heavier). Notably, Velez (2016) recently considered a more general preference domain where agents may have other-regarding preferences.

possibility-of-compensation assumption: there is an amount of money v_i such that there is indifference between $(v_i, 0)$ and $(0, 1)$. These assumptions together imply there is a unique *valuation* $v_i \in \mathbb{R}$ such that for each pair $(t_i, x_i), (t'_i, x'_i) \in \mathbb{R} \times \{0, 1\}$,

$$(t_i, x_i) R_i (t'_i, x'_i) \text{ if and only if } v_i x_i + t_i \geq v_i x'_i + t'_i.$$

Abusing notation, we identify each agent's preference relation by its valuation v_i . We use $V_i \subseteq \mathbb{R}$ to denote the class of admissible preference relations for i , and we use $V \equiv \times V_i$ to denote the class of admissible preference profiles. In both special settings we consider, agents share a common class of admissible preference profiles (that is, for each pair $i, j \in N$, $V_i = V_j$); our notation simply allows us to improve clarity.

The agents wish to select an *allocation* $(t, W) \in \mathbb{R}^N \times 2^N$, which specifies both a list of monetary transfers t and a group of winners W . An allocation does not run a deficit if the winning group is produced using a sufficient amount of money as input. This is expressed using an exogenous *cost function* $C : 2^N \rightarrow \mathbb{R}_+ \cup \{\infty\}$, with $C(\emptyset) = 0$, which specifies the cost of production for each winning group.

Though an allocation that is favorable to the agents should be selected, preferences are private information. We consider the direct mechanism approach to this problem: each agent reports his preferences, then these reports jointly determine the allocation. Because we are interested in dominant strategy equilibria, focusing on direct mechanisms is without loss of generality (Hurwicz, 1972; Gibbard, 1973). Our problem is to recommend a direct mechanism, or a *rule*:

DEFINITION: A *transfer policy* is a mapping $\tau : V \rightarrow \mathbb{R}^N$. A *winner policy* is a mapping $\varphi : V \rightarrow 2^N$. A *rule* (τ, φ) consists of a transfer policy τ and a winner policy φ .

Given a winner policy φ and an agent i , we sometimes speak of the associated policy for determining whether or not i is a winner, using $\varphi_i : V \rightarrow \{0, 1\}$ for the mapping

$$\varphi_i(v) \equiv \begin{cases} 1, & i \in \varphi(v), \\ 0, & i \notin \varphi(v). \end{cases}$$

In this article, we seek rules that satisfy desirable properties, or *axioms*.

2.2 Axioms

An axiom is a mathematical property of a rule that captures some aspect of our intuition. Each of the axioms we consider in this article is standard in the literature.

Our first axiom is the feasibility requirement that production is funded entirely by the agents:

NO-DEFICIT: For each $v \in V$,

$$0 \geq \sum \tau_i(v) + C(\varphi(v)).$$

Our second axiom is an efficiency notion which is weaker than Pareto-efficiency as it allows money to be burned. Informally: for a no-deficit allocation (t, W) , the money burned $m \geq 0$ solves $0 = \sum t_i + C(W) + m$. The requirement is that if (t, W) is selected, and before anything is produced m units of money are burned and the no-deficit constraint is updated, then there should be no allocation (t', W') that is still no-deficit,

that everybody finds at least as desirable as (t, W) , and that one agent prefers to (t, W) . This can be seen as an ex-post notion: though certain resources may be destroyed, thereafter the allocation is Pareto-efficient even if production is reversible.

Because preferences are quasi-linear, the amount of money burned has no impact on whether or not this axiom is satisfied. Moreover, this notion is equivalent to selecting a group of winners that maximizes a particular objective function: the aggregate consumer surplus, or the sum across agents of the difference between willingness-to-pay and payment (Marshall, 1890), that is attainable if no money is burned. For convenience, we take this as our definition.

Formally, for each $v \in V$, define the (*attainable Marshallian consumer*) surplus $\sigma_v : 2^N \rightarrow \mathbb{R}$ by:

$$\sigma_v(W) \equiv \sum_{i \in W} v_i - C(W).$$

PRODUCTION EFFICIENCY: For each $v \in V$ and each $W \subseteq N$,

$$\sigma_v(\varphi(v)) \geq \sigma_v(W).$$

Our third axiom asserts that a certain notion of property rights should be respected. In particular, without participating in the community and using the production technology, an agent could receive zero transfer and lose; the axiom requires that this is not preferred to what is received through participation. In other words, the rule should not be coercive:

VOLUNTARISM:⁹ For each $i \in N$ and each $v \in V$,

$$v_i \varphi_i(v) + \tau_i(v) \geq 0.$$

Each rule takes as input preference reports, but our normative axioms are only compelling when stated in terms of true preferences. Our justification for conflating the reports with the truth is our fourth axiom, which requires that it is a dominant strategy for each agent to report his valuation honestly.¹⁰

STRATEGY-PROOFNESS: For each $i \in N$, each $v \in V$, and each $v'_i \in V_i$,

$$v_i \varphi_i(v) + \tau_i(v) \geq v_i \varphi_i(v'_i, v_{-i}) + \tau_i(v'_i, v_{-i}).$$

Our fifth axiom is the equal-opportunity requirement that no agent prefers another's net change in resources to his own (Kolm, 1971; Schmeidler and Vind, 1972):

NO-ENVY-IN-TRADES: For each pair $i, j \in N$ and each $v \in V$,

$$v_i \varphi_i(v) + \tau_i(v) \geq v_i \varphi_j(v) + \tau_j(v).$$

⁹This is sometimes called *individual rationality*, which is appropriate when agents are in fact free to not participate.

¹⁰This standard justification for assuming honest reporting is unfortunately not well-supported by experimental evidence (Kagel, Harstad, and Levin, 1987; Kagel and Levin, 1993; Attiyeh, Franciosi, and Isaac, 2000; Kawagoe and Mori, 2001). This has motivated stronger implementation notions, such as *secure implementation* (Saijo, Sjöström, and Yamato, 2007) and *obvious strategy-proofness* (Li, 2016), which we do not consider here.

Finally, we consider the standard incomplete ordering of rules in which a first rule dominates a second if, at each profile, the recommendation of the first Pareto-dominates the recommendation of the second:

DEFINITION: For each pair of rules (τ^*, φ^*) , (τ, φ) , we say (τ^*, φ^*) *weakly Pareto-dominates* (τ, φ) if for each $i \in N$ and each $v \in V$,

$$v_i \varphi_i^*(v) + \tau_i^*(v) \geq v_i \varphi_i(v) + \tau_i(v).$$

If (τ^*, φ^*) weakly Pareto-dominates (τ, φ) but (τ, φ) does not weakly Pareto-dominate (τ^*, φ^*) , then we say (τ^*, φ^*) *(strictly) Pareto-dominates* (τ, φ) . If (τ^*, φ^*) and (τ, φ) weakly Pareto-dominate each other, then we say they are *Pareto-indifferent*.

3 The service setting

Our analysis focuses on two special cases of the general model, each of which involves additional assumptions about the structure of the cost function and the domain of admissible preferences.

In this section, we consider the *service setting*: no agents can be served for free, and there are weak returns to scale in the sense that some group's cost is less than the sum of costs for its members individually (see Figure 2 in introduction). Beyond these assumptions, the club good is not required to actually be a service; we simply use *service* to concisely contrast this setting from our second setting, and to suggest a few possible applications (such as a public utility or an exclusive postal service). The incompatibility of our axioms holds even if it is common knowledge that the service is undesirable, and strategic agents are unable to report otherwise:

DEFINITION: The *service setting* entails the following assumptions:

- POSITIVE COSTS: For each nonempty $W \subseteq N$, $C(W) > 0$.
- WEAK RETURNS: There is $W \subseteq N$ such that $\sum_{W} C(\{i\}) > C(W)$.
- DESIRABLE DOMAIN: For each $i \in N$, $V_i = \mathbb{R}_+$.

As discussed in the introduction, when serving nobody is free, serving everybody has some finite positive cost, and serving any other group has infinite cost, no rule satisfies our axioms (Groves, 1973; Green and Laffont, 1977; Holmström, 1979; Green and Laffont, 1979). On the other hand, when costs are positive and concave, our axioms are incompatible when the divisible resource cannot be transferred from one agent to another (Moulin and Shenker, 2001).

Unfortunately, we find that these negative results are part of a broader, pervasive impossibility. The incompatibility of our axioms is not caused by non-excludability of the indivisible output, nor by non-transferability of the divisible input, and moreover this incompatibility holds across a large class of cost functions:

THEOREM 1: In the service setting, there is no rule satisfying *no-deficit*, *production efficiency*, *voluntarism*, and *strategy-proofness*.

The proof is in [Appendix 1](#). For these cost functions, the *free-rider problem* cannot be solved by excluding non-contributors: any direct mechanism is either unaffordable, inefficient, manipulable, or coercive.

We remark that this impossibility is tight. If any of the four axioms is dropped, there is a rule that satisfies the other three:

- *No-Deficit*: For each $v \in V$ and each $i \in N$, define $\sigma_v^* \equiv \max_{W \subseteq N} \sigma_v(W)$ to be the maximal surplus, and define $\sigma_v^{-i} \equiv \max_{W \subseteq N \setminus \{i\}} \sigma_v(W)$ to be the maximal surplus among groups that do not include i . Let $\varphi^*(v)$ be such that $\sigma_v(\varphi^*(v)) = \sigma_v^*$, and for each $i \in N$, define $\tau_i^1(v) \equiv -v_i \varphi_i^*(v) + (\sigma_v^* - \sigma_v^{-i})$. Then (τ^1, φ^*) is (welfare-equivalent to) the *marginal cost pricing mechanism* of [Moulin and Shenker \(2001\)](#), and therefore satisfies *production efficiency*, *voluntarism*, and *strategy-proofness*, but fails *no-deficit*.
- *Production efficiency*: For each $v \in V$ and each $i \in N$, define $\varphi^2(v) \equiv \emptyset$ and define $\tau^2(v) \equiv 0$. It is easy to see that (τ^2, φ^2) satisfies *no-deficit*, *voluntarism*, and *strategy-proofness*, but fails *production efficiency*.
- *Voluntarism*: Let $\tau^4(v)$ be such that for each $i \in N$, $\tau_i^4(v) = -\max_{W \subseteq N} C(W)$. It is easy to see that (τ^4, φ^*) satisfies *no-deficit*, *production efficiency*, and *strategy-proofness*, but fails *voluntarism*.
- *Strategy-proofness*: For each $i \in N$, define $\tau_i^3(v) \equiv -\varphi_i^*(v)v_i$. It is easy to see that (τ^3, φ^*) satisfies *production efficiency* and *voluntarism*, but fails *strategy-proofness*. To see that *no-deficit* is satisfied, assume, by way of contradiction, that there is $v \in V$ such that $\sum \tau_i^3(v) + C(\varphi^*(v)) > 0$. Then $0 > \sum_{\varphi^*(v)} v_i - C(\varphi^*(v))$, so $\sigma_v(\emptyset) > \sigma_v(\varphi^*(v))$, contradicting that φ^* maximizes surplus at v .

Moreover, if either of the cost function restrictions is dropped, there is a rule satisfying all four axioms:

- *Weak Returns*: This is the topic of [Section 4](#).
- *Positive Costs*: Consider $N = \{1, 2\}$ and C given by $C(\{1\}) = 0$, $C(\{2\}) = 2$ and $C(\{1, 2\}) = 1$. Let $\varphi(v) = \{1, 2\}$ if $v_2 \geq 1$ and $\varphi(v) = \{1\}$ otherwise, and let $\tau(v) = (0, 1)$ if $2 \in \varphi(v)$ and $\tau(v) = (0, 0)$ otherwise. It is easy to see that (τ, φ) satisfies all four axioms.

4 The object setting

We move onward from the disappointing findings of [Section 3](#), and now turn our attention to the *object setting*: costs are finite, the cost of serving a group depends only on its size (not on the specific identities of its members), and costs are convex. The simplest interpretation is that we are producing and distributing identical indivisible objects, and producing additional objects becomes progressively more expensive. The compatibility of our axioms holds even if any given agent might view the object as either a good or a bad:

DEFINITION: The *object setting* entails the following assumptions:

- **FINITE COSTS**: For each $W \subseteq N$, $C(W) < \infty$.

- **SYMMETRY:** For each pair $W, W' \subseteq N$, $|W| = |W'|$ implies $C(W) = C(W')$.
- **CONVEXITY:** For each $i \in N$ and each pair $W, W' \subseteq N \setminus \{i\}$ such that $W \subseteq W'$,

$$C(W' \cup \{i\}) - C(W') \geq C(W \cup \{i\}) - C(W).$$

- **FULL DOMAIN:** For each $i \in N$, $V_i = \mathbb{R}$.

As costs are symmetric, we abuse notation, writing $C : \{0, 1, 2, \dots, n\} \rightarrow \mathbb{R}_+$, where $C(w)$ denotes the monetary cost of producing w winners. Moreover, as costs are finite and convex, there are *marginal costs* $\mu_1, \mu_2, \dots, \mu_n \in \mathbb{R}_+$ such that

- (i) $\mu_n \geq \mu_{n-1} \geq \dots \geq \mu_1$, and
- (ii) for each $w \in \{1, 2, \dots, n\}$, $C(w) = \sum_{i=1}^w \mu_i$.

We begin by introducing an index that ranks agents from highest valuation to lowest, breaking ties using the original index: for each $v \in V$, define $A_1(v), A_2(v), \dots, A_n(v) \in N$ such that (1) $\{A_i(v)\} = N$, and (2) for each pair $r, r' \in \{1, 2, \dots, n\}$ such that $r' > r$, either

- (i) $v_{A_r(v)} > v_{A_{r'}(v)}$, or
- (ii) $v_{A_r(v)} = v_{A_{r'}(v)}$ and $A_{r'}(v) > A_r(v)$.

We say that r is the *rank-index of agent $A_r(v)$ at v* . We also introduce an index for listing the valuations themselves from highest to lowest, thinking of them as “bids” when ranked: for each $v \in V$ and each $r \in \{1, 2, \dots, n\}$, define $B_r(v) \equiv v_{A_r(v)}$. We say that r is the *rank-index of valuation $B_r(v)$ at v* . For emphasis, $B_1(v) \geq B_2(v) \geq \dots \geq B_n(v)$.

Analogous notation is also used for each $i \in N$ and each $v_{-i} \in V_{-i}$. Put another way, $N \setminus \{i\} = \{A_1(v_{-i}), A_2(v_{-i}), \dots, A_{n-1}(v_{-i})\}$, $B_1(v_{-i}) \geq B_2(v_{-i}) \geq \dots \geq B_{n-1}(v_{-i})$, and the re-indexed agents correspond to the re-indexed values at v_{-i} (with ties broken using the original index as above).

Our first result characterizes the rules satisfying *production efficiency*, whose defining feature is that the winners have the high valuations, justifying the term “bid” for preference report. More precisely, the winners are the high bidders, provided that each winning valuation is at least the marginal cost of the final object and each losing valuations does not exceed the marginal cost of another:

BIDDING LEMMA: In the object setting, a rule (τ, φ) satisfies *production efficiency* if and only if for each $v \in V$ with $w \equiv |\varphi(v)|$,

- (i) for each $i \in \varphi(v)$ and each $j \in N \setminus \varphi(v)$, $v_i \geq v_j$,
- (ii) $w \neq 0$ implies $B_w(v) \geq \mu_w$, and
- (iii) $w \neq n$ implies $\mu_{w+1} \geq B_{w+1}(v)$.

The proof is in [Appendix 2](#). By the [Bidding Lemma](#), at each profile v , a rule satisfying *production efficiency* first selects a number of winners w satisfying two constraints, then selects a group of that size whose valuations are highest. This is suggestive of an “equilibrium quantity,” though there is no corresponding suggestion of an “equilibrium price.”

Our second result characterizes the rules satisfying *strategy-proofness*, whose defining feature is that each agent’s attainable set is described by a consolation award and a price. In particular, each agent can either lose for the consolation award or win for the

consolation award minus the price. Moreover, the agent wins when he bids more than the price and loses when he bids less:

PRICE LEMMA: In the object setting, a rule (τ, φ) satisfies *strategy-proofness* if and only if for each $i \in N$ and each $v_{-i} \in V_{-i}$, there are $\tau_i^+(v_{-i}), \tau_i^-(v_{-i}) \in \mathbb{R}$ and $p_i(v_{-i}) \in \mathbb{R} \cup \{-\infty, \infty\}$ such that for each $v_i \in V_i$,

- (i) $(\tau_i(v), \varphi_i(v))$ is either $(\tau_i^+(v_{-i}), 1)$ or $(\tau_i^-(v_{-i}), 0)$,
- (ii) $p_i(v_{-i}) \notin \{-\infty, \infty\}$ implies $\tau_i^+(v_{-i}) = \tau_i^-(v_{-i}) - p_i(v_{-i})$,
- (iii) $v_i > p_i(v_{-i})$ implies $i \in \varphi(v)$, and
- (iv) $p_i(v_{-i}) > v_i$ implies $i \notin \varphi(v)$.

The proof is in [Appendix 3](#). Each agent faces a personalized price determined by the reports of his peers, though this is not quite enough to suggest an “equilibrium price.”

Our next objective is to characterize the rules satisfying both *production efficiency* and *strategy-proofness*. As the winner policy is fixed, what remains is to describe the transfer policy. By the [Price Lemma](#), each agent i faces a price $p_i(v_{-i})$ such that he demands an object with a higher report and demands no object with a lower report. For such a rule to satisfy *production efficiency*, these n prices must form an “equilibrium” at which honest agents independently demand objects in a manner that maximizes surplus. This dramatically restricts the manner in which an agent’s price can depend on the reports of his peers.

For each $i \in N$, define the *price for i* , $\pi : V_{-i} \rightarrow [\mu_1, \mu_n]$, by:

$$\begin{aligned} \Omega(v_{-i}) &\equiv |\{r \in \{1, 2, \dots, n-1\} \mid B_r(v_{-i}) \geq \mu_r\}|, \\ \pi(v_{-i}) &\equiv \begin{cases} \min\{B_{\Omega(v_{-i})}(v_{-i}), \mu_{\Omega(v_{-i})+1}\}, & \Omega(v_{-i}) > 0, \\ \mu_1, & \Omega(v_{-i}) = 0. \end{cases} \end{aligned}$$

Thus $\Omega(v_{-i})$ is the maximum number of i ’s peers that may win, and $\pi(v_{-i})$ is either the lowest of the bids of these peers or the marginal cost of an additional object—whichever is lower. Note that π is not indexed by any agent and that it processes reports anonymously; each agent’s price is determined by this common function that only depends on the list of peer bids, but not on which peers submitted which bids.

We use “auction” to refer to a rule in which both (i) winners are high bidders, and (ii) each agent’s bid is used to determine whether or not he demands an object at a particular price. The following class of auctions use prices, and are characterized by *production efficiency* and *strategy-proofness*:

DEFINITION: For each $i \in N$, a *peer-consolation schedule* is a function $\alpha_i : V_{-i} \rightarrow \mathbb{R}$. A rule (τ, φ) is a *peer-consolation rule* if (I) it satisfies *production efficiency*, and (II) for each $i \in N$, there is a peer-consolation schedule $\alpha_i : V_{-i} \rightarrow \mathbb{R}$ such that for each $v_{-i} \in V_{-i}$,

- (i) $(\tau_i(v), \varphi_i(v))$ is either $(\alpha_i(v_{-i}) - \pi(v_{-i}), 1)$ or $(\alpha_i(v_{-i}), 0)$,
- (ii) $v_i > \pi(v_{-i})$ implies $i \in \varphi(v)$, and
- (iii) $\pi(v_{-i}) > v_i$ implies $i \notin \varphi(v)$.

AUCTION LEMMA: In the object setting, a rule (τ, φ) satisfies *production efficiency* and *strategy-proofness* if and only if it is a peer-consolation rule.

The proof is in [Appendix 4](#). By a superficial adaptation of [Holmström \(1979\)](#) (see

footnote 3), these are precisely the [Groves \(1973\)](#) rules; the additional structure in our representation is due to the additional structure in our model.

We have thus far emphasized that each agent's price is determined by the reports of his peers. But when the entire profile of reports is considered, the prices are collectively harmonious in a particular way: winners face one price and losers face another. Moreover, the lowest winning bid is at least the losers' price, which is at least the winners' price, which is at least the highest losing bid. Formally, define:

$$\begin{aligned}\Omega(v) &\equiv |\{r \in \{1, 2, \dots, n\} | B_r(v) \geq \mu_r\}|, \\ \pi_+(v) &\equiv \begin{cases} \mu_n, & \Omega(v) = n, \\ \max\{B_{\Omega(v)+1}(v), \mu_{\Omega(v)}\}, & n > \Omega(v) > 0 \\ \mu_1, & \Omega(v) = 0. \end{cases} \\ \pi_-(v) &\equiv \begin{cases} \mu_n, & \Omega(v) = n, \\ \min\{B_{\Omega(v)}(v), \mu_{\Omega(v)+1}\}, & n > \Omega(v) > 0 \\ \mu_1, & \Omega(v) = 0. \end{cases}\end{aligned}$$

WINNER-LOSER PRICES LEMMA: In the object setting, for each $v \in V$ and each $i \in N$,

(i) $v_i \geq \pi(v_{-i})$ implies $\pi(v_{-i}) = \pi_+(v)$, and

(ii) $\pi(v_{-i}) \geq v_i$ implies $\pi(v_{-i}) = \pi_-(v)$.

Moreover, either $v_i \geq \pi_-(v) \geq \pi_+(v)$ or $\pi_-(v) \geq \pi_+(v) \geq v_i$.

The proof is in [Appendix 5](#).

Which peer-consolation rules satisfy *no-envy-in-trades*? In order to provide a complete answer, we first observe that *no-envy-in-trades* imposes a strong restriction on peer-consolation schedules: an agent's consolation award cannot depend on the reports of his peers in any way, but rather can only depend on a particular summary statistic of their reports. Moreover, this particular summary statistic is his own price:

CONSOLATION LEMMA: In the object setting, if a rule (τ, φ) satisfies *production efficiency*, *strategy-proofness*, and *no-envy-in-trades*, then (τ, φ) is a peer-consolation rule with peer-consolation schedules (α_i) such that for each $i \in N$ and each pair $v_{-i}, v'_{-i} \in V_{-i}$,

$$\pi(v_{-i}) = \pi(v'_{-i}) \text{ implies } \alpha_i(v_{-i}) = \alpha_i(v'_{-i}).$$

The proof, which is the central proof of this article, is in [Appendix 6](#).

In fact, *no-envy-in-trades* further implies that agents share a common peer-consolation schedule that depends only on price, and moreover that this common schedule is nondecreasing, Lipschitz-continuous, and without a slope greater than 1 at any point. We call a communal peer-consolation schedule with these features a *social-dividend schedule*:

DEFINITION: A *social-dividend schedule* is a (Lipschitz-continuous) function $\alpha_0 : [\mu_1, \mu_n] \rightarrow \mathbb{R}$ such that for each distinct pair $\pi', \pi \in [\mu_1, \mu_n]$,

$$\frac{\alpha_0(\pi') - \alpha_0(\pi)}{\pi' - \pi} \in [0, 1].$$

A rule (τ, φ) is *citizen-shareholder rule* if (I) it satisfies *production efficiency*, and (II) there is a social-dividend schedule α_0 such that for each $i \in N$ and each $v_{-i} \in V_{-i}$,

- (i) $(\tau_i(v), \varphi_i(v))$ is either $(\alpha_0(\pi(v_{-i})) - \pi(v_{-i}), 1)$ or $(\alpha_0(\pi(v_{-i})), 0)$,
- (ii) $v_i > \pi(v_{-i})$ implies $i \in \varphi(v)$, and
- (iii) $\pi(v_{-i}) > v_i$ implies $i \notin \varphi(v)$.

THEOREM 2: In the object setting, a rule (τ, φ) satisfies *production efficiency*, *strategy-proofness*, and *no-envy-in-trades* if and only if it is a citizen-shareholder rule.

The proof is in [Appendix 7](#).

Remarkably, each citizen-shareholder rule can be viewed as a kind of equal-income Walrasian rule. In particular, consider a citizen-shareholder rule with social-dividend schedule α_0 , and let $v \in V$. Let w denote the number of winners, and define

$$\begin{aligned} Y(v) &\equiv \alpha_0(\pi_-(v)), \\ p_+^{Walras}(v) &\equiv \pi_+(v) + (\alpha_0(\pi_-(v)) - \alpha_0(\pi_+(v))), \text{ and} \\ p_-^{Walras}(v) &\equiv 0. \end{aligned}$$

Then the income $Y(v)$ and the price vector $(p_+^{Walras}(v), p_-^{Walras}(v))$ together determine a budget set. By construction, each winner receives $Y(v) - p_+^{Walras}(v)$ and each loser receives $Y(v) - p_-^{Walras}(v)$. If $n > w > 0$, then there is at least one winner and one loser, so by *no-envy-in-trades*, each winner selects his bundle from the budget set and each loser does as well. If $w = n$, then each agent wins and receives $\alpha_0(\mu_n) - \mu_n$, which he finds at least as desirable as losing and receiving $\alpha_0(\mu_n)$ as his value is at least μ_n . If $w = 0$, then each agent loses and receives $\alpha_0(\mu_1)$, which he finds at least as desirable as winning and receiving $\alpha_0(\mu_1) - \mu_1$ as his value is at most μ_1 . Thus in all cases, each agent receives a bundle he most-prefers from the common budget set.

Unsurprisingly, the price of an object is never negative; in fact it is always at least μ_1 . More surprisingly, the income $Y(v)$ can be positive, and in particular the total income received by the agents may exceed the amount of money initially in the economy. This is possible when the winners overpay; their overpayments can be used to finance the income for themselves and for the losers.

We now turn our attention to the constraints imposed by the *no-deficit* axiom.

DEFINITION: A *cost-constrained social-dividend schedule* is a social-dividend schedule such that:

- (i) $\alpha_0(\mu_1) \leq 0$,
- (ii) for each $w \in \{1, 2, \dots, n-1\}$,

$$(n-w)\alpha_0(\mu_{w+1}) \leq (w\mu_w - C(w)) - w\alpha_0(\mu_w), \text{ and}$$

$$\text{(iii) } \alpha_0(\mu_n) \leq \frac{n\mu_n - C(n)}{n}.$$

A rule (τ, φ) is a *cost-constrained citizen-shareholder rule* if (I) (τ, φ) is standard and (II) there is a cost-constrained social-dividend schedule α_0 such that for each $i \in N$ and each $v_{-i} \in V_{-i}$, $(\tau_i(v), \varphi_i(v))$ is either $(\alpha_0(\pi(v_{-i})) - \pi(v_{-i}), 1)$ or $(\alpha_0(\pi(v_{-i})), 0)$.

THEOREM 3: In the object setting, a rule (τ, φ) satisfies *no-deficit*, *production efficiency*, *strategy-proofness* and *no-envy-in-trades* if and only if it is a cost-constrained citizen-shareholder rule.

The proof is in [Appendix 8](#).

Finally, we turn our attention to (i) rules that are not Pareto-dominated, and (ii) our final axiom, *voluntarism*. First, we define the family of social-dividend schedules which most-aggressively increase to hit the social-dividend awards at the marginal costs:

DEFINITION: A social-dividend schedule α_0 is *unrelenting* if there is $(a_w) \in \mathbb{R}^N$ such that

- (i) for each $w \in \{1, 2, \dots, n\}$, $\alpha_0(\mu_w) = a_w$, and
- (ii) for each $w \in \{1, 2, \dots, n-1\}$ and each $\delta \in [0, \mu_{w+1} - \mu_w]$,

$$\alpha_0(\mu_w + \delta) = \min\{\alpha_0(\mu_w) + \delta, \alpha_0(\mu_{w+1})\}.$$

In this case, we say that (a_w) is the *marginal cost award profile* of α_0 .

As each unrelenting social-dividend schedule can be described by its marginal cost award profile, so too can the class of citizen-shareholder rules with such schedules that are not Pareto-dominated be described by its associated class of marginal cost award profiles. In particular, define the set of marginal cost award profiles $\mathcal{A} \subseteq \mathbb{R}^N$ by:

$$\begin{aligned} \mathcal{A} \equiv \{a \in \mathbb{R}^N \mid & a_1 \leq 0; \\ & \text{for each } w \in \{1, 2, \dots, n-1\}, \\ & a_{w+1} - a_w \leq \mu_{w+1} - \mu_w, \\ & a_{w+1} - a_w \geq 0, \text{ and} \\ & (n-w)a_{w+1} + wa_w \leq w\mu_w - C(w); \text{ and} \\ & a_n \leq \frac{n\mu_n - C(n)}{n}\}. \end{aligned}$$

Define the subset of profiles that (we claim, and will show) are compatible with *voluntary* rules $\mathcal{A}^V \subseteq \mathcal{A}$ by:

$$\mathcal{A}^V \equiv \{a \in \mathcal{A} \mid a_1 = 0\}.$$

To define the vector domination relation $\geq$, for each pair $a, a' \in \mathbb{R}^N$, define $a \geq a'$ if and only if for each $w \in \{1, 2, \dots, n\}$, $a_w \geq a'_w$. If $a \geq a'$ and $a \neq a'$, we write $a > a'$. Define the respective ‘frontier’ sets $\mathcal{F} \subseteq \mathcal{A}$ and $\mathcal{F}^V \subseteq \mathcal{A}^V$ by:

$$\begin{aligned} \mathcal{F} &\equiv \{a \in \mathcal{A} \mid a' \in \mathcal{A} \text{ and } a' \geq a \text{ implies } a' = a\}, \text{ and} \\ \mathcal{F}^V &\equiv \{a \in \mathcal{A}^V \mid a' \in \mathcal{A} \text{ and } a' \geq a \text{ implies } a' = a\}. \end{aligned}$$

THEOREM 4: In the object setting,

(I) a rule (τ, φ) (i) satisfies *no-deficit*, *production efficiency*, *strategy-proofness*, and *no-envy-in-trades*, and (ii) is not Pareto-dominated by another such rule, if and only if it is a cost-constrained citizen-shareholder rule with an unrelenting social-dividend schedule whose marginal cost award profile belongs to $\mathcal{F}$.

(II) a rule (τ, φ) (i) satisfies *no-deficit*, *production efficiency*, *voluntarism*, *strategy-proofness*, and *no-envy-in-trades*, and (ii) is not Pareto-dominated by another such rule, if and only if it is a cost-constrained citizen-shareholder rule with an unrelenting social-dividend schedule whose marginal cost award profile belongs to $\mathcal{F}^V$.

The proof is in [Appendix 9](#).

5 Conclusion

We conclude by discussing several observations about our results.

5.1 Asymmetric costs

For our positive results, we imposed that the cost function is symmetric, but analogous results hold if costs are asymmetric in a structured way and *no-envy-in-trades* is adapted accordingly. In particular, suppose that—as before—production costs are given by a symmetric, finite, and convex function C . Now, however, in addition to these production costs, each agent $i \in N$ has a *personal cost* κ_i . For example, this might be the cost of shipping the object to agent i , or it might be the cost of further customizing the object to his specifications. Altogether, the cost of serving group W is now given by:

$$\hat{C}(W) = C(W) + \sum_{i \in W} \kappa_i.$$

In this setting, it is straightforward to modify our rules so that each agent necessarily pays his own personal cost should he win. In particular, for each $i \in N$, define $\hat{v}_i \equiv v_i + \kappa_i$. For each rule (τ, φ) , define the rule $(\hat{\tau}, \hat{\varphi})$ by defining, for each $v \in V$ and each $i \in N$,

$$\begin{aligned} \hat{\varphi}(\hat{v}) &\equiv \varphi(v), \text{ and} \\ \hat{\tau}_i(\hat{v}) &\equiv \tau_i(v) - \varphi_i(v)\kappa_i. \end{aligned}$$

For brevity, we use the past tense to refer to (τ, φ) and the present tense to refer to $(\hat{\tau}, \hat{\varphi})$. We claim that if (τ, φ) satisfied the axioms for C , then $(\hat{\tau}, \hat{\varphi})$ ‘inherits’ these axioms for $\hat{C}$ when *no-envy-in-trades* is properly adapted. Indeed, for each $v \in V$,

- since *no-deficit* was satisfied at v , it is now satisfied at $\hat{v}$ because each winner now covers his personal cost,
- since *production efficiency* was satisfied at v , it is now satisfied at $\hat{v}$ because each group has the same Marshallian surplus as before (since valuation increases are equal to cost increases), and
- since *voluntarism* was satisfied at v , it is now satisfied at $\hat{v}$ because a loser receives the same non-negative transfer, while a winner’s valuation increases by the same amount as his contribution.

For *strategy-proofness*, it suffices to observe that for each $i \in N$, each $v \in V$, and each $v'_i \in V_i$,

$$\begin{aligned} v_i \varphi_i(v) + \tau_i(v) &= \hat{v}_i \hat{\varphi}_i(\hat{v}) + \hat{\tau}_i(\hat{v}), \text{ and} \\ v_i \varphi_i(v'_i, v_{-i}) + \tau_i(v'_i, v_{-i}) &= \hat{v}_i \hat{\varphi}_i(\hat{v}_i, \hat{v}_{-i}) + \hat{\tau}_i(\hat{v}_i, \hat{v}_{-i}). \end{aligned}$$

from which the inheritance of *strategy-proofness* immediately follows.

To adapt *no-envy-in-trades*, we require that when an agent considers a peer’s assignment, he first adjusts it by replacing his peer’s personal cost with his own before comparing:

NO-ENVY-IN-ADJUSTED-TRADES: For each pair $i, j \in N$ and each $v \in V$,

$$v_i \varphi_i(v) + \tau_i(v) \geq v_i \varphi_j(v) + \tau_j(v) + \varphi_j(v)(\kappa_j - \kappa_i).$$

In other words, when i considers the assignment of j , if j wins, then in the adjustment κ_j is returned and then κ_i is deducted. It is straightforward to verify that if (τ, φ) satisfies *no-envy-in-trades*, then $(\hat{\tau}, \hat{\varphi})$ satisfies *no-envy-in-adjusted-trades*.

Altogether, then, in this special class of cost functions with asymmetric costs (which includes all additive cost functions), these natural analogues of our rules satisfy our requirements when the equal opportunity requirement is adjusted to take into account differences in personal costs.

5.2 Two-sided economies

Consider an economy with money and indivisible objects, where each agent desires at most one object and has preferences summarized by his valuation. Moreover, each agent belongs to one of two *sides*: each *buyer* is initially endowed with no object, while each *seller* is initially endowed with one object.

In such an economy, there are Walrasian price vectors (Koopmans and Beckmann, 1957; Gale, 1960), which have associated outcomes that form the core of a cooperative game for that economy (Shapley and Shubik, 1972). Moreover, the set of Walrasian price vectors is a bounded lattice (Shapley and Shubik, 1972), and when seller valuations are common knowledge, the *minimum-price Walrasian rule*, which uses reported buyer valuations to calculate the minimum Walrasian price vector and assign the associated core outcome, is *strategy-proof* (Demange, 1982; Leonard, 1983).¹¹

Our analysis applies to the special case where the objects are identical. Indeed, though we originally presented $\mu_1, \mu_2, \dots, \mu_n$ as marginal costs for the production technology, it would be just as legitimate to interpret these numbers as commonly-known seller valuations. In this special case, there is a single Walrasian price instead of a Walrasian price vector (Shapley and Shubik, 1972). Note that in this interpretation, a Walrasian outcome is Pareto-efficient: it does not involve burned money, but rather seller profit, and thus there is no violation of the First Welfare Theorem.

A rule in our class uses reported buyer valuations to calculate the minimum Walrasian price, but it does not necessarily recommend the associated core outcome, where each buyer receives his most-preferred bundle from the budget set given by this price and zero transfer. Instead, it may recommend an outcome where each buyer receives his most-preferred bundle from the budget set given by this price and some positive transfer, which is an outcome outside of the core.

In the cooperative game for our Big Show example, the two fans who go to the show and the two scalpers who sell them tickets would together block our rule's recommendation. Thus it is only if the buyers can commit to honoring our rule's recommendation and not cooperating with the sellers that they might benefit, with respect to the Walrasian core outcome, at the sellers' expense. Similarly, in the market for horses considered by Böhm-Bawerk (1888), the horse-less might benefit over the classic solutions through such

¹¹Demange and Gale (1985) showed that these facts persist even when preferences are not quasi-linear—namely, that Walrasian prices form a lattice, and that the minimum-price Walrasian rule is *strategy-proof*—and this rule was recently characterized by Morimoto and Serizawa (2015).

a pact. Whether such an agreement constitutes cooperation or collusion is a matter of perspective, varying from one side of the market to the other.

We remark that if, in our Big Show example, the fans are viewed as a nation that exports money, while the scalpers are viewed as a nation that exports tickets, then we have a (small) economy of international trade. For such economies, our rules suggest a role for a government that is dedicated only to the welfare of its own citizens.

5.3 Markets versus central planning

As notably emphasized by Hayek (1945), the solution to an economic problem typically depends on information that is scattered across different individuals, and historically, two institutions for reaching solutions have dominated economic analysis. In a *market*, a public summary of the scattered information is generated, and this summary shapes the opportunities available to individuals so that their independent choices are harmonious. In a (*direct*) *mechanism*, the scattered information is reported to a central authority, who calculates and administers a solution. While today, it is perhaps generally recognized that markets may be imperfect and mechanisms may help, historically (and especially during the Cold War), the issue was often painted in black and white.¹² The relative merits of these institutions, cast as capitalism and socialism, were notably investigated during the *socialist calculation debate* beginning with von Mises (1920).

An important observation from that debate is that our (standard) definitions of markets and mechanisms do not preclude a hybrid institution (Lange 1936; Lerner 1944). Indeed, a market in Walrasian equilibrium summarizes information and shapes opportunities with a list of Walrasian (clearing) prices, which exist in general settings (Walras 1896; McKenzie 1954; Arrow and Debreu 1954), but it is not specified how this summary is generated; a mechanism calculates and administers a solution (Hurwicz 1972; Gibbard 1973), but it is not specified how this solution is administered. A Vickrey direct mechanism (Vickrey, 1961) fits both descriptions.

We believe that our analysis contributes to this debate. On the one hand, for a large class of industries, we find that centrally-planned production is necessarily unaffordable, inefficient, manipulable, or coercive. On the other hand, for a special class of industries, we find that central planning that uses market principles can in fact outperform the market, and to do so it need not resort to coercion. In such an institution, agents are incentivized to report their private information, then these reports are used to calculate an efficient production plan, and finally the agents voluntarily contribute the necessary funding in exchange for promised output. The central planner plays the role of the Walrasian auctioneer by calculating the Walrasian prices, but then goes further, calculating whether or not the agents over-contribute and returning social dividends to them when doing so does not interfere with incentives. Remarkably, precisely such an institution is described by Lange (1936):

It seems, therefore, convenient to regard the income of consumers as being composed of two parts: one part being the receipts for the labor services performed and the other part being a social dividend constituting the individual's

¹²For example, Milton Friedman (Friedman, 1962) writes: “Fundamentally, there are only two ways of co-ordinating the economic activities of millions. One is central direction involving the use of coercion—the technique of the army and of the modern totalitarian state. The other is voluntary co-operation of individuals—the technique of the market place.”

share in the income derived from the capital and the natural resources owned by society.

This notion of a *social dividend* continues to play a role in models of market socialism (Bardhan and Roemer, 1992).

While our analysis suggests that for many industries, there would be some serious problems with such an institution, our analysis also suggests that for some very special industries, such an institution could help improve market outcomes—precisely by treating agents as citizen-shareholders and awarding social dividends. Theoretically, in such industries, a central computer could democratically and transparently guide production according to the interests of voluntary participants.

Appendix 1

In this appendix, we prove Theorem 1.

THEOREM 1: In the service setting, there is no rule satisfying *no-deficit*, *production efficiency*, *voluntarism*, and *strategy-proofness*.

PROOF: Assume, by way of contradiction, (τ, φ) satisfies the given axioms. Define $\mathcal{W}_f$ to be the set of nonempty groups with finite cost:

$$\mathcal{W}_f \equiv \{W \in 2^N \setminus \{\emptyset\} \mid C(W) < \infty\}.$$

By *weak returns*, $\mathcal{W}_f$ is nonempty. We consider two cases.

CASE 1: For each $W \in \mathcal{W}_f$ and each $i \in W$, $C(\{i\}) < \infty$.

For each $W \in \mathcal{W}_f$, define the *group rate for* W , $\Gamma(W)$, by:

$$\Gamma(W) \equiv \frac{C(W)}{\sum_{i \in W} C(\{i\})}.$$

By *positive costs* and the case's assumption, for each $W \in \mathcal{W}_f$, $\Gamma(W)$ is well-defined. Thus we can define the minimal group rate γ by:

$$\gamma \equiv \min_{W \in \mathcal{W}_f} \Gamma(W).$$

By *positive costs*, $\gamma > 0$, and by *weak returns*, $\gamma < 1$.

Define the collection of groups with group rate γ , $\mathcal{W}_\gamma$, by:

$$\mathcal{W}_\gamma \equiv \{W \in \mathcal{W}_f \mid \Gamma(W) = \gamma\}.$$

By construction, $\mathcal{W}_\gamma$ is nonempty. Let W^* be an inclusion-minimal member of $\mathcal{W}_\gamma$ (that is, for each $W \in \mathcal{W}_\gamma$, $W \not\subset W^*$). Since $\gamma < 1$, thus $|W^*| \geq 2$. By *positive costs* and the case's assumption, for each $i \in W^*$, $C(\{i\}) \in (0, \infty)$.

We proceed by first considering two successive profiles $v', v \in V$, then considering two families of profiles $\{v^{i'}\}_{i \in W^*}$, $\{v^i\}_{i \in W^*}$. First, define $v' \in V$ by:

$$v'_i = \begin{cases} \gamma C(\{i\}), & i \in W^*, \\ 0, & \text{else.} \end{cases}$$

We claim $\emptyset$ and W^* are the surplus-maximizers at v' . Indeed:

- for $\emptyset$, $\sigma_{v'}(\emptyset) = 0$,
- for W^* ,

$$\begin{aligned}\sigma_{v'}(W^*) &= \sum_{W^*} \gamma C(\{i\}) - C(W^*) \\ &= \gamma \sum_{W^*} C(\{i\}) - C(W^*) \\ &= \frac{C(W^*)}{\sum_{W^*} C(\{i\})} \sum_{W^*} C(\{i\}) - C(W^*) \\ &= 0,\end{aligned}$$

• for each $W \in \mathcal{W}_\gamma \setminus \{W^*\}$, by construction there is $i' \in W \setminus W^*$, and by *positive costs* and the case's assumption $C(\{i'\}) \in (0, \infty)$, so

$$\begin{aligned}\sigma_{v'}(W) &= \sum_{W \cap W^*} \gamma C(\{i\}) - C(W) \\ &\leq \sum_{W \setminus \{i'\}} \gamma C(\{i\}) - C(W) \\ &< \sum_W \gamma C(\{i\}) - C(W) \\ &= \gamma \sum_W C(\{i\}) - C(W) \\ &\leq \frac{C(W)}{\sum_W C(\{i\})} \sum_W C(\{i\}) - C(W) \\ &= 0,\end{aligned}$$

- for each $W \in \mathcal{W}_f \setminus \mathcal{W}_\gamma$,

$$\begin{aligned}\sigma_{v'}(W) &= \sum_{W \cap W^*} \gamma C(\{i\}) - C(W) \\ &\leq \sum_W \gamma C(\{i\}) - \frac{C(W)}{\sum_W C(\{i\})} \sum_W C(\{i\}) \\ &= (\gamma - \frac{C(W)}{\sum_W C(\{i\})}) \sum_W C(\{i\}) \\ &< 0, \text{ and}\end{aligned}$$

- for each $W \in 2^N \setminus \mathcal{W}_f$, $\sigma_{v'}(W) = -\infty$.

Thus there is $\epsilon \in (0, 1)$ such that

- (i) for each $W \in 2^N \setminus \{\emptyset, W^*\}$, $\sigma_{v'}(W^*) - \epsilon > \sigma_{v'}(W)$, and
- (ii) for each $i \in W^*$, $v'_i - \epsilon \geq 0$.

Define $v \in V$ by:

$$v_i = \begin{cases} \gamma C(\{i\}) + 1, & i \in W^*, \\ 0, & \text{else.} \end{cases}$$

We claim W^* is the unique surplus-maximizer at v . Indeed, each group's surplus increases by at most $|W^*|$ from v' to v , and the surplus of W^* increases by this amount while the surplus of $\emptyset$ does not. Thus by *production efficiency*, $\varphi(v) = W^*$.

For each $i \in W^*$, define $v^{i'}, v^i \in V$ by:

$$v_j^{i'} = \begin{cases} v_i' - \epsilon, & j = i, \\ v_j', & \text{else.} \end{cases}$$

$$v_j^i = \begin{cases} v_i' - \epsilon, & j = i, \\ v_j, & \text{else.} \end{cases}$$

By construction of ϵ , these are indeed profiles in V . Let $i \in W^*$. We claim W^* is the unique surplus-maximizer at v^i . Indeed:

- for $\emptyset$, since $|W^*| \geq 2$ and by construction of ϵ ,

$$\begin{aligned} \sigma_{v^i}(W^*) &= \sigma_{v^{i'}}(W^*) + (|W^*| - 1) \\ &\geq \sigma_{v^{i'}}(W^*) + 1 \\ &= (\sigma_{v'}(W^*) - \epsilon) + 1 \\ &> \sigma_{v'}(W^*) \\ &= 0 \\ &= \sigma_{v^i}(\emptyset), \text{ and} \end{aligned}$$

- for $W \in 2^N \setminus \{\emptyset, W^*\}$, by construction of ϵ ,

$$\begin{aligned} \sigma_{v^{i'}}(W^*) &= \sigma_{v'}(W^*) - \epsilon \\ &> \sigma_{v'}(W) \\ &\geq \sigma_{v^{i'}}(W), \end{aligned}$$

so since only members of W^* have higher valuations at v^i than $v^{i'}$, thus $\sigma_{v^i}(W^*) > \sigma_{v^i}(W)$.

By *production efficiency*, $\varphi(v^i) = W^*$, so by *voluntarism*, $\tau_i(v^i) \geq -v_i^i > -\gamma C(\{i\})$, so by *strategy-proofness*, $\tau_i(v) \geq \tau_i(v^i) > -\gamma C(\{i\})$.

Since $i \in W^*$ was arbitrary, thus for each $i \in W^*$, $\tau_i(v) > -\gamma C(\{i\})$. By *voluntarism*, for each $i \in N \setminus W^*$, $\tau_i(v) \geq 0$. But then

$$\begin{aligned} \sum \tau_i(v) &> -\gamma \sum_{W^*} C(\{i\}) \\ &= -C(W^*) \\ &= -C(\varphi(v)), \end{aligned}$$

contradicting *no-deficit*.

CASE 2: There is $W \in \mathcal{W}_f$ and $i \in W$ such that $C(\{i\}) = \infty$.

Define M to be the maximum finite cost:

$$M \equiv \max_{\mathcal{W}_f} C(W).$$

For each $i \in \cup \mathcal{W}_f$ such that $C(\{i\}) = \infty$, define $C^*(\{i\}) \equiv M + 1$. For all other groups, define $C^*(W) \equiv C(W)$. It is easy to verify that C^* satisfies *weak returns* and *positive costs*, and thus satisfies the Case 1 assumptions.

For each axiom, let us distinguish the version involving C and the version involving C^* by writing C -axiom and C^* -axiom; for example, C -no-deficit and C^* -no-deficit. Similarly, we write C -surplus and C^* -surplus.

By assumption, (τ, φ) satisfies the C -axioms. We claim that furthermore, (τ, φ) satisfies the C^* -axioms. Indeed, both versions of *strategy-proofness* and *voluntarism* are equivalent, as neither axiom makes reference to the cost function. Moreover, C^* -no-deficit cannot be violated because no group has a higher cost under C^* than under C . Finally, all groups have the same C -surplus and C^* -surplus except the singletons from $\cup \mathcal{W}_f$, so a violation of C^* -production efficiency must involve a valuation profile v where such a singleton $\{i\}$ has the highest C^* -surplus, but is not selected. But there is $W \in \mathcal{W}_f$ such that $i \in W$, and by construction the C^* -surplus of W is higher than the C^* -surplus of i at any profile, contradicting that $\{i\}$ has the highest C^* -surplus at v .

Altogether, (τ, φ) satisfies the C^* -axioms, contradicting that there is no such rule by Case 1. ■

Appendix 2

In this appendix, we prove the [Bidding Lemma](#).

BIDDING LEMMA: In the object setting, a rule (τ, φ) satisfies *production efficiency* if and only if for each $v \in V$ with $w \equiv |\varphi(v)|$,

- (i) for each $i \in \varphi(v)$ and each $j \in N \setminus \varphi(v)$, $v_i \geq v_j$,
- (ii) $w \neq 0$ implies $B_w(v) \geq \mu_w$, and
- (iii) $w \neq n$ implies $\mu_{w+1} \geq B_{w+1}(v)$.

PROOF: We establish both implications.

[$\Rightarrow$] Let (τ, φ) satisfy *production efficiency*, let $v \in V$, and define $w \equiv |\varphi(v)|$.

(i) Assume, by way of contradiction, there are $i \in \varphi(v)$ and $j \in N \setminus \varphi(v)$ such that $v_j > v_i$. Then

$$\begin{aligned} \sigma_v((\varphi(v) \setminus \{i\}) \cup \{j\}) &= \sum_{(\varphi(v) \setminus \{i\}) \cup \{j\}} v_k - C(|(\varphi(v) \setminus \{i\}) \cup \{j\}|) \\ &= \sum_{\varphi(v)} v_k + (v_j - v_i) - C(|\varphi(v)|) \\ &> \sum_{\varphi(v)} v_k - C(|\varphi(v)|) \\ &= \sigma_v(\varphi(v)), \end{aligned}$$

contradicting that φ satisfies *production efficiency*.

(ii) Assume, by way of contradiction, $w \neq 0$ and $\mu_w > B_w(v)$. By (i), there is $i \in \varphi(v)$

with $v_i = B_w(v)$. But then

$$\begin{aligned}
\sigma_v(\varphi(v) \setminus \{i\}) &= \sum_{\varphi(v) \setminus \{i\}} v_k - C(|\varphi(v) \setminus \{i\}|) \\
&= \sum_{\varphi(v)} v_k - B_w(v) - (C(w) - \mu_w) \\
&= \sigma_v(\varphi(v)) + (\mu_w - B_w(v)) \\
&> \sigma_v(\varphi(v)),
\end{aligned}$$

contradicting that φ satisfies *production efficiency*.

(iii) Assume, by way of contradiction, $w \neq n$ and $B_{w+1}(v) > \mu_{w+1}$. By (i), there is $i \in N \setminus \varphi(v)$ with $v_i = B_{w+1}(v)$. But then

$$\begin{aligned}
\sigma_v(\varphi(v) \cup \{i\}) &= \sum_{\varphi(v) \cup \{i\}} v_k - C(|\varphi(v) \cup \{i\}|) \\
&= \sum_{\varphi(v)} v_k + B_{w+1}(v) - (C(w) + \mu_{w+1}) \\
&= \sigma_v(\varphi(v)) + (B_{w+1}(v) - \mu_{w+1}) \\
&> \sigma_v(\varphi(v)),
\end{aligned}$$

contradicting that φ satisfies *production efficiency*.

[$\Leftarrow$] Assume, by way of contradiction, (τ, φ) satisfies the hypotheses but does not satisfy *production efficiency*. Then there are $v \in V$ and $W \subseteq N$ such that $\sigma_v(W) > \sigma_v(\varphi(v))$. Let $W^* \subseteq N$ be a group of $|W|$ agents such that $i \in W^*$ and $v_j > v_i$ implies $j \in W^*$. Since $C(W^*) = C(W)$, thus $\sigma_v(W^*) \geq \sigma_v(W) > \sigma_v(\varphi(v))$.

Let $\sigma^*(w)$ denote the surplus at v for serving w highest-valuation agents. Formally, define $\sigma^* : \{0, 1, \dots, n\} \rightarrow \mathbb{R}$ by: $\sigma^*(0) = 0$, and for each $w \in \{1, 2, \dots, n\}$,

$$\sigma^*(w) = \sum_{r=1}^w (B_r(v) - \mu_r).$$

By construction of W^* and by (i), $\sigma^*(|W^*|) = \sigma_v(W^*) > \sigma_v(\varphi(v)) = \sigma^*(|\varphi(v)|)$.

For each $w \in \{1, 2, \dots, n\}$, since $B_w(v)$ is non-increasing in w and μ_w is non-decreasing in w , thus $B_w(v) - \mu_w$ is non-increasing in w . Then σ^* is maximized at w if $\sigma^*(w)$ sums across all positive terms and no negative terms, or if:

- $w \neq 0$ implies $B_w(v) - \mu_w \geq 0$ and
- $w \neq n$ implies $0 \geq B_{w+1}(v) - \mu_{w+1}$.

But by (i), (ii), and (iii), these conditions are met for $|\varphi(v)|$, so σ^* is maximized at $|\varphi(v)|$, contradicting $\sigma^*(|W^*|) > \sigma^*(|\varphi(v)|)$. ■

Appendix 3

In this appendix, we prove the [Price Lemma](#).

PRICE LEMMA: In the object setting, a rule (τ, φ) satisfies *strategy-proofness* if and only if for each $i \in N$ and each $v_{-i} \in V_{-i}$, there are $\tau_i^+(v_{-i}), \tau_i^-(v_{-i}) \in \mathbb{R}$ and $p_i(v_{-i}) \in \mathbb{R} \cup \{-\infty, \infty\}$ such that for each $v_i \in V_i$,

- (i) $(\tau_i(v), \varphi_i(v))$ is either $(\tau_i^+(v_{-i}), 1)$ or $(\tau_i^-(v_{-i}), 0)$,
- (ii) $p_i(v_{-i}) \notin \{-\infty, \infty\}$ implies $\tau_i^+(v_{-i}) = \tau_i^-(v_{-i}) - p_i(v_{-i})$,
- (iii) $v_i > p_i(v_{-i})$ implies $i \in \varphi(v)$, and
- (iv) $p_i(v_{-i}) > v_i$ implies $i \notin \varphi(v)$.

PROOF: We establish both implications.

[$\Rightarrow$] Let (τ, φ) satisfy *strategy-proofness*, let $i \in N$, and let $v_{-i} \in V_{-i}$. We begin by defining $\tau_i^+(v_{-i})$, $\tau_i^-(v_{-i})$, and $p_i(v_{-i})$.

If there are $v_i, v'_i \in V_i$ such that $i \in \varphi(v)$ and $i \in \varphi(v'_i, v_{-i})$, then by *strategy-proofness* $\tau_i(v) = \tau_i(v'_i, v_{-i})$. Thus if there is $v_i^+ \in V_i$ such that $i \in \varphi(v_i^+, v_{-i})$, then there is a unique $\tau_i^+(v_{-i}) \in \mathbb{R}$ such that for each $v_i \in V_i$, $i \in \varphi(v)$ implies $\tau_i(v) = \tau_i^+(v_{-i})$. If there is no such $v_i^+ \in V_i$, let $\tau_i^+(v_{-i}) \in \mathbb{R}$ be arbitrary.

Similarly, if there is $v_i^- \in V_i$ such that $i \notin \varphi(v_i^-, v_{-i})$, then there is a unique $\tau_i^-(v_{-i}) \in \mathbb{R}$ such that for each $v_i \in V_i$, $i \notin \varphi(v)$ implies $\tau_i(v) = \tau_i^-(v_{-i})$; if there is no such $v_i^- \in V_i$, let $\tau_i^-(v_{-i}) \in \mathbb{R}$ be arbitrary.

If there are $v_i^+, v_i^- \in V_i$ such that $i \in \varphi(v_i^+, v_{-i})$ and $i \notin \varphi(v_i^-, v_{-i})$, then define $p_i(v_{-i}) \equiv \tau_i^-(v_{-i}) - \tau_i^+(v_{-i})$. If for each $v_i \in V_i$, $i \in \varphi(v)$, then define $p_i(v_{-i}) \equiv -\infty$. If for each $v_i \in V_i$, $i \notin \varphi(v)$, then define $p_i(v_{-i}) \equiv \infty$.

By construction, (i) and (ii) hold. If $p_i(v_{-i}) \in \{-\infty, \infty\}$, then it is immediate that (iii) and (iv) hold as well, so assume $p_i(v_{-i}) \in \mathbb{R}$. Then there are $v_i^+, v_i^- \in V_i$ such that $i \in \varphi(v_i^+, v_{-i})$ and $i \notin \varphi(v_i^-, v_{-i})$. Moreover, $p_i(v_{-i}) = \tau_i^-(v_{-i}) - \tau_i^+(v_{-i})$, so with valuation $p_i(v_{-i}) \in V_i$, i is indifferent between $(\tau_i^+(v_{-i}), 1)$ and $(\tau_i^-(v_{-i}), 0)$. If $v_i > p_i(v_{-i})$, then $(\tau_i^+(v_{-i}), 1)$ is preferred and is attainable by reporting v_i^+ , so by *strategy-proofness* it must be attained by reporting v_i , so $i \in \varphi(v)$. Similarly, $p_i(v_{-i}) > v_i$ implies $i \notin \varphi(v)$; thus (iii) and (iv) hold, as desired.

[$\Leftarrow$] Assume (τ, φ) satisfies the hypotheses. Let $i \in N$, $v \in V$, and $v'_i \in V_i$. If $p_i(v_{-i}) \in \{-\infty, \infty\}$, then by (iii) and (iv) $(\tau_i(v), \varphi_i(v)) = (\tau_i(v'_i, v_{-i}), \varphi_i(v'_i, v_{-i}))$ and we are done. If not, by (ii) $p_i(v_{-i}) = \tau_i^-(v_{-i}) - \tau_i^+(v_{-i})$. If $\varphi_i(v) = \varphi_i(v'_i, v_{-i})$, then by (i) $(\tau_i(v), \varphi_i(v)) = (\tau_i(v'_i, v_{-i}), \varphi_i(v'_i, v_{-i}))$ and we are done, so assume either $i \in \varphi(v) \setminus \varphi(v'_i, v_{-i})$ or $i \in \varphi(v'_i, v_{-i}) \setminus \varphi(v)$.

If $i \in \varphi(v) \setminus \varphi(v'_i, v_{-i})$, then by (iv) $v_i \geq p_i(v_{-i})$, so by (i)

$$\begin{aligned}
v_i \varphi_i(v) + \tau_i(v) &\geq p_i(v_{-i}) + \tau_i^+(v_{-i}) \\
&= (\tau_i^-(v_{-i}) - \tau_i^+(v_{-i})) + \tau_i^+(v_{-i}) \\
&= \tau_i^-(v_{-i}) \\
&= v_i \varphi_i(v'_i, v_{-i}) + \tau_i(v'_i, v_{-i}),
\end{aligned}$$

as desired. If $i \in \varphi(v'_i, v_{-i}) \setminus \varphi(v)$, then by (iii) $p_i(v_{-i}) \geq v_i$, so by (i)

$$\begin{aligned}
v_i \varphi_i(v) + \tau_i(v) &= \tau_i^-(v_{-i}) \\
&= (\tau_i^-(v_{-i}) - \tau_i^+(v_{-i})) + \tau_i^+(v_{-i}) \\
&= p_i(v_{-i}) + \tau_i^+(v_{-i}) \\
&\geq v_i \varphi_i(v'_i, v_{-i}) + \tau_i(v'_i, v_{-i}),
\end{aligned}$$

as desired. ■

Appendix 4

In this appendix, we prove the [Auction Lemma](#).

AUCTION LEMMA: In the object setting, a rule (τ, φ) satisfies *production efficiency* and *strategy-proofness* if and only if it is a peer-consolation rule.

PROOF: We establish both implications.

[$\Rightarrow$] Let (τ, φ) be a rule satisfying *production efficiency* and *strategy-proofness*. By the [Price Lemma](#), there are $\tau_i^+(v_{-i}), \tau_i^-(v_{-i}) \in \mathbb{R}$ and $p_i(v_{-i}) \in \mathbb{R}$ satisfying the stated requirements.¹³

Let $i \in N$ and let $v_{-i} \in V_{-i}$. Define $\pi \equiv \pi(v_{-i})$ and define $w \equiv \Omega(v_{-i})$. By definition, $B_w(v_{-i}) \geq \mu_w$ and, if $w < n - 1$, $\mu_{w+1} > B_{w+1}(v_{-i})$. We claim that $v_i > \pi$ implies $i \in \varphi(v)$ and $\pi > v_i$ implies $i \notin \varphi(v)$.

CASE 1: $w > 0$ and $v_i > B_w(v_{-i})$. If $w > |\varphi(v)|$, then by the definition of w and by the [Bidding Lemma](#),

$$\begin{aligned} v_i &> B_w(v_{-i}) \\ &\geq \mu_w \\ &\geq \mu_{|\varphi(v)|+1} \\ &\geq B_{|\varphi(v)|+1}(v). \end{aligned}$$

Thus i is in any group of $|\varphi(v)|$ highest bidders, so by the [Bidding Lemma](#), $i \in \varphi(v)$.

If $|\varphi(v)| \geq w$, then since i is in any group of w highest bidders, thus i is in any group of $|\varphi(v)|$ highest bidders, so by the [Bidding Lemma](#) $i \in \varphi(v)$.

CASE 2: $v_i > \mu_{w+1}$. If $w = n - 1$, then by *production efficiency* $i \in \varphi(v)$, so assume $n - 1 > w$.

- If $w = 0$, then $v_i > \mu_1 > B_1(v_{-i})$, so by *production efficiency* $\varphi(v) = \{i\}$.
- If $w > 0$ and $v_i > B_w(v_{-i})$, then by Case 1, $i \in \varphi(v)$.
- Assume $w > 0$ and $B_w(v_{-i}) \geq v_i$. Then $B_w(v_{-i}) \geq v_i > \mu_{w+1} > B_{w+1}(v_{-i})$, so $B_{w+1}(v) = v_i > \mu_{w+1}$ and $\mu_{w+2} \geq \mu_{w+1} > B_{w+1}(v_{-i}) = B_{w+2}(v)$. Thus by the [Bidding Lemma](#), $|\varphi(v)| = w + 1$ and $i \in \varphi(v)$.

CASE 3: $\pi > v_i$. Assume, by way of contradiction, $i \in \varphi(v)$.

If $w = 0$, then $\mu_1 > v_i$, so $\sigma_v(\varphi(v) \setminus \{i\}) > \sigma_v(\varphi(v))$, contradicting that φ satisfies *production efficiency*.

Assume $w > 0$. By the [Bidding Lemma](#), i is one of the $|\varphi(v)|$ highest bidders, so as $B_w(v_{-i}) > v_i$, thus $|\varphi(v)| > w$. But then $|\varphi(v)| \geq w + 1$, so by the [Bidding Lemma](#), $v_i \geq \mu_{|\varphi(v)|} \geq \mu_{w+1}$, contradicting $\mu_{w+1} > v_i$.

In each case our claim is true, so $p_i(v_{-i}) = \pi = \pi(v_{-i})$. Define $\alpha_i(v_{-i}) \equiv \tau_i^-(v_{-i})$ and by the [Price Lemma](#) we are done.

¹³Using *production efficiency*, we can rule out $p_i(v_{-i}) \in \{-\infty, \infty\}$: since $0 \leq \mu_1 < \infty$, we can always find v_i^+ so that i wins and v_i^- so that i loses.

[$\Leftarrow$] Assume that (τ, φ) satisfies the requirements and let (α_i) be its list of peer-consolation schedules. Since (τ, φ) is a peer-consolation rule, it satisfies *production efficiency*. For each $i \in N$ and each $v_{-i} \in V_{-i}$, define $\tau_i^+(v_{-i}) \equiv \alpha_i(v_{-i}) - \pi(v_{-i})$, $\tau_i^-(v_{-i}) \equiv \alpha_i(v_{-i})$, and $p_i(v_{-i}) \equiv \pi(v_{-i})$. It is straightforward to verify that these satisfy the requirements of the [Price Lemma](#), so (τ, φ) is *strategy-proof*. ■

Appendix 5

In this appendix, we prove the [Winner-Loser Prices Lemma](#).

WINNER-LOSER PRICES LEMMA: In the object setting, for each $v \in V$ and each $i \in N$,

- (i) $v_i \geq \pi(v_{-i})$ implies $\pi(v_{-i}) = \pi_+(v)$, and
- (ii) $\pi(v_{-i}) \geq v_i$ implies $\pi(v_{-i}) = \pi_-(v)$.

Moreover, either $v_i \geq \pi_-(v) \geq \pi_+(v)$ or $\pi_-(v) \geq \pi_+(v) \geq v_i$.

PROOF: Let $v \in V$ and let $i \in N$. Define $\pi_i \equiv \pi(v_{-i})$, define $w \equiv \Omega(v)$, and define $w_i \equiv \Omega(v_{-i})$. By definition, in N at least w agents bid at least μ_w and (if $n > w$) less than $w + 1$ agents bid at least μ_{w+1} . Similarly, in $N \setminus \{i\}$ at least w_i agents bid at least μ_{w_i} and less than $w_i + 1$ agents bid at least μ_{w_i+1} .

If $w = n$, then $w_i = n - 1$ and $B_{n-1}(v_{-i}) \geq \mu_n$, so $\pi_i = \mu_n$. If $w = 0$, then $w_i = 0$, so $\pi_i = \mu_1$. Assume $n > w > 0$.

CASE 1: $\mu_{w_i+1} > B_{w_i}(v_{-i})$. Then $\pi_i = B_{w_i}(v_{-i})$, and in $N \setminus \{i\}$ at least w_i agents bid at least π_i and less than w_i agents bid more than π_i . Thus in N , regardless of what i bids, at least w_i agents bid at least π_i , which is at least μ_{w_i} , and less than $w_i + 1$ agents bid more than π_i , which is at most μ_{w_i+1} , so $w = w_i$.

If $v_i \geq \pi_i$, then $v_i \geq B_w(v_{-i})$, so $B_{w+1}(v) = B_w(v_{-i}) = \pi_i$. Since $B_{w+1}(v) = B_{w_i}(v_{-i}) \geq \mu_{w_i} = \mu_w$, altogether $\pi_i = \max\{B_{w+1}(v), \mu_w\}$, as desired.

If $\pi_i \geq v_i$, then $B_w(v_{-i}) \geq v_i$, so $B_w(v) = B_w(v_{-i}) = \pi_i$. Since $\mu_{w+1} > B_{w_i}(v_{-i}) = B_w(v)$, altogether $\pi_i = \min\{B_w(v), \mu_{w+1}\}$, as desired.

CASE 2: $B_{w_i}(v_{-i}) \geq \mu_{w_i+1}$. Then $\pi_i = \mu_{w_i+1}$, and in $N \setminus \{i\}$ exactly w_i agents bid at least π .

If $v_i \geq \pi_i$, then in N exactly $w_i + 1$ agents bid at least $\pi_i = \mu_{w_i+1}$, so $w = w_i + 1$ and $\pi_i = \mu_w$. Since $v_i \geq \pi_i = \mu_{w_i+1} > B_{w_i+1}(v_{-i})$, thus $B_{w_i+1}(v_{-i}) = B_{w_i+2}(v) = B_{w+1}(v)$. Since $\mu_w = \mu_{w_i+1} > B_{w_i+1}(v_{-i}) = B_{w+1}(v)$, altogether $\pi_i = \max\{B_{w+1}(v), \mu_w\}$, as desired.

In the special case that $v_i = \pi_i = \mu_w$, we have that $v_i = B_{w_i+1}(v) = B_w(v)$. Since $\mu_{w+1} \geq \mu_w = v_i = B_w(v)$, thus $\pi_i = \min\{B_w(v), \mu_{w+1}\}$, as desired.

If $\pi > v_i$, then in N exactly w_i agents bid at least $\pi_i = \mu_{w_i+1}$, so $w = w_i$ and $\pi_i = \mu_{w+1}$. Since $B_{w_i}(v_{-i}) \geq \mu_{w_i+1} = \pi_i > v_i$, thus $B_{w_i}(v_{-i}) = B_{w_i}(v) = B_w(v)$. Since $B_w(v) = B_{w_i}(v_{-i}) \geq \mu_{w_i+1} = \mu_{w+1}$, altogether $\pi_i = \min\{B_w(v), \mu_{w+1}\}$, as desired.

CONCLUSION: If $w \in \{0, n\}$, then $\pi_-(v) = \pi_+(v)$. If $n > w > 0$, then $B_w(v) \geq B_{w+1}(v)$, $B_w(v) \geq \mu_w$, $\mu_{w+1} > B_{w+1}(v)$, and $\mu_{w+1} \geq \mu_w$, so altogether $\pi_-(v) \geq \pi_+(v)$.

Assume, by way of contradiction, $\pi_-(v) > v_i > \pi_+(v)$. Then $n > w > 0$ and $B_w(v) > v_i > B_{w+1}(v)$, which is impossible. ■

Appendix 6

In this appendix, we prove the [Consolation Lemma](#).

CONSOLATION LEMMA: In the object setting, if a rule (τ, φ) satisfies *production efficiency*, *strategy-proofness*, and *no-envy-in-trades*, then (τ, φ) is a peer-consolation rule with peer-consolation schedules (α_i) such that for each $i \in N$ and each pair $v_{-i}, v'_{-i} \in V_{-i}$,

$$\pi(v_{-i}) = \pi(v'_{-i}) \text{ implies } \alpha_i(v_{-i}) = \alpha_i(v'_{-i}).$$

PROOF: Assume that (τ, φ) satisfies *production efficiency*, *strategy-proofness* and *no-envy-in-trades*. By the [Auction Lemma](#), (τ, φ) is a peer-consolation rule. Let (α_i) be its peer-consolation schedules. By *no-envy-in-trades*, for each $v \in V$, there are $\tau_+(v), \tau_-(v) \in \mathbb{R}$ such that $i \in \varphi(v)$ implies $\tau_i(v) = \tau_+(v)$ and $i \notin \varphi(v)$ implies $\tau_i(v) = \tau_-(v)$.

Let $i \in N$ and let $\pi \in [\mu_1, \mu_n]$. We claim that for each pair $v_{-i}, v'_{-i} \in V_{-i}$ such that $\pi(v_{-i}) = \pi(v'_{-i}) = \pi$, $\alpha_i(v_{-i}) = \alpha_i(v'_{-i})$.

CASE 1:¹⁴ $\pi \in [\mu_1, \mu_n] \setminus \{\mu_1, \mu_2, \dots, \mu_n\}$. Let $v_{-i} \in V_{-i}$ with $\pi(v_{-i}) = \pi$, and define $w \equiv \Omega(v_{-i})$. By construction, $B_w(v_{-i}) = \pi$, $\mu_{w+1} \geq B_w(v_{-i})$, and $B_w(v_{-i}) \geq \mu_w$. Since $\pi \notin \{\mu_w, \mu_{w+1}\}$, thus $\pi \in (\mu_w, \mu_{w+1})$. Define $v^* \in V$ to be the profile where each agent reports π .

Define $v_i \equiv \max\{B_1(v_{-i}), \mu_n\} + 1$ and define $v^0 \equiv (v_i, v_{-i})$. For each $r \in \{1, 2, \dots, n\}$, define $v^r \in V$ by:

$$v_k^r \equiv \begin{cases} \pi, & k = A_r(v^0), \\ v_k^{r-1}, & \text{else.} \end{cases}$$

Observe that $v^n = v^*$. Observe also that at v^0 , w agents report more than π and one agent reports π . We prove three claims, then conclude.

First, we claim that for each $r \in \{1, 2, \dots, n\}$, $\tau_+(v^r) = \tau_-(v^r) - \pi$. Indeed, let $r \in \{1, 2, \dots, n\}$. Since at v^r , less than w agents report more than π but more than w agents bid at least π , and since $\pi \in (\mu_w, \mu_{w+1})$, thus by the [Bidding Lemma](#) $|\varphi(v^r)| = w$, and moreover there are $j, j' \in N$ such that $v_j^r = \pi = v_{j'}^r$, $j \in \varphi(v^r)$, and $j' \in N \setminus \varphi(v^r)$. By *no-envy-in-trades* for j and j' , $\pi + \tau_+(v^r) = \tau_-(v^r)$, as desired.

Second, we claim $\tau_+(v^1) = \tau_+(v^0)$. Since $v_i > B_1(v_{-i})$, thus $i = A_1(v^0)$. Since $v_i > \mu_n$, thus by *production efficiency* $i \in \varphi(v^0)$, so $\tau_+(v^0) = \tau_i(v^0)$. By the [Auction Lemma](#), either $i \in \varphi(v^1)$ and $\tau_i(v^1) = \tau_+(v^0)$ or $i \notin \varphi(v^1)$ and $\tau_i(v^1) = \tau_+(v^0) + \pi$. Thus either $\tau_+(v^1) = \tau_+(v^0)$ or $\tau_-(v^1) = \tau_+(v^0) + \pi$. By the first claim, in either case $\tau_+(v^1) = \tau_+(v^0)$, as desired.

Third, we claim that for each $r \in \{1, 2, \dots, n-1\}$, $\tau_+(v^{r+1}) = \tau_+(v^r)$. Indeed, let $r \in \{1, 2, \dots, n-1\}$ and define $j \equiv A_r(v^r)$. Since at v_{-j} less than w agents report more than π but at least w agents bid at least π , and since $\pi \in (\mu_w, \mu_{w+1})$, thus $\pi(v_{-j}) = \pi$. We consider four exhaustive possibilities:

- $j \in \varphi(v^{r+1})$ and $j \in \varphi(v^r)$. Then by *strategy-proofness* $\tau_+(v^{r+1}) = \tau_+(v^r)$ as desired.
- $j \in \varphi(v^{r+1})$ and $j \notin \varphi(v^r)$. Then by the [Auction Lemma](#) $\tau_+(v^{r+1}) = \tau_-(v^r) - \pi$, so by the first claim $\tau_+(v^{r+1}) = \tau_+(v^r)$ as desired.

¹⁴This is an adaptation of a similar argument from [Ohseto \(2006\)](#).

- $j \notin \varphi(v^{r+1})$ and $j \in \varphi(v^r)$. Then by the [Auction Lemma](#) $\tau_-(v^{r+1}) = \tau_+(v^r) + \pi$, so by the first claim $\tau_+(v^{r+1}) = \tau_+(v^r)$ as desired.
- $j \notin \varphi(v^{r+1})$ and $j \notin \varphi(v^r)$. Then by *strategy-proofness* $\tau_-(v^{r+1}) = \tau_-(v^r)$, so by the first claim $\tau_+(v^{r+1}) = \tau_+(v^r)$ as desired.

To conclude, by the first claim either $i \in \varphi(v^*)$ and $\tau_i(v^*) = \tau_+(v^*)$ or $i \notin \varphi(v^*)$ and $\tau_i(v^*) = \tau_+(v^*) + \pi$. Since at v_{-i}^* less than w agents report more than π but at least w agents bid at least π , and since $\pi \in (\mu_w, \mu_{w+1})$, thus $\pi(v_{-i}^*) = \pi$. Since $v_i > \mu_n$, thus by *production efficiency*, $i \in \varphi(v_i, v_{-i}^*)$. Altogether, by the [Auction Lemma](#) $\tau_i(v_i, v_{-i}^*) = \tau_+(v^*)$.

By the second and third claim, $\tau_+(v^*) = \tau_+(v^n) = \tau_+(v^0) = \tau_+(v)$. Since $v_i > \mu_n$, thus by *production efficiency*, $\tau_i(v) = \tau_+(v) = \tau_+(v^*) = \tau_i(v_i, v_{-i}^*)$. Since $i \in \varphi(v)$, $i \in \varphi(v_i, v_{-i}^*)$, and $\pi(v_{-i}) = \pi(v_{-i}^*)$, thus by the [Auction Lemma](#), $\alpha_i(v_{-i}) = \alpha_i(v_{-i}^*)$.

Since $v_{-i} \in V_{-i}$ with $\pi(v_{-i}) = \pi$ was arbitrary, thus for each pair $v_{-i}, v'_{-i} \in V_{-i}$ with $\pi(v_{-i}) = \pi = \pi(v'_{-i})$, $\alpha_i(v_{-i}) = \alpha_i(v_{-i}^*) = \alpha_i(v'_{-i})$, as desired.

CASE 2: $\pi \in \{\mu_1, \mu_2, \dots, \mu_n\}$. Since several marginal costs may be equal, define

$$w_- \equiv \min\{w \in \{1, 2, \dots, n\} | \pi = \mu_w\}, \text{ and} \\ w_+ \equiv \max\{w \in \{1, 2, \dots, n\} | \pi = \mu_w\}.$$

We wish to partition V_{-i} based on (i) the number h of agents with reports higher than π , and (ii) the number e of agents with reports equal to π . Formally, for each $v_{-i} \in V_{-i}$, define

$$H(v_{-i}) \equiv |\{j \in N \setminus \{i\} | v_j > \pi\}|, \text{ and} \\ E(v_{-i}) \equiv |\{j \in N \setminus \{i\} | v_j = \pi\}|.$$

For each pair $h, e \in \{0, 1, \dots, n-1\}$, define

$$V_{-i}(h, e) \equiv \{v_{-i} \in V_{-i} | H(v_{-i}) = h \text{ and } E(v_{-i}) = e\}.$$

Note that these classes are mutually exclusive and collectively exhaustive, though some are empty (in particular, when $h + e > n - 1$). Define the subcollection $\mathcal{V}$ of these classes over which price is constantly π by:

$$\mathcal{V} \equiv \{V_{-i}(h, e) | V_{-i}(h, e) \neq \emptyset \text{ and } v_{-i} \in V_{-i}(h, e) \text{ implies } \pi(v_{-i}) = \pi\}.$$

We claim that $\pi(v_{-i}) = \pi$ if and only if $v_{-i} \in \cup_{\mathcal{V}} V_{-i}(h, e)$.

CLAIM 1: For each $v_{-i} \in V_{-i}$, $\pi(v_{-i}) > \pi$ if and only if $H(v_{-i}) \geq w_+$.

[$\Rightarrow$] We prove the contrapositive. Assume $w_+ > H(v_{-i})$. If $w_+ = n$, then $\pi = \mu_n \geq \pi(v_{-i})$ as desired, so assume $n > w_+ > H(v_{-i})$.

If $w_+ = n - 1$, then $w_+ \geq \Omega(v_{-i})$. If $n - 1 > w_+$, then $\mu_{w_++1} > \mu_{w_+} \geq B_{w_+}(v_{-i}) \geq B_{w_++1}(v_{-i})$, so $w_+ \geq \Omega(v_{-i})$.

If $w_+ > \Omega(v_{-i})$, then $\pi = \mu_{w_+} \geq \mu_{\Omega(v_{-i})+1} \geq \pi(v_{-i})$.

If $w_+ = \Omega(v_{-i})$, then $\pi = \mu_{w_+} \geq B_{w_+}(v_{-i}) = B_{\Omega(v_{-i})}(v_{-i}) \geq \pi(v_{-i})$.

Altogether, $\pi(v_{-i}) > \pi$ implies $H(v_{-i}) \geq w_+$, as desired.

[$\Leftarrow$] Assume $H(v_{-i}) \geq w_+$. Then $B_{w_+}(v_{-i}) > \mu_{w_+}$, so $\Omega(v_{-i}) \geq w_+$.
 If $\pi(v_{-i}) = B_{\Omega(v_{-i})}(v_{-i})$ and $\Omega(v_{-i}) = w_+$, then $\pi(v_{-i}) > \mu_{w_+} = \pi$ directly from above.
 If $\pi(v_{-i}) = B_{\Omega(v_{-i})}(v_{-i})$ and $\Omega(v_{-i}) > w_+$, then $\pi(v_{-i}) = B_{\Omega(v_{-i})}(v_{-i}) \geq \mu_{\Omega(v_{-i})} \geq \mu_{w_++1} > \mu_{w_+} = \pi$.
 If $\pi(v_{-i}) = \mu_{\Omega(v_{-i})+1}$, then $\pi(v_{-i}) = \mu_{\Omega(v_{-i})+1} \geq \mu_{w_++1} > \mu_{w_+} = \pi$.
 Altogether, $\pi(v_{-i}) > \mu_{w_+} = \pi$, as desired.

CLAIM 2: For each $v_{-i} \in V_{-i}$, $\pi > \pi(v_{-i})$ if and only if $w_- - 1 > H(v_{-i}) + E(v_{-i})$.

[$\Rightarrow$] We prove the contrapositive. Assume $H(v_{-i}) + E(v_{-i}) \geq w_- - 1$. Then $B_{w_- - 1}(v_{-i}) \geq \mu_{w_-} > \mu_{w_- - 1}$, so $\Omega(v_{-i}) \geq w_- - 1$.

If $\pi(v_{-i}) = B_{\Omega(v_{-i})}(v_{-i})$ and $\Omega(v_{-i}) = w_- - 1$, then $\pi(v_{-i}) > \mu_{w_-} = \pi$ directly from above.

If $\pi(v_{-i}) = B_{\Omega(v_{-i})}(v_{-i})$ and $\Omega(v_{-i}) > w_- - 1$, then $\pi(v_{-i}) = B_{\Omega(v_{-i})}(v_{-i}) \geq \mu_{\Omega(v_{-i})} \geq \mu_{w_-} = \pi$.

If $\pi(v_{-i}) = \mu_{\Omega(v_{-i})+1}$, then $\pi(v_{-i}) = \mu_{\Omega(v_{-i})+1} \geq \mu_{w_-} = \pi$.

Altogether, $\pi > \pi(v_{-i})$ implies $w_- - 1 > H(v_{-i}) + E(v_{-i})$, as desired.

[$\Leftarrow$] Assume $w_- - 1 > H(v_{-i}) + E(v_{-i})$. Then $\mu_{w_-} > B_{w_- - 1}(v_{-i}) \geq B_{w_-}(v_{-i})$, so $w_- - 1 \geq \Omega(v_{-i})$.

If $w_- - 1 = \Omega(v_{-i})$, then $\pi = \mu_{w_-} > B_{w_- - 1}(v_{-i}) = B_{\Omega(v_{-i})}(v_{-i}) \geq \pi(v_{-i})$.

If $w_- - 1 > \Omega(v_{-i})$, then $\pi = \mu_{w_-} > \mu_{w_- - 1} \geq \mu_{\Omega(v_{-i})+1} \geq \pi(v_{-i})$.

Altogether, $w_- - 1 > H(v_{-i}) + E(v_{-i})$ implies $\pi > \pi(v_{-i})$.

CLAIM 3: For each $h \in \{0, 1, \dots, n-1\}$ such that $V_{-i}(h, 0) \in \mathcal{V}$, α_i is constant on $V_{-i}(h, 0)$.

Let $h \in \{0, 1, \dots, n-1\}$ such that $V_{-i}(h, 0) \in \mathcal{V}$.

SUBCASE 1: $h = 0$. Let $v_{-i}, v'_{-i} \in V_{-i}(h, 0)$, and define $v, v' \in V$ by $v \equiv (\mu_1 - 1, v_{-i})$ and $v' \equiv (\mu_1 - 1, v'_{-i})$. Define $v^0 \equiv v$, and for each $r \in \{1, 2, \dots, n\}$, recursively define $v^r \in V$ by $v^r \equiv (v'_r, v_{-r}^{r-1})$. For each $r \in \{0, 1, \dots, n\}$, at v^r all agents report less than μ_1 and lose, so all agents receive the same transfer. By *strategy-proofness*, for each $r \in \{1, 2, \dots, n\}$, $\tau_r(v^r) = \tau_r(v^{r-1})$. Altogether, $\tau_1(v) = \tau_1(v^0) = \tau_1(v^n) = \tau_1(v')$, so by the [Auction Lemma](#), $\alpha_i(v_{-i}) = \alpha_i(v'_{-i})$.

SUBCASE 2: $h > 0$. First, we claim that for each $v_{-i} \in V_{-i}(h, 0)$, each $v_i \in V_i$ such that $\pi > v_i$, and each $j \in N$ such that $\pi > v_j$, $\pi_j(v_{-j}) = \pi$. Indeed, let $v_{-i} \in V_{-i}(h, 0)$, $v_i \in V_i$, and $j \in N$ satisfy these requirements. Since $E(v_{-i}) = 0$, thus $B_{\Omega(v_{-i})}(v_{-i}) \neq \pi$, so $B_{\Omega(v_{-i})}(v_{-i}) > \pi = \mu_{\Omega(v_{-i})+1}$. By construction, $\mu_{\Omega(v_{-i})+1} > B_{\Omega(v_{-i})+1}(v_{-i})$. Thus at v , there are $\Omega(v_{-i})$ agents with bids over $\mu_{\Omega(v_{-i})+1}$, and the others (including i and j) have bids under $\mu_{\Omega(v_{-i})+1}$. It follows that $\pi(v_{-j}) = \mu_{\Omega(v_{-i})+1} = \pi$, as desired.

Next, we introduce a definition. For each $v_{-i} \in V_{-i}(h, 0)$, we say that $v''_{-i} \in V_{-i}(h, 0)$ is a *swap* of v_{-i} if there are $j, k \in N \setminus \{i\}$ such that $v_k > \pi > v_j$, $v''_j > \pi > v''_k$, and

$$v''_{i'} \equiv \begin{cases} v''_j, & i' = j, \\ v''_k, & i' = k, \\ v_{i'}, & \text{else.} \end{cases}$$

Observe that if v''_{-i} is a swap of v_{-i} , then $v''_{-i} \in V_{-i}(h, 0)$ and v_{-i} is a swap of v''_{-i} .

Second, we claim that for each $v_{-i} \in V_{-i}(h, 0)$ and each v''_{-i} that is a swap of v_{-i} , $\alpha_i(v_{-i}) = \alpha_i(v''_{-i})$. Indeed, let $v_{-i}, v''_{-i} \in V_{-i}(h, 0)$ and $j, k \in N \setminus \{i\}$ be as in the definition. Define $v'_{-i} \in V_{-i}$ by $v'_{-i} \equiv (v''_j, v_{-i,j})$, and define $v, v', v'' \in V$ by $v \equiv (\mu_1 - 1, v_{-i})$, $v' \equiv (\mu_1 - 1, v'_{-i})$, and $v'' \equiv (\mu_1 - 1, v''_{-i})$. As argued for the first claim, $\pi = \mu_{\Omega(v_{-i})+1}$. By the first claim, $\pi(v_{-i}) = \pi(v_{-j}) = \pi$, so at v both i and j lose, so by *no-envy-in-trades* $\tau_j(v) = \tau_i(v)$. By the [Auction Lemma](#), $\tau_j(v') = \tau_j(v) - \pi = \tau_i(v) - \pi$. At v' , there are $\Omega(v_{-i}) + 1$ agents (including j and k) with bids above π and the others (including i) have bids below π . Since $\pi = \mu_{\Omega(v_{-i})+1}$, thus $\pi(v'_{-j}) = \pi(v'_{-k}) = \pi$, so at v' both j and k win, so by *no-envy-in-trades* $\tau_k(v') = \tau_j(v') = \tau_i(v) - \pi$. By the [Auction Lemma](#), $\tau_k(v'') = \tau_k(v') + \pi = \tau_i(v)$. By the first claim, $\pi(v''_{-i}) = \pi(v''_{-k}) = \pi$, so at v'' both i and k lose, so by *no-envy-in-trades* $\tau_i(v'') = \tau_k(v'') = \tau_i(v)$. Thus by the [Auction Lemma](#), $\alpha_i(v_{-i}) = \alpha_i(v''_{-i})$, as desired.

To conclude, let $v_{-i}, v'_{-i} \in V_{-i}(h, 0)$. If $h \geq 1$, then with either one or two swaps from v_{-i} , we can arrive at v^1_{-i} for which the first rank-index agent and his bid are as in v'_{-i} . Similarly, if $h \geq 2$, with either one or two swaps from v^1_{-i} , we can arrive at v^2_{-i} for which the first two rank-index agents and their bids are as in v'_{-i} . Proceeding in this fashion, we can arrive at v^h_{-i} for which the first h rank-index agents and their bids are as in v'_{-i} . By our second claim, $\alpha_i(v_{-i}) = \alpha_i(v^h_{-i})$. At this point we use an argument similar to that in Subcase 1. Define $v^h, v' \in V$ by $v^h \equiv (\mu_1 - 1, v^h_{-i})$ and $v' \equiv (\mu_1 - 1, v'_{-i})$. If there is $j \in N \setminus \{i\}$ with rank-index $h + 1$ at v'_{-i} , then j bids less than π at both v^h and v' , so by the first claim, both i and j face price π and lose at both v^h and v' . By *strategy-proofness* j receives the same transfer at both profiles, and by *no-envy-in-trades* i receives the same transfer as j at each profile, so altogether i receives the same transfer at both profiles. By repeating this argument, i receives the same transfer as we proceed from v^h to v' by changing each agent's bid to match his bid at v' one at a time, so $\tau_i(v^h) = \tau_i(v')$. By the [Auction Lemma](#), $\alpha_i(v^h_{-i}) = \alpha_i(v'_{-i})$, so altogether $\alpha_i(v_{-i}) = \alpha_i(v'_{-i})$, as desired.

CLAIM 4: For each pair $h, e \in \{0, 1, \dots, n-1\}$ such that $V_{-i}(h, e), V_{-i}(h, e+1) \in \mathcal{V}$, if α_i is constant on $V_{-i}(h, e)$, then α_i is constant on $V_{-i}(h, e+1) \cup V_{-i}(h, e)$.

Assume $V_{-i}(h, e), V_{-i}(h, e+1) \in \mathcal{V}$ and α_i is constant on $V_{-i}(h, e)$. Then there is $a \in \mathbb{R}$ such that for each $v_{-i} \in V_{-i}(h, e)$, $\alpha_i(v_{-i}) = a$.

Let $v_{-i} \in V_{-i}(h, e+1)$. Since $v_{-i} \in V_{-i}(h, e+1) \in \mathcal{V}$, thus $\pi(v_{-i}) = \pi$. Define $v \in V$ by (π, v_{-i}) . By definition of $V_{-i}(h, e+1)$, there is $j \in N \setminus \{i\}$ such that $v_j = \pi$. Since $v_i = v_j$, thus $\pi(v_{-j}) = \pi(v_{-i}) = \pi$. By *no-envy-in-trades*, no matter whether both i and j win, or both lose, or one wins and the other loses, $\alpha_i(v_{-i}) = \alpha_j(v_{-j})$.

For each $\epsilon > 0$, define $v^\epsilon \in V$ by $(\pi - \epsilon, v_{-j})$. Consider an arbitrary v^ϵ . Since $v^\epsilon_{-i} \in V_{-i}(h, e)$, thus either i wins and receives $a - \pi$ or i loses and receives a . Moreover, $v^\epsilon_{-j} = v_{-j}$. By the [Auction Lemma](#), j loses and receives $\alpha_j(v_{-j})$. If i wins, then by *no-envy-in-trades*, $\alpha_j(v_{-j}) \geq (\pi - \epsilon) + (a - \pi)$ and $\pi + (a - \pi) \geq \alpha_j(v_{-j})$. If i loses, then by *no-envy-in-trades*, $\alpha_j(v_{-j}) = a$. Thus in both cases, $\alpha_j(v_{-j}) \in [a - \epsilon, a]$.

Since $\epsilon > 0$ was arbitrary, thus $\alpha_j(v_{-j}) = a$, so $\alpha_i(v_{-i}) = a$. Since $v_{-i} \in V_{-i}(h, e+1)$ was arbitrary, thus α_i is constant on $V_{-i}(h, e+1) \cup V_{-i}(h, e)$, as desired.

CLAIM 5: For each pair $h, e \in \{0, 1, \dots, n-1\}$ such that $V_{-i}(h, e), V_{-i}(h-1, e+1) \in \mathcal{V}$, if α_i is constant on $V_{-i}(h, e)$, then α_i is constant on $V_{-i}(h-1, e+1) \cup V_{-i}(h, e)$.

Assume $V_{-i}(h, e), V_{-i}(h-1, e+1) \in \mathcal{V}$ and α_i is constant on $V_{-i}(h, e)$. Then there is $a \in \mathbb{R}$ such that for each $v_{-i} \in V_{-i}(h, e)$, $\alpha_i(v_{-i}) = a$.

Let $v_{-i} \in V_{-i}(h-1, e+1)$. Since $v_{-i} \in V_{-i}(h-1, e+1) \in \mathcal{V}$, thus $\pi(v_{-i}) = \pi$. Define $v \in V$ by (π, v_{-i}) . By definition of $V_{-i}(h-1, e+1)$, there is $j \in N \setminus \{i\}$ such that $v_j = \pi$. Since $v_i = v_j$, thus $\pi(v_{-j}) = \pi(v_{-i}) = \pi$. By *no-envy-in-trades*, no matter whether both i and j win, or both lose, or one wins and the other loses, $\alpha_i(v_{-i}) = \alpha_j(v_{-j})$.

For each $\epsilon > 0$, define $v^\epsilon \in V$ by $(\pi + \epsilon, v_{-j})$. Consider an arbitrary v^ϵ . Since $v_{-i}^\epsilon \in V_{-i}(h, e)$, thus either i wins and receives $a - \pi$ or i loses and receives a . Moreover, $v_{-j}^\epsilon = v_{-j}$. By the [Auction Lemma](#), j wins and receives $\alpha_j(v_{-j}) - \pi$. If i wins, then by *no-envy-in-trades*, $\alpha_j(v_{-j}) = a$. If i loses, then by *no-envy-in-trades*, $(\pi + \epsilon) + (\alpha_j(v_{-j}) - \pi) \geq a$ and $a \geq \pi + (\alpha_j(v_{-j}) - \pi)$. Thus in both cases, $\alpha_j(v_{-j}) \in [a, a + \epsilon]$.

Since $\epsilon > 0$ was arbitrary, thus $\alpha_j(v_{-j}) = a$, so $\alpha_i(v_{-i}) = a$. Since $v_{-i} \in V_{-i}(h-1, e+1)$ was arbitrary, thus α_i is constant on $V_{-i}(h-1, e+1) \cup V_{-i}(h, e)$, as desired.

CONCLUSION: The proof technique is illustrated by [Figure 3](#). By Claim 1 and Claim 2, for each $v_{-i} \in V_{-i}$, $\pi(v_{-i}) = \pi$ if and only if

$$v_{-i} \in \bigcup_{\mathcal{V}} V_{-i}(h, e).$$

Moreover, $\mathcal{V}$ is the collection of classes $V_{-i}(h, e)$ such that

- (i) $h \leq w_+ - 1$,
- (ii) $h + e \geq w_- - 1$,
- (iii) $h \geq 0$,
- (iv) $e \geq 0$, and
- (v) $h + e \leq n - 1$.

Thus $V_{-i}(w_- - 1, 0) \in \mathcal{V}$ and $V_{-i}(w_- - 2, 0) \notin \mathcal{V}$. By Claim 3, α_i is constant on $V_{-i}(w_- - 1, 0)$, so by Claim 4, α_i is constant on $\cup\{V_{-i}(w_- - 1, e) \in \mathcal{V}\}$, so by Claim 5, α_i is constant on $\cup\{V_{-i}(h, e) \in \mathcal{V} | w_- - 1 \geq h\}$.

Suppose there are $h, e \in \{0, 1, \dots, n-1\}$ such that $V_{-i}(h, e) \in \mathcal{V}$ and $h > w_- - 1$. Then $V_{-i}(h, 0) \in \mathcal{V}$. By Claim 3, α_i is constant on $V_{-i}(h, 0)$, so by Claim 4 α_i is constant on $V_{-i}(h, 0) \cup V_{-i}(h, e)$. Moreover, for each $m \in \{0, 1, \dots, h - (w_- - 1)\}$, $V(h - m, e + m) \in \mathcal{V}$, so by Claim 5 α_i is constant on $V_{-i}(h, 0) \cup V_{-i}(w_- - 1, h - (w_- - 1))$. Thus by the previous paragraph, α_i is constant on $V_{-i}(w_- - 1, 0) \cup V_{-i}(h, e)$.

Altogether, α_i is constant on $\{V_{-i}(h, e) \in \mathcal{V}\}$. Thus for each pair $v_{-i}, v'_{-i} \in V_{-i}$ such that $\pi(v_{-i}) = \pi(v'_{-i})$, $\alpha_i(v_{-i}) = \alpha_i(v'_{-i})$, as desired. ■

Appendix 7

In this appendix, we prove [Theorem 2](#).

THEOREM 2: In the object setting, a rule (τ, φ) satisfies *production efficiency*, *strategy-proofness*, and *no-envy-in-trades* if and only if it is a citizen-shareholder rule.

PROOF: We establish both implications.

[$\Rightarrow$] Let (τ, φ) satisfy the axioms. By the [Consolation Lemma](#), then (τ, φ) is a peer-consolation rule with peer-consolation schedules (α_i) such that for each $i \in N$ and each

Figure 3: In this example, there are five agents, $\mu_5 > \mu_4 = \mu_3 > \mu_2 > \mu_1$, and $\pi = \mu_4 = \mu_3$. We have selected an agent i , and wish to show that α_i is constant across the peer-profiles in V_{-i} that offer him price π . The horizontal axis measures the number of peers who bid higher than π and the vertical axis measures the number of peers who bid exactly π . Each box in the grid $V_{-i}(h, e)$ is the corresponding class of peer profiles; those crossed out are empty. By Claim 1 and Claim 2, $\mathcal{V}$ is the collection of white (unshaded) boxes, and the profiles in these boxes are precisely those that offer price π . By Claim 3, α_i is constant over each white box in the bottom row. By Claim 4, if α_i is constant over one white box with a second white box above it, then α_i is constant over both. By Claim 5, if α_i is constant over one white box with a second white box directly to its upper-left, then α_i is constant over both. These observations together imply that α_i is constant over the white boxes.

pair $v_{-i}, v'_{-i} \in V_{-i}$,

$$\pi(v_{-i}) = \pi(v'_{-i}) \text{ implies } \alpha_i(v_{-i}) = \alpha_i(v'_{-i}).$$

Thus there are functions $\beta_i : [\mu_1, \mu_n] \rightarrow \mathbb{R}$ such that for each $i \in N$ and each $v_{-i} \in V_{-i}$,

- (i) $(\tau_i(v), \varphi_i(v))$ is either $(\beta_i(\pi(v_{-i})) - \pi(v_{-i}), 1)$ or $(\beta_i(\pi(v_{-i})), 0)$,
- (ii) $v_i > \pi(v_{-i})$ implies $i \in \varphi(v)$, and
- (iii) $\pi(v_{-i}) > v_i$ implies $i \notin \varphi(v)$.

Let $\pi \in [\mu_1, \mu_n]$, and let $v \in V$ be the profile where each agent reports π . Assume, by way of contradiction, there is $i \in N$ such that $\pi(v_{-i}) \neq \pi$. Then $B_{\Omega(v_{-i})}(v_{-i}) > \mu_{\Omega(v_{-i})+1} = \pi(v_{-i})$. But if $n-1 \geq \Omega(v_{-i})+1$, then $B_{\Omega(v_{-i})+1}(v_{-i}) \geq \mu_{\Omega(v_{-i})+1}$, which is impossible by definition of $\Omega(v_{-i})$. Thus $\Omega(v_{-i}) = n-1$. But then $B_{n-1}(v_{-i}) > \mu_n$, contradicting that each agent reports $\pi \in [\mu_1, \mu_n]$. Altogether, at v , each winner i receives $\beta_i(\pi) - \pi$ and each loser j receives $\beta_j(\pi)$. By *no-envy-in-trades*, for each pair $i, j \in N$, $\beta_i(\pi) = \beta_j(\pi)$. Since π was arbitrary, for each pair $i, j \in N$, $\beta_i = \beta_j$.

Thus there is $\alpha_0 : [\mu_1, \mu_n] \rightarrow \mathbb{R}$ such that for each $i \in N$ and each $v_{-i} \in V_{-i}$, $(\tau_i(v), \varphi_i(v))$ is either $(\alpha_0(\pi(v_{-i})) - \pi(v_{-i}), 1)$ or $(\alpha_0(\pi(v_{-i})), 0)$.

Let $w \in \{1, 2, \dots, n-1\}$ such that $\mu_{w+1} > \mu_w$ and let $\pi, \pi' \in [\mu_w, \mu_{w+1}]$ such that $\pi' > \pi$. Let v be a profile where w agents report π' and the others report π ; note that there are agents in both groups. Let i be an agent with $v_i = \pi'$ and let j be an agent with $v_j = \pi$. It can be easily verified that $\pi(v_{-i}) = \pi$ and $\pi(v_{-j}) = \pi'$; thus i wins and receives $\alpha_0(\pi) - \pi$ and j loses and receives $\alpha_0(\pi')$. By *no-envy-in-trades*, $\pi' + (\alpha_0(\pi) - \pi) \geq \alpha_0(\pi')$ and $\alpha_0(\pi') \geq \pi + (\alpha_0(\pi) - \pi)$. Thus

$$1 \geq \frac{\alpha_0(\pi') - \alpha_0(\pi)}{\pi' - \pi}, \text{ and} \\ \alpha_0(\pi') \geq \alpha_0(\pi).$$

Since π, π' were arbitrary, thus α_0 is Lipschitz-continuous with the desired restriction on $[\mu_w, \mu_{w+1}]$. Since w was arbitrary, thus α_0 is Lipschitz-continuous with the desired restriction on $[\mu_1, \mu_n]$, so it is a social-dividend schedule. Altogether, (τ, φ) is a citizen-shareholder rule.

[$\Leftarrow$] Suppose (τ, φ) is a citizen-shareholder rule, with α_0 its social-dividend schedule. By the [Bidding Lemma](#) and the [Auction Lemma](#), (τ, φ) satisfies *production efficiency* and *strategy-proofness*. Assume, by way of contradiction, (τ, φ) violates *no-envy-in-trades*. Then there are $v \in V$ and $i, j \in N$ such that

$$v_i \varphi_j(v) + \tau_j(v) > v_i \varphi_i(v) + \tau_i(v).$$

This is clearly impossible if both agents win or if both agents lose.

If i wins and j loses, then by the [Winner-Loser Prices Lemma](#), $v_i \geq \pi(v_{-j}) \geq \pi(v_{-i}) \geq v_j$, so as α_0 is a social-dividend schedule,

$$\begin{aligned} v_i \varphi_i(v) + \tau_i(v) &= v_i + (\alpha_0(\pi(v_{-i})) - \pi(v_{-i})) \\ &= \alpha_0(\pi(v_{-i})) + (v_i - \pi(v_{-i})) \\ &\geq \alpha_0(\pi(v_{-i})) + (\alpha_0(v_i) - \alpha_0(\pi(v_{-i}))) \\ &\geq \alpha_0(\pi(v_{-i})) + \alpha_0(\pi(v_{-j})) - \alpha_0(\pi(v_{-i})) \\ &= \alpha_0(\pi(v_{-j})) \\ &= v_i \varphi_j(v) + \tau_j(v), \end{aligned}$$

as desired.

If i loses and j wins, then by the [Winner-Loser Prices Lemma](#), $v_j \geq \pi(v_{-i}) \geq \pi(v_{-j}) \geq v_i$, so as α_0 is a social-dividend schedule,

$$\begin{aligned}
v_i \varphi_i(v) + \tau_i(v) &= \alpha_0(\pi(v_{-i})) \\
&\geq \alpha_0(\pi(v_{-j})) \\
&= v_i + (\alpha_0(\pi(v_{-j})) - v_i) \\
&\geq v_i + (\alpha_0(\pi(v_{-j})) - \pi(v_{-j})) \\
&= v_i \varphi_j(v) + \tau_j(v),
\end{aligned}$$

as desired. ■

Appendix 8

In this appendix, we prove [Theorem 3](#).

THEOREM 3: In the object setting, a rule (τ, φ) satisfies *no-deficit*, *production efficiency*, *strategy-proofness* and *no-envy-in-trades* if and only if it is a cost-constrained citizen-shareholder rule.

PROOF: We establish both implications.

[$\Rightarrow$] Let (τ, φ) satisfy the axioms. By [Theorem 2](#), (τ, φ) is a citizen-shareholder rule with social-dividend schedule α_0 . We verify that α_0 satisfies the restrictions:

(i) For each $i \in N$, define $v_i \equiv -1$. By the [Bidding Lemma](#), $\varphi(v) = \emptyset$. Moreover, for each $i \in N$, $\Omega(v_{-i}) = 0$, so $\pi(v_{-i}) = \mu_1$. Thus by *no-deficit*,

$$\begin{aligned}
0 &= -C(\varphi(v)) \\
&\geq \sum \tau_i(v) \\
&= n\alpha_0(\mu_1),
\end{aligned}$$

so $\alpha_0(\mu_1) \leq 0$, as desired.

(ii) Let $w \in \{1, 2, \dots, n-1\}$. Define $v \in V$ by:

$$v_i \equiv \begin{cases} \mu_n + 1, & i \in \{1, 2, \dots, w\}, \\ -1, & \text{else.} \end{cases}$$

By the [Bidding Lemma](#), $\varphi(v) = \{1, 2, \dots, w\}$. Moreover, $\Omega(v) = w$, $\pi_+(v) = \mu_w$, and $\pi_-(v) = \mu_{w+1}$, so by the [Winner-Loser Prices Lemma](#), so for each $i \in \varphi(v)$, $\pi(v_{-i}) = \mu_w$ and for each $i \in N \setminus \varphi(v)$, $\pi(v_{-i}) = \mu_{w+1}$. Thus by *no-deficit*,

$$\begin{aligned}
-C(w) &= -C(\varphi(v)) \\
&\geq \sum \tau_i(v) \\
&= w(\alpha_0(\mu_w) - \mu_w) + (n-w)\alpha_0(\mu_{w+1}),
\end{aligned}$$

so $(n - w)\alpha_0(\mu_{w+1}) \leq (w\mu_w - C(w)) - w\alpha_0(\mu_w)$, as desired.

(iii) For each $i \in N$, define $v_i \equiv \mu_n + 1$. By the [Bidding Lemma](#), $\varphi(v) = N$. Moreover, for each $i \in N$, $\Omega(v_{-i}) = n - 1$, so $\pi(v_{-i}) = \mu_n$. Thus by *no-deficit*,

$$\begin{aligned} -C(n) &= -C(\varphi(v)) \\ &\geq \sum \tau_i(v) \\ &= n(\alpha_0(\mu_n) - \mu_n), \end{aligned}$$

so $\alpha_0(\mu_n) \leq \frac{n\mu_n - C(n)}{n}$, as desired.

[$\Leftarrow$] Let (τ, φ) be a cost-constrained citizen-shareholder rule with social-dividend schedule α_0 . By [Theorem 2](#), (τ, φ) satisfies *production efficiency*, *strategy-proofness*, and *no-envy-in-trades*.

Let $v \in V$. Define $w \equiv |\varphi(v)|$, define $\pi_+ \equiv \pi_+(v)$, and define $\pi_- \equiv \pi_-(v)$.

CASE 1: $w = 0$. By the [Bidding Lemma](#), $\mu_1 \geq B_1(v)$. If $\Omega(v) = 0$, then $\pi_- = \mu_1$. If $\Omega(v) > 0$, then $\mu_1 \geq B_1(v) \geq B_{\Omega(v)}(v) \geq \pi_-$, so $\pi_- = \mu_1$. Thus in both cases, $\pi_- = \mu_1$, so by (i),

$$\begin{aligned} \sum \tau_i(v) &= n\alpha_0(\mu_1) \\ &\leq 0 \\ &= -C(\varphi(v)), \end{aligned}$$

as desired.

CASE 2: $w \in \{1, 2, \dots, n-1\}$ and $w < \Omega(v)$. Then $w+1 \leq \Omega(v)$, so by the [Bidding Lemma](#) and the [Winner-Loser Prices Lemma](#), $\mu_{w+1} \geq B_{w+1}(v) \geq B_{\Omega(v)}(v) \geq \pi_- \geq \pi_+ \geq \mu_{\Omega(v)} \geq \mu_{w+1}$, so $\mu_{w+1} = \pi_- = \pi_+$.

If $\mu_{w+1} = \mu_w$, then $(\alpha_0(\mu_{w+1}) - \mu_{w+1}) = (\alpha_0(\mu_w) - \mu_w)$. If $\mu_{w+1} > \mu_w$, then since α_0 is a social-dividend schedule, $\frac{\alpha_0(\mu_{w+1}) - \alpha_0(\mu_w)}{\mu_{w+1} - \mu_w} \leq 1$, so $\alpha_0(\mu_{w+1}) - \alpha_0(\mu_w) \leq \mu_{w+1} - \mu_w$, so $(\alpha_0(\mu_{w+1}) - \mu_{w+1}) \leq (\alpha_0(\mu_w) - \mu_w)$. Thus in both cases, $(\alpha_0(\mu_{w+1}) - \mu_{w+1}) \leq (\alpha_0(\mu_w) - \mu_w)$, so by (ii),

$$\begin{aligned} \sum \tau_i(v) &= w(\alpha_0(\pi_+) - \pi_+) + (n - w)\alpha_0(\pi_-) \\ &= w(\alpha_0(\mu_{w+1}) - \mu_{w+1}) + (n - w)\alpha_0(\mu_{w+1}) \\ &\leq w(\alpha_0(\mu_w) - \mu_w) + (n - w)\alpha_0(\mu_{w+1}) \\ &\leq -C(w) \\ &= -C(\varphi(v)), \end{aligned}$$

as desired.

CASE 3: $w \in \{1, 2, \dots, n-1\}$ and $w = \Omega(v)$. Then $\mu_{w+1} = \mu_{\Omega(v)+1} \geq \pi_-$ and $\pi_+ \geq \mu_{\Omega(v)} = \mu_w$.

If $\pi_+ = \mu_w$, then $(\alpha_0(\pi_+) - \pi_+) = (\alpha_0(\mu_w) - \mu_w)$. If $\pi_+ > \mu_w$, then since α_0 is a social-dividend schedule, $\frac{\alpha_0(\pi_+) - \alpha_0(\mu_w)}{\pi_+ - \mu_w} \leq 1$, so $\alpha_0(\pi_+) - \alpha_0(\mu_w) \leq \pi_+ - \mu_w$, so $(\alpha_0(\pi_+) - \pi_+) \leq (\alpha_0(\mu_w) - \mu_w)$. Thus in both cases, $(\alpha_0(\pi_+) - \pi_+) \leq (\alpha_0(\mu_w) - \mu_w)$.

Since α_0 is a social-dividend schedule, it is non-decreasing, so $\alpha_0(\pi_-) \leq \alpha_0(\mu_{w+1})$. Thus by (ii),

$$\begin{aligned} \sum \tau_i(v) &= w(\alpha_0(\pi_+) - \pi_+) + (n - w)\alpha_0(\pi_-) \\ &\leq w(\alpha_0(\mu_w) - \mu_w) + (n - w)\alpha_0(\mu_{w+1}) \\ &\leq -C(w) \\ &= -C(\varphi(v)), \end{aligned}$$

as desired.

CASE 4: $w = n$. By the [Bidding Lemma](#), $B_n(v) \geq \mu_n$, so $\Omega(v) = n$, so $\pi_+ = \mu_n$. Thus by (iii),

$$\begin{aligned} \sum \tau_i(v) &= n(\alpha_0(\mu_n) - \mu_n) \\ &\leq -C(n) \\ &= -C(\varphi(v)), \end{aligned}$$

as desired.

Thus in each case, $\sum \tau_i(v) \leq -C(\varphi(v))$. Since $v \in V$ was arbitrary, thus (τ, φ) satisfies *no-deficit*, as desired. ■

Appendix 9

In this appendix, we prove [Theorem 4](#).

To prove this theorem, it is convenient to be able to abstract from the problem of selecting winners altogether. Let us say that a rule is *standard* if it first selects as many winners as possible subject to *production efficiency*, then chooses the winners according to their rank-indices. Formally, define:

$$\varphi^\oplus(v) \equiv \{A_r(v) \in N \mid \Omega(v) \geq r\}.$$

DEFINITION: A rule (τ, φ) is *standard* if $\varphi = \varphi^\oplus$, in which case we also say φ is *standard*.

Note that each *standard* rule indeed satisfies *production efficiency*. Next, we define an operator that associates each rule with the *standard* rule that is Pareto-indifferent to it, called its *standardization*.¹⁵

¹⁵The reader may wonder why we did not abstract from this problem earlier; we did not do so because even though our axioms are preserved by standardization, establishing this fact relies crucially on [Theorem 2](#). To put it another way, *standardness* can be viewed as a strong continuity axiom for the winner policy, and it is not at all obvious that this requirement can be imposed without loss of generality.

DEFINITION: For each rule (τ, φ) , the *standardization* of (τ, φ) is the rule $(\tau^\oplus, \varphi^\oplus)$, where

$$\tau_i^\oplus(v) \equiv \begin{cases} \tau_i(v) - v_i, & i \in \varphi^\oplus(v) \setminus \varphi(v), \\ \tau_i(v) + v_i, & i \in \varphi(v) \setminus \varphi^\oplus(v), \\ \tau_i(v), & \text{else.} \end{cases}$$

We now prove the theorem:

THEOREM 4: In the object setting,

(I) a rule (τ, φ) (i) satisfies *no-deficit*, *production efficiency*, *strategy-proofness*, and *no-envy-in-trades*, and (ii) is not Pareto-dominated by another such rule, if and only if it is a cost-constrained citizen-shareholder rule with an unrelenting social-dividend schedule whose marginal cost award profile belongs to $\mathcal{F}$.

(II) a rule (τ, φ) (i) satisfies *no-deficit*, *production efficiency*, *voluntarism*, *strategy-proofness*, and *no-envy-in-trades*, and (ii) is not Pareto-dominated by another such rule, if and only if it is a cost-constrained citizen-shareholder rule with an unrelenting social-dividend schedule whose marginal cost award profile belongs to $\mathcal{F}^V$.

PROOF: By Theorem 3 and the definition of a cost-constrained social-dividend schedule, a rule (τ, φ) satisfies the (I) axioms if and only if it is a citizen-shareholder rule with a social-dividend schedule whose marginal cost award profile belongs to $\mathcal{A}$. We claim such a rule is *voluntary* if and only if its marginal cost award profile belongs to $\mathcal{A}^V$.

Indeed, let (τ, φ) satisfy the (I) axioms, let α_0 be its social-dividend schedule, and let $a \in \mathcal{A}$ be its marginal cost award profile. Since $a \in \mathcal{A}$, thus $a_1 \leq 0$. If $a \in \mathcal{A}^V$, then $a_0 = 0$, so since α_0 is non-decreasing, α_0 only makes non-negative assignments. Thus at each profile, each agent receives something at least as good as losing with a nonnegative social-dividend award, so (τ, φ) is *voluntary*. If $a \notin \mathcal{A}^V$, then $a_0 < 0$, so when each agent reports valuation -1 , each agent loses and receives $a_0 < 0$, so (τ, φ) is not *voluntary*.

From here, we prove three claims, then conclude.

CLAIM 1: Each rule is Pareto-indifferent to its standardization.

Let $i \in N$ and let $v \in V$. If $(\tau_i^\oplus, \varphi^\oplus(v)) = (\tau_i(v), \varphi(v))$, then we are done. Otherwise, there is $t_i \in \mathbb{R}$ such that $\{(\tau_i^\oplus, \varphi^\oplus(v)), (\tau_i(v), \varphi(v))\} = \{(t_i, 0), (t_i - v_i, 1)\}$. With preferences v_i , i is indifferent between the two. Since $i \in N$ was arbitrary, all agents are indifferent between the two rules at v . Since $v \in V$ was arbitrary, thus $(\tau^\oplus, \varphi^\oplus)$ is Pareto-indifferent to (τ, φ) .

CLAIM 2: Let (τ^*, φ^*) and (τ, φ) be *standard* citizen-shareholder rules with social-dividend schedules α_0^* and α_0 , respectively. If (i) α_0^* and α_0 share marginal cost award profile a , and (ii) α_0^* is unrelenting while α_0 is not, then α_0^* strictly Pareto-dominates α_0 .

Let (τ^*, φ^*) , (τ, φ) , α_0^* , α_0 , and a satisfy the hypotheses. Then $a \in \mathcal{A}$. Let $\pi \in [\mu_1, \mu_n]$.

We claim that $\alpha_0(\pi) \leq \alpha_0^*(\pi)$. Indeed, if $\pi \in \{\mu_1, \mu_2, \dots, \mu_n\}$, then $\alpha_0(\pi) = \alpha_0^*(\pi)$. Otherwise, there is $w \in \{1, 2, \dots, n\}$ and $\delta \in (0, \mu_{w+1} - \mu_w)$ such that $\pi = \mu_w + \delta$. If $\delta < a_{w+1} - a_w$, then $\alpha_0(\pi) \leq \alpha_0^*(\pi)$; otherwise $\alpha_0(\pi) - \alpha_0(\mu_w) > \alpha_0^*(\pi) - a_w = \pi - \mu_w$, contradicting $a \in \mathcal{A}$. If $\delta \geq a_{w+1} - a_w$, then $\alpha_0(\pi) \leq \alpha_0^*(\pi)$; otherwise $\alpha_0(\mu_{w+1}) - \alpha_0(\pi) < a_{w+1} - \alpha_0^*(\pi) = 0$, contradicting $a \in \mathcal{A}$. Thus in all cases, $\alpha_0(\pi) \leq \alpha_0^*(\pi)$, as desired.

Since $\pi \in [\mu_1, \mu_n]$ was arbitrary, thus for each $\pi \in [\mu_1, \mu_n]$, $\alpha_0(\pi) \leq \alpha_0^*(\pi)$. Moreover, since α is not unrelenting, $\alpha_0 \neq \alpha_0^*$, so there is some $\pi \in [\mu_1, \mu_n]$ such that $\alpha_0(\pi) < \alpha_0^*(\pi)$. From here, since (τ^*, φ^*) and (τ, φ) are *standard*, it is straightforward to show that (τ^*, φ^*) strictly Pareto-dominates (τ, φ) .

CLAIM 3: Let (τ^*, φ^*) and (τ, φ) be *standard* citizen-shareholder rules with unrelenting social-dividend schedules α_0^* and α_0 , respectively, and let a^* and a be their respective marginal cost award profiles. Then (τ^*, φ^*) strictly Pareto-dominates (τ, φ) if and only if $a^* > a$.

Let (τ^*, φ^*) , (τ, φ) , α_0^* , α_0 , a^* , and a satisfy the hypotheses. Then $a^*, a \in \mathcal{A}$.

If $a^* > a$, then let $\pi \in [\mu_1, \mu_n]$. Then there are $w \in \{1, 2, \dots, n\}$ and $\delta \in [0, \mu_{w+1} - \mu_w]$ such that $\pi = \mu_w + \delta$. Since $\alpha_0^*(\mu_w) + \delta = a_w^* + \delta \geq a_w + \delta = \alpha_0(\mu_w) + \delta$ and $\alpha_0^*(\mu_{w+1}) = a_{w+1}^* \geq a_{w+1} = \alpha_0(\mu_{w+1})$, and since α_0^* and α_0 are unrelenting, thus $\alpha_0^*(\pi) \geq \alpha_0(\pi)$. Since $\pi \in [\mu_1, \mu_n]$ was arbitrary, thus for each $\pi \in [\mu_1, \mu_n]$, $\alpha_0(\pi) \leq \alpha_0^*(\pi)$. Moreover, since $a^* > a$, there is $w \in \{1, 2, \dots, n\}$ such that $\alpha_0(\mu_w) < \alpha_0^*(\mu_w)$. From here, since (τ^*, φ^*) and (τ, φ) are *standard*, it is straightforward to show that (τ^*, φ^*) strictly Pareto-dominates (τ, φ) .

If it is not the case that $a^* > a$, then either (i) $\alpha_0 = \alpha_0^*$ or (ii) there is $w \in \{1, 2, \dots, n\}$ such that $\alpha_0(\mu_w) > \alpha_0^*(\mu_w)$. If (i), then since (τ^*, φ^*) and (τ, φ) are *standard*, they are equal, so neither strictly Pareto-dominates the other. If (ii), then since (τ^*, φ^*) and (τ, φ) are *standard*, it is straightforward to show that (τ^*, φ^*) does not strictly Pareto-dominate (τ, φ) .

We begin by proving (I). First, suppose (τ, φ) is not Pareto-dominated by another such rule. By Claim 1, its standardization $(\tau^\oplus, \varphi^\oplus)$ is not Pareto-dominated by another such rule. By Claim 2, $(\tau^\oplus, \varphi^\oplus)$ is unrelenting. By Claim 3, $(\tau^\oplus, \varphi^\oplus)$ has marginal cost award profile in $\mathcal{F}$. Since (τ, φ) and its standardization share a social-dividend schedule, altogether, (τ, φ) has an unrelenting social-dividend schedule whose marginal cost award profile is in $\mathcal{F}$, as desired.

Second, suppose (τ, φ) has an unrelenting social-dividend schedule whose marginal cost award profile is in $\mathcal{F}$. Assume, by way of contradiction, (τ', φ') is a rule satisfying the axioms of (I) that strictly Pareto-dominates (τ, φ) . Let $a \in \mathcal{A}$ be the marginal cost award profile of (τ, φ) , let $a' \in \mathcal{A}$ be the marginal cost award profile of (τ', φ') , let $(\tau^\oplus, \varphi^\oplus)$ be the standardization of (τ, φ) , let $(\tau'^\oplus, \varphi'^\oplus)$ be the standardization of (τ', φ') , and let (τ^*, φ^*) be the *standard* citizen-shareholder rule with the unrelenting social-dividend schedule whose marginal cost award profile is a' . By Claim 1, $(\tau'^\oplus, \varphi'^\oplus)$ strictly Pareto-dominates (τ, φ) . By Claim 2, (τ^*, φ^*) Pareto-dominates $(\tau'^\oplus, \varphi'^\oplus)$, so (τ^*, φ^*) strictly Pareto-dominates (τ, φ) . But then by Claim 3, $a' > a$, contradicting $a \in \mathcal{F}$.

The proof of (II) is analogous. ■

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