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A paradox of rationality à la von Neumann-Morgenstern*

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Abstract

We show that there are games and decision situations in which it is not possible for the decision maker to be rational à la von Neumann-Morgenstern in both situations simultaneously, which is the source of the paradox presented in this note. We provide an assumption which is the necessary and sufficient condition for a decision maker to be rational in both situations.

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1 Setting

Let $G = (\Delta X, \Delta Y, \mathcal{U}_1, \mathcal{U}_2)$ be a game in mixed extension which is played by Alice and Bob. The sets ΔX and ΔY denote the set of all probability distributions over the pure strategy set $X = \{x_1, x_2, ..., x_m\}$ of Alice and the pure strategy set $Y = \{y_1, y_2, ..., y_n\}$ of Bob, respectively. We assume that

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$$G = \begin{array}{c} \\ L \\ R \end{array}\!\!\begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 5,10 & 5,0 \\ 5,0 & 5,10 \end{pmatrix} \end{array} \quad D = \begin{array}{c} \\ L \\ R \end{array}\!\!\begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \end{array}$$

Figure 1: A game and the associated decision problem, respectively. Numbers represent (non-utility) monetary payoffs.

players are rational in the sense that their preferences in G can be represented by a von Neumann-Morgenstern (vN-M) expected utility function $\mathcal{U}_i : \Delta X \times \Delta Y \to \mathbb{R}$ for $i \in \{1, 2\}$. The function $\mathcal{U}_i$ is the bilinear extension of the Bernoulli utility function $\hat{\mathcal{U}_i} : X \times Y \to \mathbb{R}$ for $i \in \{1, 2\}$.

Let $A_{m\times n}$ and $B_{m\times n}$ be two matrices whose elements a_{ij} and b_{ij} denote (non-utility) payoffs of Alice and Bob, respectively. We assume that when Alice chooses x_i and Bob chooses y_j, Alice receives a_{ij} Euros[1] and Bob receives b_{ij} Euros. Note that for a player, say for Alice, $\mathcal{U}_1(x_i, y_j)$ is not necessarily equal to the corresponding monetary payoff a_{ij} unless we assume that utility is linear in money and players have purely selfish preferences.

Next, we introduce the decision problem $D = (\Delta X, \Delta Y, u_1)$ of Alice which is associated to the game G; Alice's monetary payoffs in D are identical to those in G. We assume that Alice is rational in D, that is, her preferences can be represented by a von Neumann-Morgenstern expected utility function $u_1 : \Delta X \times \Delta Y \to \mathbb{R}$ which is the bilinear extension of the Bernoulli utility function $\hat{u}_1 : X \times Y \to \mathbb{R}$. The associated decision problem can be thought of as a one-person game version of G in which Bob is replaced by an unconscious lottery choosing actions. So there is no payoff for another player in the decision problem. For example, Alice receives a_{ij} Euros when the outcome is (x_i, y_j). See Figure 1 for an illustration of a game and its associated decision problem.

Since Alice's payoffs in the game are completely identical to her payoffs in the associated decision problem, the difference in her preferences come from the fact that she plays the game against a human player Bob, who also receives payoffs. For each pair (a_{ij}, b_{ij}), define the function $\hat{v}_1$ as $\hat{v}_1(a_{ij}, b_{ij}) := \mathcal{U}_1(x_i, y_j) - u_1(x_i, y_j)$. The function $\hat{v}_1$ represents Alice's social preferences; it represents the additional utility (or disutility) that Alice receives from the outcome that Bob receives b_{ij} when she receives a_{ij}. We define the set of pair

[1] We assume monetary payoffs for simplicity.

of feasible expected monetary payoffs as follows: $M = \{(a_{pq}, b_{pq}) \in \mathbb{R}^2 |\, a_{pq} = p^\intercal Aq$ and $b_{pq} = p^\intercal Bq$ for $(p,q) \in \Delta X \times \Delta Y\}$. Then, we extend $\hat{v}_1$ to M by defining $v_1 : M \to \mathbb{R}$ as

$$v_1(a_{pq}, b_{pq}) = \mathcal{U}_1(p,q) - u_1(p,q).$$

Example 1. *Consider the game in Figure 1 in which Alice is inequality averse; she suffers a disutility from uneven expected payoff distributions but does not suffer any disutility from even ones. Suppose for the sake of argument that* $u_1(L,L) = u_1(R,L) = 5$*, and that Alice faces a utility loss of* 1 *both from the outcome* $(5,10)$ *and from the outcome* $(5,0)$*, i.e.* $v_1(5,10) = v_1(5,0) = -1$*. It implies that we have* $\mathcal{U}_1(L,L) = \mathcal{U}_1(R,L) = 4$*. Now consider the profile* $((\frac{1}{2},\frac{1}{2}),L)$ *in which Alice mixes* 50-50 *between* L *and* R*, and Bob plays* L*. Since* $\mathcal{U}_1$ *and* u_1 *are vN-M utility functions, we have* $\mathcal{U}_1((\frac{1}{2},\frac{1}{2}),L) = 4$ *and* $u_1((\frac{1}{2},\frac{1}{2}),L) = 5$ *which imply that* $v_1(5,5) = -1$*. This, however, contradicts with our assumption that Alice suffers less disutility in this case, that is,* $v_1(5,5)$ *should have been zero.*

The same conclusion can be obtained without assuming any specific value for the v *function as follows. We have* $v_1(5,10) = \mathcal{U}_1(L,L) - u_1(L,L)$, $v_1(5,0) = \mathcal{U}_1(R,L) - u_1(R,L)$ *and* $v_1(5,5) = \mathcal{U}_1((\frac{1}{2},\frac{1}{2}),L) - u_1((\frac{1}{2},\frac{1}{2}),L)$*. Since* $\mathcal{U}_1$ *and* u_1 *are vN-M utility functions, the latter equation is equal to* $\frac{1}{2}\mathcal{U}_1(L,L) + \frac{1}{2}\mathcal{U}_1(L,R) - \frac{1}{2}u_1(L,L) - \frac{1}{2}u_1(L,R) = \frac{1}{2}v_1(5,10) + \frac{1}{2}v_1(5,0)$*. As a result, we have* $v_1(5,5) = \frac{1}{2}v_1(5,10) + \frac{1}{2}v_1(5,0)$ *which is in conflict with our supposition that Alice faces a utility loss from outcomes* $(5,10)$ *and* $(5,0)$*.*

The paradoxical result described above occurs due to the assumption that Alice is rational both in the game and in the decision problem. To put it differently, assuming rationality of Alice in one domain forces her to be irrational[2] in the other domain. To resolve the paradox, we introduce the following restriction on the social preferences function.

Definition 1. A function v is called *doubly linear* if for all $(a_{pq}, b_{pq}) \in M$,

$$v(a_{pq}, b_{pq}) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j v(a_{ij}, b_{ij}).$$

The following theorem shows that doubly linearity of the social preferences function is a necessary and sufficient condition for a decision maker to be rational both in games and in the associated decision problems.

[2] A person who is not rational.

$$\begin{array}{c} \\ T \\ M \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 10,20 & 10,0 \\ 10,0 & 10,40 \\ 9,0 & 9,0 \end{pmatrix} \end{array}$$

Figure 2: A game without a Nash equilibrium. Numbers represent monetary payoffs.

Theorem 1. *Suppose that u_1 is a vN-M utility function. Then, $\mathcal{U}_1$ is a vN-M utility function if and only if v_1 is doubly linear.*

Proof. '$\Rightarrow$' By definition of v_1, for each pair (a_{pq}, b_{pq}) in M we have

$$v_1(a_{pq}, b_{pq}) = \mathcal{U}_1(p,q) - u_1(p,q) \stackrel{(1)}{=} \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j \mathcal{U}_1(x_i, y_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j u_1(x_i, y_j)$$

$$\stackrel{(2)}{=} \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j (u_1(x_i, y_j) + v_1(a_{ij}, b_{ij})) - \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j u_1(x_i, y_j)$$

$$\stackrel{(3)}{=} \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j v_1(a_{ij}, b_{ij}).$$

where Equation (1) is obtained by the assumption that $\mathcal{U}_1$ and u_1 are vN-M utility functions, and (2) is obtained by definition of v_1. Finally, cancelling out the terms we obtain (3).

'$\Leftarrow$' By definition of v_1 we have $\mathcal{U}_1(p,q) = u_1(p,q) + v_1(a_{pq}, b_{pq})$.

$$\stackrel{(4)}{=} \sum_{i=1}^{m}\sum_{i=1}^{n} p_i q_j u_1(x_i, y_j) + \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j v_1(a_{ij}, b_{ij}) \stackrel{(5)}{=} \sum_{i=1}^{m}\sum_{i=1}^{n} p_i q_j \mathcal{U}_1(x_i, x_j).$$

where (4) is obtained by using our supposition that u_1 is a vN-M utility function and v_1 is doubly linear. By definition of v_1, we obtain (5). $\square$

Corollary 1. *If a player is rational in a game, then he cannot be rational in the associated decision problem unless his social preference function satisfies doubly linearity.*

The existence of mixed strategy Nash equilibrium in a game might be in conflict with the rationality of a player in the associated decision problem. The following example illustrates this situation.

Example 2. *Suppose that Alice is inequality averse as in Example 1, and that Bob is self-regarding in the game in Figure 2. If we assume that Alice is rational in the associated decision problem, then this game does not possess a Nash equilibrium in mixed strategies. If we assume that this game has a Nash equilibrium, then Alice cannot be rational in the associated decision problem.*

First, suppose that Alice is rational in the decision problem. Notice that there is no Nash equilibrium in which Bob plays a pure strategy. Because, if Bob plays L*, then the best response of Alice to* L *denoted by* $BR_1(L)$ *is* $(\frac{1}{2}, \frac{1}{2}, 0)$*, but* $BR_2((\frac{1}{2}, \frac{1}{2}, 0))$ *is* R*. Similarly,* $BR_1(R)$ *is* $(\frac{3}{4}, \frac{1}{4}, 0)$ *however* $BR_2((\frac{3}{4}, \frac{1}{4}, 0))$ *is* L*. To reach a contradiction, suppose that* p *is a Nash equilibrium strategy of Alice. Then, we have* $\mathcal{U}_2(p, L) = \mathcal{U}_2(p, R)$ *which implies* $20p_1 = 40p_2$ *and* $p_2 = \frac{p_1}{2}$*. Since probabilities sum to 1, we obtain* $p_3 = 1 - p_1 - \frac{p_1}{2} = \frac{2-3p_1}{2}$*. Since* $\frac{2-3p_1}{2} \geq 0$*, we have* $p_1 \leq \frac{2}{3}$*. For all values* $p_1 < \frac{2}{3}$*, Bob's unique best response is* R*, but we showed that there is no Nash equilibrium corresponding to this case. If* $p_1 = \frac{2}{3}$*, then* $p_2 = \frac{1}{3}$*. But,* $(\frac{2}{3}, \frac{1}{3}, 0)$ *is never best response, because it forces the outcome to be inequitable as the* v *function will be negative; no matter what Bob plays, the expected payoffs will be* $(10, \frac{40}{3})$*. Alice, however, can decrease Bob's payoff sufficiently by deviating to a strategy in which she puts some probability on* B*. For example, if Bob plays* $(\frac{2}{3}, \frac{1}{3})$*, Alice's best response would be* $(\frac{27}{37}, 0, \frac{10}{37})$ *whose expected payoffs would be approximately* $(9.73, 9.73)$*.*

Second, one may construct a Nash equilibrium in this game by altering the utilities as desired in the associated decision problem, since we do not specify any particular type of irrationality of Alice in the decisin problem.

The main result in this note characterizes game and decision situations in which the same person who makes a decision can be rational and irrational simultaneously. This may challenge the belief that rationality is a personal trait.

References

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