Strategy-proof location of public bads

STRATEGY-PROOF LOCATION OF PUBLIC BADS

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STRATEGY-PROOF LOCATION OF PUBLIC BADS

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To my mother, for tolerating me irrespective of anything.

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CONTENTS

1 INTRODUCTION

"'A camel', it has been said, 'is a horse designed by a committee'." - Amartya Sen

A committee, formed with the objective to design a "horse" while trying to reflect the diverse wishes of its members, may end up with designing something different: a "camel" perhaps. This problem of social choice theory or collective decision making has historically been pointed out on several occations. A seminal result is the impossibility theorem in the social welfare context by Arrow (1950). An equally fundamental impossibility in the social choice context was pointed out by Gibbard (1973) and Satterthwaite (1975). The Gibbard - Satterthwaite impossibility theorem says that there is no non-dictatorial choice rule, with at least three alternatives in its range, which satisfies strategy-proofness. In particular, if there are at least three alternatives, then there exists no Pareto optimal and strategy-proof non-dictatorial rule. Pareto optimality says that given a collective choice, if one tries to make some agents better off, then this would make some other agent(s) worse off. Strategy-proofness means that no agent should gain from the collective choice rule by lying about his preferences. These two seemingly ideal properties have been shown to lead to dictatorship. A major factor behind this result is the fact that an agent can have any preference over the set of alternatives. This in turn implies that a restricted domain of preferences may give rise to possibilities. The domain of single-peaked preferences is an example of such a restricted domain. Inada (1964) has shown that restricted to this domain, there exist Pareto optimal, strategy-proof and non-dictatorial rules. This dissertation investigates another restricted domain - the domain of single-dipped preferences.

Single-dipped domains of preferences are quintessential in analysing collective decisions for locating noxious public facilities or public bads. These preferences are characterised by a unique point (the back yard of an agent for example), and his preference increases as the public facility moves away from this point - a precise way to capture the "Not in My Back Yard" syndrome. This unique point could be the place where the resident is living, but it could also be some natural resort, or place of historical interest; so it makes sense to assume that preferences are private knowledge. This in turn, implies that an agent might have some incentive to misreport his preference, in the hope of benefiting from the collective choice rule. This is the reason behind emphasising on strategy-proofness. Collective decision making for locating one such facility has already been studied. The common theme is that under strategy-proofness and Pareto optimality, the facility has to be placed at the boundary of the region under consideration. The main theme of this dissertation is to consider the problem of locating two such public bads, under different circumstances. The objective is to see along with Pareto optimality and strategy-proofness what other conditions are necessary to guarantee that a rule can never select an inner point. In each situation, we provide characterisations of the class of rules.

This dissertation is partitioned into three parts. In the first part, we consider two neighbouring countries modelled by adjoining unit intervals. Our objective is to see if the two countries can jointly decide upon the location of two public bads, one in each country; while maintaining their sovereignty to a large extent. Our main emphasis is on strategy-proofness, but to ensure the sovereignty of the countries we assume a stronger version of Pareto optimality - namely, country-wise Pareto optimality. In the first chapter of this part, we consider a myopic extension of single-dipped preferences, where agents compare two pairs of locations by comparing only the locations of the closest bads to their dips. In this case, to compare two pairs of locations, the locations of the farther public bads play no role, and therefore myopic preferences allow for many indifferences. Other than strategy-proofness and country-wise Pareto optimality, we also assume non-corruptibility and the far away condition. Non-corruptibility ensures that no agent can change the collective decision, without affecting himself. Far away condition is a tie-breaking condition in favour of the boundary points. In particular given the decision about the location of the bad of one country, ties are broken in favour of placing the other bad in the non-common boundary of the other country. In this chapter we show that an inner point cannot be chosen under these four properties and characterise the class of rules satisfying these conditions. This class primarily contains rules where both countries decide jointly about the locations of the bads. In the second chapter of this part, we consider lexicographic extension of single-dipped preferences, where an agent compares two pairs of locations by first comparing the locations of the closest bads to his dip. If they are at the same utility level, then he compares the location pairs by comparing the locations of the farthest bads to his dip. In this chapter, we show that under country-wise Pareto optimality and strategy-proofness inner points cannot be chosen and characterise the class of rules satisfying these two properties. This class contains joint decision rules as well as independent decision rules, where each country independently decides about the location of its own bad. We characterise the class of rules, corresponding to both myopic preferences and lexicographic preferences, in terms of collections of coalitions; so they are comparable. We have shown that, neither the former class of rules contains the later; nor the later class of rules contains the former.

In the second part, we consider the problem of locating two public bads in a region modelled by a unit interval. Each agent has a strict Euclidean single-dipped preference as a marginal preference. We allow agents to report any complete, strict, transitive and separable extensions of such a marginal preference as a preference over the pairs of locations. We show that under strategy-proofness and Pareto optimality, inner points cannot be chosen and also both bads have to be located at the same place. We characterise the class of rules satisfying these two properties. The class consists of non-constant monotonic voting rules.

The third part is a deviation from the previous parts. Here we consider the problem of locating a public facility, which is a public good for some agents but a public bad for the remaining agents. A public good is a public facility that entails positive externalities. It is natural to assume that agents, for whom the facility in question is a public good, would want it to be close to them. In other words, each of these agents has a single-peaked preference characterised by a unique point, which is his optimal point for the facility. His utility would decrease as facility moves away from that point. On the other hand, agents, for whom the facility in question is a public bad, will have single-dipped preferences. We assume that there are two agents and the designer knows whether an agent has single-peaked preference or single-dipped preference. We assume that there are finitely many alternatives. We characterise the class of rules satisfying strategy-proofness and Pareto optimality. Other than dictatorial rules, the class consists of rules where an agent is a dictator except some situations where he reports some particular preferences. In these situations the preference of the other agent is taken into account.

Each of the chapters can be read independently and contains its own conclusion. Although notations are as uniform as possible across chapters, slight differences may occur. As a consequence, relevant notations are defined within each chapter.

Part I

TWO PUBLIC BADS IN TWO NEIGHBOURING COUNTRIES

2 MYOPIC PREFERENCES

2.1 INTRODUCTION

We consider two neighbouring countries which jointly decide on where to locate two public bads, i.e., public provisions that are beneficial for the countries but that no one likes to have in his backyard. We assume that these countries maintain their independence to a large extent: one public bad will be located in each country, but the purpose of deciding jointly is to take also the preferences of the residents of the other country into account. As an example specific for what we do in this paper, one could think of Belgium and the Netherlands jointly deciding on the location of a windmill park in each country.

In our analysis joint decision making will be based on voting by the residents, who report their preferences. The emphasis is on strategy-proofness, which means that the joint decision rule should provide no incentives to report insincere preferences. We assume that each preference is characterised by a single dip, representing the worst location of the public bad in the resident country. This could be the place where the resident is living, but it could also be some natural resort, or place of historical interest; so it makes sense to assume that preferences are private knowledge and, moreover, each person should be free in expressing a preference.

In our stylised model for this situation we assume that the two countries are represented by the real intervals $A = [-1, 0]$ and $B = [0, 1]$. A (social choice) *rule* then assigns a location in country A and one in country B, based on the reported dips of the (finitely many) residents or *agents* in each country. Apart from strategy-proofness, we impose that the rule is country-wise Pareto optimal: for each country the location of the public bad should be Pareto optimal given the reported preferences and the location of the public bad in the other country. Since the agents will report truthfully, this means that the assigned locations will be country-wise Pareto optimal *ex post*. Country-wise Pareto optimality is a modification of the usual Pareto optimality condition, and reflects the assumption that countries keep their sovereignty in the decision making process.

In this chapter, we consider the following specification of the single-dipped preferences. A preference is *myopic* if it is completely determined by the distance between the dip and the nearer public bad (thus, preference increases with this distance). In this case, the location of the other public bad plays no role as long as it is farther away, and therefore myopic preferences allow for many indifferences. To break some of those indifferences we impose two further conditions. Non-corruptibility (Ritz (1985)) says that if an agent both before and after a change of preference is indifferent between the location pairs, then the locations themselves should not change. The 'far away' condition says that if all agents in a country, given the location in the other country, weakly prefer the non-shared border as location (i.e., -1 in country A and 1 in country B) then this should be assigned. Under these two additional conditions we show that only border locations can be chosen, i.e., one of the pairs $(-1, 1)$, $(-1, 0)$, $(0, 1)$, and $(0, 0)$.

We show that a social choice *rule* satisfying these four properties cannot place any bad in the interior of a country. Further we characterise all rules satisfying the four mentioned conditions by so-called decisive pairs of coalitions. These rules range from majority voting to almost dictatorial rules. As an example, consider the following rule (for simplicity, we assume that agents cannot have dips at -0.5 or at 0.5; and the total number of agents is odd). The rule selects $(-1, 1)$ if there is a majority in favour of $(-1, 1)$ over $(0, 0)$. If that is not the case and there are agents with dips in $[-1, -0.5)$, but there are no agents with dips in $(0.5, 1]$, then the rule selects $(0, 1)$. For the symmetrically opposite case where there are no agents with dips in $[-1, -0.5)$, but there are agents with dips in $(0.5, 1]$, the rule selects $(-1, 0)$. In all other cases, the rule selects $(0, 0)$.

This is a positive result as compared to the seminal impossibility theorem of Gibbard (1973) and Satterthwaite (1975) which says that if there are three or more alternatives, then it is impossible to find a non-dictatorial social choice rule which is also strategy proof and Pareto optimal. One way out from this impossibility result is to consider restricted preference domains. One possible restricted domain is the single-dipped preference domain. Peremans and Storcken (1999) have shown the equivalence between individual and group strategy proofness in sub domains of single-dipped preferences. Manjunath (2014) has characterised the class of all non-dictatorial, strategy-proof and Pareto optimal social choice functions when preferences are single-dipped over an interval. Barberà, Berga, and Moreno (2012) have characterised the class of all non-dictatorial, group strategy-proof and Pareto optimal social choice functions when preferences are single-dipped over a line. The rules in the present paper bear similarities to the rules in the last two papers.

But there are impossibility results in this domain as well. Öztürk, Peters, and Storcken (2013) have shown that there does not exist any non-dictatorial social choice rule that is strategy-proof and Pareto optimal when preferences are single-dipped over a disk. Öztürk, Peters, and Storcken (2014) have shown that there does not exist any non-dictatorial social choice rule that is strategy-proof and Pareto optimal when preferences are single-dipped over some, but not all, convex polytopes in the plane.

All these results are about strategy-proof location of one public bad. There is also a literature adopting a mechanism design approach to the location of public bads, that is, including monetary sidepayments: e.g., recently, Lescop

(2007) and Sakai (2012), but we are not aware of general results in this area addressing the location in more than one region.

This chapter is organised as follows. Section 2.2 introduces the model and some preliminary results. Section 2.3 shows that internal locations are excluded, and Section 2.4 provides the characterisation of all rules satisfying our conditions. Section 2.5 concludes with some examples, including examples showing logical independence of the four conditions.

2.2 THE TWO COUNTRY MODEL

Let country A be represented by the interval $[-1, 0]$ and country B by $[0, 1]$. The set of possible alternatives is denoted by $\mathcal{A} = [-1, 0] \times [0, 1]$. The set of agents N is partitioned in the set N_A of inhabitants of country A and the set N_B of inhabitants of country B. Let the cardinalities of N, N_A and N_B be natural numbers n, n_A, n_B, with $n = n_A + n_B$.

Each agent $i \in N$ has a preference $R_{x(i)}$ over $\mathcal{A}$, characterised by its *dip* $x(i)$ in $[-1, 1]$, such that alternative (a_1, b_1) is at least as good as alternative (a_2, b_2) at $R_{x(i)}$, with the usual notation $(a_1, b_1)R_{x(i)}(a_2, b_2)$, if

$$\min\{|a_1 - x(i)|, |b_1 - x(i)|\} \geqslant \min\{|a_2 - x(i)|, |b_2 - x(i)|\}.$$

As usual, $P_{x(i)}$ denotes the strict or asymmetric part of $R_{x(i)}$ and $I_{x(i)}$ denotes the indifference or symmetric part. Preference $R_{x(i)}$ is completely determined by its dip $x(i)$. Therefore, preferences are identified with their dips and denoted by $x(i)$ instead of $R_{x(i)}$.

A preference profile z assigns to each agent i in N a preference $z(i)$ such that $z(i)$ is a dip in $[-1, 0]$ if $i \in N_A$ and in $[0, 1]$ if $i \in N_B$. The set of all preference profiles is denoted by $\mathcal{R}$.

For a profile z and a non-empty set $S \subseteq N$, let $z_S = (z(i))_{i \in S}$. For $i \in N$, profile z' is an i-deviation of z if $z_{N\setminus\{i\}} = z'_{N\setminus\{i\}}$. For $a \in A$ and $S \subseteq N_A$, $(a^S, z_{N\setminus S})$ denotes the profile where all $i \in N\setminus S$ have preference $z(i)$ and all $i \in S$ have preference a. Similar notations will have similar meaning.

A *rule* φ assigns to each preference profile z an alternative $\varphi(z) = (\alpha(z), \beta(z)) \in \mathcal{A}$. For $x, y \in \mathbb{R}$, $\mu(x, y) = \frac{x+y}{2}$ denotes the midpoint of x and y. In case there is no confusion, for a profile z we write $\mu(z)$ instead of $\mu(\alpha(z), \beta(z))$.

We consider the following properties for a rule φ.

Strategy-Proofness (SP) φ is strategy-proof if $\varphi(z)R_{z(i)}\varphi(z')$ for every $z \in \mathcal{R}$, every $i \in N$, and every i-deviation z' of z.

Strategy-proofness says that truth-telling is a weakly dominant strategy.

Non-Corruptibility (NC) φ is non-corruptible if $\varphi(z) = \varphi(z')$ for every $z \in \mathcal{R}$, $i \in N$, and i-deviation z' of z such that $\varphi(z)I_{z(i)}\varphi(z')$ and $\varphi(z)I_{z'(i)}\varphi(z')$.

At non-corruptible rules a unilateral deviation either affects the deviator's preference somewhere or has no effect at all. This condition, introduced by Ritz (1985), eliminates tie-breaking caused by individual indifferences.

Country-wise Pareto optimality (CPO) Rule φ is Pareto optimal for country A if for every profile z there does not exist an $a \in [-1,0]$ such that $(a, \beta(z))R_{z(i)}\varphi(z)$ for all $i \in N_A$ and $(a, \beta(z))P_{z(k)}\varphi(z)$ for at least one $k \in N_A$. It is Pareto optimal for country B if for every profile z there does not exist a $b \in [0,1]$ such that $(\alpha(z), b)R_{z(i)}\varphi(z)$ for all $i \in N_B$ and $(\alpha(z), b)P_{z(i)}\varphi(z)$ for at least one $k \in N_B$. Rule φ is country-wise Pareto optimal if it is both Pareto optimal for country A and Pareto optimal for country B.

Far Away Condition (FA) Rule φ satisfies the far away condition if for very profile z:

- if $(\alpha(z), 1)R_{z(i)}\varphi(z)$ for all $i \in N$, then $\beta(z) = 1$, and
- if $(-1, \beta(z))R_{z(i)}\varphi(z)$ for all $i \in N$, then $\alpha(z) = -1$.

Monotonicity (MON) Rule φ is monotone if $\varphi(z) = \varphi(z')$ for all $z, z' \in \mathcal{R}$ such that for all agents $i \in N$:

- $z'(i) \leqslant z(i) \leqslant \alpha(z)$ or
- $\alpha(z) \leqslant z(i) \leqslant z'(i) \leqslant \mu(z)$ or
- $\mu(z) \leqslant z'(i) \leqslant z(i) \leqslant \beta(z)$ or
- $\beta(z) \leqslant z(i) \leqslant z'(i)$.

Thus, φ is monotone if outcomes do not change whenever agents increase their minimal distance to these locations while not jumping across any of these locations.

Remark 2.2.1. For an agent i with dip $z(i)$; the weak lower contour set at location $(a, b) \in \mathcal{A}$ is defined by $L((a, b), z(i)) = \{(x, y) \in \mathcal{A}\colon (a, b)R_{z(i)}(x, y)\}$ and the strict lower contour set at location $(a, b) \in \mathcal{A}$ is defined by $L'((a, b), z(i)) = \{(x, y) \in \mathcal{A}\colon (a, b)P_{z(i)}(x, y)\}$. Rule φ is said to be Maskin monotone if $\varphi(z) = \varphi(z')$ for all profiles z and z' such that $L(\varphi(z), z(i)) \subseteq L(\varphi(z), z'(i))$ and $L'(\varphi(z), z(i)) \subseteq L'(\varphi(z), z'(i))$ for all $i \in N$. Evidently, Maskin monotonicity implies monotonicity as defined above. To see that the reverse does not hold consider the following rule.

$$g(z) = \begin{cases} (-1,1) & \text{if } |\{i \in N : z(i) \in [-\frac{3}{4}, \frac{3}{4}]\}| > \frac{n}{2} \\ (-1,0) & \text{if } |\{i \in N : z(i) \in [-\frac{3}{4}, \frac{3}{4}]\}| \leqslant \frac{n}{2} \\ & \quad \text{and } \{i \in N_A : z(i) \in [-\frac{1}{2}, 0]\} = N_A \\ (0,1) & \text{if } |\{i \in N : z(i) \in [-\frac{3}{4}, \frac{3}{4}]\}| \leqslant \frac{n}{2} \\ & \quad \text{and } \{i \in N_B : z(i) \in [0, \frac{1}{2}]\} = N_B \\ (0,0) & \text{otherwise.} \end{cases}$$

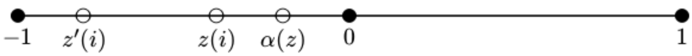
Note that, this rule satisfies monotonicity as defined above. Now consider the case when $N_A = \{i, j\}$ and $N_B = \{k, l\}$. Then for the profile z, where $z(i) = -1$, $z(j) = z(k) = 0$ and $z(l) = 1$; we have $g(z) = (0, 0)$. Now consider another profile z', which is an $i-$deviation from z such that $z'(i) = -\frac{3}{4}$ and $z_{N \setminus \{i\}} = z'_{N \setminus \{i\}}$. Note that $L(g(z), z(m)) \subseteq L(g(z), z'(m))$ and $L'(g(z), z(m)) \subseteq L'(g(z), z'(m))$ for all $m \in N$. But contrary to Maskin monotonicity, $g(z') = (-1, 1) \neq g(z)$. This shows that the monotonicity condition defined above is a weaker condition than Maskin monotonicity.

The following lemma shows that monotonicity is implied by strategy-proofness and non-corruptibility.

Lemma 2.2.2. *Let φ satisfy SP and NC. Then φ satisfies MON.*

Proof. It is sufficient to prove monotonicity for an i-deviation from $z \in \mathcal{R}$ to $z' \in \mathcal{R}$ for an agent $i \in N_A$. There are three cases.

(a) $z'(i) < z(i) \leqslant \alpha(z)$.



If agent i manipulates from $z(i)$ to $z'(i)$ then SP implies

$$|z(i) - \alpha(z)| \geqslant |z(i) - \alpha(z')| \tag{1}$$

and if agent i manipulates from $z'(i)$ to $z(i)$ then SP implies

$$|z'(i) - \alpha(z')| \geqslant |z'(i) - \alpha(z)| \,. \tag{2}$$

If $x \in \mathbb{R}$ is such that $z(i) = \mu(x, \alpha(z))$ then by (1): $\alpha(z') \in [\max\{x, -1\}, \alpha(z)]$, hence by (2): $\alpha(z') = \alpha(z)$. By NC, $\varphi(z) = \varphi(z')$.

(b) $\alpha(z) \leqslant z(i) < z'(i) \leqslant \mu(z)$.

If agent i manipulates from $z(i)$ to $z'(i)$ then SP implies

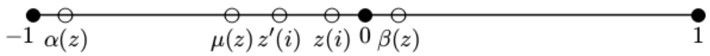
$$|z(i) - \alpha(z)| \geqslant |z(i) - \alpha(z')| \text{ or } |z(i) - \alpha(z)| \geqslant |z(i) - \beta(z')| \tag{3}$$

and if agent i manipulates from $z'(i)$ to $z(i)$ then SP implies

$$|z'(i) - \alpha(z')| \geqslant |z'(i) - \alpha(z)| \text{ and } |z'(i) - \beta(z')| \geqslant |z'(i) - \alpha(z)| \,. \tag{4}$$

If the first inequality in (3) holds then the first inequality in (4) implies $\alpha(z') = \alpha(z)$. By the second inequality in (4), $\alpha(z) = \alpha(z')$ is closer to both $z(i)$ and $z'(i)$ than $\beta'(z)$ is, so by NC, $\varphi(z) = \varphi(z')$. If the second inequality in (3) holds then the second inequality in (4) is violated, a contradiction.

(c) $\alpha(z) \leqslant \mu(z) \leqslant z'(i) < z(i)$.



If agent i manipulates from $z(i)$ to $z'(i)$ then SP implies

$$|z(i) - \beta(z)| \geqslant |z(i) - \alpha(z')| \text{ or } |z(i) - \beta(z)| \geqslant |z(i) - \beta(z')| \tag{5}$$

and if agent i manipulates from $z'(i)$ to $z(i)$ then SP implies

$$|z'(i) - \alpha(z')| \geqslant |z'(i) - \beta(z)| \text{ and } |z'(i) - \beta(z')| \geqslant |z'(i) - \beta(z)| \,. \tag{6}$$

If the first inequality in (5) holds then the first inequality in (6) implies $\alpha(z') = \beta(z)$, hence $\alpha(z') = \beta(z) = 0$. Hence by NC, $\varphi(z) = \varphi(z')$. If the second inequality in (5) holds then the second inequality in (6) implies $\beta(z) = \beta(z')$. Then both at $z(i)$ and $z'(i)$ agent i is indifferent between $\varphi(z)$ and $\varphi(z')$, so that by NC, $\varphi(z) = \varphi(z')$. □

2.3 NO INTERNAL SOLUTIONS

We first show that under the conditions imposed in this paper a rule never assigns locations in the interiors of the two countries.

Theorem 2.3.1. *Let rule φ satisfy SP, CPO, NC, and FA. Then $\varphi(z) \in \{(-1,1), (0,0), (-1,0), (0,1)\}$ for every $z \in \mathcal{R}$.*

The proof of this theorem uses the two lemmas below. For the rest of this section we assume that φ is a rule satisfying the four conditions in Theorem 2.3.1. For a profile $z \in \mathcal{R}$ we define $S(z) = \{i \in N_A : z(i) \geqslant \alpha(z)\}$ and $T(z) = \{i \in N_B : z(i) \leqslant \beta(z)\}$. By Lemma 2.2.2, φ is monotone. Therefore we may assume that

(a) $z = (-1^{N_A \setminus S(z)}, 0^{S(z)}, \mu(z)^{T(z)}, 1^{N_B \setminus T(z)})$ if $\mu(z) \in [0,1]$, and

(b) $z = (-1^{N_A \setminus S(z)}, \mu(z)^{S(z)}, 0^{T(z)}, 1^{N_B \setminus T(z)})$ if $\mu(z) \in [-1,0]$.

The following lemma shows that if one of the two bads is located at 0, then the other one cannot be located at an interior point of its country.

Lemma 2.3.2.

(a) Let $\alpha(z) = 0 < \beta(z)$. Then $z(i) \leqslant \frac{1}{2}$ for all $i \in N$, and $\beta(z) = 1$.

(b) Let $\alpha(z) < 0 = \beta(z)$. Then $z(i) \geqslant -\frac{1}{2}$ for all $i \in N$, and $\alpha(z) = -1$.

Proof. We only prove part (a), part (b) is analogous. Let $\alpha(z) = 0 < \beta(z)$. Then $\mu(z) \in [0,1]$, so $z = (-1^{N_A \setminus S(z)}, 0^{S(z)}, \mu(z)^{T(z)}, 1^{N_B \setminus T(z)})$. Since all agents $i \in T(z)$ are indifferent between $(0,0)$ and $(0,\beta(z))$ and all agents $i \in N_B \setminus T(z)$ strictly prefer $(0,0)$ to $(0,\beta(z))$, CPO implies that $T(z) = N_B$. From this, $\beta(z) = 1$ follows by FA, and thus $z(i) \leqslant \mu(z) = \frac{1}{2}$ for all $i \in N$. □

The next lemma shows that if one of the two bads is located at the extreme end, then the other is located at an extreme end as well.

Lemma 2.3.3. $\alpha(z) \in \{-1,0\}$ *if and only if* $\beta(z) \in \{0,1\}$.

Proof. We show the if-direction, the other direction is analogous. By Lemma 2.3.2(b) it is sufficient to prove that $\alpha(z) \in \{-1,0\}$ if $\beta(z) = 1$. To the contrary suppose $-1 < \alpha(z) < 0$ and $\beta(z) = 1$. Then $0 < \mu(z) < \frac{1}{2}$ and $T(z) = N_B$ by definition of $T(z)$, so that $z = (-1^{N_A \setminus S(z)}, 0^{S(z)}, \mu(z)^{N_B})$. By FA, $\beta(z') = 1$ for all profiles $z' \in \mathcal{R}$ with $z'(i) = \mu(z)$ for all $i \in N_B$. We will compare the four profiles in the following table:

	$N_A \setminus S(z)$	$S(z)$	N_B	α	β
z	-1	0	$\mu(z)$	$\alpha(z)$	1
z^-	$\frac{\alpha(z)-1}{2}$	0	$\mu(z)$	$\alpha(z^-)$	1
z^*	$\frac{\alpha(z)-1}{2}$	$\frac{\alpha(z)}{2}$	$\mu(z)$	$\alpha(z^*)$	1
z^+	-1	$\frac{\alpha(z)}{2}$	$\mu(z)$	$\alpha(z^+)$	1

(For instance, the first line of this table means that $z = (-1^{N_A \setminus S(z)}, 0^{S(z)}, \mu(z)^{N_B})$ and that $\beta(z) = 1$. Note that to all these profiles β assigns location

1 by FA.) Consider profiles z and z^-. SP implies that $\alpha(z^-) \in \{\alpha(z), -1\}$, and then CPO implies that $\alpha(z^-) = -1$. Applying SP at profiles z^- and z^* now yields that $\alpha(z^*) = -1$. Considering SP at the profiles z and z^+ yields that $\alpha(z^+) \in \{\alpha(z), 0\}$, and then CPO implies $\alpha(z^+) = 0$. Finally, comparing profiles z^+ and z^* yields a contradiction with SP since $(0,1) = \varphi(z^+)$ is better for dip $\frac{\alpha(z)-1}{2}$ than $(-1,1) = \varphi(z^*)$. □

Proof of Theorem 2.3.1. Let $z \in \mathcal{R}$ and suppose that $-1 < \alpha(z) < 0$ and $0 < \beta(z) < 1$. We will derive a contradiction: then the proof is complete by Lemma 2.3.3. We assume without loss of generality that $\mu(z) \in [0,1]$, so that $z = (-1^{N_A \setminus S(z)}, 0^{S(z)}, \mu(z)^{T(z)}, 1^{N_B \setminus T(z)})$.

First note that $S(z) \neq \emptyset, N_A$ and $T(z) \neq \emptyset, N_B$ by CPO. Consider the following profiles, where $t \in \mathbb{N}$:

	$N_A \setminus S(z)$	$S(z)$	$T(z)$	$N_B \setminus T(z)$
$z^0 = z$	-1	0	$\mu(z)$	1
z^t $(t \geqslant 1)$	-1	$\frac{\alpha(z^{t-1})}{2}$	$\mu(z)$	1
v^1	$\frac{\alpha(z^0)-1}{2}$	0	$\mu(z)$	1
v^t $(t \geqslant 2)$	$\frac{\alpha(z^{t-1})-1}{2}$	$\frac{\alpha(z^{t-2})}{2}$	$\mu(z)$	1
w^t $(t \geqslant 1)$	$\frac{\alpha(z^{t-1})-1}{2}$	$\frac{\alpha(z^{t-1})}{2}$	$\mu(z)$	1

The proof proceeds in a few steps.

Step 1. Let $t \geqslant 1$ and suppose that $-1 < \alpha(z^{t-1}) < 0$. Then $\varphi(v^t) = \varphi(w^t) = (-1,1)$.

Proof. Comparing z^{t-1} and v^t, SP implies $\alpha(v^t) \leqslant \alpha(z^{t-1})$ since otherwise $N_A \setminus S(z)$ can manipulate from z^{t-1} to v^t, and therefore $\alpha(v^t) \in \{-1, \alpha(z^{t-1})\}$ since otherwise $N_A \setminus S(z)$ can manipulate from v^t to z^{t-1}. By CPO, this implies $\alpha(v^t) = -1$. Therefore, by Lemma 2.3.3, $\beta(v^t) \in \{0,1\}$. Now Lemma 2.3.2(b) implies that $\beta(v^t) \neq 0$. Thus, $\beta(v^t) = 1$ and $\varphi(v^t) = (-1,1)$. By SP, going from w^t to v^t, we obtain $\alpha(w^t) = -1$. By Lemma 2.3.3, $\beta(w^t) \in \{0,1\}$. Since $\frac{\alpha(z^{t-1})-1}{2} < -\frac{1}{2}$, Lemma 2.3.2(b) then implies $\beta(w^t) = 1$. Thus, $\varphi(w^t) = (-1,1)$.

Step 2. For all $t \geqslant 1$:

$$-1 < \alpha(z^t) < \alpha(z^{t-1}) < 0 \leqslant 2\mu(z) < \beta(z^t) \leqslant -\alpha(z^{t-1}) < 1 .$$

Proof. By assumption we have $-1 < \alpha(z^0) = \alpha(z) < 0 \leqslant 2\mu(z) < \beta(z^0) = \beta(z) < 1$. We prove the statement in Step 2 by induction. Assume it is true for all $s < t$, where $t \geqslant 1$. By going from z^t to z^{t-1}, SP implies $\alpha(z^t) \in [-1, \alpha(z^{t-1})]$ or $\alpha(z^t) = 0$. If $\alpha(z^t) = 0$, then $N_A \setminus S(z)$ could manipulate from w^t to z^t since $\varphi(w^t) = (-1,1)$, as established in Step 1. Hence, $\alpha(z^t) \in [-1, \alpha(z^{t-1})]$. Now CPO applied to the profile z^t implies

$\alpha(z^t) \neq \alpha(z^{t-1})$, so $\alpha(z^t) \in [-1, \alpha(z^{t-1}))$. In turn, this and the induction hypothesis imply by SP, going from z^{t-1} to z^t, that $\beta(z^t) \leqslant -\alpha(z^{t-1})$. Then $\alpha(z^t) > -1$, since otherwise by Lemma 2.3.3, $\beta(z^t) = 0$, contradicting Lemma 2.3.2(b), or $\beta(z^t) = 1$, contradicting that $-\alpha(z^{t-1}) < 1$ by the induction hypothesis. Finally, $2\mu(z) < \beta(z^t)$ follows since otherwise CPO would imply that $\beta(z^t) = 0$.

Step 3. $\mu(z) = 0$.
Proof. First suppose that for some $t > 1$, $\mu(z^t) \geqslant \frac{\alpha(z^{t-2})}{2}$. Then, by Step 2, for all $i \in S(z)$ we have $\alpha(z^t) < \frac{\alpha(z^{t-1})}{2} = z^t(i) < z^{t-1}(i) = \frac{\alpha(z^{t-2})}{2} \leqslant \mu(z^t)$, so that by MON we have $\varphi(z^{t-1}) = \varphi(z^t)$, a contradiction. Thus, $\mu(z^t) < \frac{\alpha(z^{t-2})}{2}$ for all $t > 1$. Hence, $\alpha(z^t) + \beta(z^t) < \alpha(z^{t-2})$ for all $t > 1$. By Step 2 this implies $2\mu(z) < \beta(z^t) < \alpha(z^{t-2}) - \alpha(z^t)$, which implies that $\mu(z) = 0$ since $\alpha(z^{t-2}) - \alpha(z^t)$ converges to 0 for t going to infinity.

Step 4. If $\mu(z) = 0$ then $\mu(z^1) < 0$.
Proof. Follows from Step 2 by taking $t = 1$.

We can now complete the proof of the theorem. Step 3 implies that $\mu(z) = 0$ for *any* profile z with $\alpha(z) \in (-1, 0)$ and $\beta(z) \in (0, 1)$ – observe that this indeed does not depend on our initial assumption $\mu(z) \geqslant 0$. This contradicts Step 4 since z^1 is also such a profile. □

2.4 THE RULES

In this section, we describe the class of rules φ that satisfy CPO, SP, NC, and FA. Theorem 2.3.1 says that the range of such rules φ is equal to $\mathcal{B} = \{-11, -10, 01, 00\}$, where -11 denotes $(-1, 1)$, etc. For a profile $z \in \mathcal{R}$, the restriction of z to $\mathcal{B}$ is denoted by $z|_{\mathcal{B}}$. We first show that for such rules φ only these restricted preferences matter.

Lemma 2.4.1. *Let rule φ satisfy CPO, SP, NC, and FA, and let $z, z' \in \mathcal{R}$ such that $z|_{\mathcal{B}} = z'|_{\mathcal{B}}$. Then $\varphi(z) = \varphi(z')$.*

Proof. Without loss of generality we may assume that z and z' are i-deviations. Theorem 2.3.1 implies that $\varphi(z)$ and $\varphi(z')$ are both in $\mathcal{B}$. Since $z|_{\mathcal{B}} = z'|_{\mathcal{B}}$, agent i in N has at $z(i)$ the same preference between $\varphi(z)$ and $\varphi(z')$ as at $z'(i)$. By SP, agent i must be indifferent between $\varphi(z)$ and $\varphi(z')$ both at $z(i)$ and $z'(i)$. Hence $\varphi(z) = \varphi(z')$ by NC. □

On $\mathcal{B}$ there are just four different single-dipped preferences. These preferences, with dip x and symmetric and asymmetric parts $\sim$ and $\succ$, are the following:

- If $-1 \leqslant x < 0.5$, then $00 \sim 01 \succ -11 \sim -10$.
- If $x \in \{-0.5, 0.5\}$, then $00 \sim 01 \sim -11 \sim -10$.

- If $-0.5 < x < 0.5$, then $-11 \succ -10 \sim 00 \sim 01$.
- If $0.5 < x \leqslant 1$, then $-10 \sim 00 \succ -11 \sim 01$.

We will show that set of rules satisfying CPO, SP, NC, and FA, consists of monotonic voting between -11 and 00, except for cases where -11 and 00 cannot be selected because of FA or CPO. These voting rules are characterised by families of decisive pairs of coalitions of agents. The first coalition of such a pair contains the agents with dip strictly between -0.5 and 0.5: these agents strictly prefer outcome -11 over outcome 00. The second coalition contains the agents with dip either -0.5 or 0.5: these agents are indifferent between -11 and 00. We will now make this more precise.

Definition 2.4.2. $\mathcal{W} \subseteq 2^N \times 2^N$ is a *family of decisive pairs* if

(d1) $(U, V) \in \mathcal{W}$ for all $U, V \subseteq N$ with $U \cup V = N$,

(d2) $(U', V') \in \mathcal{W}$ for all $U', V' \subseteq N$ for which there is $(U, V) \in \mathcal{W}$ with $U \subseteq U'$ and $U \cup V \subseteq U' \cup V'$.

(d3) $U \cap N_A \neq \emptyset$ or $N_A \subseteq V$ for all $(U, V) \in \mathcal{W}$.

(d4) $U \cap N_B \neq \emptyset$ or $N_B \subseteq V$ for all $(U, V) \in \mathcal{W}$.

With a rule φ that has the properties CPO, SP, NC, and FA, we will associate a family of decisive pairs, as follows. For a profile $z \in \mathcal{R}$ let $U(z) = \{i \in N : -\frac{1}{2} < z(i) < \frac{1}{2}\}$, which is the set of agents who strictly prefer -11 to 00; and let $V(z) = \{i \in N : z(i) \in \{-\frac{1}{2}, \frac{1}{2}\}\}$, which is the set of agents who are indifferent between -11 and 00. We define

$$\mathcal{W}_\varphi = \{(U, V) \in 2^N \times 2^N : \text{there exists } z \in \mathcal{R} \text{ with } (U, V) = (U(z), V(z)) \text{ and } \varphi(z) = -11\}.$$

Observe that, by Lemma 2.4.1, $\varphi(z) = -11$ for *all* $z \in \mathcal{R}$ such that $(U(z), V(z)) \in \mathcal{W}_\varphi$. We now have:

Lemma 2.4.3. *Let φ satisfy SP, CPO, NC, and FA. Then $\mathcal{W}_\varphi$ is a family of decisive pairs.*

Proof. For condition (d1), let $U, V \subseteq N$ with $U \cup V = N$. Take a profile z with $z(i) \in (-\frac{1}{2}, \frac{1}{2})$ for all $i \in U$ and $z(i) \in \{-\frac{1}{2}, \frac{1}{2}\}$ for all $i \in V$. Then $\varphi(z) = -11$ by FA (or CPO), hence $(U, V) = (U(z), V(z)) \in \mathcal{W}_\varphi$.

For condition (d2), let $z \in \mathcal{R}$ with $\varphi(z) = -11$. Consider a j-deviation z' of z such that $U(z) \subseteq U(z')$ and $U(z) \cup V(z) \subseteq U(z') \cup V(z')$. It is sufficient to prove that $\varphi(z') = -11$. Without loss of generality assume that $j \in N_A$. Assume $U(z) \neq U(z')$ or $V(z) \neq V(z')$, otherwise we are done by Lemma 2.4.1. Since $U(z) \subseteq U(z')$ and $U(z) \cup V(z) \subseteq U(z') \cup V(z')$ it follows that $z(j) < z'(j) \leqslant 0$. So by MON, $\varphi(z') = \varphi(z) = -11$.

For condition (d3), let $(U, V) \in 2^N \times 2^N$ and suppose that $U \cap N_A = \emptyset$ and $N_A \not\subseteq V$ (the other case is similar). Let z be any profile with $(U(z), V(z)) = (U, V)$. Then $z(i) \leqslant -\frac{1}{2}$ for all $i \in N_A$ and $z(i) < -\frac{1}{2}$ for some $i \in N_A$. By CPO, $\alpha(z) = 0$. Hence, $(U, V) \notin \mathcal{W}_\varphi$. □

Conversely, for a family of decisive pairs $\mathcal{W}$ we define a rule $\varphi_\mathcal{W}$ as follows. For every $z \in \mathcal{R}$:

$$\varphi_\mathcal{W}(z) = \begin{cases} -11 & \text{if } (U(z), V(z)) \in \mathcal{W} \\ -10 & \text{if } (U(z), V(z)) \notin \mathcal{W} \text{ and } N_A \subseteq U(z) \cup V(z) \\ 01 & \text{if } (U(z), V(z)) \notin \mathcal{W} \text{ and } N_B \subseteq U(z) \cup V(z) \\ 00 & \text{otherwise.} \end{cases}$$

In words, $\varphi_\mathcal{W}$ assigns -11 to a profile z if the pair $(U(z), V(z))$ is decisive. Otherwise, it assigns 00 unless FA demands otherwise, that is, -10 or 01. Next, we prove that $\varphi_\mathcal{W}$ satisfies our four conditions.

Lemma 2.4.4. *Let $\mathcal{W}$ be a family of decisive pairs. Then $\varphi_\mathcal{W}$ satisfies SP, CPO, NC, and FA.*

Proof. We first prove SP of $\varphi_\mathcal{W}$. Consider $z \in \mathcal{R}$ and an i-deviation z' of z for $i \in N_A$. It is sufficient to prove that i weakly prefers $\varphi_\mathcal{W}(z)$ to $\varphi_\mathcal{W}(z')$. This is evidently the case if $\varphi_\mathcal{W}(z) = \varphi_\mathcal{W}(z')$ or if $z(i) = -\frac{1}{2}$. Therefore assume that $\varphi_\mathcal{W}(z) \neq \varphi_\mathcal{W}(z')$ and that $z(i) \neq -\frac{1}{2}$. We distinguish the following two cases.

- $-1 \leqslant z(i) < -\frac{1}{2}$. Then $U(z) \subseteq U(z')$ and $U(z) \cup V(z) \subseteq U(z') \cup V(z')$ and because of $\varphi_\mathcal{W}(z) \neq \varphi_\mathcal{W}(z')$ at least one of these inclusions is strict. Hence, (d2) and $\varphi_\mathcal{W}(z) \neq \varphi_\mathcal{W}(z')$ imply $(U(z), V(z)) \notin \mathcal{W}$ and $\varphi_\mathcal{W}(z) \neq -11$. Since $N_A \not\subseteq U(z) \cup V(z)$, we have $\varphi_\mathcal{W}(z) \neq -10$. Hence, $\varphi_\mathcal{W}(z) \in \{00, 01\}$, so that i at $z(i)$ weakly prefers $\varphi_\mathcal{W}(z)$ to $\varphi_\mathcal{W}(z')$.

- $-\frac{1}{2} < z(i) \leqslant 0$. Then $U(z') \subseteq U(z)$ and $U(z') \cup V(z') \subseteq U(z) \cup V(z)$, and because of $\varphi_\mathcal{W}(z) \neq \varphi_\mathcal{W}(z')$ at least one of these inclusions is strict. If $\varphi_\mathcal{W}(z) \in \{00, 01\}$, then by (d2) of $\mathcal{W}$ and the definition of $\varphi_\mathcal{W}$ we have $\varphi_\mathcal{W}(z) = \varphi_\mathcal{W}(z')$, a contradiction. If $\varphi_\mathcal{W}(z) = -10$ and $z'(i) < -\frac{1}{2}$, then $\varphi_\mathcal{W}(z') = 00$ and in that case agent i does not manipulate. Otherwise, $\varphi_\mathcal{W}(z) = -11$, which is the single best outcome at $z(i)$.

We next prove CPO of $\varphi_\mathcal{W}$. It is sufficient to prove this for country A. To the contrary, suppose that all agents in N_A weakly prefer $(a, \beta_\mathcal{W}(z))$ to $\varphi_\mathcal{W}(z) = (\alpha_\mathcal{W}(z), \beta_\mathcal{W}(z))$ and some j in N_A strictly. We distinguish three cases.

- $\alpha_\mathcal{W}(z) = 0$. Then $(U(z), V(z)) \notin \mathcal{W}$ and all agents in N_A have their dip equal to or greater than $\frac{a}{2} \geqslant -\frac{1}{2}$. Hence $N_A \subseteq U(z) \cup V(z)$, which implies $\alpha_\mathcal{W}(z) = -1$, a contradiction.

- $\beta_{\mathcal{W}}(z) = 0$ and $\alpha_{\mathcal{W}}(z) = -1$. Then $(U(z), V(z)) \notin \mathcal{W}$ and $N_A \subseteq U(z) \cup V(z)$. But then all agents in N_A have their dip greater than or equal to $-\frac{1}{2}$, which contradicts the existence of agents j who strictly prefer $(a, 0)$ to -10.

- $\beta_{\mathcal{W}}(z) = 1$ and $\alpha_{\mathcal{W}}(z) = -1$. Then $(U(z), V(z)) \in \mathcal{W}$. So, by condition (d3) of $\mathcal{W}$ there are agents $i \in N_A \cap U(z)$ or all agents in N_A have their dip at $-\frac{1}{2}$. Since agents in $N_A \cap U(z)$ strictly prefer -11 to every other outcome $(a, 1)$, we must have that that all agents in N_A have their dip at $-\frac{1}{2}$. At dip $-\frac{1}{2}$, however, outcome -11 is weakly preferred to every outcome $(a, 1)$ for $-1 \leqslant a \leqslant 0$. This contradicts the existence of agents j who strictly prefer $(a, 1)$ to -11.

Third, we next prove NC of $\varphi_{\mathcal{W}}$. Consider $z \in \mathcal{R}$ and an i-deviation z' of z for $i \in N_A$, and suppose that i is indifferent between $\varphi_{\mathcal{W}}(z)$ and $\varphi_{\mathcal{W}}(z')$ both at $z(i)$ and at $z'(i)$. It is sufficient to prove that $\varphi_{\mathcal{W}}(z) = \varphi_{\mathcal{W}}(z')$. We may assume that $z(i) < z'(i)$ and the ordering at $z(i)$ of $\mathcal{B}$ is different from that of $z'(i)$. We distinguish two cases.

- $-1 \leqslant z(i) < -\frac{1}{2}$. Then $\varphi_{\mathcal{W}}(z) \neq -10$ since $N_A \nsubseteq U(z) \cup V(z)$. If $\varphi_{\mathcal{W}}(z) = -11$ then by $z(i) < z'(i)$ and (d2) of $\mathcal{W}$ we have $\varphi_{\mathcal{W}}(z') = -11$ and we are done. Suppose $\varphi_{\mathcal{W}}(z) \in \{00, 01\}$. Then since at $z(i)$ outcomes $\varphi_{\mathcal{W}}(z)$ and $\varphi_{\mathcal{W}}(z')$ are indifferent we have $\varphi_{\mathcal{W}}(z') \in \{00, 01\}$. Then, since $N_B \subseteq U(z) \cup V(z)$ if and only if $N_B \subseteq U(z') \cup V(z')$, it follows that $\varphi_{\mathcal{W}}(z) = 01$ if and only if $\varphi_{\mathcal{W}}(z') = 01$.

- $z(i) = -\frac{1}{2}$ and $-\frac{1}{2} < z'(i) \leqslant 0$. If $\varphi_{\mathcal{W}}(z') = -11$, then the indifference between $\varphi_{\mathcal{W}}(z)$ and $\varphi_{\mathcal{W}}(z')$ at $z'(i)$ yields that $\varphi_{\mathcal{W}}(z) = \varphi_{\mathcal{W}}(z') = -11$. Since $N_A \subseteq U(z') \cup V(z')$ if and only if $N_A \subseteq U(z) \cup V(z)$, it follows that $\varphi_{\mathcal{W}}(z) = -10$ if and only if $\varphi_{\mathcal{W}}(z') = -10$. Further, since $N_B \subseteq U(z) \cup V(z)$ if and only if $N_B \subseteq U(z') \cup V(z')$, it follows that $\varphi_{\mathcal{W}}(z) = 01$ if and only if $\varphi_{\mathcal{W}}(z') = 01$. Hence, we also have $\varphi_{\mathcal{W}}(z) = 00$ if and only if $\varphi_{\mathcal{W}}(z') = 00$ since that is the only remaining case. Thus, $\varphi_{\mathcal{W}}(z) = \varphi_{\mathcal{W}}(z')$.

Finally, we prove FA of φ. Suppose that all agents weakly prefer $(\alpha_{\mathcal{W}}(z), 1)$ to $\varphi_{\mathcal{W}}(z) = (\alpha_{\mathcal{W}}(z), \beta_{\mathcal{W}}(z))$. It is sufficient to prove that $\beta_{\mathcal{W}}(z) = 1$. To the contrary suppose $\beta_{\mathcal{W}}(z) = 0$. Then all agents in N_B have their dip smaller than or equal to $\frac{1}{2}$. So, $N_B \subseteq U(z) \cup V(z)$. As $\beta_{\mathcal{W}}(z) = 0$, we have $(U(z), V(z)) \notin \mathcal{W}$. This however contradicts the definition of $\varphi_{\mathcal{W}}$ because if $(U(z), V(z)) \notin \mathcal{W}$ and $N_B \subseteq U(z) \cup V(z)$, then $\varphi_{\mathcal{W}}(z) = 01$. □

We can now formulate the main result of this section, which is a corollary to the preceding two lemmas.

Corollary 2.4.5. *Let φ be a rule. Then φ satisfies SP, CPO, NC, and FA, if and only if there is a family $\mathcal{W}$ of decisive pairs such that $\varphi = \varphi_{\mathcal{W}}$.*

Proof. If $\mathcal{W}$ is a family of decisive pairs, then $\varphi_{\mathcal{W}}$ satisfies SP, CPO, NC, and FA by Lemma 2.4.4. Conversely, let φ satisfy these four conditions. We show that $\varphi = \varphi_{\mathcal{W}_\varphi}$, which completes the proof by Lemma 2.4.3. Let $z \in \mathcal{R}$. Then $\varphi(z) = -11 \Leftrightarrow (U(z), V(z)) \in \mathcal{W}_\varphi \Leftrightarrow \varphi_{\mathcal{W}_\varphi}(z) = -11$. If $\varphi_{\mathcal{W}_\varphi}(z) = 01$ then (by the previous step) $\varphi(z) \neq -11$ and moreover, $N_B \subseteq U(z) \cup V(z)$, so that $\varphi(z) = 01$ by FA. Similarly, $\varphi_{\mathcal{W}_\varphi}(z) = -10$ implies $\varphi(z) = -10$. Hence, we also have $\varphi_{\mathcal{W}_\varphi}(z) = 00$ if and only if $\varphi(z) = 00$. □

2.5 DISCUSSION

Corollary 2.4.5 determines the class of all SP, CPO, NC, and FA rules for locating two bads in two neighbouring countries. The examples below show that this class leaves room for a spectrum of monotone voting rules between combinations of boundary points ranging from majority decisions to joint dictatorship. Although not exemplified here, combinations of those over the different countries are possible as well. It shows that the restrictive nature of strategy-proof rules found in the one country case (e.g., Peremans and Storcken (1999)) generalises to the case of two countries.

Example 2.5.1 (Majority decisions)**.** Let the pair (U, V) be decisive if and only if $|U| \geqslant |N \backslash (U \cup V)|$. Thus, a pair is decisive if there are at least as many agents who strictly prefer -11 over the other outcomes than there are agents strictly preferring 00, 01 or -10 over -11. Since the agents in V are indifferent between all outcomes we have a (weak) majority for -11 over all other outcomes: -11 is a (weak) Condorcet winner. If $|U| < |N \backslash (U \cup V)|$, then there is a (weak) majority of 00 over all other outcomes, except when $N_A \subseteq U \cup V$ or $N_B \subseteq U \cup V$. In these cases we have, respectively, that -10 and 00 or 01 and 00 are indifferent to all agents, and FA then imposes the outcome to be -10 or 01, respectively.

The following example shows that also rules that are highly asymmetrical in agents' decision power, are possible.

Example 2.5.2 (An almost dictatorial rule)**.** Fix agents $i_A \in N_A$ and $i_B \in N_B$. Let the pair (U, V) be decisive if both i_A and i_B are in U or if $N = U \cup V$. If (U, V) is not decisive the outcome is -10, 01, or 00, depending on whether, respectively, $N_A \subseteq U \cup V$, $N_B \subseteq U \cup V$, or neither of these. In this rule, i_A and i_B exercise a kind of joint dictatorship. Observe that we cannot have dictatorial rules since these would violate CPO.

The following example shows that the class of rules characterised in Corollary 2.4.5 include rules that allow for independent decisions in the two countries.

Example 2.5.3 (Independent decision)**.** Let $\mathcal{W} = \{(U, V) \in 2^N \times 2^N : U \cup V = N\}$, then $\mathcal{W}$ satisfies (d1)–d(4) and therefore is a family of decisive pairs. Then

$\varphi_{\mathcal{W}}(z) = (\alpha(z), \beta(z))$, where $\alpha(z) = -1$ if $z(i) \geqslant -\frac{1}{2}$ for all $i \in N_A$ and $\alpha(z) = 0$ otherwise; and $\beta(z) = 1$ if $z(i) \leqslant \frac{1}{2}$ for all $i \in N_B$ and $\beta(z) = 0$ otherwise. This rule satisfies the four conditions.

On the other hand, the class of rules characterised in Corollary 2.4.5 does *not* include rules that allow for more general independent decisions in the two countries, as we illustrate in the next example.

Example 2.5.4 (Generalised independent decision)**.** For a profile z denote $U_A(z) = U(z) \cap N_A$, $V_A(z) = V(z) \cap N_A$, $U_B(z) = U(z) \cap N_B$, and $V_B = V(z) \cap N_B$. At profile z define $\alpha(z) = -1$ if there is a (weak) majority in N_A for -1 against 0, i.e., $|U_A(z)| \geqslant |N_A \backslash (U_A(z) \cup V_A(z))|$ and in all other cases let $\alpha(z) = 0$. Define $\beta(z) = 1$ if $|U_B(z)| \geqslant |N_B \backslash (U_B(z) \cup V_B(z))|$. Thus, $\varphi(z)$ is based on (weak) majorities, but now for each country separately. We show that this rule corruptible. Consider a profile w at which there is a strict majority for 0 in country B, that is, $|U_B(w)| < |N_B \backslash (U_B(w) \cup V_B(w))|$; and a strict majority for 0 in country A, but with a swing voter at $-\frac{1}{2}$. Then $\varphi(w) = 00$. If the swing voter deviates, yielding profile v such that $|U_A(v)| = |N_A \backslash (U_A(v) \cup V_A(v))|$, then $\varphi(v) = -10$. But this swing voter is indifferent between 00 and -10 both at $-\frac{1}{2}$ and at his new dip.

Both FA and NC are needed to deduce that the outcomes of a rule are combinations of country border points. Dropping these conditions may lead to interior solutions – see below. For an analysis of the one agent per country case see Öztürk (2013).

Finally, we provide examples showing the logical independence of the characterising conditions in Corollary 2.4.5.

Example 2.5.5 (Logical independence of SP, CPO, NC, FA)**.**

SP Consider a family of decisive pairs $\mathcal{W}$ with associated rule $\varphi_{\mathcal{W}}$. Let i_A, j_A be two different agents in N_A and let i_B, j_B be two different agents in N_B. Define the rule $\widehat{\varphi}$ for all profiles equal to $\varphi_{\mathcal{W}}$ except for profiles q at which $q(i_A) = -1$, $q(i_B) = 1$, and $q(j_A) = q(j_B) = 0$: then let $\widehat{\varphi}(q) = (-\frac{1}{2}, \frac{1}{2})$. Rule $\widehat{\varphi}$ satisfies FA and CPO, as is not difficult to see. Non-corruptibility can be seen by considering the preferences of the agents i_A, j_A, i_B, j_B at such a profile q, as given in the following table.

$$\begin{aligned} i_A: &\quad 00 \sim 01 \succ -\tfrac{1}{2}\tfrac{1}{2} \succ -11 \sim -10 \\ j_A, j_B: &\quad -11 \succ -\tfrac{1}{2}\tfrac{1}{2} \succ 00 \sim -10 \sim 01 \\ i_B: &\quad 00 \sim -10 \succ -\tfrac{1}{2}\tfrac{1}{2} \succ -11 \sim 01 \end{aligned}$$

Thus, these four agents are not indifferent between the outcome at q and any of the outcomes -11, -10, 00, or 01 of $\varphi_{\mathcal{W}}$ at a profile not having the structure of q. It is not difficult to see that $\widehat{\varphi}$ is not strategy-proof.

CPO The constant rule that assigns -11 to every profile satisfies SP, NC, and FA, but not CPO.

NC The rule in Example 2.5.3 satisfies SP, CPO, and FA, but not NC.

FA Consider a family of decisive pairs $\mathcal{W}$ with associated rule $\varphi_{\mathcal{W}}$. Let rule $\widetilde{\varphi}$ be equal to $\varphi_{\mathcal{W}}$ except that $\widetilde{\varphi}((-\frac{1}{2})^{N_A}, (\frac{1}{2})^{N_B}) = -10$. Then $\widetilde{\varphi}$ violates the far away condition at profile $((-\frac{1}{2})^{N_A}, (\frac{1}{2})^{N_B})$ for country B. Clearly, $\widetilde{\varphi}$ satisfies CPO. For SP and NC consider a unilateral deviation z from $((-\frac{1}{2})^{N_A}, (\frac{1}{2})^{N_B})$ by agent i. First suppose $z(i) < -\frac{1}{2}$. Then CPO implies that, with $\varphi_{\mathcal{W}}(z) = (\alpha(z), \beta(z))$, $\alpha(z) = 0$, so that $(U(z), V(z)) \notin \mathcal{W}$. Since $N_B \subseteq U(z) \cup V(z)$, it follows that $\widetilde{\varphi}(z) = 01$. At $((-\frac{1}{2})^{N_A}, (\frac{1}{2})^{N_B})$ agent i is indifferent between -10 and 01, but at $z(i)$ agent i strictly prefers 01 to -10. It follows that at these deviations the requirements of SP and NC hold.

Next consider the case that $0 \geqslant z(i) > -\frac{1}{2}$. Then $(N_A \cup N_B) \subseteq (U(z) \cup V(z))$ and $(U(z), V(z)) \in \mathcal{W}$, which implies that $\widetilde{\varphi}(z) = -11$. At $((-\frac{1}{2})^{N_A}, (\frac{1}{2})^{N_B})$ agent i is indifferent between -10 and -11 and at $z(i)$ agent i strictly prefers outcome -11 to -10. Hence, also at these deviations the requirements of SP and NC hold.

Now consider the case that $0 \leqslant z(i) < \frac{1}{2}$. Then $(N_A \cup N_B) \subseteq (U(z) \cup V(z))$ and $(U(z), V(z)) \in \mathcal{W}$, which implies that $\widetilde{\varphi}(z) = -11$. At $((-\frac{1}{2})^{N_A}, (\frac{1}{2})^{N_B})$ agent i is indifferent between -10 and -11 and at $z(i)$ agent i strictly prefers -11 to -10. Hence, at these deviations again the requirements of SP and NC hold.

Finally, consider $z(i) > \frac{1}{2}$. CPO implies $\beta(z) = 0$. Hence, $(U(z), V(z)) \notin \mathcal{W}$. Since $N_A \subseteq U(z) \cup V(z)$, it follows that $\widetilde{\varphi}(z) = -10$. Again the requirements of SP and NC hold at these deviations.

3 LEXICOGRAPHIC PREFERENCES

3.1 INTRODUCTION

As in chapter 2, we consider a model of two neighbouring countries represented by two adjoining unit intervals. The difference from chapter 2 is in the assumptions regarding the preferences of agents. As before, we assume that each preference is characterised by a single dip, representing the worst location of the public bad in the resident country. But in this chapter we consider a different specification of the single-dipped preference, namely *lexmin preferences*. That is, preference is determined by the distance to the nearer public bad and, in case of a tie, by the distance to the other public bad. We show that in this case, where there are fewer indifferences as compared to myopic preferences in chapter 2, strategy-proofness and country-wise Pareto optimality are sufficient for a rule to pick only from the four corner points, which are the combinations of boundary points of each country. Next we show that restricted to these four outcomes, strategy-proofness is equivalent to Maskin monotonicity and independence of irrelevant alternatives. Lastly, we characterise the class of rules under these preferences. In chapter 2, we showed that that the class of rules, that satisfies strategy-proofness, non-corruptibility, country-wise Pareto optimality and the far away condition, primarily contains joint decision rules where the location of one bad of a country depends on the preference of all the agents in the two countries. The class of rules we characterise in this chapter contains some rules of this type and also some rules where each country decides independently about the location of its bad.

The organisation of the chapter is as follows. Section 3.2 introduces the formal model and some preliminary results. Section 3.3 shows that internal locations are excluded, and Section 3.4 provides a characterisation of strategy-proofness restricted to the corner points. Section 3.5 gives the formal characterisation of the class. Section 3.6 provides some examples to compare the class of rules described in chapter 2 with the class described in this section. This section also includes examples showing logical independence of Maskin monotonicity, independence of irrelevant alternatives and country-wise Pareto optimality. Section 3.7 concludes.

3.2 THE MODEL

Let country A be represented by the interval $[-1, 0]$ and country B by $[0, 1]$. The set of possible *alternatives* is denoted by $\mathcal{A} = [-1, 0] \times [0, 1]$. The set of *agents* N is partitioned into the set N_A of inhabitants of country A and the set N_B of inhabitants of country B. Let the cardinalities of N, N_A and N_B be natural numbers n, n_A, n_B, with $n = n_A + n_B$.

Each agent $i \in N$ has a *lexmin* preference $R_{x(i)}$ over $\mathcal{A}$, characterised by its *dip* $x(i) \in [-1, 1]$ as follows. A preference $R_{x(i)}$ of agent $i \in N$ is a *lexmin* preference if for all (a_1, b_1), $(a_2, b_2) \in \mathcal{A}$, (a_1, b_1) is at least as good as (a_2, b_2) at $R_{x(i)}$, with the usual notation $(a_1, b_1)R_{x(i)}(a_2, b_2)$, if

$$\min\{|a_1 - x(i)|, |b_1 - x(i)|\} > \min\{|a_2 - x(i)|, |b_2 - x(i)|\}, \text{ or}$$

$$\min\{|a_1 - x(i)|, |b_1 - x(i)|\} = \min\{|a_2 - x(i)|, |b_2 - x(i)|\} \text{ and}$$
$$\max\{|a_1 - x(i)|, |b_1 - x(i)|\} \geqslant \max\{|a_2 - x(i)|, |b_2 - x(i)|\}.$$

As usual, $P_{x(i)}$ denotes the strict or asymmetric part of $R_{x(i)}$ and $I_{x(i)}$ denotes the indifference or symmetric part. Preference $R_{x(i)}$ is completely determined by its dip $x(i)$. Therefore, preferences are identified with their dips and denoted by $x(i)$ instead of $R_{x(i)}$.

A (preference) *profile* z assigns to each agent $i \in N$ a preference $z(i)$ such that $z(i) \in [-1, 0]$ if $i \in N_A$ and $z(i) \in [0, 1]$ if $i \in N_B$. The set of all profiles is denoted by $\mathcal{R}$.

For a profile z and a non-empty set $S \subseteq N$, let $z_S = (z(i))_{i \in S}$. For $i \in N$, profile z' is an i-*deviation* of z if $z_{N\setminus\{i\}} = z'_{N\setminus\{i\}}$. For $a \in A$ and $S \subseteq N_A$, $(a^S, z_{N\setminus S})$ denotes the profile where all $i \in N\setminus S$ have preference $z(i)$ and all $i \in S$ have preference a. Define the restriction of a preference $R_{z(i)}$ to a subset $\mathcal{C}$ of $\mathcal{A}$ by $R_{z(i)}|_{\mathcal{C}} = (\mathcal{C} \times \mathcal{C}) \cap R_{z(i)}$. Further, define the restriction of a profile z to $\mathcal{C}$ component wise; i.e; $z|_{\mathcal{C}} = (R_{z(i)}|_{\mathcal{C}})_{i \in N}$. For $a, b \in \mathcal{A}$, we denote $z|_{\{a,b\}}$ also by $z|_{a,b}$. For an agent i with dip at $z(i)$ and an out come $(a, b) \in \mathcal{A}$ define $L((a, b), z(i)) = \{(c, d) \in \mathcal{A} \colon (a, b)R_{z(i)}(c, d)\}$ as the weak lower contour set of $R_{z(i)}$ at (a, b) and $L'((a, b), z(i)) = \{(c, d) \in \mathcal{A} \colon (a, b)P_{z(i)}(c, d)\}$ as the strict lower contour set.

A *rule* φ assigns to each profile z an alternative $\varphi(z) = (\alpha(z), \beta(z)) \in \mathcal{A}$.[1]

For $x, y \in \mathbb{R}$, $\mu(x, y) = \frac{x+y}{2}$ denotes the midpoint of x and y. In case there is no confusion, for a profile z we write $\mu(z)$ instead of $\mu(\alpha(z), \beta(z))$ to indicate the midpoint of the interval with endpoints given by $\varphi(z)$.

The main properties of a rule φ considered in this paper are the following.

STRATEGY-PROOFNESS (SP) φ is *strategy-proof* if $\varphi(z)R_{z(i)}\varphi(z')$ for every $z \in \mathcal{R}$, every $i \in N$, and every i-deviation z' of z.

COUNTRY-WISE PARETO OPTIMALITY (CPO) φ is *Pareto optimal for country* A (resp. B) if for every profile z there does not exist an $a \in [-1, 0]$ (resp. $b \in [0, 1]$) such that $(a, \beta(z))R_{z(i)}\varphi(z)$ for all $i \in N_A$ and $(a, \beta(z))$ $P_{z(k)}$ $\varphi(z)$ for at least one $k \in N_A$ (resp. $(\alpha(z), b)R_{z(i)}\varphi(z)$ for all $i \in$

1 Of course, $\alpha(\cdot)$ and $\beta(\cdot)$ depend on φ.

N_B and $(\alpha(z), b)P_{z(i)}\varphi(z)$ for at least one $k \in N_B$). Rule φ is *country-wise Pareto optimal* if it is both Pareto optimal for country A and Pareto optimal for country B.

INDEPENDENT OF IRRELEVANT ALTERNATIVES (I.I.A.) Rule φ is I.I.A. if $\varphi(z) = \varphi(z')$ for all $z, z' \in \mathcal{R}$ with $z|_{\varphi(z),\varphi(z')} = z'|_{\varphi(z),\varphi(z')}$.

MASKIN MONOTONICITY (MMON) Rule φ is Maskin monotone if $\varphi(z) = \varphi(z')$ for all $z, z' \in \mathcal{R}$ such that $L(\varphi(z), z(i)) \subseteq L(\varphi(z), z'(i))$ and $L'(\varphi(z), z(i)) \subseteq L'(\varphi(z), z'(i))$ for all $i \in N$.

WEAK MONOTONICITY (WMON) φ is *weakly monotone* if $\varphi(z) = \varphi(z')$ for all $z, z' \in \mathcal{R}$ such that for all agents $i \in N$:

- $z'(i) \leqslant z(i) \leqslant \alpha(z)$, or
- $\alpha(z) \leqslant z(i) \leqslant z'(i) \leqslant \mu(z)$ and $\alpha(z) < 0$, or
- $\mu(z) \leqslant z'(i) \leqslant z(i) \leqslant \beta(z)$ and $\beta(z) > 0$, or
- $\beta(z) \leqslant z(i) \leqslant z'(i)$.

Maskin monotonicity is the usual condition introduced by Maskin (1985), which says that the outcomes of a rule at two different profiles are same if the lower contour sets of each agent at the first profile is contained in the lower contour of each agent at the new profile. In other words, if a deviating agent makes his lower contour set larger by deviation, then this property says that the outcome at these two profiles should be the same.

Independence of irrelevant alternatives is stronger than the non-corruptibility condition introduced in Chapter 2. This property ensures that after a unilateral deviation of a resident, if his preference between the old and the new outcome does not change, then both these outcomes must be the same. This condition is based on the I.I.A. condition introduced by Arrow (1950) in his seminal dictatorial result in the context of aggregating preferences to come up with a social welfare function. Weak monotonicity condition is a weaker version of the monotonicity condition introduced in Chapter 2.

Next, we show that WMON is implied by SP.

Lemma 3.2.1. *Let $\varphi : \mathcal{R} \to \mathcal{A}$ satisfy SP. Then φ satisfies WMON.*

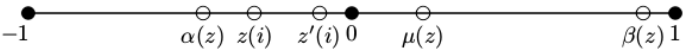
Proof. It is sufficient to prove weak monotonicity for an i-deviation from $z \in \mathcal{R}$ to $z' \in \mathcal{R}$ for an agent $i \in N$. We consider the following two cases (the remaining two cases are analogous).

(a) $z'(i) < z(i) \leqslant \alpha(z)$.

(i) If $\alpha(z) < \alpha(z') \leqslant 0$, then agent i manipulates from $z(i)$ to $z'(i)$. (ii) Now suppose $-1 \leqslant \alpha(z') < \alpha(z)$. Then we must have $\alpha(z') \leqslant z'(i) - r$, where $r = \alpha(z) - z'(i)$, otherwise i manipulates from $z'(i)$ to $z(i)$. In turn this implies $|z(i) - \alpha(z')| > |z(i) - \alpha(z)|$, so we must have $\alpha(z) = \beta(z') = 0$, otherwise i manipulates from $z(i)$ to $z'(i)$. Now $\beta(z) \leqslant z'(i) + (z'(i) - \alpha(z'))$, otherwise i manipulates from $z'(i)$ to $z(i)$; and $\beta(z) \geqslant z(i) + (z(i) - \alpha(z'))$, otherwise i manipulates from $z(i)$ to $z'(i)$. These two inequalities combined, however, contradict the assumption that $z(i) > z'(i)$. (iii) The only remaining possibility is $\alpha(z) = \alpha(z')$, and by strategy-proofness this implies $\beta(z) = \beta(z')$.

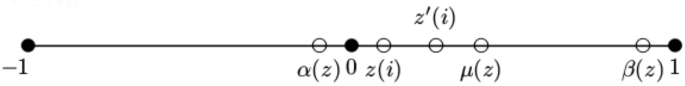
(b) $\alpha(z) \leqslant z(i) < z'(i) \leqslant \mu(z)$ and $\alpha(z) < 0$. We consider two subcases.

(b1) $i \in N_A$.



(i) Suppose $\alpha(z') < \alpha(z)$. Then $\beta(z') < z(i) + (z(i) - \alpha(z))$ otherwise i manipulates from $z(i)$ to $z'(i)$. In turn, this implies $\beta(z') - z'(i) < \beta(z') - z(i) < z(i) - \alpha(z) < z'(i) - \alpha(z')$ and therefore, $z'(i) - \alpha(z) \leqslant \beta(z') - z'(i)$, otherwise i manipulates from $z'(i)$ to $z(i)$. These two inequalities imply $z(i) > z'(i)$, contradicting our assumption. (ii) Suppose $\alpha(z') > \alpha(z)$. Then $\alpha(z) < \alpha(z') \leqslant z(i) + (z(i) - \alpha(z))$, otherwise i manipulates from $z(i)$ to $z'(i)$. Thus, $|z'(i) - \alpha(z')| < z'(i) - \alpha(z)$, hence $\beta(z) < z'(i) + |z'(i) - \alpha(z)|$, otherwise i manipulates from $z'(i)$ to $z(i)$. This, however, contradicts the assumption that $z'(i) \leqslant \mu(z)$. (iii) The only remaining possibility is $\alpha(z) = \alpha(z')$, and by strategy-proofness this implies $\beta(z) = \beta(z')$.

(b2) $i \in N_B$.



(i) Suppose $\alpha(z') < \alpha(z)$. Then $\beta(z') < z(i) + (z(i) - \alpha(z))$ otherwise i manipulates from $z(i)$ to $z'(i)$. In turn, this implies $z'(i) - \alpha(z) \leqslant |\beta(z') - z'(i)|$, otherwise i manipulates from $z'(i)$ to $z(i)$. If $\beta(z') \geqslant z'(i)$ then $z(i) + (z(i) - \alpha(z)) > \beta(z') \geqslant z'(i) + (z'(i) - \alpha(z)$, which contradicts the assumption $z(i) < z'(i)$. If $\beta(z) < z'(i)$ then $\beta(z') \leqslant \alpha(z)$, which implies $\alpha(z) = \beta(z') = 0$, a contradiction since $\alpha(z) < 0$. (ii) If $\alpha(z') > \alpha(z)$ then i manipulates from $z'(i)$ to $z(i)$. (iii) The only remaining possibility is $\alpha(z) = \alpha(z')$, and by strategy-proofness this implies $\beta(z) = \beta(z')$. □

3.3 NO INTERNAL SOLUTION

Let $\varphi : \mathcal{R} \to \mathcal{A}$ be a rule satisfying strategy-proofness and country-wise Pareto optimality. We now have the following result.

Theorem 3.3.1. $\varphi(z) \in \{(-1,1),(0,0),(-1,0),(0,1)\}$ *for every* $z \in \mathcal{R}$.

We prove this theorem with the help of the following two lemmas. Throughout φ is a rule satisfying SP and CPO. Let $z \in \mathcal{R}$. The first lemma shows that if one of the two bads is located at 0, then the other one cannot be located at an interior point of its country.

Lemma 3.3.2. $\alpha(z) = 0$ *implies* $\beta(z) \in \{0,1\}$ *and* $\beta(z) = 0$ *implies* $\alpha(z) \in \{-1,0\}$.

Proof. We prove that $\alpha(z) = 0$ implies $\beta(z) \in \{0,1\}$, the other part of the lemma is analogous. Suppose $\alpha(z) = 0$ but to the contrary $\beta(z) \in (0,1)$. By Pareto optimality for country B, it follows that $T_1 := \{i \in N_B : z(i) \geqslant \beta(z)\} \neq \emptyset$ and $T_2 := \{i \in N_B : z(i) \leqslant \mu(z)\} \neq \emptyset$. By weak monotonicity (Lemma 3.2.1), we may assume that $z(i) = 1$ for all $i \in T_1$ and $z(i) = -1$ for all $i \in N_A$. Without loss of generality, assume that $T_2 := \{1,\ldots,m\}$ where $m < n_B$. Consider the following profiles for all $j = 1,\ldots,m$:

$$\begin{aligned} z_1 &= (z_{N\setminus T_1}, \mu(\beta(z),1)^{T_1}) \\ z_2^j &= (z_{N\setminus\{1,\ldots,j\}}, \mu(z)^{\{1,\ldots,j\}}) \\ z_3^j &= (z_{N\setminus(T_1\cup\{1,\ldots,j\})}, \mu(\beta(z),1)^{T_1}, \mu(z)^{\{1,\ldots,j\}}) \end{aligned}$$

By Pareto optimality for country A, it follows that $\alpha(z_1) = \alpha(z_2^j) = \alpha(z_3^j) = 0$ for every $j \in T_2$.

First, consider the deviation from z to z_1. Since $\alpha(z_1) = 0$, strategy-proofness implies that $\beta(z_1) \in \{\beta(z), 1\}$. Pareto optimality for country B then implies $\beta(z_1) = 1$. So $\varphi(z_1) = (0,1)$. Now consider the deviation from z_1 to z_3^1. As $\mu(z) < \frac{1}{2}$, strategy-proofness implies that $\varphi(z_3^1) = (0,1)$. Now suppose $\varphi(z_3^j) = (0,1)$ for some $j < m$. Consider the deviation from z_3^j to z_3^{j+1}. Since $\mu(z) < \frac{1}{2}$, strategy-proofness implies that $\varphi(z_3^{j+1}) = (0,1)$. Hence, $\varphi(z_3^m) = (0,1)$.

Next, consider the deviation from z to z_2^1. Since $\alpha(z_2^1) = 0$, strategy-proofness implies that $\beta(z_2^1) \in \{0, \beta(z)\}$. So $\varphi(z_2^1) \in \{f(z), (0,0)\}$. Consider the deviation from z_2^1 to z_2^2. Since $\alpha(z_2^2) = 0$, we have that $f(z_2^2) = (0,0)$ if $\varphi(z_2^1) = (0,0)$ and $f(z_2^2) \in \{\varphi(z),(0,0)\}$ if $\varphi(z_2^1) = \varphi(z)$. Continuing this way, we conclude that $\varphi(z_2^m) \in \{(0,0), \varphi(z)\}$. Then Pareto optimality for country B implies that $\varphi(z_2^m) = (0,0)$. Consider the deviation from z_2^m to z_3^m. Since $\mu(\beta(z),1) > \frac{1}{2}$, strategy-proofness implies that $\varphi(z_3^m) = (0,0)$, which contradicts the fact that $\varphi(z_3^m) = (0,1)$ and concludes the proof of the lemma. □

The second lemma shows that if one of the two bads is located at the noncommon border of a country, then the other one cannot be located at an interior point of the other country.

Lemma 3.3.3. *$\alpha(z) = -1$ implies $\beta(z) \notin (0,1)$ and $\beta(z) = 1$ implies $\alpha(z) \notin (-1,0)$.*

Proof. We prove that $\alpha(z) = -1$ implies $\beta(z) \notin (0,1)$, the other part of the lemma is analogous. Suppose $\alpha(z) = -1$ but to the contrary $\beta(z) \in (0,1)$. So, in this case $-\frac{1}{2} < \mu(z) < 0$. By Pareto optimality for country B, it follows that $T := \{i \in N_B : z(i) \geqslant \beta(z)\} \neq \emptyset$ and $N_B \setminus T \neq \emptyset$. By weak monotonicity (Lemma 3.2.1), we may assume that $z(i) = \mu(z)$ for all $i \in N_A$, $z(i) = 1$ for all $i \in T$ and $z(i) = 0$ for all $i \in N_B \setminus T$. Consider now the following three profiles.

	N_A	T	$N_B \setminus T$
z_1	$\mu(z)$	$\mu(\beta(z),1)$	0
z_2	$\mu(z)$	1	$\mu(0,\beta(z))$
z_3	$\mu(z)$	$\mu(\beta(z),1)$	$\mu(0,\beta(z))$

Since $-\frac{1}{2} < \mu(z) < 0$, Pareto optimality for country A implies that $\alpha(z_1) = \alpha(z_2) = \alpha(z_3) = -1$.

First, consider the deviation from z to z_1. Since $\alpha(z_1) = -1$, strategy-proofness implies $\beta(z_1) \in \{\beta(z), 1\}$. Pareto optimality for country B implies $\beta(z_1) = 1$. So $\varphi(z_1) = (-1,1)$. Now consider the deviation from z_1 to z_3. Since $\mu(0,\beta(z)) < \frac{1}{2}$, strategy-proofness implies that $\varphi(z_3) = (-1,1)$.

Next, consider the deviation from z to z_2. As $\alpha(z_2) = -1$, strategy-proofness implies $\beta(z_2) \in \{0, \beta(z)\}$. Pareto optimality for country B implies that $\varphi(z_2) = (-1,0)$. Now consider the deviation from z_2 to z_3. Since $\mu(\beta(z),1) > \frac{1}{2}$, strategy-proofness implies that $\varphi(z_3) = (-1,0)$, which contradicts the fact that $\varphi(z_3) = (-1,1)$ and concludes the proof of the lemma. □

Proof of Theorem 3.3.1. Let $z \in \mathcal{R}$. In view of Lemmas 3.3.2 and 3.3.3 it is sufficient to show that $\varphi(z) \notin (-1,0) \times (0,1)$. Suppose, to the contrary, that $\varphi(z) \in (-1,0) \times (0,1)$. Without loss of generality assume that $\mu(z) \geqslant 0$. Because of country-wise Pareto optimality $S(z) := \{i \in N_A : z(i) \geqslant \alpha(z)\}$ and $T(z) := \{i \in N_B : z(i) \leqslant \beta(z)\}$ are nonempty strict subsets of N_A and N_B, respectively. Consider the following profiles, where $t \in \mathbb{N}$:

	$N_A \setminus S(z)$	$S(z)$	$T(z)$	$N_B \setminus T(z)$
$z^0 = z$	-1	0	$\mu(z)$	1
z^t $(t \geqslant 1)$	-1	$\frac{\alpha(z^{t-1})}{2}$	$\mu(z)$	1
v^1	$\frac{\alpha(z^0)-1}{2}$	0	$\mu(z)$	1
v^t $(t \geqslant 2)$	$\frac{\alpha(z^{t-1})-1}{2}$	$\frac{\alpha(z^{t-2})}{2}$	$\mu(z)$	1
w^t $(t \geqslant 1)$	$\frac{\alpha(z^{t-1})-1}{2}$	$\frac{\alpha(z^{t-1})}{2}$	$\mu(z)$	1

The proof proceeds in a few steps.

Step 1. Let $t \geqslant 1$ and suppose that $-1 < \alpha(z^{t-1}) < 0$. Then $\varphi(v^t) = \varphi(w^t) = (-1, 1)$.
Proof. Considering strategy-proofness at z^{t-1} and v^t yields that $\alpha(v^t) \in \{-1, \alpha(z^{t-1})\}$. Pareto optimality for country A now implies that $\alpha(v^t) = -1$. Then strategy-proofness implies $\beta(v^t) \in [\beta(z^{t-1}), 1]$. Since $0 \notin [\beta(z^{t-1}), 1]$, Lemma 3.3.3 implies $\beta(v^t) = 1$. Thus, $\varphi(v^t) = (-1, 1)$. Comparing v^t and w^t and noting that $-1 < \alpha(z^{t-1})$ and therewith $-\frac{1}{2} < \frac{\alpha(z^{t-1})}{2}$, strategy-proofness implies that $\varphi(w^t) = (-1, 1)$.

Step 2. For all $t \geqslant 1$:

$$-1 < \alpha(z^t) < \alpha(z^{t-1}) < 0 \leqslant 2\mu(z) < \beta(z^t) \leqslant -\alpha(z^{t-1}) < 1 \,.$$

Proof. By assumption we have $-1 < \alpha(z^0) = \alpha(z) < 0 \leqslant 2\mu(z) < \beta(z^0) = \beta(z) < 1$. We prove the statement in Step 2 by induction. Assume it is true for all $s < t$, where $t \geqslant 1$. Consider the deviation from z^{t-1} to z^t. Then strategy-proofness implies at least one of $\alpha(z^t)$ and $\beta(z^t)$ is in the closed interval $[\alpha(z^{t-1}), -\alpha(z^{t-1})]$. By considering the deviation from z^t to z^{t-1} it follows that $\alpha(z^t), \beta(z^t) \notin (\alpha(z^{t-1}), 0)$. Since $\alpha(z^{t-1})$ is not a Pareto optimal location for country A at z^t we have $\alpha(z^t) \neq \alpha(z^{t-1})$. Further, $\alpha(z^t) \neq 0$ otherwise by considering w^t we would have a contradiction with strategy-proofness, since $(0, \beta(z^t)) = \varphi(z^t)$ is better for dip $\frac{\alpha(z^{t-1})-1}{2}$ than $(-1, 1) = \varphi(w^t)$ (by Step 1). So, $\alpha(z^t) < \alpha(z^{t-1})$ and $0 \leqslant \beta(z^t) \leqslant -\alpha(z^{t-1})$. Now suppose $\beta(z^t) = 0$. Then $\alpha(z^t) < \alpha(z^{t-1})$ and Lemma 3.3.2 imply that $\alpha(z^t) = -1$. But $(-1, 1) = \varphi(w^t)$ (by Step 1) is better for dip -1 than $\varphi(z^t) = (-1, 0)$, which is a violation of strategy-proofness. So $\beta(z^t) \neq 0$. Country-wise Pareto optimality now implies $\beta(z^t) > 2\mu(z)$. Since $0 < \beta(z^t) < 1$, Lemma 3.3.3 implies $\alpha(z^t) \neq -1$. Altogether we have $-1 < \alpha(z^t) < \alpha(z^{t-1}) < 0 \leqslant 2\mu(z) < \beta(z^t) \leqslant -\alpha(z^{t-1}) < 1$. Hence Step 2 follows by induction.

Step 3. $\mu(z) = 0$.
Proof. First suppose that for some $t > 1$, $\mu(z^t) \geqslant \frac{\alpha(z^{t-2})}{2}$. Then, by Step 2, for all $i \in S(z)$ we have $\alpha(z^t) < \frac{\alpha(z^{t-1})}{2} = z^t(i) < z^{t-1}(i) = \frac{\alpha(z^{t-2})}{2} \leqslant \mu(z^t)$, so that by monotonicity we have $\varphi(z^{t-1}) = \varphi(z^t)$, a contradiction. Thus, $\mu(z^t) < \frac{\alpha(z^{t-2})}{2}$ for all $t > 1$. Hence, $\alpha(z^t) + \beta(z^t) < \alpha(z^{t-2})$ for all $t > 1$. By Step 2 this implies $2\mu(z) < \beta(z^t) < \alpha(z^{t-2}) - \alpha(z^t)$, which implies that $\mu(z) = 0$ since $\alpha(z^{t-2}) - \alpha(z^t)$ converges to 0 for t going to infinity.

Step 4. If $\mu(z) = 0$ then $\mu(z^1) \neq 0$.
Proof. Follows from Step 2 by taking $t = 1$.

We can now complete the proof. Step 3 implies that $\mu(z) = 0$ for *any* profile z with $\alpha(z) \in (-1, 0)$ and $\beta(z) \in (0, 1)$. This contradicts Step 3 since z^1 is also such a profile. □

3.4 CHARACTERISATION OF STRATEGY-PROOFNESS

In this section, we are going to characterise strategy-proofness. Notice that, Theorem 3.3.1 shows that the set of possible alternatives is $\mathcal{B} = \{-11, -10, 01, 00\}$, where -11 denotes $(-1, 1)$, etc. We are going to show that restricted to $\mathcal{B}$, a social choice rule is strategy-proof if and only if it is Maskin monotone and satisfies I.I.A.

Lemma 3.4.1. *If $\varphi : \mathcal{R} \longrightarrow \mathcal{B}$ is a strategy-proof social choice rule then it is Maskin monotone and satisfies I.I.A.*

Proof. The proof is presented as follows.

Proof of Maskin monotonicity. Let $z, z' \in \mathcal{R}$ be such that $L(\varphi(z), z(i)) \subseteq L(\varphi(z), z'(i))$ and $L'(\varphi(z), z(i)) \subseteq L'(\varphi(z), z'(i))$ for all $i \in N$. To show $\varphi(z) = \varphi(z')$. Without loss of generality, assume that there exists an $i \in N_A$ such that $z|_{N\setminus\{i\}} = z'|_{N\setminus\{i\}}$. Strategy-proofness implies that $\varphi(z)R_{z(i)}\varphi(z')$ and $\varphi(z')R_{z'(i)}\varphi(z)$. Now $L(\varphi(z), z(i)) \subseteq L(\varphi(z), z'(i))$ implies that $\varphi(z)R_{z'(i)}\varphi(z')$. So, it follows that $\varphi(z)I_{z'(i)}\varphi(z')$.
Now suppose $\varphi(z)P_{z(i)}\varphi(z')$. Then as $L'(\varphi(z), z(i)) \subseteq L'(\varphi(z), z'(i))$ holds, so it follows that $\varphi(z)P_{z'(i)}\varphi(z')$, which contradicts $\varphi(z)I_{z'(i)}\varphi(z')$..
So we have $\varphi(z)I_{z(i)}\varphi(z')$.
Now suppose for contradiction $\varphi(z) \neq \varphi(z')$. Then there does not exist any $\varphi(z), \varphi(z') \in \mathcal{B}$ such that $\varphi(z)I_{z(i)}\varphi(z')$ and $\varphi(z)I_{z(i)}\varphi(z')$ holds simultaneously. This can be seen in the following table.

Dips	Preferences
$z(i) \in [-1, -0.5)$	$01P_{z(i)}00P_{z(i)} - 11P_{z(i)} - 10$
$z(i) = -0.5$	$01I_{z(i)} - 11P_{z(i)}00I_{z(i)} - 10$
$z(i) \in (-0.5, 0)$	$-11P_{z(i)}01P_{z(i)} - 10P_{z(i)}00$
$z(i) = 0$	$-11P_{z(i)}01I_{z(i)} - 10P_{z(i)}00$

So we have $\varphi(z) = \varphi(z')$, and this concludes the proof of Maskin monotonicity.

Proof of I.I.A. Let $z, z' \in \mathcal{R}$ be such that $z|_{\varphi(z),\varphi(z')} = z'|_{\varphi(z),\varphi(z')}$. To show $\varphi(z) = \varphi(z')$. To the contrary, assume that $\varphi(z) \neq \varphi(z')$. Without loss of generality, assume that there exists an $i \in N_A$ such that $z|_{N\setminus\{i\}} = z'|_{N\setminus\{i\}}$. Strategy-proofness implies that $\varphi(z)R_{z(i)}\varphi(z')$ and $\varphi(z')R_{z'(i)}\varphi(z)$. Now $z|_{\varphi(z),\varphi(z')} = z'|_{\varphi(z),\varphi(z')}$ implies that $\varphi(z)R_{z'(i)}\varphi(z')$ and $\varphi(z')R_{z(i)}\varphi(z)$. So, it follows that $\varphi(z)I_{z(i)}\varphi(z')$ and $\varphi(z)I_{z'(i)}\varphi(z')$. Then there does not exist any $\varphi(z), \varphi(z') \in \mathcal{B}$ such that $\varphi(z)I_{z(i)}\varphi(z')$ and $\varphi(z)I_{z(i)}\varphi(z')$ holds simultaneously. This can be seen in the above table. So we have $\varphi(z) = \varphi(z')$, and this concludes the proof of I.I.A.

This Concludes the proof of Lemma 3.4.1 □

Next, we are going to show the other direction.

Lemma 3.4.2. *If $\varphi : \mathcal{R} \longrightarrow \mathcal{B}$ is a Maskin monotone social choice rule satisfying I.I.A., then it is strategy-proof.*

Proof. Consider $z, z' \in \mathcal{R}$ where z' is an $i-$deviation from z for some $i \in N$. Without loss of generality assume that $\varphi(z) \neq \varphi(z')$. It is sufficient to show that $f(z)R_{z(i)}f(z')$. To the contrary, suppose $f(z')P_{z(i)}f(z)$. So, I.I.A. implies that $f(z)R_{z'(i)}f(z')$. Now for the deviation from z to z', Maskin monotonicity restricted to $\mathcal{B}$ implies that there exists $a \in \mathcal{B}$ such that $a \neq f(z)$ and

either $f(z)P_{z(i)}a$ and $aR_{z'(i)}f(z)$ or $f(z)R_{z(i)}a$ and $aP_{z'(i)}f(z)$.

Also for the deviation from z' to z, Maskin monotonicity restricted to $\mathcal{B}$ implies that there exists $b \in \mathcal{B}$ such that $b \neq f(z')$ and

either $f(z')P_{z'(i)}b$ and $bR_{z(i)}f(z')$ or $f(z')R_{z'(i)}b$ and $bP_{z(i)}f(z')$.

Combining we have

$$bR_{z(i)}f(z')P_{z(i)}f(z)R_{z(i)}a \tag{7}$$
$$aR_{z'(i)}f(z)R_{z'(i)}f(z')R_{z'(i)}b \tag{8}$$

Note that, restricted to $\mathcal{B}$, we have the following preferences.

Dips	Preferences
$z(i) \in [-1, -0.5)$	$01P_{z(i)}00P_{z(i)} - 11P_{z(i)} - 10$
$z(i) = -0.5$	$01I_{z(i)} - 11P_{z(i)}00I_{z(i)} - 10$
$z(i) \in (-0.5, 0)$	$-11P_{z(i)}01P_{z(i)} - 10P_{z(i)}00$
$z(i) = 0$	$-11P_{z(i)}01I_{z(i)} - 10P_{z(i)}00$
$z(i) \in (0, 0.5)$	$-11P_{z(i)} - 10P_{z(i)}01P_{z(i)}00$
$z(i) = 0.5$	$-10I_{z(i)} - 11P_{z(i)}00I_{z(i)}01$
$z(i) \in (0.5, 1]$	$-10P_{z(i)}00P_{z(i)} - 11P_{z(i)}01$

From the table, it follows that for any choice of $a, b, f(z), f(z') \in \mathcal{B}$ such that $a \neq f(z) \neq f(z') \neq b$, there does not exist $z(i), z'(i) \in A$ or $z(i), z'(i) \in B$ such that $z(i) \neq z'(i)$, equations 7 and 8 holds simultaneously. This violates our assumption that z' is an $i-$deviation from z for some $i \in N$ and concludes the proof of Lemma 3.4.2. □

This brings us to the following theorem

Theorem 3.4.3. *Suppose $f : \mathcal{R} \longrightarrow \mathcal{B}$ be a social choice rule. Then f is strategy-proof if and only if it is Maskin monotone and I.I.A.*

Proof. If direction follows from Lemma 3.4.2. Only if direction follows from Lemma 3.4.1. □

Corollary 3.4.4. Suppose $f : \mathcal{R} \longrightarrow \mathcal{A}$ be a country-wise Pareto optimal social choice rule. Then f is strategy-proof if and only if it is Maskin monotone, I.I.A. and $f(z) \in \mathcal{B}$ for all $z \in \mathcal{R}$.

Proof. Follows from Theorems 3.3.1 and 3.4.3. □

3.5 RULES

As we did not assume anonymity, we can characterise the class in terms of coalitions. Define $\overline{N} = (N_1, N_2, \ldots, N_7) \in \prod_{i=1}^{7} 2^N = \mathcal{N}$. Define $N_{ij} = N_i \cup N_j$ and $N_{ijk} = N_i \cup N_j \cup N_k$ and so on for any $i \neq j \neq k... \in \{1, 2, \ldots, 7\}$. Now consider the collection $\mathcal{W} = \{\mathcal{W}_a : a \in \mathcal{B}\}$, where $\mathcal{W}_a \subset \mathcal{N}$. We define $\mathcal{W}$ to be decisive if it satisfies the following properties.

MONOTONICITY $\mathcal{W}$ is monotone if it satisfies the following properties.

- If $\overline{N} \in \mathcal{W}_{(-1,1)}$, then $\overline{N'} \in \mathcal{W}_{(-1,1)}$, where $N_{345} \subseteq N'_{345}$ and $N_{23456} \subseteq N'_{23456}$.
- If $\overline{N} \in \mathcal{W}_{(0,0)}$, then $\overline{N'} \in \mathcal{W}_{(0,0)}$, where $N_{17} \subseteq N'_{17}$ and $N_{1267} \subseteq N'_{1267}$.
- If $\overline{N} \in \mathcal{W}_{(0,1)}$, then $\overline{N'} \in \mathcal{W}_{(0,1)}$, where $N_1 \subseteq N'_1, N_{12} \subseteq N'_{12}, \ldots, N_{123456} \subseteq N'_{123456}$.
- If $\overline{N} \in \mathcal{W}_{(-1,0)}$, then $\overline{N'} \in \mathcal{W}_{(-1,0)}$, where $N_7 \subseteq N'_7$, $N_{67} \subseteq N'_{67}, \ldots, N_{234567} \subseteq N'_{234567}$.

TRANSITION CONDITIONS Consider an $i \in N$ and $\overline{N}, \overline{N'} \in \mathcal{N}$ such that $N_t \backslash \{i\} = N'_t \backslash \{i\}$ for all $t \in \{1, 2, \ldots, 7\}$.

- If $i \in N_{34}$ and $i \in N'_2$ and $\overline{N} \in \mathcal{W}_{(-1,1)}$, then $\overline{N'} \notin \mathcal{W}_{(0,0)}$.
- If $i \in N_3$ and $i \in N'_{12}$ and $\overline{N} \in \mathcal{W}_{(-1,0)}$, then $\overline{N'} \notin \mathcal{W}_{(0,1)}$.
- If $i \in N_2$ and $i \in N'_1$ and $\overline{N} \in \mathcal{W}_{(-1,0)}$, then $\overline{N'} \notin \mathcal{W}_{(0,1)}$.
- If $i \in N_{54}$ and $i \in N'_6$ and $\overline{N} \in \mathcal{W}_{(-1,1)}$, then $\overline{N'} \notin \mathcal{W}_{(0,0)}$.
- If $i \in N_5$ and $i \in N'_{67}$ and $\overline{N} \in \mathcal{W}_{(0,1)}$, then $\overline{N'} \notin \mathcal{W}_{(-1,0)}$.
- If $i \in N_6$ and $i \in N'_7$ and $\overline{N} \in \mathcal{W}_{(0,1)}$, then $\overline{N'} \notin \mathcal{W}_{(-1,0)}$.

BOUNDARY CONDITIONS $\mathcal{W}$ satisfies the following properties.

- If $N_1 \cap N_A \neq \emptyset$ and $N_A \subseteq N_{12}$, then $\overline{N} \in \mathcal{W}_{(0,0)} \cup \mathcal{W}_{(0,1)}$.
- If $N_A \subseteq N_{234}$ and $N_{34} \cap N_A \neq \emptyset$, then $\overline{N} \in \mathcal{W}_{(-1,0)} \cup \mathcal{W}_{(-1,1)}$.
- If $N_B \subseteq N_{67}$ and $N_7 \cap N_B \neq \emptyset$, then $\overline{N} \in \mathcal{W}_{(0,0)} \cup \mathcal{W}_{(-1,0)}$.
- If $N_B \subseteq N_{456}$ and $N_{45} \cap N_B \neq \emptyset$, then $\overline{N} \in \mathcal{W}_{(0,1)} \cup \mathcal{W}_{(-1,1)}$.

DISJOINT CONDITION For any $a \in \mathcal{B}$, if $\overline{N} \in \mathcal{W}_a$, then $\overline{N} \notin \bigcup_{b \in \mathcal{B} \setminus \{a\}} \mathcal{W}_b$.

COMPLETENESS For any $\overline{N} \in \mathcal{N}$ such that $N_i \cap N_j = \emptyset$ for any $i, j \in \{1, 2, \ldots, 7\}$ with $i \neq j$; we have $\overline{N} \in \bigcup_{a \in \mathcal{B}} \mathcal{W}_a$.

Now, using an arbitrary social choice rule φ, we may define $\mathcal{W}^\varphi = \{\mathcal{W}_a^\varphi : a \in \mathcal{B}\}$ as follows.

For every profile $z \in \mathcal{R}$, define

$$N_1(z) = \{i \in N : z(i) \in [-1, -0.5)\} \subseteq N.$$
$$N_2(z) = \{i \in N : z(i) = -0.5\} \subseteq N.$$
$$N_3(z) = \{i \in N : z(i) \in (-0.5, 0)\} \subseteq N.$$
$$N_4(z) = \{i \in N : z(i) = 0\} \subseteq N.$$
$$N_5(z) = \{i \in N : z(i) \in (0, 0.5)\} \subseteq N.$$
$$N_6(z) = \{i \in N : z(i) = 0.5\} \subseteq N.$$
$$N_7(z) = \{i \in N : z(i) \in (0.5, 1]\} \subseteq N.$$
$$\overline{N(z)} = (N_1(z), N_2(z), \ldots, N_7(z)) \in \mathcal{N}.$$

For any $a \in \mathcal{B}$, define $\mathcal{W}_a^\varphi = \{\overline{N(z)} \in \mathcal{N} : \text{there exists } z \in \mathcal{R} \text{ with } \varphi(z) = a\}$.

Let us define $N_{ij}(z) = N_i(z) \cup N_j(z)$ and $N_{ijk}(z) = N_i(z) \cup N_j(z) \cup N_k(z)$ and so on for any $i \neq j \neq k \ldots \in \{1, 2, \ldots, 7\}$. This brings us to the following lemma.

Lemma 3.5.1. *Suppose φ is strategy proof and country-wise Pareto optimal. Then $\mathcal{W}^\varphi$ is decisive.*

Proof. Note that for any two profiles $z, z' \in \mathcal{R}$, $\overline{N(z)} = \overline{N(z')}$ implies and implied by $z|_{\mathcal{B}} = z'|_{\mathcal{B}}$. As φ is strategy proof and country-wise Pareto optimal, so Theorem 3.3.1 implies that $\varphi(z) \in \mathcal{B}$ for all $z \in \mathcal{R}$. Then strategy-proofness implies that if $\overline{N(z)} = \overline{N(z')}$, then $\varphi(z) = \varphi(z')$. So it follows that $\mathcal{W}_a^\varphi$ is well defined for all $a \in B$. From the definition of $\mathcal{W}^\varphi$ it follows that $\mathcal{W}^\varphi$ satisfies the disjoint condition. Next we show that $\mathcal{W}^\varphi$ is monotone.

Proof of Monotonicity. Consider a profile $z \in \mathcal{R}$. It is sufficient to show that if $\overline{N(z)} \in \mathcal{W}^\varphi_{(-1,1)}$ then for any other $z' \in \mathcal{R}$, we have $\overline{N(z')} \in \mathcal{W}^\varphi_{(-1,1)}$, where $N_{345}(z) \subseteq N_{345}(z')$ and $N_{23456}(z) \subseteq N_{23456}(z')$. As φ is strategy-proof and country-wise Pareto optimal, Theorem 3.3.1 shows that $\varphi(z) \in \mathcal{B}$ for all $z \in \mathcal{R}$. Then we have the following 7 types of preferences.

Dips	Preferences
$z(i) \in [-1, -0.5)$	$01P_{z(i)}00P_{z(i)} - 11P_{z(i)} - 10$
$z(i) = -0.5$	$01I_{z(i)} - 11P_{z(i)}00I_{z(i)} - 10$
$z(i) \in (-0.5, 0)$	$-11P_{z(i)}01P_{z(i)} - 10P_{z(i)}00$
$z(i) = 0$	$-11P_{z(i)}01I_{z(i)} - 10P_{z(i)}00$
$z(i) \in (0, 0.5)$	$-11P_{z(i)} - 10P_{z(i)}01P_{z(i)}00$
$z(i) = 0.5$	$-10I_{z(i)} - 11P_{z(i)}00I_{z(i)}01$
$z(i) \in (0.5, 1]$	$-10P_{z(i)}00P_{z(i)} - 11P_{z(i)}01$

As $\varphi(z) = (-1, 1)$ and $N_{345}(z) \subseteq N_{345}(z')$ and $N_{23456}(z) \subseteq N_{23456}(z')$ holds, we have $L(\varphi(z), z(i)) \subseteq L(\varphi(z), z'(i))$ and $L'(\varphi(z), z(i)) \subseteq L'(\varphi(z), z'(i))$ for all $i \in N$. As φ is strategy-proof and the range of φ is $\mathcal{B}$ (from Theorem 3.3.1), so Lemma 3.4.1 implies that φ is Maskin monotone. So it follows that $\varphi(z') = (-1, 1)$ and $\overline{N(z')} \in \mathcal{W}^{\varphi}_{(-1,1)}$. This concludes the proof of monotonicity of $\mathcal{W}^{\varphi}$.

Note that the transition conditions are direct consequence of strategy proofness and the boundary conditions follows from country-wise Pareto optimality. Completeness follows from Theorem 3.3.1 and the fact that φ is a well defined function. □

Now suppose we have an arbitrary $\mathcal{W} = \{\mathcal{W}_a : a \in \mathcal{B}\}$, where $\mathcal{W}_a \subset \mathcal{N}$. Using $\mathcal{W}$, we may define a social choice rule $\varphi^{\mathcal{W}}$ as follows. For every $z \in \mathcal{R}$:

$$\psi_{\mathcal{W}}(z) = \begin{cases} (-1,1) & \text{if } \overline{N(z)} \in \mathcal{W}_{(-1,1)} \\ (-1,0) & \text{if } \overline{N(z)} \in \mathcal{W}_{(-1,0)} \\ (0,1) & \text{if } \overline{N(z)} \in \mathcal{W}_{(0,1)} \\ (0,0) & \text{if } \overline{N(z)} \in \mathcal{W}_{(0,0)} \end{cases}$$

From the definition of $\psi_{\mathcal{W}}(.)$ it follows that for any two profiles $z, z' \in \mathcal{R}$ such that $\overline{N(z)} = \overline{N(z')}$, we have $\psi_{\mathcal{W}}(z) = \psi_{\mathcal{W}}(z')$. This brings us to the following lemma.

Lemma 3.5.2. *Suppose $\mathcal{W}$ is decisive. Then $\psi_{\mathcal{W}}$ is strategy-proof and country-wise Pareto optimal.*

Proof. As $\mathcal{W}$ is decisive, so the disjoint condition and the completeness condition implies that $\psi_{\mathcal{W}}$ is a well defined function. Next we show that $\psi_{\mathcal{W}}$ satisfies Maskin monotonicity and I.I.A. As $\psi_{\mathcal{W}} : \mathcal{R} \longrightarrow \mathcal{B}$, so Lemma 3.4.2 would imply that $\psi_{\mathcal{W}}$ is strategy proof.

Proof of Maskin monotonicity. Consider two profiles $z, z' \in \mathcal{R}$ such that $L(\psi_{\mathcal{W}}(z), z(i)) \subseteq L(\psi_{\mathcal{W}}(z), z'(i))$ and $L'(\psi_{\mathcal{W}}(z), z(i)) \subseteq L'(\psi_{\mathcal{W}}(z), z'(i))$ for all $i \in N$. It is sufficient to show that if $\psi_{\mathcal{W}}(z) = (-1, 1)$ then $\psi_{\mathcal{W}}(z') = (-1, 1)$.

Now due to the conditions of Maskin monotonicity and the fact that $\psi_{\mathcal{W}}(z) \in \mathcal{B}$ for all $z \in \mathcal{B}$; we have $N_{345}(z) \subseteq N_{345}(z')$ and $N_{23456}(z) \subseteq N_{23456}(z')$. As $\overline{N(z)} \in \mathcal{W}_{(-1,1)}$ and $\mathcal{W}$ is monotone, so it follows that $\overline{N(z')} \in \mathcal{W}_{(-1,1)}$ and $\psi_{\mathcal{W}}(z') = (-1,1)$. This concludes the proof of Maskin monotonicity.

Proof of I.I.A. Consider two profiles $z, z' \in \mathcal{R}$ such that $z|_{\psi_{\mathcal{W}}(z),\psi_{\mathcal{W}}(z')} = z'|_{\psi_{\mathcal{W}}(z),\psi_{\mathcal{W}}(z')}$; but to the contrary assume that $\psi_{\mathcal{W}}(z) \neq \psi_{\mathcal{W}}(z')$. So we have $\overline{N(z)} \neq \overline{N(z')}$. Without loss of generality, assume that z' is an $i-$deviation from z, where $i \in N_A$. As $\psi_{\mathcal{W}}(z) \in \mathcal{B}$ for all $z \in \mathcal{R}$, agent i can have one of the following preferences.

Dips	Preferences
$z(i) \in [-1, -0.5)$	$01P_{z(i)}00P_{z(i)} - 11P_{z(i)} - 10$
$z(i) = -0.5$	$01I_{z(i)} - 11P_{z(i)}00I_{z(i)} - 10$
$z(i) \in (-0.5, 0)$	$-11P_{z(i)}01P_{z(i)} - 10P_{z(i)}00$
$z(i) = 0$	$-11P_{z(i)}01I_{z(i)} - 10P_{z(i)}00$

Now consider the following cases.

$\psi_{\mathcal{W}}(z) = (-1, 1)$ AND $\psi_{\mathcal{W}}(z') = (0, 0)$ In this case, due to $z|_{\psi_{\mathcal{W}}(z),\psi_{\mathcal{W}}(z')} = z'|_{\psi_{\mathcal{W}}(z),\psi_{\mathcal{W}}(z')}$, we have the following sub cases.

SUB CASE 1 : If $i \in N_{34}(z)$ then $i \in N_{234}(z')$.

SUB CASE 2 : If $i \in N_2(z)$ then $i \in N_{34}(z')$.

Note that, in all these cases $N_t(z)\backslash\{i\} = N_t(z')\backslash\{i\}$ for all $t \in \{1, 2, \ldots, 7\}$ holds. As $\psi_{\mathcal{W}}(z') = (0, 0)$, transition condition implies that if $i \in N_{34}(z)$ then $i \notin N_2(z')$. So combining these sub cases we have $N_{345}(z) \subseteq N_{345}(z')$ and $N_{23456}(z) \subseteq N_{23456}(z')$. So monotonicity of $\mathcal{W}$ would imply that $\psi_{\mathcal{W}}(z') = (-1, 1)$, which contradict our assumption that $\psi_{\mathcal{W}}(z') = (0, 0)$.

$\psi_{\mathcal{W}}(z) = (-1, 1)$ AND $\psi_{\mathcal{W}}(z') = (-1, 0)$ In this case, due to $z|_{\psi_{\mathcal{W}}(z),\psi_{\mathcal{W}}(z')} = z'|_{\psi_{\mathcal{W}}(z),\psi_{\mathcal{W}}(z')}$, we have the following sub cases.

SUB CASE 1 : If $i \in N_1(z)$ then $i \in N_{234}(z')$.

SUB CASE 2 : If $i \in N_2(z)$ then $i \in N_{34}(z')$.

SUB CASE 3 : If $i \in N_3(z)$ then $i \in N_4(z')$.

SUB CASE 4 : If $i \in N_4(z)$ then $i \in N_{123}(z')$.

SUB CASE 5 : If $i \in N_3(z)$ then $i \in N_{12}(z')$.

SUB CASE 6 : If $i \in N_2(z)$ then $i \in N_1(z')$.

Note that, in all these cases $N_t(z)\backslash\{i\} = N_t(z')\backslash\{i\}$ for all $t \in \{1, 2, \ldots, 7\}$ holds. Note that in sub cases 1 to 3 we have $N_{345}(z) \subseteq N_{345}(z')$ and $N_{23456}(z) \subseteq N_{23456}(z')$. So monotonicity of $\mathcal{W}$ would imply that $\psi_{\mathcal{W}}(z') = (-1, 1)$, which contradict our assumption that $\psi_{\mathcal{W}}(z') = (-1, 0)$. Also in sub cases 4 to 6 we have $N_7(z') \subseteq N_7(z), N_{67}(z') \subseteq N_{67}(z), \ldots, N_{234567}(z') \subseteq N_{234567}(z)$. As $\psi_{\mathcal{W}}(z') = (-1, 0)$, so monotonicity of $\mathcal{W}$ would imply that $\psi_{\mathcal{W}}(z) = (-1, 0)$, which contradicts our assumption that $\psi_{\mathcal{W}}(z) = (-1, 1)$.

$\psi_{\mathcal{W}}(z) = (-1, 1)$ AND $\psi_{\mathcal{W}}(z') = (0, 1)$ In this case, due to $z|_{\psi_{\mathcal{W}}(z), \psi_{\mathcal{W}}(z')} = z'|_{\psi_{\mathcal{W}}(z), \psi_{\mathcal{W}}(z')}$, we have the following sub cases.

SUB CASE 1 : If $i \in N_4(z)$ then $i \in N_3(z')$.

SUB CASE 2 : If $i \in N_3(z)$ then $i \in N_4(z')$.

Note that, in all these cases $N_t(z)\backslash\{i\} = N_t(z')\backslash\{i\}$ for all $t \in \{1, 2, \ldots, 7\}$ holds. So combining these sub cases we have $N_{345}(z) \subseteq N_{345}(z')$ and $N_{23456}(z) \subseteq N_{23456}(z')$. So monotonicity of $\mathcal{W}$ would imply that $\psi_{\mathcal{W}}(z') = (-1, 1)$, which contradict our assumption that $\psi_{\mathcal{W}}(z') = (0, 0)$.

$\psi_{\mathcal{W}}(z) = (-1, 0)$ AND $\psi_{\mathcal{W}}(z') = (0, 1)$ In this case, due to $z|_{\psi_{\mathcal{W}}(z), \psi_{\mathcal{W}}(z')} = z'|_{\psi_{\mathcal{W}}(z), \psi_{\mathcal{W}}(z')}$, we have the following sub cases.

SUB CASE 1 : If $i \in N_3(z)$ then $i \in N_{12}(z')$.

SUB CASE 2 : If $i \in N_2(z)$ then $i \in N_{13}(z')$.

SUB CASE 3 : If $i \in N_1(z)$ then $i \in N_{23}(z')$.

Note that, in all these cases $N_t(z)\backslash\{i\} = N_t(z')\backslash\{i\}$ for all $t \in \{1, 2, \ldots, 7\}$ holds. As $\psi_{\mathcal{W}}(z') = (0, 1)$, transition condition implies that if $i \in N_3(z)$ then $i \notin N_{12}(z')$ and if $i \in N_2(z)$ then $i \notin N_{11}(z')$. So combining these sub cases we have $N_7(z) \subseteq N_7(z')$, $N_{67}(z) \subseteq N_{67}(z'), \ldots, N_{234567}(z) \subseteq N_{234567}(z')$.. So monotonicity of $\mathcal{W}$ would imply that $\psi_{\mathcal{W}}(z') = (-1, 0)$, which contradict our assumption that $\psi_{\mathcal{W}}(z') = (0, 1)$.

$\psi_{\mathcal{W}}(z) = (-1, 0)$ AND $\psi_{\mathcal{W}}(z') = (0, 0)$ In this case, due to $z|_{\psi_{\mathcal{W}}(z), \psi_{\mathcal{W}}(z')} = z'|_{\psi_{\mathcal{W}}(z), \psi_{\mathcal{W}}(z')}$, we have the following sub cases.

SUB CASE 1 : If $i \in N_4(z)$ then $i \in N_3(z')$.

SUB CASE 2 : If $i \in N_3(z)$ then $i \in N_4(z')$.

Note that, in all these cases $N_t(z)\backslash\{i\} = N_t(z')\backslash\{i\}$ for all $t \in \{1, 2, \ldots, 7\}$ holds. So combining these sub cases we have $N_{17}(z') \subseteq N_{17}(z)$ and $N_{1267}(z') \subseteq N_{1267}(z)$. As $\psi_{\mathcal{W}}(z') = (0, 0)$, monotonicity of $\mathcal{W}$ would imply that $\psi_{\mathcal{W}}(z) = (0, 0)$, which contradict our assumption that $\psi_{\mathcal{W}}(z) = (-1, 0)$.

$\psi_{\mathcal{W}}(z) = (0, 1)$ AND $\psi_{\mathcal{W}}(z') = (0, 0)$ In this case, due to $z|_{\psi_{\mathcal{W}}(z), \psi_{\mathcal{W}}(z')} = z'|_{\psi_{\mathcal{W}}(z), \psi_{\mathcal{W}}(z')}$, we have the following sub cases.

SUB CASE 1 : If $i \in N_1(z)$ then $i \in N_{234}(z')$.

SUB CASE 2 : If $i \in N_2(z)$ then $i \in N_{34}(z')$.

SUB CASE 3 : If $i \in N_3(z)$ then $i \in N_4(z')$.

SUB CASE 4 : If $i \in N_4(z)$ then $i \in N_{123}(z')$.

SUB CASE 5 : If $i \in N_3(z)$ then $i \in N_{12}(z')$.

SUB CASE 6 : If $i \in N_2(z)$ then $i \in N_1(z')$.

Note that, in all these cases $N_t(z) \backslash \{i\} = N_t(z') \backslash \{i\}$ for all $t \in \{1, 2, \dots, 7\}$ holds. Note that in sub cases 1 to 3 we have $N_{17}(z') \subseteq N_{17}(z)$ and $N_{1267}(z') \subseteq N_{1267}(z)$. As $\psi_{\mathcal{W}}(z') = (0, 0)$, monotonicity of $\mathcal{W}$ would imply that $\psi_{\mathcal{W}}(z) = (0, 0)$, which contradict our assumption that $\psi_{\mathcal{W}}(z) = (0, 1)$. Also in sub cases 4 to 6 we have $N_1(z) \subseteq N_1(z'), N_{12}(z) \subseteq N_{12}(z'), \dots, N_{123456}(z) \subseteq N_{123456}(z')$. As $\psi_{\mathcal{W}}(z) = (0, 1)$, so monotonicity of $\mathcal{W}$ would imply that $\psi_{\mathcal{W}}(z') = (0, 1)$, which contradicts our assumption that $\psi_{\mathcal{W}}(z') = (0, 0)$.

This concludes the proof of I.I.A.

So we conclude using Lemma 3.4.2 that $\psi_{\mathcal{W}}$ is strategy-proof. Next we show that $\psi_{\mathcal{W}}$ is country-wise Pareto optimal.

Proof of country-wise Pareto optimality. It is sufficient to show that $\psi_{\mathcal{W}}$ is Pareto optimal for country A. So consider a profile $z \in \mathcal{R}$ such that $(\alpha, \beta^{\psi_{\mathcal{W}}}(z))R_{z(i)}\psi_{\mathcal{W}}(z) = (\alpha^{\psi_{\mathcal{W}}}(z), \beta^{\psi_{\mathcal{W}}}(z))$ for all $i \in N_A$ with $(\alpha, \beta^{\psi_{\mathcal{W}}}(z))P_{z(j)}\psi_{\mathcal{W}}(z)$ for atleast one $j \in N_A$; where $\alpha \neq \alpha^{\psi_{\mathcal{W}}}(z)$. Now consider the following cases.

$\alpha^{\psi_{\mathcal{W}}}(z) = 0$ In this case, boundary condition implies that either $N_A \subseteq N_{12}(z)$ or $N_1(z) \cap N_A \neq \emptyset$. As there exists a $j \in N_A$ such that $(\alpha, \beta^{\psi_{\mathcal{W}}}(z))P_{z(j)}(0, \beta^{\psi_{\mathcal{W}}}(z))$, we have $N_A \not\subseteq N_{12}(z)$. This implies that $N_1(z) \cap N_A \neq \emptyset$, which contradicts our assumption that $(\alpha, \beta^{\psi_{\mathcal{W}}}(z))R_{z(i)}(0, \beta^{\psi_{\mathcal{W}}}(z))$ for all $i \in N_A$.

$\alpha^{\psi_{\mathcal{W}}}(z) = -1$ In this case, boundary condition implies that either $N_A \subseteq N_{234}(z)$ or $N_{34}(z) \cap N_A \neq \emptyset$. As there exists a $j \in N_A$ such that $(\alpha, \beta^{\psi_{\mathcal{W}}}(z))P_{z(j)}(-1, \beta^{\psi_{\mathcal{W}}}(z))$, we have $N_A \not\subseteq N_{234}(z)$. This implies that $N_{34}(z) \cap N_A \neq \emptyset$, which contradicts our assumption that $(\alpha, \beta^{\psi_{\mathcal{W}}}(z))R_{z(i)}(-1, \beta^{\psi_{\mathcal{W}}}(z))$ for all $i \in N_A$.

This concludes the proof of country-wise Pareto optimality.

This concludes the proof of Lemma 3.5.2. □

We can now formulate the main result of this section, which is a corollary to the preceding two lemmas.

Corollary 3.5.3. *Let φ be a rule. Then φ is strategy-proof and country-wise Pareto optimal if there is a decisive $\mathcal{W}$ such that $\varphi = \psi_{\mathcal{W}}$.*

Proof. If $\mathcal{W}$ is decisive, then $\psi_{\mathcal{W}}$ satisfies strategy-proofness and country-wise Pareto optimality by Lemma 3.5.2. Conversely, let φ satisfy these two conditions. We show that $\varphi = \psi_{\mathcal{W}^{\varphi}}$, which completes the proof by Lemma 3.5.1. Let $z \in \mathcal{R}$. Then for any $a \in \mathcal{B}$, we have $\varphi(z) = a \Leftrightarrow \overline{N(z)} \in \mathcal{W}^{\varphi}_{a} \Leftrightarrow \psi_{\mathcal{W}_{\varphi}}(z) = a$. □

3.6 EXAMPLES

This section compares the class of rules described in chapter 2 with the class described in this chapter. We start with identifying the rules in section 2.4 of chapter 2 that are strategy-proof and country-wise Pareto optimal on the class of profiles with lexmin preferences.

Corollary 3.6.1. *Let $\mathcal{R}^{lex}$ be the set of all profiles of lexmin preferences and let $\mathcal{W}$ be a family of decisive pairs and let $\varphi_{\mathcal{W}} : \mathcal{R}^{lex} \to \mathcal{A}$ be the associated rule as defined in section 2.4 of chapter 2. Then $\varphi_{\mathcal{W}}$ is country-wise Pareto optimal. Rule $\varphi_{\mathcal{W}}$ is strategy-proof if and only if $\mathcal{W}$ satisfies the following condition: for all $(U, V) \in \mathcal{W}$, if $i \in N_A \cap U$ and $N_B \not\subseteq U \cup V$ or if $i \in N_B \cap U$ and $N_A \not\subseteq U \cup V$, then $(U \setminus \{i\}, V \cup \{i\}) \in \mathcal{W}$.*

Proof. CPO of $\varphi_{\mathcal{W}}$ follows by exactly the same arguments as in the proof of Lemma 2.4.4.

For the only-if direction concerning strategy-proofness, suppose without loss of generality that there are $(U, V) \in \mathcal{W}$ and $i \in N_A \cap U$ such that $N_B \not\subseteq U \cup V$ and $(U \setminus \{i\}, V \cup \{i\}) \notin \mathcal{W}$. Consider a profile z with $U(z) = U \setminus \{i\}$, $V(z) = V \cup \{i\}$, and $z(j) > \frac{1}{2}$ for some $j \in N_B$. Then $N_B \not\subseteq (U \setminus \{i\}) \cup V \cup \{i\}$, so that $\varphi_{\mathcal{W}}(z) \in \{-10, 00\}$. Consider an i-deviation z' with $z'(i) > -\frac{1}{2}$. Then $(U(z'), V(z')) = (U, V) \in \mathcal{W}$, so that $\varphi_{\mathcal{W}}(z') = -11$, which implies that i manipulates from z to z'.

For the if-direction concerning strategy-proofness, again the same arguments as in the proof of Lemma 2.4.4 apply, with one exception, namely the case where $z(i) = -\frac{1}{2}$: this case follows by using the additional condition on $\mathcal{W}$ in the corollary. □

Example 2.5.3 is also strategy-proof and country-wise Pareto optimal under lexmin preferences. But example 2.5.4 shows a rule which is not of the form $\varphi_{\mathcal{W}}$ for some family of decisive pairs $\mathcal{W}$, but which is nevertheless strategy-proof and country-wise Pareto optimal both for myopic and for lexmin preferences. In both these examples, countries make their decisions independently. An example of a rule satisfying all conditions both for myopic and lexmin preferences, but in which the countries do not decide independently, is as follows.

Example 3.6.2. For convenience suppose that $n_A = n_B \geqslant 2$, and let $\mathcal{W}$ be the set of all $(U, V) \in 2^N \times 2^N$ satisfying at least one of the following three conditions: (i) $U \cup V = N$, (ii) $N_A \subseteq U \cup V$ and $N_B \cap U \neq \emptyset$, (iii) $N_B \subseteq U \cup V$ and $N_A \cap U \neq \emptyset$. Then $\mathcal{W}$ satisfies (d1)–(d4) of definition 2.4.2 as well as the condition in Corollary 3.6.1, so that $\varphi_{\mathcal{W}}$ satisfies SP, CPO, NC, and FA for myopic preferences as well as lexmin preferences. Let z be a profile of preferences such that $z(i) \geqslant -\frac{1}{2}$ for all $i \in N_A$, $z(j) < \frac{1}{2}$ for some agent $j \in N_B$, and $z(k) > \frac{1}{2}$ for some $k \in N_B$. Then $\varphi_{\mathcal{W}}(z) = -11$. Pick an agent $i_1 \in N_A$ and consider the profile $\hat{z}$ equal to z except that $\hat{z}(i_1) < -\frac{1}{2}$. Then $\varphi_{\mathcal{W}}(\hat{z}) = 00$, so that the decision in country B has been altered by an agent in country A.

An example of a generic rule in the class described in this chapter is as follows.

Example 3.6.3 (Combined voting)**.** For convenience suppose that $n_A = n_B \geqslant 2$. Now we define a decisive $\mathcal{W}$ as follows.

$$\mathcal{W}_{(-1,1)} = \left\{ \overline{N} \in \mathcal{N} : \begin{array}{lll} \text{either} & |N_{23456}| \geqslant \frac{n}{2} \text{ and} & \begin{array}{ll} \text{either} & N_{34} \cap N_A \neq \emptyset \\ \text{or} & N_{45} \cap N_B \neq \emptyset \end{array} \\ \text{or} & N_2 = N_A \text{ and } N_6 = N_B & \end{array} \right\}.$$

$$\mathcal{W}_{(-1,0)} = \left\{ \overline{N} \in \mathcal{N} \backslash \mathcal{W}_{(-1,1)} : \begin{array}{l} |N_A \cap N_{234}| \geqslant \frac{n}{4} \text{ and } N_{34} \cap N_A \neq \emptyset \\ \text{and } |N_B \cap N_{456}| < \frac{n}{4} \end{array} \right\}.$$

$$\mathcal{W}_{(0,1)} = \left\{ \overline{N} \in \mathcal{N} \backslash \mathcal{W}_{(-1,1)} : \begin{array}{l} |N_B \cap N_{456}| \geqslant \frac{n}{4} \text{ and } N_{45} \cap N_B \neq \emptyset \\ \text{and } |N_A \cap N_{234}| < \frac{n}{4} \end{array} \right\}.$$

$$\mathcal{W}_{(0,0)} = \{\overline{N} \in \mathcal{N} : |N_1| > \tfrac{n}{4} \text{ and } |N_7| > \tfrac{n}{4}\}.$$

$$\mathcal{W} = \{\mathcal{W}_a : a \in \mathcal{B}\}.$$

It can be seen that $\mathcal{W}$ is decisive. So it follows that $\psi_{\mathcal{W}}$ is strategy-proof and country-wise Pareto optimal. Consider a situation where $n_A = n_B = 3$. Then for the profile $z \in \mathcal{R}$ where $|N_1(z)| = 1$, $|N_3(z)| = 2$, $N_5(z) = 1$ and $|N_7(z)| = 2$, we have $\psi_{\mathcal{W}}(z) = (-1, 1)$. Now consider an i–deviation z' of z, where $i \in N_3(z)$ and $z'(i) = -\frac{3}{4}$. Note that $\psi_{\mathcal{W}}(z') = (0, 0)$. So the decision in country B has been altered by an agent in country A, which implies that in this rule the countries do not decide independently. But on the other hand, this rule also do not belong to the class described under myopic preferences. Consider the situation where $n = 20$ and a profile z such that $|N_1(z)| = |N_3(z)| = 5$, $|N_5(z)| = 4$ and $|N_7(z)| = 6$. In this case, $\psi_{\mathcal{W}}(z') = (-1, 0)$, which is a violation of country-wise Pareto optimality under myopic preferences but not under lexmin preferences.

Finally we show that country-wise Pareto optimality, Maskin monotonicity and I.I.A. are logically independent by means of the following examples.

VIOLATION OF COUNTRY-WISE PARETO OPTIMALITY Any constant rule that selects one outcome from $\mathcal{B}$ for all profiles will violate country-wise Pareto optimality; but it will satisfy Maskin monotonicity and I.I.A.

VIOLATION OF MASKIN MONOTONICITY Consider the rule g as follows.

$$\alpha'(z) = \begin{cases} -1 & \text{if } N_A \subseteq N_{234}(z) \text{ and } N_A \not\subseteq N_2(z) \\ 0 & \text{if } N_A \subseteq N_{12}(z) \text{ and } N_A \not\subseteq N_2(z) \\ \alpha(z) & \text{otherwise} \end{cases}$$

$$\beta'(z) = \begin{cases} 1 & \text{if } N_B \subseteq N_{456}(z) \text{ and } N_B \not\subseteq N_6(z) \\ 0 & \text{if } N_B \subseteq N_{67}(z) \text{ and } N_B \not\subseteq N_6(z) \\ \beta(z) & \text{otherwise} \end{cases}$$

$$\alpha(z) = \begin{cases} -1 & \text{if } |N_{34}(z) \cap N_A| < \frac{1}{2}n_A \\ 0 & \text{otherwise} \end{cases}$$

$$\beta(z) = \begin{cases} 1 & \text{if } |N_{45}(z) \cap N_B| < \frac{1}{2}n_B \\ 0 & \text{otherwise} \end{cases}$$

$$g(z) = (\alpha'(z), \beta'(z)) \text{ for all } z \in \mathcal{R}.$$

Note that in this rule, an agent in N_A, cannot change the location of the bad in country B. Similar conditions hold for agents in country B.

To show that g satisfies country-wise Pareto optimality, it is sufficient to show the following conditions.

Suppose $N_A \subseteq N_{12}(z)$ and $N_1(z) \cap N_A \neq \emptyset$. Then $g(z) \in \{(0,0),(0,1)\}$.

Suppose $N_B \subseteq N_{67}(z)$ and $N_7(z) \cap N_B \neq \emptyset$. Then $g(z) \in \{(0,0),(-1,0)\}$.

Suppose $N_A \subseteq N_{234}(z)$ and $N_{34}(z) \cap N_A \neq \emptyset$. Then $g(z) \in \{(-1,1),(-1,0)\}$.

Suppose $N_B \subseteq N_{456}(z)$ and $N_{45}(z) \cap N_B \neq \emptyset$. Then $g(z) \in \{(-1,1),(0,1)\}$.

Proof of these properties follows from the definition of g.
Next we show that this rule satisfies I.I.A.

Proof of I.I.A. Consider two profiles $z, z' \in \mathcal{R}$ where z' is a $i-$deviation from z with $i \in N_A$ and $z|_{g(z),g(z')} = z'|_{g(z),g(z')}$. To the contrary, assume that $g(z) \neq g(z')$. As agents in N_A, cannot change the location of the bad in country B, we have the following cases.

CASE 1 : $g(z) = (-1, 1)$ and $g(z') = (0, 1)$.

CASE 2 : $g(z) = (-1, 0)$ and $g(z') = (0, 0)$.

Note that if $z(i), z'(i) \in (-\frac{1}{2}, 0]$, then $g(z) = g(z')$. Also if $z(i), z'(i) \in [-1, -\frac{1}{2}]$, then $g(z) = g(z')$. Otherwise we have $z(i) \in (-\frac{1}{2}, 0]$ and $z'(i) \in [-1, -\frac{1}{2}]$ or the reverse. Then in both case 1 and 2, we have $z|_{g(z),g(z')} \neq z'|_{g(z),g(z')}$. So it follows that this rule satisfies I.I.A. □

Next, to show that g violates Maskin monotonicity, we provide the following example. We consider a situation, where $n_A = 4$ and $n_B = 4$. In profile z we have

$|N_1(z)| = 3$ and $|N_{34}(z) \cap N_A| = 1$.

$|N_{45}(z) \cap N_B| = 1$ and $|N_7(z)| = 3$.

There exists $i \in N_A$ such that $z(i) = -\frac{3}{4}$

In profile z', which is an $i-$deviation from z, we have

$z'(i) = -\frac{1}{4}$.

$z|_{N-\{i\}} = z'|_{N-\{i\}}$.

Note that $g(z) = (-1, 1)$ and $g(z') = (0, 1)$. Note that $L(g(z), z(i)) \subset L(g(z), z'(i))$ and $L'(g(z), z(i)) \subset L'(g(z), z'(i))$. Also $L(g(z), z(j)) = L(g(z), z'(j))$ and $L'(g(z), z(j)) = L'(g(z), z'(j))$ for all $j \in N - \{i\}$. So $g(z) \neq g(z')$ is a violation of Maskin monotonicity.

VIOLATION OF I.I.A. For convenience suppose that $n_A = n_B \geqslant 2$. Now we define $\mathcal{V}$ as follows.

$$\mathcal{V}_{(-1,1)} = \left\{ \overline{N} \in \mathcal{N} : |N_{23456}| \geqslant \frac{n}{2} \text{ and } \begin{array}{ll} \text{either} & N_{34} \cap N_A \neq \emptyset \\ \text{or} & N_{45} \cap N_B \neq \emptyset \end{array} \right\}.$$

$$\mathcal{V}_{(-1,0)} = \left\{ \overline{N} \in \mathcal{N} \backslash \mathcal{V}_{(-1,1)} : \begin{array}{l} |N_A \cap N_{234}| \geqslant \frac{n}{4} \text{ and } N_{34} \cap N_A \neq \emptyset \\ \text{and } |N_B \cap N_{456}| < \frac{n}{4} \end{array} \right\}.$$

$$\mathcal{V}_{(0,1)} = \left\{ \overline{N} \in \mathcal{N} \backslash \mathcal{V}_{(-1,1)} : \begin{array}{l} |N_B \cap N_{456}| \geqslant \frac{n}{4} \text{ and } N_{45} \cap N_B \neq \emptyset \\ \text{and } |N_A \cap N_{234}| < \frac{n}{4} \end{array} \right\}.$$

$$\mathcal{V}_{(0,0)} = \left\{ \overline{N} \in \mathcal{N} : \begin{array}{ll} \text{either} & |N_1| > \frac{n}{4} \text{ and } |N_7| > \frac{n}{4} \\ \text{or} & N_A = N_2 \text{ and } N_B = N_6 \end{array} \right\}.$$

$$\mathcal{V} = \{\mathcal{V}_a : a \in \mathcal{B}\}.$$

Consider the decisive $\mathcal{W}$ and the associated rule $\psi_{\mathcal{W}}$ as defined in the combined voting example in this section. Note that $\psi_{\mathcal{V}} = \psi_{\mathcal{W}}$ in all cases except for the profile z such that $N_2(z) = N_A$ and $N_6(z) = N_B$. So it follows that $\mathcal{V}$ satisfies monotonicity, boundary conditions, disjoint condition and completeness as defined in section 3.5. So we can conclude that the rule $\psi_{\mathcal{V}}$ is country-wise Pareto optimal and Maskin monotone. Now consider a profile z such that $N_6(z) = N_B$, $N_3(z) = \{i\}$ and $N_2(z) = N_A \backslash \{i\}$. Note that $\psi_{\mathcal{V}}(z) = (-1, 1)$. Now consider an i–deviation z' of z, where $z'(i) = -\frac{1}{2}$. This implies that $N_2(z') = N_A$ and $N_6(z') = N_B$ and $\psi_{\mathcal{V}}(z') = (0, 0)$. This is a violation of I.I.A., as $z|_{\psi_{\mathcal{V}}(z),\psi_{\mathcal{V}}(z')} = z'|_{\psi_{\mathcal{V}}(z),\psi_{\mathcal{V}}(z')}$ but $\psi_{\mathcal{V}}(z) \neq \psi_{\mathcal{V}}(z')$.

3.7 CONCLUSION

To conclude the part on two neighbouring countries, first note that in view of examples 2.5.3, 2.5.4, 3.6.2 and 3.6.3; the class described in chapter 2 is neither contained in nor contains the class described in chapter 3. The reason behind this variation comes from the fact that myopic preferences allow for more indifferences than lexmin preferences. For example, we may imagine that each agent's marginal preference is a strict Euclidean single dipped preference with arbitrary tie breaking over the interval $[-1, 1]$. We denote such a marginal preference by $\succ_i$ with dip $d(\succ_i) \in [-1, 1]$. As preference over $\mathcal{A}$, agents are allowed to report lexmin extensions of such a marginal preferences. Let such a preference be denoted by P_i and correspondingly a profile of such preferences be denoted by P. Other notations can be defined in a similar way. We are going to show that in this situation, each country decide about the location of their own public bad independently of the other country. It can be seen that under these preferences Theorem 3.3.1 still holds. Now restricted to -1, 0 and 1, we have the following marginal preferences.

Dips ($d(\succ_i)$)	$[-1, -\frac{1}{2})$	$-\frac{1}{2}$		$(-\frac{1}{2}, 0)$	0		$(0, \frac{1}{2})$	$\frac{1}{2}$		$(\frac{1}{2}, 1]$
Marginal Preferences	1	1	1	1	1	−1	−1	−1	−1	−1
	0	0	−1	−1	−1	1	1	1	0	0
	−1	−1	0	0	0	0	0	0	1	1

Since we can restrict our attention to the corner points of $\mathcal{A}$, from the restricted marginal preferences shown in the above table, we have the following preferences over the corner points of $\mathcal{A}$.

Agent	$i \in N_A$			$i \in N_B$		
Marginal Preferences	1	1	−1	1	−1	−1
	0	−1	1	−1	1	0
	−1	0	0	0	0	1
Lexmin Extensions	$(0, 1)$	$(-1, 1)$	$(-1, 1)$	$(-1, 1)$	$(-1, 1)$	$(-1, 0)$
	$(0, 0)$	$(0, 1)$	$(-1, 0)$	$(0, 1)$	$(-1, 0)$	$(0, 0)$
	$(-1, 1)$	$(-1, 0)$	$(0, 1)$	$(-1, 0)$	$(0, 1)$	$(-1, 1)$
	$(-1, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 1)$

Next we show that the location of the bad of one country is determined by the preference of the agents in that country only with the help of the following two propositions.

Proposition 3.7.1. *Suppose* P *be a profile and suppose* f *be a strategy-proof, country-wise Pareto optimal rule.*

(a) If $f(P) = (0,1)$*, then* $f(P') \in \{(0,0),(0,1)\}$ *for any* P' *such that* $P_{N_A} = P'_{N_A}$.

(b) If $f(P) = (-1,0)$*, then* $f(P') \in \{(-1,0),(-1,1)\}$ *for any* P' *such that* $P_{N_A} = P'_{N_A}$.

Proof. First we prove part (a). Suppose $f(P) = (0,1)$. So using Maskin monotonicity (implied by strategy-proofness as a result of strict preferences), we may assume that $(-1,1)P_i(0,1)P_i(-1,0)P_i(0,0)$ for all $i \in N_B$. Consider another profile P' such that $P_{N_A} = P'_{N_A}$. With out loss of generality, we may assume (X,Y,Z) as a non-trivial disjoint partition of agents in N_B such that,

For all $i \in X$, $(-1,1)P'_i(0,1)P'_i(-1,0)P'_i(0,0)$.

For all $i \in Y$, $(-1,1)P'_i(-1,0)P'_i(0,1)P'_i(0,0)$.

For all $i \in Z$, $(-1,0)P'_i(0,0)P'_i(-1,1)P'_i(0,1)$.

Now consider a profile P^1 where $P_{N_A} = P^1_{N_A}$ as follows.

	X	Y	Z
P	$(-1,1)$	$(-1,1)$	$(-1,1)$
	$(0,1)$	$(0,1)$	$(0,1)$
	$(-1,0)$	$(-1,0)$	$(-1,0)$
	$(0,0)$	$(0,0)$	$(0,0)$
P^1	$(-1,1)$	$(-1,1)$	$(-1,1)$
	$(0,1)$	$(-1,0)$	$(-1,0)$
	$(-1,0)$	$(0,1)$	$(0,1)$
	$(0,0)$	$(0,0)$	$(0,0)$
P'	$(-1,1)$	$(-1,1)$	$(-1,0)$
	$(0,1)$	$(-1,0)$	$(0,0)$
	$(-1,0)$	$(0,1)$	$(-1,1)$
	$(0,0)$	$(0,0)$	$(0,1)$

As $f(P) = (0,1)$, strategy-proofness for the deviation from P to P^1 implies that $f(P^1) \in \{(-1,0),(0,1)\}$. Country-wise Pareto optimality at P^1 implies that $f(P^1) \in \{(-1,1),(0,1)\}$. Combining we have $f(P^1) = (0,1)$. Then for the deviation from P^1 to P', strategy-proofness implies that $f(P') \in \{(0,0),(0,1)\}$. This concludes the proof of part (a) of Proposition 3.7.1.

Using part (a), proof of part (b) follows in a similar way. This concludes the proof of Proposition 3.7.1. □

Remark 3.7.2. Due to symmetry, it follows from Proposition 3.7.1 that, if for some profile P and a strategy-proof, country-wise Pareto optimal rule f, we have

(a) If $f(P) = (0, 1)$, then $f(P') \in \{(-1, 1), (0, 1)\}$ for any P' such that $P_{N_B} = P'_{N_B}$.

(b) If $f(P) = (-1, 0)$, then $f(P') \in \{(-1, 0), (0, 0)\}$ for any P' such that $P_{N_B} = P'_{N_B}$.

Remark 3.7.3. Using Proposition 3.7.1 and Remark 3.7.2 we can show the following.

Suppose P be a profile and suppose f be a strategy-proof, country-wise Pareto optimal rule.

- (a) If $f(P) = (-1, 1)$, then $f(P') \in \{(-1, 0), (-1, 1)\}$ for any P' such that $P_{N_A} = P'_{N_A}$.
- (b) If $f(P) = (0, 0)$, then $f(P') \in \{(0, 0), (0, 1)\}$ for any P' such that $P_{N_A} = P'_{N_A}$.

Suppose P be a profile and suppose f be a strategy-proof, country-wise Pareto optimal rule.

- (a) If $f(P) = (-1, 1)$, then $f(P') \in \{(-1, 1), (0, 1)\}$ for any P' such that $P_{N_B} = P'_{N_B}$.
- (b) If $f(P) = (0, 0)$, then $f(P') \in \{(0, 0), (-1, 0)\}$ for any P' such that $P_{N_B} = P'_{N_B}$.

Using Proposition 3.7.1 and Remarks 3.7.2, 3.7.3, we may characterise the class of rules as a non-constant monotone voting between $-1, 0$ for the location of the left country and another non-constant monotone voting between $1, 0$ for the location of the right country.

Part II

TWO PUBLIC BADS IN A REGION

4 SEPARABLE PREFERENCES

4.1 INTRODUCTION

In this part, we consider the problem of locating two noxious facilities in one region. For example consider the problem of locating two garbage dumping sites along a road. We model this region by the unit line segment. We assume that there are finitely many agents. A decision rule selects two points from the interval for every reported profile of preferences. Each agent has a strict Euclidean single-dipped preference with arbitrary tie breaking over the line segment as his marginal preference. We allow agents to report any complete, strict, transitive and separable extensions of such a marginal preference as a preference over the pairs of locations. We elaborate and formalise the preferences in section 4.2.

In this situation, a decision rule will take the extensions of the single-dipped preferences of all the residents as input, and give a pair of locations as output. In this paper we define the class of all decision rules that simultaneously satisfy strategy-proofness and Pareto optimality. Strategy-proofness ensures that for every agent, truth telling is the weakly dominant strategy, and Pareto optimality says that given a decision about the locations of the bads, improving one agent would result in worsening other agents.

Under these properties, first we show that no bad can be placed in the interior of the interval. Next we show that a decision rule satisfying these properties can only select either $(1, 1)$ or $(0, 0)$. The class of rules consists of any non-constant monotone voting rule between these two alternatives. This class is similar to the class described by Manjunath (2014), where he considers the problem of locating one public bad in a unit interval under single-dipped preferences.

Our results can be seen as positive results compared to the seminal impossibility theorem of Gibbard (1973) and Satterthwaite (1975) which says that if there are three or more alternatives, then it is impossible to find a non-dictatorial social choice rule which is also strategy proof and Pareto optimal. One way out from this impossibility result is to consider restricted preference domains. One possible restricted domain is the single-dipped preference domain. Peremans and Storcken (1999) have shown the equivalence between individual and group strategy-proofness in sub domains of single-dipped preferences. Manjunath (2014) has characterised the class of all non-dictatorial, strategy-proof and Pareto optimal social choice rules when preferences are single-dipped on an interval. Barberà et al. (2012) have characterised the class of all non-dictatorial, group strategy-proof and Pareto optimal social choice rules when preferences are single-dipped on a line. The rules in the present paper bear similarities to the rules in the last two papers.

But there are impossibility results in this domain as well. Öztürk et al. (2013) and Öztürk et al. (2014) have shown that there does not exist a non-dictatorial social choice rule that is strategy-proof and Pareto optimal when preferences are single-dipped on a disk, and on some, but not all, convex polytopes in the plane. Chatterjee, Peters, and Storcken (2016) have extended these results to social choice rules on a sphere, when preferences are single-dipped or, equivalently in this case, single-peaked.

All these results are about strategy-proof location of one public bad. As far as we know, the present paper is the first one to consider the location of two public bads in a region, apart from an analysis of the one agent per country case in Öztürk (2013). There is also a literature adopting a mechanism design approach to the location of public bads, that is, including monetary side payments: e.g., recently, Lescop (2007) and Sakai (2012), but we are not aware of results in this area addressing more than one noxious facilities. On the other hand, the problem of locating two public goods on an interval has been considered previously. Ehlers (2002) considered lexicographic extension of single-peaked preference over pairs of locations where agent first considers the best good, then the worst one. This paper characterises the class of rules that satisfies Pareto-optimality and replacement-domination.

This chapter is organised as follows. Section 4.2 introduces the model and some preliminary results. Section 4.3 shows that internal locations are excluded, and Section 4.4 provides the characterisation of all rules satisfying our conditions. Section 4.5 concludes by speculating how these results can be weakened.

4.2 MODEL

Let A represented by the interval $[0, 1]$ be the set of possible locations of public bads. Our objective is to locate two bads in A. We assume that the location pair (α, β) and the location pair (β, α) are the same. So the set of possible alternatives is $\mathcal{A} = \{(\alpha, \beta) \in A \times A : \alpha \leqslant \beta\}$. The set of agents is N with cardinality n.

Each agent $i \in N$ has a strict single-dipped preference ordering P_i with dip $d(P_i) \in A$ over the elements of A such that, for any two distinct points $\alpha, \beta \in A$, $\alpha P_i \beta$ if $|d(P_i) - \alpha| > |d(P_i) - \beta|$. Ties are broken arbitrarily. As usual $\neg(\alpha P_i \beta)$ denotes either $\beta P_i \alpha$ or $\alpha = \beta$. Suppose $\mathcal{P}_i$ is the set of all such single-dipped preferences over A. We extend each $P_i \in \mathcal{P}_i$ to a strict preference ordering (complete and transitive relation) on $\mathcal{A}$, denoted by $Q(P_i)$ as follows: for any two distinct pairs $(\alpha, \beta), (\alpha', \beta') \in \mathcal{A}$, if $\alpha = \alpha'$ then $\beta P_i \beta'$ implies $(\alpha, \beta) Q(P_i) (\alpha', \beta')$, and if $\beta = \beta'$ then $\alpha P_i \alpha'$ implies $(\alpha, \beta) Q(P_i) (\alpha', \beta')$. Let $\mathcal{Q}(P_i)$ be the set of all possible preference extensions of $P_i \in \mathcal{P}_i$.

Define $\mathcal{Q}_i = \bigcup_{P_i \in \mathcal{P}_i} \mathcal{Q}(P_i)$. For $Q_i \in \mathcal{Q}_i$, define $\mathcal{P}(Q_i) \subseteq \mathcal{P}_i$ as $\mathcal{P}(Q_i) = \{P_i \in \mathcal{P}_i : Q_i \in \mathcal{Q}(P_i)\}$. We denote by $P(Q_i)$ an element of $\mathcal{P}(Q_i)$. The next lemma shows that for any $Q_i \in \mathcal{Q}_i$, $P(Q_i)$ is unique.

Lemma 4.2.1. *For all $P_i^1, P_i^2 \in \mathcal{P}_i$, we have $\mathcal{Q}(P_i^1) \cap \mathcal{Q}(P_i^2) \neq \emptyset$ if and only if $P_i^1 = P_i^2$.*

Proof. The if part follows directly. For the only if part, suppose there exists a $Q \in \mathcal{Q}(P_i^1) \cap \mathcal{Q}(P_i^2)$. Now for any $\beta_1 \neq \beta_2 \in A$ and any $\alpha \in A$, we have $(\beta_1, \beta_2) \in P_i^1 \Leftrightarrow ((\alpha, \beta_1), (\alpha, \beta_2)) \in Q \Leftrightarrow (\beta_1, \beta_2) \in P_i^2$. This concludes the proof of Lemma 4.4.1. □

Remark 4.2.2. Using Lemma 4.2.1, it follows that for any $Q_i \in \mathcal{Q}_i$, $P(Q_i)$ is unique.

Remark 4.2.3. Here we provide three examples of such extensions.

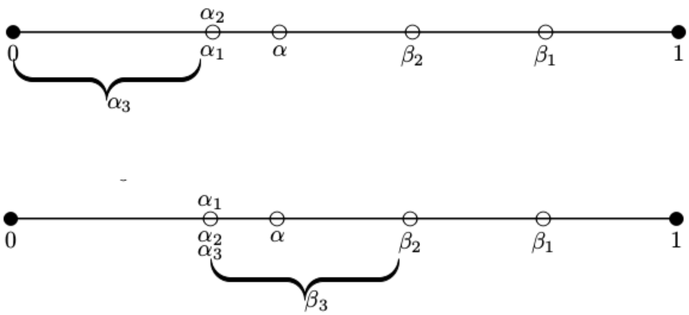
FIRST COMPONENT α BOUNDED EXTENSION This is an extension of $P_i^0 \in \mathcal{P}_i$ where $d(P_i^0) = 0$. We define an extension of P_i^0 as first component α bounded $(Q_\alpha^1(P_i^0))$, where $\alpha \in A$, as follows. For any $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2) \in \mathcal{A}$, we have (α_1, β_1) is strictly preferred to (α_2, β_2) according to $Q_\alpha^1(P_i^0)$ $((\alpha_1, \beta_1) Q_\alpha^1(P_i^0)(\alpha_2, \beta_2))$ if

1. either $\alpha_2 < \alpha$ and $\alpha_2 < \alpha_1$,
2. or $\alpha_2 < \alpha$ and $\alpha_2 = \alpha_1$ and $\beta_2 < \beta_1$,
3. or $\alpha \leqslant \alpha_2$ and $\alpha \leqslant \alpha_1$ and $\beta_2 < \beta_1$,
4. or $\alpha \leqslant \alpha_2 < \alpha_1$ and $\beta_2 = \beta_1$.

In other words if $\alpha_k < \alpha$ for some $k \in \{1, 2\}$, then the ordering between the pairs (α_1, β_1) and (α_2, β_2) is determined on the basis of α_1 and α_2. On the other hand, if $\alpha \leqslant \alpha_1, \alpha_2$, then it is determined on the basis of β_1 and β_2. Next we show that $Q_\alpha^1(P_i^0) \in \mathcal{Q}(P_i^0)$. Note that from the definition it follows that $Q_\alpha^1(P_i^0)$ is complete and anti-symmetric. Next we show that $Q_\alpha^1(P_i^0)$ is transitive.

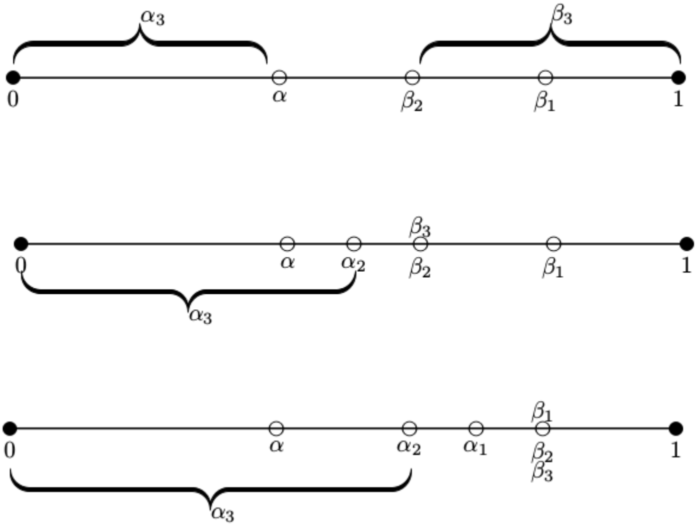
Proof of $Q_\alpha^1(P_i^0)$ is transitive. Consider three distinct pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3) \in \mathcal{A}$ such that $(\alpha_1, \beta_1) Q_\alpha^1(P_i^0)(\alpha_2, \beta_2)$ and $(\alpha_2, \beta_2) Q_\alpha^1(P_i^0)(\alpha_3, \beta_3)$. Now consider the following cases.

$\min\{\alpha_1, \alpha_2\} < \alpha$:



In this case, as $(\alpha_1, \beta_1)Q^1_\alpha(P^0_i)(\alpha_2, \beta_2)$, we have either $\alpha_1 > \alpha_2$ or $\alpha_1 = \alpha_2$ and $\beta_1 > \beta_2$. So, we have $\min\{\alpha_1, \alpha_2\} = \alpha_2 < \alpha$. Now $(\alpha_2, \beta_2)Q^1_\alpha(P^0_i)(\alpha_3, \beta_3)$ implies either $\alpha_2 > \alpha_3$ or $\alpha_2 = \alpha_3$ and $\beta_2 > \beta_3$. This in turn implies that either $\alpha_1 > \alpha_3$ and $\alpha_3 < \alpha$, or $\alpha_1 = \alpha_3 < \alpha$ and $\beta_1 > \beta_3$. So we have $(\alpha_1, \beta_1)Q^0(P^0_i)(\alpha_3, \beta_3)$.

$\min\{\alpha_1, \alpha_2\} \geqslant \alpha$:



In this case, if $\alpha_3 < \alpha$, then $(\alpha_1, \beta_1)Q^1_\alpha(P^0_i)(\alpha_3, \beta_3)$. So suppose $\alpha_3 \geqslant \alpha$. So $\alpha \leqslant \min\{\alpha_1, \alpha_2, \alpha_3\}$. Because of $(\alpha_1, \beta_1)Q^1_\alpha(P^0_i)(\alpha_2, \beta_2)$, we have $\beta_2 \leqslant \beta_1$. Also because of $(\alpha_2, \beta_2)Q^1_\alpha(P^0_i)(\alpha_3, \beta_3)$, we have $\beta_3 \leqslant \beta_2$. So $\beta_3 \leqslant \beta_2 \leqslant \beta_1$. We are done if $\beta_3 < \beta_1$. So suppose $\beta_3 = \beta_1$. Then $\beta_1 = \beta_2 = \beta_3$ and $(\alpha_1, \beta_1)Q^1_\alpha(P^0_i)(\alpha_2, \beta_2)$ implies $\alpha_1 > \alpha_2$ and $(\alpha_2, \beta_2)Q^1_\alpha(P^0_i)(\alpha_3, \beta_3)$ implies $\alpha_2 > \alpha_3$. So $\alpha_1 > \alpha_3$ and $(\alpha_1, \beta_1)Q^1_\alpha(P^0_i)(\alpha_3, \beta_3)$.

Proof of $Q^1_\alpha(P^0_i) \in \mathcal{Q}(P^0_i)$. Consider any $(\gamma_1, \delta_1), (\gamma_2, \delta_2) \in \mathcal{A}$ such that $\gamma_1 = \gamma_2$. Then $\delta_1 P^0_i \delta_2$ implies $\delta_1 > \delta_2$. So we have $(\gamma_1, \delta_1)Q^1_\alpha(P^0_i)(\gamma_2, \delta_2)$ irrespective of whether $\min\{\gamma_1, \gamma_2\} < \alpha$ or not.

This shows that $Q^1_\alpha(P^0_i) \in \mathcal{Q}(P^0_i)$.

SECOND COMPONENT β BOUNDED EXTENSION This extension is similar to first component α bounded extension. Here we define an extension of $P^1_i \in \mathcal{P}_i$, where $d(P^1_i) = 1$, as second component β bounded $(Q^2_\beta(P^1_i))$, where $\beta \in A$, as follows. For any $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2) \in \mathcal{A}$, we have (α_1, β_1) is strictly preferred to (α_2, β_2) according to $Q^2_\beta(P^1_i)$ $((\alpha_1, \beta_1)Q^2_\beta(P^1_i)(\alpha_2, \beta_2))$ if

1. either $\beta_2 > \beta$ and $\beta_1 < \beta_2$,
2. or $\beta_2 > \beta$ and $\beta_1 = \beta_2$ and $\alpha_1 < \alpha_2$,
3. or $\beta \geqslant \beta_1$ and $\beta \geqslant \beta_2$ and $\alpha_1 < \alpha_2$,
4. or $\beta \geqslant \beta_1$ and $\beta \geqslant \beta_2$ and $\alpha_1 = \alpha_2$ and $\beta_1 < \beta_2$.

In other words if $\beta_k > \beta$ for some $k \in \{1, 2\}$, then the ordering between the pairs (α_1, β_1) and (α_2, β_2) is determined on the basis of β_1 and β_2. On the other hand, if $\beta \geqslant \beta_1, \beta_2$, then it is determined on the basis of α_1 and α_2. Similarly it follows that $Q^2_\beta(P^1_i) \in \mathcal{Q}(P^1_i)$.

LEXMIN EXTENSION For any $P_i \in \mathcal{P}_i$, we define an extension of P_i as LexMin $(Q^{LexMin}(P_i))$ as follows. Consider any two pairs $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2) \in \mathcal{A}$. Suppose $\{x_1, y_1\} := \{\alpha_1, \beta_1\}$ be such that $\neg(y_1 P_i x_i)$. Also suppose $\{x_2, y_2\} := \{\alpha_2, \beta_2\}$ be such that $\neg(y_2 P_i x_2)$. We define $(\alpha_1, \beta_1)Q^{LexMin}(P_i)(\alpha_2, \beta_2)$ if

1. either $y_1 P_i y_2$,
2. or $y_1 = y_2$ and $x_1 P_i x_2$.

In other words, to compare two different pairs of locations according to LexMin extension of a single-dipped preference, an agent first compares the worst location from the two pairs according to the single-dipped preference and prefers that pair for which the worst location is better. Otherwise if the worst locations are the same then agent compares the best locations and prefers that pair for which the best location is better. Note that for any P_i, we have $Q^{LexMin}(P_i) \in \mathcal{Q}(P_i)$.

A preference profile Q assigns to each agent i in N a preference $Q_i \in \mathcal{Q}_i$. The set of all preference profiles is denoted by $\mathcal{Q}$.

For a profile Q and a non-empty set $S \subseteq N$, let $Q_S = (Q_i)_{i \in S}$. For $i \in N$, profile Q' is an i-deviation of Q if $Q_{N\setminus\{i\}} = Q'_{N\setminus\{i\}}$. For a profile $Q \in \mathcal{Q}$, suppose $d(Q) = (d(P(Q_i)))_{i \in N}$. Define the restriction of a preference Q_i to a subset $\mathcal{C}$ of $\mathcal{A}$ by $Q_i|_{\mathcal{C}} = (\mathcal{C} \times \mathcal{C}) \cap Q_i$. Further, define the restriction of a profile Q to $\mathcal{C}$ component wise; i.e., $Q|_{\mathcal{C}} = (Q_i|_{\mathcal{C}})_{i \in N}$. For $a, b \in \mathcal{A}$, we denote $Q|_{\{a,b\}}$ and $Q_i|_{\{a,b\}}$ also by $Q|_{a,b}$ and $Q_i|_{a,b}$ respectively. For an outcome $(\alpha, \beta) \in \mathcal{A}$ and a dip $x \in A$, define the conflict set $C((\alpha, \beta), x) \subseteq \mathcal{A}$ as

$$C((\alpha,\beta),x) = \left\{ (\alpha',\beta') \in \mathcal{A} : \begin{array}{r} \text{either } |\alpha' - x| > |\alpha - x| \\ \text{and } |\beta - x| > |\beta' - x| \\ \text{or } |\alpha - x| > |\alpha' - x| \\ \text{and } |\beta' - x| > |\beta - x| \end{array} \right\}.$$

For an agent i with preference Q_i and an outcome $a \in \mathcal{A}$ define $L(a, Q_i) = \{b \in \mathcal{A} \colon aQ_ib\}$ as the lower contour set of Q_i at a.

Remark 4.2.4. Consider an alternative $(\alpha, \beta) \in \mathcal{A}$ and a preference $Q_i \in \mathcal{Q}_i$. Then we have the following cases.

Suppose $0 \leqslant d(P(Q_i)) \leqslant \alpha$. Now consider the preference $Q^1_\alpha(P^0_i) \in \mathcal{Q}_i$, where $Q^1_\alpha(P^0_i)$ is as defined before. We are going to show that $L((\alpha,\beta), Q_i) \subseteq L((\alpha,\beta), Q^1_\alpha(P^0_i))$. So consider an $(\alpha', \beta') \in L((\alpha,\beta), Q_i)$. If $\alpha' < \alpha$ or $\alpha' \geqslant \alpha$ and $\beta' < \beta$, then by definition $(\alpha,\beta)Q^1_\alpha(P^0_i)(\alpha',\beta')$. Note that if $\alpha' \geqslant \alpha$ and $\beta' \geqslant \beta$, then $(\alpha',\beta') \notin L((\alpha,\beta), Q_i)$. This shows the inclusion.

Suppose $\beta \leqslant d(P(Q_i)) \leqslant 1$. Now consider the preference $Q^2_\beta(P^1_i) \in \mathcal{Q}_i$, where $Q^2_\beta(P^1_i)$ is as defined before. We are going to show that $L((\alpha,\beta), Q_i) \subseteq L((\alpha,\beta), Q^2_\beta(P^1_i))$. So consider an $(\alpha', \beta') \in L((\alpha,\beta), Q_i)$. If $\beta' > \beta$ or $\beta' \leqslant \beta$ and $\alpha' > \alpha$, then by definition $(\alpha,\beta)Q^2_\beta(P^1_i)(\alpha',\beta')$. Note that if $\alpha' \leqslant \alpha$ and $\beta' \leqslant \beta$, then $(\alpha',\beta') \notin L((\alpha,\beta), Q_i)$. This shows the inclusion.

A *rule* f assigns to each preference profile Q an alternative $f(Q) = (\alpha^f(Q), \beta^f(Q)) \in \mathcal{A}$ such that for all $Q \in \mathcal{Q}$, $\alpha^f(Q) \leqslant \beta^f(Q)$. For $x, y \in \mathbb{R}$, $\mu(x,y) = \frac{x+y}{2}$ denotes the midpoint of x and y. For any profile $Q \in \mathcal{Q}$, in case there is no confusion we write $\mu(Q)$ instead of $\mu(\alpha^f(Q), \beta^f(Q))$.

We consider the following properties for a rule f.

Strategy-Proofness f is strategy-proof if for any $i \in N$ and any $Q \in \mathcal{Q}$ and any i-deviation Q' of Q, we have either $f(Q') = f(Q)$ or $f(Q)Q_if(Q')$.

Strategy-proofness says that truth-telling is a weakly dominant strategy.

Pareto optimality Rule f is Pareto optimal if for every profile Q there does not exist an $a \in \mathcal{A}$ such that $aQ_i f(Q)$ for all $i \in N$.

Maskin Monotonicity Rule f is Maskin monotone if $f(Q) = f(Q')$ for all $Q, Q' \in \mathcal{Q}$ such that $L(f(Q), Q_i) \subseteq L(f(Q), Q'_i)$ for all $i \in N$.

The following lemma shows an implication of Maskin monotonicity. For any $Q \in \mathcal{Q}$ and any rule f, define

$$S(Q) = \{i \in N : d(P(Q_i)) \leqslant \alpha^f(Q)\}$$

$$T(Q) = \{i \in N : d(P(Q_i)) \geqslant \beta^f(Q)\}$$

Lemma 4.2.5. *Suppose* f *satisfies Maskin monotonicity. Then for any profile* $Q \in \mathcal{Q}$*, we have* $f(Q) = f(Q^\star)$ *for some* $Q^\star \in \mathcal{Q}$ *where*

$$Q^\star_i = \begin{cases} Q^1_{\alpha^f(Q)}(P^0_i) & \textit{if } i \in S(Q) \\ Q^2_{\beta^f(Q)}(P^1_i) & \textit{if } i \in T(Q) \end{cases}$$

and $Q^\star_{N\setminus(S(Q)\cup T(Q))} = Q_{N\setminus(S(Q)\cup T(Q))}$.

Proof. Follows from Maskin monotonicity and Remark 4.2.4. □

Remark 4.2.6. As we consider only strict preferences, Strategy-proofness implies Maskin monotonicity. Then under strategy-proofness, for any profile $Q \in \mathcal{Q}$, using Lemma 4.2.5 we may assume that for some $S, T \subseteq N$ with $S \cap T = \emptyset$

$$Q_i = \begin{cases} Q^1_{\alpha^f(Q)}(P^0_i) & \text{if } i \in S \\ Q_i & \text{if } i \in N\setminus(S \cup T) \\ Q^2_{\beta^f(Q)}(P^1_i) & \text{if } i \in T \end{cases}$$

Where $\alpha^f(Q) < d(P(Q_i)) < \beta^f(Q)$ for all $i \in N\setminus(S \cup T)$.

4.3 NO INTERNAL SOLUTION

Fix a profile $Q \in \mathcal{Q}$. Let f be a social choice rule. We are going to prove that if a social choice rule satisfies strategy-proofness and Pareto optimality, then the outcome of that rule must be one of the corner points of $\mathcal{A}$. Formally,

Proposition 4.3.1. *If* f *is a strategy-proof and Pareto optimal social choice rule, then for any profile* $Q \in \mathcal{Q}$, $f(Q) \in \{(1,1),(0,1),(0,0)\}$.

We prove this proposition with the help of the following three lemmas. From here onwards, we shall assume that f satisfies strategy-proofness and Pareto optimality.

The first lemma shows that if one of the two bads is located at the extreme end of A, then the other one cannot be located at an interior point of A.

Lemma 4.3.2. *For the profile* Q, $\alpha^f(Q) = 0$ *implies* $\beta^f(Q) \in \{0,1\}$ *and* $\beta^f(Q) = 1$ *implies* $\alpha^f(Q) \in \{0,1\}$.

Proof. Due to symmetry, it is sufficient to prove that $\alpha^f(Q) = 0$ implies $\beta^f(Q) \in \{0,1\}$.
Suppose $\alpha^f(Q) = 0$ but to the contrary $\beta^f(Q) \in (0,1)$. Using Lemma 4.2.5, Remark 4.2.6 and Pareto optimality, it follows that there exists a non-trivial subset S of N such that

$$Q_i = \begin{cases} Q_i & \text{if } i \in N\backslash S \\ Q^2_{\beta^f(Q)}(P^1_i) & \text{if } i \in S \end{cases}$$

Where $0 \leqslant d(P(Q_i)) < \beta^f(Q)$ for all $i \in N\backslash S$. Next we define two preferences over A as follows. Consider $P^{\text{Left}}, P^{\text{Right}}$ such that $d(P^{\text{Left}}) = \mu(Q)$ and $d(P^{\text{Right}}) = \mu(\beta^f(Q),1)$. Also P^{Left} is symmetric around $\mu(Q)$ and at equal distance the left is preferred to the right. Similarly, P^{Right} is symmetric around $\mu(\beta^f(Q),1)$ and at equal distance the right is preferred to the left. Note that $P^{\text{Left}}, P^{\text{Right}} \in \mathcal{P}_i$ for any $i \in N$. Next we consider the extensions $Q^{LexMin}(P^{\text{Left}})$ and $Q^{LexMin}(P^{\text{Right}})$. Now note that Pareto optimality implies that $X = \{i \in N\backslash S : d(P(Q_i)) \leqslant \mu(Q)\} \neq \emptyset$. Without loss of generality assume that $X = \{1,2,\ldots,m\}$ where $m < n$ as $S \neq \emptyset$. For any $j \in X$, suppose $X_j = \{1,2,\ldots,j\} \subseteq X$. In other words, $X_1 = \{1\}$, $X_2 = \{1,2\}$, and so on, till $X_m = X$. Now for any $j \in X$, consider these profiles Q^{1_j}, Q^2 and Q^{3_j} as follows.

1. $Q^{1_j} = ((Q^{LexMin}(P^{\text{Left}}))_{i\in X_j}, (Q_i)_{i\in X\backslash X_j}, Q_{N\backslash(S\cup X)}, Q_S)$.
2. $Q^2 = (Q_X, Q_{N\backslash(S\cup X)}, Q^{LexMin}(P^{\text{Right}})_S)$.
3. $Q^{3_j} = ((Q^{LexMin}(P^{\text{Left}}))_{i\in X_j}, (Q_i)_{i\in X\backslash X_j}, Q_{N\backslash(S\cup X)}, Q^{LexMin}(P^{\text{Right}})_S)$.

Consider the unilateral deviation from Q to Q^{1_1}. Strategy-proofness for this deviation implies that either $\alpha^f(Q^{1_1}) = 0$ or $\beta^f(Q^{1_1}) = 0$. Also if $\alpha^f(Q^{1_1}) = 0$ then $\beta^f(Q^{1_1}) \in \{0, \beta^f(Q)\}$. So we can conclude that $f(Q^{1_1}) \in \{f(Q),(0,0)\}$. Now suppose for some $t < m$, we have $f(Q^{1_j}) \in \{f(Q),(0,0)\}$ for all $1 \leqslant j \leqslant t$. Now consider the deviation from Q^{1_t} to $Q^{1_{t+1}}$. Strategy-proofness for this deviation implies that $f(Q^{1_{t+1}}) = (0,0)$ if $f(Q^{1_t}) = (0,0)$ and $f(Q^{1_{t+1}}) \in \{f(Q),(0,0)\}$ if $f(Q^{1_t}) = f(Q)$. So $f(Q^{1_{t+1}}) \in \{f(Q),(0,0)\}$, and using induction we can conclude that $f(Q^{1_m}) \in \{f(Q),(0,0)\}$. As $(0,0)Q_i^{1_m}f(Q)$ for all $i \in N$, Pareto optimality implies that $f(Q^{1_m}) = (0,0)$. As $\mu(\beta^f(Q),1) > \frac{1}{2}$, strategy-proofness for the deviation from Q^{1_m} to Q^{3_m} implies that $f(Q^{3_m}) = (0,0)$.

On the other hand, consider the deviation from Q to Q^2. Strategy-proofness implies that $f(Q^2) \in \{f(Q),(0,1)\}$. As $(0,1)Q_i^2 f(Q)$ for all $i \in N$, Pareto optimality implies that $f(Q^2) = (0,1)$. Next we show by induction that $f(Q^{3_j}) = (0,1)$ for all $j \in X$. Consider the unilateral deviation from Q^2 to Q^{3_1}. As $d(P(Q_1)) \leqslant \mu(Q) < \frac{1}{2}$, strategy-proofness for this deviation implies that

$f(Q^{3_1}) = f(Q^2) = (0,1)$. Now suppose for some $t < m$, we have $f(Q^{3_j}) = (0,1)$ for all $1 \leqslant j \leqslant t$. Now consider the deviation from Q^{3_t} to $Q^{3_{t+1}}$. As $d(P(Q_{t+1})) \leqslant \mu(Q) < \frac{1}{2}$, strategy-proofness for this deviation implies that $f(Q^{3_{t+1}}) = f(Q^{3_t}) = (0,1)$. So, by induction we can conclude that $f(Q^{3_m}) = (0,1)$, which contradicts the fact that $f(Q^{3_m}) = (0,0)$. □

The second lemma shows that both bads cannot be located at a common inner point.

Lemma 4.3.3. *For the profile* Q, *if* $\alpha^f(Q) = \beta^f(Q) = c$, *then* $c \in \{0,1\}$.

Proof. Suppose $\alpha^f(Q) = \beta^f(Q) = c$ but to the contrary $c \in (0,1)$. Using Lemma 4.2.5, Remark 4.2.6 and Pareto optimality, it follows that there exists a non-trivial subset S of N such that

$$Q_i = \begin{cases} Q_c^1(P_i^0) & \text{if } i \in S \\ Q_c^2(P_i^1) & \text{if } i \in N \backslash S \end{cases}$$

Now consider the profile $Q^\star = (\{Q^{LexMin}(P_i^0)\}_{i \in S}, \{Q^{LexMin}(P_i^1)\}_{i \in N \backslash S})$. Note that $L((c,c), Q_c^1(P_i^0)) = C((c,c),0) \cup \{(\alpha,\beta) \in \mathcal{A} \backslash \{(c,c)\} : \alpha \leqslant \beta \leqslant c\} = L((c,c), Q^{LexMin}(P_i^0))$. Also $L((c,c), Q_c^2(P_i^1)) = C((c,c),1) \cup \{(\alpha,\beta) \in \mathcal{A} \backslash \{(c,c)\} : c \leqslant \alpha \leqslant \beta\} = L((c,c), Q^{LexMin}(P_i^1))$. So Maskin monotonicity implies that $f(Q^\star) = f(Q)$. Next we define two preference over A as follows. Consider $P^{\text{Left}}, P^{\text{Right}}$ such that $d(P^{\text{Left}}) = \mu(0,c)$ and $d(P^{\text{Right}}) = \mu(c,1)$. Also P^{Left} is symmetric around $\mu(0,c)$ and at equal distance the left is preferred to the right. Similarly, P^{Right} is symmetric around $\mu(c,1)$ and at equal distance the right is preferred to the left. Note that $P^{\text{Left}}, P^{\text{Right}} \in \mathcal{P}_i$ for any $i \in N$. Next we consider the extensions $Q^{LexMin}(P^\star)$ and $Q^{LexMin}(P')$. Now consider the three profiles Q^1, Q^2 and Q^3 as follows.

1. $Q^1 = (Q^{LexMin}(P^{\text{Left}})_S, Q^\star_{N \backslash S})$.
2. $Q^2 = (Q^\star_S, Q^{LexMin}(P^{\text{Right}})_{N \backslash S})$.
3. $Q^3 = (Q^{LexMin}(P^{\text{Left}})_S, Q^{LexMin}(P^{\text{Right}})_{N \backslash S})$.

Consider the deviation from $Q^\star$ to Q^1. Strategy-proofness for this deviation implies that $\alpha^f(Q^1) \in \{0,c\}$. Then Lemma 4.3.2 and strategy-proofness implies that $f(Q^1) \in \{(0,0),(0,1),f(Q)\}$. As $(0,0)Q_i^1 f(Q)$ and $(\mu(c,1),\mu(c,1))Q_i^1(0,1)$ for all $i \in N$, Pareto optimality implies that $f(Q^1) = (0,0)$. As $\mu(c,1) > \frac{1}{2}$, strategy-proofness for the deviation from Q^1 to Q^3 implies that $f(Q^3) = (0,0)$.

On the other hand, consider the deviation from $Q^\star$ to Q^2. Strategy-proofness for this deviation implies that $\beta^f(Q) \in \{c,1\}$. Then Lemma 4.3.2 and strategy-proofness implies that $f(Q^2) \in \{f(Q),(0,1),(1,1)\}$. As $(1,1)Q_i^2 f(Q)$ and $(\mu(0,c),\mu(0,c))Q_i^2(0,1)$ for all $i \in N$, Pareto optimality implies that $f(Q^2) = (1,1)$. As $\mu(0,c) < \frac{1}{2}$, strategy-proofness for the deviation from Q^2 to Q^3 implies that $f(Q^3) = (1,1)$, which contradicts the fact that $f(Q_3) = (0,0)$. □

The third lemma shows that none of the two bads can be in the interior of A simultaneously.

Lemma 4.3.4. *For the profile* Q, $(\alpha^f(Q), \beta^f(Q)) \notin (0,1) \times [\alpha^f(Q), 1)$.

Proof. Due to Lemma 4.3.3, it is sufficient to show that $(\alpha^f(Q), \beta^f(Q)) \notin (0,1) \times (\alpha^f(Q), 1)$. First we show that in such a case we may assume without loss of generality that $\{i \in N : \alpha^f(Q) < d(P(Q_i)) < \beta^f(Q)\} = \emptyset$.

Remark 4.3.5. Suppose $0 < \alpha^f(Q) < \beta^f(Q) < 1$ and there exists $\emptyset \neq S \subsetneq N$ such that $\alpha^f(Q) < d(P(Q_i)) < \beta^f(Q)$ for all $i \in S$. Then there exists $Q' \in \mathcal{Q}$ such that $0 < \alpha^f(Q') < \beta^f(Q') < 1$ and $\{i \in N : \alpha^f(Q') < d(P(Q'_i)) < \beta^f(Q')\} = \emptyset$.

Proof of Remark 4.3.5. Using Lemma 4.2.5, Remark 4.2.6 and Pareto optimality, it follows that there exists a non-trivial disjoint subsets T_1 and T_2 of N such that $S \cup T_1 \cup T_2 = N$ and

$$Q_i = \begin{cases} Q^1_{\alpha^f(Q)}(P^0_i) & \text{if } i \in T_1 \\ Q_i & \text{if } i \in S \\ Q^2_{\beta^f(Q)}(P^1_i) & \text{if } i \in T_2 \end{cases}$$

Now consider an $i \in S$. It follows that his top ranked alternative can be either $(1,1)$ or $(0,0)$. Suppose his top ranked alternative is $(1,1)$. Now consider the $i-$deviation Q^1 of Q, where $Q^1_i = Q^1_{\alpha^f(Q)}(P^0_i)$. Strategy-proofness for the deviation from Q to Q^1 implies that either $\alpha^f(Q^1) \in [\alpha^f(Q), \beta^f(Q)]$ or $\beta^f(Q^1) \in [\alpha^f(Q), \beta^f(Q)]$. As $0 < \alpha^f(Q) < \beta^f(Q) < 1$, so Lemma 4.3.2 implies that neither $\alpha^f(Q^1) \in \{0,1\}$ nor $\beta^f(Q^1) \in \{0,1\}$. So it follows that $0 < \alpha^f(Q^1) \leqslant \beta^f(Q^1) < 1$. Then Lemma 4.3.3 implies $0 < \alpha^f(Q^1) < \beta^f(Q^1) < 1$. If the top ranked alternative of agent i was $(0,0)$, then consider the $i-$deviation Q^1 of Q, where $Q^1_i = Q^2_{\beta^f(Q)}(P^1_i)$. Then using the same argument we have $0 < \alpha^f(Q^1) < \beta^f(Q^1) < 1$. Define $S^1 = \{i \in N : \alpha^f(Q^1) < d(P(Q^1_i)) < \beta^f(Q^1)\}$. Note that $S^1 \subsetneq S$. Now if $S^1 = \emptyset$, then this concludes the proof of the remark. Otherwise pick an agent $j \in S^1$ and repeat the procedure above and construct S^2. As $|S| < n$, so there exists a finite $k \in \mathbb{N}$ such that $S^k = \emptyset$. This concludes the proof of of this remark.

Now we prove Lemma 4.3.4. To the contrary, suppose $(\alpha^f(Q), \beta^f(Q)) \in (0,1) \times (\alpha^f(Q), 1)$. Using Lemma 4.2.5, Remark 4.2.6 and Pareto optimality, it follows that there exists a non-trivial subset S of N such that

$$Q_i = \begin{cases} Q^1_{\alpha^f(Q)}(P^0_i) & \text{if } i \in S \\ Q^2_{\beta^f(Q)}(P^1_i) & \text{if } i \in N \backslash S \end{cases}$$

Now consider two profiles Q^1 and Q^2 as follows.

1. $Q^1 = (Q_S, \{Q^{LexMin}(P_i^1)\}_{N\setminus S})$.
2. $Q^2 = (Q^{LexMin}(P_i^0)_S, \{Q^{LexMin}(P_i^1)\}_{N\setminus S})$.

Note that for the deviation from Q to Q^1, strategy-proofness implies that either $f(Q) = f(Q^1)$ or $\alpha^f(Q) < \alpha^f(Q^1) \leqslant \beta^f(Q^1) < \beta^f(Q)$. Then Lemma 4.3.3 implies that $\alpha^f(Q) < \alpha^f(Q^1) < \beta^f(Q^1) < \beta^f(Q)$. Then for the deviation from Q^1 to Q^2 strategy-proofness implies either $f(Q^1) = f(Q^2)$ or $\alpha^f(Q^1) < \alpha^f(Q^2) \leqslant \beta^f(Q^2) < \beta^f(Q^1)$. Then Lemma 4.3.3 implies that $\alpha^f(Q^1) < \alpha^f(Q^2) < \beta^f(Q^2) < \beta^f(Q^1)$. So we can conclude that $0 < \alpha^f(Q^2) < \beta^f(Q^2) < 1$. But this violates Pareto optimality, as $(\mu(Q^2), \mu(Q^2))Q_i^2 f(Q^2)$ for any $0 < \alpha^f(Q^2) < \beta^f(Q^2) < 1$ and for all $i \in N$. This concludes the proof of Lemma 4.3.4. □

Proof of Proposition 4.3.1. Follows from Lemmas 4.3.2, 4.3.3 and 4.3.4. □

4.4 CHARACTERISATION

In this section, we characterise the class of social choice rules satisfying strategy-proofness and Pareto optimality. Due to Proposition 4.3.1, the range of any such rule is $\mathcal{B} = \{(1,1), (0,1), (0,0)\}$. We define $(1,1) = a,$, $(0,1) = b$ and $(0,0) = c$. Restricted to $\mathcal{B}$, we have the following two types of preferences.

Preferences	Extensions
$P(Q_i)$ such that $1P(Q_i)0$	aQ_ibQ_ic
$P(Q_i)$ such that $0P(Q_i)1$	cQ_ibQ_ia

This brings us to the following lemma.

Lemma 4.4.1. *Suppose* f *be a strategy-proof and Pareto optimal rule. Then* $f(Q) = f(Q')$ *for any* $Q, Q' \in \mathcal{Q}$ *such that* $Q|_{\mathcal{B}} = Q'|_{\mathcal{B}}$.

Proof. Follows from Proposition 4.3.1 and strategy-proofness. □

Next we show that a strategy-proof and Pareto optimal rule can never select b as an outcome.

Lemma 4.4.2. *If* f *is strategy-proof and Pareto optimal, then* $f(Q) \neq b$ *for any* $Q \in \mathcal{Q}$.

Proof. Suppose for some profile $Q \in \mathcal{Q}$, we have $f(Q) = b$. Then due to Pareto optimality there exists a non-trivial coalition S such that aQ_ibQ_ic for all $i \in S$ and cQ_ibQ_ia for all $i \in N\setminus S$. Now consider another profile Q', where $Q' := ((Q^{LexMin}(P_i^0))_{i\in S}, (Q^{LexMin}(P_i^1))_{i\in N\setminus S})$. Note that $Q|_{\mathcal{B}} = Q'|_{\mathcal{B}}$. So Lemma4.4.1 implies that $f(Q') = b$, but this violates Pareto optimality as $(\frac{1}{2}, \frac{1}{2})Q_i'b$ for all $i \in N$. □

This brings us to the following theorem.

Theorem 4.4.3. *If* f *is a strategy-proof and Pareto optimal social choice rule, then* $f(Q) \in \{(0,0),(1,1)\}$ *for all* $Q \in \mathcal{Q}$

Proof. Follows from Proposition 4.3.1 and Lemma 4.4.2. □

As we did not assume anonymity, we will characterise the class in terms of collections of coalitions. In view of Corollary 4.4.3, define $\mathcal{W}_a \subset 2^N$ as a family of coalitions decisive for a if it satisfies the following properties

Monotonicity : If $S \in \mathcal{W}_a$, then $S' \in \mathcal{W}_a$, where $S \subseteq S'$.

Boundary Conditions : $N \in \mathcal{W}_a$ and $\emptyset \notin \mathcal{W}_a$.

Based on such a family of coalitions $\mathcal{W}_a$, we define a rule $g^{\mathcal{W}_a} : \mathcal{Q} \longrightarrow \mathcal{A}$ as follows. For every profile $Q \in \mathcal{Q}$ define

$$S(Q) = \{i \in N : aQ_ibQ_ic\}.$$

$$g^{\mathcal{W}_a}(Q) = \begin{cases} a & \text{if } S(Q) \in \mathcal{W}_a \\ c & \text{otherwise} \end{cases}$$

Remark 4.4.4. Consider two profiles $Q, Q' \in \mathcal{Q}$ such that $S(Q) = S(Q')$. Then it follows that $g^{\mathcal{W}_a}(Q) = g^{\mathcal{W}_a}(Q')$.

This brings us to the following theorem.

Theorem 4.4.5. *Suppose* $f : \mathcal{Q} \longrightarrow \mathcal{A}$ *be a social choice rule. Then* f *is strategy-proof and Pareto optimal if and only if there is a family of coalitions* $\mathcal{W}_a$ *decisive for* a*, such that for all* $Q \in \mathcal{Q}$, $f(Q) = g^{\mathcal{W}_a}(Q)$.

We prove Theorem 4.4.5 using the following two lemmas. The first lemma shows the only if direction. For any arbitrary rule f, define

$$\mathcal{W}_a(f) = \{S \in 2^N : \text{ For some } Q \in \mathcal{Q}, S = S(Q) \text{ and } f(Q) = a\}.$$

Lemma 4.4.6. *Suppose* f *is a strategy-proof and Pareto optimal rule. Then* $\mathcal{W}_a(f)$ *is decisive for* a.

Proof. Consider two profiles $Q, Q' \in \mathcal{Q}$ such that $S(Q) = S(Q')$. Then Lemma 4.4.1 implies that $f(Q) = f(Q')$. So it follows that $\mathcal{W}_a(f)$ is a well defined set. Next we show that $\mathcal{W}_a(f)$ satisfies monotonicity.

Proof of Monotonicity. Consider two profiles $Q, Q' \in \mathcal{Q}$ such that $S(Q) \subseteq S(Q')$. Let $S(Q) \in \mathcal{W}_a(f)$. It is sufficient to prove $S(Q') \in \mathcal{W}_a(f)$. As $S(Q) \in \mathcal{W}_a(f)$, so $f(Q) = a$. Suppose $S(Q) = S(Q')$. Then there is a profile $\hat{Q} \in \mathcal{Q}$ such that $S(\hat{Q}) = S(Q)$ and $f(\hat{Q}) = a$. So $S(Q') \in \mathcal{W}_a(f)$. Now suppose $S(Q) \subsetneq S(Q')$. This implies that there exists atleast one agent i such that $i \notin S(Q)$ but $i \in S(Q')$. As $i \notin S(Q)$, so we have cQ_ibQ_ia. As f is strategy-proof, we have $f(Q') = a$. So we can conclude that $S(Q') \in \mathcal{W}_a(f)$.

Next we show that $\mathcal{W}_a(f)$ satisfies boundary conditions.

Proof of Boundary Conditions. Follows from Pareto optimality.

This concludes the proof of $\mathcal{W}_a(f)$ is decisive for a □

The next lemma shows the if direction of Theorem 4.4.5.

Lemma 4.4.7. *Suppose $\mathcal{W}_a \subset 2^N$ be a family of coalitions decisive for a. Then $g^{\mathcal{W}_a}$ is strategy-proof and Pareto optimal.*

Proof. First note that, $g^{\mathcal{W}_a}$ is a well defined function.

Proof of strategy-proofness. Consider a profile $Q \in \mathcal{Q}$ and an agent i such that $i \in S(Q)$. Consider an i-deviation Q' of Q. To show $g^{\mathcal{W}_a}(Q) Q_i g^{\mathcal{W}_a}(Q')$. With out loss of generality assume that $g^{\mathcal{W}_a}(Q) \neq g^{\mathcal{W}_a}(Q')$. Using Remark 4.4.4, it follows that $i \in N \backslash S(Q')$. If $g^{\mathcal{W}_a}(Q) = a$, then it follows that $g^{\mathcal{W}_a}(Q) Q_i g^{\mathcal{W}_a}(Q')$. So suppose $g^{\mathcal{W}_a}(Q) = c$. In this case, note that $S(Q) \notin \mathcal{W}_a$. As $S(Q') \subsetneq S(Q)$, monotonicity of $\mathcal{W}_a$ implies that $S(Q') \notin \mathcal{W}_a$ and $g^{\mathcal{W}_a}(Q') = c$, which violates our assumption that $g^{\mathcal{W}_a}(Q) \neq g^{\mathcal{W}_a}(Q')$. This concludes the proof of strategy-proofness.

Proof of Pareto optimality. Suppose for contradiction there exists a profile $Q \in \mathcal{Q}$ and some $x \in \mathcal{A}$ such that for all $i \in N$, $x Q_i g^{\mathcal{W}_a}(Q)$. Now we consider the following cases.

$a Q_i b Q_i c$ for all $i \in N$
: In this case boundary condition of $\mathcal{W}_a$ implies that $g^{\mathcal{W}_a}(Q) = a$, which contradicts the existence of a $x \in \mathcal{A}$ such that $x Q_i g^{\mathcal{W}_a}(Q)$ holds for all $i \in N$.

$c Q_i b Q_i a$ for all $i \in N$
: In this case boundary condition of $\mathcal{W}_a$ implies that $g^{\mathcal{W}_a}(Q) = c$, which contradicts the existence of a $x \in \mathcal{A}$ such that $x Q_i g^{\mathcal{W}_a}(Q)$ holds for all $i \in N$.

$a Q_i b Q_i c$ for all $i \in S$ and $c Q_i b Q_i a$ for all $i \in N \backslash S$ for some non trivial subset S of N.
: In this case note that a is the top ranked alternative for all $i \in S$ and c is the top ranked alternative for all $i \in N \backslash S$. As $g^{\mathcal{W}_a}(Q) \in \{a, c\}$ for all $Q \in \mathcal{Q}$, there does not exists a $x \in \mathcal{A}$ such that $x Q_i g^{\mathcal{W}_a}(Q)$ for all $i \in N$.

This concludes the proof of Pareto optimality.

This concludes the proof of Lemma 4.4.7 □

Proof of Theorem 4.4.5. If $\mathcal{W}$ is decisive, then $g^{\mathcal{W}_a}$ satisfies strategy-proofness and Pareto optimality by Lemma 4.4.7. Conversely, let f satisfy these two conditions. We show that $g = g^{\mathcal{W}_a(f)}$, which completes the proof by Lemma 4.4.6. Let $Q \in \mathcal{Q}$. Then for any $a_1 \in \mathcal{B} \backslash \{b\}$, we have $f(Q) = a_1 \Leftrightarrow S(z) \in \mathcal{W}_a(f) \Leftrightarrow g_{\mathcal{W}_a(f)}(z) = a$. □

4.5 CONCLUSION

Theorem 4.4.5 characterises the class of rules in our model. Note that in the proof of Theorem 4.4.5, we mainly used lexmin extensions. So it follows that if we consider only lexmin extensions of strict symmetric single-dipped preferences as the whole domain, then Theorem 4.4.5 still holds. Similarly it can be seen that allowing for indifferences would also imply Theorem 4.4.5. Now suppose we consider a weakening of Pareto optimality, namely unanimity, which says that if at a given profile all the agents have the same top ranked alternative, then the rule selects that alternative. Then we can show that Theorem 4.4.5 does not hold any more. In our domain of preferences, strategy-proofness together with unanimity does not imply Pareto optimality. We show this by means of the following example.

Example 4.5.1. Consider a rule f as follows.

$$f(Q) = \begin{cases} a & \text{if } |S(Q)| \geqslant \frac{3}{4}n \\ c & \text{if } |N \setminus S(Q)| \geqslant \frac{3}{5}n \\ b & \text{otherwise} \end{cases}$$

First we show that this rule is strategy-proof. Consider an agent i and a profile $Q \in \mathcal{Q}$ such that $i \in S(Q)$. Consider any $i-$deviation Q' of Q. Without loss of generality assume that $f(Q) \neq f(Q')$. To show that $f(Q)Q_i f(Q')$. Note that $i \notin S(Q')$. Now if $f(Q) = a$, then it follows that $f(Q)Q_i f(Q')$. So, consider the following cases.

$f(Q) = c$:

In this case $|N \setminus S(Q)| \geqslant \frac{3}{5}n$. As $i \notin S(Q')$, so it follows that $|N \setminus S(Q')| > |N \setminus S(Q)| \geqslant \frac{3}{5}n$. So $f(Q') = c$, which violates our assumption that $f(Q) \neq f(Q')$.

$f(Q) = b$:

In this case $|S(Q)| < \frac{3}{4}n$. As $i \in S(Q)$ but $i \notin S(Q')$, we have $|S(Q')| < |S(Q)| < \frac{3}{4}n$. So $f(Q') \neq a$. As $f(Q) \neq f(Q')$, so $f(Q') = c$ and we can conclude that $f(Q)Q_i f(Q')$.

This concludes the proof of strategy-proofness. Note that for any preference in our domain, the top ranked alternative can be either a or c. So it follows that the rule f is unanimous. But f is not Pareto optimal. Consider a situation where n is even and a profile $Q' := ((Q^{LexMin}(P_i^0))_{i \in S}, (Q^{LexMin}(P_i^1))_{i \in N \setminus S}) \in \mathcal{Q}$, with $|S| = \frac{n}{2}$. Note that $f(Q') = b$, but this violates Pareto optimality as $(\frac{1}{2}, \frac{1}{2})Q'_i b$ for all $i \in N$.

Also weakening Pareto optimality to unanimity may allow for inner solutions as well. We illustrate this fact with the help of the following example.

Example 4.5.2. Here we consider only lexmin extensions of symmetric single-dipped preferences with indifferences as the whole domain. In other words a preference $R_{x(i)}$ of agent $i \in N$ is a lexmin preference over $\mathcal{A}$ with dip at $x(i) \in A$ if for all $(a_1, b_1), (a_2, b_2) \in \mathcal{A}$, (a_1, b_1) is at least as good as (a_2, b_2) at $R_{x(i)}$, with the usual notation $(a_1, b_1)R_{x(i)}(a_2, b_2)$, if

$$\min\{|a_1 - x(i)|, |b_1 - x(i)|\} > \min\{|a_2 - x(i)|, |b_2 - x(i)|\}, \text{ or}$$

$$\min\{|a_1 - x(i)|, |b_1 - x(i)|\} = \min\{|a_2 - x(i)|, |b_2 - x(i)|\} \text{ and}$$
$$\max\{|a_1 - x(i)|, |b_1 - x(i)|\} \geqslant \max\{|a_2 - x(i)|, |b_2 - x(i)|\}.$$

Also we assume that $N = \{i, j\}$. For any alternative $(\alpha, \beta) \in \mathcal{A} \cap \{(0,1) \times (0,1)\}$, we define a rule $h^{(\alpha,\beta)} : A^n \longrightarrow \mathcal{A}$ as follows. For any profile $X = (x(i), x(j))$, suppose $y_1 = \min_{i \in N} x(i)$ and $y_2 = \max_{i \in N} x(i)$. The rule is defined as follows.

$$h^{(\alpha,\beta)}(X) = \begin{cases} (1,1) & \text{if } x(i) \leqslant \frac{1}{2} \text{ and } x(j) \leqslant \frac{1}{2} \\ (0,0) & \text{if } x(i)| > \frac{1}{2} \text{ and } x(j) > \frac{1}{2} \\ & \quad \text{or } x(i)| = \frac{1}{2} \text{ and } x(j) > \frac{1}{2} \\ & \quad \text{or } x(i)| > \frac{1}{2} \text{ and } x(j) = \frac{1}{2} \\ (\alpha, \beta) & \text{if } 2y_1 < \alpha \leqslant \beta < 2y_2 - 1 \\ (0,1) & \text{otherwise} \end{cases}$$

Note that this rule is also unanimous. Next, we show that this rule satisfies strategy-proofness. Without loss of generality consider a profile X such that $x(i) < \frac{1}{2}$ and $y_1 = x_i$ and $y_2 = x_j$. Consider an $i-$deviation X' and assume that $h^{(\alpha,\beta)}(X) \neq h^{(\alpha,\beta)}(X')$. To show $h^{(\alpha,\beta)}(X)R_{x(i)}h^{(\alpha,\beta)}(X')$. As $x(i) < \frac{1}{2}$, so $(1,1)$ is the top ranked alternative of $R_{x(i)}$. So, if $h^{(\alpha,\beta)}(X) = (1,1)$, then it follows that $h^{(\alpha,\beta)}(X)R_{x(i)}h^{(\alpha,\beta)}(X')$. Also we can conclude that $h^{(\alpha,\beta)}(X) \neq (0,0)$ and $x(j) > \frac{1}{2}$. As X' is an $i-$deviation of X, it follows that $h^{(\alpha,\beta)}(X') \neq (1,1)$. Now we consider the following cases.

$h^{(\alpha,\beta)}(X) = (\alpha, \beta)$:
In this case, we have $2y_1 < \alpha \leqslant \beta < 2y_2 - 1$. This implies $|\alpha - x(i)| > |0 - x(i)|$. So we can conclude that $(\alpha, \beta)R_{x(i)}(0,0)$ and $(\alpha, \beta)R_{x(i)}(0,1)$. So in this case we can conclude that $h^{(\alpha,\beta)}(X)R_{x(i)}h^{(\alpha,\beta)}(X')$.

$h^{(\alpha,\beta)}(X) = (0,1)$:
Note that if $h^{(\alpha,\beta)}(X') = (0,0)$ then we have $(0,1) = h^{(\alpha,\beta)}(X)R_{x(i)}h^{(\alpha,\beta)}(X') = (0,0)$. So we may assume that $x(i) < \frac{1}{2}$. Now we consider the following sub cases.

$\alpha \leqslant 2y_1 = 2x(i)$: In this situation we have $(0,1)R_{x(i)}(\alpha, \beta)$.

$\alpha > 2y_1 = 2x(i)$: In this situation, as $h^{(\alpha,\beta)}(X) = (0,1)$, so we have $2y_2 - 1 = 2x(j) - 1 \leqslant \beta$. Then as $x(j) > \frac{1}{2}$, so from the definition of $h^{(\alpha,\beta)}$ it follows that agent i cannot change the outcome to (α, β) by unilateral deviation.

Also note that if $x(i) = \frac{1}{2}$, then $h^{(\alpha,\beta)}(X) \in \{(1,1),(0,0)\}$ which are his top ranked alternatives. So we can conclude that this rule is strategy-proof. Note that as in the previous example, this rule is not Pareto optimal.

Part III

ONE PUBLIC FACILITY IN A REGION

5 MIXED PREFERENCES

5.1 INTRODUCTION

In this part, we consider the problem of locating a facility in a region, which provides positive externalities for some agents and negative externalities for the remaining agents; i.e; some agents have single-peaked preferences, and the remaining agents have single-dipped preferences. For example, consider a situation where a private car manufacturer approaches a state government for a location to build a factory. Now the government has to decide where to place the factory within its territory by taking into account the preferences of the residents. A farmer in this state would not want the factory to be located in his farmland. More precisely, because of possible pollution effects of the factory, he would want it to be as far away from his farm land as possible. On the other hand, a labourer in this state would be in favour of this facility as it creates new job opportunities. He also might want it to be as close to his residence as possible as it might shorten his commuting time. In this example, the farmer is a single-dipped agent and the labourer is a single-peaked agent. In this chapter, we model the region by a finite subset $\mathcal{A}$ of the set of natural numbers $\mathbb{N}$. We assume that there are two agents i and j, where the designer knows that agent i is the agent with strict single-peaked preference and agent j is the agent with strict single-dipped preference; but does not know where their peak and dip are.

In this situation, a rule would select a point from $\mathcal{A}$ as the location of the facility for every profile of reported preferences. We characterise the class of rules that satisfies strategy-proofness and Pareto optimality. Strategy proofness ensures that for every agent, truth telling is a weakly dominant strategy. Pareto optimality ensures that given a decision about the location of the facility, improving one agent would result in worsening the other agent. The class we characterise here consists of almost dictatorial rules.

Suppose the designer has no information regarding whether an agent is a single-peaked or single-dipped type and he also has no information on the best or worst location of the agents. In this case, Condorcet cycles may exist in the profile, which is a strong indication of dictatorship. From here, we can partition this problem in two ways. In the first method, we can assume that the designer knows the best or worst location for each agent, but he does not know which agents are single-peaked and which agents are single-dipped. In this situation Vorsatz and Alcalde-Unzu (2015) have shown that there exists non-dictatorial rules. In the second method, we may assume that the designer knows which agents are single-dipped and which agents are single-peaked, but he does not know their locations. We discuss this method in this chapter with the restriction of two agents.

This chapter is organised as follows. Section 5.2 introduces the model and preferences. Section 5.3 shows a restriction in the range of a social choice rule under strategy-proofness and Pareto optimality. Section 5.4 provides a characterisation and section 5.5 concludes.

5.2 MODEL

Let $\mathcal{A} = \{a_1, a_2, \ldots, a_m\} \subset \mathbb{N}$ be the set of outcomes (possible locations of a public facility) with cardinality $m \geqslant 3$. Also assume that $a_1 < a_2 < \ldots < a_m$. Let $N = \{i, j\}$ be the set of agents. Agent $i \in N$ has a strict single-peaked preference R_i with peak $p(R_i) \in \mathcal{A}$ over the elements of A as follows. For any two distinct alternatives $a, b \in \mathcal{A}$, if either $b < a \leqslant p(R_i)$ or $p(R_i) \leqslant a < b$, then we have aR_ib. Agent $j \in N$ has a strict single-dipped preference R_j with dip $d(R_j) \in A$ over the elements of A as follows. For any two distinct alternatives $a, b \in A$, if either $a < b \leqslant d(R_j)$ or $d(R_j) \leqslant b < a$, then we have aP_jb. Suppose $\mathcal{P}^{Peak}$ denotes the set of all strict single-peaked preferences over $\mathcal{A}$ and $\mathcal{P}^{Dip}$ denotes the set of all strict single-dipped preferences over $\mathcal{A}$. A preference profile R assigns to agent i a strict single-peaked preference $R_i \in \mathcal{P}^{Peak}$ and to agent j a strict single-dipped preference $R_j \in \mathcal{P}^{Dip}$. Formally $R \in \mathcal{P}^{Peak} \times \mathcal{P}^{Dip} = \mathcal{P}$. For agent $k \in N$, profile R' is an k-deviation of R if $R_{N\setminus\{k\}} = R'_{N\setminus\{k\}}$. For an agent $k \in N$ with preference R_k and an outcome $a \in \mathcal{A}$ define $L(a, R_k) = \{b \in \mathcal{A} : aR_kb\}$ as the strict lower contour set of R_k at a. A rule f assigns to every profile R, an outcome $f(R) \in \mathcal{A}$. For any $k \in N$ and any profile R and any $l \in \{1, 2, \ldots, m\}$, $R_k(l)$ denotes the l^{th} ranked alternative according to R_k.

We consider the following properties for a rule f.

Strategy-Proofness f is strategy-proof if for any $k \in N$ and any $R \in \mathcal{P}$ and any k-deviation R' of R, we have either $f(R') = f(R)$ or $f(R)R_kf(R')$.

Strategy-proofness says that truth-telling is a weakly dominant strategy.

Pareto optimality Rule f is Pareto optimal if for every profile R there does not exist an $a \in \mathcal{A}$ such that $aR_kf(R)$ for all $k \in N$.

Maskin Monotonicity Rule f is Maskin monotone if $f(R) = f(R')$ for all $R, R' \in \mathcal{P}$ such that $L(f(R), R_k) \subseteq L(f(R), R'_k)$ for all $k \in N$.

Remark 5.2.1. As we consider only strict preferences, strategy-proofness implies Maskin monotonicity.

5.3 RANGE RESTRICTION

In this section, we consider two maximal conflict profiles $R^1 = (R_i^1, R_j^1)$ and $R^m = (R_i^m, R_j^m)$, where $p(R_i^1) = d(R_j^1) = a_1$ and $p(R_i^m) = d(R_j^m) = a_m$. Note that for any preference R_i in the single peaked domain $\mathcal{P}^{Peak}$, if a_2 is the second ranked alternative ($R_i(2) = a_2$), then $p(R_i) = a_1$. Similar condition holds for a_m and a_{m-1}. Kalai and Ritz (1978) has defined the pairs a_1, a_2 and a_m, a_{m-1} as inseparable top pairs in $\mathcal{P}^{Peak}$. We show that under strategy-proofness and Pareto optimality $f(R^1) \in \{a_1, a_2, a_m\}$ and $f(R^m) \in \{a_1, a_{m-1}, a_m\}$.

Lemma 5.3.1. *Suppose* f *is a strategy-proof and Pareto optimal rule. Then* $f(R^1) \in \{a_1, a_2, a_m\}$ *and* $f(R^m) \in \{a_1, a_{m-1}, a_m\}$.

Proof. Suppose $f(R^1) = a_l$ for some $2 < l < m$. Now consider the following profiles.

R_i^1	R_i'	$R_i^\star$	R_j^1	R_j'	$R_j^\star$
a_1	a_2	a_2	a_m	a_m	a_m
a_2	a_3	a_1	a_{m-1}	a_{m-1}	a_{m-1}
.	.	.	.	.	.
.	.	.	.	.	.
a_{l-2}	a_{l-1}	a_{l-2}	.	.	.
a_{l-1}	a_l	a_{l-1}	.	.	.
a_l	a_{l+1}	a_l	.	.	.
a_{l+1}	a_{l+2}	a_{l+1}	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	a_{l+1}	a_{l+1}	a_{l+1}
.	.	.	a_l	a_1	a_1
.	.	.	a_{l-1}	a_l	a_2
.	.	.	a_{l-2}	a_{l-1}	a_3
.	.	.	.	.	.
.	.	.	.	.	.
a_{m-1}	a_m	a_{m-1}	a_2	a_3	a_{l-1}
a_m	a_1	a_m	a_1	a_2	a_l

On the one hand, as $f(R^1) = a_l$, so Maskin monotonicity implies that $f(R_i^\star, R_j^1) = a_l$. Now for the deviation from $(R_i^\star, R_j^1)$ to $(R_i^\star, R_j')$, strategy-proofness implies that $f(R_i^\star, R_j') \in \{a_1, a_l\}$. As a_l is Pareto dominated by a_1 at the profile $(R_i^\star, R_j')$, Pareto optimality implies that $f(R_i^\star, R_j') = a_1$. Then Maskin monotonicity implies that $f(R_i^\star, R_j^\star) = a_1$.

On the other hand, as $f(R^1) = a_l$, so Maskin monotonicity implies that $f(R'_i, R^1_j) = a_l$. Now for the deviation from (R'_i, R^1_j) to (R'_i, R'_j), strategy-proofness implies that $f(R'_i, R'_j) \in \{a_1, a_l\}$. As a_1 is Pareto dominated by a_{l+1} at the profile (R'_i, R'_j), Pareto optimality implies that $f(R'_i, R'_j) = a_l$. Then for the deviation from (R'_i, R'_j) to $(R'_i, R^\star_j)$, strategy-proofness implies that $f(R'_i, R^\star_j) \in \{a_2, a_3, \dots, a_l\}$. As a_{l_1} is Pareto dominated by a_2 for every $l_1 \in \{3, 4, \dots, l\}$ at the profile $(R'_i, R^\star_j)$, Pareto optimality implies that $f(R'_i, R^\star_j) = a_2$. Then Maskin monotonicity implies that $f(R^\star_i, R^\star_j) = a_2$, which contradicts the fact that $f(R^\star_i, R^\star_j) = a_1$ and shows that $f(R^1) \in \{a_1, a_2, a_m\}$.

Similarly we can show that $f(R^m) \in \{a_1, a_{m-1}, a_m\}$. □

5.4 CHARACTERISATION

In this section we provide a characterisation on the basis of Lemma 5.3.1. From here onwards, we shall assume that f is a strategy-proof and Pareto optimal social choice function. The following lemma shows that if $f(R^1) = a_m$, then agent j is the dictator.

Lemma 5.4.1. *If* $f(R^1) = a_m$, *then* $f(R) = R_j(1)$ *for all* $R \in \mathcal{P}$.

Proof. Notice that for any $R \in \mathcal{P}$, $R_i(m) \in \{a_1, a_m\}$ and $R_j(1) \in \{a_1, a_m\}$. So we consider the following partition of $\mathcal{P}$. For any $k, l \in \{1, m\}$, define

$$\mathcal{P}_{k,l} = \{P \in \mathcal{P} : P_i(m) = a_k P_j(1) = a_l\}.$$

In other words, $\mathcal{P}_{k,l}$ denotes the collection of all those profiles where the worst ranked alternative of agent i is a_k and the best ranked alternative of agent j is a_l. Note that $\{\mathcal{P}_{k,l}\}_{k,l \in \{1,m\}}$ forms a partition of $\mathcal{P}$. We are going to show that under the assumption of $f(R^1) = a_m$, $f(R) = R_j(1)$ for all $R \in \mathcal{P}_{l,k}$ for any $l, k \in \{1, m\}$. So consider the following cases.

$R \in \mathcal{P}_{1,m} \cup \mathcal{P}_{m,m}$: Note that $R^1 \in \mathcal{P}_{m,m}$, and $f(R^1) = a_m$. For any $R \in \mathcal{P}_{1,m} \cup \mathcal{P}_{m,m}$, we have $L(a_m, R^1_j) = L(a_m, R_j)$ and $L(a_m, R^1_i) = \emptyset$. So in this case, Maskin monotonicity implies that $f(R) = a_m$ for all $R \in \mathcal{P}_{1,m} \cup \mathcal{P}_{m,m}$.

$R \in \mathcal{P}_{m,1}$: Consider $R' \in \mathcal{P}_{m,1}$ such that $R'_i(m-1) = a_1$ and $R'_j(2) = a_m$. As $(R'_i, R^1_j) \in \mathcal{P}_{m,m}$, $f(R'_i, R^1_j) = a_m$. Now consider the deviation from (R'_i, R^1_j) to R'. Strategy-proofness for this deviation implies that $f(R') \in \{a_1, a_m\}$. At the profile R', a_m is Pareto dominated by a_1. So it follows that $f(R') = a_1$. Now note that for all $R \in \mathcal{P}_{m,1}$, we have $L(a_1, R'_j) = L(a_1, R_j)$ and $L(a_1, R'_i) \subseteq L(a_1, R_i)$. So, Maskin monotonicity implies that $f(R) = a_1$ for all $R \in \mathcal{P}_{m,1}$.

$R \in \mathcal{P}_{1,1}$: Consider a profile $R^\star \in \mathcal{P}_{1,1}$ such that $R^\star_i(m-1) = a_m$ and $R^\star_j = R'_j$, where R'_j is as defined before. As $(R^\star_i, R^1_j) \in \mathcal{P}_{1,m}$. So $f(R^\star_i, R^1_j) = a_m$.

Now consider the deviation from $(R_i^\star, R_j^1)$ to $R^\star$. Strategy-proofness for this deviation implies that $f(R^\star) \in \{a_1, a_m\}$. Suppose for contradiction, $f(R^\star) = a_m$. Now consider the following preferences.

$R_i^\star$	$\overline{R}_i$	$R_j^\star$	$\overline{R}_j$	R_j^1
.	.	a_1	a_1	a_m
.	.	a_m	a_2	a_{m-1}
.	.	.	a_m	.
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
a_m	a_1	.	.	a_2
a_1	a_m	.	.	a_1

As $(\overline{R}_i, \overline{R}_j) \in \mathcal{P}_{m,1}$, $f(\overline{R}_i, \overline{R}_j) = a_1$. Now consider the deviation from $R^\star$ to $(R_i^\star, \overline{R}_j)$. As $f(R^\star) = a_m$, strategy-proofness implies that $f(R_i^\star, \overline{R}_j) \in \{a_2, a_m\}$. At the profile $(R_i^\star, \overline{R}_j)$ as $m \geqslant 3$, a_m is Pareto dominated by a_2. So it follows that $f(R_i^\star, \overline{R}_j) = a_2$. But $a_2 \overline{R}_i f(\overline{R}_i, \overline{R}_j) = a_1$, which is a violation of strategy-proofness. So, we can conclude that $f(R^\star) = a_1$. Then Maskin monotonicity implies that $f(R) = a_1$ for all $R \in \mathcal{P}_{1,1}$.

Combining these cases concludes the proof of Lemma 5.4.1. □

The next lemma shows that if $f(R^1) \in \{a_1, a_2\}$, then agent i is an almost dictator. More precisely, we are going to show that under the assumption of $f(R^1) \in \{a_1, a_2\}$, agent i is the dictator in all cases except when his peak is either at a_1 or at a_m. In these two cases we have either a_1, a_2 or a_{m-1}, a_m as inseparable top pairs. As shown by Kalai and Ritz (1978), in this situation non-dictatorial rules exist. Let us define $\mathcal{P}^\star = \{\mathcal{P}^{Peak} \backslash \{R_i^1, R_i^m\}\} \times \mathcal{P}^{Dip}$.

Lemma 5.4.2. *If* $f(R^1) \in \{a_1, a_2\}$, *then* $f(R) = p(R_i)$ *for all* $R \in \mathcal{P}^\star$.

Proof. First we partition $\mathcal{P}^\star$ as follows. For any $l \in \{2, 3, \ldots, m-1\}$, define

$$\mathcal{P}_l = \{R \in \mathcal{P}^\star : p(R_i) = a_l\}$$

Suppose $f(R^1) \in \{a_1, a_2\}$. Now, we prove the following proposition, which we will use to prove Lemma 5.4.2.

Proposition 5.4.3. *Suppose for any* $l \in \{2, 3, \ldots, m-1\}$ *and any* $R^\star \in \mathcal{P}_l$ *such that* $d(R_j^\star) = a_{l-1}$ *and* $R_j^\star(m-1) = a_l$, *we have* $f(R^\star) = a_l$. *Then* $f(R) = a_l$ *for any* $R \in \mathcal{P}_l$.

Proof of Proposition 5.4.3. Consider the following preferences which constitutes profiles in $\mathcal{P}_l$.

$\overline{R}_i$	$\overline{R}_j$	R'_j
a_l	.	.
a_{l+1}	.	.
a_{l-1}	.	.
.	.	.
.	.	.
.	.	.
.	a_l	a_{l-1}
.	a_{l-1}	a_l

Note that $(\overline{R}_i, \overline{R}_j) \in \mathcal{P}_l$ and $d(\overline{R}_j) = a_{l-1}$ and $\overline{R}_j(m-1) = a_l$. Then by assumption $f(\overline{R}_i, \overline{R}_j) = a_l$. Now consider the deviation from $(\overline{R}_i, \overline{R}_j)$ to $(\overline{R}_i, R'_j)$. Strategy-proofness for this deviation implies that $f(\overline{R}_i, R'_j) \in \{a_l, a_{l-1}\}$. But a_{l-1} is Pareto dominated at the profile $(\overline{R}_i, R'_j)$ by a_{l+1}. So $f(\overline{R}_i, R'_j) = a_l$. Now note that for any $R \in \mathcal{P}_l$, we have $L(a_l, \overline{R}_i) = L(a_l, R_i) = \mathcal{A} \backslash \{a_l\}$ and $\emptyset = L(a_l, R'_j) \subseteq L(a_l, R_j)$. So Maskin monotonicity implies that $f(R) = a_l$ for any $R \in \mathcal{P}_l$.

This concludes the proof of Proposition 5.4.3. Next using induction on l, we are going to show that for any $l \in \{2, 3, \ldots, m-1\}$, $f(R) = p(R_i)$ for all $R \in \mathcal{P}_l$.

BASE CASE : $l = 2$ - First we are going to show that $f(R) = a_2$ for all $R \in \mathcal{P}_2$ such that $d(R_j) = a_1$ and $R_j(m-1) = a_2$; i,e; $f(R) = a_2$ for all $R \in \mathcal{P}_2$ such that $R_j = R^1_j$. Consider $(R^2_i, R^1_j) \in \mathcal{P}_2$ such that $R^2_i(2) = a_1$. If $f(R^1) = a_2$, Maskin monotonicity implies $f(R^2_i, R^1_j) = a_2$. If $f(R^1) = a_1$, strategy-proofness implies $f(R^2_i, R^1_j) \in \{a_1, a_2\}$. At the profile (R^2_i, R^1_j), a_1 is Pareto dominated by a_2. So $f(R^2_i, R^1_j) = a_2$. Then Maskin monotonicity implies that $f(R) = a_2$ for all $R \in \mathcal{P}_2$ such that $R_j = R^1_j$. Then Proposition 5.4.3 implies that $f(R) = a_2$ for all $R \in \mathcal{P}_2$.

INDUCTIVE STEP - Suppose $f(R) = a_{k-1}$ for all $R \in \mathcal{P}_{k-1}$ for any $k-1 \in \{2, 3, \ldots, m-2\}$. First we are going to show that $f(R) = a_k$ for all $R \in \mathcal{P}_k$ such that $d(R_j) = a_{k-1}$ and $R_j(m-1) = a_k$. Now consider a profile $(R^k_i, R^\star_j) \in \mathcal{P}_k$ such that $R^k_i(2) = a_{k-1}$ and $d(R^\star_j) = a_{k-1}$ and $R^\star_j(m-1) = a_k$. Note that for any $(R_i, R^\star_j) \in \mathcal{P}_{k-1}$, we have $f(R_i, R^\star_j) = a_{k-1}$. Then for the deviation from $(R_i, R^\star_j)$ to $(R^k_i, R^\star_j)$, strategy-proofness implies that $f(R^k_i, R^\star_j) \in \{a_{k-1}, a_k\}$. At the profile $(R^k_i, R^\star_j)$, a_{k-1} is Pareto dominated by a_k. So $f(R^k_i, P^\star_j) = a_k$. Then using Maskin monotonicity, it follows that $f(R) = a_k$ for all $R \in \mathcal{P}_k$ such that $d(R_j) = a_{k-1}$ and $R_j(m-1) = a_k$. Then Proposition 5.4.3 implies that $f(R) = a_k$ for all $R \in \mathcal{P}_k$.

This concludes the proof of Lemma 5.4.2. □

The next lemma shows the situation where agent i has his peak at either a_1 or a_m.

Lemma 5.4.4. *If* $f(R^1) \in \{a_1, a_2\}$*, then* $f(R) = a_1$ *for all* $R \in \{R_i^1\} \times \{P^{Dip} \backslash \{R_j^1\}\}$*. Also if* $f(R^m) \in \{a_m, a_{m-1}\}$*, then* $f(R) = a_m$ *for all* $R \in \{R_i^m\} \times \{P^{Dip} \backslash \{R_j^m\}\}$*.*

Proof. It is sufficient to show that if $f(R^1) \in \{a_1, a_2\}$, then $f(R) = a_1$ for all $R \in \{R_i^1\} \times \{P^{Dip} \backslash \{R_j^1\}\}$. Now if $f(R^1) = a_1$, then Maskin monotonicity directly implies the result. So suppose that $f(R^1) = a_2$. Now consider the profile (R_i^1, R_j^2), where $d(R_j^2) = a_2$ and $R_j^2(m-1) = a_1$. Note that for the deviation from R^1 to (R_i^1, R_j^2), strategy-proofness implies that $f(R_i^1, R_j^2) \in \{a_1, a_2\}$. At the profile (R_i^1, R_j^2), a_2 is Pareto dominated by a_1. So we have $f(R_i^1, R_j^2) = a_1$. Note that $a_2 \in L(a_1, R_j)$ for any $R_j \in \mathcal{P}^{Dip} \backslash \{R_j^1\}$. So Maskin monotonicity implies that $f(R) = a_1$ for all $R \in \{R_i^1\} \times \{P^{Dip} \backslash \{R_j^1\}\}$. □

Now we define the following rules as follows.

$$g_1(R) = \begin{cases} p(R_i) & \text{if } p(R_i) \neq a_1 \\ a_1 & \text{if } p(R_i) = a_1 \text{ and } a_1 R_j a_2 \\ a_2 & \text{otherwise} \end{cases}$$

$$g_2(R) = \begin{cases} p(R_i) & \text{if } p(R_i) \neq a_m \\ a_m & \text{if } p(R_i) = a_m \text{ and } a_m R_j a_{m-1} \\ a_{m-1} & \text{otherwise} \end{cases}$$

$$g_3(R) = \begin{cases} p(R_i) & \text{if } p(R_i) \notin \{a_1, a_m\} \\ a_1 & \text{if } p(R_i) = a_1 \text{ and } a_1 R_j a_2 \\ a_m & \text{if } p(R_i) = a_m \text{ and } a_m R_j a_{m-1} \\ a_2 & \text{if } p(R_i) = a_1 \text{ and } a_2 R_j a_1 \\ a_{m-1} & \text{otherwise} \end{cases}$$

$$g_4(R) = p(R_i) \text{ for all } R \in \mathcal{P}$$

$$g_5(R) = R_j(1) \text{ for all } R \in \mathcal{P}$$

The following corollary combines the previous lemmas.

Corollary 5.4.5. *Suppose* f *is a rule. Then* f *is strategy-proof and Pareto optimal if and only if* $f \in \{g_1, g_2, g_3, g_4, g_5\}$*.*

Proof. Follows directly from Lemmas 5.4.1 to 5.4.4. □

5.5 CONCLUSION

This chapter shows an almost impossibility result for two agents. For more than two agents, the class is still unknown. Note that any median voter rule with fixed ballots that completely ignores the single-dipped agents would be strategy-proof and Pareto optimal in this case. Strategy-proofness follows from the fact that these rules are strategy-proof for single-peaked agents and single-dipped agents cannot influence the outcome of these rule. Pareto optimality follows from the fact that the outcome of these rules are Pareto optimal for single-peaked agents. So for any given profile, any outcome other than the one selected by this rule would be strictly worse for atleast one single-peaked agent, regardless of whether such an outcome is strictly better for atleast one single-dipped agent. Also any non-constant monotone voting between the boundary points will also be strategy-proof and Pareto optimal. As the range of this rule is 2, strategy-proofness is immediate. For Pareto optimality, note that the boundary points are the top ranked alternatives of the single-dipped agents. So for any given profile, any outcome other than the one selected by this rule would be strictly worse for atleast one single-dipped agent, regardless of whether such an outcome is strictly better for atleast one single-peaked agent. Also we can extend the almost dictatorial rule g_1 to more than two agents case as follows. A single-peaked agent would be the dictator unless his peak is at a_1. In this case we use any non-constant monotonic voting rule between a_1 and a_2 among all the agents except the dictator. Similarly we can extend g_2 and g_3 to more than two agent scenarios.
Also note that, if we allow for indifferences, then we cannot use Maskin monotonicity. But as long as agents are allowed to have any preference between two alternatives that are on the opposite side of their peaks (or dips), Lemma 5.3.1 would still hold. Then the class of rules would be similar to the one described in section 5.4. The differences would appear in cases where the dictator has more than one top ranked alternatives. On the other hand, if we consider Euclidean preferences, then Lemma 5.3.1 does not hold. So other non-dictatorial rules may be possible in this situation.

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VALORISATION

Knowledge valorisation refers to the process of creating value from knowledge, by making knowledge suitable and/or available for social (and/or economic) use and by making knowledge suitable for translation into competitive products, services, processes and new commercial activities.

This thesis investigates the problem of locating two noxious facilities as a joint decision of a group of agents. In this thesis, the class of rules that satisfy certain properties has been characterised. One such property is strategy-proofness, which says that truth telling is a weakly dominant strategy. The objective of such studies is not to determine whether the imposed properties are "relevant in the real world, or not". Our objective is to find out the set of solutions of the given problem satisfying the properties. If a governing body finds these properties as "useful", then that entity can use our result.

For example, in recent news, nine countries in the North Sea region and the European Commission have decided to enhance their collaboration in order to better utilise the potential of the North Sea as an area for wind farms. Wind farms are generally regarded as public bads. These farms are elegant sources of green energy, essential to combat the increasing danger of global warming; but they provide negative externalities. A crucial problem in this scenario is to decide where to place these wind farms. In the first two chapters, we provide solutions to this problem while imposing four properties. The first one is strategy-proofness. The second one is country-wise Pareto optimality. This is the usual efficiency property that ensures sovereignty of each country to a large extent. The third property is non-corruptibility, which ensures that no one can gain by bribing some individuals. The last property we introduced is a tie breaking condition. Now if the European Commission assumes that these properties are essential in deciding the collaboration structure, then our work might be useful for them.

In the third chapter we provide strategy-proof and Pareto optimal solutions about how to locate two public bads in a region. Any municipality, which has to decide about locating two garbage dumping sites along a road might find our work useful.

The last chapter of this dissertation is motivated by an example from the state of West Bengal in India. An Indian car manufacturer approached the state government with a proposal for building a car factory. The government's decision on the location of the said factory was turned into a controversy and could be considered as a factor in the regime change that followed in the next state election. We model this scenario as locating a public facility, which is good for some agents and bad for others. Although some work still needs to be done in this chapter, we provide some partial solutions that satisfy strategy-proofness and Pareto optimality. These might be useful to the government, if they are looking for mechanisms that satisfy these two properties.

NEDERLANDSE SAMENVATTING

In de Social Keuze Theorie bestudeert men de vraag hoe een groep mensen met persoonlijke voorkeuren tot een gemeenschappelijke keuze kan komen. Een ten grondslag liggend resultaat van dit vakgebied is de onmogelijkheidsstelling van Gibbard (1973) en Satterthwaite (1975). Deze stelling zegt dat iedere collectieve keuzeregel, met minstens drie verschillende alternatieven, die Pareto optimaal en niet-manipuleerbaar is, een dictatoriale keuzeregel moet zijn. Een collectieve keuzeregel is Pareto optimaal indien er voor ieder mogelijke combinatie van persoonlijke voorkeuren een uitkomst wordt toegewezen zodanig dat er geen ander alternatief bestaat dat voor iedere persoon de voorkeur heeft. Een collectieve keuzeregel is manipuleerbaar indien een persoon de mogelijkheid heeft om een betere uitkomst te verkrijgen door te liegen over zijn persoonlijke voorkeuren. Een belangrijke onderliggende factor voor het resultaat is dat een persoon alle mogelijke voorkeuren kan hebben. Dit betekent dat indien de mogelijke voorkeuren beperkt zijn, er een kans bestaat dat er niet-dictatoriale collectieve keuzeregels zijn die voldoen aan de twee eerder genoemde eigenschappen. De verzameling van ééntoppige voorkeuren is een voorbeeld van een beperkt domein. Inada (1964) heeft laten zien dat er voor deze verzameling van voorkeuren collectieve keuzeregels bestaan die Pareto optimaal, niet-manipuleerbaar en niet-dictatoriaal zijn. Dit proefschrift onderzoekt een ander beperkt domein, namelijk de verzameling van ééndalige voorkeuren.

Ééndalige voorkeuren spelen een rol tijdens het plaatsen van een negatief publiek goed, zoals een nucleaire kernreactor of een vuilnisstortplaats. De voorkeuren worden gekenmerkt door een uniek punt (genaamd de dip) en de voorkeur voor een locatie neemt toe zodra de afstand tot dit punt toeneemt. De dip kan iemands achtertuin zijn, maar denk ook aan de school van je kinderen of een natuurgebied. In dit proefschrift staat het probleem van het toewijzen van twee negatieve publieke goederen centraal. Het doel is om extra eigenschappen te vinden zodanig dat een Pareto optimale en niet-manipuleerbare collectieve keuzeregel altijd een grenslocatie kiest. Ieder hoofdstuk geeft een beschrijving van deze regels.

Dit proefschrift bestaat uit drie delen. In het eerste gedeelte bekijken we de situatie van twee aangrenzende landen, gemodelleerd als twee aangrenzende lijnstukken, die beiden moeten beslissen over de locatie van een negatief publiek goed. Naast niet-manipuleerbaar, gebruiken we een iets andere versie van Pareto optimaal genaamd landsgewijs Pareto optimaal. Deze eigenschap zorgt ervoor dat ieder land, tot op zekere hoogte, zijn eigen soevereiniteit behoudt. We bekijken twee verschillende specificaties van ééndalige voorkeuren: kortzichtige en lexicografische. Kortzichtige personen vergelijken twee locatieparen aan de hand van de dichtstbijzijnde locatie. Hierdoor ontstaan er veel onverschilligheden. Om dit te verhelpen worden twee extra eigenschappen aangenomen: niet-corruptief en de ver-weg-conditie. Een collectieve keuzeregel is niet-corruptief indien geen persoon de uitkomst van de regel kan veranderen zonder een effect te hebben op zichzelf. De ver-weg-conditie verkiest grenslocaties boven andere locaties indien er onverschilligheden zijn. Voor kortzichtige personen beschrijven we ieder regel die aan deze vier eigenschappen voldoet. Deze klasse bestaat voornamelijk uit regels waarbij beide landen samen beslissen over de twee locaties van de negatieve publieke goederen.

Lexicografische personen vergelijken twee locatieparen aan de hand van de dichtstbijzijnde locatie en indien deze dezelfde zijn, wordt de tweede locatie gebruikt als beslissing. Voor lexicografische personen laten we zien dat als een regel landsgewijs Pareto optimaal en niet-manipuleerbaar is, dan wordt een grenslocatie gekozen. We zijn in staat om iedere regel die aan deze twee eigenschappen voldoet te beschrijven. Deze klasse bezit regels waarin beide landen samen als ook onafhankelijk kiezen. We tonen aan dat de klasse gerelateerd aan kortzichtige personen regels bevat die niet zitten in de klasse gerelateerd aan lexicografische personen, en vice versa.

In het tweede gedeelte bekijken we het probleem van het vinden van een locatie voor twee negatieve publieke goederen in een gesloten lijnstuk. Iedere agent heeft ééndalige marginale voorkeuren op basis van de afstand tot de dip. Voor personen is iedere complete, strikte, transitieve en scheidbare uitbreiding van deze marginale voorkeuren toegestaan. We laten zien dat iedere Pareto optimale en niet-manipulatieve collectieve keuzeregel grenslocaties kiest en bovendien dezelfde grenslocatie toekent voor beide publieke goederen. De klasse van regels die deze twee eigenschappen bezit bestaat uit niet-constante monotone kiesregels.

Het laatste gedeelte wijkt af van de eerdere gedeeltes. We bekijken het probleem van het vinden van een locatie voor één publiek goed. Echter voor sommige personen is dit een positief publiek goed en voor anderen een negatief publiek goed. Dat wil zeggen, sommige spelers hebben éénpiekige voorkeuren en andere spelers hebben ééndalige voorkeuren. In ons model zijn er twee spelers en de ontwerper weet van iedere speler of hij éénpiekige of ééndalige voorkeuren heeft. We nemen aan dat er slechts een eindig aantal locaties zijn. We beschrijven iedere regel die Pareto optimaal en niet-manipuleerbaar is. In deze regels is de speler met de éénpiekige voorkeuren de dictator, met uitzondering van bepaalde situaties waarin de voorkeuren van de andere speler ook van invloed zijn.

CURRICULUM VITAE

Abhinaba Lahiri was born on March 17, 1990 in Uttarpara, India. In 2011, he received his Bachelor's degree in Statistics from University of Calcutta. In July of that same year, he started studying Master's in Statistics at Indian Statistical Institute. He obtained his MSTAT degree in 2013 with a specialisation in Quantitative Economics.

As a next step he started to pursue his Ph.D. degree at Maastricht University under the supervision of Dr. Hans Peters and Dr. Ton Storcken. The results of his research are presented in this thesis. Abhinaba presented his work at various international conferences. Some parts of this thesis have been accepted for publication in international refereed academic journals.