Strategic and normative analysis of queueing, matching, and cost allocation

Strategic and Normative Analysis of Queueing, Matching, and Cost Allocation

by

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Dedication

To my family.

Curriculum Vitae

The author was born in Istanbul, Turkey, on the sixteenth day of June, 1975. After finishing his higher secondary education there, he attended Boğaziçi University, Istanbul, from 1993 to 1998, and graduated with a Bachelor of Science degree in Mathematics. He began graduate studies in Boğaziçi University, Istanbul, and received a Master of Arts degree in Economics in 2000. He came to the University of Rochester in the fall of 2001 where he received the University of Rochester Fellowship from 2001 till 2005. Focusing his research on various aspects of economic theory under the guidance of William Thomson, he received a Master of Arts degree from the University of Rochester in 2003.

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“ ...

from ferns till coal

nothing lived is in vain ”

Abstract

This thesis is a collection of essays on the incentive, fairness, and solidarity properties of recommendations to different economic problems such as queueing, matching, and cost allocation.

In Chapter 1, we consider queueing problems. We prove that no rule is *Pareto-efficient* and *coalitional strategy-proof*. We identify the class of rules that satisfy *Pareto-efficiency*, *equal treatment of equals*, and *strategy-proofness*. Among multi-valued rules, there is a unique rule that satisfies *Pareto-efficiency*, *anonymity*, and *strategy-proofness*.

In Chapter 2, we consider two-sided matching markets with contracts. We prove that the stable correspondence is the only solution that satisfies *unanimity*, *population monotonicity*, and *Maskin-monotonicity*. If a rule satisfies *unanimity*, both forms of *population monotonicity* and a weak notion of *consistency*, then it is a subsolution of the stable correspondence. We also analyze immunity of solutions to strategic behavior such as to misreporting the availability of contracts held by the firms, and misrepresenting preferences by workers and firms. We introduce *destruction-proofness*, and study *destruction-proofness* and *strategy-proofness*. We show that if the firms' preferences satisfy the substitute condition, then the worker-optimal solution is not *destruction-proof* and the firm-optimal solution is *destruction-proof*. If the firms' prefer-

ences satisfy the substitute condition, the law of aggregate demand, and the top-dominance condition then the worker-optimal solution is the only solution satisfying *stability* and *strategy-proofness.*

In Chapter 3, we consider a class of cost sharing problems with the following features: agents are ordered in terms of their needs for a public facility; satisfying an agent implies satisfying all agents with smaller needs than his at no extra cost. The "sequential equal contributions" rule assigns each agent using a given segment to contribute equally to the cost of the segment and to pay the total of the contributions of each segment that the agent uses. We show that the sequential equal contributions rule is the only rule satisfying *equal treatment of equals*, *independence of predecessors*, and *smallest-cost consistency* and it is the only rule satisfying *individual rationality*, *cost monotonicity*, and *smallest-cost consistency.*

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Foreword

This thesis is a collection of joint and solo essays on the incentive, fairness, and solidarity properties of recommendations to different economic problems such as queueing, matching, and cost allocation. Some results are from joint works with Youngsub Chun, Eve Ramaekers, Chun-Hsien Yeh, and Duygu Yengin.

Chapter 1

Queueing

1.1 Introduction

A set of agents simultaneously arrive at a service facility that can only serve one agent at a time. Agents require service for the same length of time. The waiting cost may vary from one agent to the other. Each agent is assigned a "consumption bundle" consisting of a position in the queue and a positive or negative transfer. Each agent has quasi-linear preferences over positions and transfers. For such a *queueing problem*, a rule assigns each agent a position in the queue and a positive or negative transfer such that no two agents are assigned the same position, and the sum of transfers is not positive.

Our objective is to identify rules that are well-behaved from the normative and strategic viewpoints. In addition to efficiency, we assess the desirability of a rule from two perspectives: the fairness of the allocations it selects and the incentive it gives to agents to tell the truth about their cost parameters. The first requirement is efficiency. It says that if an allocation is selected, there should be no other feasible allocation that each agent finds at least as desirable and at least one agent prefers. Since preferences are quasi-linear, *Pareto-efficiency* can be decomposed into two axioms: on the one hand, *efficiency of queues*, which says that a queue should minimize the total waiting cost, and on the other hand, *balancedness*, which says that transfers should sum up to zero.

Second is a minimal symmetry requirement: agents with equal waiting costs should be treated equally. As agents cannot be served simultaneously, it is of course impossible for two agents with equal costs to have equal assignments. However, using monetary transfers, we can give them assignments between which they are indifferent. We require *equal treatment of equals in welfare*: agents with equal waiting costs should be indifferent their assignments. It is implied by *no-envy*, which requires that no agent should prefer another agent's assignment to her own.

Third is immunity to strategic behavior. As unit waiting costs may not be known, the rule should provide agents incentive to reveal these costs truthfully. *Strategy-proofness* requires that each agent should find her assignment when she truthfully reveals her unit waiting cost at least as desirable as her assignment when she misrepresents it. We are also concerned about possible manipulations by groups, and consider *coalitional strategy-proofness*: no group of agents should be able to make each of its members at least as well off, and at least one of them better off, by jointly misrepresenting their waiting costs. Finally is *non-bossiness*: if an agent's change in her announcement does not affect her assignment, then it should not affect any other agent's assignment.

We identify the class of rules that satisfy *efficiency of queues* and *strategy-proofness*. We show that a unique allocation rule satisfies *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*. For each problem, this rule selects a *Pareto-efficient* queue and it sets transfers as follows: consider each pair of agents in turn, make each agent in the pair pay the waiting cost incurred by the other agent in the pair, and distributes the sum of these two payments equally among the others. We refer to this rule as the *Equally Distributed Pairwise Pivotal rule*. As the name indicates, it applies the idea of the well-known Pivotal rule from the class of Groves' rules in public decision-making problems in each pair (Clarke 1971, Groves, 1973). The Equally Distributed Pairwise Pivotal rule also satisfies *no-envy*. Using this result, we also show that in combination with *Pareto-efficiency* and *strategy-proofness*, *equal treatment of equals in welfare* is equivalent to *no-envy*.

We may also be concerned about possible manipulations by groups. However, if we impose the stronger incentive property of *coalitional strategy-proofness*, even with *efficiency of queues*, we have an impossibility result. This result suggests that the previous result is tight.

We then extend the first result to possibly multi-valued rules. First, we consider fairness properties when it is possible to give two agents with equal unit waiting costs same assignments at two different allocations. Then, *symmetry* requires that agents with equal waiting costs should be treated symmetrically, that is, if there is another allocation at which two agents exchange their assignments and the other agents keep theirs, then this allocation should be selected. It is implied by *anonymity*, which requires that agents' names should not matter. However, whereas single- and multi-valued rules may satisfy *equal treatment of equals in welfare*, only multi-valued rules may satisfy *symmetry*. Thus, because agents cannot be served simultaneously, *anonymity* is possible if and only if multi-valuedness is allowed. Second, *strategy-proofness* has to be redefined for multi-valued rules. To compare the welfare levels derived from two sets of feasible allocations, we assume that an agent prefers the former to the latter if and only if for each allocation in the latter, there is an allocation in the former that she finds at least as desirable; and for each allocation in the former, there is an allocation in the latter that she does not prefer.

Next, we define the rule that selects all *Pareto-efficient* queues and for each queue, sets transfers as in the Equally Distributed Pairwise Pivotal rule. We refer to it as the *Largest Equally Distributed Pairwise Pivotal rule*. We prove that a unique allocation rule satisfies *Pareto-efficiency*, *symmetry*, and *strategy-proofness*. Moreover, it is *anonymous*. Also, as *anonymity* implies *symmetry*, and as the Largest Equally Distributed Pairwise Pivotal rule is the union of all the rules that satisfy *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*, it follows that this rule is the only rule that satisfies *Pareto-efficiency*, *equal treatment of equals in welfare*, *symmetry*, and *strategy-proofness*.

The intuition for the results is simple. Any rule can be described by selecting the queues appropriately and setting each agent's transfer equal to the cost she imposes on the others plus an appropriately chosen amount. By *Pareto-efficiency*, a desirable rule should select *Pareto-efficient* queues and as the costs agents impose on the others are always strictly positive (except for the last agent in the queue), it should redistribute the sum of these costs. By equity, it should select all *Pareto-efficient* queues and it should redistribute this sum fairly. By *strategy-proofness*, it should redistribute this sum in such a way that each agent's share only depends on the others' waiting costs. This is exactly what the Largest Equally Distributed Pairwise Pivotal rule does. It selects all *Pareto-efficient* queues (so it is efficient and fair). It sets each agent's transfer considering each pair of agents in turn, making each agent in the pair pay the cost she imposes on the pair. Then, it distributes the sum of these two payments (so it is efficient) equally (so it is fair) among the others (so it is *strategy-proof*).

Literature Review: Our results provide another example of a situation in which *Pareto-efficiency*, equity axioms such as *equal treatment of equals in welfare* and *symmetry*, and *strategy-proofness* are compatible. For general social choice problems, each equity axiom is incompatible with *strategy-proofness* (Gibbard, 1973 and Satterthwaite, 1975). For the classical problem of distributing of private goods (and even if preferences are homothetic and smooth), *Pareto-efficiency*, *equal treatment of equals*, and *strategy-proofness* are incompatible (Serizawa, 2002). In economies with indivisible goods when monetary compensations are possible, *no-envy* and *strategy-proofness* are incompatible (Alkan, Demange, and Gale, 1991, Tadenuma and Thomson, 1995); moreover, when rules exist that satisfy these axioms on more restricted classes of problems, they violate *Pareto-efficiency*.

There are some exceptions. For the problem of choosing a public good in an interval over which the agents have continuous and single-peaked preferences, *Pareto-efficiency*, *anonymity*, and *strategy-proofness* are compatible (Moulin, 1980). For the problem of distributing an infinitely divisible private good over which the agents have continuous and single-peaked preferences, *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness* are compatible, and so are *Pareto-efficiency*, *anonymity*, and *strategy-proofness* (Sprumont, 1991, Ching, 1994). For the problem of distributing infinitely divisible private goods produced by means of a linear technology, *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness* are compatible (Maniquet and Sprumont, 1999).[1]

The literature on queueing can be organized in two groups of papers. The first group concerns the identification of rules satisfying equity axioms pertaining to changes in the set of agents or in their waiting costs, in addition to the efficiency and equity axioms that we impose too (Maniquet, 2003; Chun, 2004a; Chun, 2004b; Katta and Sethuraman, 2005). Only rules that select *Pareto-efficient* queues and set each agent's transfer in such a way that her welfare is equal to the Shapley value of some associated coalitional game, satisfy these axioms (Maniquet, 2003, Chun, 2004a; Katta and Sethuraman, 2005). However, while there are rules that satisfy *Pareto-efficiency* and *no-envy* (Chun, 2004b; Katta and Sethuraman, 2005), none satisfies the solidarity requirement that if the waiting costs change, then all agents should gain together or lose together (Chun, 2004b). The second group concerns the identification of necessary and sufficient conditions for the existence of rules satisfying *Pareto-efficiency* and *strategy-proofness*. For such problems, like for any public decision-making problem in which agents have additively separable preferences, there are rules that satisfy *efficiency of queues* and *strategy-proofness* (Groves, 1973). Also, like for any public decision-making problem in which preference profiles are

[1]For an extensive survey on *strategy-proofness*, see Thomson (2006).

convex, only these rules satisfy these properties (Holmström, 1979).[2] However, these rules are not *balanced* (Green and Laffont, 1977). Unless we further restrict the domain, *Pareto-efficiency* and *strategy-proofness* are incompatible. In queueing problems, if preferences are quasi-linear over positions and transfers, there are rules that satisfy *Pareto-efficiency* and *strategy-proofness* (Suijs, 1996, Mitra and Sen, 1998).

In Section 1.2, we formally introduce the model. In Section 1.3, we define the axioms on rules. In Section 1.4, we give the results. In Section 1.5, we give concluding comments. In Appendix A, we provide all proofs.

1.2 Model

There is a finite set of agents N indexed by $i \in N$. Each agent $i \in N$ has to be assigned a *position* $\sigma_i \in \mathbb{N}$ in a queue and may receive a positive or negative monetary *transfer* $t_i \in \mathbb{R}$. Preferences are quasi-linear over $X \equiv \mathbb{N} \times \mathbb{R}$. Let $c_i \in \mathbb{R}_+$ be the *unit waiting cost of* $i \in N$. If i is served σ_i-th, her *total waiting cost* is $(\sigma_i - 1)c_i$. Her preferences can be represented by the function u_i defined as follows: for each $(\sigma_i, t_i) \in X$, $u_i(\sigma_i, t_i) = -(\sigma_i - 1)c_i + t_i$. We use the following notational shortcut. If her waiting cost is c'_i, then her preferences

[2] In fact, Holmström (1979) shows it that any public decision-making problem in which preference profiles are smoothly connected, i.e., for any profile in the domain, if there is a differentiable deformation of the profile into other then the other profile is also in the domain; only Groves' rules satisfy efficiency of assignment and *strategy-proofness*. This characterization also holds on the universal domain of preferences (Green and Laffont, 1977).

are represented by the function u'_i, defined by $u'_i(\sigma_i, t_i) = -(\sigma_i - 1)c'_i + t_i$; if it is $\tilde{c}_i$, then we use $\tilde{u}_i(\sigma_i, t_i) = -(\sigma_i - 1)\tilde{c}_i + t_i$, and so on. A *queueing problem* is defined as a list $c \equiv (c_i)_{i \in N} \in \mathbb{R}^N_+$. Let $\mathcal{C} \equiv \mathbb{R}^N_+$ be the set of all problems. Let $n = |N|$.

An *allocation for* $c \in \mathcal{C}$ is a pair $(\sigma, t) \equiv (\sigma_i, t_i)_{i \in N} \in X^N$. An allocation $(\sigma, t) \in X^N$ is *feasible for* $c \in \mathcal{C}$ if no two agents are assigned the same position in σ, (i.e., for each $\{i, j\} \subseteq N$ with $i \neq j$, we have $\sigma_i \neq \sigma_j$), and the sum of the coordinates of t is non-positive, (i.e., $\sum_{i \in N} t_i \leq 0$). Let $Z(N)$ be the set of all feasible allocations for $c \in \mathcal{C}$. An (allocation) *rule* φ is a correspondence that associates with each problem $c \in \mathcal{C}$ a non-empty set of feasible allocations $\varphi(c) \subseteq Z(N)$.

Given $c \in \mathcal{C}$ and $S \subseteq N$, $c_S \equiv (c_l)_{l \in S}$ is the restriction of c to S. Given $i \in N$, $c_{-i} \equiv (c_l)_{l \in N \setminus \{i\}}$ is the restriction of c to $N \setminus \{i\}$. Let $(\sigma, t) \in Z(N)$. Given $i \in N$, let $P_i(\sigma) \equiv \{j \in N | \sigma_j < \sigma_i\}$ be the set of agents served before i in σ, (the predecessors), and $F_i(\sigma) \equiv \{j \in N | \sigma_j > \sigma_i\}$ the set of agents served after i in σ, (the followers). Given $\{i, j\} \subseteq N$, let $B_{ij}(\sigma) \equiv \{l \in N | \min\{\sigma_i, \sigma_j\} < \sigma_l < \max\{\sigma_i, \sigma_j\}\}$ be the set of agents served *between* i and j in σ.[3] Given $S \subseteq N$, the *total waiting cost of* S is $\sum_{i \in S}(\sigma_i - 1)c_i$. Given $i \in N$, let σ^{-i} be such that for each $l \in P_i(\sigma)$, we have $\sigma^{-i}_l = \sigma_l$ and for each $l \in F_i(\sigma)$, we have $\sigma^{-i}_l = \sigma_l - 1$. Given $i \in N$, and $S \subseteq N$, the *cost that agent*

[3]For each $c \in \mathcal{C}$, each $(\sigma, t) \in Z(N)$, and each $\{i, j\} \subseteq N$, we have $B_{ij}(\sigma) = B_{ji}(\sigma)$.

i imposes on S is $\sum_{l\in S\cap F_i(\sigma)} c_l$. Thus, the cost an agent imposes on society is always equal to the sum of the unit waiting costs of her followers in an efficient queue.

1.3 Properties of rules

In this section, we define properties of rules. Let φ be a rule. First, if an allocation is selected, there should be no other feasible allocation that each agent finds at least as desirable and at least one agent prefers.

Pareto-efficiency: *For each $c \in \mathcal{C}$ and each $(\sigma, t) \in \varphi(c)$, if there is no $(\sigma', t') \in Z(N)$ such that for each $i \in N$, $u_i(\sigma'_i, t'_i) \geq u_i(\sigma_i, t_i)$ and for at least one $j \in N$, we have $u_j(\sigma'_j, t'_j) > u_j(\sigma_j, t_j)$.*

Consider a *Pareto-efficient* allocation for c, any other allocation at which the queue is the same is also *Pareto-efficient*. Therefore, it is meaningful to define *efficiency of queues*. It requires to minimize the total waiting cost. Thus, an allocation (σ, t) is *Pareto-efficient for c* if and only if for each $\sigma' \in \mathbb{N}^N$, we have $\sum_{i\in N}(\sigma'_i - 1)c_i \geq \sum_{i\in N}(\sigma_i - 1)c_i$, i.e., σ is *efficient for c* and $\sum_{i\in N} t_i = 0$, i.e., t is *balanced for c*. Let $Q^*(c)$ be the set of all *efficient* queues for c. For each $c \in \mathcal{C}$ and each $(\sigma, t) \in Z(N)$, we have $\sigma \in Q^*(c)$ if and only if for each $\{i, j\} \subset N$ with $i \neq j$, if $\sigma_i < \sigma_j$, then $c_i \geq c_j$. Thus, up to permutation of agents with equal unit waiting costs, there is only one *efficient* queue.

Summarizing the discussion above, *Pareto-efficiency* can be decomposed into two axioms:

Efficiency of queues: *For each* $c \in \mathcal{C}$ *and each* $(\sigma, t) \in \varphi(c)$*, we have* $\sigma \in Q^*(c)$.

Balancedness: *For each* $c \in \mathcal{C}$ *and each* $(\sigma, t) \in \varphi(c)$*, we have* $\sum_{i \in N} t_i = 0$.

Equity requires to treat agents with equal unit waiting costs equally. We require that equal agents should have equal welfare.

Equal treatment of equals in welfare: *For each* $c \in \mathcal{C}$*, each* $(\sigma, t) \in \varphi(c)$*, and each* $\{i, j\} \subset N$ *with* $i \neq j$ *and* $c_i = c_j$*, we have* $u_i(\sigma_i, t_i) = u_j(\sigma_j, t_j)$.

This requirement is necessary for no agent to prefer another agent's assignment to her own.

No-envy: *For each* $c \in \mathcal{C}$*, each* $(\sigma, t) \in \varphi(c)$*, and each* $i \in N$*, there is no* $j \in N \backslash \{i\}$ *such that* $u_i(\sigma_j, t_j) > u_i(\sigma_i, t_i)$.

The last requirements are motivated by strategic considerations. The planner may not know the agents' cost parameters. If agents behave strategically when announcing them, neither efficiency nor equity may be attained. Thus, we require that each agent should find her assignment when she truthfully reveals her unit waiting cost at least as desirable as her assignment when she misrepresents it.

Strategy-proofness: *For each* $c \in \mathcal{C}$*, each* $i \in N$*, and each* $c'_i \in \mathbb{R}_+$*, if* $(\sigma, t) = \varphi(c)$ *and* $(\sigma', t') = \varphi(c'_i, c_{-i})$*, then* $u_i(\sigma_i, t_i) \geq u_i(\sigma'_i, t'_i)$*.*

We also consider the requirement that no group of agents should be able to make each of its members at least as well off, and at least one of them better off, by jointly misrepresenting its members waiting costs.

Coalitional strategy-proofness: *For each* $c \in \mathcal{C}$ *and each* $S \subseteq N$*, there is no* $c'_S \in \mathbb{R}^S_+$ *such that if* $(\sigma, t) = \varphi(c)$ *and* $(\sigma', t') = \varphi(c'_S, c_{N \setminus S})$*, then for each* $i \in S$*, we have* $u_i(\sigma'_i, t'_i) \geq u_i(\sigma_i, t_i)$ *and for some* $j \in S$*, we have* $u_j(\sigma'_j, t'_j) > u_j(\sigma_j, t_j)$*.*

The next requirement is that if an agent's change in her announcement does not affect her assignment, then it should not affect any other agent's assignment.

Non-bossiness: *For each* $c \in \mathcal{C}$*, each* $i \in N$*, and each* $c'_i \in \mathbb{R}_+$*, if* $\varphi_i(c) = \varphi_i(c'_i, c_{-i})$*, then* $\varphi(c) = \varphi(c'_i, c_{-i})$*.*

1.4 Results

In this section, we first characterize the class of *single-valued* rules that satisfy *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness.* We then show that these rules in fact satisfy the stronger fairness property of *no-envy* (Theorem 1.2). Then, we extend these results to multi-valued rules and we prove that there is a unique rule that satisfies *Pareto-efficiency*, *symmetry*, and *strategy-proofness*. Also, this rule satisfies *anonymity* (Theorem 1.4).[4]

1.4.1 Single-valued rules

We first prove that a *single-valued* rule satisfies *efficiency of queues* and *strategy-proofness* if and only if for each problem, it selects an *efficient* queue (of course) and sets each agent's transfer as prescribed in Groves (1973), i.e., equal to the total waiting cost of all other agents plus an amount only depending on these agents' unit waiting costs (Theorem 1.1). As the domain of preference profiles is convex, it is smoothly connected. Thus, this result follows from Holmström's (1979). However, we are able to give a simpler proof by exploiting the special features of our model. Formally, let $D \equiv \{d|$ for each $c \in \mathcal{C}$, we have $d(c) \in Q^*(c)\}$. Let $H \equiv \{(h_i)_{i \in N}|$ for each $i \in N$, we have $h_i : \mathbb{R}_+^{N\setminus\{i\}} \to \mathbb{R}\}$. A *single-valued* rule φ is a *Groves' rule* if and only if there are $d \in D$ and $h \in H$ such that for each $c \in \mathcal{C}$, $\varphi(c) = (\sigma, t) \in Z(N)$ with $\sigma = d(c)$, and for each $i \in N$, $t_i = -\sum_{l \in N\setminus\{i\}}(\sigma_l - 1)c_l + h_i(c_{-i})$.

Theorem 1.1. *A* single-valued *rule is a Groves rule if and only if it satisfies* efficiency of queues *and* strategy-proofness*.*

The class of Groves' rules is large. We distinguish subclasses according to their

[4]By extending Theorem 1.2 to multi-valued rules, we prove that what holds in the special case of *single-valued* rules still holds in the general case of single- and multi-valued rules. Thus, single-valuedness and Theorem 1.4 imply Theorem 1.2.

h function. For instance, the *Pivotal rules* are the Groves' rules associated with $h \in H$ such that for each $c \in \mathcal{C}$, for each $i \in N$, $h_i(c_{-i}) = \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_l$.[5] By Theorem 1.1, a *single-valued* rule satisfies *Pareto-efficiency* and *strategy-proofness* if and only if it is a Groves rule and it is *balanced*. However, for two-agent problems, no Groves rule is *balanced* (Suijs, 1996). From now on, we focus on problems with more than two agents.

We now introduce another class of *single-valued* rules. A rule in this class selects for each problem a *Pareto-efficient* queue and sets transfers considering each pair of agents in turn, making each agent in the pair pay what a Pivotal rule recommends for the subproblem consisting of these two agents, and distributing the sum of these two payments equally among the others. Thus, for each problem and each selected queue, each agent's transfer is such that she pays the cost she imposes on the other agent and she receives $\frac{1}{n-2}$-th of the cost each agent imposes on the other agent in the pair that she is not part of.

Equally Distributed Pairwise Pivotal rule, φ^*: For each $c \in \mathcal{C}$, if $(\sigma, t) = \varphi^*(c)$, then $\sigma \in Q^*(c)$ and for each $i \in N$, we have

$t_i = -\sum_{j \in N \setminus \{i\}} \sum_{l \in \{i,j\} \cap F_i(\sigma)} c_l + \frac{1}{(n-2)} \sum_{j \in N \setminus \{i\}} \sum_{k \in N \setminus \{i,j\}} \sum_{l \in \{j,k\} \cap F_j(\sigma)} c_l.$

An example of a problem illustrating the rule:

Let $N = \{1, 2, 3, 4\}$ and $c \in \mathbb{R}^N_+$ such that $c_1 > c_2 > c_3 > c_4$. The *efficiency*

[5] Pivotal rules are also known as *Clarke's rules* (Clarke, 1971).

of queues implies that agents should be served in decreasing order of their waiting costs. Thus, the efficient queue is $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (1, 2, 3, 4)$. Then, consider each pair of agents, and make each agent in the pair pay the cost that the agent imposes on the other agent. Then, distribute the sum of these two payments equally among the others. The following table shows how payments are calculated. For example, for the pair $\{2, 4\}$, by *Pareto-efficiency* agent 2 should be served before agent 4. The cost agent 2 imposes on agent 4 is c_4 but agent 4 does not impose any cost on agent 2. So, agent 2 pays c_4 and agent 4 pays nothing. The amount collected in total is distributed among agents 1 and 3 equally: each of them receives $c_4/2$.

	1	2	3	4
12	$-c_2$	0	$c_2/2$	$c_2/2$
13	$-c_3$	$c_3/2$	0	$c_3/2$
14	$-c_4$	$c_4/2$	$c_4/2$	0
23	$c_3/2$	$-c_3$	0	$c_3/2$
24	$c_4/2$	$-c_4$	$c_4/2$	0
34	$c_4/2$	$c_4/2$	$-c_4$	0

The final monetary consumption is the sum of all the transfers for each possible pair. Then, $t = (t_1, t_2, t_3, t_4) = (-c_2 - c_3/2, -c_3/2, c_2/2, c_2/2 + c_3)$. The allocation selected by the *Equally Distributed Pairwise Pivotal rule* is *Pareto-*

efficient. The rule satisfies *equal treatment of equals*, *no-envy*, and *strategy-proofness*. However, the rule is not *coalitionally strategy-proof* because agent 2 and agent 3 have incentive to misrepresent their waiting costs jointly.

As there may be several *Pareto-efficient* queues for a problem, there are several Equally Distributed Pairwise Pivotal rules. Proposition 1.1 states that for each problem and each *Pareto-efficient* queue, the transfers set by any Equally Distributed Pairwise Pivotal rule can be obtained in three other ways. First, making each agent pay what the Pivotal rule recommends for the problem, giving each agent $\frac{1}{n-2}$-th of what the others pay. Second, giving each agent $\frac{1}{n-2}$-th of her predecessors' total waiting cost and making each agent pay $\frac{1}{n-2}$-th of her followers' gain from not being last (Mitra and Sen, 1998, Mitra, 2001). Third, giving each agent one half of her predecessors' unit waiting cost and making each agent pay one half of her followers' unit waiting cost plus $\frac{1}{2(n-2)}$-th of the difference between two unit waiting costs of any other agent and this agent's predecessors's (Suijs, 1996).

Proposition 1.1. *Let* φ *be a* single-valued *rule. Then, the following statements are equivalent.*

1. φ *is an Equally Distributed Pairwise Pivotal rule.*

2. φ *is a Groves rule associated with* $h \in H$ *such that for each* $c \in \mathcal{C}$, *if* $(\sigma, t) = \varphi(c)$, *then for each* $i \in N$, $h_i(c_{-i}) = \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_l +$

$\frac{1}{(n-2)} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_l$.

3. *φ is such that for each $c \in \mathcal{C}$, if $(\sigma, t) = \varphi(c)$, then $\sigma \in Q^*(c)$ and for each $i \in N$, $t_i = \sum_{l \in P_i(\sigma)} \frac{(\sigma_l - 1)}{(n-2)} c_l - \sum_{l \in F_i(\sigma)} \frac{(n - \sigma_l)}{(n-2)} c_l$.*

4. *φ is such that for each $c \in \mathcal{C}$, if $(\sigma, t) = \varphi(c)$, then $\sigma \in Q^*(c)$ and for each $i \in N$, $t_i = \sum_{l \in P_i(\sigma)} \frac{c_l}{2} - \sum_{l \in F_i(\sigma)} \frac{c_l}{2} - \sum_{l \in N \setminus \{i\}} \sum_{k \in P_l(\sigma) \setminus \{i\}} \frac{c_k - c_l}{2(n-2)}$.*

Next, we prove that requiring *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness* implies choosing an Equally Distributed Pairwise Pivotal rule.

Theorem 1.2. *A* single-valued *rule satisfies* Pareto-efficiency*,* equal treatment of equals in welfare*, and* strategy-proofness *if and only if it is an Equally Distributed Pairwise Pivotal rule.*

The following paragraphs establish the independence of the axioms in Theorem 1.2.

(i) Let φ be a rule such that for each $c \in \mathcal{C}$, if $(\sigma, t) = \varphi(c)$, then $\sigma \in Q^*(c)$. Let $i \in N$ if $\sigma_i \neq 1$ and for each $\{j, k\} \subset N$ are such that $\sigma_j = \sigma_i - 1$ and $\sigma_k = \sigma_i + 1$, then $\alpha_i \in [c_j, c_k]$ and $t_i = \sum_{l \in P_i(\sigma) \cup \{i\}} \alpha_l$, and if $\sigma_i = 1$, then $t_i = \alpha_i$ where in each case $\alpha \in \mathbb{R}^N$ is chosen so as to achieve $\sum_{l \in N} t_l = 0$. Any such rule satisfies all the axioms of Theorem 1.2 but *strategy-proofness* (Chun, 2004b).

(ii) Let φ be a Groves rule associated with $h \in H$ such that for each $c \in \mathcal{C}$ and let $\lambda \in \mathbb{R}$ be such that $\lambda \neq 0$ and $h_1 = \sum_{l \in N \setminus \{1\}} (\sigma_l^{-1} - 1) c_l + \frac{1}{n-2} \sum_{l \in N \setminus \{1\}} (\sigma_l^{-1} - 1) c_l + \lambda$, and for each $i \in N \setminus \{1\}$, we have $h_i = \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_i + \frac{1}{n-2} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_l - \frac{\lambda}{(n-1)}$. Any such rule satisfies all the axioms of Theorem 1.2 but *equal treatment of equals in welfare*.

(iii) Let φ be a Groves rule associated with $h \in H$ such that $c \in \mathcal{C}$ and let $\lambda \in \mathbb{R}_+$ be such that for each $i \in N$, $h_i = \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_i + \frac{1}{n-2} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_l - \lambda$ satisfies all axioms but *Pareto-efficiency*.

Remark 1.1. *Equally Distributed Pairwise Pivotal rules satisfy* no-envy*. Thus, as* no-envy *implies* equal treatment of equals in welfare*, we prove that only* single-valued *Equally Distributed Pairwise Pivotal rules satisfy* Pareto-efficiency*,* equal treatment of equals in welfare*, and* strategy-proofness.

Next, we show that if we impose the stronger condition of *coalitional strategy-proofness*, even with *efficiency of queues*, then we have a negative result.

Theorem 1.3. *No rule satisfies* efficiency of queues *and* coalitional strategy-proofness.

The following paragraphs establish examples of rules that satisfy only one of the axioms in Theorem 1.3.

(i) Equally Distributed Pairwise Pivotal rule satisfies *efficiency of queues* but not *coalitional strategy-proofness*.

(ii) Any rule that selects the same arbitrary queue and sets the transfer to each agent equal to zero satisfies *coalitional strategy-proofness*, but not *efficiency of queues*.

Theorem 1.3 implies that no rule satisfies *Pareto-efficiency* and *coalitional strategy-proofness*. It also implies that no rule satisfies *Pareto-efficiency*, *non-bossiness*, and *strategy-proofness*. Since *no-envy* implies *Pareto-efficiency of queues*[6], it follows that no rule satisfies *no-envy*, *non-bossiness*, and *strategy-proofness*.

1.4.2 Multi-valued rules

Let Φ be a rule. When we allow multi-valuedness of rules, it is possible to give two agents with equal unit waiting costs equal assignments. We require that this be the case: If there is another allocation at which two agents exchange their assignments and the other agents keep theirs, then this allocation should be selected.

Symmetry: *For each $c \in \mathcal{C}$, each $(\sigma, t) \in \Phi(c)$, and each $\{i, j\} \subset N$ with $i \neq j$ and $c_i = c_j$, if $(\sigma', t') \in Z(N)$ such that $(\sigma'_i, t'_i) = (\sigma_j, t_j)$, $(\sigma'_j, t'_j) = (\sigma_i, t_i)$,*

[6] Assume that φ satisfies *no-envy*. Let $c \in \mathcal{C}$, $(\sigma, t) = \varphi(c)$, $\{i, j\} \subset N$, with $i \neq j$ be such that $c_i > c_j$ but $\sigma_i > \sigma_j$. By *no-envy*, we have $u_i(\sigma_i, t_i) \geq u_i(\sigma_j, t_j)$ and $u_j(\sigma_j, t_j) \geq u_j(\sigma_i, t_i)$. Then, $(\sigma_i - \sigma_j)c_i + t_j \leq t_i \leq (\sigma_i - \sigma_j)c_i$ that contradicts $c_i > c_j$.

and for each $l \in N \setminus \{i, j\}$, *we have* $(\sigma'_l, t'_l) = (\sigma_l, t_l)$, *then* $(\sigma', t') \in \Phi(c)$.

This second requirement is that if we permute agents' unit waiting costs, we should permute the selected assignments accordingly. Formally, let Π be the set of all permutations on N. For each $\pi \in \Pi$ and each $c \in \mathbb{R}^N_+$, let $\pi(c) \equiv (c_{\pi(i)})_{i \in N}$ and $\pi(\sigma, t) \equiv (\sigma_{\pi(i)}, t_{\pi(i)})_{i \in N}$.

Anonymity: *For each* $c \in \mathcal{C}$, *each* $(\sigma, t) \in \Phi(c)$, *and each* $\pi \in \Pi$, *we have* $\pi(\sigma, t) \in \Phi(\pi(c))$.

Single- and multi-valued rules may satisfy *equal treatment of equals in welfare.* However, only multi-valued rules may satisfy *symmetry.* Indeed, *symmetry* is necessary for agents' names not to matter. Thus, the presence of indivisibilities implies that we may require *anonymity* of rules only if we allow multi-valuedness.

For multi-valued rules, *strategy-proofness* has to be redefined. To compare the welfare levels derived from two sets of feasible allocations, we assume that an agent prefers the former to the latter if and only if for each allocation in the latter, there is an allocation in the former that she finds at least as desirable; and for each allocation in the former, there is an allocation in the latter that she does not prefer.[7] Formally, let $\mathcal{X}_i$ be the set of

[7] Determining how agents rank non-empty sets given their preferences over singletons has been studied in, e.g., Pattanaik (1973), Barberá (1977), Dutta (1977), Kelly (1977), Feldman (1979, 1980), Gärdenfors (1979), Thomson (1979), Ching and Zhou (2000), Duggan and Schwartz (2000), and Barberá, Dutta, and Sen (2001).

positions and transfers in $\mathbb{N} \times \mathbb{R}$. Given $c_i \in \mathbb{R}_+$, let $R_i(c_i)$ be the preference relation on subsets $\mathcal{X}_i$ defined as follows: for each $\{X_i, X_i'\} \subseteq \mathcal{X}_i$, we have $X_i \; R_i(c_i) \; X_i'$ if and only if $\min_{(\sigma_i,t_i)\in X_i} u_i(\sigma_i, t_i) \geq \min_{(\sigma_i',t_i')\in X_i'} u_i(\sigma_i', t_i')$ and $\max_{(\sigma_i,t_i)\in X_i} u_i(\sigma_i, t_i) \geq \max_{(\sigma_i',t_i')\in X_i'} u_i(\sigma_i', t_i')$. Let $\mathcal{Z}$ be the set of all non-empty subsets of $Z(N)$. For each $Z \in \mathcal{Z}$, and each $i \in N$, let $Z_i \equiv \bigcup_{(\sigma,t)\in Z}(\sigma_i, t_i)$.

Strategy-proofness: *For each $c \in \mathcal{C}$, each $i \in N$, and each $c_i' \in \mathbb{R}_+$, if $Z = \Phi(c)$ and $Z' = \Phi(c_i', c_{-i})$, then $Z_i \; R_i(c_i) \; Z_i'$.*

Thus, as in Pattanaik (1973), Dutta (1977), and Thomson (1979), *strategy-proofness* requires each agent to find the worst assignment she may receive when she reveals her unit waiting cost at least as desirable as the worst assignment she may receive when she misrepresents it. Furthermore, it requires each agent to find the best assignment she may receive when she reveals her unit waiting cost at least as desirable as the best assignment she may receive when she misrepresents it. The second requirement is also implied by further basic incentive compatibility requirements. In particular, it is a necessary condition for implementation in undominated strategies by bounded mechanisms (Jackson, 1992, Ching and Zhou, 2002).[8]

[8]Formally, an agent does not find misrepresenting her unit waiting cost more desirable as revealing it if there is no $c \in \mathcal{C}$, each $i \in N$, and each $c_i' \in \mathbb{R}_+$ such that for $(\sigma', t') \in \Phi(c_i', c_{-i}) \backslash \Phi(c)$, we have $u_i(\sigma_i', t_i') > \min_{(\sigma,t)\in\Phi(c)} u_i(\sigma_i, t_i)$ or for $(\sigma, t) \in \Phi(c) \backslash \Phi(c_i', c_{-i})$, we have $max_{(\sigma',t')\in\Phi(c_i',c_{-i})} u_i(\sigma_i', t_i') < u_i(\sigma_i, t_i)$.

Next, we show that a unique rule satisfies *Pareto-efficiency*, *symmetry*, and *strategy-proofness* and identify the rule. Moreover, this rule satisfies *no-envy* and *anonymity*. For each problem, the rule selects all *Pareto-efficient* queues and for each queue, it sets transfers as in the Equally Distributed Pairwise Pivotal rule. Thus, it is the union of the desirable rules introduced in the previous subsection. Formally,

The Largest Equally Distributed Pairwise Pivotal rule, Φ^*: For each $c \in \mathcal{C}$, we have $(\sigma, t) \in \Phi^*(c)$ if and only if $\sigma \in Q^*(c)$ and for each $i \in N$, we have $t_i = -\sum_{j \in N \setminus \{i\}} \sum_{l \in \{i,j\} \cap F_i(\sigma)} c_l + \frac{1}{(n-2)} \sum_{j \in N \setminus \{i\}} \sum_{k \in N \setminus \{i,j\}} \sum_{l \in \{j,k\} \cap F_j(\sigma)} c_l$.

For each problem and each *Pareto-efficient* queue, the transfers set by the Largest Equally Distributed Pairwise Pivotal rule can be obtained as the transfers set by any rule described in Proposition 1.1. Thus, for each $c \in \mathcal{C}$, each $(\sigma, t) \in \Phi^*(c)$, and each $i \in N$, we have

$t_i = -\sum_{l \in N \setminus \{i\}} (\sigma_l - 1) c_l + \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_l + \frac{1}{(n-2)} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_l$

$= \sum_{l \in P_i(\sigma)} \frac{(\sigma_l - 1)}{(n-2)} c_l - \sum_{l \in F_i(\sigma)} \frac{(n - \sigma_l)}{(n-2)} c_l$

$= \sum_{l \in P_i(\sigma)} \frac{c_l}{2} - \sum_{l \in F_i(\sigma)} \frac{c_l}{2} - \sum_{l \in N \setminus \{i\}} \sum_{k \in P_l(\sigma) \setminus \{i\}} \frac{c_k - c_l}{2(n-2)}$.

Furthermore, Theorem 1.4 states that only subcorrespondences of the Largest Equally Distributed Pairwise Pivotal rule can satisfy *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*. They also satisfy *no-*

envy. Then, we prove that the Largest Equally Distributed Pairwise Pivotal rule is the only rule that satisfies *Pareto-efficiency*, *symmetry*, and *strategy-proofness*.

Theorem 1.4.

1. *A rule satisfies* Pareto-efficiency*,* equal treatment of equals in welfare*, and* strategy-proofness *if and only if it is a subcorrespondence of the Largest Equally Distributed Pairwise Pivotal rule.*

2. *A rule satisfies* Pareto-efficiency*,* symmetry*, and* strategy-proofness *if and only if it is the Largest Equally Distributed Pairwise Pivotal rule.*

The following paragraphs establish the independence of axioms in the second statement in Theorem 1.4.

(i) Consider a rule that selects all *Pareto-efficient* queues and sets each agent's transfer equal to the Shapley value of the associated coalitional game, where the worth of a coalition is the minimum possible sum of its members waiting costs (Maniquet, 2003). Such a rule satisfies all the axioms of the second statement of Theorem 1.4 but *strategy-proofness*.

(ii) Consider any proper subcorrespondence of a rule that is the union of all the *single-valued* rules that are Groves' rules associated with $h \in H$ and satisfy *balancedness*. Such a rule satisfies all the axioms of the second statement of Theorem 1.4 but *symmetry*.

(iii) Consider a rule such that for each queueing problem and each $\lambda \in \mathbb{R}_+$, it selects a fixed queue and sets each agent's transfer equal to $-\lambda$. Such a rule satisfies all the axioms of the second statement of Theorem 1.4 but *Pareto-efficiency*.

Remark 1.2. *The Largest Equally Distributed Pairwise Pivotal rule also satisfies* anonymity*. Since* anonymity *implies* symmetry *and the Largest Equally Distributed Pairwise Pivotal rule is the union of all the rules that satisfy* Pareto-efficiency*,* equal treatment of equals in welfare*, and* strategy-proofness*, it follows that this rule is the only rule that satisfies* Pareto-efficiency*,* equal treatment of equals in welfare*,* symmetry*, and* strategy-proofness*.*

1.5 Conclusion

Our objective was to identify allocation rules for queueing problems that satisfy efficiency, equity, and incentive requirements simultaneously on the domain of quasi-linear preferences in positions and transfers. We proved that the Largest Equally Distributed Pairwise Pivotal rule is the only such rule. It is the only rule, together with any of its subcorrespondences, that satisfies *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness.* It is the only rule that satisfies *Pareto-efficiency*, *symmetry*, and *strategy-proofness.* As any of it subcorrespondences, it satisfies *no-envy.* Furthermore, it satisfies *anonymity*.

We draw three lessons from these results. First and foremost, queueing problems are among the few problems in which *Pareto-efficiency*, a weak equity axiom such as *equal treatment of equals in welfare* or *symmetry*, and *strategy-proofness*, are compatible. The natural next step is to determine if this compatibility extends to other problems, in particular to ones in which agents have different processing times. However, in queueing problems in which waiting costs vary non-linearly across positions, no rule satisfies *Pareto-efficiency* and *strategy-proofness* (Mitra, 2002).

Second, while *efficiency of queues* and *strategy-proofness* leave us with a large class of *single-valued* rules, adding as weak equity axiom as *equal treatment of equals in welfare* imposes a unique way of setting transfers. The open question is to determine the class of multi-valued rules that satisfy *Pareto-efficiency* and *strategy-proofness*.

Finally, in the queueing problems we studied, simply requiring *equal treatment of equals* in addition to *Pareto-efficiency* and *strategy-proofness*, guarantees further basic fairness requirements. First, it prevents agents from envying one another. In allocation problems of private goods, *equal treatment of equals in welfare* and *coalition strategy-proofness* together imply *no-envy* (Moulin, 1993). In general public decision-making problems in which the domain of preferences is strictly monotonically closed, *equal treatment of equals in wel-*

fare, *strategy-proofness*, and *non-bossiness* together imply *no-envy* (Fleurbaey and Maniquet, 1997). However, these results do not apply to the problems we studied. Indeed, here no rule satisfies *Pareto-efficiency* and *coalition strategy-proofness*. Also, as preferences are quasi-linear in positions and transfers, they are not monotonically closed. In fact, no rule satisfies *Pareto-efficiency*, *non-bossiness*, and *strategy-proofness*. Second, it guarantees that agents' names do not matter. Finally, it guarantees each agent a minimal welfare level. Indeed, in allocation problems of at most one indivisible private good per agent, *no-envy* implies the *identical-preferences lower bound*, i.e., each agent should find her assignment at least as desirable as any assignment recommended by *Pareto-efficiency* and *equal treatment of equals in welfare* when the other agents have her preferences (Bevia, 1996).

Chapter 2

Matching

2.1 Introduction

We consider two-sided matching markets with contracts as modeled by Roth (1984) and studied in Hatfield and Milgrom (2005). The model in which the contracts are introduce into general matching markets, includes classical marriage markets, entry-level labor markets, school admission markets, and auction markets. Here, we discuss a medical job market with doctors and hospitals as a matching market with contracts. One of the concerns in matching theory is the stability. A matching is *stable* if there is no individual or a pair of set of agents who can arrange a new matching preferred to the original matching. We are interested in the stable solution that assigns to each matching market with contracts its set of stable matchings. We consider a *substitutes* condition which requires when an agent chooses from an expanded set of contracts, the set of contracts it rejects also expands. If the preferences are *substitutable*, then the stable solution is well-defined, (Hatfield and Milgrom, 2005). Therefore, we impose *substitutability* on preferences.

We follow the axiomatic approach and analyze solutions under variable populations and preferences. A test for finding desirable rules involves variations in the number of agents and we provide two properties. First, we consider *population-monotonicity* (Thomson, 1983). When there is an exogenous change in the population, it would be unfair if the agents who were not responsible for this change were treated unequally. *Population monotonicity* represents the idea of solidarity, and requires that if some agents leave, then as a result either all remaining agents (weakly) gain or they all (weakly) lose. In particular, we introduce *own-side population-monotonicity*, which requires that no agent on one side of the market should benefit from an increase of population in its own side, and *other-side population monotonicity*, which requires that no agent on one side of the market should lose from an increase of population of the other side. Second, we consider *the consistency principle*[1] which has been applied to a wide class of economic problems. Suppose an allocation has been chosen by the solution for the problem we consider, and then a subset of agents leaves with what they are allocated. Now, consider the remaining

[1]For an extensive survey on consistency, see Thomson (2005).

agents and all resources they have collectively received, which form the new problem. The solution is *consistent* if it recommends the same bundle to be allocated to each agent as initially. In other words, *consistency* is a property in which some agents are leaving with what they are prescribed by the solution and if we apply the solution to the reduced economy, then the prescribed allocation does not change for the remaining ones. When applying the consistency principle, it is necessary to define a new problem with the remaining agents, which is called a reduced problem. Here, we consider two different ways of reducing a problem. We introduce *strong consistency* where we allow any set of agents to leave unless an agent who has been already matched with some agents is left alone and *weak consistency* where we only allow agents to leave in blocks, i.e., in pair of sets originally matched. Consistency has been studied in matching markets and Sasaki and Toda (1992) obtain characterizations of the core of one-to-one matching market. They show that the stable solution is the only solution which satisfies *Pareto-efficiency*, *anonymity*, *consistency*, and *converse consistency*[2]. Toda (2006) obtains characterization of the core of many-to-one matching markets.

We also consider *Maskin monotonicity* studied in Haake and Klaus (2005). This property requires that if the selected allocation has improved in the new

[2] A solution is *conversely-consistent* if for each problem and each allocation for that problem, if the restriction of the allocation to each subgroup of two matched pairs is among the recommendations made by the solution for the four-agent reduced problem and the allocation, then the allocation should be one of the recommendations for the original problem.

preference relation, then the rule should select the same allocation when the new preference relation is announced. *Maskin monotonicity* is a necessary but not sufficient condition for Nash implementability. Haake and Klaus (2005) shows that the stable solution is *Maskin monotonic* and any solution that is *Pareto-efficient*, *individual rational*, *Maskin monotonic* contains the stable solution.

We prove that the stable solution is the only rule that satisfies *unanimity*, *own-side population-monotonicity*, and *Maskin-monotonicity*. Moreover, if a rule satisfies *unanimity*, *own-side population-monotonicity*, *other-side population monotonicity*, and *weak consistency*, then it is a subsolution of the stable solution. In a recent paper, Toda (2006) obtains similar results in two-sided many-to-one matching markets.

We also analyze immunity of solutions to strategic behavior such as to misreporting the availability of contracts held by the hospitals, and misrepresenting preferences by doctors and hospitals. Postlewaite (1979) introduces *destruction-proofness* in the context of exchange economies and shows that there is no solution which is *Pareto-efficient*, *individually rational*, and *destruction-proof*. There is no solution which is *no-envy in trades* and *ε-witholding-proof* (Thomson, 1987). It is easy to find examples that the Walrasian rule is not *destruction-proof* in classical private goods economies and the Lindahl rule is not *destruction-proof* in classical public goods economies. In exchange markets with heterogeneous indivisible goods and agents with *separable* preferences, *Pareto-efficiency* and *hiding-proofness* together imply *individual rationality* and there is no rule which is *Pareto-efficient*, *individually rational*, and *destruction-proof* (Atlamaz and Klaus, 2005). When there are at least two firms and three workers, there exists no matching rule that is stable and *non-manipulable via capacities* in two-sided many to one matchings (Sönmez, 1997). However, there are some rules that are *Pareto-efficient*, *individually rational*, and *non-manipulable via capacities*.

We introduce *destruction-proofness*, and study *destruction-proofness* and *strategy-proofness* in matching markets with contracts. We show that if the hospitals' preferences satisfy the substitute condition, then the doctor-optimal solution is not *destruction-proof* and the hospital-optimal solution is *destruction-proof*.

Alcalde and Barberà (1994) show that there is no matching rule that is *Pareto-efficient*, *individually rational*, and *strategy-proof*. They consider domain of preferences satisfying top dominance condition. Consider a pair of preference relations and a pair of individually rational alternatives. Assume that first alternative is preferred to the second under the first preference relation and the second alternative is preferred to the first under the second preference relation. The condition requires that there is no other alternative such that it is preferred to the first alternative under the first preference relation and is preferred to the second alternative under the second preference relation. They also show that if the preference of firms are *responsive* and satisfy top-dominance condition, then the worker-optimal solution is *Pareto-efficient*, *individually rational*, and *strategy-proof*. We also impose another restriction on preferences of firms, the law of aggregate demand. This law states that as more contracts are available, the firms should take on (weakly) more contracts. In other words, if the set of possible contracts expands, then the total number of contracts chosen by firm either rises or stays the same. This property for a worker is implied by revealed preference, since each worker chooses at most one contract. If the firms' preferences satisfy the law of aggregate demand and the substitutes condition, then for worker-optimal solution, it is dominant strategy for workers to reveal truthfully their preferences over contracts. However, it is not a dominant strategy for firms to truthfully reveal, even for firm-optimal solution (Hatfield and Milgrom (2005)). We also consider top-dominance condition. We show that if the hospitals' preferences satisfy the substitute condition, the law of aggregate demand, and the top-dominance condition then the doctor-optimal solution is the only solution satisfying *stability* and *strategy-proofness.* [3] In two-sided matching markets with contracts

[3]Haeringer and Wooders (2004) studies a decentralized job market model where firms propose sequentially a (unique) position to some workers. Successful candidates then decide whether to accept the offers, and departments whose positions remain unfilled propose to other candidates. They provide a complete characterization of the Nash equilibrium outcomes and the subgame perfect equilibria. While the set of Nash equilibria outcomes contain all individually rational matchings, it turns out that in most cases considered all subgame

and more than two agents the stable correspondence is Nash implementable (Haake and Klaus, 2005). Note that they only impose substitute condition, whereas in order to have dominant strategy implementation, we need to impose the substitute condition, the law of aggregate demand, and the top-dominance condition.

In Section 2.2, we introduce the model where we discuss a medical job market with doctors and hospitals as a matching market with contracts. Sections 3.3 and 2.4 introduce the properties of solutions and some well-known solutions in the literature. Section 2.5 provides the results and all proofs are provided in the Appendix B. Section 2.6 concludes.

2.2 Model

There are $\mathbb{D}$ the infinite set of potential doctors and $\mathbb{H}$ the infinite set of potential hospitals. Let $\mathcal{D}$ and $\mathcal{H}$ be the sets of all finite subsets of $\mathbb{D}$ and $\mathbb{H}$, and let D and H be generic sets of doctors and hospitals. Let d and h be a generic doctor and hospital. Also, let i, j be generic agents. A *contract* is a match between a doctor and a hospital that specifies the conditions of employment. Let $\mathbb{X}$ be the infinite set of potential contracts and $\mathcal{X}$ be the set of all finite subsets of $\mathbb{X}$. Let X be a generic set of contracts and x a generic contract.

perfect equilibria yield a unique outcome, the worker-optimal solution.

Let $\mu : \mathbb{X} \to \mathbb{D} \times \mathbb{H}$ be the function that specifies the bilateral structure of each contract. That is, μ assigns to each contract $x \in \mathbb{X}$ the ordered pair $(d, h) \in \mathbb{D} \times \mathbb{H}$ that lists the doctor and the hospital between whom contract x is established. Clearly, for each pair $\{x, x'\} \subseteq \mathbb{X}$, if $x \neq x'$ and $\mu(x) = \mu(x')$, then x and x' are contracts between the same doctor and hospital but under different terms. Also, each agent may stay unmatched, i.e., each doctor may stay unemployed and each hospital may employ no doctor. We refer to this situation as the *null contract*. We denote it by $\emptyset \in \mathbb{X}$. Also, by abuse of language, we say that the null contract matches the agent to herself. Then, for each $i \in \mathbb{D} \cup \mathbb{H}$, let $\mathcal{X}_i$ be the set of all sets of contracts of $\mathcal{X}$ in which i is matched (including to herself), i.e., $\mathcal{X}_i \equiv \{X \in \mathcal{X} :$ for each $x \in X$, there is $j \in \mathbb{D} \cup \mathbb{H}$ such that $\mu(x) = (i, j)$ or $\mu(x) = (j, i)\}$. Also, for each $i \in \mathbb{D} \cup \mathbb{H}$ and each $X \in \mathcal{X}$, let X_i be the set of all contracts of X in which i is matched, i.e., $X_i \equiv \{x \in X :$ there is $j \in \mathbb{D} \cup \mathbb{H}$ such that $\mu(x) = (i, j)$ or $\mu(x) = (j, i)\}$. We assume that each doctor is matched at most one hospital, whereas each hospital may be matched to several doctors. That is, we assume that, for each $d \in \mathbb{D}$ and each $X \in \mathcal{X}_d$, we have $|X| \leq 1$.

Each doctor $d \in \mathbb{D}$ has preferences over $\mathbb{X} \cup \{\emptyset\}$, described by a total order $\mathcal{R}_d$.[4] Let $\mathfrak{R}_d$ be the set of all preferences of agent $d \in \mathbb{D}$. For each $d \in \mathbb{D}$

[4]That is, $\mathcal{R}_d$ is a binary relation that satisfies *completeness* (for each $x', x'' \in \mathbb{X} \cup \{\emptyset\}$, either $x' \mathcal{R}_d x''$ or $x'' \mathcal{R}_d x'$), *transitivity* (for each $x', x'', x''' \in \mathbb{X} \cup \{\emptyset\}$, if $x' \mathcal{R}_d x''$ and $x'' \mathcal{R}_d x'''$, then $x' \mathcal{R}_d x'''$), and *antisymmetry* (for each $x', x'' \in \mathbb{X} \cup \{\emptyset\}$ with $x' \neq x''$, either $x' \mathcal{P}_d x''$ or $x'' \mathcal{P}_d x'$).

and each $\mathcal{R}_d \in \mathfrak{R}_d$, $C(., \mathcal{R}_d) : \mathcal{X} \to \mathcal{X}_d$ is the *choice function of agent d with preferences* $\mathcal{R}_d$ that assigns to each set of contracts $X \in \mathcal{X}$ the most preferred contract $C(X, \mathcal{R}_i) \in \mathcal{X}_d$. Formally, for each $X \in \mathcal{X}$, we have $C(X, \mathcal{R}_d) \equiv max_{\mathcal{R}_d}\{x \in X_d\}$. Each hospital $h \in \mathbb{H}$ has preferences over $\mathcal{X}_h$, described by a total order $\mathcal{R}_h$.[5] Let $\mathfrak{R}_h$ be the set of all preferences of agent $h \in \mathbb{H}$. For each $h \in \mathbb{H}$ and each $\mathcal{R}_h \in \mathfrak{R}_h$, $C(., \mathcal{R}_h) : \mathcal{X} \to \mathcal{X}_h$ is the *choice set correspondence of agent h with preferences* $\mathcal{R}_h$ that assigns to each set of contracts $X \in \mathcal{X}$ the most preferred subset of contracts $C(X, \mathcal{R}_h) \subseteq X_h$ (including the null contract). Formally, for each $X \in \mathcal{X}$, we have $C(X, \mathcal{R}_h) \equiv max_{\mathcal{R}_h}\{X' \subseteq X_h\}$. We impose two conditions on hospitals' preferences.

- $\mathcal{R}_h$ is *substitutable* if for each $X, X' \in \mathcal{X}_h$ with $X' \subsetneq X$, we have $X' \cap C(X, \mathcal{R}_h) \subseteq C(X', \mathcal{R}_h)$.

- $\mathcal{R}_h$ satisfy the law of aggregate demand if for all $X'' \subseteq X'$, we have $|C(X'', \mathcal{R}_h)| \leq |C(X', \mathcal{R}_h)|$.

Also, for each $i \in \mathbb{D} \cup \mathbb{H}$ and each $X \in \mathcal{X}$, let $\mathfrak{R}_i|_{X_i}$ be the reduced set of all preferences of $\mathfrak{R}_i$ relative to X_i and, each $X \in \mathcal{X}$, let $\mathfrak{R}|_X = \prod_{i \in \mathbb{D} \cup \mathbb{H}} \mathfrak{R}_i|_{X_i}$ be the reduced set of all preferences profiles of $\mathfrak{R}$ relative to X. Next, we consider a domain restriction.

[5]That is, $\mathcal{R}_h$ is a binary relation that satisfies *completeness* (for each $X', X'' \in \mathcal{X}_h$, either $X' \mathcal{R}_h X''$ or $X'' \mathcal{R}_h X'$), *transitivity* (for each $X', X'', X''' \in \mathcal{X}_h$, if $X' \mathcal{R}_h X''$ and $X'' \mathcal{R}_h X'''$, then $X' \mathcal{R}_h X'''$), and *antisymmetry* (for each $X', X'' \in \mathcal{X}_h$ with $X' \neq X''$, either $X' \mathcal{P}_h X''$ or $X'' \mathcal{P}_h X'$).

- The domain of preferences satisfies top-dominance condition if for each pair $\{R_h, R'_h\} \subseteq \mathfrak{R}$, and each pair $\{X', X''\} \subseteq X$ such that $X'\ R_h\ \emptyset$, $X''\ R'_h\ \emptyset$, $X'\ R_h\ X''$ and $X''\ R'_h\ X'$, then there is no X''' such that $X'''\ R_h\ X'$ and $X'''\ R'_h\ X''$

A *matching market with contracts* is a quadruple $M \equiv (D, H, X, R)$ such that:

(i) $D \in \mathcal{D} \backslash \emptyset$,

(ii) $H \in \mathcal{H} \backslash \emptyset$,

(iii) $X \in \mathcal{X}$ such that for each $x \in X$, there are $d \in D$and $h \in H$ such that $\mu(x) = (d, h)$,

(iv) $R = (R_i)_{i \in D \cup H}$ such that for each $i \in D \cup H$, there is $\mathcal{R}_i \in \mathfrak{R}_i$ such that $R_i = \mathcal{R}_i|_{X_i}$.

Let $\mathcal{M}$ be the set of all matching markets with contracts. An *allocation A for* $M = (D, H, X, R) \in \mathcal{M}$ is a list of subsets of contracts $((A_d)_{d \in D}; (A_h)_{h \in H})) \in \prod_{i \in D \cup H} 2^{X_i}$ such that, for each $d \in D$ and each $h \in H$, we have $|A_d \cap A_h| \leq 1$ and if there is $x \in A_d \cup A_h$ with $\mu(x) = (d, h)$, then $\{x\} = A_d \cap A_h$. Let $\mathcal{A}(M)$ be the set of all allocations for $M \in \mathcal{M}$. A *solution* φ is a correspondence that assigns to each matching market with contracts $M \in \mathcal{M}$ a non-empty set of allocations $\varphi(M) \subseteq \mathcal{A}(M)$.

2.3 Properties of solutions

Our objective is to formulate desirable properties and identify appealing solutions that satisfy these properties together in this section. The efficiency requirement is standard. There should be no feasible allocation other than the one selected that each agent finds at least as well as and at least one agent prefers.

An allocation $A \in \mathcal{A}$ is *Pareto-efficient* for $M \in \mathcal{M}$ if there is no $A' \in \mathcal{A}$ with $A' \neq A$ such that, for each $i \in D \cup H$, $C(A \cup A', R_i) = A'_i$. Let $P(M)$ be the set of *Pareto-efficient* allocations for M. Then,

Pareto-efficiency: *For each $M = (D, H, X, R) \in \mathcal{M}$, $\varphi(M) \subseteq P(M)$.*

Next, we require a very mild property which requires that if there is a feasible allocation which is ranked as the top choice for everyone, then it should be the only allocation chosen by the rule.

Unanimity: *For each $M = (D, H, X, R) \in \mathcal{M}$, if there is $A \in \mathcal{A}(M)$ such that for each $i \in D \cup H$, $C(X, R_i) = A_i$, then $\varphi(M) = \{A\}$.*

One of the concerns in matching theory is the stability and stability of an allocation requires that there is no individual or a pair of set of agents who can arrange a new allocation preferred to the original allocation.

An allocation $A \in \mathcal{A}(M)$ is *weakly individually rational* for $M = (D, H, X, R) \in$

$\mathcal{M}$ if, for each $i \in D \cup H$ such that $A_i \neq \emptyset$, we have $C(A_i, R_i) \neq \emptyset$. Let $WIR(M)$ be the set of *weakly individually rational* allocations for $M \in \mathcal{M}$. Then,

Weak individual rationality: *For each* $M \in \mathcal{M}$, *we have* $\varphi(M) \subseteq WIR(M)$.

An allocation $A \in \mathcal{A}(M)$ is *individually rational* for $M = (D, H, X, R) \in \mathcal{M}$ if, for each $i \in D \cup H$, we have $C(A_i, R_i) = A_i$. Let $IR(M)$ be the set of *individually rational* allocations for $M \in \mathcal{M}$. Then,

Individual rationality: *For each* $M \in \mathcal{M}$, *we have* $\varphi(M) \subseteq IR(M)$.

Clearly, for each $M = (D, H, X, R) \in \mathcal{M}$ and each $A \in \mathcal{A}(M)$, if an allocation A is *individually rational* for M, then it is *weakly individually rational* for M. Besides, under substitutable preferences, for each $M = (D, H, X, R) \in \mathcal{M}$ and each $A \in \mathcal{A}(M)$, if A is *individually rational* for M, then, for each $i \in D \cup H$ and each $x \in A_i$, we have $C(\{x\}, R_i) = \{x\}$, i.e., x is acceptable.[6]

A doctor $d \in D$ blocks $A \in \mathcal{A}(M)$ only if $C(A_d, R_d) = \emptyset$. Also, a pair of subsets $(D', H') \subseteq D \times H$ blocks $A \in \mathcal{A}(M)$ only if, for each $h \in H'$, there

[6]Moreover, consider a smaller domain of *separable* preferences, i.e., $\mathcal{R}_h$ is *separable* if, for each $X \in \mathcal{X}_h$ and each $\{x\} \in \mathcal{X}_h \backslash X$, we have $X \cup \{x\} \; \mathcal{R}_h \; X$ if and only if $\{x\} \; \mathcal{R}_h \; \emptyset$. Under separable preferences, for each $M = (D, H, X, R) \in \mathcal{M}$ and each $A \in \mathcal{A}(M)$, we have that A is *individually rational* for M if and only if for each $i \in D \cup H$ and each $x \in A_i$, we have $C(\{x\}, R_i) = \{x\}$, i.e., x is acceptable. We define *individual rationality* as in Haake and Klaus (2005) because this definition does not depend on the assumptions made on preferences. One may find other definitions that do. In particular, Toda's (2006) requires that, for each $i \in D \cup H$ and each $x \in A_i$, we have $C(\{x\}, R_i) = \{x\}$. Thus, our definition and his are equivalent only under separable preferences.

are $D'' \subseteq D'$ and $X' \in \mathcal{X}_h$ with $|D''| = |X'|$ and $X' \nsubseteq A_h$ such that, for each $d \in D''$, there is $x \in X'$ with $\{x\} = C(A_d \cup \{x\}, R_d)$ and $X' = C(A_h \cup X', R_h)$.[7] An allocation $A \in \mathcal{A}(M)$ is *stable* for M if and only if

(i) there is no $d \in D$ such that $C(A_d, R_d) = \emptyset$,

(ii) there is no $h \in H$ such that $C(A_h, R_h) \subsetneq A_h$, and

(iii) there is no pair of subsets $(D', \{h\}) \subseteq D \times H$ such that there is $X' \in \mathcal{X}_h$ with $|D'| = |X'|$ and $X' \nsubseteq A_h$ such that, for each $d \in D'$, there is $x \in X'$ with $\{x\} = C(A_d \cup \{x\}, R_d)$ and $X' = C(A_h \cup X', R_h)$.

Also, if $A \in \mathcal{A}(M)$ is *weakly individually rational* for M, then no doctor $d \in D$ blocks $A \in \mathcal{A}(M)$. Let $S(M)$ be the set of *stable* allocations for $M \in \mathcal{M}$. Then,

Stability: *For each $M \in \mathcal{M}$, we have $\varphi(M) \subseteq S(M)$.*

Next, we turn to the changes in population. The idea of population monotonicity which is introduced in Thomson (1983), requires that when there is an exogenous change in the population, the agents who were not responsible for this change should not be treated unequally. Population monotonicity expresses the idea of solidarity, and requires that if some agents leave, then as a result either all remaining agents gain or they all lose. We apply population monotonicity when there is only a change in population of one side of the mar-

[7] Under substitutable preferences, if there is a *blocking pair of subsets*, then there is $(d, h) \in D \times H$ such that there is $x \in (\mathcal{X}_d \cap \mathcal{X}_h) \setminus (A_d \cup A_h)$ with $C(A_d \cup \{x\}, R_d)\ P_d\ C(A_d, R_d)$ and $C(A_h \cup \{x\}, R_h)\ P_h\ C(A_h, R_h)$.

ket. We formulate two properties. The first property requires that no agent on one side of the market should benefit from an increase of population in its own side. For $M = (D, H, X, R) \in \mathcal{M}$, we say that $M' = (D', H', X', R') \in \mathcal{M}$ is a *restriction* of M if $D' \subseteq D$, $H' \subseteq H$, $X' = X|_{D' \cup H'} \equiv \{x \in X :$ there are $d \in D'$ and $h \in H'$ such that $\mu(x) = (d, h)\}$, $R' = (R_i|_{2^{X'} \cap \mathcal{X}_i})_{i \in D' \cup H'}$. In particular, if $D' \neq D$ and $H' = H$, then M' is a D*-restriction* of M, and if $D' = D$ and $H' \neq H$, then M' is a H*-restriction* of M. Then,

Own-side population-monotonicity: *For each $M = (D, H, X, R) \in \mathcal{M}$, each $M' = (D', H', X', R') \in \mathcal{M}$, and each $A' \in \varphi(M')$, if M' is a D-restriction of M, then there is $A \in \varphi(M)$ such that, for each $d \in D'$, $A'_d \; R_d \; A_d$, and if M' is a H-restriction of M, then there is $A \in \varphi(M)$ such that, for each $h \in H'$, $A'_h \; R_h \; A_h$.*

The second property requires that no agent on one side of the market should lose from an increase of population in the other side.

Other-side population-monotonicity: *For each $M = (D, H, X, R) \in \mathcal{M}$, each $M' = (D', H', X', R') \in \mathcal{M}$, and each $A' \in \varphi(M')$, if M' is a D-restriction of M, then there is $A \in \varphi(M)$ such that, for each $h \in H'$, $A_h \; R_h \; A'_h$, and if M' is a H-restriction of M, then there is $A \in \varphi(M)$ such that, for each $d \in D'$, $A_d \; R_d \; A'_d$.*

Next, we consider *the consistency principle.* Suppose an allocation has been chosen by the solution for the problem we consider, and then a subset of agents leaves with what they are allocated. Now, consider the remaining agents and all resources they have collectively received, which forms the new problem. The solution is consistent if it recommends the same bundle to be allocated to each agent as initially. When applying the consistency principle, it is necessary to define a new problem with the remaining agents, which is called a reduced problem. Here, we consider two different ways of reducing a problem. We introduce two properties. The first property allows agents to leave in blocks only, in pair of sets originally matched. For $M = (D, H, X, R) \in \mathcal{M}$, $M' = (D', H', X', R') \in \mathcal{M}$ and $A \in \varphi(M)$, we say that $r^A_{D' \cup H'}(M)$ is the *type-1 reduced economy of M relative to $D' \cup H'$ at A* if:

- $D' \subseteq D$ and $H' \subseteq H$ are such that, for each $i \in D' \cup H'$, either $A_i = \emptyset$ or for each $x \in A_i$, there is no $j \in (D \cup H) \backslash (D' \cup H')$ with $\{x\} = A_i \cap A_j$,
- $X' = X|_{D' \cup H'} \equiv \{x \in X :$ there are $d \in D'$ and $h \in H'$ such that $\mu(x) = (d, h)\}$, and
- R' is such that, for each $i \in D' \cup H'$, we have $R'_i = R_i|_{X'_i}$.

Then,

Weak consistency: *For each $M = (D, H, X, R) \in \mathcal{M}$, each $M' = (D', H', X', R') \in \mathcal{M}$, and each $A \in \varphi(M)$, we have $A|_{D' \cup H'} \in \varphi(r^A_{D' \cup H'}(M))$.*

The second property allows any set of agents to leave unless an agent who has been already matched with some agents is left alone. For $M = (D, H, X, R) \in \mathcal{M}$, $M' = (D', H', X', R') \in \mathcal{M}$ and $A \in \varphi(M)$, we say that $r^A_{D' \cup H'}(M)$ is the *type-2 reduced economy of M relative to $D' \cup H'$ at A* if:

- $D' \subseteq D$ and $H' \subseteq H$ are such that, for each $i \in D' \cup H'$, either $A_i = \emptyset$ or for each $x \in A_i$, there is $j \in D' \cup H'$ with $\{x\} = A_i \cap A_j$,
- $X' = X|_{D' \cup H'} \equiv \{x \in X : \text{ there are } d \in D' \text{ and } h \in H' \text{ such that } \mu(x) = (d, h)\}$, and
- R' is such that, for each $i \in D' \cup H'$, we have $R'_i = R_i|_{X'_i}$.

Then,

Strong consistency: *For each $M = (D, H, X, R)$, each $M' = (D', H', X', R')$, and each $A \in \varphi(M)$, we have $A|_{D' \cup H'} \in \varphi(r^A_{D' \cup H'}(M))$.*

Next, we return to changes in preferences. Let $M = (D, H, X, R) \in \mathcal{M}$. For each $i \in D \cup H$, each $R_i \in \mathfrak{R}_i|_{X_i}$, and each $A \in \mathcal{A}(M)$, if, for each $X' \subseteq X$ with $A_i \; P_i \; X'$, we have $A_i \; P'_i \; X'$, then R'_i is a *Maskin-monotonic transformation of R_i at A_i*. Let $MT(R_i, A_i)$ be the set of all *Maskin-monotonic transformations of R_i at A_i*. Also, if for each $i \in D \cup H$, $R'_i \in MT(R_i, A_i)$, then R' is a *Maskin-monotonic transformation of R at A*. Let $MT(R, A)$ be the set of all *Maskin-monotonic transformations of R at A*. Now, we introduce a property which requires that a preference profile changes in such a way that the selected allocation has improved in the new preference profile, then the rule should select the same allocation when the new preference profile is announced.

Maskin-monotonicity: *For each* $M = (D, H, X, R) \in \mathcal{M}$*, each* $M' = (D, H, X, R')$*, and each* $A \in \varphi(M)$*, if* $R' \in MT(R, A)$*, then* $A \in \varphi(M')$*.*

Finally, we focus on the strategic issues. We assume that for each problem the solution is single-valued. First, we will consider the case where the hospital will destruct some of the contracts and be matched in the market according to the subset of contracts it had initially. A solution is *non-manipulable via destructing contracts* or *destruction-proof* if

Destruction-proofness: *For each* $M = (D, H, X, R) \in \mathcal{M}$*, each* $h \in H$*, and each* $X'_h \subseteq X_h$*, we have* $\varphi_h(D, H, X, R)\ R_h\ \varphi_h(D, H, X_{-h}, X'_h, R_{-h}, R_h|_{X'_h})$*.*

Next, we will consider the case where an agent behave strategically when announcing the preference. We require that each agent should find her assignment when she truthfully reveals her preference at least as desirable as her assignment when she misrepresents it. A solution is *strategy-proof* if

Strategy-proofness: *For each* $M = (D, H, X, R) \in \mathcal{M}$*, each* $i \in D \cup H$*, and each* $R'_i \in \mathfrak{R}_i$*, we have* $\varphi_i(D, H, X, R)\ R_i\ \varphi_i(D, H, X, R_{-i}, R'_i)$*.*

2.4 Solutions

The solutions identified next have played an important role in the literature. First, we consider the solution which chooses all *stable* matchings.

The stable solution: *For each $M \in \mathcal{M}$, we have $\varphi^S(M) = S(M)$.*

Second, we consider the doctor-optimal solution and hospital-optimal solutions. The doctor-optimal solution associates with each profile of preferences the stable allocation which is preferred by all doctors to all other stable allocations.

The doctor-optimal solution: *For each $M \in \mathcal{M}$, we have $\varphi^D(M) \in S(M)$ such that for each $d \in D$ and $A \in S(M)$, we have $\varphi^D(M)_d \; R_d \; A_d$.*

The doctors-proposing deferred acceptance algorithm of Gale and Shapley (1962) can be used to find the doctor-optimal solution. This algorithm can be outlined as:

Step 1: Each doctor d makes an offer to the firm with her best contract in X. Each hospital h that receives one or more offers holds the best set of contracts and rejects the rest. The algorithm terminates if no contract is rejected. Otherwise, doctors skip to the next step.

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Step t: Each doctor d whose contract was rejected at Step t-1 proposes to the hospital with the best acceptable contract to which she has not proposed before. Each hospital d holds the best set of contracts among the ones it receives at this step and the ones it was holding from the previous step. It rejects the rest. The algorithm terminates if no offer is rejected by any hospital. Otherwise, doctors skip to the next step.

When the algorithm terminates, the tentatively held contracts are realized as assignments.

Symmetrically, we can define the hospital-optimal solution. It associates with each profile of preferences the stable allocation which is preferred by all hospitals to all other stable allocations. The hospitals-proposing deferred acceptance algorithm can be used to find the hospital-optimal solution.

The hospital-optimal solution: *For each $M \in \mathcal{M}$, we have $\varphi^H(M) \in S(M)$ such that for each $h \in H$ and $A \in S(M)$, we have $\varphi^H(M)_h \; R_h \; A_h$.*

2.5 Results

First, we show that the stable solution satisfies the properties listed above.

Proposition 2.1. *The stable solution satisfies* unanimity*,* individual rationality*,* own-side population-monotonicity*,* other-side population-monotonicity*,* Maskin monotonicity*, and* weak consistency*.*

Next, we show that if any two agents' best choice is to match together, then they should match at each problem whenever the solution satisfies *unanimity* and *own-side population-monotonicity.*

Lemma 2.1. *Let φ be a rule satisfying* unanimity *and* own-side population-monotonicity*. Then, for each $M = (D, H, X, R) \in \mathcal{M}$, each $A \in \varphi(M)$, and each $h \in H$ such that for each $x \in C(X, R_h)$, there is $d \in D$ with $\{x\} = C(X, R_d)$, we have $A_h = C(X, R_h)$.*

The next proposition establishes a relationship between the properties. When we impose an efficiency property, a solidarity property and an incentive property, we get a necessary condition for stability.

Proposition 2.2. *If a rule φ satisfies* unanimity*,* own-side population-monotonicity*, and* Maskin-monotonicity*, then φ satisfies* weak individual rationality*.*

Proposition 2.3. *A rule φ satisfies* weak individual rationality*,* unanimity*,* own-side population-monotonicity*, and* Maskin-monotonicity *if and only if $\varphi = \varphi^S$.*

Using Propositions 2.1 and 2.3, we can now state our first result.

Theorem 2.1. *A rule φ satisfies* unanimity*,* own-side population-monotonicity*, and* Maskin-monotonicity *if and only if $\varphi = \varphi^S$.*

The axioms of Theorem 2.1 are independent of each other. The solution that chooses all possible matchings satisfies *own-side population-monotonicity*, *Maskin-monotonicity*, but not *unanimity*. The solution that chooses all Pareto-efficient matchings satisfies *unanimity*, *Maskin-monotonicity*, but not *own-side population-monotonicity*. The union of the doctor-optimal solution and the hospital-optimal solution satisfies *unanimity*, *own-side population-monotonicity* but not *Maskin-monotonicity*.

Next, we turn to *weak consistency* and show that a symmetric result to Proposition 2.2 holds when we impose *weak consistency* and *other-side population-monotonicity* instead of *Maskin-monotonicity*.

Proposition 2.4. *Let φ be a rule satisfying* weak individual rationality*,* unanimity*,* own-side population-monotonicity*,* other-side population-monotonicity*, and* weak consistency*. Then, $\varphi \subseteq \varphi^S$.*

The next proposition establishes a relationship between the properties. When we impose an efficiency property, a solidarity property and an invariance property, we get a necessary condition for stability.

Proposition 2.5. *Let φ be a rule satisfying* unanimity*,* own-side population-monotonicity*, and* weak consistency*. Then, it satisfies* weak individual rationality*.*

Using Propositions 2.1 and 2.4, we can now state our second result.

Theorem 2.2. *Let* φ *be a rule satisfying* unanimity*,* own-side population-monotonicity*,* other-side population-monotonicity*, and* weak consistency*. Then,* $\varphi \subseteq \varphi^S$.

Finally, we show that why we impose *weak consistency* rather than any other form of consistency. The following example shows that if we allow any subset of agents to leave, then a *stable* allocation restricted to the new economy is not a *stable* allocation.

Example 2.1. *Let* $M = (D, H, X, R)$ *be such that* $D = \{d_1, d_2, d_3\}$*,* $H = \{h_1, h_2\}$*,* $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$*, where* $\mu(x_1) = (d_1, h_1)$*,* $\mu(x_2) = (d_1, h_2)$*,* $\mu(x_3) = (d_2, h_1)$*,* $\mu(x_4) = (d_2, h_2)$*,* $\mu(x_5) = (d_3, h_1)$*,* $\mu(x_6) = (d_3, h_2)$*, and* R *be as follows:*

R_{d_1}	R_{d_2}	R_{d_3}	R_{h_1}	R_{h_2}
x_1	x_3	x_5	$\{x_1, x_3\}$	$\{x_2, x_6\}$
x_2	x_4	x_6	$\{x_3, x_5\}$	$\{x_6\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\{x_3\}$	$\emptyset$
			$\{x_5\}$	
			$\{x_1\}$	
			$\emptyset$	

Let $\varphi = \varphi^S$*. Consider the* stable *allocation* $A = (\{x_1\}, \{x_3\}, \{x_6\}, ; \{x_1, x_3\}, \{x_6\})$*.* *Let* $M' = (D', H', X', R')$ *be such that* $D' = \{d_3\}$*,* $H' = \{h_1, h_2\}$*,* $X' =$

$\{x_5, x_6\}$, *and* $R' = (R_i|_{X'})_{i \in D' \cup H'}$. *Then,* $A|_{D' \cup H'} = \{\{x_6\}; \emptyset, \{x_6\}\}$ *is not* stable *because* (d_3, h_1) *is a blocking pair, since* d_3 *prefers* x_5 *to* x_6 *and* h_1 *prefers* $\{x_5\}$ *to* $\emptyset$. *Let* $M'' = (D'', H'', X'', R'')$ *be such that* $D' = \{d_1, d_2, d_3\}$, $H' = \{h_2\}$, $X' = \{x_2, x_4, x_6\}$, *and* $R' = (R_i|_{X'})_{i \in D' \cup H'}$. *Then,* $A|_{D' \cup H'} = \{\emptyset, \emptyset, \{x_6\}; \{x_6\}\}$ *is not* stable *because* $(\{d_1, d_3\}, h_2)$ *is a blocking pair, since* d_1 *prefers* x_2 *to* $\emptyset$ *and* h_2 *prefers* $\{x_2, x_6\}$ *to* $\{x_6\}$.

The next example shows that the stable solution is not *strongly consistent.*

Example 2.2. *Let* $M = (D, H, X, R)$ *be such that* $D = \{d_1, d_2, d_3\}$, $H = \{h_1, h_2\}$, $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, *where* $\mu(x_1) = (d_1, h_1)$, $\mu(x_2) = (d_1, h_2)$, $\mu(x_3) = (d_2, h_1)$, $\mu(x_4) = (d_2, h_2)$, $\mu(x_5) = (d_3, h_1)$, $\mu(x_6) = (d_3, h_2)$, *and* R *be as follows:*

R_{d_1}	R_{d_2}	R_{d_3}	R_{h_1}	R_{h_2}
x_1	x_3	x_5	$\{x_1, x_3\}$	$\{x_6\}$
x_2	x_4	x_6	$\{x_3, x_5\}$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\{x_3\}$	
			$\{x_1\}$	
			$\{x_5\}$	
			$\emptyset$	

Let $A \equiv (\{x_1\}, \{x_3\}, \{x_6\}; \{x_1, x_3\}, \{x_6\})$. *Clearly,* $A \in \varphi^S(M)$. *Now, let* $M' = (D', H', X', R')$ *be such that* $D' = \{d_2, d_3\}$, $H' = \{h_1, h_2\}$, $X' =$

$\{x_3, x_4, x_5, x_6\}$, *and* $R' = (R_i|X')_{i\in D'\cup H'}$. *Clearly,* M' *is type-2 reduced economy of* M *relative to* $D' \cup H'$ *at* A. *Also,* $A|_{D'\cup H'} = \{\{x_3\}, \{x_6\}; \{x_3\}, \{x_6\}\} \notin S(M')$. *Indeed, for the pair of subsets* $(\{d_2, d_3\}, \{h_1\}) \subset D \times H$, *there is* $\{x_3, x_5\} \in \mathcal{X}_{h_1}$ *with* $|\{d_2, d_3\}| = |\{x_3, x_5\}|$ *and* $\{x_3, x_5\} \nsubseteq A_{h_1}$ *such that* $\{x_3\} = C(A_{d_2} \cup \{x_3\}, R_{d_2})$, $\{x_5\} = C(A_{d_3} \cup \{x_3\}, R_{d_3})$, *and* $\{x_3, x_5\} = C(A_{h_1} \cup \{x_3, x_5\}, R_{h_1})$. *Thus,* $(\{d_2, d_3\}, \{h_1\})$ *blocks* $A|_{D'\cup H'}$.

Next, consider the domain of separable preference which is a subdomain of substitutable preferences. The preference relation $\mathcal{R}_h$ is *separable* if, for each $X \in \mathcal{X}_h$ and each $\{x\} \in \mathcal{X}_h \backslash X$, we have $X \cup \{x\}\ \mathcal{R}_h\ X$ if and only if $\{x\}\ \mathcal{R}_h\ \emptyset$. We have the following result:

Proposition 2.6. *On the domain of separable preferences, the stable solution is* strongly consistent.

Next, we show that the doctor-optimal solution $\varphi^D(M)$ is not *destruction-proof.*

Proposition 2.7. *The doctor-optimal solution* $\varphi^D(M)$ *is not* destruction-proof.

Proof. Let $M = (D, H, X, R) \in \mathcal{M}$ be such that $D = \{d_1, d_2\}$, $H = \{h_1, h_2\}$, $X_{h_1} = \{x_{11}, x'_{11}, x_{21}\}$, $X_{h_2} = \{x_{21}, x_{22}\}$, $X = X_{h_1} \cup X_{h_2}$ where $\mu(x_{11}) = \mu(x'_{11}) = (d_1, h_1)$, $\mu(x_{12}) = (d_1, h_2)$, $\mu(x_{21}) = (d_2, h_1)$, $\mu(x_{22}) = (d_2, h_2)$, and R

be as follows:

$$\begin{array}{cccc} R_{d_1} & R_{d_2} & R_{h_1} & R_{h_2} \\ x_{11} & x_{21} & \{x'_{11}, x_{21}\} & \{x_{12}\} \\ x'_{11} & x_{22} & \{x_{11}, x_{21}\} & \{x_{22}\} \\ x_{12} & \emptyset & \{x'_{11}\} & \emptyset \\ \emptyset & & \{x_{11}\} & \\ & & \{x_{21}\} & \\ & & \emptyset & \end{array}$$

Then, $\varphi^D(M) = (x_{11}, x_{21}; \{x_{11}, x_{21}\}, \emptyset)$. Now assume that $X'_{h_1} = \{x'_{11}, x_{21}\}$ and consider $M' = (D, H, X', R') \in \mathcal{M}$ such that $X' = X'_{h_1} \cup X'_{h_2}$ and $R'_{h_1} = R_{h_1}|_{X'_{h_1}}$ and $R'_{h_2} = R_{h_2}$. Then, $\varphi^D(M') = (x'_{11}, x_{21}; \{x'_{11}, x_{21}\}, \emptyset)$ and $\varphi^D_{h_1}(M')\ R_{h_1}\ \varphi^D_{h_1}(M)$. □

Next, we state our results about immunity of solutions to strategic behavior.

Proposition 2.8. *The hospital-optimal solution $\varphi^F(M)$ is* destruction-proof.

Now, we state our result on *strategy-proofness*.

Theorem 2.3. *If the hospitals' preferences satisfy substitutes condition, law of aggregate demand, and top-dominance condition then a rule φ satisfies* stability *and* strategy-proofness *if and only if $\varphi = \varphi^D$.*

2.6 Conclusion

We looked for the solutions for matching with contracts that satisfy efficiency, solidarity and incentives requirements simultaneously under variable populations and preferences. We defined *population-monotonicity* and *consistency* axioms, and we also considered *Maskin-monotonicity.* Although *Maskin-monotonicity* is a requirement on variable preferences, whereas *weak consistency* and *other-side population-monotonicity* are requirements on variable populations, Theorems 2.1 and 2.2 show that *Maskin-monotonicity* plays the same role as *weak consistency* and *other-side population-monotonicity.* Also, an interesting question concerns the relationship between efficiency and solidarity conditions.

We are also interested in immunity of solutions to strategic behavior such as to misreporting the availability of contracts held by the hospitals, and misrepresenting preferences by doctors and hospitals. Although Theorem 2.3 seems to be a positive result, one should keep in mind that top dominance condition is a strong condition.

Chapter 3

Cost Allocation

3.1 Introduction

We consider the problem of sharing the cost of a public facility among agents who have different needs for it. An example is the so-called "irrigation problem". Ranchers are distributed along an irrigation ditch which they use jointly. A rancher only needs the part of the ditch from his field to the headgate. To accommodate all ranchers, the ditch should reach the furthest field from the headgate. How should the cost of maintaining the ditch be shared among the ranchers? A "rule" is a function that associates with each irrigation problem an allocation of the cost of maintaining the ditch, which we call a "contributions vector". For a comprehensive survey of this literature, initiated by Littlechild and Owen (1973), see Thomson (2005).

A well-known rule has been employed by ranchers in south-central Montana for over 100 years. For an empirical and axiomatic analysis on this subject, see Aadland and Kolpin (1998). It is the "sequential equal contributions" (SEC) rule[1], which works as follows. Imagine that the ditch is composed of "segments": the rancher closest to the headgate only needs the part of the ditch from the headgate to his field, the first segment; the second closest rancher needs the first segment and the part of the ditch from first segment to his field, second segment; and so on. All ranchers using a given segment contribute equally to the cost of the segment, and thus pay the total of the contributions of each segment that they use. Littlechild and Owen (1973) show that the contributions vector recommended by the SEC rule coincides with that prescribed by the "Shapley value" (Shapley, 1953) applied to the TU game associated to the problem in a natural way. Given an irrigation problem, we first transform the problem into a TU game, which we call the associated irrigation game, by defining the worth of each coalition as the cost of maintaining the ditch used by a rancher with the furthest field from the headgate in that coalition. We then apply a TU game solution to solve the game. This yields a payoff vector. Finally, we take this payoff vector as the contributions vector for the irrigation problem.

Our purpose here is to base axiomatic characterizations of the SEC rule

[1] The terminology we adopt is borrowed from Thomson (2005). Aadland and Kolpin (1998) refer to it as the serial cost-share rule.

on the following variable-population property, which is an application for irrigation problems of a general principle of "consistency". For a survey of the literature on *consistency* and its converse, see Thomson (2000). A number of authors have provided several axiomatic characterizations of the SEC rule. Readers are referred to Dubey (1982), Moulin and Shenker (1992), Aadland and Kolpin (1998), and Potters and Sudhölter (1999). Suppose that there are n ranchers indexed by $1, \ldots, n$ and rancher i's cost parameter c_i represents the cost of maintaining the irrigation ditch that rancher i uses. For simplicity, assume that $c_1 < \cdots < c_n$. Consider a contributions vector x chosen by a rule for the problem just defined. Imagine that rancher 1 pays his contribution x_1 and "leaves", and reassess the situation from the viewpoint of the remaining ranchers. It is of course natural to think of x_1 as a contribution to the part of the ditch that rancher 1 uses. Since contributing to the part of the ditch that rancher 1 uses implies contributing to the part of the ditch that all other ranchers use, the cost parameters of the remaining ranchers are then revised down by the amount x_1. *Smallest-cost consistency* (Potters and Sudhölter, 1999) of the rule requires that for the reduced problem just defined, each of the remaining ranchers should contribute the same amount as he did initially. When other rancher leaves, it is not easy to define the reduced problem. Thus, the benefit of a characterization is based on *smallest-cost consistency.* Potters and Sudhölter (1999) propose two types of consistency requirements for irrigation problems, "ν-consistency" and "ψ-consistency". Depending on which formation of a reduced problem is adopted, we are led to different consistency properties. However, when we focus on the departure of a rancher with the smallest cost parameter, *ν-consistency* and *ψ-consistency* coincide with *smallest-cost consistency*. Compared to *ν-consistency* and *ψ-consistency*, *smallest-cost consistency* is natural and no controversial at all.

In addition to *smallest-cost consistency*, we consider the following desirable properties. The first property is a symmetry property, "equal treatment of equals": two ranchers with the same cost parameters should contribute equal amounts. The second property is an independence property, "independence of followers": if the cost parameters of all ranchers other than the ranchers before a segment increase by the same positive amount, then the ranchers before the segment should contribute the same amounts as they did initially. The third one is a lower bound requirement on each rancher's contribution. Imagine, for each rancher separately, that his cost parameter is the smallest. We then divide his cost parameter by the numbers of ranchers. (Note that any segment the rancher uses is jointly used by all other ranchers. Thus, an equal share of the cost parameter of the rancher is very natural.) *Equal share lower bound* of the rule requires that each rancher should contribute at least as much as an equal share of his cost parameter. The last one is a monotonicity property, "cost monotonicity": if a rancher's cost parameter increases, all other ranchers should contribute at most as much as they did initially.

We establish two characterizations of the SEC rule: (i) it is the only rule satisfying *equal treatment of equals*, *independence of followers*, and *smallest-cost consistency*, and (ii) it is the only rule satisfying *equal share lower bound*, *cost monotonicity*, and *smallest-cost consistency.*

In Section 3.2, we formally introduce the model and the properties of solutions. Section 3.4 provides the results, proofs and the independence of axioms.

3.2 Model

There is a universe of "potential" agents, denoted by $\mathcal{I} \subsetneqq \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers. Let $\mathcal{N}$ be the class of non-empty and finite subsets of $\mathcal{I}$. Given $N \in \mathcal{N}$ and $i \in N$, let $c_i \in \mathbb{R}_+$ be agent$\boldsymbol{i}$'s cost parameter, and $c \equiv (c_i)_{i \in N}$ the profile of cost parameters. An irrigation problem for N, or simply a problem for $\boldsymbol{N}$, is a list $c \in \mathbb{R}^N_+$. Let $\mathcal{C}^N$ be the class of all problems for N. A contributions vector for $c \in \mathcal{C}^N$ is a vector $x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = \max_{i \in N} c_i$, a condition we call "efficiency", and for each $i \in N$, $0 \leq x_i \leq c_i$, a condition we call "reasonableness". Let $X(c)$ be the set of all contributions vectors for $c \in \mathcal{C}^N$. A rule is a function defined on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ that associates with each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$ a vector in $X(c)$. Let n denote the number of agents in N and $\eta : \{1, \ldots, n\} \to N$ be a bijection such that

$c_{\eta(1)} \leq \cdots \leq c_{\eta(n)}$. Thus, the agents in N are ordered in terms of their cost parameters. Note that if several agents have the same cost parameters, then the order is not unique. By convention, we assume that $N \equiv \{1, \ldots, n\}$ and $c_1 \leq \cdots \leq c_n$. Our generic notation for rules is S. For each coalition $N' \subset N$, we denote $(c_i)_{i \in N'}$ by $c_{N'}$, $(S_i(c))_{i \in N'}$ by $S_{N'}(c)$, and so on. The terminology we adopt in this paper is borrowed from Thomson (2005)

3.3 Sequential equal contributions rule and properties of rules

We now introduce the sequential equal contributions rule.

Sequential equal contributions rule, SEC: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$,

$$SEC_i(c) \equiv \frac{c_1}{n} + \frac{c_2 - c_1}{n-1} + \cdots + \frac{c_i - c_{i-1}}{n-i+1}.$$

The SEC rule satisfies the following properties informally defined in the introduction.

Equal treatment of equals: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each pair $\{i, j\} \subseteq N$, if $c_i = c_j$, then $S_i(c) = S_j(c)$.

Equal share lower bound: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$, $S_i(c) \geq \frac{c_i}{n}$.

Independence of followers: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, each $c' \in \mathcal{C}^N$, each $i \in N$, and each $\delta \geq 0$, if for each $j \in \{1, \ldots, i\}$, $c'_j = c_j$ and for each $j \in N \backslash \{1, \ldots, i\}$, $c'_j = c_j + \delta$, then for each $j \in \{1, \ldots, i\}$, $S_j(c) = S_j(c')$.[2]

Cost monotonicity: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, each $c' \in \mathcal{C}^N$, and $i \in N$, if $c'_i \geq c_i$ and for each $j \in N \backslash \{i\}$, $c'_j = c_j$, then for each $j \in N \backslash \{i\}$, $S_j(c') \leq S_j(c)$.[3]

Next is the central property to our analysis. Let $N \in \mathcal{N}$, $c \in \mathcal{C}^N$, $x \in X(c)$, and $i^* \in \{i \in N \mid \text{for each } k \in N,\, c_i \leq c_k\}$. The reduced problem of $\boldsymbol{c}$ with respect to $\boldsymbol{N' \equiv N \backslash \{i^*\}}$ and $\boldsymbol{x}$, $r^x_{N'}(c)$, is defined by setting for each $j \in N'$,

$$\left(r^x_{N'}(c)\right)_j \equiv c_j - x_{i^*}.$$

Smallest-cost consistency: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $N' \subset$

[2] A weaker version of *independence of followers* can be obtained by restricting attention to an agent with the largest cost parameter. This new property together with *efficiency* implies an additivity property, "last-agent cost additivity": if the cost parameter of an agent with the largest cost parameter increases by a positive amount, the agent's contribution should increase by an equal amount. A stronger version of *independence of followers*, "independence of at-least-as-large costs" (Moulin and Shenker, 1992), can be obtained by only requiring that the cost parameters of those agents with larger cost parameters than agent i's increase by a positive amount rather than an equal amount.

[3] This property is introduced by Thomson (2005). The property is a complement of "individual cost monotonicity" (Potters and Sudhölter, 1999), which is defined as follow: under the same hypotheses, agent i should pay at least as much as he did initially.

N, if $x \equiv S(c)$, then $r^x_{N'}(c) \in \mathcal{C}^{N'}$ and $x_{N'} = S(r^x_{N'}(c))$.

Remark 1: *Reasonableness* and *smallest-cost consistency* together imply *efficiency.*

3.4 Results

Our first result is that the SEC rule is the only rule satisfying *equal treatment of equals*, *independence of followers*, and *smallest-cost consistency.* To prove this characterization, we use the fact that the SEC rule satisfies the following monotonicity property. If the cost parameters of all agents increase by the same positive amount, each rancher should contribute at least as much as he did initially.

Uniform-cost-increase monotonicity: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, each $c' \in \mathcal{C}^N$, and each $\delta > 0$, if for each $i \in N$, $c'_i = c_i + \delta$, then for each $i \in N$, $S_i(c) \leq S_i(c')$.

Since the fact that the SEC rule satisfies *uniform-cost-increase monotonicity* is an immediate consequence of the definition of the SEC rule, we omit its proof. We are now ready to prove the announced assertion.

Theorem 3.1. *The SEC rule is the only rule satisfying* equal treatment of equals, independence of followers, *and* smallest-cost consistency.

Proof. Clearly, the SEC rule satisfies the three properties above.[4] Conversely, let S be a rule satisfying the properties. Without loss of generality, let $N \equiv \{1, \ldots, n\}$ and suppose that $c_1 \leq \cdots \leq c_n$.[5] Let $x \equiv S(c)$ and $y \equiv SEC(c)$. We show that $x = y$. The proof is by induction on n.

Case 1: $\mathbf{n = 1}$. By *efficiency* of the rule, $x = y$.

Case 2: $\mathbf{n > 1}$. The induction hypothesis is that for each $N' \in \mathcal{N}$ and $c^* \in \mathcal{C}^{N'}$ with $N' \subset N$ and $|N'| \leq n - 1$, we have $S(c^*) = SEC(c^*)$. We first show that $x_1 = y_1$. By *smallest-cost consistency* and the induction hypothesis, we then conclude that $x = y$. Let $N' \equiv N \backslash \{1\}$. We distinguish two cases.

Subcase 2.1: $\mathbf{c_1 = c_2}$. Suppose, by contradiction, that $x_1 \neq y_1$. Thus, either $x_1 > y_1$ or $x_1 < y_1$. If $x_1 > y_1$, then by *equal treatment of equals*, $x_2 > y_2$. By *smallest-cost consistency*, $x_2 = S_2(r^x_{N'}(c))$ and $y_2 = SEC_2(r^y_{N'}(c))$. Since $x_1 > y_1$, then for each $i \in N'$, $(r^x_{N'}(c))_i \leq (r^y_{N'}(c))_i$. Note that $|N'| < n$. By the induction hypothesis, $S_2(r^x_{N'}(c)) = SEC_2(r^x_{N'}(c))$. Since for each $i \in N'$, $(r^y_{N'}(c))_i - (r^x_{N'}(c))_i = x_1 - y_1 > 0$, the two reduced problems $r^x_{N'}(c)$ and $r^y_{N'}(c)$ satisfy the hypotheses of *uniform-cost-increase monotonicity*. Since the SEC rule satisfies *uniform-cost-increase monotonicity*, it follows that $x_2 \leq y_2$,

[4] In the proof, we only use a weaker version of *independence of followers* obtained by restricting attention to the situation when $i = 2$, and a weaker version of *uniform-cost-increase monotonicity* obtained by restricting attention to an agent with the smallest cost parameter.

[5] Note that if several agents have the same cost parameters, then of course the order is not unique. However, our Theorem 1 and the next characterization of the SEC rule do not rely on any particular ordering of agents.

in violation of $x_2 > y_2$. If $x_1 < y_1$, then by a similar argument, we derive the desired contradiction.

Subcase 2.2: $\mathbf{c_1 < c_2}$. Let c' be such that $c'_1 = c_1$ and for each $i \in N'$, $c'_i \equiv c_i - (c_2 - c_1)$. Let $x' \equiv S(c')$ and $y' \equiv SEC(c')$. Note that $c'_1 = c'_2$. By Subcase 2.1, $x'_1 = y'_1$. By *independence of followers*, $x'_1 = x_1$ and $y'_1 = y_1$. Thus, $x_1 = y_1$. □

Yeh (2006) shows that the "nucleolus" is the only rule satisfying *equal treatment of equals*, *independence of followers*, and "largest-cost consistency"[6]. This result reveals the interest of focusing on an agent with the largest cost parameter in characterizing the nucleolus. In contrast to Yeh's result, our Theorem 1 reveals the interest of focusing on an agent with the smallest cost parameter in characterizing the SEC rule.

Our second result is another characterization of the SEC rule on the basis of *smallest-cost consistency.*

Theorem 3.2. *The SEC rule is the only rule satisfying equal share lower bound, cost monotonicity, and smallest-cost consistency.*

[6] It is a weaker version of ν-*consistency* (Potters and Sudhölter, 1999) obtained by restricting attention to the departure of an agent with the largest cost parameter.

Proof. Clearly, the SEC rule satisfies *equal share lower bound*, *cost monotonicity*, and *smallest-cost consistency.* Conversely, let S be a rule satisfying the three properties. Without loss of generality, let $N \equiv \{1, \dots, n\}$ and suppose that $c_1 \leq \cdots \leq c_n$. Let $x \equiv S(c)$ and $y \equiv SEC(c)$. We show that $x = y$. The proof is by induction on n.

Case 1: $\mathbf{n = 1}$. By *efficiency* of the rule, $x = y$.

Case 2: $\mathbf{n > 1}$. The induction hypothesis is that for each (N', c^*) such that $N' \subset N$ and $|N'| \leq n - 1$, we have $S(c^*) = SEC(c^*)$. We first show that $x_1 = y_1$. By *smallest-cost consistency* and the induction hypothesis, we then conclude that $x = y$. By *equal share lower bound*, $x_1 \geq \frac{c_1}{n}$. Suppose that $x_1 > \frac{c_1}{n}$. Let $\bar{c} \in \mathcal{C}^N$ be such that for each $i \in N \backslash \{1\}$, $\bar{c}_i = c_1$. Let $\bar{x} \equiv S(\bar{c})$. By *equal share lower bound* and *efficiency*, $\bar{x}_1 = \frac{c_1}{n}$. Now, let $\bar{c}^n \in \mathcal{C}^N$ be such that $\bar{c}^n_n = c_n$ and for each $i \in N \backslash \{n\}$, $\bar{c}^n_i = \bar{c}_i$. Let $\bar{x}^n \equiv S(\bar{c}^n)$. By *cost monotonicity*, $\bar{x}^n_1 \leq \bar{x}_1$. Let $\bar{c}^{n-1} \in \mathcal{C}^N$ be such that $\bar{c}^{n-1}_{n-1} = c_{n-1}$ and for each $i \in N \backslash \{n-1\}$, $\bar{c}^{n-1}_i = \bar{c}^n_i$. Let $\bar{x}^{n-1} \equiv S(\bar{c}^{n-1})$. By *cost monotonicity*, $\bar{x}^{n-1}_1 \leq \bar{x}^n_1$. Continuing this process, we have $\bar{c}^2 \equiv c$ and $\bar{x}^2_1 \leq \bar{x}^3_1$. Since $x = \bar{x}^2$, we have $x_1 = \bar{x}^2_1 \leq \bar{x}^3_1 \leq \cdots \leq \bar{x}^n_1 = \frac{c_1}{n}$, which contradicts to the assumption of $x_1 > \frac{c_1}{n}$. Thus, $x_1 = \frac{c_1}{n}$. □

The following paragraphs establish the independence of axioms in Theorem 3.1 and Theorem 3.2. To show this, we consider several rules. The first rule is the constrained egalitarian rule (Aadland and Kolpin, 1998).[7] Start by requiring equal contributions from all agents in N until there are $\gamma^1 \in \mathbb{R}_+$ and $k^1 \in N$ such that $k^1\gamma^1 = c_{k^1}$ (if there are several such k^1, select the largest). Then, each $i \in \{1, ..., k^1\}$ pays γ^1. Continue by requiring equal contributions from members of $\{k^1+1, ..., n\}$ until there are $\gamma^2 \in \mathbb{R}_+$ and $k^2 \in N$ such that $k^1\gamma^1 + (k^2 - k^1)\gamma^2 = c_{k^2}$ (if there are several such k^2, select the largest). Then each $i \in \{k^1+1, ..., k^2\}$ pays γ^2. Continue in this way until the total amount collected is c_n. This algorithm can be expressed as follows.

Constrained Egalitarian rule, $\boldsymbol{CE}$: For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$,

$$
\begin{aligned}
&CE_1(c) \equiv \min\left\{\tfrac{c_1}{1}, \cdots, \tfrac{c_n}{n}\right\}\\
&CE_i(c) \equiv \min\left\{\tfrac{c_k - \sum_{p=1}^{i-1} CE_p(c)}{k-i+1} \mid i \le k \le n\right\} 2 \le i \le n-1\\
&CE_n(c) \equiv c_n - \textstyle\sum_{p=1}^{n-1} CE_p(c).
\end{aligned}
$$

The second rule makes one of the agents with the largest cost parameter pay the entire cost (Potters and Sudhölter, 1999). Let Π denote the class of strict and complete order on $\mathcal{N}$, with generic element $\prec$. The notation $i \prec j$ means that agent i has priority over agent j. Given $N \in \mathcal{N}$

[7] Aadland and Kolpin (1998) name this rule as the restricted average cost-share rule. Aadland and Kolpin show that the contributions vector the rule chooses coincides with that prescribed by the "egalitarian rule" (Dutta and Ray, 1989) applied to the associated airport game.

and $c \in \mathcal{C}^N$, let $m(c) \equiv \{j \in N \,|\, \text{for each } k \in N\backslash\{j\}, c_j \geq c_k\}$ and $d_N(c) \equiv \{\, k \in m(c) \;|\; \text{for each } j \in m(c),\, k \prec j\}$.

Last-agent rule, ***LA***: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$,

$$LA_i\,(c) \equiv \begin{cases} 0 \text{ if } i \neq d_N(c); \\ c_i \text{ otherwise.} \end{cases}$$

The next rule is a "modified sequential equal contributions" rule. When there are three agents and their cost parameters differ, the rule assigns each agent an equal share of the smallest cost parameter plus the difference between his cost parameter and the cost parameter of his immediate predecessor; otherwise, the rule assigns agents the contributions made by the sequential equal contributions rule.

Modified sequential equal contributions rule, ***SEC****: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$,

$$SEC_i^*\,(c) \equiv \begin{cases} S_i\,(c) \quad \text{if } |N| = 3 \text{ and } c_1 < c_2 < c_3; \\ SEC_i\,(c) \text{ otherwise,} \end{cases}$$

where S is defined as follows: Let $N \equiv \{1, 2, 3\}$.

$$S_1(c) = \frac{c_1}{3}$$
$$S_2(c) = \frac{c_1}{3} + c_2 - c_1$$
$$S_3(c) = \frac{c_1}{3} + c_3 - c_2$$

The last rule assigns each agent an equal share of his cost parameter, and then one of the agents with the largest cost parameter pays the remaining amount to be collected.

$\boldsymbol{S^*}$: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$,

$$S_i^*(c) \equiv \begin{cases} \frac{c_i}{n} & \text{if } i \in N \setminus \{d_N(c)\}; \\ c_{d_N(c)} - \sum_{j \in N \setminus \{d_N(c)\}} \frac{c_j}{n} & \text{otherwise,} \end{cases}$$

Table 3.1 shows that the properties listed in Theorems 3.1 and 3.2 are independent.[8] For instance, the last-agent rule satisfies *independence of followers* and *smallest-cost consistency* but violates *equal treatment of equals*. The constrained egalitarian rule satisfies *equal treatment of equals* and *smallest-cost consistency* but violates *independence of followers*. The modified sequential

[8]The notation "+" ("−") means that a certain rule satisfies (violates) a certain property.

Property/Rule	CE	LA	SEC^*	S^*	SEC
Equal treatment of equals	+	−	+	−	+[1]
Independence of followers	−	+	+	+	+[1]
Smallest-cost consistency	+	+	−	−	+[1,2]
Equal share lower bound	+	−	−	+	+[2]
Cost monotonicity	−	+	−	+	+[2]

Table 3.1: Independence of the properties in Theorems 3.1 and 3.2.

equal contributions rule satisfies *equal treatment of equals* and *independence of followers* but violates *smallest-cost consistency*.

3.5 Conclusion

Our objective was to identify allocation rules for irrigation problems that satisfy normative requirements simultaneously. We proved that the sequential equal contributions rule is the only rule satisfying *equal treatment of equals*, *independence of predecessors*, and *smallest-cost consistency*, and it is the only rule satisfying *equal share lower bound*, *cost monotonicity*, and *smallest-cost consistency*. These results provide justification for the usage of the sequential equal contributions rule by ranchers in south-central Montana for over 100 years. The open question is to consider strategic issues in irrigation problems.

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Appendix

Appendix A

Proof of Theorem 1.1.

Let φ be a *single-valued* rule. Then,

If part:

Efficiency of queues: Let φ be a Groves rule. Let $c \in \mathcal{C}$ and $(\sigma, t) = \varphi(c)$. Then, by definition of a Groves rule, there is $d \in D$ such that $\sigma = d(c) \in Q^*(c)$.

Strategy-proofness: Let φ be a Groves rule. Let $c \in \mathcal{C}$, $i \in N$, $c_i' \in \mathbb{R}_+$, $(\sigma, t) = \varphi(c)$, and $(\sigma', t') = \varphi(c_i', c_{-i})$. By definition of a Groves rule, there is $d \in D$ such that $\sigma = d(c) \in Q^*(c)$. Also, there is $h \in H$ such that $t_i = -\sum_{l \in N\setminus\{i\}}(\sigma_l - 1)c_l + h_i(c_{-i})$ and $t_i' = -\sum_{l \in N\setminus\{i\}}(\sigma_l' - 1)c_l + h_i(c_{-i})$. By contradiction, suppose $u_i(\sigma_i', t_i') > u_i(\sigma_i, t_i)$. Then, $-(\sigma_i' - 1)c_i - \sum_{l \in N\setminus\{i\}}(\sigma_l' - 1)c_l + h_i(c_{-i}) > -(\sigma_i - 1)c_i - \sum_{l \in N\setminus\{i\}}(\sigma_l - 1)c_l + h_i(c_{-i})$. Thus, $-\sum_{l \in N}(\sigma_l' - 1)c_l > -\sum_{l \in N}(\sigma_l - 1)c_l$, contradicting $\sigma \in Q^*(c)$.

Only if part: Let φ be a rule satisfying *efficiency of queues* and *strategy-proofness*. Then, by *efficiency of queues*, for each $c \in \mathcal{C}$, if $(\sigma, t) = \varphi(c)$, then $\sigma \in Q^*(c)$. Thus, there is $d \in D$ such that for each $c \in \mathcal{C}$, if $(\sigma, t) = \varphi(c)$, then $\sigma = d(c)$. In what follows, we prove that there is $h \in H$ such that for each $c \in \mathcal{C}$, if $(\sigma, t) = \varphi(c)$, then for each $i \in N$, $t_i = -\sum_{l \in N\setminus\{i\}}(\sigma_l - 1)c_l + h_i(c_{-i})$. Let $c \in \mathcal{C}$, $i \in N$, and $g_i : \mathbb{R}^N_+ \to \mathbb{R}$ be a real-valued function such that ***(i)*** if $(\sigma, t) = \varphi(c)$, then $t_i = -\sum_{l \in N\setminus\{i\}}(\sigma_l - 1)c_l + g_i(c)$. By contradiction, suppose that $c'_i \in \mathbb{R}_+$, we have ***(ii)*** $g_i(c) - g_i(c'_i, c_{-i}) > 0$. (The symmetric case is immediate.) Let $(\sigma, t) = \varphi(c)$ and $(\sigma', t') = \varphi(c'_i, c_{-i})$. By *strategy-proofness*, the following inequalities hold:

- $u_i(\sigma_i, t_i) - u_i(\sigma'_i, t'_i) \geq 0.$
- $u'_i(\sigma'_i, t'_i) - u'_i(\sigma_i, t_i) \geq 0.$

By ***(i)***,

$[-(\sigma_i - 1)c_i - \sum_{l \in N\setminus\{i\}}(\sigma_l - 1)c_l + g_i(c)] - [-(\sigma'_i - 1)c_i - \sum_{l \in N\setminus\{i\}}(\sigma'_l - 1)c_l + g_i(c'_i, c_{-i})] \geq 0.$

Thus, $g_i(c) - g_i(c'_i, c_{-i}) \geq (\sigma_i - \sigma'_i)c_i + \sum_{l \in N\setminus\{i\}}(\sigma_l - \sigma'_l)c_l$.

By ***(i)***,

$[-(\sigma'_i - 1)c'_i - \sum_{l \in N\setminus\{i\}}(\sigma'_l - 1)c_l + g_i(c'_i, c_{-i})] - [-(\sigma_i - 1)c'_i - \sum_{l \in N\setminus\{i\}}(\sigma_l - 1)c_l + g_i(c)] \geq 0.$

Thus, $g_i(c'_i, c_{-i}) - g_i(c) \geq (\sigma'_i - \sigma_i)c'_i + \sum_{l \in N\setminus\{i\}}(\sigma'_l - \sigma_l)c_l$.

Altogether,

(iii) $(\sigma_i - \sigma'_i)c'_i + \sum_{l \in N\setminus\{i\}}(\sigma_l - \sigma'_l)c_l \geq g_i(c) - g_i(c'_i, c_{-i}) \geq (\sigma_i - \sigma'_i)c_i + \sum_{l \in N\setminus\{i\}}(\sigma_l - \sigma'_l)c_l$.

Let us rewrite this expression. By *efficiency of queues*, for each $S \subseteq N$, if for each $\{k, k'\} \subseteq S$ with $k \neq k'$, we have $c_k = c_{k'}$ and there is no $k'' \in N\setminus S$ such that $k'' \in B_{kk'}(\sigma) \cup B_{kk'}(\sigma')$, then $-\sum_{l \in S}(\sigma_l - 1)c_l = -\sum_{l \in S}(\sigma'_l - 1)c_l$. Also, there is $j \in N$ such that $\sigma_j = \sigma'_i$. Thus, $\sum_{l \in N\setminus\{i\}}(\sigma_l - \sigma'_l)c_l = -sign(\sigma_i - \sigma'_i)\sum_{l \in B_{ij}(\sigma)\cup\{j\}} c_l$.[9] Thus, we may rewrite ***(iii)*** as

(iv) $(\sigma_i - \sigma'_i)c'_i - sign(\sigma_i - \sigma'_i)\sum_{l \in B_{ij}(\sigma)\cup\{j\}} c_l \geq g_i(c) - g_i(c'_i, c_{-i}) \geq (\sigma_i - \sigma'_i)c_i - sign(\sigma_i - \sigma'_i)\sum_{l \in B_{ij}(\sigma)\cup\{j\}} c_l$.

Then, we distinguish three cases:

Case 1: $(\sigma_i - \sigma'_i) = 0$. Then, $-sign(\sigma_i - \sigma'_i)\sum_{l \in B_{ij}(\sigma)\cup\{j\}} c_l = 0$. Thus, by ***(iv)***, $g_i(c) - g_i(c'_i, c_{-i}) = 0$ contradicting ***(ii)***.

Case 2: $|\sigma_i - \sigma'_i| = 1$. Suppose $c'_i > c_i$. (The symmetric case is immediate.) Then, $(\sigma_i - \sigma'_i) = 1$ and $-sign(\sigma_i - \sigma'_i)\sum_{l \in B_{ij}(\sigma)\cup\{j\}} c_l = -c_j$. Thus, by ***(iv)***, $c'_i - c_j \geq g_i(c) - g_i(c'_i, c_{-i}) \geq c_i - c_j$. Thus, as $c'_i > c_i$, either $c'_i - c_j > g_i(c) - g_i(c'_i, c_{-i})$ or $g_i(c) - g_i(c'_i, c_{-i}) > c_i - c_j$. Suppose $g_i(c) - g_i(c'_i, c_{-i}) > c_i - c_j$. (The other case is also immediate.) Let $c''_i \in \mathbb{R}_+$ be such that ***(v)***$g_i(c) - g_i(c'_i, c_{-i}) > c''_i - c_j > 0$. Let $(\sigma'', t'') = \varphi(c''_i, c_{-i})$. By ***(iv)*** and ***(v)***, $c'_i > c''_i > c_j > c_i$.

[9] For each $a \in \mathbb{R}$, let $sign(a) = -1$ if and only if $a < 0$, $sign(a) = 0$ if and only if $a = 0$, and $sign(a) = 1$ if and only if $a > 0$.

Thus, by *efficiency of queues*, $\sigma''_i = \sigma'_i$. Thus, $(\sigma_i - \sigma''_i) = (\sigma_i - \sigma'_i) = 1$ and $\sum_{l\in N\setminus\{i\}}(\sigma_l - \sigma''_l)c_l = \sum_{l\in N\setminus\{i\}}(\sigma_l - \sigma'_l)c_l = -c_j$. Also, by the logic of Case 1, $g_i(c''_i, c_{-i}) = g_i(c'_i, c_{-i})$, implying $g_i(c) - g_i(c''_i, c_{-i}) = g_i(c) - g_i(c'_i, c_{-i})$. Thus, by $\boldsymbol{(v)}$, $g_i(c) - g_i(c''_i, c_{-i}) > (\sigma_i - \sigma''_i)c''_i + \sum_{l\in N\setminus\{i\}}(\sigma_l - \sigma''_l)c_l$. Thus, $-(\sigma_i - 1)c''_i - \sum_{l\in N\setminus\{i\}}(\sigma_l - 1)c_l + g_i(c) > -(\sigma''_i - 1)c''_i - \sum_{l\in N\setminus\{i\}}(\sigma''_l - 1)c_l + g_i(c''_i, c_{-i})$. Thus, by $\boldsymbol{(i)}$, $u''_i(\sigma_i, t_i) > u''_i(\sigma''_i, t''_i)$, contradicting *strategy-proofness.*

Case 3: $|\sigma_i - \sigma'_i| > 1$. By the logic of Case 2, starting from σ'_i, we can find $\tilde{c}_i$ such that $\tilde{\sigma}_i$ is one position closer to σ_i. We continue by one position at a time and at each step we obtain $g_i(c) = g_i(\tilde{c}_i, c_{-i})$. Thus, $g_i(c) = g_i(c'_i, c_{-i})$ contradicting $\boldsymbol{(ii)}$. $\square$

Proof of Proposition 1.1.

Let φ be a *single-valued* rule. Let $c \in \mathcal{C}$, $(\sigma, t) = \varphi(c)$, and $i \in N$. Let $h \in H$ be as in Statement 2. Then,

$$
\begin{aligned}
t_i &= -\textstyle\sum_{j\in N\setminus\{i\}} \sum_{l\in\{i,j\}\cap F_i(\sigma)} c_l + \frac{1}{(n-2)} \sum_{j\in N\setminus\{i\}} \sum_{k\in N\setminus\{i,j\}} \sum_{l\in\{j,k\}\cap F_j(\sigma)} c_l \\
&= -\textstyle\sum_{l\in F_i(\sigma)} c_l + \frac{1}{(n-2)} \sum_{j\in N\setminus\{i\}} \sum_{l\in F_j(\sigma^{-i})} c_l \\
&= -\textstyle\sum_{l\in F_i(\sigma)} c_l + \frac{1}{(n-2)} \sum_{l\in N\setminus\{i\}} (\sigma_l^{-i} - 1)c_l \\
&= -\textstyle\sum_{l\in N\setminus\{i\}}(\sigma_l - 1)c_l + \sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l + \frac{1}{(n-2)} \sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l \text{ (a Groves rule)} \\
&= -\textstyle\sum_{l\in F_i\sigma} c_l + \frac{1}{n-2} \sum_{l\in P_i(\sigma)}(\sigma_l - 1)c_l + \frac{1}{(n-2)} \sum_{l\in F_i(\sigma)}(\sigma_l - 2)c_l \\
&= \textstyle\sum_{l\in P_i(\sigma)} \frac{(\sigma_l - 1)}{(n-2)} c_l + \sum_{l\in F_i(\sigma)} \frac{(\sigma_l - 2) - (n-2)}{(n-2)} c_l \\
&= \textstyle\sum_{l\in P_i(\sigma)} \frac{(\sigma_l - 1)}{(n-2)} c_l - \sum_{l\in F_i(\sigma)} \frac{(n - \sigma_l)}{(n-2)} c_l \text{ (rule in Mitra and Sen, 1998, and Mitra, 2001)} \\
&= \textstyle\sum_{l\in P_i(\sigma)} \frac{c_l}{2} - \sum_{l\in F_i(\sigma)} \frac{c_l}{2} - \sum_{l\in N\setminus\{i\}} \frac{(n - 2\sigma_l)c_l}{2(n-2)}
\end{aligned}
$$

$= \sum_{l \in P_i(\sigma)} \frac{c_l}{2} - \sum_{l \in F_i(\sigma)} \frac{c_l}{2} - [\sum_{l \in N\setminus\{i\}} \frac{(n-\sigma_l-1)c_l}{2(n-2)} - \sum_{l \in N\setminus\{i\}} \frac{(\sigma_l-1)c_l}{2(n-2)}]$

$= \sum_{l \in P_i(\sigma)} \frac{c_l}{2} - \sum_{l \in F_i(\sigma)} \frac{c_l}{2} - \sum_{l \in N\setminus\{i\}} \sum_{k \in P_l(\sigma)\setminus\{i\}} \frac{c_k - c_l}{2(n-2)}$ (rule in Suijs, 1996).

□

Proof of Theorem 1.2.

Let φ be a *single-valued* rule. Then,

If part: Let φ be a rule satisfying the axioms in the first statement of Theorem 1.2. Let $c \in \mathcal{C}$ and $(\sigma, t) = \varphi(c)$. Then, by *Pareto-efficiency*, $\sigma \in Q^*(c)$. By Theorem 1.1, *Pareto-efficiency* and *strategy-proofness* imply that φ is a Groves rule, i.e., there is $(h_i)_{i \in N} \in H$ such that for each $i \in N$, $t_i = -\sum_{l \in N\setminus\{i\}}(\sigma_l - 1)c_l + h_i(c_{-i})$. For $(\gamma_i)_{i \in N} \in H$ such that $t_i = -\sum_{l \in F_i(\sigma)} c_l + \gamma_i(c_{-i})$. In what follows, we prove by induction that $\gamma_i(c_{-i}) = \frac{1}{(n-2)} \sum_{l \in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l$. Then, $t_i = -\sum_{l \in N\setminus\{i\}}(\sigma_l - 1)c_l + \sum_{l \in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l + \frac{1}{(n-2)} \sum_{l \in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l$. Thus, by Proposition 1.1,

$t_i = -\sum_{j \in N\setminus\{i\}} \sum_{l \in \{i,j\} \cap F_i(\sigma)} c_l + \frac{1}{(n-2)} \sum_{j \in N\setminus\{i\}} \sum_{k \in N\setminus\{i,j\}} \sum_{l \in \{j,k\} \cap F_j(\sigma)} c_l.$

Without loss of generality, suppose $N = \{1, 2, ..., n\}$ and $c_1 \geq c_2 \geq ... \geq c_n$. Let $i \in N$. Then,

Basis Step: $c = (c_n, ..., c_n)$.

By *Pareto-efficiency*, $\gamma_1(c_n, ..., c_n) + ... + \gamma_n(c_n, ..., c_n) = \frac{n(n-1)}{2} c_n$. By *equal treatment of equals in welfare*, $\gamma_1(c_n, ..., c_n) = ... = \gamma_n(c_n, ..., c_n)$. Thus, for each $j \in N$, $\gamma_j(c_n, ..., c_n) = \frac{(n-1)}{2} c_n$.

Step 1: $c = (c_1, c_n, ..., c_n)$.

By *Pareto-efficiency*, $\gamma_1(c_n, ..., c_n) + \gamma_2(c_1, c_n, ..., c_n) + ... + \gamma_n(c_1, c_n, ..., c_n) = \frac{(n-1)n}{2}c_n$. By the basis step, $\gamma_1(c_n, ..., c_n) = \frac{(n-1)}{2}c_n$. By *equal treatment of equals in welfare*, $\gamma_2(c_1, c_n, ..., c_n) = ... = \gamma_n(c_1, c_n, ..., c_n)$. Thus, for each $j \in N\backslash\{1\}$, we have $\gamma_j(c_1, c_n, ..., c_n) = \frac{(n-1)}{2}c_n$. This holds for each $k \in N\backslash\{n\}$. Thus, for each $j \in N$:

- if $j = k$, then $\gamma_j(c_n, ..., c_n) = \frac{(n-1)}{2}c_n$;
- if $j \in N\backslash\{k\}$, then $\gamma_j(c_k, c_n, ..., c_n) = \frac{(n-1)}{2}c_n$.

$\vdots$

Step *s*: (Induction step) $c = (c_1, c_2, ..., c_s, c_n, ..., c_n)$.

By *Pareto-efficiency*, $\gamma_1(c_2, c_3, ..., c_s, c_n, ..., c_n) + \gamma_2(c_1, c_3, ..., c_s, c_n, ..., c_n) + ... + \gamma_n(c_1, c_2, ..., c_s, c_n, ..., c_n)$
$= \sum_{l\in\{1,2,...,s\}}(\sigma_l - 1)c_l + \frac{(n-s)(n+s+1)}{2}c_n$.

By Step $s-1$, for $j \in \{1, 2, ..., s\}$, we have
$\gamma_j(c_1, c_2, ..., c_s, c_n, ..., c_n) = \sum_{l\in\{1,2,...,s\}\backslash\{j\}} \frac{(\sigma_l^{\{1,2,...,s\}\backslash\{j\}}-1)}{(n-2)}c_l + \frac{(n-1-(s-1))(n-2+(s-1))}{2(n-2)}c_n$.

By *equal treatment of equals in welfare*, $\gamma_{s+1}(c_1, c_2, ..., c_s, c_n, ..., c_n) = ... = \gamma_n(c_1, c_2, ..., c_s, c_n, ..., c_n)$.

Thus, for each $j \in N\backslash\{1, 2, ..., s\}$, we have
$\gamma_j(c_1, c_2, ..., c_s, c_n, ..., c_n) = \sum_{l\in\{1,2,...,s\}} \frac{(\sigma_l^{\{1,2,...,s\}}-1)}{(n-2)}c_l + \frac{(n-1-(s))(n-2+(s))}{2(n-2)}c_n$.

This holds for each $S \subset N\backslash\{n\}$ with $|S| = s$. Thus, for each $j \in N$:

- if $j \in S$, then $\gamma_j(c_{S\backslash\{j\}}, c_n, ..., c_n) = \sum_{l\in S\backslash\{j\}} \frac{(\sigma_l^{S\backslash\{j\}}-1)}{(n-2)}c_l + \frac{(n-1-|S\backslash\{j\}|)(n-2+|S\backslash\{j\}|)}{2(n-2)}c_n$;

- if $j \in N\backslash S$, then $\gamma_j(c_S, c_n, ..., c_n) = \sum_{l\in S} \frac{(\sigma_l^S-1)}{(n-2)}c_l + \frac{(n-1-|S|)(n-2+|S|)}{2(n-2)}c_n$.

$\vdots$

Step $n-1$: $c = (c_1, c_2, ..., c_{n-1}, c_n)$.

By *Pareto-efficiency*, $\gamma_1(c_2, c_3, ..., c_{n-1}, c_n)+\gamma_2(c_1, c_3, ..., c_{n-1}, c_n)+...+\gamma_n(c_1, c_2, ..., c_{n-1})$

$= \sum_{l\in\{1,2,...,n-1\}}(\sigma_l - 1)c_l$.

By Step $n-2$, for $j \in \{1, 2, ..., n-1\}$, we have

$\gamma_i(c_1, c_2, ..., c_{n-1}, c_n) = \sum_{l\in\{1,2,...,n-1\}\backslash\{i\}} \frac{(\sigma_l^{\{1,2,...,n-1\}\backslash\{i\}}-1)}{(n-2)}c_l + c_n$.

Thus, $\gamma_n(c_1, c_2, ..., c_{n-1}) = \sum_{l\in\{1,2,...,n-1\}} \frac{(\sigma_l^{\{1,2,...,n-1\}}-1)}{(n-2)}c_l$. Thus, we have

$\gamma_i(c_{-i}) = \sum_{l\in N\backslash\{i\}} \frac{(\sigma_l^{N\backslash\{i\}}-1)}{(n-2)}c_l = \sum_{l\in N\backslash\{i\}} \frac{(\sigma_l^{-i}-1)}{(n-2)}c_l$.

Only if part:

Pareto-efficiency: Let $c \in \mathcal{C}$ and $(\sigma, t) = \varphi^*(c)$. By definition of φ^* rule, $\sigma \in Q^*(c)$ and by Proposition 1.1, for each $i \in N$,

$t_i = -\sum_{l\in N\backslash\{i\}}(\sigma_l - 1)c_l + \sum_{l\in N\backslash\{i\}}(\sigma_l^{-i} - 1)c_l + \frac{1}{(n-2)}\sum_{l\in N\backslash\{i\}}(\sigma_l^{-i} - 1)c_l$.

Thus,

$\sum_{i\in N} t_i = \sum_{i\in N}[-\sum_{l\in N\backslash\{i\}}(\sigma_l-1)c_l+\sum_{l\in N\backslash\{i\}}(\sigma_l^{-i}-1)c_l+\frac{1}{(n-2)}\sum_{l\in N\backslash\{i\}}(\sigma_l^{-i}-1)c_l]$

$$= \sum_{i\in N}[-\sum_{l\in F_i(\sigma)} c_l + \frac{1}{(n-2)}\sum_{l\in N\backslash\{i\}}(\sigma_l^{-i} - 1)c_l]$$

$$= -\sum_{i\in N}\sum_{l\in F_i(\sigma)} c_l + \frac{1}{(n-2)}\sum_{i\in N}\sum_{l\in N\backslash\{i\}}(\sigma_l^{-i} - 1)c_l$$

$= -\sum_{i\in N}(\sigma_i - 1)c_i + \frac{1}{(n-2)}\sum_{i\in N}(n-2)(\sigma_i - 1)c_i$

$= 0.$

Strategy-proofness: By Proposition 1.1, φ^* is a Groves rule. Thus, by Theorem 1.1, φ^* is *strategy-proof.* $\square$

Proof of Remark 1. *No-envy:*[10] Let $c \in \mathcal{C}$, $(\sigma, t) = \varphi^*(c)$, and $\{i, j\} \subset N$ with $i \neq j$. Then, by definition of φ^*, $\sigma \in Q^*(c)$ and by Proposition 1.1, $t_i = \sum_{l\in P_i(\sigma)} \frac{(\sigma_l-1)}{(n-2)}c_l - \sum_{l\in F_i(\sigma)} \frac{(n-\sigma_l)}{(n-2)}c_l$ and $t_j = \sum_{l\in P_j(\sigma)} \frac{(\sigma_l-1)}{(n-2)}c_l - \sum_{l\in F_j(\sigma)} \frac{(n-\sigma_l)}{(n-2)}c_l$. Then, we distinguish two cases:

Case 1: $\sigma_i < \sigma_j$. Let $d \in \mathbb{N}$ be such that $\sigma_j = \sigma_i + d$. Then, as, by assumption, $1 \leq \sigma_i < \sigma_j \leq n$, we have $d \leq n - \sigma_i$. Also, as $\sigma \in Q^*(c)$, for each $l \in B_{ij}(\sigma)$, we have $c_i \geq c_l \geq c_j$. Thus,

$$\begin{aligned}
u_i(\sigma_i, t_i) - u_i(\sigma_j, t_j) &= (-(\sigma_i - 1)c_i - \textstyle\sum_{l\in B_{ij}(\sigma)} \frac{(n-\sigma_l)}{(n-2)}c_l - \frac{(n-\sigma_j)}{(n-2)}c_j) \\
&\quad - (-(\sigma_j - 1)c_i + \tfrac{(\sigma_i-1)}{(n-2)}c_i + \textstyle\sum_{l\in B_{ij}(\sigma)} \frac{(\sigma_l-1)}{(n-2)}c_l) \\
&= \tfrac{(n-2)d-(\sigma_i-1)}{(n-2)}c_i - \tfrac{(n-1)}{(n-2)}\textstyle\sum_{l\in B_{ij}(\sigma)} c_l - \frac{(n-\sigma_i-d)}{(n-2)}c_j \\
&\geq (\tfrac{(n-2)d-(\sigma_i-1)-(n-1)(d-1)-(n-\sigma_i-d)}{(n-2)})c_i \\
&= 0.
\end{aligned}$$

Case 2: $\sigma_i > \sigma_j$. Let $d \in \mathbb{N}$ be such that $\sigma_i = \sigma_j + d$. Then, as, by assumption,

[10] Chun (2004b) provides a necessary and sufficient condition for a rule φ to satisfy *Pareto-efficiency* and *no-envy*: For each $c \in \mathcal{N} \times \mathbb{R}^N_+$ and each $(\sigma, t) \in \varphi(c)$, we have $\sigma \in Q^*(c)$, $\sum_{i\in N} t_i = 0$, and for each $\{i, j\} \subset N$, if $\sigma_j = \sigma_i + 1$, then $c_i \geq t_j - t_i \geq c_j$. An alternative proof thus consists in proving that φ^* satisfies this condition. In fact, rules in Suijs (1996) satisfy this condition (Katta and Sethuraman, 2005). Thus, by Proposition 1.1, φ^* satisfies this condition.

$n \geq \sigma_i > \sigma_j \geq 1$. Also, as $\sigma \in Q^*(c)$, for each $l \in B_{ji}(\sigma)$, we have $c_i \leq c_l \leq c_j$. Thus,

$$\begin{aligned} u_i(\sigma_i, t_i) - u_i(\sigma_j, t_j) &= (-(\sigma_i - 1)c_i + \tfrac{(\sigma_j - 1)}{(n-2)}c_j + \textstyle\sum_{l \in B_{ji}(\sigma)} \tfrac{(\sigma_l - 1)}{(n-2)}c_l) \\ &\quad - (-(\sigma_j - 1)c_i - \textstyle\sum_{l \in B_{ji}(\sigma)} \tfrac{(n-\sigma_l)}{(n-2)}c_l - \tfrac{(n-\sigma_i)}{(n-2)}c_i) \\ &= \tfrac{-(n-2)d + (n-\sigma_j - d)}{(n-2)}c_i + \tfrac{(n-1)}{(n-2)}\textstyle\sum_{l \in B_{ji}(\sigma)} c_l + \tfrac{(\sigma_j - 1)}{(n-2)}c_j \\ &\geq (\tfrac{-(n-2)d + (n-\sigma_j - d) + (n-1)(d-1) + (\sigma_j - 1)}{(n-2)})c_i \\ &= 0. \ \square \end{aligned}$$

Proof of Theorem 1.3. By contradiction, let φ be a rule satisfying the axioms of Theorem 1.3. We show that φ satisfies *non-bossiness*. Let $c \in \mathcal{C}$, $i \in N$, $c'_i \in \mathbb{R}_+$, $(\sigma, t) = \varphi(c)$, and $(\sigma', t') = \varphi(c'_i, c_{-i})$ be such that $(\sigma_i, t_i) = (\sigma'_i, t'_i)$. Suppose that there is $j \in N$ such that $(\sigma_j, t_j) \neq (\sigma'_j, t'_j)$. Since $(\sigma_i, t_i) = (\sigma'_i, t'_i)$, we have $u_i(\sigma_i, t_i) = u_i(\sigma', t'_i)$. By *efficiency of queues*, $\sigma_j = \sigma'_j$. Since $(\sigma_j, t_j) \neq (\sigma'_j, t'_j)$, we have $t_j \neq t'_j$. First, suppose $t_j > t'_j$. Then, $u_j(\sigma_j, t_j) > u_j(\sigma'_j, t'_j)$. Then, there is $(c'_i, c_j) \in \mathbb{R}_+^{\{i,j\}}$ such that $u_i(\sigma_i, t_i) = u_i(\sigma'_i, t'_i)$ and $u_j(\sigma_j, t_j) > u_j(\sigma'_j, t'_j)$ contradicting *coalitional strategy-proofness*. Second, suppose $t_j < t'_j$. Then, $u_j(\sigma_j, t_j) < u_j(\sigma'_j, t'_j)$. Then, there is $(c_i, c_j) \in \mathbb{R}_+^{\{i,j\}}$ such that $u'_i(\sigma_i, t_i) = u'_i(\sigma'_i, t'_i)$ $u_j(\sigma'_j, t'_j) > u_j(\sigma_j, t_j)$ contradicting *coalitional strategy-proofness*.

Now, we establish two claims:

Claim 1: For each $c \in \mathbb{R}_+^N$, each $i \in N$, and each $c'_i \in \mathbb{R}_+$, if $(\sigma, t) = \varphi(c)$ and $(\sigma', t') = \varphi(c'_i, c_{-i})$ are such that $\sigma_i = \sigma'_i$, then $(\sigma, t) = (\sigma', t)$.

Let $c \in \mathbb{R}^N_+$, $i \in N$, $c'_i \in \mathbb{R}_+$, $(\sigma, t) = \varphi(c)$, and $(\sigma', t') = \varphi(c'_i, c_{-i})$ be such that $\sigma_i = \sigma'_i$. By *strategy-proofness*, $-(\sigma_i - 1)c_i + t_i \geq -(\sigma'_i - 1)c_i + t'_i$ and $-(\sigma - 1)c'_i + t_i \leq -(\sigma'_i - 1)c'_i + t'_i$. Thus, as $\sigma_i = \sigma'_i$, we have $t_i = t'_i$. By *non-bossiness*, $(\sigma, t) = (\sigma', t')$.

Claim 2: For each $c \in \mathbb{R}^N_+$ such that for each $\{j, k\} \subseteq N$, we have $c_j \neq c_k$ if and only if $j \neq k$, for each $i \in N$, and each $c'_i \in \mathbb{R}_+$ such that for each $j \in N \backslash \{i\}$, we have $c'_i > c_j$ if and only if $c_i > c_j$, if $(\sigma, t) = \varphi(c)$, then $(\sigma, t) = \varphi(c'_i, c_{-i})$.

Let $c \in \mathbb{R}^N_+$, $i \in N$, $c'_i \in \mathbb{R}_+$ be such that for each $j \in N \backslash \{i\}$, we have $c'_i \neq c_j$ and $c'_i > c_j$ if and only if $c_i > c_j$, and $(\sigma, t) = \varphi(c)$, $(\sigma', t') = \varphi(c'_i, c_{-i})$. By *efficiency of queues*, we have $\sigma'_i = \sigma_i$. By Claim 1, $(\sigma, t) = \varphi(c'_i, c_{-i})$.

Claims 1 and 2 being proved, we now come to a contradiction. Without loss of generality, suppose $N = \{1, 2, ..., n\}$. Let $\{c, c'\} \subseteq \mathbb{R}^N_+$ be such that

(i) $c_1 > c_2 > c_3 ... > c_n$,

(ii) $c'_2 > c'_1 > c'_3 > ... > c'_n$, and

(iii) for each $i \in N \backslash \{1\}$, $c'_i = c_i$.

Let $(\sigma, t) = \varphi(c)$ and $(\sigma', t') = \varphi(c')$. By *efficiency of queues*, for each $i \in N$, we have $\sigma_i = i$, whereas $\sigma'_1 = 2$, $\sigma'_2 = 1$, and for each $i \in N \backslash \{1, 2\}$, we have $\sigma_i = \sigma'_i = i$. Thus, $(\sigma, t) \neq (\sigma', t')$. By *strategy-proofness*, $u_1(\sigma_1, t_1) = t_1 \geq -c_1 + t'_1 = u_1(\sigma'_1, t'_1)$ and $u'_1(\sigma'_1, t'_1) = -c'_1 + t'_1 \geq t_1 = u'_1(\sigma_1, t_1)$. That is,

$t_1' \in [t_1+c_1', t_1+c_1]$. Thus, agent 1's transfer depends either on a constant, i.e., $\bar{t}_1 = t_1+c$ with $c \in [c_1, c_1']$, or on its own announcement, i.e., $\bar{t}_1 = t_1 + f(c_1', c_1)$ with $f(c_1', c_1) \in [c_1, c_1']$. Clearly, this contradicts *strategy-proofness.* $\square$

Proof of Statement 1 in Theorem 1.4. Let φ be a rule. Then,

If Part Let φ be a rule satisfying the axioms of Theorem 1.4.1. Let $c \in \mathcal{C}$ and $(\sigma, t) \in \varphi(c)$. By *Pareto-efficiency*, $\sigma \in Q^*(c)$. The Claims 1 and 2 state that *Pareto-efficiency* and *strategy-proofness* imply that there is $\{\underline{h}, \overline{h}\} \subseteq H$ such that for each $i \in N$,

- if $(\underline{\sigma}, \underline{t}) \in \arg\min_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$, then $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \underline{h}_i(c_{-i})$ and
- if $(\overline{\sigma}, \overline{t}) \in \arg\max_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$, then $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)c_l + \overline{h}_i(c_{-i})$.

Thus, repeating the proof by induction of Theorem 1.2, for each $i \in N$,

- $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l^{-i} - 1)c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l^{-i} - 1)c_l$ and
- $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)c_l + \sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l^{-i} - 1)c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l^{-i} - 1)c_l$.

By *Pareto-efficiency*, for each $i \in N$, $u_i(\underline{\sigma}, \underline{t}) = u_i(\overline{\sigma}, \overline{t})$. Thus, for each $i \in N$, $t_i = -\sum_{l\in N\setminus\{i\}}(\sigma_l - 1)c_l + \sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l$. Thus, by Proposition 1.1, for each $i \in N$,

$t_i = -\sum_{j\in N\setminus\{i\}} \sum_{l\in\{i,j\}\cap F_i(\sigma)} c_l + \frac{1}{(n-2)} \sum_{j\in N\setminus\{i\}} \sum_{k\in N\setminus\{i,j\}} \sum_{l\in\{j,k\}\cap F_j(\sigma)} c_l.$

Claim 1: There is $\overline{h} \in H$ such that for each $c \in \mathcal{C}$ and each $i \in N$, if $(\overline{\sigma}, \overline{t}) \in \arg\max_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$, then $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)c_l + \overline{h}_i(c_{-i})$.

For each $i \in N$, let $\overline{g}_i : \mathbb{R}^N_+ \to \mathbb{R}$ be a function such that ***(i)*** for each $c \in \mathbb{R}^N_+$ if $(\underline{\sigma}, \underline{t}) \in \arg\max_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$, then $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)c_l + \overline{g}_i(c)$. By contradiction, suppose that for $c \in \mathbb{R}^N_+$ and $c'_i \in \mathbb{R}$, we have ***(ii)*** $\overline{g}_i(c) - \overline{g}_i(c'_i, c_{-i}) \neq 0$. Let $(\overline{\sigma}, \overline{t}) \in \arg\max_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$ and $(\overline{\overline{\sigma}}, \overline{\overline{t}}) \in \arg\max_{(\sigma,t)\in\varphi(c'_i,c_{-i})} u'_i(\sigma_i, t_i)$. Then, by *strategy-proofness*,

- $u_i(\overline{\sigma}_i, \overline{t}_i) \geq \max_{(\sigma',t')\in\varphi(c'_i,c_{-i})} u_i(\sigma'_i, t'_i)$,
- $\max_{(\sigma,t)\in\varphi(c)} u'_i(\sigma_i, t_i) \leq u'_i(\overline{\overline{\sigma}}_i, \overline{\overline{t}}_i)$,
- $max_{(\sigma',t')\in\varphi(c'_i,c_{-i})} u_i(\sigma'_i, t'_i) \geq u_i(\overline{\overline{\sigma}}_i, \overline{\overline{t}}_i)$,
- $u'_i(\overline{\sigma}_i, \overline{t}_i) \leq max_{(\sigma,t)\in\varphi(c)} u'_i(\sigma_i, t_i)$.

Thus, ***(iii)*** $u_i(\overline{\sigma}_i, \overline{t}_i) - u_i(\overline{\overline{\sigma}}_i, \overline{\overline{t}}_i) \geq 0$ and $u'_i(\overline{\overline{\sigma}}_i, \overline{\overline{t}}_i) - u'_i(\overline{\sigma}_i, \overline{t}_i) \geq 0$. By the logic of Theorem 1.1, ***(i)***, ***(ii)***, and ***(iii)*** together imply a contradiction.

Claim 2: There is $\underline{h} \in H$ such that for each $c \in \mathcal{C}$ and each $i \in N$, if $(\underline{\sigma}, \underline{t}) \in \arg\min_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$, then $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \underline{h}_i(c_{-i})$.

Let $c \in \mathbb{R}^N_+$, $i \in N$, $c'_i \in \mathbb{R}_+$, $(\underline{\sigma}, \underline{t}) \in \arg\min_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$ and $(\underline{\underline{\sigma}}, \underline{\underline{t}}) \in \arg\min_{(\sigma,t)\in\varphi(c'_i,c_{-i})} u'_i(\sigma_i, t_i)$. Let $\underline{g}_i : \mathbb{R}^N_+ \to \mathbb{R}$ be a function such that

- $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \underline{g}_i(c)$ and

- $\underline{\underline{t}}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\underline{\sigma}}_l - 1)c_l + \underline{g}_i(c'_i, c_{-i})$.

In what follows, we prove that there is $\underline{h}_i : \mathbb{R}_+^{N\setminus\{i\}} \to \mathbb{R}$ such that $\underline{g}_i(c) = \underline{h}_i(c_{-i})$ and $\underline{g}_i(c'_i, c_{-i}) = \underline{h}_i(c_{-i})$. Thus, $\underline{g}_i(c) = \underline{g}_i(c'_i, c_{-i})$.

First, there is $\underline{h}_i : \mathbb{R}_+^{N\setminus\{i\}} \to \mathbb{R}$ such that

- if $(\sigma^*, t^*) \in \arg\min_{(\sigma,t)\in\varphi(c)} u'_i(\sigma_i, t_i)$ then $t^*_i = -\sum_{l\in N\setminus\{i\}}(\sigma^*_l - 1)c_l + \underline{h}_i(c_{-i})$ and

- if $(\sigma^{**}, t^{**}) \in \arg\min_{(\sigma,t)\in\varphi(c'_i,c_{-i})} u_i(\sigma_i, t_i)$, then $t^{**}_i = -\sum_{l\in N\setminus\{i\}}(\sigma^{**}_l - 1)c_l + \underline{h}_i(c_{-i})$.

Let $(\sigma^*, t^*) \in \arg\min_{(\sigma,t)\in\varphi(c)} u'_i(\sigma_i, t_i)$ and $(\sigma^{**}, t^{**}) \in \arg\min_{(\sigma,t)\in\varphi(c'_i,c_{-i})} u_i(\sigma_i, t_i)$.

Let $g^*_i : \mathbb{R}_+^N \to \mathbb{R}$ be a function such that by choosing g^*_i appropriately,,

(i) $t^*_i = -\sum_{l\in N\setminus\{i\}}(\sigma^*_l - 1)c_l + g^*_i(c)$ and $t^{**}_i = -\sum_{l\in N\setminus\{i\}}(\sigma^{**}_l - 1)c_l + g^*_i(c'_i, c_{-i})$.

By contradiction, suppose

(ii) $g^*_i(c) - g^*_i(c'_i, c_{-i}) \neq 0$.

Then, by *strategy-proofness*, $u_i(\underline{\sigma}_i, \underline{t}_i) \geq u_i(\sigma^{**}_i, t^{**}_i)$ and $u'_i(\sigma^*_i, t^*_i) \leq u'_i(\underline{\underline{\sigma}}_i, \underline{\underline{t}}_i)$.

By assumption, $u_i(\sigma^*_i, t^*_i) \geq u_i(\underline{\sigma}_i, \underline{t}_i)$ and $u'_i(\underline{\underline{\sigma}}_i, \underline{\underline{t}}_i) \leq u'_i(\sigma^{**}_i, t^{**}_i)$. Thus,

(iii) $u_i(\sigma^*_i, t^*_i) - u_i(\sigma^{**}_i, t^{**}_i) \geq 0$ and $u'_i(\sigma^{**}_i, t^{**}_i) - u'_i(\sigma^*_i, t^*_i) \geq 0$.

By the logic of Theorem 1.1, ***(i)***, ***(ii)***, and ***(iii)*** together imply a contradiction. This holds for each $c'_i \in \mathbb{R}_+$.

Second, $\underline{g}_i(c) = \underline{h}_i(c_{-i})$ and $\underline{g}_i(c'_i, c_{-i}) = \underline{h}_i(c_{-i})$. By contradiction, suppose $\underline{g}_i(c) - \underline{h}_i(c_{-i}) \neq 0$. (The other case is immediate.) First, by assumption, $u_i(\sigma_i^*, t_i^*) \geq u_i(\underline{\sigma}_i, \underline{t}_i)$. Thus, $-(\sigma_i^* - 1)c_i - \sum_{l \in N \setminus \{i\}}(\sigma_l^* - 1)c_l + \underline{h}_i(c_{-i}) \geq -(\underline{\sigma}_i - 1)c_i - \sum_{l \in N \setminus \{i\}}(\underline{\sigma}_l - 1)c_l + \underline{g}_i(c)$. Thus, $-\sum_{l \in N}(\sigma_l^* - 1)c_l + \underline{h}_i(c_{-i}) \geq -\sum_{l \in N \setminus \{i\}}(\underline{\sigma}_l - 1)c_l + \underline{g}_i(c)$. Thus, by *Pareto-efficiency*, $\underline{h}_i(c_{-i}) \geq \underline{g}_i(c)$. Second, by *strategy-proofness*, $u_i(\underline{\sigma}_i, \underline{t}_i) \geq u_i(\sigma_i^{**}, t_i^{**})$. Thus, $-(\underline{\sigma}_i - 1)c_i - \sum_{l \in N \setminus \{i\}}(\underline{\sigma}_l - 1)c_l + \underline{g}_i(c) \geq -(\sigma_i^{**} - 1)c_i - \sum_{l \in N \setminus \{i\}}(\sigma_l^{**} - 1)c_l + \underline{h}_i(c_{-i})$. Thus, $\underline{g}_i(c) \geq (\underline{\sigma}_i - \sigma_i^{**})c_i + \sum_{l \in N \setminus \{i\}}(\underline{\sigma}_l - \sigma_l^{**})c_l + \underline{h}_i(c_{-i})$. Altogether,

(iv) $\underline{h}_i(c_{-i}) \geq \underline{g}_i(c) \geq (\underline{\sigma}_i - \sigma_i^{**})c_i + \sum_{l \in N \setminus \{i\}}(\underline{\sigma}_l - \sigma_l^{**})c_l + \underline{h}_i(c_{-i})$.

By *Pareto-efficiency*, for each $S \subseteq N$, if for each $\{k, k'\} \subseteq S$ with $k \neq k'$, we have $c_k = c_{k'}$ and there is no $k'' \in N \setminus S$ such that $k'' \in B_{kk'}(\sigma) \cup B_{kk'}(\sigma')$, then $\sum_{l \in S} -(\sigma_l - 1)c_l = \sum_{l \in S} -(\sigma'_l - 1)c_l$. Also, there is $j \in N$ such that $\underline{\sigma}_j = \sigma_i^{**}$. Thus, $\sum_{l \in N \setminus \{i\}}(\underline{\sigma}_l - \sigma_l^{**})c_l = -sign(\underline{\sigma}_i - \sigma_i^{**}) \sum_{l \in B_{ij}(\sigma) \cup \{j\}} c_l$. Thus, we may rewrite ***(iv)*** as

(v) $\underline{h}_i(c_{-i}) \geq \underline{g}_i(c) \geq (\underline{\sigma}_i - \sigma_i^{**})c_i - sign(\underline{\sigma}_i - \sigma_i^{**}) \sum_{l \in B_{ij}(\sigma) \cup \{j\}} c_l + \underline{h}_i(c_{-i})$.

We distinguish three cases:

Case 1: $(\underline{\sigma}_i - \sigma_i^{**}) = 0$. Then, $-sign(\underline{\sigma}_i - \sigma_i^{**}) \sum_{l \in B_{ij}(\sigma) \cup \{j\}} c_l = 0$. Thus, by ***(v)***, $\underline{g}_i(c) = \underline{h}_i(c_{-i})$ contradicting $\underline{g}_i(c) - \underline{h}_i(c_{-i}) \neq 0$.

Case 2: $|\underline{\sigma}_i - \sigma_i^{**}| = 1$. Suppose $c'_i > c_i$. (The symmetric case is immediate.) Then, $(\underline{\sigma}_i - \sigma_i^{**}) = 1$ and $-sign(\underline{\sigma}_i - \sigma_i^{**}) \sum_{l \in B_{ij}(\sigma) \cup \{j\}} c_l = -c_j$. Thus, by ***(v)***,

$\underline{h}_i(c_{-i}) \geq \underline{g}_i(c) \geq (c_i - c_j) + \underline{h}_i(c_{-i})$. Let $c_i'' \in \mathbb{R}_+$ be such that ***(vi)*** $\underline{g}_i(c) > (c_i'' - c_j) + \underline{h}_i(c_i)$ and $c_i' > c_i'' > c_i$. Let $(\sigma^{***}, t^{***}) \in \arg\min_{(\sigma,t)\in\varphi(c_i'',c_{-i})} u_i(\sigma_i, t_i)$. Then, by *Pareto-efficiency of queues*, $\sigma_i^{***} = \sigma_i^{**}$. Then,

$(\underline{\sigma}_i - \sigma_i^{***}) = (\underline{\sigma}_i - \sigma_i^{**}) = 1$ and $\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma_l^{***})c_l = \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma_l')c_l = -c_j$. By ***(vi)***,

$\underline{g}_i(c) > (\underline{\sigma}_i - \sigma_i^{***})c_i'' + \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma_l^{***})c_l + \underline{h}_i(c_i)$.

Then, $-(\underline{\sigma}_i - 1)c_i'' - \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \underline{g}_i(c) > -(\sigma_i^{***} - 1)c_i'' - \sum_{l\in N\setminus\{i\}}(\sigma_l^{***} - 1)c_l + \underline{h}_i(c_{-i})$.

Thus, $u_i''(\underline{\sigma}_i, \underline{t}_i) > u_i''(\sigma_i^{***}, t_i^{***})$. Also, $u_i''(\sigma_i^{***}, t_i^{***}) \geq min_{(\sigma,t)\in\varphi(c_i'',c_{-i})} u_i''(\sigma_i, t_i)$. Therefore, $u_i''(\underline{\sigma}_i, \underline{t}_i) > min_{(\sigma,t)\in\varphi(c_i'',c_{-i})} u_i''(\sigma_i, t_i)$ contradicting *strategy-proofness.*

Case 3: $|\underline{\sigma}_i - \sigma_i^{**}| > 1$. By the logic of Case 2, starting from σ_i^{**}, we can find $\tilde{c}_i$ such that $\tilde{\sigma}_i$ is one position closer to $\underline{\sigma}_i$. We continue by one position at a time and at each step we obtain $\underline{g}_i(\tilde{c}_i, c_{-i}) = \underline{h}_i(c_{-i})$. Thus, $\underline{g}_i(c) = \underline{h}_i(c_{-i})$ contradicting $\underline{g}_i(c) - \underline{h}_i(c_{-i}) \neq 0$.

Only if part:

Pareto-efficiency: Straightforward from Theorem 1.2.

No-envy: Straightforward from Theorem 1.2.

Strategy-proofness: Let $c \in \mathcal{C}$, $i \in N$, $c_i' \in \mathbb{R}_+$, $(\sigma, t) \in \Phi^*(c)$, and $(\sigma', t') \in \Phi^*(c_i', c_{-i})$. Then, by definition of Φ^*, $\sigma \in Q^*(c)$ and by Proposition 1.1, there is $h \in H$ such that for each $i \in N$, $h_i(c_{-i}) = \sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l +$

$\frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\sigma_l^{-i}-1)c_l = \sum_{l\in N\setminus\{i\}}(\sigma_l'^{-i}-1)c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\sigma_l'^{-i}-1)c_l$ and $t_i = -\sum_{l\in N\setminus\{i\}}(\sigma_l-1)c_l + h_i(c_{-i})$ and $t_i' = -\sum_{l\in N\setminus\{i\}}(\sigma_l'-1)c_l + h_i(c_{-i})$. Suppose $u_i(\sigma_i', t_i') > u_i(\sigma_i, t_i)$. Thus, $-(\sigma_i'-1)c_i - \sum_{l\in N\setminus\{i\}}(\sigma_l'-1)c_l + h_i(c_{-i}) > -(\sigma_i-1)c_i - \sum_{l\in N\setminus\{i\}}(\sigma_l-1)c_l + h_i(c_{-i})$. Thus, $-\sum_{l\in N}(\sigma_l'-1)c_l > -\sum_{l\in N}(\sigma_l-1)c_l$ contradicting $\sigma \in Q^*(c)$. Thus, $u_i(\sigma_i', t_i') \leq u_i(\sigma_i, t_i)$. This holds for each $(\sigma, t) \in \varphi(c)$ and each $(\sigma', t') \in \varphi(c_i', c_{-i})$. Thus, if $Z = \varphi(c)$ and $Z' = \varphi(c_i', c_{-i})$, then $Z_i \; R_i(c_i) \; Z_i'$.

Proof of Statement 2 in Theorem 1.4.

If Part: Let φ be a rule satisfying the axioms of the third statement of Theorem 1.4. Let $c \in \mathcal{C}$ and $(\sigma, t) \in \varphi(c)$. Then, by *Pareto-efficiency*, $\sigma \in Q^*(c)$. By Statement 1, *Pareto-efficiency* and *strategy-proofness* imply that there is $\{\underline{h}, \overline{h}\} \subseteq H$ such that for each $i \in N$,

- if $(\underline{\sigma}, \underline{t}) \in \arg\min_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$, then $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \underline{h}_i(c_{-i})$,
- if $(\overline{\sigma}, \overline{t}) \in \arg\max_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$, then $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)c_l + \overline{h}_i(c_{-i})$.

By *symmetry*, for each $\{i, j\} \subset N$, if $c_{-i} = c_{-j}$, then $\underline{h}_i(c_{-i}) = \underline{h}_j(c_{-j})$ and $\overline{h}_i(c_{-i}) = \overline{h}_j(c_{-j})$. Thus, for each $\{i, j\} \subset N$, if $c_i = c_j$, then $\underline{h}_i(c_{-i}) = \underline{h}_j(c_{-j})$ and $\overline{h}_i(c_{-i}) = \overline{h}_j(c_{-j})$. This is true for each $c \in \mathbb{R}_+$. Thus, repeating the proof by induction of Theorem 1.2, for each $i \in N$, we have $\underline{t}_i = -\sum_{l\in F_i(\underline{\sigma})} c_l +$

$\frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l^{-i}-1)c_l$ and $\bar{t}_i = -\sum_{l\in F_i(\bar{\sigma})} c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l^{-i}-1)c_l$. Thus, by *Pareto-efficiency*, for each $i \in N$, $u_i(\underline{\sigma}, \underline{t}) = u_i(\overline{\sigma}, \bar{t})$. Thus, for each $i \in N$, we have $t_i = -\sum_{l\in F_i(\sigma)} c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\sigma_l^{-i}-1)c_l$. Thus, by Proposition 1.1, $t_i = -\sum_{j\in N\setminus\{i\}}\sum_{l\in\{i,j\}\cap F_i(\sigma)} c_l + \frac{1}{(n-2)}\sum_{j\in N\setminus\{i\}}\sum_{k\in N\setminus\{i,j\}}\sum_{l\in\{j,k\}\cap F_j(\sigma)} c_l$. Thus, for each $c \in \mathcal{C}$, we have $\varphi(c) \subseteq \Phi^*(c)$. Thus, by *symmetry*, $\varphi(c) = \Phi^*(c)$.

Only if part: Suppose that for each $c \in \mathcal{C}$, we have $\varphi(c) = \Phi^*(c)$. By the second statement of Theorem 1.4, φ satisfies *Pareto-efficiency* and *strategy-proofness*. Also, φ does not depend on agents' names. In particular, t_i has the same structure for each $i \in N$. Thus, φ satisfies *anonymity*. $\square$

Appendix B

Proof of Proposition 2.1. Clearly, since φ^S is *stable*, it satisfies *unanimity* and *individual rationality*. Next, we show that φ^S satisfies *own-side population-monotonicity* and *other-side population-monotonicity*. Let $M = (D, H, X, R) \in \mathcal{M}$ and $\tilde{M} = (\tilde{D}, \tilde{H}, \tilde{X}, \tilde{R}) \in \mathcal{M}$ be such that M is the D-restriction of $\tilde{M}$. Let $A^H \in \varphi^S(M)$ and $\tilde{A}^H \in \varphi^S(\tilde{M})$ be the hospital-optimal allocations of M and $\tilde{M}$, respectively. By Ostrovsky (2005), for each $d \in D$ and each $h \in H$, we have ***(i)*** $A^H_d \; R_d \; \tilde{A}^H_d$ and $\tilde{A}^H_h \; R_h \; A^H_h$. Then, by Hatfield and Milgrom (2005), for each $A \in \varphi^S(M)$, each $h \in H$, and each $d \in D$, we have ***(ii)*** $A_d \; R_d \; A^H_d$ and $A^H_h \; R_h \; A_h$. By ***(i)*** and ***(ii)***, for each

$A \in \varphi^S(M)$, each $h \in H$, and each $d \in D$, there is $A' = \tilde{A}^H \in \varphi^S(\tilde{M})$ such that $A_d \; R_d \; A'_d$ and $A'_h \; R_h \; A_h$. A symmetric result holds when M is the H-restriction of $\tilde{M}$. Thus, φ^S satisfies *weak own-side population-monotonicity* and *other-side population-monotonicity*. By Theorem 1 of Haake and Klaus (2005), φ^S satisfies *Maskin-monotonicity*. Next, we show that φ^S satisfies *weak consistency*. Let $M = (D, H, X, R) \in \mathcal{M}$ and $\boldsymbol{(i)}$ $A \in \varphi^S(M)$. Let $M' = (D', H', X', R') \in \mathcal{M}$ be the type-2 reduced economy of M relative to $D' \cup H'$ at A. Let $A|_{D' \cup H'} = A'$. By contradiction, assume $A' \notin \varphi^S(M')$:

Case 1: There is $d \in D'$ such that $C(A'_d, R'_d) = \emptyset$. Then, since for each $d \in D'$, $A'_d = A_d$, we have $C(A_d, R_d) = \emptyset$, which contradicts $\boldsymbol{(i)}$.

Case 2: There is $h \in H'$ such that $C(A'_h, R'_h) \subsetneq A'_h$. Then, since $A'_h = A_h$, then $C(A_h, R_h) \subsetneq A_h$, which contradicts $\boldsymbol{(i)}$.

Case 3: There are $h \in H'$, $\tilde{D} \subseteq D'$, and $\tilde{X} \nsubseteq A_h$ such that for each $d \in \tilde{D}$, there are $x \in \tilde{X}$ with $\{x\} = C(A'_d \cup \{x\}, R'_d)$ and $\tilde{X} = C(A'_h \cup \tilde{X}, R'_h)$. Then, since $\tilde{X} \subseteq X$ and $A'_h = A_h$, $\tilde{X} = C(A_h \cup \tilde{X}, R_h)$ and for each $d \in \tilde{D}$, $\{x\} = C(A_d \cup \{x\}, R_d)$, which contradicts $\boldsymbol{(i)}$. $\square$

Proof of Lemma 2.1.

Let $M = (D, H, X, R) \in \mathcal{M}$. Also, let $i \in H$ and $J \subseteq D$ with $J \neq \emptyset$ be such that, for each $j \in J$, there is $x_{ji} \in X$ with $\{x_{ji}\} = C(X, R_j)$ and $\bigcup_{j \in J} \{x_{ji}\} = C(X, R_i)$.

Let $M' = (D', H', X', R') \in \mathcal{M}$ be as follows:

– $D' = D \cup \hat{D}$ such that, for each $d \in \hat{D}$ and $h \in H \backslash \{i\}$, there is $x_{dh} \in X'$ with $\{x_{dh}\} = C(X, R_d) = C(X, R_h)$,

– $H' = H \cup \hat{H}$ such that, for each $d \in D \backslash J$ and $h \in \hat{H}$, there is $x_{dh} \in X'$ with $\{x_{dh}\} = C(X, R_d) = C(X, R_h)$,

– for each $x \in X' \backslash X$, $\mu(x) = (d, h)$ where either $d \in D \backslash J$ and $h \in \hat{H}$, or $h \in H \backslash \{i\}$ and $d \in \hat{D}$.

Then, ***(i)*** for each $j \in J$, $C(X', R'_j) = C(X, R_j)$ and we have $C(X', R'_i) = C(X, R_i)$. Also, by *unanimity*, $\varphi(M') = \{A'\}$ such that for each $d \in D' \cup H'$, $A'_d = C(X', R'_d)$. In particular, ***(ii)*** for each $d \in D$, we have $A'_d = C(X', R'_d)$ and $A'_i = C(X', R'_i)$.

Let $M'' = (D'', H'', X'', R'') \in \mathcal{M}$ be the D-restriction of M' with $D'' = D$. Then, for each $d \in D''$, the following statements hold:

- By ***(ii)***, $A'_d = C(X', R'_d)$.
- $C(X', R'_d) = C(X'', R''_d)$.
- By *own-side population-monotonicity*, for each $A'' \in \varphi(M'')$, we have $A''_d \; R''_d \; A'_d$.

Thus, for each $d \in D''$, we have $A''_d = A'_d$. That is, ***(iii)*** $A''_j = A'_j$.

Clearly, M is the H-restriction of M''. Then, the following statements hold:

- By ***(iii)***, $A''_i = A'_i$

- By ***(ii)***, $A'_i = C(X', R'_i)$.

- By ***(i)***, $C(X', R'_i) = C(X, R_i) = C(X'', R''_i)$.

Since φ is *own-side population-monotonic*, for each $A \in \varphi(M)$, we have $A_i \; R_i \; A''_i$. Thus, we have ***(iv)*** $A_i = A''_i$. Together, ***(i)***, ***(ii)***, ***(iii)***, and ***(iv)*** imply that, for each $A \in \varphi(M)$, we have $A_i = C(X, R_i)$. $\square$

Proof of Proposition 2.2.

Let φ be a rule satisfying the first three of Proposition 2.2. By contradiction, assume that φ is not *weakly individually rational*. That is, there are $M = (D, H, X, R) \in \mathcal{M}$, $A \in \varphi(M)$ and $i \in D \cup H$ such that ***(i)*** $A_i \neq \emptyset$ and $C(A_i, R_i) = \emptyset$. Let $J \equiv \{j \in D \cup H : \text{ there is } x \in A_i \text{ such that } \mu(x) = (i, j)\}$.

Let $M' = (D', H', X', R') \in \mathcal{M}$ be as follows:

– $D' = D$, $H' = H$, and $X' = X$,

– for each $j \in (D \cup H) \backslash \{i\}$, we have $C(X, R'_j) = A_j$, $C(X, R'_i) = \emptyset$ and for each $X' \subseteq X$ such that $X' \neq \emptyset$, $C(A_i \cup X') = A_i$.

Clearly, $R' \in MT(R, A)$. Thus, by *Maskin-monotonicity*, $A \in \varphi(M')$.

Let $M'' = (D'', H'', X'', R'') \in \mathcal{M}$ be as follows:

– $D'' \cup H'' = D \cup H \cup \{k\}$,

– $X'' = X' \cup \{X_k\}$ where $C(X'', R''_k) = X_k$, for each $j \in J$, there is $x_{jk} \in X_k$ with $\mu(x_{jk}) = (j, k)$ such that $C(X'', R''_j) = \{x_{jk}\}$, and $R''_i = R'_i$.

By *unanimity*, $\varphi(M'') = \{A''\}$ where for each $j \in D'' \cup H''$, $A''_j = C(X'', R''_j)$.

In particular, ***(ii)*** $A_i'' = \emptyset$.

Clearly, M' is a H-restriction of M'' if $i \in H$, and M' is a D-restriction of M'' if $i \in D$. By *own-side population-monotonicity* and ***(ii)***, for each $A' \in \varphi(M')$, $A_i' = \emptyset$. By ***(i)***, $A \notin \varphi(M')$, which contradicts *Maskin-monotonicity.* $\square$

Proof of Proposition 2.3.

Let φ be a rule satisfying the axioms of Proposition 2.3. Also, let $M = (D, H, X, R) \in \mathcal{M}$ and $A \in \varphi(M)$. The proof that $\varphi(M) = \varphi^S(M)$ is in two steps.

Step 1: $\boldsymbol{\varphi(M) \subseteq \varphi^S(M)}$. By contradiction, assume $A \notin \varphi^S(M)$. Since φ satisfies *weak individual rationality*, there is a blocking pair $i \in H$ and $J \subseteq D$ such that there is $X^i \in \mathcal{X}_i$ with $X^i \neq A_i$ and, for each $j \in J$, there is $x_{ji} \in X^i$ with $\{x_{ji}\} = C(A \cup X^i, R_j)$ and $X^i = C(A \cup X^i, R_i)$. Let $K \equiv \{k \in D : \mu(A_d) = (k, i)\}$.

Let $M' = (D', H', X', R') \in \mathcal{M}$ be as follows:

– $D' = D$, $H' = H$, and $X' = X$,

– R_i' is such that, for each $X' \in \mathcal{X}_i$, we have $X' \; R_i' \; \emptyset$ if and only if $X' \subseteq A_i \cup X^i$.

– for each $j \in J$, R_j' is such that, for each $x \in \mathcal{X}_j$, we have $\{x\} \; R_j' \; \emptyset$ if and only if $x \in A_j \cup X^i$ and $\{x_{ji}\} \; R_j \; A_j$.

– for each $k \in K$, R_k' is such that, for each $x \in \mathcal{X}_k$, we have $\{x\} \; R_k' \; \emptyset$ if and only if $\{x\} = A_k$.

– for each $l \in [D \backslash (J \cup K)] \cup [H \backslash \{i\}]$, we have $R'_l = R_l$.

Then, the following statements hold:

(i) $R' \in MT(R, A)$.

(ii) $C(X, R'_i) = X^i$.

(iii) For each $j \in J$, $C(X, R'_j) = \{x_{ji}\}$.

(iv) For each $k \in K$, $C(X, R'_k) = A_k$.

(v) $X^i \neq A_i$.

By Lemma 2.1, ***(ii)***, ***(iii)***, and ***(iv)*** imply that, for each $A' \in \varphi(M')$, we have $A'_i = X^i$. By ***(v)***, $A \notin \varphi(M')$, which, by ***(i)***, contradicts *Maskin-monotonicity*.

Step 2: $\boldsymbol{\varphi(M) = \varphi^S(M)}$. Since φ satisfies *Maskin-monotonicity*, the result follows from Corollary 1 of Haake and Klaus (2005).□

Proof of Proposition 2.4.

Let φ be a rule satisfying the axioms of Proposition 2.4. Also, let $M = (D, H, X, R)$ and $A \in \varphi(M)$. By contradiction, assume $A \notin \varphi^S(M)$. Then, since φ satisfies *weak individual rationality*, there are $i \in H$ and $J \subseteq D$ such that there is $X^i \in \mathcal{X}_i$ with $X^i \neq A_i$ and, for each $j \in J$, there is $\{x_{ji}\} \in X^i$ with $\{x_{ji}\} = C(A \cup \{x_{ji}\}, R_j)$ and $X^i = C(A \cup X^i, R_{h^i})$. Let $K \equiv \{k \in D : \mu(A_k) = (k, i)\}$ and $L \equiv \{l \in H :$ there is $j \in J$ such that $\mu(A_k) = (j, k)\}$. Also, let $J' \equiv \{j \in D :$ there is $l \in L$ such that $\mu(A_j) = (j, l)\}$.

Let $\hat{M} = (\hat{D}, \hat{H}, \hat{X}, \hat{R})$ be as follows:

- $\hat{D} = J \cup K$ and $\hat{H} = \{i\} \cup L$,
- $\hat{X} = \{x \in X : \text{ there is } \{j,k\} \subseteq \hat{D} \cup \hat{H} \text{ with } \mu(x) = (j,k)\}$,
- $\hat{R} = R|_{\hat{X}}$.

Then, for each $j \in J$, $C(\{x_{ji}\} \cup A_j, \hat{R}_j) = \{x_{ji}\}$ and **(*i*)** $C(X^i \cup A_i, \hat{R}_i) = X^i$. By *unanimity* and *own-side population-monotonicity*, Lemma 1 holds. Hence, for each $\hat{A} \in \varphi(\hat{M})$, $\hat{A}_i = X^i$ and for each $j \in J$, $\hat{A}_j = x_{ji}$.

Let $M' = (D', H', X', R')$ be as follows:

- $D' = J \cup K \cup K'$ and $H' = \{i\} \cup L$,
- $X' = \{x \in X : \text{ there is } \{j,k\} \subseteq D' \cup H' \text{ with } \mu(x) = (j,k)\}$,
- $R' = R|_{X'}$.

By *other-side population-monotonicity* and **(*i*)**, for each $A' \in \varphi(M')$, **(*ii*)** $A'_i \; R'_i \; X^i$. Note that M' is the reduced economy of M relative to $D' \cup H'$ at A. Thus, by *weak consistency*, there is $A' \in \varphi(M')$ such that **(*iii*)** $A'_i = A_i$. Then, **(*ii*)** and **(*iii*)** contradict **(*i*)**.$\square$

Proof of Proposition 2.5.

Let φ satisfy the first three axioms in Proposition 2.5. By contradiction, assume that φ is not *weakly individual rational*. Thus, there are $M = (D, H, X, R)$, $A \in \varphi(M)$ and $i \in D \cup H$ such that $A_i \neq \emptyset$ and $C(A_i, R_i) = \emptyset$.

Case 1: There are $j \in D$ and $k \in H$ such that $\mu(A_j) = (j,k)$, $i = j$, and

$A_k\ R_k\ \emptyset$. Then, let $J = \{d \in D :$ there is $x_{dk} \in A_k$ such that $\mu(x) = (d, k)\}$.

Let $M' = (D', H', X', R')$ be as follows:

– $D' = J$ and $H' = \{k\}$,

– $X' = \{x \in X :$ there is $j \in J$ with $\mu(x) = (j, k)\}$,

– $R' = (R_l|_{X'})_{l \in J \cup \{k\}}$.

By *weak consistency*, $A' = A|_{J \cup \{k\}} \in \varphi(M')$. Let $K = \{l \in J : A'_l\ R'_l\ \emptyset\}$. Let $l' \in \mathbb{D} \backslash J$ be such that there is $x_{l'k}$ with $\mu(x_{l'k}) = (l', k)$ and $x_{l'k}\ \mathcal{R}_{l'}\ \emptyset$. Let $\tilde{X} = \bigcup_{l \in K} \{A_l\}$.

Let $M'' = (D'', H'', X'', R'')$ be as follows:

– $D'' = J \cup \{l'\}$ and $H'' = \{k\}$,

– $X'' = X' \cup \{x_{l'k}\}$,

– $R'' = (R''_l)_{l \in D'' \cup H''}$ be such that $C(X'', R''_k) = \tilde{X} \cup \{x_{l'k}\}$, $R''_{l'} = \mathcal{R}_{l'}|_{X''}$, and for each $l \in J$, $R''_l = R'_l$.

By *unanimity*, $\varphi(M'') = \{A''\}$ where for each $l \in D'' \cup H''$, $A''_l = C(X'', R''_l)$. In particular, ***(ii)*** $A''_j = \emptyset$.

Clearly, M' is the D-restriction of M''. By *weak own-side population-monotonicity* and ***(ii)***, for each $A' \in \varphi(M')$, $A'_j = \emptyset$, which contradicts *weak consistency*.

Case 2: There is $h \in H$ such that $i = h$. Then, let $J = \{j \in D :$ there is $x \in A_i$ such that $\mu(x) = (j, i)\}$.

Let $M' = (D', H', X', R')$ be as follows:

– $D' = J$ and $H' = \{i\}$,

– $X' = \{x \in X : \text{ there is } j \in J \text{ with } \mu(x) = (j, i)\}$,

– $R' = (R_k|_{X'})_{k \in J \cup \{i\}}$.

By *weak consistency*, $A' = A|_{J \cup \{i\}} \in \varphi(M')$. Let $k \in \mathbb{H} \backslash \{i\}$.

Let $M'' = (D'', H'', X'', R'')$ be as follows:

– $D'' = J$ and $H'' = \{i, k\}$,

– $X'' = X' \cup \{X^k\}$ where $|X^k| = |J|$, $C(X'', R''_k) = X^k$, for each $j \in J$, there is $x_{jk} \in X^k$ with $\mu(x_{jk}) = (j, k)$ such that $C(X'', R''_j) = \{x_{jk}\}$, and $R''_i = R'_i$.

By Lemma 2.1, $\varphi(M'') = \{A''\}$ where for each $l \in D'' \cup H''$, $A''_l = C(X'', R''_l)$. In particular, ***(ii)*** $A''_i = \emptyset$.

Clearly, M' is the H-restriction of M''. By *own-side population-monotonicity* and ***(ii)***, for each $A' \in \varphi(M')$, $A'_i = \emptyset$, which contradicts *weak consistency*. $\square$

Proof of Proposition 2.6.

Let $M = (D, H, X, R)$ and ***(i)*** $A \in \varphi^S(M)$. Let $M' = (D', H', X', R')$ be the type-2 reduced economy of M relative to $D' \cup H'$ at A. Let $A|_{D' \cup H'} = A'$. By contradiction, assume $A' \notin \varphi^S(M')$.

Case 1: There is $d \in D'$ such that $C(A'_d, R'_d) = \emptyset$. Then, since for each $d \in D'$, $A'_d = A_d$, we have $C(A_d, R_d) = \emptyset$, which contradicts ***(i)***.

Case 2: There is $h \in H'$ such that $C(A'_h, R'_h) \subsetneq A'_h$. Then, by separability, there is $x \in A'_h$ such that $\emptyset\ R'_h\ \{x\}$ and $A_h \backslash \{x\}\ R_h\ A_h$, which contradicts ***(i)***.

Case 3: There are $h \in H'$, $\tilde{D} \subseteq D'$, and $\tilde{X} \nsubseteq A_h$ such that for each $d \in \tilde{D}$, there is $x \in \tilde{X}$ with ***(ii)*** $\{x\} = C(A'_d \cup \{x\}, R'_d)$ and $\tilde{X} = C(A'_h \cup \tilde{X}, R'_h)$. Then, we have $\tilde{X} \backslash A'_h \neq \emptyset$. Moreover, there is $d^* \in \tilde{D}$ and $x^* \in \tilde{X} \backslash A'_h$ with $\mu(x^*) = (d^*, h)$ such that ***(iv)*** $\{x^*\}\ R'_h\ \emptyset$. Otherwise, we would have $\emptyset\ R'_h\ \tilde{X}$ and since $\tilde{X}\ R'_h\ A'_h$, **Case 2** applies. Now, since $x^* \notin A'_h$, then $x^* \notin A_h$. Hence, by ***(iii)*** and separability, ***(iv)*** $A_h \cup \{x^*\}\ R_h\ A_h$. By ***(ii)*** and ***(iv)***, it contradicts ***(i)***.□

Proof of Proposition 2.8.

Assume that the hospital-optimal solution $\varphi^H(M)$ is not *destruction-proof*. Let $M = (D, H, X, R) \in \mathcal{M}$, $h \in H$, $X'_h \subseteq X_h$, and $M' = (D, H, X_{-h}, X'_h, R_{-h}, R_h|_{X'_h})$. Then, we have $\varphi^H_h(M')\ R_h\ \varphi^H_h(M)$. For any $M \in \mathcal{M}$, let $X^t_h(M)$ be the set of all contracts that h offers along the steps of the hospitals-proposing deferred acceptance algorithm at M, up to the t-th step. Let $A^t_h(M)$ be the set of the contracts accepted at the end of the t-th step. There exists a t such that $A^t_h(M')\ R_h\ A^t_h(M)$. Since the preferences are *substitutable*, we have $A^t_h(M) = C(R_h, X^t_h(M))$. Then, $C(R_h, X^t_h(M'))\ R_h$ $C(R_h, X^t_h(M))$. This yield a contradiction because $X^t_h(M') \subseteq X^t_h(M)$. □

Proof of Theorem 2.3.

Assume that the hospitals' preference satisfy all properties stated in the theorem and let φ be a rule which satisfies *stability* and *strategy-proofness*.

First, we will show that the doctor-optimal solution satisfy these properties. By Hatfield and Milgrom (2005), if the firms' preferences satisfy the law of aggregate demand and the substitutes condition, then for the doctor-optimal solution, it is dominant strategy for doctors to reveal truthfully their preferences over contracts. We need to show that hospitals reveal truthfully their preferences over set of contracts. For any $R \in \mathcal{R}$, let $X^t_{h_i}(R)$ be the set of all contracts that have offered to h_i along the steps of the doctors-proposing deferred acceptance algorithm at R, up to the t-th step. Let $A^t_i(R)$ be the set of the contracts at the end of the t-th step. Since the preferences are *substitutable*, we have $A^t_i(R) = C(R_{h_i}, X^t_{h_i}(R))$. Thus, $A_{h_i}(R)\ R_{h_i}\ A^t_{h_i}(R)$ or $A_{h_i}(R) = A^t_{h_i}(R)$. Suppose that h_i reveals R'_{h_i} instead of R_{h_i}. For some k, $X^t_{h_i}(R) = X^t_{h_i}(R_{-h_i}, R'_{h_i})$ holds for $t = 1, 2, ..., k$, and $A^k_{h_i}(R) = C(R_{h_i}, X^k_{h_i}(R)) \neq C(R'_{h_i}, X^h_i(R)) = A^k_{h_i}(R_{-h_i}, R'_{h_i})$. If φ^D is not *strategy-proof*, then $A_{h_i}(R_{-h_i}, R'_{h_i})\ R'_{h_i}\ A^k_{h_i}(R_{-h_i}, R'_{h_i})\ R'_{h_i}\ A^k_{h_i}(R)$ which yields a contradiction to the fact that the hospitals' preferences satisfy top-dominance condition. Second, we will show that any *stable* rule which is not the doctor-optimal solution is *strategy-proof*. Assume that $A(M)$ is not the doctor-optimal solution. Then, there exists a doctor d who has not the contract that he would get under the doctor-optimal solution. By Hatfield and Milgrom (2005), the set of stable allocation is a nonempty finite lattice. Then, $A^D_d(M)\ R_d\ A_d(M)$. Now, consider the following preference relation R'_d, for each $\{x, x'\} \subseteq X_d$, $x \ R_d \ x'$ implies $x \ R'_d \ x'$, for each $x \in X_d \backslash \{A^D_d(M)\}$, $x \ R_d \ A^D_d(M)$ if and only if $x \ R'_d \ \emptyset$, and $A^D_d(M) \ R'_d \ \emptyset$. The doctor-optimal solution is still stable at $M' = (D, H, R_{-d}, R'_d, X)$. If the hospitals' preferences satisfy the law of aggregate demand and the substitutes condition, then at every stable allocation, the same doctors are employed and every hospital fills the same number of positions. Since $A^D_d(M') \neq \emptyset$, at another stable allocation $A_d(M') \neq \emptyset$ and $A_d(M') \ R'_d \ \emptyset$. Then, we have $A_d(M') \ R'_d \ A_d(M)$, which is contradictory to *strategy-proofness*. $\square$