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Citation for published version (APA):

Document status and date:
Published: 01/01/2003

DOI:
10.1016/S1571-0653(04)00449-4

Document Version:
Publisher's PDF, also known as Version of record

Please check the document version of this publication:
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Download date: 16 Sep. 2023
Enumeration of Circuits
and Minimal Forbidden Sets

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Abstract

In resource-constrained scheduling, it is sometimes important to know all inclusion-minimal subsets of jobs that must not be scheduled simultaneously. These so-called minimal forbidden sets are given implicitly by a linear inequality system, and can be interpreted more generally as the circuits of a particular independence system. We present several complexity results related to computation, enumeration, and counting of the circuits of an independence system. On this account, we also propose a backtracking algorithm that enumerates all minimal forbidden sets for resource constrained scheduling problems.

Key words: Independence system, Circuit, Enumeration, Minimal forbidden set

1 Introduction

Given a finite ground set \( V \), an independence system is defined as a family \( \mathcal{I} \) of subsets of \( V \) with two properties. First, \( \emptyset \in \mathcal{I} \), and second, any subset of any member of \( \mathcal{I} \) also belongs to \( \mathcal{I} \). The sets in \( \mathcal{I} \) are called independent sets, and the inclusion-maximal independent sets are the bases \( \mathcal{B} \) of \( \mathcal{I} \). The sets not in \( \mathcal{I} \) are called dependent sets, and inclusion-minimal dependent sets are the circuits \( \mathcal{C} \) of \( \mathcal{I} \); see also [6]. Given a membership oracle for such an independence system \( \mathcal{I} \), we are interested in the problem to enumerate all circuits of \( \mathcal{I} \). It is obvious that the output size may be exponential in terms of the size of \( V \). The complexity is thus measured in terms of the size of both, in- and output. Given that \( |V| = n \), and given a \( \text{poly}(n) \) membership oracle for some collection of subsets \( S \subseteq 2^V \), we use the following definitions; see [4].

Definition 1 The enumeration problem for \( S \) is solvable in polynomial total time if there exists an algorithm and a polynomial \( p(\cdot, \cdot) \), such that the algorithm correctly outputs all members of \( S \) in \( p(n, |S|) \) time.

Preprint submitted to Elsevier Science 27 March 2003
Given a sub-collection $X \subseteq S$, the increments problem is the problem: either decide that $X = S$, otherwise output a new element in $S \setminus X$.

**Definition 2** The enumeration problem for $S$ is solvable in incremental polynomial time if there exists an algorithm and a polynomial $p(\cdot, \cdot)$, such that the algorithm correctly solves the increments problem in $p(n, |X|)$ time, for any $X \subseteq S$.

If the enumeration problem for $S$ is solvable in incremental polynomial time, it is also solvable in polynomial total time: Starting with $X = \emptyset$, the iterative solution of the increments problem yields the complete collection $S$, in time polynomial in $n$ and $|S|$. The reverse, however, need not be true.

2 The general case

The following theorem is well known; it is proved using a reduction from the NP-complete decision problem SATISFIABILITY [3].

**Theorem 3 ([5])** Unless $P=NP$, there does not exist a polynomial total time algorithm that enumerates the bases $B$ of any independence system $I$.

Likewise, using a simple duality argument, we can show the following.

**Theorem 4** Unless $P=NP$, there does not exist a polynomial total time algorithm that enumerates the circuits $C$ of any independence system $I$.

The proof uses the fact that the circuits of $I$ are the bases of the following, dual independence system: $I^D = \{ W \subseteq V \mid V \setminus W \notin I \}$. These two theorems say that, unless $P=NP$, an algorithm cannot exist which solves the problem for any independence system $I$. For particular realizations of the membership oracle of $I$ however, efficient enumeration algorithms may well exist.

3 Scheduling and linear inequality systems

In resource-constrained scheduling, the input consists of a set of partially ordered jobs $(V, \prec)$ (the partial order representing the precedence constraints) and resource constraints. The latter are given through a number of resource types $k$ with availabilities $b_k$, and resource requirements $a_{kj}$ of these resource types for all jobs $j \in V$. If a subset $S$ of jobs consumes more of a resource type than available, the respective jobs in $S$ may not be processed in parallel. The subsets of jobs that may be processed in parallel define an inde-
pendence system. The circuits of this independence system are either pairs of (precedence-)related jobs, \( \{i, j\} \) with \( i \prec j \), or the so-called \textit{minimal forbidden sets} (minimal anti-chains of \((V, \prec)\) which may not be processed in parallel). For several algorithmic purposes, e.g. in stochastic scheduling \cite{8}, a complete list of the minimal forbidden sets is required. This amounts to the computation of the circuits of the corresponding independence system. The membership oracle of this system is a linear inequality system \( Ax \leq b \), where \( A, b \) contains one row in for each resource type \( k \), and one row for each edge in the comparability graph of the partial order \((V, \prec)\). The circuits are the minimally infeasible \( \{0, 1\} \)-vectors for \( Ax \leq b \); they are the incidence vectors of either pairs of (precedence-)related jobs, or minimal forbidden sets.

Inspired by \cite{2}, and using a reduction from the NP-complete decision problem \textsc{Independent Set} in graphs \cite{3}, we show:

**Theorem 5** Unless \( P=NP \), there does not exist a polynomial total time algorithm that enumerates the minimally infeasible \( \{0, 1\} \)-vectors of an arbitrary linear inequality system \( Ax \leq b \).

In fact, in \cite{2} it is shown that the decision version of the corresponding increments problem is NP-complete. What is interesting is the fact that the 'dual' problem is apparently much easier: It follows from \cite{2} that the increments problem for the maximally feasible \( \{0, 1\} \)-vectors of an arbitrary linear inequality system \( Ax \leq b \) can be solved in quasi-polynomial time, hence there is also a quasi-polynomial total time algorithm for this problem. (Such a result is not likely for the problem considered here, because then all NP-hard problems could be solved in quasi-polynomial time.)

In terms of scheduling, Theorem 5 immediately yields:

**Corollary 6** Unless \( P=NP \), there does not exist a polynomial total time algorithm that enumerates the minimal forbidden sets for any instance of the resource-constrained project scheduling problem.

It turns out that even the computation of the \textit{number} of minimal forbidden sets is a hard problem, because we can show:

**Theorem 7** The problem to compute the number of minimally infeasible (maximally feasible) \( \{0, 1\} \)-vectors of an arbitrary linear inequality system \( Ax \leq b \) is \#P-complete.

The proof uses a reduction from the problem to compute a maximum cardinality anti-chain of a partial order. While this problem is polynomially solvable, the associated counting problem is known to be \#P-complete \cite{7}.

Nevertheless, for practical purposes we implemented a simple backtracking al-
algorithm that lists the minimal forbidden sets for any instance of the resource-constrained project scheduling problem. In general, this algorithm can have an exponential running time in terms of input and output of the problem. Yet, empirically it improves considerably upon a divide-and-conquer algorithm previously suggested in [5,1]; see [9]. Moreover, we can show that the algorithm is efficient for an important special case.

**Proposition 8** There exists an incremental polynomial time (hence also a polynomial total time algorithm) that enumerates the minimal forbidden sets for any instance of the resource-constrained project scheduling problem, given that the number of resource types is 1.

We refer to [9] for several other results related to enumeration and computation of minimal forbidden sets, as well as detailed computational results.

**Acknowledgements** The authors thank Marc Pfetsch for many insightful discussions on the topic, which eventually lead to Theorem 5.

**References**