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Measuring Polarization in Preferences

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Abstract

In this paper, we study the measurement of polarization in collective decision making problems with ordinal preferences over alternatives. We argue that polarization can be measured as an aggregation of antagonisms over pairs of alternatives in the society. We propose a measure of this sort and show that it is the only measure satisfying some normatively appealing conditions.

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1 Introduction

Higher polarization in ideologies or preferences over policies is generally considered as a bad feature in politics mainly due to representational concerns. It is argued to cause policy gridlock (Jones (2001)), decrease turnout if it is only in elite level (Hetherington (2008)) and increase economic inequality (McCarty et al. (2003)).

Due to the disagreements in measurement, we see disparity in the results of polarization analyses. For instance, the increase in polarization in the U.S. politics is somewhat unequivocal for the elite level although the literature on public polarization is inconclusive. For a review in line with this conclusion, see Hetherington (2009). This paper introduces yet another approach to the measurement of polarization. However, the major component of our contribution is in that of the subject of measurement. Although there are quite a number of articles analyzing the measurement of polarization for distributions that can be represented on a line[1], this paper is among the very first attempts for analyzing polarization measures for ordinal preference profiles.[2]

Some of the related concepts that could be found analyzed in the social choice literature could be listed as *consensus* (Herrera-Viedma et al., 2011), *assent* (Baldiga and Green, 2013) and *cohesiveness* (Alcalde-Unzu and Vorsatz, 2013). According to numerous authors, as formulated in Bosch (2006), consensus can be formulated such that it can be measured with mappings that assign to any profile of preferences a value in unit interval, which has the following two properties necessarily: first, the value given to a profile is highest, namely 1, if and only if all individuals agree on how to rank alternatives and second, the same value given to any two profiles if the only difference in between them is the names of either the alternatives or individuals. Alcalde-Unzu and Vorsatz (2008) have introduced some axiomatic characterizations in this vein. García-Lapresta and Pérez-Román (2011) analyze properties of a class of consensus measures that are based on the distances among individual weak orders.

[1] See, *inter alia*, Esteban and Ray (1994) and Montalvo and Reynal-Querol (2005).

[2] For a measure of ordinal preference polarization which adopts the methodology of Esteban and Ray (1994) with the use of a metric *à la* Kemeny (1959), see Ozkes (2013).

Baldiga and Green (2013) define conflict between two individuals as the disagreement in their top choices. They then use an aggregate-assent maximizing approach to the selection of the choice rule, where the assent between preferences is the probability that these preferences would be conflictual on a random feasible set.

Finally, Alcalde-Unzu and Vorsatz (2013) denote the level of similarity among preferences in a profile as cohesiveness and characterize a class of cohesiveness measures with a set of plausible axioms. This class of functions falls within the above definition of consensus.

In what follows we argue, first, that polarization is not necessarily the opposite of consensus and hence calls for a particular treatment. The least polarized case naturally coincides with a fully consensual state, which is easily defined as a unanimous preference profile. However, there is no unique way of framing the most polarized situation. This would entail a normative approach, which we embrace in this paper as follows. Since we investigate polarization in preferences that are represented as linear orders, we restrict the most polarized situations to societies which are divided equally into two completely opposite linear orders.

Second, we impose that the polarization level should not depend on the number of individuals in a society but stay the same if the supporting individuals of each preference is multiplied by equal terms. Furthermore, we require a form of equal treatment of marginal changes in the composition of preferences. More precisely, if a single individual changes her preference to conform with the majority view on a single issue, then the change in polarization should not depend on the size of this majority. Finally, we impose neutrality towards alternatives.

In this paper, we show that interpreting polarization as an aggregation of antagonisms in a society is the only way of measuring polarization with the properties above. In this context antagonisms are taken as disagreements over pairwise comparisons of alternatives.

We proceed as follows. In the next section we introduce basic notations and formal definitions regarding the axiomatic model. Section 3 provides our main results and proofs thereof. We conclude in Section 4 by pointing to a possible

direction for further research. We show the logical independence of axioms in the appendix.

2 Model

2.1 Preliminaries

Let A be a finite and nonempty set of m alternatives. For any finite and nonempty set of individuals N, and for any individual i in N, let $p(i)$ denote the preference of i in terms of a linear order, i.e., a complete, antisymmetric and transitive binary relation on A. Furthermore, p indicates a *profile*, a combination of such individual preferences and $\mathbb{L}$ the set of all preferences on A. So, p is an element of $\mathbb{L}^N$.

We denote by $\bar{A}$ the set of all subsets of A with cardinality 2. For a given profile p in $\mathbb{L}^N$ and different alternatives a and b in A let $n_{ab}(p)$ denote the number of individuals who prefer a to b, i.e., $n_{ab}(p) = \#\{i \in N : (a,b) \in p(i)\}$. Let $d_{ab}(p) = |n_{ab}(p) - n_{ba}(p)|$ denote the absolute difference between the number of voters preferring a to b and those preferring b to a at profile p.

For a preference R, let R^N denote the unanimous profile where all individuals have preference R. Let $-R = \{(y,x) : (x,y) \in R\}$ be the preference where all pairs in R are reversed. If π denotes a permutation on A, then the permuted preference of R is $\pi R = \{(\pi(a), \pi(b)) : (a,b) \in R\}$ which naturally defines the permuted profile πp in a coordinate-wise manner, i.e., $(\pi p)(i) = \pi(p(i))$.

For two profiles p and q of two disjoint sets of individuals, say N_1 and N_2 respectively, let (p,q) denote the profile, say r, such that $r(i) = p(i)$ if i is in N_1 and $r(i) = q(i)$ if i is in N_2. Similarly define $p^2 = (p,p)$ to be a profile where preference p is replicated once and $p^3 = (p,p,p)$ twice, and so on.

Let p and q be two profiles in $\mathbb{L}^N$. We say that p and q form an *elementary change from ab to ba* whenever there is an individual i in N who ranks a and b consecutively in p and furthermore $q(i) = \Big(p(i) \cup \{(b,a)\}\Big) \setminus \{(a,b)\}$ and for all j in $N \setminus \{i\}$, $p(j) = q(j)$. This means that $q(i)$ can be obtained from $p(i)$ by only reversing the ordered pair (a,b).

Finally, a *polarization measure* Ψ assigns to any profile p in $\mathbb{L}^N$ a real number

$\Psi(p)$, where N is any finite and nonempty set of individuals. Next we discuss a few normatively appealing conditions on the polarization measures.

2.2 Conditions on Polarization Measures

We first impose a regularity condition on polarization measures to normalize between 0 and 1. The former value is reserved for profiles wherein each individual has the same preferences, i.e., a unanimous profile. In this regard, we see the maximal consensus as a case of minimal polarization. However, we furthermore restrict the maximally polarized case. The profiles (with even number of individuals) where half of the individuals have a preference R and the rest have $-R$, for some $R \in \mathbb{L}$ are considered to be the maximally polarized profiles.

Regularity : $\Psi(R^N) = 0$ and $\Psi(R^{N_1}, (-R)^{N_2}) = 1$ for all preferences R and all finite and nonempty sets N_1 and N_2 of individuals such that N_1 and N_2 are disjoint and equal in size, i.e., $\#N_1 = \#N_2$.

Neutrality is a standard property in social choice. In this context, it requires that a renaming of the alternatives does not change the polarization level.

Neutrality : $\Psi(p) = \Psi(\pi p)$ for all permutations π on A and all profiles p.

The following condition requires that when societies are replicated by some positive integer, the polarization is unchanged. Note that this also implies *anonymity*, i.e., renaming the individuals does not change the polarization level. Formally:

Replication invariance : $\Psi(p^k) = \Psi(p)$ for all positive integers k, and all profiles p.

Finally, we introduce our final condition which we call *support independence*. This condition requires that elementary changes in favor of an alternative that has a majoritarian support against another lead to identical changes in polarization across profiles. For instance, if a majority of individuals agree that a is better than b in each of the two profiles, then an increase in the support of a over b should lead to the same amount of change in the polarization for both of these profiles.

Support independence : $\Psi(p) - \Psi(q) = \Psi(\widehat{p}) - \Psi(\widehat{q})$ for any two elementary changes p, q and $\widehat{p}, \widehat{q}$ both from ba to ab for some alternatives a and b with

$n_{ab}(p) \geq n/2$ and $n_{ab}(\widehat{p}) \geq n/2$.

3 Result

Assume for simplicity that the issue in hand is a binary choice, that there are only two alternatives. If the absolute difference between numbers of individuals preferring a to b and b to a, i.e., d_{ab}, is 0, then the polarization should intuitively be maximal. If this number is equal to n, then we have that everyone prefers a over b or vice versa, a full agreement. Hence polarization should be minimal. Therefore, the polarization can be related to $n - d_{ab}$. If we normalize by dividing by n, then we have a bound on the polarization (between 0 and 1) therefore regularity is also satisfied. For profiles on more than two alternatives, we iterate this process over all pairs of distinct alternatives. Thereafter we normalize this value with respect to the number of such pairs and the number of individuals. Hence we obtain the following polarization measure:

$$\Psi^*(p) = \sum_{\{a,b\} \in \bar{A}} \frac{n - d_{ab}(p)}{n \cdot \binom{m}{2}}.$$

It is easy to verify that Ψ^* satisfies the conditions introduced in Section 2.2. In the sequel, we will show that it is indeed the only measure that satisfies these conditions. Before, we discuss some features regarding elementary changes that are instrumental in what follows.

Let p and q form an elementary change from ab to ba, so that $n_{ab}(p) - 1 = n_{ab}(q)$ and $n_{ba}(p) + 1 = n_{ba}(q)$. This change can be of one of the following three;

(i) a minority decrement if $n_{ab} \leq n/2$,

(ii) a majority decrement if $n_{ab} \geq n/2$ and

(iii) a swing if $n_{ab}(p) > n/2$ and $n_{ab}(q) < n/2$.[3]

The first two changes are straightforward. For the third, consider the case where 4 individuals prefer a to b and 3 prefer b to a. An elementary change, in this case, from ab to ba makes the former minority a majority.

[3] Hence $n_{ab}(p) = n_{ba}(q)$.

Remark 1. *Note that if p and q form an elementary change from ab to ba that is a minority decrement, then q and p form an elementary change from ba to ab that is a majority decrement. This duality allows us to construct the forthcoming lemmas by focusing on either of the two first elementary changes.*

The following Lemma shows that all minority decrements yield an equal change in polarization regardless of what alternatives are involved. By Remark 1, the result also holds for majority decrements. Let Ψ satisfy the four conditions; regularity, neutrality, replication invariance and support independence.

Lemma 1. *Let p and q be a minority elementary change from ab to ba and let $\widehat{p}$ and $\widehat{q}$ be a minority elementary change from xy to yx. We have*

$$\Psi(p) - \Psi(q) = \Psi(\widehat{p}) - \Psi(\widehat{q}).$$

Proof. Let $p^{\{a,b\}}$ be the profile obtained from p by shifting a and b to the two top positions for each individual while leaving preference between a and b as well as those between alternative in $A\backslash\{a,b\}$ unchanged. That is for all individuals i in N let $p^{\{a,b\}}(i) = p(i)|_{\{a,b\}^2} \cup (\{a,b\} \times A\backslash\{a,b\}) \cup p(i)|_{(A\backslash\{a,b\})^2}$. Similarly define $q^{\{a,b\}}$. Then by support independence we have

$$\Psi(p) - \Psi(q) = \Psi(p^{\{a,b\}}) - \Psi(q^{\{a,b\}}).$$

Considering the permutation π on A such that $\pi(a) = x$, $\pi(x) = a$, $\pi(b) = y$, $\pi(y) = b$ and $\pi(z) = z$ for all $z \in A\backslash\{a,b,x,y\}$ neutrality implies

$$\Psi(p^{\{a,b\}}) - \Psi(q^{\{a,b\}}) = \Psi(\pi p^{\{a,b\}}) - \Psi(\pi q^{\{a,b\}}).$$

Note that $\pi p^{\{a,b\}}$ and $\pi q^{\{a,b\}}$ are preferences at which the alternatives x and y are in the two top position for every individual. Furthermore they form a minority elementary change from xy to yx. Therefore support independence implies

$$\Psi(\pi p^{\{a,b\}}) - \Psi(\pi q^{\{a,b\}}) = \Psi(\widehat{p}) - \Psi(\widehat{q}).$$

So, all in all

$$\Psi(p) - \Psi(q) = \Psi(\widehat{p}) - \Psi(\widehat{q}).$$

∎

Next we prove that all minority elementary changes yield a decrease of polarization by $\alpha = \frac{2}{n \cdot \binom{m}{2}}$. By Remark 1, then, all elementary changes in majority increase polarization by that same amount α.

Lemma 2. *Let p and q be a minority elementary change from ab to ba. Then*

$$\Psi(p) - \Psi(q) = \frac{2}{n \cdot \binom{m}{2}}.$$

Proof. By Lemma 1 it is sufficient to prove that at some minority elementary change, polarization decreases by $\alpha = \frac{2}{n \cdot \binom{m}{2}}$. Replication invariance implies that we may assume that the set of individuals is even, that is $n = 2 \cdot k$. Consider any two set of individuals $\#N_1 = \#N_2 = k$ with $N_1 \cap N_2 = \emptyset$ and a combined set of individuals $N = N_1 \cup N_2$. Given any preference R, consider the following two profiles $(R^{N_1}, (-R)^{N_2})$ and R^N. Note that there is a path of $k \cdot \binom{m}{2}$ elementary changes from the former to the latter. By regularity $\Psi(R^{N_1}, (-R)^{N_2}) = 1$ $\Psi(R^N) = 0$. By Lemma 1, each step in this path cause the same change in polarization, say α. Note that the amount of swaps from $(-R)$ to R is $\binom{m}{2}$. The number of individuals requiring this many swaps is $n/2$. Therefore each elementary change should decrease the polarization by $2/n\binom{m}{2}$. ∎

We have shown that each minority (or majority) elementary change causes the same amount of decrease (or increase) in the polarization. Next we show that swing elementary changes does not affect the polarization level.

Lemma 3. *Let p and q be a swing elementary change from ab to ba. Then*

$$\Psi(p) = \Psi(q).$$

Proof. Consider the profiles p^2, (p, q) and q^2. Both p^2 and (p, q) as well as q^2 and (p, q) form minority elementary changes. The former pair from ab to ba the latter pair from ba to ab. So, $\Psi(p^2) - \Psi(p, q) = \alpha = \Psi(q^2) - \Psi(p, q)$. Hence, $\Psi(p^2) = \Psi(q^2)$. Therefore by replication invariance we have $\Psi(p) = \Psi(q)$. ∎

Now we can state our main theorem.

Theorem 1. *A polarization measure Ψ satisfies regularity, neutrality, replication invariance and support independence if and only if $\Psi = \Psi^*$.*

Proof. Assume Ψ satisfies the conditions. Take any preference R and consider the profile R^N. By regularity, $\Psi(R^N) = \Psi^*(R^N) = 0$. Any profile p in L^N can be acquired by a sequence of elementary changes beginning from R^N by minority decrements, majority decrements or swings. By Lemmas 2 and 3, the increase (or decrease) induced by each of the elementary changes should be the same. Hence for any p in L^N, we conclude $\Psi(p) = \Psi^*(p)$. ∎

4 Conclusion

In this paper, we have modeled polarization as an aggregation of antagonisms per issues within a profile. The polarization measure we introduce simply check for each issue, i.e., pairwise comparison of alternatives, and compares the strength of a majority versus minority. These pairwise comparisons on issues are then aggregated and normalized to a real number between 0 and 1. The measure is very intuitive and is characterized by a few plausible conditions.

There are many directions for future research. The relation between the extent of polarization and the social aggregation outcomes would be a natural route of inquiry. Gurer (2008) studies the Arrovian impossibilities when the preferences in the society cluster, in some sense, around a preference, where it is also conjectured that in a bipolar society the sum of the distances from the two opposite clusters, around which the society is polarized, will be decisive concerning whether we end up with possibilities. The analysis is dependent on a metric-based approach to alienation between preferences. Thus, the relevance of polarization measures based on pairwise comparisons of alternatives to social aggregation outcomes is an open and immediate question one might ask.

Note that the current analysis treats pairs of alternatives impartially, i.e., every issue is of equal importance for polarization. Of course, in many real life situations we may have differing weights on issues. Another question for future research would be analyzing richer domains of preferences, e.g., weak orders, or restricted ones, e.g., single-peaked domains which are politically relevant and interesting.

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Appendix: Logical independence of axioms

Regularity: The following measure satisfies replication invariance, neutrality and support independence but not regularity:

$$\Psi'(p) = \sum_{\{a,b\}\in\bar{A}} \frac{d_{ab}(p)}{n}.$$

To show it satisfies the first three axioms is rather straightforward. To see violation of regularity it would suffice to consider a unanimous profile.

Support independence: Consider the function

$$\bar{d}_{ab}(p) = \begin{cases} 0 & \text{if } d_{ab}(p) = 0 \\ 1 & \text{if } d_{ab}(p) \neq 0. \end{cases}$$

Then the following measure satisfies neutrality, regularity and replication invariance but not support independence:

$$\bar{\Psi}(p) = \sum_{\{a,b\}\in\bar{A}} \frac{1 - \bar{d}_{ab}(p)}{\binom{m}{2}}.$$

Neutrality is straightforward. Replication invariance is due to the fact that $\bar{d}_{ab}$ stays the same in case of replication. To see regularity, note that $\bar{d}_{ab}(p) = 1$ for all $a, b \in A$ whenever $p = R^N$ for some R, hence we have 0. In the case where $p = (R^{N_1}, (-R)^{N_2})$ with $\#N_1 = \#N_2$, we have $\bar{d}_{ab}(p) = 0$ for all $a, b \in A$, hence 1 as the outcome of the function. To see why it fails support independence, consider two profiles p, $\hat{p}$ with 4 individuals:

$$p = \left(\begin{array}{cccc} i_1 & i_2 & i_3 & i_4 \\ \hline a & a & b & b \\ b & b & a & a \end{array}\right) \qquad \hat{p} = \left(\begin{array}{cccc} i_1 & i_2 & i_3 & i_4 \\ \hline a & a & b & a \\ b & b & a & b \end{array}\right)$$

Let q, (respectively $\hat{q}$) be constructed such that from p to q (respectively from $\hat{p}$ to $\hat{q}$), the third agent changes its preference to a over b. Support independence

requires that the change of polarization should be the same for p, q and $\hat{p}, \hat{q}$, which is not the case under $\bar{\Psi}$.

Neutrality: For any set of alternatives A, let $x, y \in A$ be a predefined choice of pairs. The following measure satisfies replication invariance, regularity and support independence but not neutrality:

$$\dot{\Psi}(p) = \frac{n - d_{xy}(p)}{n}$$

Replication Invariance: Let $m = 2, n = 3$. We first construct a function $\bar{K}$ such that $\bar{K}(p) = 0$ for all unanimous profiles, and $\bar{K}(p) = 1$ for all other profiles. Consider the measure below which for $n = 3$ and $m = 2$ equals $\bar{K}(p)$ and in all other cases equals $\Psi^*(p)$:

$$\hat{\Psi}(p) = \begin{cases} \bar{K}(p) & \text{if } m = 2 \text{ and } n = 3 \\ \Psi^*(p) & \text{otherwise.} \end{cases}$$

This measure satisfies neutrality, support independence, regularity but not replication invariance.