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Abstract

Incomplete preferences over lotteries on a finite set of alternatives satisfying, besides independence and continuity, a property called *bad outcome aversion* are considered. These preferences are characterized in terms of their specific multi-expected utility representations (cf. Dubra *et al.*, 2004), and can be seen as generalized stochastic dominance preferences.

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1 Introduction

A familiar way to order probability distributions on a set of alternatives is to use (first or higher degree) stochastic dominance. A typical property of such an (incomplete) ordering or preference is that a positive probability on a bad alternative cannot be compensated by putting high probabilities on better alternatives. For instance, a probability distribution that puts positive probability on the worst alternative can never stochastically dominate another probability distribution that puts less probability on that alternative, not even by putting all remaining probability on a best alternative.

In this paper we study and characterize this typical property, which we call *bad outcome aversion* (BOA). Specifically, we consider incomplete preferences over lotteries on a finite set of alternatives and assume the classical conditions of (von Neumann and Morgenstern) independence and continuity, so that the 'multi-expected utility' theorem of Dubra *et al.* (2004) applies. This result characterizes such preferences in terms of representing closed and convex sets of functions. Our main result (Theorem 4.3) characterizes BOA for such preferences in terms of specific elements contained in these representing sets of functions.

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Thus, our paper offers a broad generalization of stochastic dominance preferences. The literature on stochastic dominance is vast: see Levy (1992) for an overview of theory and applications. For our paper, in particular Fishburn (1976) is of interest. One other direct source of inspiration is our recent work on an application of stochastic dominance preferences in two-person non-cooperative games, see Perea *et al.* (2006); that paper, in turn, builds on Fishburn (1978).

Section 2 formulates the model and recalls the multi-expected utility characterization of Dubra *et al.* (2004). Section 3 studies stochastic dominance preferences and in particular adapts Fishburn (1976) to our context. The bad outcome aversion condition is introduced in Section 4, which also contains our main results. Section 5 concludes.

2 Preliminaries

Let $X := \{x_1, \ldots, x_n\}$, where $n \geq 3$, be a finite set of *alternatives* and let $\Delta(X)$ denote the set of probability distributions (lotteries) over X. We also use the letters $x, y, \ldots$ to denote elements of X. A *preference* $\succeq$ is a reflexive and transitive binary relation on $\Delta(X)$. If $(p, q) \in \succeq$ we say that p is (weakly) preferred over q. Instead of $(p, q) \in \succeq$ we often use the notation $p \succeq q$. We write $p \succ q$ if $p \succeq q$ and $q \not\succeq p$, and $p \sim q$ if $p \succeq q$ and $q \succeq p$. For $p \in \Delta(X)$ and $i \in \{1, \ldots, n\}$, p_i denotes the probability that p assigns to x_i, and $p(x)$ the probability that p assigns to $x \in X$. The degenerate lottery that assigns probability one to the alternative $x \in X$ is identified with x. Observe that we do not require completeness of $\succeq$.

The following possible conditions on $\succeq$ are well-known.

Axiom 2.1 (Independence) *For all $p, q, r \in \Delta(X)$ and $0 \leq \lambda \leq 1$,*

$$p \succeq q \Rightarrow \lambda p + (1 - \lambda) r \succeq \lambda q + (1 - \lambda) r.$$

Axiom 2.2 (Continuity) *For all $q \in \Delta(X)$, the sets $\{p \in \Delta(X) \mid p \succeq q\}$ and $\{p \in \Delta(X) \mid q \succeq p\}$ are closed in $\Delta(X)$.*

Let $U \subseteq \mathbb{R}^X$ be a set of real-valued functions on X. For $u \in \mathbb{R}^X$ and a lottery $p \in \Delta(X)$ denote by

$$\mathbb{E}_u(p) := \sum_{i=1}^{n} p_i u(x_i)$$

the expectation of p under u. We say that U *represents* the preference $\succeq$ if for all $p, q \in \Delta(X)$,

$$p \succeq q \Leftrightarrow \mathbb{E}_u(p) \geq \mathbb{E}_u(q) \text{ for all } u \in U.$$

The following 'multi-expected utility' theorem follows from Dubra *et al.* (2004) and generalizes the familiar von Neumann-Morgenstern expected utility theorem to incomplete preferences.

Theorem 2.3 *Let $\succeq$ be a preference. Then $\succeq$ satisfies independence and continuity if and only if there is a closed and convex set $U \subseteq \mathbb{R}^X$ that represents $\succeq$.*

3 Stochastic dominance

First degree stochastic dominance is a well-known example of a preference to which Theorem 2.3 applies. For any permutation π of $\{1, \ldots, n\}$, the first degree stochastic dominance preference $\succeq_\pi$ is defined by

$$p \succeq_\pi q \Leftrightarrow \sum_{i=1}^{j} p_{\pi(i)} \le \sum_{i=1}^{j} q_{\pi(i)} \text{ for all } j = 1, \ldots, n$$

for all $p, q \in \Delta(X)$. Note that $\succeq_\pi$ strictly orders all alternatives of X, specifically, $x_{\pi(n)} \succ_\pi \ldots \succ_\pi x_{\pi(1)}$. So first degree stochastic dominance preferences are complete on degenerate lotteries. Conversely, if a preference satisfies independence and strictly orders all alternatives of X, then it contains a first degree stochastic dominance preference. To show this formally, we introduce the following condition. This condition says that, *ceteris paribus*, shifting probability to a better alternative makes a lottery preferable.

Axiom 3.1 (Improvement) *For all $p, q \in \Delta(X)$ and all $x, y \in X$ such that (i) $x \succeq y$, (ii) $p(z) = q(z)$ for all $z \in X \setminus \{x, y\}$, and (iii) $p(x) > q(x)$ and (hence) $p(y) < q(y)$, we have $p \succeq q$.*

Lemma 3.2 *Let $\succeq$ satisfy independence. Then $\succeq$ satisfies improvement.*

Proof. Let p, q, x, y satisfy the conditions in the statement of the improvement axiom. Let $\varepsilon := p(x) - q(x) > 0$. Define $\tilde{p} \in \Delta(X)$ by

$$\tilde{p}(z) = \begin{cases} \frac{1}{1-\varepsilon} p(z) & \text{if } z \neq x, y \\ \frac{1}{1-\varepsilon} q(x) & \text{if } z = x \\ \frac{1}{1-\varepsilon} p(y) & \text{if } z = y. \end{cases}$$

Then $x \succeq y$ implies $p = \varepsilon x + (1-\varepsilon)\tilde{p} \succeq \varepsilon y + (1-\varepsilon)\tilde{p} = q$ by independence. ■

Lemma 3.3 *Let $\succeq$ be a preference with either $x \succeq y$ or $y \succeq x$ for all $x, y \in X$, $x \neq y$. Then:*

(i) There is a permutation π of $\{1, \ldots, n\}$ such that $x_{\pi(n)} \succ \ldots \succ x_{\pi(1)}$.

(ii) $\succeq$ satisfies improvement if and only if $\succeq_\pi \subseteq \succeq$.

Proof. Part (i) is obvious. For the 'if' part of (ii), consider p, q, x, y satisfying the conditions in the statement of the improvement axiom. In particular, $x = x_{\pi(i)}$ and $y = x_{\pi(j)}$ for some $i > j$. This implies $p \succeq_\pi q$, and therefore $p \succeq q$.

For the 'only if' part, for simplicity assume π is identity, so that $x_n \succ \ldots \succ x_1$. Let $p, q \in \Delta(X)$ such that $p \neq q$ and $p \succeq_\pi q$. Then, by definition,

$$\sum_{i=1}^{j} p_i \leq \sum_{i=1}^{j} q_i \text{ for all } j = 1, \ldots, n. \tag{1}$$

Hence, for $i^* := \min\{i \mid p_i \neq q_i\}$ we have $q_{i^*} - p_{i^*} > 0$. For $i^* \leq k \leq n$ define $q^k \in \Delta(X)$ by

$$q_i^k := \begin{cases} p_i & \text{if } i \leq k-1 \\ q_k + \sum\limits_{l=0}^{k-1} (q_l - p_l) & \text{if } i = k \\ q_i & \text{if } i \geq k+1. \end{cases}$$

By (1), q^k is well-defined, and $q^{i^*} = q$, $q^n = p$. Improvement and $\sum\limits_{i=1}^{k} p_i \leq \sum\limits_{i=1}^{k} q_i$ imply $q^{k+1} \succeq q^k$ for each $i^* \leq k \leq n-1$, so $p \succeq q$ by transitivity of $\succeq$. ■

Lemmas 3.2 and 3.3 imply the following corollary.

Corollary 3.4 *Let the preference $\succeq$ satisfy independence. If there is a permutation π of $\{1, \ldots, n\}$ such that $x_{\pi(n)} \succ \ldots \succ x_{\pi(1)}$, then $\succeq_\pi \subseteq \succeq$.*

For both illustrative purposes and later reference we now extend first degree stochastic dominance to t-degree stochastic dominance, for any $t \in \mathbb{R}$ with $t \geq 1$. This extension adapts Fishburn (1976) to our setting. We start with the following definition.

Definition 3.5 *Let $t \in \mathbb{R}$, $t \geq 1$. The $n \times n$-matrix A_t is defined by $(a_t)_{ij} := 0$ for all $i, j \in \{1, \ldots, n\}$ with $i > j$ and by*

$$(a_t)_{ij} := \frac{\Gamma(t + j - i)}{(j-i)!\,\Gamma(t)}$$

for all $i, j \in \{1, \ldots, n\}$ with $i \leq j$. Here, Γ denotes the Γ-function

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds.$$

The proof of the following lemma is given in the Appendix to this paper.

Lemma 3.6 *For all $t, t' \in \mathbb{R}$, $t, t' \geq 1$, we have*

$$A_{t+t'} = A_t \, A_{t'}.$$

Lemma 3.6 implies that, if we define $A := A_1$, then we can obtain A_t as A to the power t for all $t \geq 1$, and therefore we write A^t instead of A_t in the remainder of this paper.

For $t \geq 1$ we define the t-degree stochastic dominance preference $\succeq_t$ by

$$p \succeq_t q \Leftrightarrow pA^t \leq qA^t$$

for all $p, q \in \Delta(X)$, where the inequality is coordinate-wise. It is not hard to verify that $x_n \succ_t \ldots \succ_t x_1$, thus $\succeq_t$ strictly orders the elements of X.[1] Also, $\succeq_1$ is the first degree stochastic dominance preference associated with this ordering of the elements of X, i.e., $\succeq_1$ is equal to $\succeq_\pi$ for π being the identity.

The following lemma collects some further facts about t-degree stochastic dominance. We omit the (straightforward) proofs and only note that these facts extend similar observations about the discrete case $t \in \mathbb{N}$, see Perea *et al.* (2006).

Lemma 3.7

(i) For all $t, t' \in \mathbb{R}$ with $t \geq t' \geq 1$, we have $\succeq_{t'} \subseteq \succeq_t$.

(ii) For all $p, q \in \Delta(X)$ with $p \neq q$, if $p_{i^*} < q_{i^*}$ where $i^* := \min\{i \mid p_i \neq q_i\}$, then there is a $t' \geq 1$ such that $p \succeq_t q$ for all $t \geq t'$.

This lemma implies that t-degree stochastic dominance preferences become more complete inclusion-wise if t becomes large, and that any pair of lotteries is eventually ordered.

The (complete) lexicographic preference $\succeq_{\text{LM}}$ is defined by

$$p \succeq_{\text{LM}} q \Leftrightarrow p = q \text{ or } p_{i^*} < q_{i^*} \tag{2}$$

for all $p, q \in \Delta(X)$, where as before $i^* := \min\{i \mid p_i \neq q_i\}$.

The preferences $\succeq_t$ have the property that as t increases a (potentially small) probability that is put on a bad outcome must be compensated by an increasing weight on a good outcome. In the limit compensation is impossible. This is also expressed by the following corollary to Lemma 3.7.

Corollary 3.8 *For all $t \in \mathbb{R}$, $t \geq 1$, it holds that $\succeq_t \subseteq \succeq_{LM}$, and*

$$\bigcup_{t=1}^{\infty} \succeq_t = \succeq_{LM} \ .$$

Lemma 3.7 and Corollary 3.8 also imply that for any $p, q \in \Delta(X)$ with $p \neq q$ and satisfying the condition on the right hand side of (2), we have $q \not\succeq_t p$ for all $t \geq 1$. Hence, this is a characteristic property of stochastic dominance preferences. In the next section we formalize this property as an axiom of 'bad outcome aversion' and characterize preferences that, beside continuity and independence, satisfy this axiom.

[1] t-degree stochastic dominance can also be defined for different orderings of the alternatives in X, but for simplicity we restrict all definitions here to the ordering $x_n \succ_t \ldots \succ_t x_1$.

4 Bad outcome aversion

The following axiom captures and generalizes a basic property of stochastic dominance preferences, as explained at the end of the preceding section.

Axiom 4.1 (Bad outcome aversion, BOA) *For all $p, q \in \Delta(X)$ and all $x \in X$, if $p(x) > q(x)$ and $x \not\succeq z$ for all $z \in X \setminus \{x\}$ for which $p(z) \neq q(z)$, then $p \not\succeq q$.*

The interpretation of this axiom is as follows. Think of x as a 'bad' alternative, on which p puts more weight than q. Then the axiom says that this can never be compensated by the weights put by p on alternatives that are better or at least not worse than x.

Clearly, by Lemma 3.7 and Corollary 3.8, stochastic dominance preferences and the lexicographic preference $\succeq_{\mathrm{LM}}$ satisfy BOA. More generally, for each permutation π of $\{1, \ldots, n\}$ define the lexicographic preference $\succeq_{\mathrm{LM},\pi}$ by

$$p \succeq_{\mathrm{LM},\pi} q \Leftrightarrow p = q \text{ or } p_{\pi(i^*)} < q_{\pi(i^*)} \tag{3}$$

for all $p, q \in \Delta(X)$, where $i^* := \min\left\{i \mid p_{\pi(i)} \neq q_{\pi(i)}\right\}$. The following proposition says that a preference satisfies BOA and there there are no indifferences between the alternatives in X if and only if it is a subset of some lexicographic preference.

Proposition 4.2 *Let $\succeq$ be a preference. Then the following statements are equivalent.*

(i) $\succeq$ satisfies BOA, and $x \not\sim y$ for all $x, y \in X$ with $x \neq y$.

(ii) $\succeq \subseteq \succeq_{LM,\pi}$ for some permutation π of $\{1, \ldots, n\}$.

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. For the converse implication, take a permutation π such that for all $i, j \in \{1, \ldots, n\}$, if $i > j$ then $x_{\pi(j)} \not\succeq x_{\pi(i)}$. (Such a permutation can be seen to exist since $x \not\sim y$ for all $x \neq y$ in X, although it may not be unique.) Let $p, q \in \Delta(X)$ with $p \neq q$ and $p \succeq q$. Let $i^* := \min\left\{i \mid p_{\pi(i)} \neq q_{\pi(i)}\right\}$. Since $x_{\pi(i^*)} \not\succeq x_{\pi(i)}$ for all $i > i^*$, BOA of $\succeq$ implies $p_{\pi(i^*)} < q_{\pi(i^*)}$. Thus, $p \succeq_{\mathrm{LM},\pi} q$. ■

We now characterize BOA for preferences that satisfy independence and continuity and that strictly order all elements of X, by using the multi-expected utility theorem, Theorem 2.3. More precisely, we show that such a preference satisfies BOA if and only if the representing class of functions contains specific elements.

Theorem 4.3 *Let the preference $\succeq$ satisfy independence and continuity, and suppose $x_n \succ x_{n-1} \succ \ldots \succ x_1$. Let $U \subseteq \mathbb{R}^X$ represent $\succeq$. Then $\succeq$ satisfies BOA if and only if for each $i = 1, \ldots, n-1$ there is a sequence $(u^k)_{k\in\mathbb{N}}$, $u^k \in U$ and $u^k(x_i) < u^k(x_n)$ for each $k \in \mathbb{N}$, such that*

$$\lim_{k\to\infty} \frac{u^k(x_n) - u^k(x_{i+1})}{u^k(x_n) - u^k(x_i)} = 0 \; . \tag{4}$$

Proof. We may normalize any $u \in U$ such that $u(x_1) = 0$ and $u(x_n) = 1$.

For the 'if' part, let p, q, x satisfy the conditions in the statement of BOA, so $x = x_i$ for some $i \in \{1, \ldots, n-1\}$. Let $(u^k)_{k\in\mathbb{N}}$ be a sequence with $u^k \in U$ and $u^k(x_i) < u^k(x_n)$ for each $k \in \mathbb{N}$ such that (4) is satisfied. Without loss of generality we may assume that the sequence $(u^k)_{k\in\mathbb{N}}$ converges. With $\alpha := \sum_{j=1}^{i-1} q_j = \sum_{j=1}^{i-1} p_j$ we can write

$$\lim_{k\to\infty} \mathbb{E}_{u^k}(q) = \lim_{k\to\infty} \sum_{j=1}^{i-1} q_j u^k(x_j) + q_i u^k(x_i) + (1 - q_i - \alpha) u^k(x_{i+1})$$

and

$$\lim_{k\to\infty} \mathbb{E}_{u^k}(p) = \lim_{k\to\infty} \sum_{j=1}^{i-1} p_j u^k(x_j) + p_i u^k(x_i) + (1 - p_i - \alpha)\ .$$

We claim that for k sufficiently large, $\mathbb{E}_{u^k}(q) > \mathbb{E}_{u^k}(p)$. To show this, it is sufficient to prove that

$$q_i u^k(x_i) + (1 - q_i - \alpha) u^k(x_i + 1) > p_i u^k(x_i) + (1 - p_i - \alpha)$$

or, equivalently

$$(1 - q_i - \alpha)[1 - u^k(x_{i+1})] < (p_i - q_i)[1 - u^k(x_i)]$$

for k sufficiently large. This, however, follows by (4). So $p \not\succeq q$.

For the converse, assume that $\succeq$ satisfies BOA. Fix $i \in \{1, \ldots, n-1\}$. Fix $0 < \pi < 1$, consider the lottery $p = \pi x_i + (1-\pi) x_{i+1}$, and for each $0 < \varepsilon < 1 - \pi$ consider the lottery $p^\varepsilon = (\pi + \varepsilon) x_i + (1 - \pi - \varepsilon) x_n$. By BOA, $p^\varepsilon \not\succeq p$, hence there is a $u^\varepsilon \in U$ such that $u^\varepsilon(p) > u^\varepsilon(p^\varepsilon)$, i.e.,

$$(1 - \pi)(1 - u^\varepsilon(x_{i+1})) < \varepsilon(1 - u^\varepsilon(x_i))\ .$$

This implies the existence of a sequence $(u^k)_{k\in\mathbb{N}}$ with $u^k \in U$ and $u^k(x_i) < u^k(x_n)$ for each $k \in \mathbb{N}$ such that (4) is satisfied. ■

We conclude this section with some remarks and a corollary for the case of three alternatives.

Remark 4.4 A consequence of Theorem 4.3 is that, under the additional conditions in the theorem, BOA implies incompleteness of the preference. This is so because a complete preference is represented by a unique $0-1$ normalized function and, thus, a sequence satisfying (4) cannot exist for every i.

Remark 4.5 If, in Theorem 4.3, for some $i \in \{1, \ldots, n-1\}$ there is a function $u \in U$ with $u(x_i) < u(x_{i+1}) = \ldots = u(x_n)$, then (4) can be trivially satisfied by taking $u^k = u$ for all $k \in \mathbb{N}$. For example, let $n = 4$ and consider the set of functions

$$U = \operatorname{conv}\{(0, \eta, \eta(2-\eta), 1) \in \mathbb{R}^4 \mid 0 \leq \eta \leq 1\}\ ,$$

where 'conv' denotes the convex hull operator. This set is closed and convex. It contains the elements $(0,1,1,1)$ and (e.g.) $(0,\frac{1}{2},\frac{1}{2},1)$, so that (4) is satisfied for $i=1$ and $i=3$. It does not contain an element of the form $(0,\alpha,1,1)$ for some $0\leq\alpha<1$, but (e.g.) the sequence $(0,\frac{k}{k+1},\frac{k(k+2)}{(k+1)^2},1)_{k\in\mathbb{N}}$ satisfies (4) for $i=2$. It follows that the preference represented by U satisfies BOA.

Remark 4.6 Theorem 4.3 applies in particular to stochastic dominance preferences $\succeq_t$, $t\in\mathbb{R}$, $t\geq 1$. In that case for the set U we can take the convex hull of the columns of the matrix $-A^t$. This is a case in which for each $i=1,\ldots,n-1$ there is a function u with $u(x_i)<u(x_{i+1})=\ldots=u(x_n)$ as in Remark 4.5, namely the i-th column of $-A^t$.

Remark 4.7 For the case of three alternatives the consequences of Theorem 4.3 are as follows. Since we can assume that every $u\in U$ has the form $(0,\alpha,1)$, the theorem applied for $i=1$ implies $(0,1,1)\in U$ and the theorem applied for $i=2$ implies $(0,\alpha,1)\in U$ for some $\alpha<1$. Let $\alpha^*:=\inf\{\alpha\mid(0,\alpha,1)\in U\}$, then convexity and closedness of U imply that U is the convex hull of $(0,1,1)$ and $(0,\alpha^*,1)$. Consider, on the other hand, the 3×3-matrix $-A^t$ and normalize its columns. This results in the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & \frac{t-1}{t} & \frac{t}{t+2} \\ 1 & 1 & 1 \end{pmatrix},$$

implying that the class of functions U^t representing $\succeq_t$ is the convex hull of $(0,1,1)$ and $(0,(t-1)/t,1)$ if $t\geq 2$ and of $(0,1,1)$ and $(0,t/(t+2),1)$ if $t\leq 2$. In turn, this implies that $\succeq$ coincides with $\succeq_t$, where $t=1/(1-\alpha^*)$ if $\alpha^*\geq /2$ and $t=2\alpha^*/(1-\alpha^*)$ if $\alpha^*\leq 1/2$. Thus, we have the following corollary to Theorem 4.3.

Corollary 4.8 *Let $X=\{x_1,x_2,x_3\}$ and let $\succeq$ satisfy independence, continuity, and BOA. Let $x_3\succ x_2\succ x_1$. Then $\succeq=\succeq_t$ for some $t\in\mathbb{R}$, $t\geq 1$.*

5 Concluding remarks

(1) Although there is no direct formal connection, bad outcome aversion is somewhat similar in flavor to risk aversion. It is well-known that first degree stochastic dominance has the set of all functions (respecting the order on the alternatives in X) as representing set – this can be seen, for instance, by considering the matrix $-A^1$ – and that for second degree stochastic dominance this is the set of all concave, 'risk averse' functions, i.c., the set of all functions exhibiting decreasing differences.

(2) One could say that higher degree stochastic dominance corresponds to higher bad outcome aversion or, more generally, that $\succeq'$ is more bad outcome averse than $\succeq$ if $\succeq_1\subseteq\succeq\subseteq\succeq'\subseteq\succeq_{\mathrm{LM}}$, cf. Corollary 3.4 and Proposition 4.2. As

an application of this, Perea *et al.* (2006) study the effects of this notion of 'increasing bad outcome aversion', expressed by t-degree stochastic dominance preferences for increasing t, on the equilibria of finite two-player games, where the players have incomplete preferences on the probability distributions resulting from their mixed strategies. It is shown that such equilibria converge to a pair of pure strategies that are max-min in a specific sense: in the limit, each player plays a pure strategy that is max-min in terms of the preferences of the opponent. It can be shown that the results in that paper and in particular the asymptotic result remain true if players become 'more bad outcome averse' in the more general sense described above. Proofs can be found in Schulteis (2007).

Appendix: proof of Lemma 3.6

Let $i, j \in \{1, ..., n\}$ and $j \geq i$. We have to show

$$\frac{\Gamma(t+t'+j-i)}{(j-i)!\Gamma(t+t')} = \sum_{l=i}^{j} \frac{\Gamma(t+l-i)}{(l-i)!\Gamma(t)} \frac{\Gamma(t'+j-l)}{(j-l)!\Gamma(t')}.$$

We introduce the shift of indices $k := l - i$ and the variable $\xi := j - i$. Hence we have to show

$$\frac{\Gamma(t+t'+\xi)}{\xi!\Gamma(t+t')} = \sum_{k=0}^{\xi} \frac{\Gamma(t+k)}{k!\Gamma(t)} \frac{\Gamma(t'+\xi-k)}{(\xi-k)!\Gamma(t')}. \tag{5}$$

Since ξ is a natural number and the Gamma-function has the property

$$x\Gamma(x) = \Gamma(x+1)$$

for all real and positive x, we have

$$\frac{\Gamma(x+\xi)}{\Gamma(x)} = (x+\xi-1)(x+\xi-2) \cdot \ldots \cdot x,$$

which is a polynomial of degree ξ in the variable x. Hence, the RHS of (5) is a polynomial of degree ξ in $(t+t')$. In the LHS of (5) there are terms $t^k (t')^{\xi-k}$ for all $k = 0, \ldots, \xi$. Hence, this LHS is a polynomial of degree ξ in the variables t and t'. Thus, both sides are polynomials of degree ξ in the two variables t and t'. Furthermore, from the representation for natural t as given in Perea *et al.* (2006) it follows that the polynomials are equal for $t, t' \in \mathbb{N}$. Take any fixed $t \in \mathbb{N}$. We obtain two polynomials of degree ξ in t' that coincide at infinitely many points (all natural t') and hence, these polynomials are equal for all $t' \in \mathbb{R}$, $t \geq 0$. Next, we take any $t' \in \mathbb{R}$, $t' \geq 0$, and obtain two polynomials of degree ξ in t that coincide on infinitely many points (all natural t) and hence, these polynomials are equal for all $t \in \mathbb{R}$, $t \geq 0$. This completes the proof. ■

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