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Existence of justifiable equilibrium
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Abstract

We present a general existence result for a type of equilibrium in normal-form games. We consider nonzero-sum normal-form games with an arbitrary number of players and arbitrary action spaces. We impose merely one condition: the payoff function of each player is bounded. We allow players to use finitely additive probability measures as mixed strategies.

Since we do not assume any measurability conditions, for a given strategy profile the expected payoff is generally not uniquely defined, and integration theory only provides an upper bound, the upper integral, and a lower bound, the lower integral. A strategy profile is called a justifiable equilibrium if each player evaluates this profile by the upper integral, and each player evaluates all his possible deviations by the lower integral. We show that a justifiable equilibrium always exists.

Our equilibrium concept and existence result are motivated by Vasquez (2017), who defines a conceptually related equilibrium notion, and shows its existence under the conditions of finitely many players, separable metric action spaces and bounded Borel measurable payoff functions. Our proof borrows several ideas from Vasquez (2017), but is more direct as it does not make use of countably additive representations of finitely additive measures by Yosida and Hewitt (1952).

1 Introduction

The model and main result. The main goal of the current paper is to present a general existence result for a type of equilibrium in normal-form games, with an arbitrary number of players and arbitrary action spaces. The only condition we impose is that the payoff function of each player is bounded. We allow players to use finitely additive probability measures as mixed strategies.

Since we do not pose any measurability assumptions, the payoff function of a player is not necessarily integrable. That is, a strategy profile does not always induce a unique expected payoff. In that case, the upper integral, i.e. the upper approximations of the integral by simple functions, is not the same as the lower integral, i.e. lower approximations of the integral by simple functions. So based on integration theory, the upper integral could be interpreted as the best possible expected payoff, while the lower integral as the worst expected payoff.

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We call a strategy profile a justifiable equilibrium if each player evaluates this strategy profile by the upper integral, and each player evaluates all his possible deviations by the lower integral. Our equilibrium concept is motivated by the concept of optimistic equilibrium in Vasquez (2017). The concept of justifiable equilibrium has two main conceptual advantages. First, the definition is straightforward and has an easy interpretation. Second, we only need to approximate the integral of the payoff functions at the strategy profiles under consideration. This is in stark contrast with optimistic equilibrium in Vasquez (2017), which is defined through several abstract steps and makes use of small perturbations of each strategy profile. Admittedly, both concepts have one drawback: a strategy profile is not necessarily evaluated in the same way when it is a candidate equilibrium and when it arises by a deviation of a player.

Our main result is that a justifiable equilibrium always exists, in any normal-form game with bounded payoff functions. The proof uses the Kakutani-Fan-Glicksberg fixed point theorem. Our proof borrows several ideas from Vasquez (2017), but is more direct as it does not make use of countably additive representations of finitely additive measures (comment 4.5 in Yosida and Hewitt (1952)).

Related literature. Finite additivity, instead of countable additivity, for probability measures was argued for on several grounds. For example, in decision theory conceptual arguments were given by de Finetti (1975), Savage (1942), and Dubins and Savage (2014). For a comparison between finitely additive and countably additive measures, see Bingham (2010).

In game theory, countable additivity is the usual assumption on probability measures. However, equilibria in finitely additive strategies have also gained recognition. Marinacci (1997) proves the existence of Nash equilibrium in nonzero-sum normal-form games, when the payoff functions are integrable. In this case, the lower and upper integrals coincide, and hence our result can be seen as a vast generalization of the existence result in Marinacci (1997). In a strongly related work Harris et al (2005) give different types of characterizations of the utility functions that Marinacci (1997) considers. In a different vain, Capraro and Scarsini (2013) consider some nonzero-sum games where the upper and lower integrals of utility functions do not coincide. They calculate expected payoffs through convex combinations of different orders of integration, and prove the existence of Nash equilibrium when the game has countable action spaces and can be defined through an algebraic operator. They extend their result to uncountable action spaces by adding further restrictions on the payoff functions. Generally speaking, the existence of finitely additive Nash equilibrium in normal-form games seems to require fairly restrictive assumptions on the payoff functions, but sometimes also on the action spaces.

There are various results on the existence of the value and optimal strategies in zero-sum games, see for instance Yanovskaya (1970), Heath and Sudderth (1972), Kindler (1983), Maitra and Sudderth

(1993), Schervish and Seidenfeld (1996), Maitra and Sudderth (1998), and Flesch et al (2017). For an extensive overview we refer to Flesch et al (2017).

What all the above mentioned papers have in common is that either each strategy profile induces a unique expected payoff or each strategy profile is assigned a certain expected payoff according to some rule. Then, Nash equilibrium can be defined in the usual way by requiring that each player's strategy is a best response to the strategies of his opponents. In this sense, our definition of justifiable equilibrium and the notion of optimistic equilibrium in Vasquez (2017) conceptually separate themselves from the literature and take a somewhat new direction. Indeed, as mentioned earlier, both concepts assign to a strategy profile a possibly different payoff when it is a candidate equilibrium and when it arises by a deviation of a player. We discuss later in Section 5 whether our proof and existence result could be extended to a Nash equilibrium, that is when each strategy profile is assigned the same expected payoff, irrespective of it being considered a candidate equilibrium or not.

Justifiable equilibrium uses the upper integral and the lower integral, when the payoff function is not integrable. The use of the upper and lower integrals is of course not a new idea, see for example Lehrer (2009) who uses the upper integral for the definition of a new integral for capacities, and Stinchcombe (2005) where the upper and lower integrals appear in the context of set-valued integrals.

Structure of the paper. In the next section we discuss some technical preliminaries on finitely additive probability measures. We present the model and the main existence result in Section 3. We provide the proof in Section 4. Finally, in Section 5, we demonstrate the difficulties of improving upon this existence result.

2 Preliminaries

Charges. Take a nonempty set X endowed with an algebra $\mathcal{F}(X)$. A finitely additive probability measure, or simply charge, on $(X, \mathcal{F}(X))$ is a mapping $\mu\colon \mathcal{F}(X) \to [0,1]$ such that $\mu(X) = 1$ and for all disjoint sets $E, F \in \mathcal{F}(X)$ it holds that $\mu(E \cup F) = \mu(E) + \mu(F)$.

Product charge. Let I be a nonempty set, and for each i let X_i be a nonempty set endowed with an algebra $\mathcal{F}(X_i)$, and let μ_i be a charge on $(X_i, \mathcal{F}(X_i))$. Let $X = \times_{i \in I} X_i$. A rectangle of X is a set of the form $Y = \times_{i \in I} Y_i$, where $Y_i \in \mathcal{F}(X_i)$ for all $i \in I$ and moreover $Y_i = X_i$ for all but finitely many $i \in I$. Let $\mathcal{F}(X)$ be the smallest algebra on X containing the rectangles of X, which is identical to the collection of all finite unions of rectangles of X. It is known that there is a unique charge μ on $(X, \mathcal{F}(X))$, called the product charge, that assigns probability $\Pi_{i \in I} \mu_i(Y_i)$ to each rectangle $\times_{i \in I} Y_i$ of X.

INTEGRATION WITH RESPECT TO A CHARGE. We call a function $s : X \to \mathrm{R}$ an $\mathcal{F}(X)$-measurable simple function if s is of the form $s = \sum_{m=1}^{k} c_m \mathbb{I}_{B_m}$, where $c_1, \ldots, c_k \in \mathrm{R}$, the sets $B_1, \ldots, B_k$ are rectangles of X and form a partition of X, and $\mathbb{I}_{B_m}$ is the characteristic function of the set B_m. With respect to a charge μ on $(X, \mathcal{F}(X))$, the integral of s is defined by $s(\mu) = \int_{x \in X} s(x)\, \mu(dx) = \sum_{m=1}^{k} c_m \cdot \mu(B_m)$.

Consider a bounded function $u : X \to \mathrm{R}$. The upper integral of u with respect to μ is defined as

$$\overline{u}(\mu) \;=\; \inf\left\{\int_{x \in X} s(x)\, \mu(dx) : s \geq u, s \text{ is an } \mathcal{F}(X)\text{-measurable simple function}\right\},$$

and the lower integral of u with respect to μ as

$$\underline{u}(\mu) \;=\; \sup\left\{\int_{x \in X} s(x)\, \mu(dx) : s \leq u, s \text{ is an } \mathcal{F}(X)\text{-measurable simple function}\right\}.$$

3 The model and the main result

A game has an arbitrary nonempty set I of players. Each player $i \in I$ is given an arbitrary nonempty action space A_i, endowed with an algebra $\mathcal{F}(A_i)$. Let $A = \Pi_{i \in I} A_i$. Each player $i \in I$ is given an arbitrary bounded payoff function $u_i : A \to \mathrm{R}$.

A strategy for player $i \in I$ is a charge σ_i on $(A_i, \mathcal{F}(A_i))$. We denote the set of strategies for player i by Σ_i. A strategy profile is a collection of strategies $\sigma = (\sigma_i)_{i \in I}$, where σ_i is a strategy for each player $i \in I$. We denote the set of strategy profiles by Σ. Let σ_{-i} denote the partial strategy profile $(\sigma_j)_{j \in I \setminus \{i\}}$ of the opponents of player i, and Σ_{-i} denote the set of such partial strategy profiles.

As described in Section 2, every strategy profile σ generates a unique charge on $(A, \mathcal{F}(A))$, which with a small abuse of notation we also denote by σ. For a player $i \in I$ the upper integral of his payoff function is denoted by $\overline{u}_i$, and the lower integral of his payoff function is denoted by $\underline{u}_i$.

Definition 1. A strategy profile σ is called a *justifiable equilibrium* if for each player $i \in I$ and each strategy $\tau_i \in \Sigma_i$

$$\overline{u}_i(\sigma) \geq \underline{u}_i(\tau_i, \sigma_{-i}).$$

Intuitively, at a justifiable equilibrium profile σ, each player's best possible expected payoff should be greater than or equal to his worst possible expected payoff if he deviates. Our main result is the following.

Theorem 2. *Every game with bounded payoff functions has a justifiable equilibrium.*

Note that we have no restriction on the number of players and the action spaces. The proof is based on the Kakutani-Fan-Glicksberg fixed point theorem.

The game in the following example does not admit a Nash equilibrium in countably additive strategies. However, it has a justifiable equilibrium.

Example 1. The following game is a version of Wald's game (Wald (1945)). The action sets are $A_1 = A_2 = \mathbb{N}$, endowed with the algebra $2^{\mathbb{N}}$. Player 1's payoff for $(a_1, a_2) \in \mathbb{N} \times \mathbb{N}$ is $u_1(a_1, a_2) = 1$ if $a_1 \geq a_2$ and $u_1(a_1, a_2) = 0$ if $a_1 < a_2$. Player 2's payoff is $u_2(a_1, a_2) = 1 - u_1(a_1, a_2)$ for all $(a_1, a_2) \in \mathbb{N} \times \mathbb{N}$. The payoffs given by $u = (u_1, u_2)$ are represented in the following matrix, where player 1 is the row player and player 2 is the column player.

u	1	2	3	$\ldots$
1	1,0	0,1	0,1	$\ldots$
2	1,0	1,0	0,1	$\ldots$
3	1,0	1,0	1,0	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

1. This game has no Nash equilibrium in countably additive strategies. Indeed, take any strategy profile $\sigma = (\sigma_1, \sigma_2)$. Since the sum of the expected payoffs is 1, we can assume without loss of generality that $u_1(\sigma) \leq 1/2$. However, against the strategy σ_2, player 1 can obtain an expected payoff arbitrarily close to 1 by choosing a large action $a_1 \in \mathbb{N}$. Hence, σ cannot be a Nash equilibrium.

2. There is a justifiable equilibrium in this game. A strategy σ_i for player $i \in \{1, 2\}$ is called *diffuse* if $\sigma_i(n) = 0$ for every $n \in \mathbb{N}$. Indeed, each strategy profile $\sigma = (\sigma_1, \sigma_2)$ in which at least one of the strategies is diffuse, is a justifiable equilibrium.

We show that σ is a justifiable equilibrium if σ_1 is diffuse; the proof is similar when σ_2 is diffuse. Because the payoff functions only take values 0 and 1, by the definition of justifiable equilibrium, it suffices to prove that $\overline{u}_1(\sigma) = 1$ and $\underline{u}_2(\sigma_1, \sigma_2') = 0$ for every strategy σ_2' for player 2.

2.1 We show first that $\overline{u}_1(\sigma) = 1$. As before, $\mathcal{F}(\mathbb{N} \times \mathbb{N})$ is the collection of all finite unions of rectangles of $\mathbb{N} \times \mathbb{N}$. Take any $\mathcal{F}(\mathbb{N} \times \mathbb{N})$-measurable simple function $s : \mathbb{N} \times \mathbb{N} \to \mathrm{R}$ such that $s \geq u_1$. Assume that s is of the form $s = \sum_{m=1}^{k} c_m \mathbb{I}_{B_m}$, where $c_1, \ldots, c_k \in \mathrm{R}$ and the sets $B_1, \ldots, B_k$ are rectangles of $\mathbb{N} \times \mathbb{N}$ and form a partition of $\mathbb{N} \times \mathbb{N}$.

Let M be the set of m for which $\sigma(B_m) > 0$. Note that $\sum_{m \in M} \sigma(B_m) = 1$, since $B_1, \ldots, B_k$ form a finite partition of $\mathbb{N} \times \mathbb{N}$. Let $m \in M$. Since σ_1 is diffuse, the set B_m is of the form $B_m = B_m^1 \times B_m^2$ where $B_m^1 \subseteq \mathbb{N}$ is infinite. This implies that there is $(x_m^1, x_m^2) \in B_m$ such that $x_m^1 \geq x_m^2$. Consequently, $c_m \geq u_1(x_m^1, x_m^2) = 1$. Therefore,

$$s(\sigma) \;=\; \sum_{m=1}^{k} c_m \cdot \sigma(B_m) \;=\; \sum_{m \in M} c_m \cdot \sigma(B_m) \;\geq\; \sum_{m \in M} \sigma(B_m) = 1.$$

It follows that $\overline{u}_1(\sigma) = 1$.

2.2 We show that $\underline{u}_2(\sigma_1, \sigma_2') = 0$ for every strategy σ_2' for player 2. Let σ_2' be any strategy of player 2. Take any $\mathcal{F}(\mathbb{N} \times \mathbb{N})$-measurable simple function $s : \mathbb{N} \times \mathbb{N} \to \mathrm{R}$ such that $s \leq u_2$. Similarly to step 2.1, one can verify that

$$s(\sigma_1, \sigma_2') \;\leq\; 0.$$

It follows that $\underline{u}_2(\sigma_1, \sigma_2') = 0$. $\Diamond$

We remark that in the special class of games where the lower integral and the upper integral of the payoff functions coincide, that is $\underline{u}_i(\sigma) = \overline{u}_i(\sigma)$ for every strategy profile σ and every player i, the concepts of justifiable equilibrium and Nash equilibrium coincide. Hence, in view of Theorem 2, these games admit a Nash equilibrium. The existence of Nash equilibrium in these games was shown earlier in Marinacci (1997).

4 Proof of the existence result

In this section we prove Theorem 2. The proof is based on the Kakutani-Fan-Glicksberg fixed point theorem, stated below (cf. Corollary 17.55 in Aliprantis and Border (2005)).

Theorem 3 (Kakutani-Fan-Glicksberg)**.** *Let K be a nonempty compact convex subset of a locally convex Hausdorff topological vector space, and let the correspondence $\phi : K \rightrightarrows K$ have closed graph and nonempty convex values. Then the set of fixed points of ϕ is nonempty.*

We endow Σ with the topology of pointwise convergence. That is, we see Σ as a subset of $C := \times_{i\in I} \times_{A_i'\in\mathcal{F}(A_i)} \mathrm{R}$, where C is endowed with the product topology and Σ is given its relative topology. By Tychonoff's theorem, $C_{[0,1]} := \times_{i\in I} \times_{A_i'\in\mathcal{F}(A_i)} [0,1]$ is a compact subset of C. As Σ is a closed subset of $C_{[0,1]}$, the set Σ is compact. This way Σ is a nonempty compact convex subset of the locally convex Hausdorff topological vector space C.

A mapping $f : \Sigma \to \mathrm{R}$ is called *upper semicontinuous* if for every net $(\sigma^\alpha)_{\alpha\in D}$ in Σ, where D is a directed set, converging to some $\sigma \in \Sigma$, we have $\limsup_\alpha f(\sigma^\alpha) \leq f(\sigma)$. Similarly, a mapping $f : \Sigma \to \mathrm{R}$ is called *lower semicontinuous* if for every net $(\sigma^\alpha)_{\alpha\in D}$ in Σ, where D is a directed set, converging to some $\sigma \in \Sigma$, we have $\liminf_\alpha f(\sigma^\alpha) \geq f(\sigma)$.

Lemma 4. *For every player $i \in I$, the mapping $\sigma \to \overline{u}_i(\sigma)$ from Σ to R is upper semicontinuous, and the mapping $\sigma \to \underline{u}_i(\sigma)$ from Σ to R is lower semicontinuous.*

Proof. We only prove that the mapping $\sigma \to \overline{u}_i(\sigma)$ from Σ to R is upper semicontinuous. The proof of the second part is similar.

Take a net $(\sigma^\alpha)_{\alpha\in D}$ in Σ, where D is a directed set, converging to some $\sigma \in \Sigma$.

First we show that $\lim_\alpha s(\sigma^\alpha) \;=\; s(\sigma)$ for every $\mathcal{F}(A)$-measurable simple function s. Take an $\mathcal{F}(A)$-measurable simple function s of the form $s = \sum_{m=1}^{k} c_m \mathbb{I}_{B_m}$. Since each B_m is a rectangle of A, the net $(\sigma^\alpha(B_m))_{\alpha\in D}$ of probabilities converges to $\sigma(B_m)$. Therefore $\lim_\alpha s(\sigma^\alpha) \;=\; s(\sigma)$.

Let $\varepsilon > 0$. By the definition of $\overline{u}_i(\sigma)$, there is an $\mathcal{F}(A)$-measurable simple function s such that $s \geq u_i$ and

$$\overline{u}_i(\sigma) \geq s(\sigma) - \varepsilon.$$

Since s is an $\mathcal{F}(A)$-measurable simple function, we have by the argument above that $\lim_\alpha s(\sigma^\alpha) = s(\sigma)$. Because $s \geq u_i$, we also have $s(\sigma^\alpha) \geq \overline{u}_i(\sigma^\alpha)$ for each $\alpha \in D$. Hence

$$\overline{u}_i(\sigma) \ \geq \ s(\sigma) - \varepsilon \ = \ \lim_\alpha s(\sigma^\alpha) - \varepsilon \ \geq \ \limsup_\alpha \overline{u}_i(\sigma^\alpha) - \varepsilon.$$

As $\varepsilon > 0$ was arbitrary the proof is complete. $\square$

Now we prove Theorem 2 in a number of steps. We will define a correspondence from the set of strategy profiles Σ to the power set of Σ such that this correspondence has a fixed point, by the Kakutani-Fan-Glicksberg theorem, and each fixed point is a justifiable equilibrium. To define this correspondence we need a number of auxiliary steps. Some of these steps are fairly similar to steps taken by Vasquez (2017).

Step 1. Consider a player i and let γ_i be a strategy for player i. For each strategy profile σ, we define the set

$$BR_i^{\gamma_i}(\sigma) \ = \ \{\tau \in \Sigma : \ \overline{u}_i(\tau_i, \sigma_{-i}) \geq \underline{u}_i(\gamma_i, \sigma_{-i})\}.$$

Note that $BR_i^{\gamma_i}$ is a subset of Σ and not of Σ_i. It is not essential for the proof to define $BR_i^{\gamma_i}$ as a set of strategy profiles, however it makes the exposition somewhat simpler. Intuitively, $BR_i^{\gamma_i}$ consists of all strategy profiles τ such that τ_i with the upper integral is a better reply to σ_{-i} than γ_i with the lower integral. If a strategy profile τ belongs to $BR_i^{\gamma_i}$, then (τ_i, τ'_{-i}) also belongs to $BR_i^{\gamma_i}$ for every $\tau'_{-i} \in \Sigma_{-i}$.

We show that for each strategy profile σ, the set $BR_i^{\gamma_i}(\sigma)$ is nonempty and convex.

Proof of step 1.

Take a strategy profile σ. Since $(\gamma_i, \sigma_{-i}) \in BR_i^{\gamma_i}(\sigma)$, the set $BR_i^{\gamma_i}(\sigma)$ is nonempty.

We show that $BR_i^{\gamma_i}(\sigma)$ is convex. As a first step, we argue that $\overline{u}_i$ is linear in the strategy of player i. Take two strategy profiles $\tau, \mu \in \Sigma$ such that $\tau_{-i} = \mu_{-i}$ and $\lambda \in (0,1)$. We prove that

$$\overline{u}_i(\lambda \cdot \tau + (1-\lambda)\cdot \mu) \ = \ \lambda \cdot \overline{u}_i(\tau) + (1-\lambda)\cdot \overline{u}_i(\mu). \tag{1}$$

Let S_i denote the set of $\mathcal{F}(A)$-measurable simple functions s satisfying $s \geq u_i$. Clearly, for every $s \in S_i$ we have

$$s(\lambda \cdot \tau + (1-\lambda)\cdot \mu) \ = \ \lambda \cdot s(\tau) + (1-\lambda)\cdot s(\mu).$$

Hence,

$$\begin{aligned} \inf_{s \in S_i} s(\lambda \cdot \tau + (1-\lambda)\cdot \mu) \ &= \ \inf_{s \in S_i} [\lambda \cdot s(\tau) + (1-\lambda)\cdot s(\mu)] \\ &\geq \ \lambda \cdot \inf_{s \in S_i} s(\tau) + (1-\lambda)\cdot \inf_{s \in S_i} s(\mu). \end{aligned} \tag{2}$$

Let $\varepsilon > 0$, and let $s', s'' \in S_i$ such that

$$s'(\tau) \;\leq\; \inf_{s \in S_i} s(\tau) + \varepsilon \quad \text{and} \quad s''(\mu) \;\leq\; \inf_{s \in S_i} s(\mu) + \varepsilon.$$

Define $s''' = \min\{s', s''\}$. Clearly, $s''' \in S_i$. Thus

$$\begin{aligned}
\lambda \cdot \inf_{s \in S_i} s(\tau) + (1-\lambda) \cdot \inf_{s \in S_i} s(\mu) \;&\geq\; \lambda \cdot s'(\tau) + (1-\lambda) \cdot s''(\mu) - \varepsilon \\
&\geq\; \lambda \cdot s'''(\tau) + (1-\lambda) \cdot s'''(\mu) - \varepsilon \\
&=\; s'''(\lambda \cdot \tau + (1-\lambda) \cdot \mu) - \varepsilon \\
&\geq\; \inf_{s \in S_i} s(\lambda \cdot \tau + (1-\lambda) \cdot \mu) - \varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we conclude

$$\lambda \cdot \inf_{s \in S_i} s(\tau) + (1-\lambda) \cdot \inf_{s \in S_i} s(\mu) \;\geq\; \inf_{s \in S_i} s(\lambda \cdot \tau + (1-\lambda) \cdot \mu). \tag{3}$$

By (2) and (3), we have shown

$$\inf_{s \in S_i} s(\lambda \cdot \tau + (1-\lambda) \cdot \mu) \;=\; \lambda \cdot \inf_{s \in S_i} s(\tau) + (1-\lambda) \cdot \inf_{s \in S_i} s(\mu).$$

Hence, (1) holds, which shows that $\overline{u}_i$ is linear in the strategy of player i.

Take two strategy profiles $\tau, \mu \in BR_i^{\gamma_i}(\sigma)$ and $\lambda \in (0,1)$. Therefore,

$$\overline{u}_i(\lambda \cdot \tau_i + (1-\lambda) \cdot \mu_i, \sigma_{-i}) = \lambda \cdot \overline{u}_i(\tau_i, \sigma_{-i}) + (1-\lambda) \cdot \overline{u}_i(\mu_i, \sigma_{-i}) \geq \underline{u}_i(\gamma_i, \sigma_{-i}).$$

As a consequence, $BR_i^{\gamma_i}(\sigma)$ is convex.

Step 2. Consider a player i and let γ_i be a strategy for player i. We prove that the correspondence $\sigma \rightrightarrows BR_i^{\gamma_i}(\sigma)$ from Σ to 2^Σ has a closed graph.

Proof of step 2. With a directed set D, take two nets $(\sigma^\alpha)_{\alpha \in D}$ and $(\tau^\alpha)_{\alpha \in D}$ in Σ converging to respectively some $\sigma \in \Sigma$ and $\tau \in \Sigma$. Assume that for every $\alpha \in D$, we have $\tau^\alpha \in BR_i^{\gamma_i}(\sigma^\alpha)$. We show that $\tau \in BR_i^{\gamma_i}(\sigma)$. Then, the proof of step 2 will be complete (cf. also Theorems 17.16 and 17.10 in Aliprantis and Border (2005)).

For every $\alpha \in D$, as $\tau^\alpha \in BR_i^{\gamma_i}(\sigma^\alpha)$, we have $\overline{u}_i(\tau_i^\alpha, \sigma_{-i}^\alpha) \geq \underline{u}_i(\gamma_i, \sigma_{-i}^\alpha)$. By Lemma 4, taking limits yields

$$\overline{u}_i(\tau_i, \sigma_{-i}) \;\geq\; \limsup_\alpha \overline{u}_i(\tau_i^\alpha, \sigma_{-i}^\alpha) \;\geq\; \liminf_\alpha \underline{u}_i(\gamma_i, \sigma_{-i}^\alpha) \;\geq\; \underline{u}_i(\gamma_i, \sigma_{-i}).$$

Thus, $\tau \in BR_i^{\gamma_i}(\sigma)$ as desired.

Step 3. Consider a player i. For each strategy profile σ, we define the set

$$BR_i(\sigma) \;=\; \bigcap_{\gamma_i \in \Sigma_i} BR_i^{\gamma_i}(\sigma).$$

Intuitively, BR_i consists of all strategy profiles τ such that τ_i with the upper integral is a better reply to σ_{-i} than any other strategy of player i with the lower integral. We prove that for each strategy profile σ, the set $BR_i(\sigma)$ is nonempty and convex.

Proof of step 3. Take $\sigma \in \Sigma$. Convexity of $BR_i(\sigma)$ directly follows from Step 1, where we showed the convexity of $BR_i^{\gamma_i}(\sigma)$ for each $\gamma_i \in \Sigma_i$.

Now we show that $BR_i(\sigma)$ is nonempty. By the finite intersection property (cf. Theorem 2.31 in Aliprantis and Border (2005)) it is sufficient to check for finitely many strategies $\gamma_i^1, \ldots, \gamma_i^k \in \Sigma_i$ that $\cap_{j=1}^k BR_i^{\gamma_i^j}(\sigma)$ is not empty. Choose $m \in \{1, \ldots, k\}$ such that $\underline{u}_i(\gamma_i^m, \sigma_{-i}) \geq \underline{u}_i(\gamma_i^j, \sigma_{-i})$ for all $j \in \{1, \ldots, k\}$. Since $\overline{u}_i(\gamma_i^m, \sigma_{-i}) \geq \underline{u}_i(\gamma_i^m, \sigma_{-i})$, the strategy profile $(\gamma_i^m, \sigma_{-i})$ belongs to $\cap_{j=1}^k BR_i^{\gamma_i^j}(\sigma)$. So $BR_i(\sigma)$ is nonempty.

Step 4. For each strategy profile σ, we define the set

$$BR(\sigma) \;=\; \bigcap_{i \in I} BR_i(\sigma).$$

Intuitively, $BR(\sigma)$ consists of all strategy profiles τ such that, for any player i, the strategy τ_i with the upper integral is a better reply to σ_{-i} than any other strategy of player i with the lower integral. We prove that for each strategy profile σ, the set $BR(\sigma)$ is nonempty and convex.

Proof of step 4. Take $\sigma \in \Sigma$. Convexity of $BR(\sigma)$ directly follows from Step 3, where we showed the convexity of $BR_i(\sigma)$ for each $i \in I$.

Now we show that $BR(\sigma)$ is nonempty. By Step 3, $BR_i(\sigma)$ is nonempty for each player $i \in I$. Choose a strategy profile $\tau^i \in BR_i(\sigma)$ for each player $i \in I$. As usual, τ_i^i denotes the strategy of player i in the strategy profile τ^i. Construct a new strategy profile τ such that $\tau_i = \tau_i^i$ for each player $i \in I$. Then $\tau \in BR_i(\sigma)$ for all $i \in I$. This implies that $\tau \in \cap_{i \in I} BR_i(\sigma)$, and hence $BR(\sigma)$ is nonempty.

Step 5. We argue that the correspondence $\phi : \sigma \rightrightarrows BR(\sigma)$ from Σ to 2^{Σ} has a fixed point. Moreover, any fixed point of ϕ is a justifiable equilibrium.

Proof of step 5. The graph of the correspondence ϕ is the intersection of the graphs of the correspondences $\sigma \rightrightarrows BR_i^{\gamma_i}(\sigma)$ over all players $i \in I$ and strategies γ_i of player i. Hence, by Step 2, the correspondence ϕ has a closed graph. Moreover, by Step 4, ϕ has nonempty and convex values. Due to Theorem 3 the correspondence ϕ has a fixed point. It is clear that any fixed point of ϕ is a justifiable equilibrium.

5 Attempts to refine the concept of justifiable equilibrium

The concept of justifiable equilibrium has a straightforward definition, and importantly, as shown in Theorem 2, it always exists in games with bounded payoff functions, regardless the number of players and the action spaces. The purpose of this section is to demonstrate the difficulty to find any refinement of the concept of justifiable equilibrium for which the general existence is preserved, or at least guaranteed by arguments along the proof in Section 4.

In the first part of this section we examine whether the current proof could be generalised to obtain a Nash equilibrium. That is, we would like to assign one specific payoff to each strategy profile regardless whether it is a candidate equilibrium or it arises as a deviation.

In the second part of the section we investigate a possible refinement of justifiable equilibrium where instead of the lower integral we take a higher, and therefore more restrictive mapping that assigns an expected payoff to each strategy profile.

I. One natural attempt would be to take, for each player $i \in I$, a selector f_i of the correspondence $\sigma \rightrightarrows [\underline{u}_i(\sigma), \overline{u}_i(\sigma)]$ from Σ to R, try to replace both $\overline{u}_i$ and $\underline{u}_i$ by f_i in the proof, and thus find a strategy profile σ^* such that $f_i(\sigma^*) \geq f_i(\sigma_i', \sigma_{-i}^*)$ for every payer i and every strategy σ_i' of player i. So, this strategy profile σ^* would not only be a justifiable equilibrium, but even a Nash equilibrium with respect to the payoffs given by $f = (f_i)_{i \in I}$.

Since we try to replace both $\overline{u}_i$ and $\underline{u}_i$ by f_i in the proof, this approach would only work if the selector f_i, for each player i, satisfies those properties of both $\underline{u}_i$ and $\overline{u}_i$ that we used in the proof of Section 4. To be precise, in that proof we made use of the following properties of $\underline{u}_i$ and $\overline{u}_i$ for each player i: (1) $\underline{u}_i(\sigma) \leq \overline{u}_i(\sigma)$ for every strategy profile σ, (2) the mapping $\sigma \to \underline{u}_i(\sigma)$ is lower semicontinuous (cf. Lemma 4), (3) the mapping $\sigma \to \overline{u}_i(\sigma)$ is upper semicontinuous (cf. Lemma 4) and it is linear in player i's strategy σ_i (cf. Step 1 in Section 4). Even though the mapping $\sigma \to \underline{u}_i(\sigma)$ is also linear in player i's strategy σ_i, this was not needed in the proof.

So, for each player i, the selector f_i should be continuous and in addition linear in player i's strategy. However, in general, such a selector does not exist. In fact, not even a selector that is only required to be continuous. We illustrate it by showing that there is no continuous selector for player 1 in the game of Example 1. Let $\sigma = (\sigma_1, \sigma_2)$ be a strategy profile in which both strategies are diffuse charges, that is $\sigma_1(n) = \sigma_2(n) = 0$ for every $n \in \mathbb{N}$, and both strategies are 0–1 valued, that is they only assign to each set probability 0 or 1 (such strategies correspond to ultrafilters on the action sets).

Consider any open neighborhood U of σ in Σ. Then, there is a finite collection $\{B_1, \ldots, B_k\}$ of rectangles of $A = A_1 \times A_2 = \mathbb{N} \times \mathbb{N}$ and positive numbers $\varepsilon_1, \ldots, \varepsilon_k$ such that the set

$$W = \{\tau \in \Sigma : \ |\tau(B_m) - \sigma(B_m)| < \varepsilon_m \ \forall m = 1, \ldots, k\}$$

is a subset of U. By adding more constraints (splitting the sets $B_1, \ldots, B_k$ if necessary and adding more sets), we can even assume that there is a finite partition P_1 of A_1 and a finite partition P_2 of A_2 such that $\{B_1, \ldots, B_k\}$ is the same as $\{X \times Y : X \in P_1, Y \in P_2\}$.

Let X be the unique element of P_1 for which $\sigma_1(X) = 1$, and let Y be the unique element of P_2

for which $\sigma_2(Y) = 1$. Since σ_1 and σ_2 are diffuse, X and Y are both infinite. This implies that there are $(x^1, y^1), (x^2, y^2) \in X \times Y$ such that $x^1 \geq y^1$ and $x^2 < y^2$.

So we define two strategy profiles $\mu = \delta(x^1, y^1)$ and $\nu = \delta(x^2, y^2)$ where δ is the Dirac measure. We have $\mu(X \times Y) = \nu(X \times Y) = 1$, and $\mu(B_m) = \nu(B_m) = \sigma(B_m)$ for all $m = 1, \ldots, k$. Hence $\mu, \nu \in W$. As $\underline{u}_1(\mu) = 1$ and $\overline{u}_1(\nu) = 0$, in conclusion, there is no continuous selector f_1 for player 1 of the correspondence $\sigma \rightrightarrows [\underline{u}_1(\sigma), \overline{u}_1(\sigma)]$ from Σ to R.

II. A second, and probably less important, attempt to refine the concept of justifiable equilibrium would be the following. As mentioned above, in the proof we only used for each player i the following properties of the lower integral $\underline{u}_i$: $\underline{u}_i \leq \overline{u}_i$ and $\underline{u}_i$ is lower semicontinuous. This brings up the idea to consider the largest function $g_i : \Sigma \to$ R such that $g_i \leq \overline{u}_i$ and g_i is lower semicontinuous. This function g_i is called the lower semicontinuous envelope of the function $\overline{u}_i$. Clearly, $\underline{u}_i \leq g_i$. The question is thus whether g_i coincides with $\underline{u}_i$. We suspect that the answer is affirmative, meaning that replacing $\underline{u}_i$ with g_i does not lead to a refinement of justifiable equilibrium. For the equality $g_i = \underline{u}_i$, if true, it would suffice to show that for each strategy profile σ, each open neighborhood U of σ and each $\varepsilon > 0$, there is a strategy profile $\tau \in U$ such that $\underline{u}_i(\sigma) \geq \overline{u}_i(\tau) - \varepsilon$. Since $\underline{u}_i(\sigma)$ can be approximated with an integral $s(\sigma)$ with respect to an $\mathcal{F}(A)$-measurable simple function $s \leq \underline{u}_i$, it would be enough to find a strategy profile $\tau \in U$ such that $s(\sigma) \geq \overline{u}_i(\tau) - \varepsilon$. Note that U also contains strategy profiles $\mu = (\mu_i)_{i \in I}$ such that each player i's strategy μ_i is a convex combination of Dirac measures on actions in A_i (i.e. μ_i has finite range in A_i), and for such strategy profiles μ the upper integral $\overline{u}_i(\mu)$ coincides with the lower integral $\underline{u}_i(\mu)$, which we can denote by $u_i(\mu)$. Hence, to show that $g_i = \underline{u}_i$, it would also suffice to find such a strategy profile $\mu \in U$ with $s(\sigma) \geq u_i(\mu) - \varepsilon$. It is an interesting question whether it is possible to find such a μ, based on σ and s.

6 Conclusions

Under rather general conditions we prove the existence of a justifiable equilibrium in finitely additive strategies. Namely, a justifiable equilibrium exists in any normal-form game with an arbitrary number of players and arbitrary action spaces, provided that the payoff functions are bounded. The proof uses the Kakutani-Fan-Glicksberg fixed point theorem. It seems difficult to find a refinement of justifiable equilibrium for which the existence can be guaranteed while using a similar line of proof.

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