

Path-monotonicity and incentive compatibility

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André Berger, Rudolf Müller,
Seyed Hossein Naeemi

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Compatibility**

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Maastricht University School of Business and Economics
Maastricht Research School of Economics
of Technology and Organization

P.O. Box 616
NL - 6200 MD Maastricht
The Netherlands

Path–Monotonicity and Incentive Compatibility¹

André Berger²

Rudolf Müller^{2,3,4}

Seyed Hossein Naeemi^{2,4,5}

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Abstract

We study the role of monotonicity in the characterization of incentive compatible allocation rules when types are multi-dimensional, the mechanism designer may use monetary transfers, and agents have quasi-linear preferences over outcomes and transfers. It is well-known that monotonicity of the allocation rule is necessary for incentive compatibility. Furthermore, if valuations for outcomes are either convex or differentiable functions in types, revenue equivalence literature tells that path-integrals of particular vector fields are path-independent. For the special case of linear valuations it is known that monotonicity plus path-independence is sufficient for implementation. We show by example that this is not true for convex or differentiable valuations, and introduce a stronger version of monotonicity, called *path-monotonicity*. We show that path-monotonicity and path-independence characterize implementable allocation rules if (1) valuations are convex and type spaces are convex; (2) valuations are differentiable and type spaces are path-connected. Next we analyze conditions under which monotonicity is equivalent to path-monotonicity. We show that an increasing difference property of valuations ensures this equivalence. Next, we show that for simply connected type spaces incentive compatibility of the allocation rule is equivalent to path-monotonicity plus incentive compatibility in some neighborhood of each type. This result is used to show that on simply connected type spaces incentive compatible allocation rules with a finite range are completely characterized by path-monotonicity, and thus by monotonicity in cases where path-monotonicity and monotonicity are equivalent. This generalizes a theorem by Saks and Yu to a wide range of settings.

Keywords: Mechanism design, multi-dimensional types, incentive compatible, monotonicity.

1 Introduction

The goal of mechanism design is to design game forms that motivate agents with private information to choose equilibrium strategies that lead to an implementation of a desired social choice

¹A previous extended abstract of a part of this work appeared in the Proceedings of SAGT 2009 [3].

²Maastricht University, Quantitative Economics, P.O.Box 616, 6200 MD Maastricht, The Netherlands. Email: {a.berger,r.muller,h.naeemi}@maastrichtuniversity.nl

³Corresponding author

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function. In this paper we assume that agents have preferences in terms of monetary valuations for outcomes and that the mechanism designer can use monetary transfers to orchestrate agent behavior. Preferences for each outcome are given by publicly known functions of private information of each player, called *valuation functions* and *types*, respectively. Types may be a single number (one-dimensional) or vectors of numbers (multi-dimensional). Agents are assumed to have quasi-linear utilities over outcomes and transfers. Whenever the revelation principle holds, the question of implementability reduces to the existence of a transfer rule such that truthful reports of agent types becomes an equilibrium in the direct mechanism given by the social choice function and the transfer rule. A social choice function is then viewed as an *allocation rule* that selects an outcome for each combination of reported types. The direct mechanism is called *incentive compatible* (IC), if truthful reports form an equilibrium. An allocation rule is called incentive compatible, if such transfers exist.

The goal of this paper is to study conditions on the structure of type spaces and the structure of valuation functions that allow for easily verifiable properties to characterize IC allocation rules. In particular, our aim is to have transfer free, local characterizations. With such a characterization incentive compatibility can be verified without the need to construct transfers, and by understanding the influence of “small” deviations from the truth.

In single-item auctions, where types are valuations for a single good and outcomes can be interpreted as probabilities to get the good, such a property is monotonicity: the probability to get the good has to be non-decreasing in the reported type (see, e.g. Myerson [16]). In multi-dimensional settings (a generalization of) monotonicity is still necessary for IC, but often not sufficient. For example, in a combinatorial auction, type vectors represent values for each bundle of goods, and outcomes may be given by a probability to win each of the bundles. An allocation rule f that assigns a probability vector $f(t)$ to report t , is monotone if for any two types s, t we have $(f(s) - f(t)) \cdot (s - t) \geq 0$. It is well known that there exist allocation rules f that satisfy monotonicity but are not IC (see Müller, Perea and Wolf [14] for an example). The goal is then to identify conditions that, if added to monotonicity, guarantee IC. A well-known condition of this type is the existence and path-independence of path integrals of a particular vector field (Jehiel and Moldovanu [9]). As such a condition is difficult to verify—a claim that will become clear in Section 3—it is helpful to understand when the latter is not needed, in other words, when monotonicity alone implies IC. A quite general result of this type is due to Saks and Yu [20], generalizing results in more restricted settings due to Jehiel, Moldovanu and Stacchetti [10], Gui et al. [7] and Bikhchandani et al. [4]. Saks and Yu show that if types are from a convex set in \mathbb{R}^d , representing valuations for d different outcomes, then monotonicity is sufficient for IC. Shorter proofs of this result were later given by Ashlagi, Braverman, Hassidim and Monderer [2] as well as by Vohra [21]. Archer and Kleinberg [1] gave yet an alternative proof based on techniques that are of interest by their own. They show first that it is essentially sufficient for being implementable that an allocation rule is monotone and *locally IC* in some arbitrary small neighborhood of each type t , even for infinite sets of outcomes. Second, they show that for finite sets of outcomes monotonicity implies local IC. We show that this kind of result can be extended to more general settings.

Saks and Yu [20], Ashlagi et al. [2], Vohra [21] and Archer and Kleinberg [1] make use of a network interpretation of incentive constraints, which has been proposed in Gui et al. [7], and was already implicitly used in Rochet [18]. The network approach defines a network whose nodes are the types, and whose arcs between any two nodes have weights such that potentials in the network coincide with truthful transfers. Such transfers exist whenever the network does not have a cycle of negative length. Monotonicity guarantees that cycles of two arcs—forward and backward arcs between two types—have non-negative length, and is thus necessary for incentive compatibility. While monotonicity has thus a straightforward interpretation in terms of networks, path-integrals of vector fields used in analytical characterizations of implementability (Jehiel and Moldovano [9], Jehiel et al. [10]) and to establish revenue equivalence (RE) results (Krishna and Maenner [12], Milgrom and Segal [13]) seemed more difficult to fold into the network approach. Archer and Kleinberg [1] closed this gap for linear valuations by defining arc lengths differently, and linking lengths of paths and cycles in the network explicitly to path integrals of the aforementioned vector field. We show that this link can be established not only for linear valuations, but also in the settings studied in Krishna and Maenner [12] and Milgrom and Segal [13].

All previous literature on the characterization of IC allocation rules assumes in one or the other way that valuations for outcomes are linear functions of types. In many applications of mechanism design this is a very restrictive assumption. For example, a type of an agent might be parameters defining the cost structure of a firm, and outcomes might be various contracts. Valuations for outcomes could in such a case be given by a complex, non-linear function of the type. Theoretically, we could still think of a type as a vector of values for each possible outcome, thus identifying functions that map types and outcomes into real numbers by vectors of real numbers, indexed by outcomes. In a direct mechanism this would mean that agents have to report their value for each possible contract, which leads to high, if not infinite, communication cost. Furthermore, the non-linearity of the valuation function could turn a convex set of types into a non-convex, and maybe even disconnected set of type vectors. It is therefore desirable that characterizations of IC are available in terms of concise representations of types. Using the link between the network approach and path-integrals, we are able to provide such characterizations.

We establish this link in two settings. In the first setting we require that types come from a convex subset in \mathbb{R}^d , and valuations for each outcome are convex functions of the type. The set of outcomes is arbitrary. In terms of RE this setting has been studied by Krishna and Maenner [12]. The second setting requires that types come from a connected subset in \mathbb{R}^d and valuations for each outcome are differentiable functions of the type. The set of outcomes can again be arbitrary. This setting has been studied with respect to RE by Milgrom and Segal [13]. For both settings we clarify the additional conditions that have to be satisfied in order to make a monotone rule implementable. However, we show that one has to introduce a stronger version of monotonicity, which we call *path-monotonicity*. It makes use of the derivative—let us assume it exists, details on this issue are explained later—of the valuation function of the chosen allocation: if allocation a is chosen at type t , we consider the valuation for a as a function of the type, and take the derivative at t . Doing this for every type defines a vector field. Path-monotonicity requires that the change

in value for the outcome a , if the agent would be of type s rather than of type t , must not be larger than the path-integral of this vector field along any path from t to s .

Next we discuss the relation between monotonicity and path-monotonicity. Generally, the latter implies the first, but not the other way round. In particular we provide a one-dimensional example of a monotone rule that is not IC. In special cases, however, which can be easily explained in terms of a network property, both notions coincide. A property of valuation functions, called increasing differences, guarantees this network property. Increasing differences holds when a single-crossing property is present in the structure of the preferences.

We then generalize the concept of local implementability introduced by Archer and Kleinberg [1] to both our settings. We get results of the flavor that an allocation rule on a simply connected type space is implementable if and only if it is (path-)monotone and locally implementable. Such characterizations allow us to generalize the Theorem by Saks and Yu [20] to the following: whenever A is finite, and T is simply connected, then (path-)monotonicity is sufficient for implementability.

Our results are stated and proven in terms of single agent models. This is w.l.o.g. for the type of theorems we are aiming for. In case of several agents, they have an interpretation that depends on the equilibrium concept used. An allocation rule will be dominant strategy incentive compatible, if the conditions of the various theorems hold for each agent and each type report of all other agents. An allocation rule will be ex-post incentive compatible, if the conditions hold for each agent and each truthful report of all other agents. Ex-post incentive compatibility in interdependent value models is covered as well, but valuation functions of the agent under consideration change now with the type of other agents. An allocation rule will be Bayes-Nash implementable, if for each agent the conditions hold for expected valuations with respect to the randomized allocation rule that is induced by the distribution of reports of the other agents when they are truth-telling. This extension requires however, that conditions on valuation functions as stated in our theorems are robust with respect to taking expected values. In particular, results on settings with finite set of outcomes might not be applicable as the set of possible outcomes becomes infinite due to randomization.

Related literature Next to the papers mentioned so far, several recent papers deal with the characterization of IC allocation rules and RE in general, multi-dimensional settings. Chung and Olszewski [6] characterize for the case of finite sets of outcomes settings in which all IC rules satisfy RE, and provide sufficient conditions for the case of countably many outcomes. Heydenreich, Müller, Uetz and Vohra [8] provide a characterization of IC rules that satisfy RE, using a similar network approach as we do here. In settings where not all IC rules satisfy RE, their characterization allows to distinguish between IC rules that satisfy RE from those which do not. Kos and Messner [11] investigate the structure of all IC transfers of an IC rule and derive the characterization of RE by Heydenreich et al. as a special case. Rahman [17] provides a new characterization of IC allocation rules in terms of the absence of profitable, non-detectable deviations. Negative cycles, as present in the characterization of Rochet, turn out to be special cases of such deviations. He elaborates as well on the structure of IC transfers. Independently of us, and extending on a preliminary version of this

paper ([3]), Carbayal and Ely [5] derive characterizations for IC allocation rules based on a relation between valuation differences and the value of path integrals of specific vector fields. They go beyond the case of convex and differentiable valuations, but require additionally some measurability properties. While their paper emphasizes a generalization of the path-integral approach, we focus on settings where necessary conditions on path-integrals are implied by monotonicity.

Organization Section 2 defines our setting and introduces necessary notation. We prove the main characterization theorems for arbitrary outcome sets and convex or differentiable valuation functions in Section 3. Section 4 identifies properties under which path-monotonicity and monotonicity are equivalent. In Section 5 we show that local IC and path-monotonicity imply IC if the type space is simply connected. Section 6 uses this result to prove generalizations of the Theorem of Saks and Yu.

2 Definitions and Setting

Let $f : T \rightarrow A$ be an allocation rule from a set of types T to a set of outcomes A . The valuation for an outcome $a \in A$ of a certain type $t \in T$ is defined by the value $v(a, t)$ given by the function $v : A \times T \rightarrow \mathbb{R}$.

Definition 1. A (direct) mechanism is a pair (f, p) of an allocation rule f and a transfer rule $p : T \rightarrow \mathbb{R}$. The mechanism is called incentive compatible (IC) or truthful if for all $s, t \in T$:

$$v(f(s), s) + p(s) \geq v(f(t), s) + p(t), \quad (1)$$

i.e. a player of type s maximizes his utility when he reports s . The allocation rule f is called incentive compatible if there exists such a transfer rule p that makes the mechanism (f, p) IC.

Our goal is to identify properties of allocation rules f that are necessary and sufficient to guarantee the existence of a transfer rule p that satisfies (1). Rochet [18] identified a property called *cyclical monotonicity*, which was later related to node potentials in *type graphs* by Gui et al. [7]. Here, and further on, a graph consists of a set of *nodes* and a set of (directed) *arcs* between pairs of nodes. Given an allocation rule f the set of nodes of the type graph T_f is equal to T . Every pair of types $s, t \in T$ is connected by arcs from s to t and from t to s . Gui et al. [7] define the arc length from s to t as⁶

$$l_p(s, t) := v(f(s), s) - v(f(t), s).$$

Archer and Kleinberg [1] proposed alternative arc lengths $l_u(s, t)$, defined as⁷:

$$l_u(s, t) := v(f(t), t) - v(f(t), s).$$

⁶In Gui et al. [7] this would be the length of the arc from t to s . Here it will be more convenient to do it the other way round.

⁷Archer and Kleinberg [1] use subtitle s instead of u in $l_u(s, t)$

We call $l_p(s, t)$ and $l_u(s, t)$ the p -length and u -length, respectively. A path from node s to node t in T_f , or (s, t) -path for short, is defined as $P = (s = s_0, s_1, \dots, s_k = t)$ such that $s_i \in T$ for $i = 0, \dots, k$. The u -length of P is defined as

$$\text{length}_u(P) = \sum_{i=0}^{k-1} l_u(s_i, s_{i+1}).$$

Similarly, $\text{length}_p(P)$ is defined with respect to p -length. A cycle is a path with $s = t$. For any t , we regard the path from t to t without any arcs as a (t, t) -path and define its length to be 0. Let $P(s, t)$ be the set of all (s, t) -paths. The distance from s to t is defined as

$$\text{dist}_u(s, t) = \inf_{P \in P(s, t)} \text{length}_u(P).$$

It is easy to see that T_f does not contain a negative cycle if and only if $\text{dist}_u(s, t)$ is finite.

We can now define cyclical monotonicity and monotonicity for allocation rules. Monotonicity is weaker, but will play a central role in our characterization efforts. Monotonicity of the type graph generalizes the role of ordinary monotonicity of the allocation rule used in Myerson [16] to characterize incentive compatible single item auctions.

Definition 2. *An allocation rule $f : T \rightarrow A$ is called monotone, if for all $s, t \in T$ it holds that $l_u(s, t) + l_u(t, s) \geq 0$. f is called cyclically monotone, if for all cycles C , $\text{length}_u(C) \geq 0$.*

We observe that p -length and u -length of any cycle in T_f are the same, as for all $s, t \in T$ we have $l_p(s, t) = l_u(s, t) + v(f(s), s) - v(f(t), t)$. Therefore monotonicity and cyclical monotonicity could have been defined in terms of p -length as well. A node potential in a (type) graph is a function π from T to \mathbb{R} such that for all $s, t \in T$ we have $\pi(t) \leq \pi(s) + l_x(s, t)$, where subscript x indicates that we might choose either p or u lengths. Rochet provided the following characterization of IC allocation rule.

Theorem 1 (Rochet [18]). *An allocation rule $f : T \rightarrow A$ is IC if and only if it is cyclically monotone.*

The simple proof employs the fact that, due to our choice of arc lengths, node potentials in the type graph with respect to p -length coincide with transfer rules that make the allocation rule IC. Node potentials exist if and only if the type graph does not have a negative cycle. From a node potential π with respect to u -lengths we get transfers by setting $p(t) = \pi(t) - v(f(t), t)$, thus node potentials with respect to u -length provide utilities of truthful reports, given an IC transfer rule p .

If $T \subseteq \mathbb{R}$, it is easy to see that monotonicity implies cyclical monotonicity if the following property of the type graph holds (see [14] and Lemma 3 in Section 4).

Definition 3. *The type graph T_f satisfies decomposition monotonicity if for all $s, t, x \in T$, where x is a convex combination of s and t , it holds $l_u(s, t) \geq l_u(s, x) + l_u(x, t)$.*

Decomposition monotonicity holds in particular if valuations are linear in types, as it is implied by monotonicity (see Section 4). For non-linear valuations and/or multi-dimensional settings neither decomposition monotonicity is implied by monotonicity, nor would it be sufficient in combination with monotonicity for IC. In the remainder of the paper we will strengthen the notion of monotonicity and identify conditions under which monotonicity is sufficient for IC in general settings. To do so we will establish a link between the network approach to IC (and RE) and the traditional, analytical approach as developed in Jehiel, Moldovanu, and Stacchetti [10], Jehiel and Moldovanu [9], Krishna and Maenner [12] and Milgrom and Segal [13]. In the analytical approach, it has been observed that IC rules give raise to a vector field, such that path-integrals of this vector field exist and are equal to 0 on closed paths. Our strengthened notion of monotonicity relates path integrals to u -lengths of arcs. The following definitions will be needed.⁸

A *path*⁹ σ is a mapping $\sigma : [0, 1] \rightarrow T$. A path $\sigma \in C^1$ is called *smooth*¹⁰. For simplicity of notation we drop the adjective *smooth* throughout the paper. We write $t \in \sigma$ if $t \in \sigma([0, 1])$. A path is said to go from s to t if $\sigma(0) = s$ and $\sigma(1) = t$. Where relevant this is indicated by σ_{st} . T is called *path-connected* if any two types $s, t \in T$ are connected by a path. Our theorems will frequently refer to the following assumption:

Assumption 1. *The type space $T \subseteq \mathbb{R}^d$ is path connected.*

A *closed path* is a path such that $s = t$. Observe that closed paths can be parameterized by mappings $\sigma : S^1 \rightarrow T$, where S^1 is the unit cycle in \mathbb{R}^2 . We denote by $\sigma'(\lambda)$ the vector of derivatives of σ at λ . A vector field is a mapping $g : T \rightarrow \mathbb{R}^d$. Given a smooth path σ , the *path integral* of g along σ , if it exists, is defined as:

$$\int_{\sigma} g \cdot d\sigma = \int_0^1 g(\sigma(\lambda)) \cdot \sigma'(\lambda) d\lambda.$$

We denote by $L_{s,t} := \{s + \lambda(t - s) : \lambda \in [0, 1]\}$ the line segment between two types $s, t \in T$, by $\blacktriangle_{s_1, s_2, s_3}$ the convex hull of $s_1, s_2, s_3 \in T$, all three distinct, and by Δ_{s_1, s_2, s_3} the path describing the boundary of $\blacktriangle_{s_1, s_2, s_3}$, i.e. $L_{s_1, s_2} \cup L_{s_2, s_3} \cup L_{s_3, s_1}$, with direction $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_1$.

A line segment $L_{s,t} \subseteq T$ is the image of the smooth path $\sigma(\lambda) = s + \lambda(t - s)$. We denote the path integral of a vector field g along $L_{s,t}$ by

$$\int_{L_{s,t}} g \cdot d\sigma = \int_{L_{s,t}} g(s + \lambda(t - s)) \cdot (t - s) d\lambda.$$

Consistently, we define

$$\int_{\Delta_{s_1, s_2, s_3}} g \cdot d\sigma = \int_{L_{s_1, s_2}} g \cdot d\sigma + \int_{L_{s_2, s_3}} g \cdot d\sigma + \int_{L_{s_3, s_1}} g \cdot d\sigma.$$

⁸Mathematical definitions and theorems used in this paper but not stated can be found in Royden [19].

⁹The term *path* refers in this paper as well to paths in type graphs as to smooth paths in \mathbb{R}^d . The meaning will always be clear from the context.

¹⁰Usually a path is called smooth if it is in C^∞ . Here it will be sufficient if σ is piecewise contained in C^1 .

A continuous vector field g for which there exists a differentiable function $F : T \rightarrow \mathbb{R}^d$ such that $g = \nabla F$ is called conservative. For a conservative vector field, the path–integral of g along every smooth path σ exists, and for all $s, t \in T$, path integrals for all paths from s to t are equal. Since we do not want to bother whether vector fields as defined later are continuous, our theorems will be stated in terms of existence of path–integrals and their values on closed paths, and not in terms of vector fields having to be conservative.

In each of the two settings studied in the following sections we will derive a specific vector field Ψ from an allocation rule f and valuations v . Given this Ψ , the following definition provides the key property of IC allocation rules.

Definition 4. *Given T , v and f , and a vector field Ψ (to be specified later), f is called path–monotone if path–integrals of Ψ exist and for all s, t and all paths σ from s to t :*

$$l_u(s, t) \geq \int_{\sigma} \Psi \cdot d\sigma. \quad (2)$$

3 Characterizing Incentive Compatibility

In this section we fully characterize IC allocation rules for two settings. First, we study the case where valuations for outcomes are convex functions of types. Second, we consider the case where valuations for outcomes are differentiable functions of types.

3.1 Convex Valuations

We provide a characterization under the following two assumptions on types and valuations.

Assumption 2. *The type space $T \subseteq \mathbb{R}^d$ is a convex set.*

Assumption 3. *For any $a \in A$, $v(a, t) : T \rightarrow \mathbb{R}$ is convex in t .*

Recall that a vector $\nabla \in \mathbb{R}^d$ is a subgradient of a function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ at t if $h(s) \geq h(t) + \nabla \cdot (s - t)$ for all $s \in T$. By Assumption 3 for every $t \in T$ the allocation rule f defines a convex function $v(f(t), \cdot) : T \rightarrow \mathbb{R}$, $s \mapsto v(f(t), s)$. We make the following technical assumption:

Assumption 4. *For every $t \in T$ and $a \in A$ the set of subgradients of $v(a, \cdot)$ at t is nonempty.*

As convex functions have a subgradient at each point in the interior of T , this assumption restricts the set of valuations for which our results hold only in terms of their behavior on the boundary of T . We can now define a vector field $\Psi : T \rightarrow \mathbb{R}^d$ by selecting for each $t \in T$ an element from the subgradient of $v(f(t), \cdot)$ at t . Any such vector field satisfies for all $s, t \in T$

$$v(f(t), s) \geq v(f(t), t) + \Psi(t) \cdot (s - t). \quad (3)$$

The main result of this section is the following theorem.

Theorem 2. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \rightarrow \mathbb{R}$ satisfy Assumptions 2, 3, and 4. Then, $f : T \rightarrow A$ is IC if and only if for any $s, t \in T$, $\int_{L_{s,t}} \Psi \cdot d\sigma$ exists and the following hold:*

(a) for any $s, t \in T$:

$$l_u(s, t) \geq \int_{L_{s,t}} \Psi \cdot d\sigma$$

(b) for all $s_1, s_2, s_3 \in T$, all three distinct:

$$\int_{\Delta_{s_1, s_2, s_3}} \Psi \cdot d\sigma = 0$$

In order to proof Theorem 2 we start with a lemma that relates monotonicity to the existence of path integrals on line segments, which is of interest on its own.¹¹

Lemma 1. *Let $s, t \in T$ and assume that $f : T \rightarrow A$ is monotone. Then the following hold for all $s, t \in T$:*

(a) $\Psi(s) \cdot (t - s) \leq l_u(s, t) \leq \Psi(t) \cdot (t - s),$

(b) the path integral of Ψ on $L_{s,t}$ exists.

Proof. The first property follows immediately from monotonicity and the definitions of $l_u(s, t)$ and Ψ . For the second property define $g : [0, 1] \rightarrow \mathbb{R}$ by $g(\lambda) = \Psi(s + \lambda(t - s)) \cdot (t - s)$ and let $0 \leq \lambda_1 < \lambda_2 \leq 1$, $r_1 = s + \lambda_1(t - s)$ and $r_2 = s + \lambda_2(t - s)$. Then, by using monotonicity and Property (a), we get that

$$\begin{aligned} 0 &\leq l_u(r_1, r_2) + l_u(r_2, r_1) \\ &\leq \Psi(r_2) \cdot (r_2 - r_1) + \Psi(r_1) \cdot (r_1 - r_2) \\ &= (\lambda_2 - \lambda_1)(g(\lambda_2) - g(\lambda_1)), \end{aligned}$$

i.e. $g(\lambda_1) \leq g(\lambda_2)$. Since g is non-decreasing, g is integrable on $[0, 1]$ and

$$\begin{aligned} \int_0^1 g(\lambda) d\lambda &= \int_0^1 \Psi(s + \lambda(t - s)) \cdot (t - s) d\lambda \\ &= \int_{L_{s,t}} \Psi \cdot d\sigma. \end{aligned}$$

Thus the path integral of Ψ along line segment $L_{s,t}$ exists. □

The following lemma establishes the relation between the path integral of Ψ and u -lengths of paths in the type graph of f , again for the case of monotone f .

Lemma 2. *Let $s, t \in T$ and assume that $f : T \rightarrow A$ is monotone. For every $n \geq 1$ we let $S_n = \sum_{i=0}^{n-1} l_u(r_i^n, r_{i+1}^n)$, where $r_k^n := s + \frac{k}{n}(t - s)$ for $0 \leq k \leq n$. Then*

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \int_{L_{s,t}} \Psi \cdot d\sigma, \text{ and} \\ \text{dist}_u(s, t) &\leq \int_{L_{s,t}} \Psi \cdot d\sigma. \end{aligned}$$

¹¹Archer and Kleinberg [1] make even in the case of linear valuations functions the assumption that the allocation rule is locally path integrable in order to get this property. Our lemma shows that this is not necessary.

Proof. Fix $n \geq 1$. According to Lemma 1 we have that for $0 \leq i \leq n-1$

$$\Psi(r_i^n) \cdot (r_{i+1}^n - r_i^n) \leq l_s(r_i^n, r_{i+1}^n) \leq \Psi(r_{i+1}^n) \cdot (r_{i+1}^n - r_i^n).$$

If we sum up the inequalities we get that

$$\sum_{i=0}^{n-1} \Psi(r_i^n) \cdot (r_{i+1}^n - r_i^n) \leq S_n \leq \sum_{i=0}^{n-1} \Psi(r_{i+1}^n) \cdot (r_{i+1}^n - r_i^n).$$

For every $n \in \mathbb{N}$ we define $L_n := \sum_{i=0}^{n-1} \Psi(r_i^n) \cdot (r_{i+1}^n - r_i^n)$ and $U_n := \sum_{i=0}^{n-1} \Psi(r_{i+1}^n) \cdot (r_{i+1}^n - r_i^n)$. By Lemma 1 Ψ is path-integrable on the path $L_{s,t}$. Therefore we have that

$$\lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} U_n = \int_{L_{s,t}} \Psi \cdot d\sigma.$$

Furthermore, since $L_n \leq S_n \leq U_n$, we conclude that

$$\lim_{n \rightarrow \infty} S_n = \int_{L_{s,t}} \Psi \cdot d\sigma.$$

The second property follows from the fact that:

$$\text{dist}_u(s, t) \leq S_n, \quad \forall n.$$

□

We are now ready to prove Theorem 2.

Proof. (\Rightarrow) As f is IC it is monotone. From Lemma 1 it follows that for any $s, t \in T$, the path integral of Ψ exists and

$$\text{dist}_u(s, t) \leq \int_{L_{s,t}} \Psi \cdot d\sigma.$$

As $\text{dist}_u(s, t) < \int_{L_{s,t}} \Psi \cdot d\sigma$ would imply $\text{dist}_u(t, s) + \text{dist}_u(s, t) < 0$, which would contradict cyclical monotonicity, we get:

$$\text{dist}_u(s, t) = \int_{L_{s,t}} \Psi \cdot d\sigma. \tag{4}$$

Since $l_u(s, t) \geq \text{dist}_u(s, t)$, we conclude that the first property holds. To show the second property, observe that

$$\int_{\Delta_{s_1, s_2, s_3}} \Psi \cdot d\sigma = \text{dist}_u(s_1, s_2) + \text{dist}_u(s_2, s_3) + \text{dist}_u(s_3, s_1) \geq 0.$$

By the same reasoning

$$- \int_{\Delta_{s_1, s_2, s_3}} \Psi \cdot d\sigma \geq 0.$$

This proves the second property.

(\Leftarrow) Fix $x \in T$. For every $w \in T$ define the transfers as:

$$p(w) = \int_{\sigma_{x,w}} \Psi \cdot d\sigma - v(f(w), w),$$

where $\sigma_{x,w}$ is an arbitrary path from x to w . Now for every $s, t \in T$ we have:

$$p(t) - p(s) = \int_{\sigma_{x,w}} \Psi \cdot d\sigma - v(f(t), t) - \int_{\sigma_{x,s}} \Psi \cdot d\sigma + v(f(s), s)$$

Since path-integrals of Ψ on closed paths are equal to 0 and $l_u(s, t) \geq \int_{\sigma_{s,t}} \Psi \cdot d\sigma$ we get

$$p(t) - p(s) \leq v(f(s), s) - v(f(t), s).$$

which means f is IC. □

Note that Lemma 1 and Lemma 2 required only that f is monotone, which is implied by condition (a) of Theorem 2. We will see in Section 4 that both are not equivalent, and monotonicity together with condition (b) of Theorem 2 is not necessarily sufficient for IC.

In Heydenreich et al. [8] it is shown that for any IC rule f RE holds if and only if $dist_p(s, t) = -dist_p(t, s)$ in T_f . By the relation between p -lengths and u -lengths, the same characterization can be stated in terms of distances with respect to u -lengths. From (4) it follows that

$$dist_u(s, t) + dist_u(t, s) = 0.$$

Thus we get as a corollary a RE result by Krishna and Maenner.

Corollary 1. (Krishna and Maenner [12]) *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \rightarrow \mathbb{R}$ satisfy Assumptions 2, 3, and 4. If f is IC, then any two transfers that implement f differ by at most a constant.*

The way Krishna and Maenner prove their theorem yields an alternative proof for \Rightarrow of Theorem 2, however in a less elementary way.

3.2 Differentiable Valuations

Instead of Assumptions 2 and 3 we now make the following assumptions on types and valuations.

Assumption 5. *For any $a \in A$, $v(a, t) : T \rightarrow \mathbb{R}$ is differentiable in t .*

Assumption 6. *The function $\mathbf{r} : T \rightarrow \mathbb{R}$, $\mathbf{r}(t) \rightarrow \sup_{a \in A} \|\nabla_t v(a, t)\|$ is bounded on T .*

By Assumption 5 the allocation rule f defines for all $t \in T$ a differentiable function $v(f(t), \cdot) : T \rightarrow \mathbb{R}$, $s \mapsto v(f(t), s)$. By taking the gradient of this function at $s = t$ we get a vector field Ψ , more formally:

$$\begin{aligned} \Psi : T &\rightarrow \mathbb{R}^d \\ \Psi : t &\mapsto \nabla_s v(f(t), s)|_{s=t} \end{aligned}$$

Recall that f is said to be path-monotone (with respect to some Ψ) if (2) holds.¹²

Theorem 3. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \rightarrow \mathbb{R}$ satisfy Assumptions 1, 5 and 6. Then, $f : T \rightarrow A$ is IC if and only if path integrals of Ψ exist and*

(a) f is path-monotone,

(b) for every closed path σ

$$\int_{\sigma} \Psi \cdot d\sigma = 0.$$

Proof. (\Rightarrow) IC implies existence of transfers p such that:

$$v(f(s), s) + p(s) \geq v(f(t), s) + p(t) \quad \forall s, t \in T. \quad (5)$$

Fix $s, t \in T$ and a path σ from s to t . Define $M : [0, 1] \rightarrow \mathbb{R}$ as:

$$M(\lambda) = \sup_{x \in \sigma} \{v(f(x), \sigma(\lambda)) + p(x)\}. \quad (6)$$

Since p implements f , the supremum is finite and

$$\sigma(\lambda) \in \arg \max_{x \in \sigma} \{v(f(x), \sigma(\lambda)) + p(x)\}, \quad (7)$$

i.e. the player maximizes his utility with declaring $\sigma(\lambda)$, given his true type is $\sigma(\lambda)$. Using (6), for any $\alpha_1, \alpha_2 \in [0, 1]$ we have:

$$\begin{aligned} |M(\alpha_2) - M(\alpha_1)| &\leq \sup_{x \in \sigma} |v(f(x), \sigma(\alpha_2)) - v(f(x), \sigma(\alpha_1))| \\ &= \sup_{x \in \sigma} \left| \int_{\alpha_1}^{\alpha_2} \frac{\partial}{\partial \lambda} v(f(x), \sigma(\lambda)) d\lambda \right| \\ &= \sup_{x \in \sigma} \left| \int_{\alpha_1}^{\alpha_2} \nabla_s v(f(x), s)|_{s=\sigma(\lambda)} \cdot \sigma'(\lambda) d\lambda \right| \\ &\leq |\alpha_2 - \alpha_1| \sup_{\lambda} \|\sigma'(\lambda)\| \sup_{x \in \sigma} \|\nabla_s v(f(x), s)|_{s=\sigma(\lambda)}\|. \end{aligned}$$

This implies that M is Lipschitz continuous and therefore differentiable almost everywhere¹³ and

$$M(1) - M(0) = \int_0^1 M'(\lambda) d\lambda.$$

By the Envelope Theorem (we make use of the version in Milgrom and Segal [13]) it follows from (7) that for every $\lambda \in (0, 1)$:

$$\begin{aligned} M'(\lambda) &= \frac{\partial}{\partial \mu} (v(f(\sigma(\lambda)), \mu) + p(\sigma(\lambda)))|_{\mu=\sigma(\lambda)} \\ &= \frac{\partial}{\partial \mu} v(f(\sigma(\lambda)), \mu)|_{\mu=\sigma(\lambda)} \cdot \sigma'(\lambda) \\ &= \Psi(\sigma(\lambda)) \cdot \sigma'(\lambda). \end{aligned}$$

¹²The reason that we do not refer to path-monotonicity in Theorem 2 is that we need the relation only for paths that are line-segments.

¹³In fact, every Lipschitz continuous function is absolutely continuous and therefore it is differentiable almost everywhere

So, Ψ is integrable along the path σ and

$$\begin{aligned} \int_{\sigma} \Psi \cdot d\sigma &= M(1) - M(0) \\ &= v(f(t), t) + p(t) - v(f(s), s) - p(s). \end{aligned} \tag{8}$$

It follows that path integrals of Ψ on closed paths are equal to 0. Furthermore,

$$\begin{aligned} l_u(s, t) &= v(f(t), t) - v(f(t), s) \\ &= v(f(t), t) - v(f(t), s) + v(f(s), s) - v(f(t), s) \\ &\geq v(f(t), t) - v(f(t), s) + p(t) - p(s) \\ &= \int_{\sigma} \Psi \cdot d\sigma, \end{aligned}$$

where the inequality follows from (5). Hence f is path-monotone.

(\Leftarrow) is proven in the same way as in Theorem 2. \square

That the conditions of Theorem 3 are necessary for IC follows also from Milgrom and Segal [13], who show that under Assumptions 5 and 6, IC transfers of IC allocation rules satisfy (8). They provide a proof for one-dimensional T and explain how to extend it to the multi-dimensional case. Their paper is silent on sufficient conditions for IC.

We observe that for convex T we may replace in Theorem 3 general path integrals by path integrals on line segments, and we obtain the following analogue of Theorem 2.

Corollary 2. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \rightarrow \mathbb{R}$ satisfy Assumptions 5 and 6 and let T be convex. Then, $f : T \rightarrow A$ is IC if and only if for any $s, t \in T$, $\int_{L_{s,t}} \Psi \cdot d\sigma$ exists and the following hold:*

(a) for any $s, t \in T$:

$$l_u(s, t) \geq \int_{L_{s,t}} \Psi \cdot d\sigma,$$

(b) for all $s_1, s_2, s_3 \in T$, all three distinct:

$$\int_{\Delta_{s_1, s_2, s_3}} \Psi \cdot d\sigma = 0.$$

The following example shows that path-monotonicity does not imply condition (b) of Theorem 3. As the allocation rule below is also convex, the same holds for the setting of Theorem 2.

Example 1. *Let $s = (1, 0, 0)$, $t = (0, 1, 0)$ and $u = (0, 0, 1)$ and let $T \subseteq \mathbb{R}^3$ be the convex hull of $\{s, t, u\}$. For any $\mathbf{t} \in T$, $f(\mathbf{t}) = \mathbf{t}$ and the valuation functions are given by:*

$$v(\mathbf{a}, \mathbf{t}) = (a_1 + a_2)t_1 + (a_2 + a_3)t_2 + (a_1 + a_3)t_3,$$

where a_i and t_i are the i -th component of the vector \mathbf{a} and \mathbf{t} , respectively. The valuations are linear and thus also differentiable and convex. We now have that

$$\Psi(\mathbf{t}) = (t_1 + t_2, t_2 + t_3, t_1 + t_3).$$

While f is path-monotone, the integral of Ψ over $\Delta_{s,t,u}$ is not equal to zero:

$$\begin{aligned} \int_{\Delta_{s,t,u}} \Psi \cdot d\sigma &= \int_{L_{s,t}} \Psi \cdot d\sigma + \int_{L_{t,u}} \Psi \cdot d\sigma + \int_{L_{u,s}} \Psi \cdot d\sigma \\ &= \frac{-1}{2} + \frac{-1}{2} + \frac{-1}{2} \\ &= \frac{-3}{2}. \end{aligned}$$

As for convex valuations we get also a result on RE. Suppose transfers p and q both implement f . Fix $s \in T$ and let $k = q(s) - p(s)$. According to (8), for any $t \in T$ we have:

$$\begin{aligned} q(t) &= \int_{\sigma_{s,t}} \Psi \cdot d\sigma - v(f(t), t) + v(f(s), s) + q(s) \\ &= \int_{\sigma_{s,t}} \Psi \cdot d\sigma - v(f(t), t) + v(f(s), s) + p(s) + k \\ &= p(t) + k. \end{aligned}$$

This shows

Corollary 3. (Milgrom and Segal [13]) *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \rightarrow \mathbb{R}$ satisfy Assumptions 1, 5 and 6.. If f is IC, then any two transfers that implement f differ by at most a constant.*

4 Monotonicity, Path-Monotonicity and Linear Valuations

In the previous section we have characterized IC by two properties of path-integrals of particular vector fields, induced by valuation functions and the allocation rule. One of them, path-monotonicity, obviously implies monotonicity, which is a common property in the theory of IC rules. The goal of this section is to understand when monotonicity implies path-monotonicity. We will assume throughout that T is convex, and that we are either in the setting of convex valuations as in Section 3.1, or differentiable valuations as in the setting of Section 3.2. Furthermore, we assume in our discussion below that path integrals of Ψ exist. We start by an example that shows that monotonicity does not always imply path-monotonicity.

Example 2. *Suppose $T = [0, 1]$. Consider the following allocation rule:*

$$f(t) = \begin{cases} a & 0 \leq t \leq \frac{1}{3} \\ b & \frac{1}{3} < t \leq \frac{2}{3} \\ c & \frac{2}{3} < t \leq 1. \end{cases}$$

Define the valuation functions by

$$v(a, t) = \begin{cases} 0 & t \leq \frac{2}{3} \\ 3t - 2 & t > \frac{2}{3}, \end{cases}$$

$v(b, t) = 3t$ and

$$v(c, t) = \begin{cases} 2 - 3t & t \leq \frac{1}{3} \\ 3t & t > \frac{1}{3}. \end{cases}$$

Then we have that

$$\Psi(t) = \begin{cases} 0 & 0 \leq t \leq \frac{1}{3} \\ 3 & \frac{1}{3} < t \leq 1. \end{cases}$$

One can easily verify that the allocation rule f is monotone¹⁴, but not path-monotone:

$$1 = l_u(0, 1) < \int_{L_{0,1}} \Psi \cdot d\sigma = 2.$$

Example 2 raises the question when condition (a) in Theorems 2 or 3 or Corollary 2 can be replaced by monotonicity, in other words when does monotonicity imply path-monotonicity? The key will be decomposition monotonicity. We first recall that under the assumption of decomposition monotonicity, all monotone allocation rules are IC when restricted to line segments.

Lemma 3. *Let $T \subseteq \mathbb{R}^n$, A a set of outcomes, $v : T \times A \rightarrow \mathbb{R}$, and $f : T \rightarrow \mathbb{R}$ be decomposition monotone. Let $s, t \in T$. Then f is IC on $L_{s,t}$ if and only if f is monotone on $L_{s,t}$.*

The proof is straightforward. It uses the simple observation that the length of any cycle with nodes in $L_{s,t}$ can be lower-bounded by the sum of lengths of two-cycles. The latter is non-negative if f is monotone.

Now assume that we are in the setting of Corollary 2 or Theorem 2, and f is decomposition monotone. By Lemma 3 we know that a monotone f is IC on lines. By Corollary 2 and Theorem 2, respectively, we get that f must be path-monotone when restricted to such lines. This proves:

Lemma 4. *Let T , v and f be such that Assumption 2 and either Assumptions 5 and 6 or Assumptions 3 and 4 hold, and let f be decomposition monotone. Then f is monotone if and only if it is path-monotone.*

This yields the following versions of Theorem 2 and Corollary 2.

Theorem 4. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \rightarrow \mathbb{R}$ and $f : T \rightarrow A$ satisfy Assumptions 2, 3, and 4, and assume $f : T \rightarrow A$ is monotone and decomposition monotone. Then for any $s, t \in T$, $\int_{L_{s,t}} \Psi \cdot d\sigma$ exists. Furthermore f is IC if and only if for all $s_1, s_2, s_3 \in T$, all three distinct*

$$\int_{\Delta_{s_1, s_2, s_3}} \Psi \cdot d\sigma = 0.$$

¹⁴Take types $x \in [0, \frac{1}{3}]$, $y \in (\frac{1}{3}, \frac{2}{3}]$, $z \in (\frac{2}{3}, 1]$ and verify for each combination of them monotonicity.

Theorem 5. Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \rightarrow \mathbb{R}$ and $f : T \rightarrow A$ satisfy Assumptions 2, 5 and 6, and assume $f : T \rightarrow A$ is monotone and decomposition monotone. Then path integrals of Ψ exist. Furthermore f is IC if and only if for all $s_1, s_2, s_3 \in T$, all three distinct

$$\int_{\Delta_{s_1, s_2, s_3}} \Psi \cdot d\sigma = 0.$$

A direct proof of Theorem 4 can also be found in Wolf [22]. The following example illustrates how these theorems may be used for an easy verification of IC.

Example 3. Suppose $T = A = [0, 1]^2$ and for every t and a in $[0, 1]^2$ we define the valuation function $v(a, t)$ as the Euclidean distance of these two points in the plane: $v(a, t) = \|a - t\| = \sqrt{(a_1 - t_1)^2 + (a_2 - t_2)^2}$. Let the allocation rule be defined by $f(t_1, t_2) = (1 - t_1, 1 - t_2)$ for every $(t_1, t_2) \in T$.

In order to have monotonicity, for every s and t in $[0, 1]^2$ we must have $\|f(s), s\| + \|f(t), t\| \geq \|f(t), s\| + \|f(s), t\|$. This fact, however, follows easily from the triangle inequality, as the line segments from s to $f(s)$ and from t to $f(t)$ always cross in the “midpoint” $(1/2, 1/2)$ of T (cf. Fig. 1 (left)).

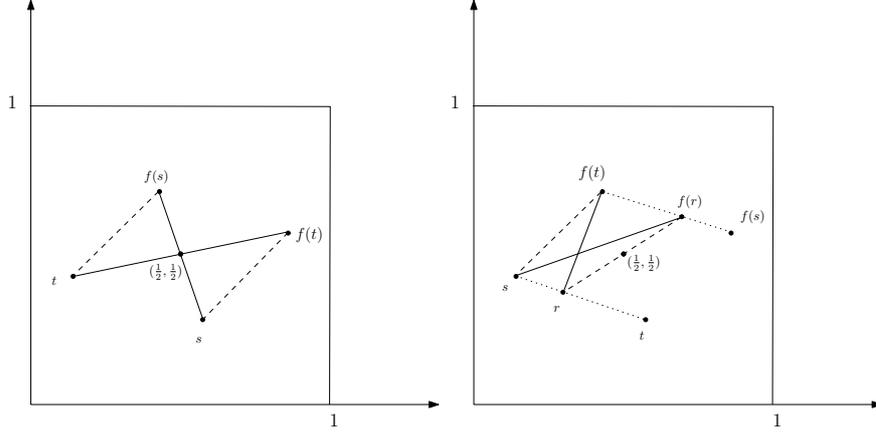


Figure 1: Left: The segment $(t, f(t))$ passes through $(\frac{1}{2}, \frac{1}{2})$ for every $t \in T$. Right: Segments $(s, f(r))$ and $(r, f(t))$ always cross each other.

Decomposition monotonicity of f can be shown similarly (cf. Fig. 1 (right)). Let us now verify the condition from Theorem 4 that will ensure that f is IC. We may choose for Ψ

$$\Psi(t_1, t_2) = \begin{cases} \left(\frac{2t_1-1}{\sqrt{(1-2t_1)^2+(1-2t_2)^2}}, \frac{2t_2-1}{\sqrt{(1-2t_1)^2+(1-2t_2)^2}} \right) & (t_1, t_2) \neq \left(\frac{1}{2}, \frac{1}{2} \right) \\ (-2, 0) & (t_1, t_2) = \left(\frac{1}{2}, \frac{1}{2} \right). \end{cases}$$

For every s and t in $[0, 1]^2$ such that $(\frac{1}{2}, \frac{1}{2}) \notin L_{s,t}$ observe that

$$\begin{aligned} \Psi(s + \lambda(t - s)) \cdot (t - s) &= \frac{(2(s_1 + \lambda(t_1 - s_1)) - 1)(t_1 - s_1) + (2(s_2 + \lambda(t_2 - s_2)) - 1)(t_2 - s_2)}{\sqrt{(1 - 2(s_1 + \lambda(t_1 - s_1)))^2 + (1 - 2(s_2 + \lambda(t_2 - s_2)))^2}} \\ &= \frac{d}{d\lambda} \frac{1}{2} \sqrt{(1 - 2(s_1 + \lambda(t_1 - s_1)))^2 + (1 - 2(s_2 + \lambda(t_2 - s_2)))^2}, \end{aligned}$$

which implies

$$\int_{L_{s,t}} \Psi \cdot d\sigma = \frac{1}{2}(\sqrt{(1-2t_1)^2 + (1-2t_2)^2} - \sqrt{(1-2s_1)^2 + (1-2s_2)^2}).$$

A simple limit argument shows that the same holds if $(\frac{1}{2}, \frac{1}{2}) \in L_{s,t}$. Since the integral depends only on the end points of $L_{s,t}$ we can conclude that $\int_{\Delta_{s_1, s_2, s_3}} \nabla f(\sigma) \cdot d\sigma = 0$ for all $s_1, s_2, s_3 \in [0, 1]^2$. Therefore, according to Theorem 2, f is IC. If we normalize the utility of type $\bar{s} = (\frac{1}{2}, \frac{1}{2})$ to be equal to 0, the utility of any other type t will be equal to the distance to \bar{s} . Observing that the valuation for reporting truthful is twice the distance to \bar{s} , the payment has to be equal to minus the distance to \bar{s} .

Decomposition monotonicity is not a necessary condition for an allocation rule to be IC, as can be seen by the following example.

Example 4. Let $T = [0, 1]$, $A = \{a, b, c\}$, and valuation functions given by: $v(a, t) = |2t - 1|$,

$$v(b, t) = \begin{cases} 1 - t & t \leq \frac{1}{2} \\ t & t > \frac{1}{2} \end{cases}$$

and $v(c, t) = 1$. Consider the following allocation rule:

$$f(t) = \begin{cases} a & 0 \leq t \leq \frac{1}{8} \\ b & \frac{1}{8} < t \leq \frac{3}{8} \\ c & \frac{3}{8} < t \leq \frac{5}{8} \\ b & \frac{5}{8} < t \leq \frac{7}{8} \\ a & \frac{7}{8} < t \leq 1. \end{cases}$$

It is easy to check that the function is monotone and that it does not satisfy decomposition monotonicity, but that it is IC.

Decomposition monotonicity puts a condition on f . It is illustrative to understand which conditions on T , v , and A let this condition hold for any monotone f .

Definition 5. Let T be convex, A a set of outcomes, and $v : T \times A \rightarrow \mathbb{R}$. We say that the increasing difference property holds if for all $s, t \in T$, $x \in L_{s,t}$, $a, b \in A$, we have that $v(b, t) - v(a, t) \geq v(b, x) - v(a, x)$ implies $v(b, x) - v(a, x) \geq v(b, s) - v(a, s)$.

We note that the increasing difference property is implied by the well-known single-crossing property, though the definition of the latter requires assumptions on differentiability which we do not need to make here. A simple verification shows that the following lemma holds:

Lemma 5. Let T be convex, A a set of outcomes, and $v : T \times A \rightarrow \mathbb{R}$ such that the increasing difference property holds. Then every monotone f is decomposition monotone.

Let us finally consider the case where $v(a, t) : T \rightarrow \mathbb{R}$ is linear in t and T is convex. Obviously this is a special case of differentiable valuations and convex valuations. Furthermore, it is easy to see that the increasing difference property holds in such a setting and Theorem 4 applies.

Corollary 4. *Let T be convex. Assume that for every fixed $a \in A$ the function $v(a, \cdot) : T \rightarrow \mathbb{R}$ is linear. Let $f : T \rightarrow A$ be monotone. Then for any $s, t \in T$, $\int_{L_{s,t}} \Psi \cdot d\sigma$ exists. Furthermore, f is IC if and only if for all $s_1, s_2, s_3 \in T$, all three distinct*

$$\int_{\Delta_{s_1, s_2, s_3}} \Psi \cdot d\sigma = 0.$$

Theorems in the spirit of Corollary 4 have been obtained by Jehiel and Moldovanu [9], Müller et al. [14], and Archer and Kleinberg [1].

5 Local Incentive Compatibility

Motivated by results in Archer and Kleinberg [1] for the case of valuations that are linear in the type, we introduce in this section the notion of *local IC*. We want to understand when local IC implies IC. This understanding will be crucial to show that in the case of finite A (path-)monotonicity alone implies IC. Having said this we use the following definition (here S_1 denotes the unit circle in \mathbb{R}^2).

Definition 6. *$T \subseteq \mathbb{R}^d$ is said to be simply connected if for every closed smooth¹⁵ path $\sigma : S_1 \rightarrow T$ there exists a smooth extension $\tau : D_2 \rightarrow T$ from the unit disk $D_2 \subseteq \mathbb{R}^2$ to T such that $\tau|_{S_1} = \sigma$.*

Definition 7. *An allocation rule $f : T \rightarrow A$ is called locally IC if for every $t \in T$ there exists an open neighborhood $U(t)$ such that $f : T \cap U(t) \rightarrow f(U(t) \cap T)$ is IC.*

Theorem 6. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \rightarrow \mathbb{R}$ satisfy Assumptions 1, 5 and 6. Furthermore, assume that T is simply connected. Then, $f : T \rightarrow A$ is implementable if and only if path integrals of Ψ exist and:*

1. f is path-monotone,
2. f is locally implementable.

Proof. Applying Theorem 3 proves (\Rightarrow) . In order to prove the other direction, it is, due to the same Theorem, sufficient to show that (2) implies

$$\int_{\sigma} \Psi \cdot d\sigma = 0.$$

for any closed path σ .

¹⁵We are aware that we use terminology from topology and differential geometry quite freely, and omit to work out the difference between connectedness of a set T with respect to smooth paths, as common in differential geometry, and connectedness with respect to continuous paths, as common in topology. We assume that in cases of economic relevance our definition will be of sufficient generality.

Fix $\sigma : S_1 \rightarrow T$, and consider an extension τ of σ to D_2 . Since $\tau(D_2)$ is closed and bounded it is compact. Local IC implies that for every $t \in \tau(D_2)$ there exists $\varepsilon(t)$ and an open neighborhood $U(t, \varepsilon(t))$ such that f is IC on $U(t, \varepsilon(t)) \cap T$. For every $x \in D_2$ there exists a $\delta(x)$ such that $\tau(U(x, \delta(x)) \cap D_2) \subseteq U(\tau(x), \varepsilon(\tau(x)))$.

The Lebesgue Number Lemma (see, e.g. [15]) implies that there is a δ' such that every subset of D_2 of diameter less than δ' is contained in at least one of the sets $U(x, \delta(x))$, $x \in D_2$. Now partition D_2 by a grid of which each cell has diameter strictly smaller than δ' . This decomposes D_2 into cells (of which some have a part of S_1 as border) with the property that any border B of these cells is completely contained in $U(x, \delta(x))$ for some x . The path $\tau(B)$ is contained in the neighborhood $U(\tau(x), \varepsilon(\tau(x)))$. As f is IC on this neighborhood the path integral of Ψ with respect to this path is equal to 0.

By construction the path integral of Ψ with respect to σ equals the sum of the path integrals on the border of the cells, as path integrals on lines originating from the grid cancel each other out. \square

Note that every convex set T is simply connected. Using the same arguments as in the proof of Theorem 6 we get an analogous theorem for the setting from Section 3.1.

Theorem 7. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \rightarrow \mathbb{R}$ satisfy Assumptions 3, 2, and 4. Then, $f : T \rightarrow A$ is IC if and only if for any $s, t \in T$ $\int_{L_{s,t}} \Psi \cdot d\sigma$ exists and:*

1. for any $s, t \in T$:

$$l_u(s, t) \geq \int_{L_{s,t}} \Psi \cdot d\sigma.$$

2. f is locally IC.

Obviously, there are similar variants of the characterization theorems in Section 4. We state only the following.

Theorem 8. *Let T be convex. Assume that for every fixed $a \in A$ the function $v(a, \cdot) : T \rightarrow \mathbb{R}$ is linear. Then f is IC if and only if it is monotone and locally IC.*

Theorem 8 has previously been proven in Archer and Kleinberg [1]. However, their proof needed the assumption that path integrals on line segments exist. We have shown in Lemma 1 that this property follows from monotonicity, even in the case of convex valuation functions.

6 Finite Outcome Space

Saks and Yu [20] were the first to show that in case of finite A , convex T and valuations that are linear in types, monotonicity alone already implies IC. Such a result simplifies significantly the identification of IC rules. We show in this section that such simplification is not limited to the linear case, but works for all other settings studied in the previous sections. Key will be the following lemma, which has been proven for the case of linear valuations in Ashlagi et al. [2].

Lemma 6. Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \rightarrow \mathbb{R}$ be continuous in t for fixed a . For all $a \in A$ let

$$D_a := \overline{f^{-1}(a)}.$$

If $f : T \rightarrow A$ is monotone and $\bigcap_{a \in A} D_a \neq \emptyset$, then f is IC.¹⁶

Proof. Let $\{s_1, \dots, s_k\} \subseteq T$ for some $k \geq 3$ and $t \in \bigcap_{a \in A} D_a$. Fix $1 \leq i \leq k$. Since $t \in D_{f(s_{i+1})}$, there is a sequence $(t_j)_{j \in \mathbb{N}}$, such that $f(t_j) = f(s_{i+1})$ for every $j \in \mathbb{N}$ and $\lim_{j \rightarrow \infty} t_j = t$.¹⁷ Note that

$$\begin{aligned} l_p(s_i, s_{i+1}) &= v(f(s_i), s_i) - v(f(s_{i+1}), s_i) = v(f(s_i), s_i) - v(f(t_j), s_i) \\ &\geq v(f(s_i), t_j) - v(f(t_j), t_j) = v(f(s_i), t_j) - v(f(s_{i+1}), t_j). \end{aligned}$$

By continuity of v in t we get

$$l_p(s_i, s_{i+1}) \geq v(f(s_i), t) - v(f(s_{i+1}), t).$$

If we sum up all inequalities, we have:

$$\sum_{i=1}^k l_p(s_i, s_{i+1}) \geq \sum_{i=1}^k v(f(s_i), t) - v(f(s_{i+1}), t) = 0.$$

Invoking Theorem 1 completes the proof. \square

Theorem 9. Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \rightarrow \mathbb{R}$ satisfy Assumptions 1, 5 and 6. Furthermore let T be simply connected and A be finite. Then, $f : T \rightarrow A$ is IC if and only if path integrals of Ψ exist and f is path-monotone.

Proof. We prove that f is locally IC, then according to Theorem 6 f is IC. Fix $t \in T$. For all $a \in A$ let $\varepsilon_a(t) := \inf_{x \in D_a} \|x - t\|_2$ if $D_a \neq \emptyset$ and $\varepsilon_a(t) = \infty$ otherwise. Then,

$$t \in D_a \Leftrightarrow \varepsilon_a(t) = 0.$$

We show existence of a neighborhood $U(t)$ around t such that $t \in D_a$ for all $a \in f(U(t))$. Set $A(t) := \{a \in A : \varepsilon_a(t) = 0\}$. As $t \in D_{f(t)}$, we have that $A(t) \neq \emptyset$ and $t \in \bigcap_{a \in A(t)} D_a$. If $A(t) = A$ we

let $U(t) = \mathbb{R}^d$, otherwise let

$$\varepsilon = \min\{\varepsilon_a(t) : a \in A \setminus A(t)\}.$$

Note that $\varepsilon > 0$. Define $U(t) = \{x \in \mathbb{R}^d : \|x - t\|_2 < \varepsilon\}$. Since $v(a, t)$ is continuous in t for all a , we can invoke Lemma 6 to prove that f is IC on $U(t)$. In other words, f is locally IC. \square

Theorem 10. Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \rightarrow \mathbb{R}$ satisfy Assumptions 2, 3, and 4. Furthermore let A be finite. Then, $f : T \rightarrow A$ is IC if and only if for any $s, t \in T$, $\int_{L_{s,t}} \Psi \cdot d\sigma$ exists and

$$l_u(s, t) \geq \int_{L_{s,t}} \Psi \cdot d\sigma.$$

¹⁶ \overline{X} denotes the topological closure of a set $X \subseteq \mathbb{R}^d$.

¹⁷Indices are taken modulo k .

Proof. Analogues to the proof of Theorem 9. □

In light of the results from Section 4 we get the Theorem by Saks and Yu as a corollary.

Corollary 5. (Saks and Yu [20])

Let T be convex. Assume that A is finite and for every fixed $a \in A$ the function $v(a, \cdot) : T \rightarrow \mathbb{R}$ is linear. Then f is IC if and only if it is monotone.

Ashlagi et al. [?] have shown that convexity of T is also necessary in the following sense: whenever the closure of a type space T is not convex one can construct a finite set A , linear valuations v , and a monotone allocation rule f that is not IC.

Using the Lebesgue Number Lemma, one can easily show that under the additional assumption of decomposition monotonicity, monotonicity is implied by local monotonicity, i.e., monotonicity in some neighborhood of each type. Therefore, we may replace in settings where decomposition monotonicity holds path-monotonicity by local monotonicity in Theorem 10.

7 Conclusions

We have shown that for multi-dimensional type spaces and either differentiable or convex valuation functions, IC allocation rules are characterized by two conditions: the allocation rule has to be IC on one-dimensional paths (line-segments), and path-integrals of a particular vector field on closed paths (triangles) are equal to 0. The vector field measures essentially marginal value differences from deviations from truthful reports. IC on one-dimensional paths is a stronger condition than monotonicity. However we have identified a general class of settings where both are equivalent. Next we have shown that the condition on path-integrals may be replaced by local IC, if the type space is simply connected. From this it follows that for a broad range of settings allocation rules with finite set of outcomes are implementable if and only if they are IC on one-dimensional paths. In particular we get for convex type spaces with valuation functions for which decreasing differences holds on every line, that an allocation rule is implementable if and only if it is monotone. We have shown by examples that all results are best possible in the sense that none of the conditions in our characterizations imply each other.

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References

- [1] A. Archer and R. Kleinberg. Truthful germs are contagious: a local to global characterization of truthfulness. In *EC '08: Proceedings of the 9th ACM conference on Electronic commerce*, pages 21–30, New York, NY, USA, 2008. ACM.
- [2] I. Ashlagi, M. Braverman, A. Hassidim, and D. Monderer. Monotonicity and implementability. *Econometrica (to appear)*, Online first, 2010.
- [3] A. Berger, R. Müller, and S. H. Naeemi. Characterizing incentive compatibility for convex valuations. In M. Mavronicolas and V. G. Papadopoulou, editors, *Algorithmic Game Theory*, volume 5814 of *Lecture Note in Computer Science*, pages 24–35. Springer, 2009.
- [4] S. Bikhchandani, S. Chatterjee, R. Lavi, A. Mu’alem, N. Nisan, and A. Sen. Weak monotonicity characterizes deterministic dominant strategy implementation. *Econometrica*, 74(4):1109–1132, 2006.
- [5] J. Carbayal and J. Ely. Implementation and revenue equivalence without differentiability. Technical report, University of Queensland and University of Minnesota, April 2010.
- [6] K.-S. Chung and W. Olszewski. A non-differentiable approach to revenue equivalence. *Theoretical Economics*, 2:469–487, 2007.
- [7] H. Gui, R. Müller, and R. Vohra. Dominant strategy mechanisms with multidimensional types. Discussion Paper 1392, The Center for Mathematical Studies in Economics & Management Sciences, Northwestern University, Evanston, IL, 2004.
- [8] B. Heydenreich, R. Müller, M. Uetz, and R. Vohra. Characterization of revenue equivalence. *Econometrica*, 77:307–316, 2009.
- [9] P. Jehiel and B. Moldovanu. Efficient design with interdependent valuations. *Econometrica*, 69(5):1237–1259, 2001.
- [10] P. Jehiel, B. Moldovanu, and E. Stacchetti. Multidimensional mechanism design for auctions with externalities. *Journal of Economic Theory*, 85(2):258–293, 1999.
- [11] N. Kos and M. Messner. Extremal incentive compatible transfers. Technical Report 359, IGIER, Bocconi University, January 2010.
- [12] V. Krishna and E. Maenner. Convex potentials with an application to mechanism design. *Econometrica*, 69(4):1113–1119, 2001.
- [13] P. Milgrom and I. Segal. Envelope theorems for arbitrary choice sets. *Econometrica*, 70(2):583–601, 2002.
- [14] R. Müller, A. Perea, and S. Wolf. Weak monotonicity and Bayes-Nash incentive compatibility. *Games and Economic Behavior*, 61(2):344–358, 2007.

- [15] J. R. Munkres. *Topology*. Prentice Hall, 2nd edition, 2000.
- [16] R. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, 1981.
- [17] D. Rahman. Detecting profitable deviations. Technical report, University of Minnesota, February 2010.
- [18] J.-C. Rochet. A necessary and sufficient condition for rationalizability in a quasi-linear context. *Journal of Mathematical Economics*, 16(2):191–200, 1987.
- [19] H. L. Royden. *Real analysis*. Macmillan Publishing Company, New York, third edition, 1988.
- [20] M. Saks and L. Yu. Weak monotonicity suffices for truthfulness on convex domains. In *Proc. 6th ACM conference on Electronic Commerce*, pages 286 – 293, 2005.
- [21] R. Vohra. Mechanism design: A linear programming approach. Technical report, Northwestern University, 2009.
- [22] S. Wolf. *Monotonicity and Bayes Nash Implementation, PhD Thesis*. Maastricht University, 2010.