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Chapter 9

The Shapley Transfer Procedure for NTU-Games

BY GERT-JAN OTTEN AND HANS PETERS

9.1 Introduction

A cooperative game is described by sets of feasible utility vectors, one set for each coalition. Such a game may arise from each situation where involved parties can achieve gains from cooperation. Examples range from exchange economies to cost allocation between divisions of multinationals or power distribution within political systems. The two central questions are: which coalitions will form; and on which payoffs will each formed coalition agree. Since an answer to the latter question seems a prerequisite to study the former question of coalition formation, most of the literature has concentrated on the question of payoff distribution. Specifically, the usual assumption is that the grand coalition of all players will form and then the question is which payoff vector(s) this coalition will agree upon.

This question has been studied extensively for two special cases: games with transferable utility, and pure bargaining games.

In a game with transferable utility, what each coalition can do is described by just one number: the total utility or payoff, which that coalition can distribute among its members in any way it wants. The underlying assumption is the presence of a common medium of exchange in which the players' utilities are linear. For instance, the payoff is in monetary units and the players have linear utility for money.

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In a pure bargaining game intermediate coalitions—coalitions other than the grand coalition or individual players—play no role. Because of these simplifying features, both types of games are easier to analyse than general cooperative games, also called games with nontransferable utility or NTU-games.

A solution is a map that assigns to every game within a certain subclass of NTU-games a feasible payoff vector or set of feasible payoff vectors for the grand coalition. In an important article, Shapley (1969) proposed a procedure to extend single-valued solutions defined on the class of games with transferable utility and satisfying a few minimal conditions, to NTU-games. This procedure works as follows. For a given NTU-game consider any vector of nonnegative weights for the players. For every coalition, maximize the correspondingly weighted sum of the utilities of its members over the set of feasible payoffs of that coalition. Regard these coalitional maxima as a game with transferable utility, and apply the given solution for transferable utility games to this game: those payoff vectors of the original NTU-game that, when similarly weighted, belong to the solution of the TU-game, are defined to be in the solution of the NTU-game. The complete solution of the NTU-game is then obtained by repeating this procedure for every possible weight vector.

This Shapley transfer procedure has been applied in particular to the Shapley value for games with transferable utility (Shapley, 1953), resulting in the 'nontransferable utility value' (Aumann, 1985) for NTU-games. For pure bargaining games, this solution coincides with the Nash bargaining solution, proposed by Nash (1950) for the case of two players. As indicated, however, by Shapley (1969) the procedure can be applied to a variety of solutions; also the existence result established by Shapley holds under quite mild conditions. An explicit example of this is the NTU τ -value studied in Borm *et al.* (1992). Another example is the so called 'inner core' studied by Qin (1994) but proposed earlier by Shapley (1984).

The main objective of the present contribution is twofold. First we review and extend the Shapley transfer procedure; the extension is to any compact and convex valued, continuous solution. Shapley's existence result will be re-established for this extension. Second, we characterize solutions that are obtained by this procedure. This characterization can be seen as an alternative description of the Shapley transfer procedure. The price for obtaining existence and characterization within the same framework is the occurrence of zero weights.

Section 2 introduces the Shapley transfer procedure. Section 3 contains the existence result and a digression on a well known and earlier procedure proposed by Harsanyi (1959, 1963)—to which the existence result applies equally. In Section 4 the announced characterization is presented. Section 5 discusses applications to several TU-solutions: the Shapley value, the core, the nucleolus, and the τ -value. Section 6 concludes.

Notations. For a finite subset S of the natural numbers let $\mathbb{R}_+^S$ denote the nonnegative orthant of $\mathbb{R}^S$. For $x, y \in \mathbb{R}^S$ denote $x > y$ if $x_i > y_i$ for every $i \in S$, and denote $x \geq y$ if $x_i \geq y_i$ for every $i \in S$. The vector inequalities $<, \leq$ are defined analogously. The $\cdot$ denotes the usual inner product: $x \cdot y = \sum_{i \in S} x_i y_i$. The product xy denotes the vector in $\mathbb{R}^S$ with i th coordinate equal to $x_i y_i$ ($i \in S$). For $Y \subseteq \mathbb{R}^S$, $xY := \{xy : y \in Y\}$; for a real number a , $aY := (a, \dots, a)Y$. For another finite set of natural numbers M with $M \cap S \neq \emptyset$, $x \in \mathbb{R}^S$ and $Y \subseteq \mathbb{R}^S$ let $x_M \in \mathbb{R}^{M \cap S}$ be defined by $(x_M)_i := x_i$ for every $i \in M \cap S$; let $Y_M := \{y_M \in \mathbb{R}^{M \cap S} : y \in Y\}$. Thus, the set Y_M is the projection of Y on the $M \cap S$ -plane.

9.2 Main Concepts

Let $N = \{1, 2, \dots, n\}$ denote the set of *players*. A *coalition* is a nonempty subset of N . A subset $D \subseteq \mathbb{R}_+^S$ is *comprehensive* if $x \in D$ and $y \leq x$ imply $y \in D$ for all $x, y \in \mathbb{R}_+^S$. The *Pareto optimal* subset of D is the set

$$PO(D) := \{x \in D : \text{for all } y \in D, \text{ if } x \leq y \text{ then } x = y\}$$

and the *weakly Pareto optimal* subset of D is the set

$$WPO(D) := \{x \in D : \text{for all } y \in \mathbb{R}_+^S, \text{ if } x < y \text{ then } y \notin D\}.$$

A *nontransferable utility game* or NTU-game is a pair (N, V) where V assigns to every coalition S a *feasible set* $V(S)$ such that

- (N1) $V(\{i\}) = \{0\} \subseteq \mathbb{R}$ for every $i \in N$;
- (N2) $V(S)$ is a nonempty compact convex and comprehensive subset of $\mathbb{R}_+^S$ for every coalition S ;
- (N3) $PO(V(S)) = WPO(V(S))$ for every coalition S .

These assumptions, though restrictive, are still quite standard. The normalization in (N1) is mainly for convenience; it is not innocent because together with (N2) it implies, for instance, that every coalition can at least as good as singleton coalitions. The convexity assumption in (N2) may arise from the players having von Neumann-Morgenstern utility functions over uncertain outcomes, or concave ordinal utility functions over bundles of goods. It is essential to what follows. Condition (N3) means that, for every coalition, every weakly Pareto optimal point is also Pareto optimal: there are no flat segments in the weakly Pareto optimal boundary of $V(S)$. One consequence is that if $x \in PO(V(S))$ with $x_i > 0$ for some $i \in S$, then there is a $y \in PO(V(S))$ with $0 < y_i < x_i$ and $y_j > x_j$ for all $j \neq i$; note that in that case $y > 0$. It follows, in particular, that either $V(S) = \{0\}$ or there is a $y \in V(S)$ with $y > 0$.

If $V(S) = \{0\} \subseteq \mathbb{R}_+^S$ for every $S \neq N$, then (N, V) is called a *pure bargaining game*. If for every coalition S there is a nonnegative real number $v(S)$ such that $V(S) = \{x \in \mathbb{R}_+^S : \sum_{i \in S} x_i \leq v(S)\}$, then (N, V) is called a *game with transferable utility* or TU-game. Such a TU-game is sometimes also denoted by (N, v) . Our definition deviates from the usual one in that all payoff vectors are restricted to the nonnegative orthant.

Instead of (N, V) or (N, v) we will usually write V or v , with the understanding that the player set is N .

The class of NTU-games [TU-games, pure bargaining games] with player set N is denoted by $\mathcal{G}^N [T^N, B^N]$. Often the superscript ' N ' is omitted. Subclasses are denoted by $\mathcal{G}'$, T' , etc.

Let $\mathcal{G}' \subseteq \mathcal{G}$ be a subclass of NTU-games. An *NTU-solution* is a correspondence $\Phi : \mathcal{G}' \rightarrow \mathbb{R}_+^N$ that assigns to each NTU-game $V \in \mathcal{G}'$ a set $\Phi(V) \subseteq V(N)$. (We use $\rightarrow$ to denote a correspondence, i.e., a set-valued function.) If $\mathcal{G}' \subseteq T$ then Φ is also called a TU-solution. Usually TU-solutions are denoted by small characters, e.g., φ .

A TU-solution φ defined on a class $T' \subseteq T$ is *regular* if it satisfies the following three conditions. In condition (T3), for an NTU-game V and a real number $a > 0$ we denote by aV the NTU-game with $(aV)(S) := aV(S)$ for every coalition S .

- (T1) $\varphi(V)$ is a nonempty, compact, and convex subset of $PO(V(N))$ for every $V \in T'$;
- (T2) φ is continuous on T' ;
- (T3) φ is *homogeneous*, that is, for every $V \in T'$ and real number $a > 0$,

$$aV \in T' \text{ implies } \varphi(aV) = a\varphi(V).$$

Here, continuity is meant with respect to the restriction to T' of the Euclidean metric on $\mathbb{R}^{2^N \setminus \{0\}} \cong T^N$ and the Hausdorff metric for compact sets in $\mathbb{R}^N$. Conditions (T1), (T2) and (T3) are not very restrictive. Most known single-valued solutions (e.g., Shapley value, nucleolus, τ -value) are continuous and homogeneous on the classes of TU-games on which they are defined. The best known multi-valued concept, the core, satisfies (T1), (T2), and (T3) on the class of balanced games. See Section 5 for some of the details.

For an arbitrary NTU-game V and an arbitrary vector $\lambda \in \mathbb{R}_+^N \setminus \{0\}$ the associated λ -transfer game V^λ is the transferable utility game defined by

$$V^\lambda(S) := \{x \in \mathbb{R}_+^S : \sum_{i \in S} x_i \leq v^\lambda(S)\}$$

where

$$v^\lambda(S) := \max\{\lambda_S \cdot x : x \in V(S)\}$$

for every coalition S . By (N1) and (N2) these numbers $v^\lambda(S)$ are well defined. For a class $\mathcal{G}'$ of NTU-games denote by

$$T(\mathcal{G}') := \{V^\lambda \in T^N : V \in \mathcal{G}', \lambda \in \mathbb{R}_+^N \setminus \{0\}\}$$

the class of all TU-games that arise as transfer games associated with NTU-games in $\mathcal{G}'$. Let $\varphi : T' \rightarrow \mathbb{R}^N$ be a regular TU-solution defined on $T(\mathcal{G}')$. We extend φ to an NTU-solution $\bar{\varphi} : \mathcal{G}' \rightarrow \mathbb{R}^N$, as follows. For each $V \in \mathcal{G}'$ and $x \in PO(V(N))$,

$$x \in \bar{\varphi}(V) :\Leftrightarrow \exists \lambda \in \mathbb{R}_+^N \setminus \{0\}, y \in \varphi(V^\lambda) : y = \lambda x.$$

One way to understand this procedure (cf. Qin, 1994) is to think of the players as countries and of the coordinates of λ as exchange rates between these countries. Then the transfer game V^λ expresses what coalitions of countries can do in real monetary terms, and the vector y represents a payoff distribution in real monetary terms. If this payoff distribution is feasible in terms of the original individual currencies, then it is a solution of the game.

We call $\bar{\varphi}$ the *transfer solution* associated with φ . Observe that not all of the properties in (T1) and (T2) are trivially inherited by $\bar{\varphi}$. In this contribution we will be mainly concerned with existence, i.e., nonemptiness.

The transfer procedure can also be applied to TU-games, resulting in an extension of a TU-solution to the associated transfer solution on TU-games. We observe:

Lemma 9.1 *Let φ be a regular TU-solution on a class T' . Let $V \in T'$ with $T(\{V\}) \subseteq T'$. Then $\varphi(V) \subseteq \bar{\varphi}(V)$.*

Proof. Let $x \in \varphi(V)$ and $\lambda = (1, 1, \dots, 1)$. Then $V^\lambda(S) = V(S)$ for every coalition S , so $\lambda x = x \in \varphi(V) = \varphi(V^\lambda)$. Hence, $x \in \bar{\varphi}(V)$. $\square$

Observe that actually we do not need regularity for Lemma 9.1 to hold. The inclusion in the lemma, however, can be strict, even for regular solutions, as the following example shows.

Example 9.2 Let $N = \{1, 2, 3\}$ and for every $v \in T^N$ let $\epsilon(v) := \frac{1}{3} \min\{v(N)/2v(\{2, 3\}), 1\}$ if $v(\{2, 3\}) > 0$ and $\epsilon(v) := \frac{1}{3}$ otherwise. Define $\varphi : T^N \rightarrow \mathbb{R}^N$ by $\varphi(v) := \{(\frac{1}{3} + 2\epsilon(v), \frac{1}{3} - \epsilon(v), \frac{1}{3} - \epsilon(v))v(N)\}$ for every $v \in T^N$. Observe that φ is a regular TU-solution. Consider the game w with $w(S) = 1$ for all coalitions S with more than one player. Then $\epsilon(w) = \frac{1}{6}$ and $\varphi(w) = \{(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})\}$. Let $\lambda = (4, 1, 1)$. Then $w^\lambda(N) = 4$, $w^\lambda(\{2, 3\}) = 1$, $\epsilon(w^\lambda) = \frac{1}{3}$, and $\varphi(w^\lambda) = \{(4, 0, 0)\}$. Since $\lambda(1, 0, 0) = (4, 0, 0)$, it follows that $(1, 0, 0) \in \bar{\varphi}(w)$. Hence, $\bar{\varphi}(w)$ is strictly larger than $\varphi(w)$. $\triangleleft$

Note that the possibility of the strict inclusion $\varphi(w) \subset \bar{\varphi}(w)$ is not due to the possibility of zero coordinates of λ , but to the occurrence of boundary solution points with zero coordinates.

In Section 5 we present examples showing that also for TU-solutions such as the core and the nucleolus the inclusion in Lemma 9.1 can be strict: hence, the transfer procedure applied to TU-games may add additional solution outcomes. This will not be the case for the Shapley value or the τ -value.

We conclude this section by reviewing a well known application of this procedure. Let, as above, $\mathcal{B} = \mathcal{B}^N$ be the class of pure bargaining games. The class $T(\mathcal{B})$ of associated transfer games consists of all TU-games v with $v(S) = 0$ for all $S \neq N$. Consider the equal-split solution ϵ on $T(\mathcal{B})$, that is,

$$\epsilon(v) := \left\{ \frac{1}{n}(v(N), \dots, v(N)) \right\} \text{ for every } v \in T(\mathcal{B}).$$

Let V be a pure bargaining game such that $x \in V(N)$ for some $x > 0$. (This is without loss of generality since the only alternative case is $V(N) = \{0\}$, see above.) Then $v^\lambda(N) > 0$ for every $\lambda \in \mathbb{R}_+^N \setminus \{0\}$. Let $z \in PO(V(N))$, then $z \in \bar{\epsilon}(V)$ if, and only if, there is a $\lambda \in \mathbb{R}_+^N \setminus \{0\}$ such that $\lambda z = (v^\lambda(N)/n)(1, \dots, 1)$. Since $v^\lambda(N) > 0$ for every λ it follows that both z and λ are positive and hence $\lambda = k(1/z_1, \dots, 1/z_n)$, where $k = v^\lambda(N)/n$, and by definition of $v^\lambda(N)$, there is a hyperplane supporting $V(N)$ at z with normal λ . Consider the product $\prod_{i \in N} x_i$ on $V(N)$. At z , the supporting hyperplane to the level curve of this product has normal $(1/z_1, \dots, 1/z_n)$ as follows straightforwardly by partial differentiation; hence this hyperplane also supports $V(N)$. It follows that the product $\prod_{i \in N} x_i$ is maximized on $V(N)$ at z . So z is the (n -person) Nash bargaining solution outcome for V (Nash, 1950). Thus, $\bar{\epsilon}(V)$ contains exactly one point, which is the Nash bargaining solution outcome.

9.3 Nonemptiness of Transfer Solutions

In this section we show that under the imposed conditions a transfer solution assigns a nonempty set of payoff vectors to any NTU-game for which it is defined. This result extends Shapley's (1969) existence result to the case where the TU-solution under consideration may be a correspondence. The proof closely follows Shapley's proof.

In order to obtain a compact set we normalize the λ -transfer vectors. Specifically, let $\Delta^N := \{\lambda \in \mathbb{R}_+^N : \sum_{i \in N} \lambda_i = 1\}$ denote the $(n-1)$ -dimensional simplex in $\mathbb{R}^N$. Elements of Δ^N are also called *weight vectors*.

Theorem 9.3 *Let $V \in \mathcal{G}^N$ and let $T' := \{V^\lambda : \lambda \in \Delta^N\}$. Let $\varphi : T' \rightarrow \mathbb{R}^N$ be a regular TU-solution. Then $\bar{\varphi}(V) \neq \emptyset$.*

Proof. Define the correspondence $P : \Delta^N \rightarrow \mathbb{R}^N$ by

$$P(\lambda) := \left\{ \pi \in \mathbb{R}^N : \sum_{i \in N} \pi_i = 0, \exists x \in \lambda PO(V(N)), y \in \varphi(V^\lambda) [\pi = y - x] \right\}.$$

So P assigns to a weight vector the set of all 'sidepayments' by which solution payoff vectors of the transfer game are carried over to feasible elements of the correspondingly weighted NTU-game. Note that P is nonempty, convex and compact valued. Moreover, it is upper semicontinuous since $\lambda \mapsto \varphi(V^\lambda)$ is continuous in the Hausdorff metric, and

$\lambda \mapsto PO(V^\lambda(N)) \cap \lambda V(N)$ is upper semicontinuous. These properties are inherited by the correspondence $L: \Delta^N \rightarrow \mathbb{R}^N$ defined by

$$L: \lambda \mapsto \lambda + P(\lambda) := \{\lambda + \pi : \pi \in P(\lambda)\}.$$

In particular this implies compactness of the set $\bigcup_{\lambda \in \Delta^N} L(\lambda)$. Therefore we can find a compact and convex subset D of $\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = 1\}$ such that $D \supseteq \Delta^N \cup (\bigcup_{\lambda \in \Delta^N} L(\lambda))$. Extend the correspondence L to D by defining, for every $d \in D$

$$L(d) := L(f(d))$$

where $f_i(d) := \max\{0, d_i\} / \sum_{j \in N} \max\{0, d_j\}$ for every $i \in N$.

Since the projection f is also continuous, Kakutani's fixed point theorem implies that there is a $d^* \in D$ with $d^* \in L(d^*)$. Let $\lambda^* := f(d^*) \in \Delta^N$.

If $d^* = \lambda^*$, then $0 \in P(\lambda^*)$, hence by definition of P there exists $y \in \varphi(V^{\lambda^*}) \cap \lambda^* PO(V(N))$. So $y \in \bar{\varphi}(V)$ and the proof is complete.

Suppose $d^* \neq \lambda^*$. We will show that this is not possible. In this case there is an $i \in N$ with $\lambda_i^* = 0 > d_i^*$. Since $d^* \in L(d^*) = L(f(d^*)) = L(\lambda^*) = \lambda^* + P(\lambda^*)$, there exists a $\pi^* \in P(\lambda^*)$ with $\pi_i^* < 0$. Moreover, by definition we have $x \geq 0$ for all $x \in \varphi(V^{\lambda^*})$; take $x^* \in \varphi(V^{\lambda^*})$ with $x^* - \pi^* \in \lambda^* PO(V(N))$ then in particular $x_i^* - \pi_i^* > 0$, in contradiction with $\lambda_i^* = 0$. $\square$

By checking the proof of this theorem one observes that it would be valid for any other transfer procedure as long as the resulting TU-games for a specific NTU-game depend continuously on the weight vector. One example of such a procedure is the one underlying the definition of the Harsanyi NTU-value (Harsanyi, 1959, 1963).

In order to define this procedure let V be an NTU-game and let λ be a weight vector. We first assume that λ has only positive coordinates. We recursively define the dividends $d_S \in \mathbb{R}$ ($\emptyset \neq S \subseteq N$) as follows: $d_{\{i\}} := 0$ for every $i \in N$, and for S with more than one player:

$$d_S := \max\{t \in \mathbb{R} : x^S(t) \in V(S)\},$$

where $x_i^S(t)$ for every $i \in S$ is given by

$$x_i^S(t) = \lambda_i^{-1} \left(t + \sum_{T: i \in T \subseteq S, T \neq S} d_T \right).$$

Observe that $x_i^{\{i,j\}}(t) = t/\lambda_i$ for all $i, j \in N$, so that $d_{\{i,j\}}(\lambda_i^{-1}, \lambda_j^{-1})$ is the maximal element in the direction reciprocal to $\lambda_{\{i,j\}}$ on the boundary

of $V(\{i, j\})$. For coalitions of three and more players the idea is similar, but the starting point is, generally speaking, no longer the origin. Now define, for each coalition S , the vector $x^S := x^S(d_S)$, and let v_d^λ be defined by

$$v_d^\lambda(S) := \begin{cases} \lambda_S \cdot x^S & \text{if } S \neq N \\ \max\{\lambda \cdot x : x \in V(N)\} & \text{if } S = N. \end{cases}$$

Note that there is an asymmetry in this definition between the grand coalition and the smaller coalitions. We comment on this below.

That v_d^λ is actually a TU-game follows immediately since $x^S \in V(S) \subseteq \mathbb{R}_+^S$ by definition. For positive λ the game v_d^λ depends continuously on λ . We still have to define the games v_d^λ for the case where λ has one or more coordinates equal to zero. The following lemma shows that this can be done by taking limits.

Lemma 9.4 *Let $\lambda \in \Delta^N$ and let V be an NTU-game. Then there is a TU-game v such that $v_d^{\lambda^k} \rightarrow v$ for any sequence $\lambda^1 > 0, \lambda^2 > 0, \dots$ in Δ^N with $\lambda^k \rightarrow \lambda$.*

Proof. Let $\lambda^1 > 0, \lambda^2 > 0, \dots$ in Δ^N with $\lambda^k \rightarrow \lambda$. Then obviously $v_d^{\lambda^k}(N)$ converges: call the limit $v(N)$. For $S \neq N$ and $k \geq 1$, let $(x^S)^k$ be the vector as defined above associated with λ^k . Since $(x^S)^k$ is in the compact set $V(S)$ for every k , we have $\lim_{k \rightarrow \infty} \lambda_S^k \cdot (x^S)^k = \lim_{k \rightarrow \infty} \lambda_S^k \cdot (x^S)^k_{S'}$, where $S' := \{i \in S : \lambda_i > 0\}$. Since $(x^S)^k_{S'}, (x^S)^k_{S'}, \dots$ converges to $x^S_{S'}$ for some $x^S \in V(S)$, the proof is complete by defining $v(S) := \lambda_S \cdot x^S$. $\square$

In view of this lemma we can define a collection of Harsanyi transfer games V_d^λ for every $\lambda \in \Delta^N$. The extension of a TU-solution can be defined completely analogous as in the case of the Shapley transfer procedure. Since the Harsanyi transfer games again depend continuously on λ the existence result, Theorem 9.3, also holds for this case:

Theorem 9.5 *Under the assumptions in Theorem 9.3, $\hat{\varphi}(V) \neq \emptyset$, where $\hat{\varphi}$ is the extension of φ according to the Harsanyi procedure.*

One application is to take the Shapley value as a TU-solution: this results in the so-called Harsanyi NTU-value under the Harsanyi transfer procedure. Specifically, if in the definition of the TU-game v_d^λ above we would take $v_d^\lambda(N) = \lambda \cdot x^N$, then x^N would be the Shapley value of the

resulting TU-game. If, additionally, x^N maximizes $\lambda \cdot x$ on $V(N)$ then x^N is said to be in the Harsanyi solution. Hence, such points x^N result by applying the Harsanyi transfer procedure to the Shapley TU-solution. See Hart (1985) for a characterization.

9.4 A Characterization

In this section we present a general characterization of NTU-solutions that are obtained by extending regular TU-solutions through the Shapley transfer procedure.

Let Φ be an NTU-solution defined on a class $\mathcal{G}'$ of NTU-games. We list the following possible properties of Φ .

Property 9.6 Φ is *Pareto optimal* if $\Phi(V) \subseteq PO(V(N))$ for every $V \in \mathcal{G}'$.

This property needs no further explanation.

In the following property, for a game V and a positive vector $\alpha \in \mathbb{R}^N$, αV denotes the game defined by $(\alpha V)(S) := \alpha_S V(S)$ for every nonempty coalition $S \subseteq N$.

Property 9.7 Φ is *scale covariant* if for every $V \in \mathcal{G}'$, every $x \in \Phi(V)$, and every $\alpha \in \mathbb{R}^N$ with $\alpha > 0$ we have: if $\alpha V \in \mathcal{G}'$, then $\alpha x \in \Phi(\alpha V)$.

One possible interpretation of this property is that the players have cardinal utility functions that are unique only up to a positive affine transformation.

For the next two properties and the ensuing characterization result we need to introduce some additional notation. For a game V , a nonempty coalition S and a nonnegative vector $\alpha \in \mathbb{R}^N$ denote by $H(V, S, \alpha)$ the halfspace of $\mathbb{R}^S$ of all points on or below the hyperplane with normal α_S supporting $V(S)$ from above, and by $\partial H(V, S, \alpha)$ its boundary. Thus, for any point of tangency $y \in PO(V(S)) \cap H(V, S, \alpha)$,

$$H(V, S, \alpha) = \{x \in \mathbb{R}^S : \alpha_S \cdot x \leq \alpha_S \cdot y\},$$

and

$$\partial H(V, S, \alpha) = \{x \in \mathbb{R}^S : \alpha_S \cdot x = \alpha_S \cdot y\}.$$

For $\alpha \in \mathbb{R}^N$ with $\alpha > 0$, an NTU-game V is called an (α) -hyperplane game if for every nonempty coalition S the set $PO(V(S))$ coincides with the nonnegative part of a hyperplane with normal α_S . Note that every TU-game is a hyperplane game.

Property 9.8 Φ is *expansion independent* if for every $V \in \mathcal{G}'$ and $x \in \Phi(V) \cap PO(V(N))$ there is a $\lambda \in \mathbb{R}_+^N \setminus \{0\}$ with $x \in \partial(H(V, N, \lambda))$ such that, for $L := \{i \in N : \lambda_i > 0\}$ and for all $V' \in \mathcal{G}'$ satisfying $V(S)_L \subseteq V'(S)_L \subseteq H(V, S, \lambda)_L$ for all S with $S \cap L \neq \emptyset$, we have: $x \in \Phi(V')$.

This property captures the essence of the Shapley transfer procedure. If a game is extended by allowing sidepayments that preserve the utility comparison ratios between the players at a certain solution outcome, then that outcome should still belong to the solution of the extended game. The property, naturally, requires this to hold also for games in between the original game and the game extended by sidepayments, although this is not needed for the characterization theorem below. If the utility comparison ratios are not uniquely determined—which is the case if there is no unique supporting hyperplane at the solution outcome—then the requirement applies to at least one set of ratios. A proviso is made for the zero components of the vector λ in the formulation of Property 9.8; by property (N3) of an NTU-game it follows that players outside the set L must have zero at x , and then Property 9.8 implies that these players will stay at zero in the solution.

The expansion independence property was first introduced by Thomson (1981) in the context of pure bargaining problems.

The fourth property is in a sense the 'dual' of Property 9.8.

Property 9.9 Φ is *contraction independent* if for every hyperplane game $V \in \mathcal{G}'$, every $x \in \Phi(V)$, and every $V' \in \mathcal{G}'$ with $x \in V'(N)$ such that, for $L := \{i \in N : x_i > 0\}$, the set $PO(V(S)_L)$ supports $V'(S)_L$ from above for every coalition S with $S \cap L \neq \emptyset$, we have: $x \in \Phi(V')$.

This property is a variant of the well known 'independence of irrelevant alternatives' condition proposed by Nash (1950).

The characterization result is as follows.

Theorem 9.10 Let Φ be an NTU-solution defined on a class $\mathcal{G}'$ of NTU-games containing all hyperplane games, and let φ be a regular TU-solution on $T(\mathcal{G}')$. The following two statements are equivalent:

- (i) Φ satisfies Properties 9.6–9.9 and $\Phi(V) = \varphi(V)$ for every $V \in T(\mathcal{G}')$.
- (ii) $\Phi(V) = \varphi(V)$ for every $V \in \mathcal{G}'$.

Proof. In this proof we use the following fact, the proof of which is left to the reader.

Fact. Let $V \in \mathcal{G}'$ and let $\alpha, \lambda \in \mathbb{R}_+^N$ with $\alpha > 0$. Then $(\alpha V)^\lambda = V^{\alpha\lambda}$.

Proof of (ii) $\Rightarrow$ (i). Assume that (ii) holds. To prove (i), we have to show that $\bar{\varphi}$ satisfies Properties 9.6–9.9.

Pareto optimality of $\bar{\varphi}$ follows by definition.

For scale covariance, let $V \in \mathcal{G}'$ and $\alpha \in \mathbb{R}^N$ with $\alpha > 0$ and $\alpha V \in \mathcal{G}'$. Let $x \in \bar{\varphi}(V)$ and $\mu \in \mathbb{R}_+^N \setminus \{0\}$ with $\mu x \in \varphi(V^\mu)$. Define $\lambda \in \mathbb{R}_+^N \setminus \{0\}$ by $\lambda_i := \mu_i/\alpha_i$ for every $i \in N$. Then $\lambda\alpha = \mu$, so $\lambda\alpha x \in \varphi(V^{\lambda\alpha}) = \varphi((\alpha V)^\lambda)$ by Fact (i). Hence, $\alpha x \in \bar{\varphi}(\alpha V)$. This implies scale covariance of $\bar{\varphi}$.

For expansion independence, take $V \in \mathcal{G}'$ and $x \in \bar{\varphi}(V)$. Let $\lambda \in \mathbb{R}_+^N \setminus \{0\}$ with $\lambda x \in \varphi(V^\lambda)$. Then $x \in \partial(H(V, N, \lambda))$. Let L and V' as in the definition of Property 9.8. Then by (N3) $x_{N \setminus L} = 0$, $(V')^\lambda = V^\lambda$ and $x \in PO(V'(N))$. Hence, $x \in \bar{\varphi}(V')$, which proves expansion independence.

Finally, for contraction independence, let $V \in \mathcal{G}'$ be an α -hyperplane game ($\alpha > 0$). Let $x \in \bar{\varphi}(V)$, hence there is a $\lambda \in \mathbb{R}_+^N \setminus \{0\}$ with $\lambda x \in \varphi(V^\lambda)$. Observe that for $i \in L := \{j \in N : x_j > 0\}$ we have $\lambda_i > 0$, otherwise there would be a y in $V(N)$ with $\lambda \cdot y > \lambda \cdot x$, contradicting $\lambda x \in \varphi(V^\lambda)$. Then for V' as in Property 9.9 it follows that $(V')^\lambda = V^\lambda$. Together with $x \in V'(N)$ this implies $x \in \bar{\varphi}(V')$.

Proof of (i) $\Rightarrow$ (ii). Assume that (i) holds. Let $V \in \mathcal{G}'$ and $x \in \Phi(V)$. We prove that $x \in \bar{\varphi}(V)$ and, thus, that $\Phi(V) \subseteq \bar{\varphi}(V)$. By Pareto optimality and expansion independence of Φ we can take λ and L as in Property 9.8. Define $\lambda' \in \mathbb{R}^N$ by $\lambda'_i := \lambda_i$ if $i \in L$ and $\lambda'_i := 1$ if $i \in N \setminus L$. For V' as in Property 9.8 take a λ' -hyperplane game. Property 9.8, expansion independence, implies $x \in \Phi(V')$. By scale covariance, $(\lambda')^{-1}x \in \Phi((\lambda')^{-1}V')$, where $(\lambda')^{-1}_i := 1/\lambda'_i$ for every $i \in N$. Since $(\lambda')^{-1}V'$ is a TU-game, this implies $(\lambda')^{-1}x \in \bar{\varphi}((\lambda')^{-1}V')$. Hence, by scale covariance, $x \in \bar{\varphi}(V')$, and by Property 4: $x \in \bar{\varphi}(V)$.

For the converse implication, let $x \in \bar{\varphi}(V)$. We show that $x \in \Phi(V)$, which completes the proof of the theorem. Let $\lambda \in \mathbb{R}_+^N \setminus \{0\}$ with $\lambda x \in \varphi(V^\lambda)$. By Lemma 9.1, $\varphi(V^\lambda) \subseteq \bar{\varphi}(V^\lambda)$; and since $\bar{\varphi}(V^\lambda) = \Phi(V^\lambda)$, we have $\lambda x \in \Phi(V^\lambda)$. Define $\mu \in \mathbb{R}^N$ by $\mu_i := \lambda_i^{-1}$ if $\lambda_i > 0$ and $\mu_i := 1$ if $\lambda_i = 0$. By scale covariance and noting that $L := \{i \in N : x_i > 0\} \subseteq \{i \in N : \lambda_i > 0\}$, we obtain $x = \mu\lambda x \in \Phi(\mu V^\lambda)$. Now the game μV^λ is a μ^{-1} -hyperplane game and V satisfies the requirements for V' with respect to this hyperplane game as in Property 9.9, contraction

independence. Hence, this property implies $x \in \Phi(V)$. $\square$

9.5 Applications

The Shapley transfer procedure and the corresponding results on existence and characterization can be applied to most known solutions for TU-games. Here, we consider applications to the Shapley value, the core, the nucleolus, and the τ -value.

First we state a lemma characterizing the transfer games associated with TU-games. The proof is straightforward and left to the reader. For a vector $\lambda \in \mathbb{R}_+^N \setminus \{0\}$ and a coalition $S \subseteq N$ denote $\bar{\lambda}_S := \max\{\lambda_i : i \in S\}$.

Lemma 9.11 *Let $v \in \mathcal{T}$ and $\lambda \in \mathbb{R}_+^N \setminus \{0\}$. Then $v^\lambda(S) = \bar{\lambda}_S v(S)$ for every coalition $S \subseteq N$. If x is efficient in v , i.e., $\sum_{i \in N} x_i = v(N)$, and $\lambda \cdot x = v^\lambda(N)$, then $x_i = 0$ for every $i \in N$ for which $\lambda_i < \bar{\lambda}_N$.*

9.5.1 The Shapley Value

For a TU-game v the Shapley value σ (Shapley, 1953) is defined by

$$\sigma_i(v) := \sum_{S \subseteq N: i \notin S} \frac{|S|!(n-|S|-1)!}{n!} [v(S \cup \{i\}) - v(S)]$$

for every $i \in N$, where $v(\emptyset) := 0$, and $|\cdot|$ denotes the cardinality of a finite set. The Shapley TU-solution assigns the set $\{\sigma(v)\}$ to a game v . With some abuse of notation we use σ for the Shapley solution and omit the set-brackets. An alternative definition using dividends (cf. Section 3) is also possible. Within our framework, the Shapley value is well defined as long as $\sigma(v) \geq 0$. Call an NTU-game V *monotonic* if $V(S) \subseteq V(T)_S$ whenever $S \subseteq T$. Hence, a TU-game v is monotonic if $v(S) \leq v(T)$ whenever $S \subseteq T$. Denote the corresponding classes of games by $\mathcal{G}_{\text{mon}}$ and $\mathcal{T}_{\text{mon}}$. Then the Shapley value is well defined on $\mathcal{T}_{\text{mon}}$, and $\mathcal{T}_{\text{mon}} = \mathcal{T}(\mathcal{G}_{\text{mon}})$. It is also straightforward to check that σ is a regular TU-solution on $\mathcal{T}_{\text{mon}}$. Moreover, the inclusion in Lemma 9.1 turns out to be an equality, as the following lemma shows.

Lemma 9.12 *Let $v \in \mathcal{T}_{\text{mon}}$. Then $\bar{\sigma}(v) = \sigma(v)$.*

Proof. Let $\lambda \in \mathbb{R}_+^N \setminus \{0\}$ and $x \in \mathbb{R}_+^N$ with $\sum_{i \in N} x_i = v(N)$ such that $\lambda x = \sigma(v^\lambda)$. In view of Lemma 9.1 it is sufficient to prove that $x = \sigma(v)$.

Let $M := \{i \in N : \lambda_i = \bar{\lambda}_N\}$. Then, by Lemma 9.11, $v^\lambda(S) = \bar{\lambda}_N v(S)$ for every coalition S with $S \cap M \neq \emptyset$, and $x_i = 0$ for all $i \in N \setminus M$. By monotonicity of v^λ we have $v^\lambda(S) \geq v^\lambda(S \cap M)$ for every coalition S with $S \cap M \neq \emptyset$. Suppose $v^\lambda(S) > v^\lambda(S \cap M)$ for some S , then by definition of the Shapley value there must be an $i \in N \setminus M$ with $\sigma_i(v^\lambda) > 0$ and hence $\lambda_i x_i > 0$, a contradiction. Hence $v^\lambda(S) = v^\lambda(S \cap M)$ for every coalition S with $S \cap M \neq \emptyset$.

We claim that $v(S) = 0$ for every coalition S with $S \subseteq N \setminus M$. Suppose not, then there is a $T \subseteq N \setminus M$ with $v(T) > 0$. Take $i \in M$ arbitrary. By monotonicity, $v(T \cup \{i\}) > 0$, hence $v^\lambda(T \cup \{i\}) = \bar{\lambda}_N v(T \cup \{i\}) > 0$. Therefore, $v^\lambda(\{i\}) = v^\lambda((T \cup \{i\}) \cap M) = v^\lambda(T \cup \{i\}) > 0$, a contradiction since $v(\{i\}) = 0$ by definition. This proves our claim.

Since $v(S) = 0$ whenever $S \cap M = \emptyset$ and $v^\lambda(S) = \bar{\lambda}_N v(S)$ whenever $S \cap M \neq \emptyset$, we have $v^\lambda = \bar{\lambda}_N v$. Also, $\lambda x = \bar{\lambda}_N x$ since $x_i = 0$ for all $i \in N \setminus M$. Hence $\bar{\lambda}_N x = \sigma(\bar{\lambda}_N v)$, so by homogeneity of σ we have $x = \sigma(v)$. $\square$

Application of the results in the preceding sections now yields:

Corollary 9.13 $\bar{\sigma}(V) \neq \emptyset$ for every $V \in \mathcal{G}_{\text{mon}}$. Moreover, $\bar{\sigma}$ is the unique NTU-solution on $\mathcal{G}_{\text{mon}}$ that satisfies Pareto optimality, scale covariance, expansion and contraction independence and coincides with the Shapley value on $\mathcal{T}_{\text{mon}}$.

Proof. Theorem 9.3 implies nonemptiness of $\bar{\sigma}$ on $\mathcal{G}_{\text{mon}}$. The second part follows from Theorem 9.10 and Lemma 9.12. $\square$

On the subclass of pure bargaining games $\mathcal{B} \subseteq \mathcal{G}_{\text{mon}}$, $\bar{\sigma}$ coincides with the Nash bargaining solution: see the last part of Section 2.

An earlier characterization of $\bar{\sigma}$ (also called the Shapley NTU-value) was given by Aumann (1985). This characterization presumes existence and makes use of specific properties of the Shapley value; it also uses the standard concept of unbounded TU-games.

9.5.2 The Core

The *core* of a TU-game v is defined by

$$c(v) := \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for every coalition } S\}.$$

More generally, the *core* of an NTU-game V is defined by

$$C(V) := \{x \in PO(V(N)) : \forall \emptyset \neq S \neq N [x_S \notin \text{int} V(S)]\}$$

where 'int' denotes the topological interior. Nonemptiness of the core is closely connected to the idea of balancedness. A collection of nonnegative numbers $\{\gamma_S : S \text{ a coalition}\}$ is called *balanced* if $\sum_{S: i \in S} \gamma_S = 1$ for every player $i \in N$. An NTU-game V is called *balanced* if for every balanced collection $\{\gamma_S\}$ we have $\sum_S \gamma_S \bar{V}(S) \subseteq V(N)$, where $\bar{V}(S) \subset \mathbb{R}^N$ is constructed from $V(S)$ by adding zeros for players outside S . It is well known (Bondareva, 1963; Shapley, 1967) that a TU-game has a nonempty core if and only if it is balanced. For an NTU-game, balancedness—and even a weaker balancedness condition, cf. Scarf (1967)—implies nonemptiness of the core, but not the other way around.

Let $\mathcal{T}_{\text{bal}}$ denote the class of balanced TU-games, i.e., TU-games with nonempty cores. The TU-solution c is regular, as is easy to verify. Let $\mathcal{G}_{\text{bal}}$ denote the class of balanced NTU-games. By slightly adapting an argument of Qin (1994)¹ it can be shown that $V \in \mathcal{G}_{\text{bal}}$ if and only if $T(\{V\}) \subseteq \mathcal{T}_{\text{bal}}$. In words, an NTU-game is balanced if and only if all associated transfer games are balanced.

Corollary 9.14 $\bar{c}(V) \neq \emptyset$ for every $V \in \mathcal{G}_{\text{bal}}$. Moreover, $\bar{c}$ is the unique NTU-solution on $\mathcal{G}_{\text{bal}}$ that satisfies Pareto optimality, scale covariance, expansion and contraction independence and coincides with $\bar{c}$ on $\mathcal{T}_{\text{bal}}$.

Proof. Theorem 9.3 implies nonemptiness of $\bar{c}$ on $\mathcal{G}_{\text{bal}}$. The second part follows from Theorem 9.10. $\square$

In this case, applying the transfer procedure on TU-games may add solution outcomes, as the following example shows.

Example 9.15 Consider the four-person TU-game v with player set $N = \{1, 2, 3, 4\}$ and with $v(N) = 2$, $v(S) = 1$ if $|S| = 3$ or $S \in \{\{1, 2\}, \{3, 4\}\}$, and $v(S) = 0$ otherwise. This is a monotonic game with core equal to

$$c(v) = \{x \in \mathbb{R}_+^N : x_1 + x_2 = 1, x_3 + x_4 = 1\}.$$

Let $\lambda = (1, 1, 0, 0)$, then v^λ is equal to v except that $v^\lambda(\{3, 4\}) = 0$. Now, for $x = (1, 1, 0, 0)$, $\lambda x = (1, 1, 0, 0) \in c(v^\lambda)$, hence $x = (1, 1, 0, 0) \in \bar{c}(v)$ but $x \notin c(v)$. $\triangleleft$

¹Attributed to Shapley.

Example 9.15 also implies that the core $C(V)$ of an NTU-game V does not have to contain $\bar{c}(V)$. Also the converse is not true:

Example 9.16 Consider the three-person NTU-game V with player set $N = \{1, 2, 3\}$ and with $V(N) = \{x \in \mathbb{R}_+^N : x_1 + x_2 + x_3 \leq 3\}$, $V(\{1, 2\}) = \{x \in \mathbb{R}_+^{\{1,2\}} : x_1 + 2x_2 \leq 3\}$, and $V(S) = \{0\}$ otherwise. Note that $(1, 1, 1) \in C(V)$. The only possible transfer game through which we could obtain $(1, 1, 1) \in \bar{c}(V)$ would be one corresponding to $\lambda = (1, 1, 1)$ (or a positive multiple of that vector). For this transfer game we have $v^\lambda(N) = 3$ and $v^\lambda(\{1, 2\}) = 3$, so that $\lambda(1, 1, 1) = (1, 1, 1) \notin c(v^\lambda)$, hence $(1, 1, 1) \notin \bar{c}(V)$. $\triangleleft$

Example 9.16 still works if we replace the game V by V' with as the only difference that now $V'(\{1, 2\}) = \{x \in \mathbb{R}_+^{\{1,2\}} : x_1 + 3x_2 \leq 4\}$. In that case, however, the resulting $(1, 1, 1)$ -transfer game has an empty core and therefore $V' \notin \mathcal{G}_{\text{bal}}$. The latter fact follows also directly by considering the collection $\gamma_{\{1,2\}} = 1$, $\gamma_{\{3\}} = 1$, and $\gamma_S = 0$ otherwise. This shows that if an NTU-game has a nonempty core, then this property is not necessarily inherited by the associated transfer games.²

9.5.3 The Nucleolus

The *nucleolus* (Schmeidler, 1969) for a TU-game v is defined as follows. For every Pareto optimal payoff vector x , arrange the so-called excesses $v(S) - \sum_{i \in S} x_i$ ($\emptyset \neq S \neq N$) in a nonincreasing order. Then compute x such that the thus associated vector of excesses is lexicographically minimal: the resulting payoff vector is the nucleolus of the game. If the game has a nonempty core, then the nucleolus is in the core.

The nucleolus on $\mathcal{T}$ is a regular TU-solution, so Theorems 9.3 and 9.10 apply again. Consequently, denoting the nucleolus by ν , we have:

Corollary 9.17 $\bar{\nu}(V) \neq \emptyset$ for every $V \in \mathcal{G}$. Moreover, $\bar{\nu}$ is the unique NTU-solution on $\mathcal{G}$ that satisfies Pareto optimality, scale covariance, expansion and contraction independence and coincides with $\bar{\nu}$ on $\mathcal{T}$.

Just as was the case with the Shapley value, also the transfer solution associated with the nucleolus coincides with the Nash bargaining solution

²Qin (1994) also studies an extension of the core to NTU-games by applying the concept of transfer games. These transfer games are (λ) -hyperplane games rather than TU-games, and his approach also differs from ours since in the transfer game the feasible set of a coalition of players with zero weights is unbounded. In particular, such a game will always have an empty core.

on the subclass of pure bargaining games (see the last part of Section 2).

Like in the case of the core the transfer procedure may add outcomes to TU-games, as is illustrated by the next example.

Example 9.18 Consider the four-person TU-game with $N = \{1, 2, 3, 4\}$, $v(N) = v(\{1, 2\}) = v(\{3, 4\}) = 1$, and $v(S) = 0$ otherwise. Then $\nu(v) = \frac{1}{4}(1, 1, 1, 1)$ as is easily derived by symmetry. Take $\lambda = (1, 1, 0, 0)$, then v^λ is equal to v except that now $v^\lambda(\{3, 4\}) = 0$. By symmetry and the fact that the nucleolus is in the core, $\nu(v^\lambda) = \frac{1}{2}(1, 1, 0, 0)$. Hence $\nu(v^\lambda) = \lambda \frac{1}{2}(1, 1, 0, 0)$ so that $\frac{1}{2}(1, 1, 0, 0) \in \bar{\nu}(v)$. $\triangleleft$

Observe that the game v in this example is not balanced. It is an open question to find an example with a balanced TU-game.

9.5.4 The τ -Value

The τ -value for TU-games (Tijs, 1981, Borm *et al.*, 1992) is defined as follows. For a TU-game v , define the 'utopia vector' $M(v) \in \mathbb{R}^N$ by $M_i(v) = v(N) - v(N \setminus \{i\})$ and the 'minimal right vector' $m(v) \in \mathbb{R}^N$ by $m_i(v) = \max_{S: i \in S} (v(S) - \sum_{j \in S \setminus \{i\}} M_j(v))$ for every $i \in N$. Then the τ -value $\tau(v)$ is the unique Pareto optimal point on the line segment with $m(v)$ and $M(v)$ as endpoints, if such a point exists and if $m(v) \leq M(v)$. Games for which these two conditions are satisfied are called quasi-balanced. It can be shown that every balanced game is quasi-balanced. By $\mathcal{T}_{\text{qbal}}$ we denote the class of quasi-balanced TU-games.

We will show that transfer games associated with quasi-balanced TU-games are again quasi-balanced. First, we derive some inequalities concerning the utopia and minimal right vectors of transfer games.

Lemma 9.19 Let v be a TU-game and $\lambda \in \mathbb{R}_+^N \setminus \{0\}$. Then

$$M_i(v^\lambda) \geq \bar{\lambda}_N M_i(v) \text{ for all } i \in N$$

and

$$m_i(v^\lambda) \leq \bar{\lambda}_N m_i(v) \text{ for all } i \in N.$$

Proof. Let $i \in N$. Then, by Lemma 9.11,

$$\begin{aligned} M_i(v^\lambda) &= v^\lambda(N) - v^\lambda(N \setminus \{i\}) \\ &= \bar{\lambda}_N v(N) - \bar{\lambda}_{N \setminus \{i\}} v(N \setminus \{i\}) \\ &\geq \bar{\lambda}_N [v(N) - v(N \setminus \{i\})] \\ &= \bar{\lambda}_N M_i(v) \end{aligned}$$

and

$$\begin{aligned}
 m_i(v^\lambda) &= \max_{S: S \ni i} [v^\lambda(S) - \sum_{j \in S \setminus \{i\}} M_j(v^\lambda)] \\
 &\leq \max_{S: S \ni i} [\bar{\lambda}_S v(S) - \sum_{j \in S \setminus \{i\}} \bar{\lambda}_N M_j(v)] \\
 &\leq \max_{S: S \ni i} \bar{\lambda}_N [v(S) - \sum_{j \in S \setminus \{i\}} M_j(v)] \\
 &= \bar{\lambda}_N m_i(v).
 \end{aligned}$$

Here, the before-last inequality follows from the first part of the proof. $\square$

Lemma 9.20 Let $v \in \mathcal{T}_{\text{qbal}}$ and $\lambda \in \mathbb{R}_+^N \setminus \{0\}$. Then $v^\lambda \in \mathcal{T}_{\text{qbal}}$.

Proof. Let $i \in N$. Then by Lemma 9.19 and the fact that $v \in \mathcal{T}_{\text{qbal}}$,

$$m_i(v^\lambda) \leq \bar{\lambda}_N m_i(v) \leq \bar{\lambda}_N M_i(v) \leq M_i(v^\lambda),$$

and

$$\sum_{j \in N} m_j(v^\lambda) \leq \bar{\lambda}_N \sum_{j \in N} m_j(v) \leq \bar{\lambda}_N v(N) \leq \bar{\lambda}_N \sum_{j \in N} M_j(v) \leq \sum_{j \in N} M_j(v^\lambda),$$

hence $\sum_{j \in N} m_j(v^\lambda) \leq v^\lambda(N) \leq \sum_{j \in N} M_j(v^\lambda)$. So $v^\lambda \in \mathcal{T}_{\text{qbal}}$. $\square$

We next show that the Shapley transfer procedure does not add solution outcomes to TU-games. Cf. Lemma 9.12, where we prove this for the Shapley value.

Lemma 9.21 Let $v \in \mathcal{T}_{\text{qbal}}$. Then $\bar{\tau}(v) = \tau(v)$.

Proof. Let $\lambda \in \mathbb{R}_+^N \setminus \{0\}$ and $x \in \mathbb{R}_+^N$ with $\sum_{i \in N} x_i = v(N)$ such that $\lambda x = \tau(v^\lambda)$. In view of Lemma 9.1 it is sufficient to prove that $x = \tau(v)$. Let $M := \{i \in N : \lambda_i = \bar{\lambda}_N\}$ then, by Lemma 9.11, $x_i = 0$ for all $i \in N \setminus M$. Hence,

$$\tau_i(v^\lambda) = 0 \text{ for all } i \in N \setminus M. \quad (9.1)$$

Let $\alpha \in [0, 1]$ such that $\tau(v^\lambda) = \alpha M(v^\lambda) + (1 - \alpha)m(v^\lambda)$.

Case (a): $\alpha = 0$.

Then

$$\tau(v^\lambda) = m(v^\lambda), \quad (9.2)$$

and

$$\bar{\lambda}_N v(N) \geq \bar{\lambda}_N \sum_{j \in N} m_j(v) \geq \sum_{j \in N} m_j(v^\lambda) = v^\lambda(N) = \bar{\lambda}_N v(N), \quad (9.3)$$

where the second inequality follows from Lemma 9.19, the second equality from Lemma 9.11, and the first equality from (9.2). Hence, all inequalities in (9.3) are equalities. In particular, $m(v)$ is efficient in v , so $\tau(v) = m(v) = (1/\bar{\lambda}_N)\tau(v^\lambda)$. So by (9.1), $x_i = 0$ and $\tau_i(v) = 0$ for all $i \in N \setminus M$, so $x_i = \tau_i(v)$ for these i . For $i \in M$, $x_i = (1/\lambda_i)\tau_i(v^\lambda) = (1/\bar{\lambda}_N)\tau_i(v^\lambda)$, hence $x_i = \tau_i(v)$. Altogether, $x = \tau(v)$.

Case (b): $\alpha > 0$.

Then for $i \in N \setminus M$, by (9.1), $m_i(v^\lambda) = M_i(v^\lambda) = 0$, hence by Lemma 9.11, $\bar{\lambda}_N v(N) = v^\lambda(N) = v^\lambda(N \setminus \{i\}) = \bar{\lambda}_N v(N \setminus \{i\})$ so that $M_i(v) = 0$. Thus,

$$M_i(v) = \tau_i(v) = 0 = x_i \text{ for all } i \in N \setminus M. \quad (9.4)$$

Now let $i \in M$. First suppose $|M| > 1$. Then $M_i(v^\lambda) = \bar{\lambda}_N v(N) - \bar{\lambda}_N v(N \setminus \{i\}) = \bar{\lambda}_N M_i(v)$ and $m_i(v^\lambda) = \max_{S: S \ni i} [v^\lambda(S) - \sum_{j \in S \setminus \{i\}} M_j(v^\lambda)] = \max_{S: S \ni i} [\bar{\lambda}_N v(S) - \sum_{j \in S \setminus \{i\}} \bar{\lambda}_N M_j(v)] = \bar{\lambda}_N m_i(v)$. Hence $\bar{\lambda}_N x_i = \tau_i(v^\lambda) = \bar{\lambda}_N \tau_i(v)$, so that $x_i = \tau_i(v)$. This concludes the proof for $|M| > 1$. If $M = \{i\}$, then by (9.4) and efficiency of the τ value, $\tau_i(v) = v(N)$. Because of $\lambda x = \tau(v^\lambda)$, (9.4) and efficiency, we have $\tau_i(v^\lambda) = v^\lambda(N) = \bar{\lambda}_N x_i$, hence $x_i = v(N) = \tau_i(v)$. This concludes the proof of the lemma. $\square$

Denote by $\mathcal{G}_{\text{qbal}}$ the class of NTU-games such that $T(\mathcal{G}_{\text{qbal}}) = \mathcal{T}_{\text{qbal}}$. By Lemma 9.20 this class contains $\mathcal{T}_{\text{qbal}}$. Moreover, $\mathcal{G}_{\text{qbal}}$ contains the class $\mathcal{G}_{\text{bal}}$ of all balanced NTU-games, since every transfer game associated with a balanced NTU-game is balanced and therefore also quasi-balanced.

We have:

Corollary 9.22 $\bar{\tau}(V) \neq \emptyset$ for every $V \in \mathcal{G}_{\text{qbal}}$. Moreover, $\bar{\tau}$ is the unique NTU-solution on $\mathcal{G}_q$ that satisfies Pareto optimality, scale covariance, expansion and contraction independence and coincides with the τ -value on $\mathcal{T}_{\text{qbal}}$.

Proof. Theorem 9.3 implies nonemptiness of $\bar{\tau}$ on $\mathcal{G}_{\text{qbal}}$. The second part follows from Theorem 9.10 and Lemma 9.21. $\square$

Since on $T(\mathcal{B})$, where as before $\mathcal{B}$ is the class of pure bargaining games, the τ -value coincides with the equal-split solution, it follows again that the transfer solution $\bar{\tau}$ coincides with the Nash bargaining solution on the class of pure bargaining games.

9.6 Concluding Remarks

The main objective of this contribution was to provide existence and characterization of NTU-solutions obtained from TU-solutions by the Shapley transfer procedure within one and the same framework. The price paid for this is the allowance of zero weights and the associated technical problems. The benefit is that the results can be applied to many TU-solutions: see Corollaries 9.13–9.22.

The approach followed above can be modified in many ways. If existence is less of an issue then we may restrict attention to only positive weights and consider other classes of games: this is the approach usually adopted in the literature. Also the transfer procedure may be varied: cf. the Harsanyi procedure as discussed in Section 3, or the procedure used to extend the so-called consistent value—which coincides with the Shapley value on TU-games—to NTU-games (see Maschler and Owen, 1992).

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