Making Solutions Invariant

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Making solutions invariant[†]

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Abstract. In this paper a procedure is developed to modify a non-invariant solution in such a way that the resulting solution is invariant. Furthermore it is investigated which properties of the solution are inherited by the modified solution.

Key words: normal form games, invariance of solution concepts

1. Introduction

In their seminal paper Kohlberg and Mertens (1986) argued that a solution (a rule that assigns to each game a collection of closed, nonempty sets of strategy profiles) should satisfy certain prescribed requirements. They proposed a list of seven such requirements and their aim in this paper was to find a solution that satisfied each and every one of them. In this search they used what might be called a standard method to generate such solutions. First of all, they defined for each game a collection of perturbations, or perturbed games, of that game. A set of strategy profiles of the game is then said to *survive* small perturbations if for every (sufficiently small) perturbation there is an equilibrium of the perturbed game close to the set under consideration. Now, a set of strategy profiles is *stable* if it survives small perturbations and is also minimal with respect to this property. A solution that assigns to a game its collection of stable sets is called a stability concept. By choosing different types of perturbations of a game Kohlberg and Mertens defined three different types of stability concepts, namely hyperstability, full stability and KM-stability.

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One of the requirements proposed by Kohlberg and Mertens stated that the sets assigned to a given game should only depend on the reduced normal form of that game. In other words, if two games have the same reduced normal form, then the solution should assign the same (solution) sets to these two games.[1]

In this paper we try to address the following question. Since invariance is one of the more prominent axioms on the wish list of Kohlberg and Mertens it is important to know to what extent invariance really is a restrictive requirement. Several solution concepts, such as full stability, quasi-stability and BR-stability (Hillas 1990), were discarded in literature because they violated invariance. We however will try to convince the reader that, although there may well be several *other* reasons to discard these concepts, it is in fact not too complicated to "sculpt" invariant versions of those concepts. In other words, invariance is far less often the bottleneck than literature seems to suggest.

Related to this "sculpting" of solution concepts, we will also address a method used by Kohlberg and Mertens to turn non-invariant solution sets into invariant ones. We will explain why their approach does not work in general and present an alternative, fairly straightforward, procedure that does work for a wide class of solution concepts. Thus we try to argue that invariance is relatively easy to enforce for many solution concepts including the above-mentioned ones such as full stability, quasi-stability and BR-stability.

Contents of the paper

Of course, there may be any number of ways to establish our aim[2]. The method presented here works for a fairly large class of potential candidates. We only require the original solution to satisfy four fairly mild conditions. The first three are

(1) each game has at least one solution set,
(2) the Hausdorff limit of solution sets is also a solution set and
(3) a closed set containing a solution set is a solution set itself.

These conditions guarantee for example that the modification of the original solution assigns at least one solution set to each game. Moreover, a solution that assigns to a game the collection of all sets that survive small perturbations (a stability concept without the final minimality requirement) will naturally satisfy these three conditions.

The fourth condition directly concerns the question to which degree the original solution already is invariant. Very vaguely speaking, it is of course clear that the further a solution concept is removed from being invariant, the more we need to change (or even mutilate) it in order to get something invariant. The fourth condition simply states that the original solution should be close enough to being invariant for the modified solution to be still recognizably similar to the original one. In particular we will show that the modified solution inherits all other requirements of the Kohlberg-Mertens program sat-

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[1] Stronger versions of the invariance conditions are usually preferred nowadays due to a better understanding of the implications of these conditions in relation to normal forms of extensive form games.

[2] An alternative method is presented in Vermeulen and Jansen (1999). However, this particular way of doing things only works for a very specific type of solution concepts.

isfied by the original solution concept. The condition we use is not very restrictive and is satisfied for example by essentiality, full stability, quasi-stability and BR-stability.

One final remark we would like to make is that there are probably many other possible ways to modify solutions that will give similar results. In that sense the specific choices we make are not always inevitable and that makes it sometimes hard to argue why these choices are the best choices possible (and maybe they *are* not always the best choices possible). Nevertheless, the method presented here is, to our knowledge, the first modification of non-invariant solution concepts into invariant ones that combines mathematical consistency with preservation of all requirements of the Kohlberg-Mertens program.

Organization of the paper

In section 2 we will introduce the standard game-theoretical notions used in this paper, as well as the reduced normal form and reduced strategic form of a game. In section 3 we will subsequently present two notions of invariance of a solution that are based on these notions of a reduced game. In section 4 we will explain the difficulties with modifying non-invariant solutions into invariant ones. In section 5 we present one way of doing this for the special class of what we call regular solutions. We will show in section 6 that essentiality, best-reply stability and quasi-stability are regular and in section 7 that our method preserves the requirements of the Kohlberg-Mertens program. Section 8 concludes with an example of an alternative way of modification that does not preserve backwards induction.

Notation. For $x \in \mathbb{R}^n$ and $\eta > 0$, $\|x\|_\infty := \max_{i \in N}|x_i|$ and $B_\eta(x) := \{y \in \mathbb{R}^n \mid \|x - y\|_\infty < \eta\}$. The Hausdorff distance of two compact subsets S and T of $\mathbb{R}^n$ is defined as $d_H(S, T) := \inf\{\eta > 0 \mid S \subset B_\eta(T), T \subset B_\eta(S)\}$, where $B_\eta(S) := \bigcup_{x \in S} B_\eta(x)$.

2. Preliminaries

A(n *n-person normal form*) *game* is a pair $\Gamma = \langle A, u\rangle$ such that $A := \prod_{i \in N} A_i$ is a product of n non-empty, finite sets and $u = (u_i)_{i \in N}$ is an n-tuple of functions $u_i : A \to \mathbb{R}$. Here N is the set of players and A_i is the set of *pure strategies* of player i and u_i is his *payoff function*. The player set N is assumed to be fixed throughout the paper. The elements of the set $\Delta(A_i)$ of probability distributions on A_i are the *mixed strategies* of player i. For a (mixed) strategy profile $x = (x_i)_{i \in N} \in \Delta_A := \prod_j \Delta(A_j)$, the (expected) payoff function of player i is defined by $u_i(x) := \sum_{a \in A} \prod_j x_{ja_j} u_i(a)$.

By abuse of notation we will identify a pure strategy $a \in A_i$ with the mixed strategy in $\Delta(A_i)$ that puts all weight on a. So, A_i will simply be viewed as a subset of $\Delta(A_i)$. Also the pure strategy profiles will be denoted by $a \in A$. In case confusion might occur we will write $a_i \in A_i$ instead of simply $a \in A_i$.

We also write Δ (Δ_i) instead of Δ_A $(\Delta(A_i))$ and $\Delta_{-i} := \prod_{j \neq i} \Delta_j$ is the set of strategy profiles for the opponents of player i. Furthermore, $(x_{-i}|y_i) \in \Delta$ is the strategy profile where player i uses $y_i \in \Delta_i$ and his opponents use the strategies in $x_{-i} \in \Delta_{-i}$.

A game in *strategic form* is a pair $\langle P, v\rangle$ in which $P = \prod_{i \in N} P_i$ is a product of polytopes $P_i \subset \mathbb{R}^{m_i}$ and $v = (v_i)_{i \in N}$ is the collection of payoff functions $v_i : P \to \mathbb{R}$ for each player $i \in N$. These payoff functions are assumed to be multi-affine, which means that

$$v_i(p_{-i} \mid \lambda p_i + (1-\lambda)q_i) = \lambda v_i(p_{-i}|p_i) + (1-\lambda)v_i(p_{-i}|q_i)$$

holds for all p_{-i} in P_{-i}, all p_i, q_i in P_i and all λ in $[0,1]$. Notice that a normal form game is a special type of strategic form game.

For player i and a strategy profile $x \in \Delta$

$$\beta_i(x) := \{y_i \in \Delta_i \mid u_i(x_{-i}|y_i) \geq u_i(x_{-i}|z_i) \text{ for all } z_i \in \Delta_i\}$$

is the set of *best replies* to x. A strategy profile in the set $\beta(x) := \prod_i \beta_i(x)$ is called a *best reply* to x. Note that a Nash equilibrium is a strategy profile $x \in \Delta$ with $x \in \beta(x)$. The set of all Nash equilibria of the game Γ is denoted by $E(\Gamma)$.

For player i, a strategy $y_i \in \Delta_i$ is an *admissible* best reply to a strategy profile $x \in \Delta$ if there is a sequence $(x^k)_{k \in \mathbb{N}}$ of completely mixed strategy profiles in Δ (i.e. all coordinates are positive) converging to x such that $y_i \in \beta_i(x^k)$, for all k. For a strategy profile $x \in \Delta$, $B_i^a(x)$ denotes the set of admissible **pure** best replies to x. If S is a subset of Δ, $B_i^a(S) := \bigcup_{x \in S} B_i^a(x)$ is the set of admissible pure best replies to S.

A completely mixed strategy profile x is called ε-perfect ($\varepsilon > 0$) if $x_{ia} \leq \varepsilon$ for every $a \in A_i$ that is not a (pure) best reply to x. A strategy profile x is called (normal form) *perfect* if there is a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ of positive real numbers converging to zero and a sequence $(x^k)_{k \in \mathbb{N}}$ converging to x such that x^k is ε_k-perfect for all $k \in \mathbb{N}$. The set of *perfect* equilibria of the game Γ is denoted by $PE(\Gamma)$.

3. Various definitions of invariance

In this paper the invariance of a solution plays a central role. Invariance reflects the intuitive feeling that two equivalent games should in some sense have the same solution sets. In this section we will briefly discuss two possible interpretations of this informal statement.

A *solution* is a mapping σ that assigns to a game Γ a collection of closed, non-empty subsets of Δ. The elements of $\sigma(\Gamma)$ are called *solution sets*.

Generally speaking, a definition of invariance of a solution σ requires an answer to the following two questions:

(1) when are two games equivalent?
(2) when two games Γ and Γ' are equivalent, how do we compare the solution sets in $\sigma(\Gamma)$ with those in $\sigma(\Gamma')$?

The first variant of invariance we will discuss is what we will call KM-invariance. This type of invariance considers two games to be equivalent if they have the same *reduced strategic form*. The reduced strategic form of a normal form game $\Gamma = \langle A, u\rangle$ is a smaller game that is in a specific sense – having to do with the notion of payoff-equivalence of strategies – strategically equivalent to the original game. It is defined as follows.

Two strategies y_i and z_i of player i are *payoff-equivalent* if for all j and all $x \in \Delta$

$$u_j(x_{-i}|y_i) = u_j(x_{-i}|z_i).$$

Since the identification of **all** payoff-equivalent strategies in $\Delta(A_i)$ can be seen as taking the quotient of $\Delta(A_i)$ with respect to a linear subspace, the result is a polytope, say P_i. Furthermore, the quotient maps $\pi_i : \Delta(A_i) \to P_i$ are linear. Thus, there exist unique multi-affine maps v_i from $P := \prod_i P_i$ to $\mathbb{R}$ such that $v_i = u_i \circ \pi$ (where $\pi : \Delta \to P$ is the map $(\pi_i)_{i \in N}$). It can be checked that the pair $\langle P, v \rangle$ is a strategic form game, and it will be denoted by Γ_{rsf}. The reduced strategic form is unique up to affine isomorphisms[4] (cf. Vermeulen and Jansen (1997b, 1998)).

Example 1. For the 2×4-bimatrix game

$$\begin{bmatrix} (1,1) & (-1,-1) & (2,-2) & (-2,2) \\ (-1,-1) & (1,1) & (-2,2) & (2,-2) \end{bmatrix}$$

two strategies x and y of player 2 are payoff equivalent if and only if

$$\begin{cases} e_iAx = e_iAy \\ e_iBx = e_iBy \end{cases} \quad \text{for all } i \Leftrightarrow x - y \in L_2 := \left\{ \alpha \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}.$$

The strategy space of player 2 in the reduced strategic form game is the quotient space

$$\{x + L_2 \mid x \in \Delta_4\}.$$

The associated quotient map π_2 is simply the map $\pi_2 : x \to x + L_2$. Since no two pure strategies of player 2 are equivalent, it can easily be seen that the quotient space can be identified with a quadrangle. $\triangleleft$

Remark: Originally, Kohlberg and Mertens talked about the equivalence of games in terms of games having the same reduced *normal* form. This reduced form of a game can be constructed as follows. Check if there is some pure strategy of some player that is payoff-equivalent with some other strategy of that player. If there is, delete it from the game. Thus we get a new game with one pure strategy less. Repeat the procedure with this new game, and keep on repeating the procedure until no such a pure strategy is left. Obviously the result of this procedure is a normal form game in which no pure strategy is payoff-equivalent with any other strategy. It can also be shown that the resulting game, up to relabeling of pure strategies, does not depend on the order in which strategies get deleted (cf. Vermeulen and Jansen (1998)).

Nevertheless, the following result shows that we might as well identify all payoff-equivalent strategies.

Theorem 1. *Two normal form games have the same reduced normal form if and only if they have the same reduced strategic form.*

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[4] An affine isomorphism from a polytope $P_i \subset \mathbb{R}^{k_i}$ to a polytope $Q_i \subset \mathbb{R}^{m_i}$ is an affine transformation T_i from $\mathbb{R}^{k_i}$ to $\mathbb{R}^{m_i}$ such that $T_i(P_i) = Q_i$ and the restriction of T_i to P_i is one-to-one.

Sketch of the proof: We will argue that, up to relabeling of pure strategies, there is a unique way to construct the reduced strategic form of a game given its reduced normal form and vice versa.

So, suppose we are given the reduced normal form of a game. Then the strategy space of the reduced strategic form can simply be constructed by identifying all payoff-equivalent strategies with one another. The induced quotient maps yield the tool for defining the payoff functions.

Conversely, suppose that the reduced strategic form $\Gamma_{rsf} = \langle P, v \rangle$ of a game Γ is given. We can construct (a game that is up to relabeling of pure strategies identical to) the reduced normal form of Γ in the following way: take $A_i := \mathrm{ext}(P_i)$. Let $f_i : \Delta(A_i) \to P_i$ be the linear extension of the inclusion map from A_i to P_i that assigns p_i to itself for any p_i in A_i. If we write f for the map that assigns the profile $(f_i(x_i))_{i \in N}$ in P to the profile $(x_i)_{i \in N}$ in $\Delta := \prod_{i \in N} \Delta(A_i)$ we can define the payoff function $u_i : \Delta \to \mathbb{R}$ by

$$u_i((x_i)_{i \in N}) := v_i(f((x_i)_{i \in N}))$$

for $(x_i)_{i \in N}$. It is straightforward to check that the game $\langle A, u \rangle$ is indeed identical to the reduced normal form of Γ. $\triangleleft$

This shows that we can define the equivalence of games either way, and we chose to take the one that is in terms of mathematics the easiest one to work with.

Now we return to the definition of KM-invariance and the way it addresses the second question. Suppose that we have two equivalent games Γ and Γ'. In other words, Γ and Γ' have the same reduced strategic form. Let π be the associated quotient map from the strategy space of Γ to the strategy space of Γ_{rsf} and let ρ be the quotient map from Γ' to Γ_{rsf}. So we get the following diagram.

$$\begin{tikzcd} \Gamma \arrow[dr, "\pi"'] & & \Gamma' \arrow[dl, "\rho"] \\ & \Gamma_{rsf} & \end{tikzcd}$$

These maps enable us to compare the solution sets in $\sigma(\Gamma)$ and $\sigma(\Gamma')$ by means of their 'projections' under π and ρ. Formally we require that

$$\{\pi(S) \mid S \in \sigma(\Gamma)\} = \{\rho(T) \mid T \in \sigma(\Gamma')\}.$$

The solution σ is called *invariant in the sense of Kohlberg and Mertens* – KM-invariant for short – if the above equality holds for all games Γ and Γ' having the same reduced strategic form.

Remark: The reason why we christened this type of invariance after Kohlberg and Mertens is the following quote from Kohlberg and Mertens (1986) on page 1012:

> In particular, one should therefore also identify any two 'duplicate' (i.e. payoff-equivalent) mixed strategies. It is in this sense that we will interpret the reduced normal form *strategies* (i.e. as the equivalence classes given this identification).

Finally we will give the second definition of invariance of a solution, simply called invariance. This second version of invariance, among other types introduced in Mertens (1987), reflects the idea that a solution should also behave well with respect to only a partial identification of payoff equivalence. Mertens (1987) deals with ordinality of solutions (we refer to this paper for a very detailed and convincing discussion of the need to require ordinality). One of the results in that paper states that the type of invariance we use here, combined with two other properties, results in a condition that is marginally stronger than ordinality. For a detailed discussion of the precise differences between these notions we refer to Vermeulen and Jansen (2000).

The tool we use to describe equivalence of games in this definition is the notion of reduction map. A reduction map between two games establishes the link between strategies in one game and the corresponding strategies in the other game. In order to formalize this, let $\Gamma = \langle A, u \rangle$ and $\Gamma' = \langle B, v \rangle$ be two games.

Definition 1. A map $f = (f_i)_{i \in N}$ from Δ' to Δ is called a *reduction map* from Γ' to Γ, denoted by $\Gamma' \rightarrow_f \Gamma$, if for every player i,

(1) $f_i : \Delta_i' \rightarrow \Delta_i$ is affine and onto
(2) $v_i = u_i \circ f$.

The motivation for the use of reduction maps to express equivalence of games is based on the notion of payoff-equivalence. It is straightforward to show that a reduction map f from a game Γ' to a game Γ preserves payoff-equivalence, i.e. two strategy profiles x and y in Δ' are payoff-equivalent if and only if $f(x)$ and $f(y)$ are payoff-equivalent in Δ. A simple consequence of the preservation of payoff-equivalence by f is the following result.

Lemma 1. *A strategy profile z is a best reply to x in the game Γ' if and only if $f(z)$ is a best reply to $f(x)$ in the game Γ.*

Thus, two equivalent games have, in a precise sense specified by the reduction map f, the same best reply correspondence. And in that sense these games are considered strategically equivalent. Therefore it is only natural to require that such games have, also in a very precise sense specified by f, the same solution sets. This is done in

Definition 2. A solution σ is called *invariant* if for every reduction map f from a game Γ' to a game Γ

$$\sigma(\Gamma) = \{f(T) \mid T \in \sigma(\Gamma')\}.$$

This version of invariance is weaker than the type of invariance required by Mertens (1987) in Theorem 2(b) to ensure ordinality. On top of the above requirement he also needed the inverse image of a solution set to be the union of solution sets that project down onto the original solution set. See Vermeulen and Jansen (1997a, 2000) for a detailed comparison of these notions. It is however easy to show that invariance implies KM-invariance[5].

[5] The solution τ in the next section is an example of a solution that is KM-invariant, but not invariant.

4. Invariance and the method of Kohlberg and Mertens

In this section we will describe how Kohlberg and Mertens tried to generate solutions that are KM-invariant and point out the flaw in this method by means of an example.

Kohlberg and Mertens introduced a method (in the definition of hyperstability) to transform a solution (essentiality in this case) that is not KM-invariant into one that might be KM-invariant. The same method was used by Hillas (1990) in the definition of full stability and stability. On page 1022 Kohlberg and Mertens (1986) write

> ... we will say that S is a *hyperstable* set of equilibria in a game G if it is minimal with respect to the following property:
>
> ($\mathscr{H}$) S is a closed set of Nash equilibria of G such that, for any equivalent game (i.e. having the same reduced normal form), and for any perturbation of the normal form of that game, there is a Nash equilibrium close to S.

So, Kohlberg and Mertens first compute the collection of essential sets, i.e., closed sets S of strategy profiles such that any small perturbation of G has a Nash equilibrium close to S. Then they specify – using the reduced form of a game – their method to select a specific class of essential sets. Finally, they take the minimal elements of this subclass.

A little bit more general, we could describe this method as follows. Let Γ be a game, and let Γ_{rnf} be its reduced normal form. Given a solution σ we say that a solution set $S \in \sigma(\Gamma)$ is an element of $\tilde{\sigma}(\Gamma)$ if

($\mathscr{P}$) for every normal form game Γ' having the same reduced normal form as Γ and maps f and g with $\Gamma \to_f \Gamma_{rnf}$ and $\Gamma' \to_g \Gamma_{rnf}$, there exists a solution set $S' \in \sigma(\Gamma')$ with $f(S) = g(S')$ and

($\mathscr{M}$) there does not exist a proper subset $T \in \sigma(\Gamma)$ of S that also satisfies $\mathscr{P}$.

Then Kohlberg and Mertens state that $\tilde{\sigma}$ is, at least in the case of hyperstability, KM-invariant by definition. Although we believe that hyperstability indeed satisfies KM-invariance, this is not immediately clear from the above definition. Furthermore it is certainly not true for a general solution. To see this, consider the following

Example 2. We will give a KM-invariant solution τ for which the modified solution $\tilde{\tau}$ is not KM-invariant. Consider the bimatrix games

$$(A,B) = \begin{bmatrix} (0,6) & (2,0) & (0,4) \\ (4,0) & (0,6) & (0,4) \end{bmatrix} \quad \text{and} \quad (A',B') = \begin{bmatrix} (0,6) & (2,0) & (0,4) \\ (4,0) & (0,6) & (0,4) \\ (2,3) & (1,3) & (0,4) \end{bmatrix}.$$

Note that (A,B) is the reduced strategic form of (A',B'), since neither player of the game (A,B) has payoff-equivalent strategies. Now we define the solution τ as follows. For a game Γ whose reduced strategic form equals (A,B), let $F(\Gamma)$ denote the collection of reduction maps from Γ to (A,B). Then define

$$\tau(\Gamma) := \begin{cases} \{S, T\} & \text{if } \Gamma = (A, B) \\ \{S', T'\} & \text{if } \Gamma = (A', B') \\ \bigcup_{f \in F(\Gamma)} \{f^{-1}(S), f^{-1}(T)\} & \text{else} \end{cases}$$

with

$$S := \mathrm{ch}\left\{\left(\tfrac{2}{3}, \tfrac{1}{3}\right), \left(\tfrac{1}{3}, \tfrac{2}{3}\right)\right\} \times \{(0, 0, 1)\},$$

$$T := \left\{\left(\left(\tfrac{1}{2}, \tfrac{1}{2}\right), (0, 0, 1)\right)\right\}$$

and

$$S' := \mathrm{ch}\left\{\left(\tfrac{2}{3}, \tfrac{1}{3}, 0\right), \left(\tfrac{1}{3}, \tfrac{2}{3}, 0\right)\right\} \times \{(0, 0, 1)\},$$

$$T' := \{((0, 0, 1), (0, 0, 1))\}.$$

Here $\mathrm{ch}(A)$ denotes the convex hull of a set A. For the sake of convenience, we will only consider τ on the class of games that are equivalent to (A, B). Now application of the above modification to τ yields

$$\tilde{\tau}(\Gamma) := \begin{cases} \{T\} & \text{if } \Gamma = (A, B) \\ \{S', T'\} & \text{if } \Gamma = (A', B') \\ \{f^{-1}(T) \mid f \in F(\Gamma)\} & \text{else.} \end{cases}$$

It is easy to check that $\tilde{\tau}$ is not a KM-invariant solution. This highlights one of the problems concerning the above modification: the sets S and $f^{-1}(S)$ are thrown away in the process *while* $S' \in \tau(A', B')$ *remains.* $\triangleleft$

Naturally, the solution τ in the Example is a rather artificial one. It is however not immediately clear why effects like the above cannot occur for a more natural solution like e.g. essentiality.

5. Modifying solutions

Now we describe a procedure to modify certain solutions in such a way that the resulting solution is invariant. First we introduce the class of solutions for which we define our procedure. Although the characterization of this class is formulated in general terms, the intentions of this paper are definitely geared towards solutions that are developed in the theory of strategic stability, such as essentiality, stability in the sense of Hillas and quasi-stability.

The idea that these solutions do not behave too badly w.r.t. invariance is captured in the second requirement. The requirements (1) and (3) will only be used to guarantee the non-emptiness of the modified solution. The necessity of requirement (4) is harder to pin down to one argument, since it is used throughout the paper. In most cases it is needed to guarantee that the inverse images of certain solution sets are also solution sets (for example, it guarantees that $f^{-1}(f(T))$ is a solution set for any solution set S and reduction map f).

Note by the way that, if you do not include minimality, the requirements (1), (3) and (4) are quite natural for a stability-like concept[6].

Definition 3. A solution σ is *regular* if for any game Γ

(1) $\sigma(\Gamma) \neq \phi$ (non-emptiness property)
(2) the 'projection' $f(T)$ of an element T of $\sigma(\Gamma')$ is an element of $\sigma(\Gamma)$ for all pairs (Γ', f) with $\Gamma' \to_f \Gamma$ (projection property)
(3) if a sequence $S_1, S_2, \dots$ of elements of $\sigma(\Gamma)$ d_H-converges to S, then S is also an element of $\sigma(\Gamma)$ (closedness property)
(4) every closed set T containing an element of $\sigma(\Gamma)$ is also an element of $\sigma(\Gamma)$ (monotonicity property).

In order to explain the idea behind our procedure, note that the examples 8 and 10 in Vermeulen and Jansen (1997a) show that above-mentioned solutions like essentiality and quasi-stability (including the minimality condition) are not KM-invariant. This is partly due to the minimality condition and partly to the basic definitions of these solutions. Thus we need a more careful selection than just requiring minimality. Our selection method is split in two steps. In the first step we take care of a (possible) flaw in the basic definition of the solution. In the second step we adapt the minimality condition.

Step 1. In order to introduce our modification of a regular solution σ we first consider, for a game Γ, the class $\tilde{\sigma}(\Gamma)$ of those sets S in $\sigma(\Gamma)$ for which $f^{-1}(S) \in \sigma(\Gamma')$ for all pairs (Γ', f) with $\Gamma' \to_f \Gamma$. Such sets are called *extension-stable* (*e-stable* for short).

We will show that the solution $\tilde{\sigma}$ is invariant and that $\tilde{\sigma}(\Gamma)$ is not empty for all Γ.

Theorem 2. *Let σ be a regular solution and let Γ be a game. Then*

(a) *there is at least one e-stable set for Γ*
(b) *every e-stable set for Γ contains a minimal e-stable set for Γ.*

In particular, $\tilde{\sigma}(\Gamma)$ is not empty.

Proof: (a) First we will show that the strategy space Δ of Γ is an e-stable set. So let (Γ', f) be a pair with $\Gamma' \to_f \Gamma$. Then the non-emptiness property and the monotonicity of σ imply that $f^{-1}(\Delta) = \Delta' \in \sigma(\Gamma')$.

(b) Now we prove that each e-stable set S contains a minimal e-stable set. If S is not minimal, take a sequence $S_1 \supset S_2 \supset \cdots$ of e-stable sets for the game Γ contained in S. Let (Γ', f) be a pair with $\Gamma' \to_f \Gamma$. Then $f^{-1}(S_1) \supset f^{-1}(S_2) \supset \cdots$ are elements of $\sigma(\Gamma')$. Because the sequence $f^{-1}(S_1)$, $f^{-1}(S_2), \dots$ d_H-converges to $\bigcap_i f^{-1}(S_i)$, the closedness property of σ implies that $f^{-1}\left(\bigcap_i S_i\right) = \bigcap_i f^{-1}(S_i) \in \sigma(\Gamma')$. Hence, $\bigcap_i S_i$ is an e-stable set for Γ. Then Zorn's lemma implies that S contains a minimal e-stable set. ◁

[6] In the particular case of the monotonicity requirement we are specifically thinking of a stability concept. As we already argued, the minimality condition destroys invariance. Therefore we like to start with sets that satisfy the basic robustness requirement against small perturbations, and that automatically gives you a monotonic solution.

In order to prove, for a regular solution σ, the invariance of the solution $\tilde{\sigma}$, we need a lemma and some notation. Let Γ be a game. Suppose that (Γ', f) and (Γ'', g) are two pairs with $\Gamma' \to_f \Gamma$ and $\Gamma'' \to_g \Gamma$.

Lemma 2. *There are maps h and k and a game $\tilde{\Gamma}$ such that h is a reduction map from $\tilde{\Gamma}$ to Γ', k is a reduction map from $\tilde{\Gamma}$ to Γ'', and $f \circ h = g \circ k$.*

Proof: Write $\Gamma = \langle A, u \rangle$, $\Gamma' = \langle B, v \rangle$ and $\Gamma'' = \langle C, w \rangle$. We will first construct the map h and the game $\tilde{\Gamma} = \langle D, z \rangle$. Since f_i is an onto map from $\Delta(B_i)$ to $\Delta(A_i)$, we can choose for each pure strategy $c_i \in C_i$ some strategy $\zeta_i(c_i)$ in $\Delta(B_i)$ such that $f_i(\zeta_i(c_i)) = g_i(c_i)$.

Assume w.l.o.g. that B_i and C_i are disjoint. Define $D_i := B_i \cup C_i$. Let $h_i : \Delta(D_i) \to \Delta(B_i)$ be the affine (extension of the) map satisfying, for $d_i \in D_i$

$$h_i(d_i) := \begin{cases} d_i & \text{if } d_i \in B_i \\ \zeta_i(d_i) & \text{if } d_i \in C_i. \end{cases}$$

Now define the payoff function of $\tilde{\Gamma}$ by $z_i := u_i \circ f \circ h$ with $h := (h_i)_{i \in N}$. Then h is a reduction map from $\tilde{\Gamma}$ to Γ' since each h_i is affine and onto, and $z_i = u_i \circ f \circ h = v_i \circ h$.

Furthermore, if we define maps k_i for each player i in an analogous way, it is straightforward to check that each k_i is affine and onto, and $f \circ h = g \circ k$ by the construction of h and k. Finally, this observation implies that

$$z_i = u_i \circ f \circ h = u_i \circ g \circ k = w_i \circ k$$

and k is a reduction map from $\tilde{\Gamma}$ to Γ''. ◁

We need this result in the proof of

Proposition 1. *Let f be a reduction map from a game Γ' to a game Γ. Then*

(1) *$f^{-1}(S)$ is an e-stable set for Γ' whenever S is an e-stable set for Γ*
(2) *$f(T)$ is an e-stable set for Γ whenever T is an e-stable set for Γ'.*

Proof: (a) In order to prove (1), let g be a reduction map from a game Γ'' to Γ'. Then, $g^{-1}(f^{-1}(S)) = (f \circ g)^{-1}(S)$ is an element of $\sigma(\Gamma'')$ since $f \circ g$ is clearly a reduction map from Γ'' to Γ and S is e-stable. Hence, $f^{-1}(S)$ is an e-stable set for Γ'.

(b) Let g be a reduction map from a game Γ'' to Γ. Choose the game $\tilde{\Gamma}$ and reduction maps h and k as in Lemma 2. First we will show that $k(h^{-1}(T))$ is an element of $\sigma(\Gamma'')$. To this end, note that $h^{-1}(T)$ is an element of $\sigma(\tilde{\Gamma})$, since T is an e-stable set for Γ'. So $k(h^{-1}(T))$ is an element of $\sigma(\Gamma'')$ by the projection property of σ. Secondly, $k(h^{-1}(T))$ is a subset of $g^{-1}(f(T))$, since

$$g(k(h^{-1}(T))) = (g \circ k)(h^{-1}(T)) = (f \circ h)(h^{-1}(T)) = f(T).$$

From these facts and the monotonicity of σ it follows that $g^{-1}(f(T))$ is an element of $\sigma(\Gamma'')$. ◁

From the foregoing proposition it is immediately clear that, for a regular solution σ, the solution $\tilde{\sigma}$ is invariant. Yet, due to the other requirements of Kohlberg and Mertens, this modification is not satisfactory. E.g. the strategy space Δ of a game is always an e-stable set, which shows that the solution $\tilde{\sigma}$ cannot be admissible (see section 6.1), even if the solution σ is. Therefore, given a regular solution σ, we need to select from the class $\tilde{\sigma}(\Gamma)$ a number of well-behaved solution sets that will constitute our modified solution. At first sight the selection of all minimal e-stable sets seems to be a reasonable attempt. However, the 'projection' of a minimal e-stable set need not be minimal, so that this choice need not yield an invariant solution. Nevertheless, for example the admissibility requirement forces us to choose only 'fairly small' e-stable sets for the modified solution.

Step 2. For these reasons we consider the class $\sigma_{mod}(\Gamma)$ of those e-stable sets S for Γ for which there exists a pair (Γ', f) with $\Gamma' \rightarrow_f \Gamma$ and a *minimal* e-stable set T for Γ' such that $f(T) = S$.

Definition 4. For a regular solution σ, we call the solution σ_{mod} as introduced before the *modified* solution.

First of all we note that, for all games Γ, $\sigma_{mod}(\Gamma)$ is a subset of $\sigma(\Gamma)$. Further, $\tilde{\sigma}(\Gamma)$ contains at least all minimal e-stable sets for Γ. So Theorem 2 implies that $\sigma_{mod}(\Gamma)$ is not empty for any game Γ. In section 6 we will moreover show that the modification σ_{mod} of a regular solution σ inherits several properties of the original solution. Furthermore,

Theorem 3. *The modification σ_{mod} of a regular solution σ is invariant.*

Proof: Let f be a reduction map from a game Γ' to a game Γ. The proof consists of two parts.

(a) In this part we will prove that $f(S) \in \sigma_{mod}(\Gamma)$ if $S \in \sigma_{mod}(\Gamma')$.

If $S \in \sigma_{mod}(\Gamma')$, there exists a reduction map g from a game Γ'' to Γ' and a minimal e-stable set U for Γ'' such that $g(U) = S$. Hence, $f(S) = f(g(U)) = (f \circ g)(U)$ while $f \circ g$ is a reduction map from Γ'' to Γ. So $f(S)$ is the image under $f \circ g$ of the minimal e-stable set U for Γ'' and therefore an element of $\sigma_{mod}(\Gamma)$.

(b) Now we will show that for each $T \in \sigma_{mod}(\Gamma)$ there is an $S \in \sigma_{mod}(\Gamma')$ with $T = f(S)$.

If $T \in \sigma_{mod}(\Gamma)$, then there exists a reduction map g from a game Γ'' to Γ and a minimal e-stable set U for Γ'' such that $g(U) = T$. Choose $\tilde{\Gamma}$, h and k as in Lemma 2. Now, according to Proposition 1 (1), $k^{-1}(U)$ is an e-stable set for $\tilde{\Gamma}$. So, by Theorem 2, $k^{-1}(U)$ contains a minimal e-stable set, say V, for the game $\tilde{\Gamma}$. So, by the definition of σ_{mod}, $S := h(V) \in \sigma_{mod}(\Gamma')$. Furthermore, according to Proposition 1 (2), $k(V) \subset U$ is an e-stable set for Γ''. Since U is a minimal e-stable set, $k(V) = U$. Hence, by Lemma 2,

$$f(S) = f(h(V)) = (f \circ h)(V) = (g \circ k)(V) = g(k(V)) = g(U) = T. \quad \triangleleft$$

6. Some regular solutions

In this section we will give three examples of regular solutions that are based on the notions of essentiality (Wu Wen-tsün and Jiang Jia-he (1962)), *BR*-

stability and quasi-stability (Hillas (1990)), respectively. The main difference with the original definitions is the absence of a minimality requirement. The omission of this requirement immediately yields monotonic solution, one of the requirements for regularity. Of course, starting with one of these solutions *including* the minimality condition, one might as well take the monotonic closure (in the set-theoretic sense) of the solution as the basis for the modification procedure.

A closed subset S of the strategy space Δ of a game Γ is called *essential* if for any neighborhood V of S there is a neighborhood U of Γ such that $E(\tilde{\Gamma}) \cap V$ is not empty for all $\tilde{\Gamma} \in U$.

It is well known (cf. Kohlberg and Mertens (1986)) that the solution σ_{ES} that assigns to a game the collection of essential sets of that game is not invariant. However, the modification of this solution is invariant by Theorem 3 and

Theorem 4. *The solution σ_{ES} that assigns to a game the collection of essential sets of that game is regular.*

Proof: Let Γ be a game. We need to check the requirements of Definition 3. Since (4) is obvious, we will only check the other requirements.

(1) By Proposition 1 of Kohlberg and Mertens (1986), $\sigma_{ES}(\Gamma)$ is non-empty.

(2) Let f be a reduction map from a game $\Gamma' = \langle B, v\rangle$ to the game $\Gamma = \langle A, u\rangle$ and let T be an essential set of Γ'. Let V be a neighborhood of $f(T)$. Then $f^{-1}(V)$ is a neighborhood of T. Since T is an essential set of Γ', there is a number $\delta > 0$ such that $E(\Gamma'') \cap f^{-1}(V)$ is not empty for all $\Gamma'' \in B_\delta(\Gamma')$. Now take a fixed $\tilde{\Gamma} \in B_\delta(\Gamma)$, say $\tilde{\Gamma} = \langle A, \tilde{u}\rangle$. Now consider the game $\tilde{\Gamma}_f = \langle B, \tilde{v}\rangle$, where $\tilde{v}_i := \tilde{u}_i \circ f$. Then obviously, $\tilde{\Gamma}_f \to_f \tilde{\Gamma}$ and $\tilde{\Gamma}_f \in B_\delta(\Gamma')$. So $E(\tilde{\Gamma}_f) \cap f^{-1}(V)$ is not empty. For an $x \in E(\tilde{\Gamma}_f) \cap f^{-1}(V)$ however, Lemma 1 implies that $f(x) \in E(\tilde{\Gamma}) \cap V$. Hence, $f(T)$ is an essential set of Γ.

(3) Suppose that $S_1, S_2, \ldots$ is a sequence of essential sets for Γ d_H-converging to a set S. If V is a neighborhood of S, then $S_{i_0} \subset V$ for some large i_0. Since S_{i_0} is essential for Γ, there is a neighborhood U of Γ such that $E(\tilde{\Gamma}) \cap V$ is not empty for all $\tilde{\Gamma} \in U$. So S is essential for Γ. ◁

Next we describe two stability concepts introduced by Hillas (1990). We start with some definitions. For two compact- and convex-valued upper hemicontinuous (uhc) correspondences $\varphi, \psi : \Delta \twoheadrightarrow \Delta$

$$d_\infty(\varphi, \psi) := \max\{d_H(\varphi(x), \psi(x)) \mid x \in \Delta\}$$

and $\mathrm{fix}(\varphi) := \{x \in \Delta \mid x \in \varphi(x)\}$ is the set of fixed points of φ.

Following the idea of Hillas (1990), a closed set $S \subset \Delta$ is called a *BR-set* if for any neighborhood V of S, there exists a number $\delta > 0$ such that $\mathrm{fix}(\varphi) \cap V$ is not empty if $d_\infty(\beta, \varphi) < \delta$.

Theorem 5. *The solution σ_{BR} that assigns to a game the collection of BR-sets is regular.*

Proof: We only show that σ_{BR} has the projection property. To this end, let f be a reduction map from a game Γ' to Γ and suppose that $T \subset \Delta'$ is a *BR*-set of Γ'. We have to show that $f(T)$ is a *BR*-set of Γ.

Let V be a neighborhood of $f(T)$. Then $f^{-1}(V)$ is a neighborhood of T. So there is a number $\delta > 0$ such that $\text{fix}(\psi) \cap f^{-1}(V)$ is not empty if $d_\infty(\psi, \beta) < \delta$. Furthermore, by Lemma 4 of Vermeulen and Jansen (1997a), there exists a number $\eta > 0$ such that for any strategy profile $y \in \Delta'$ and $z \in \Delta$ with $\|f(y) - z\|_\infty < \eta\delta$, there is a strategy profile $z' \in \Delta'$ with $\|y - z'\|_\infty < \delta$ and $f(z') = z$.

Now let $\varphi : \Delta \twoheadrightarrow \Delta$ be a compact- and convex-valued uhc correspondence with $d_\infty(\varphi, \beta) < \eta\delta$. We will show that $\text{fix}(\varphi) \cap V$ is not empty. To this end, define the correspondence $\psi : \Delta' \twoheadrightarrow \Delta'$ by

$$\psi(x) := \{y \in \Delta' \mid f(y) \in \varphi(f(x))\}.$$

Since φ is a compact- and convex-valued uhc correspondence and f is linear, also ψ is a compact- and convex-valued uhc correspondence. Furthermore, $d_\infty(\psi, \beta) < \delta$ as we will show now.

Suppose that $y \in \psi(x)$. Then $f(y) \in \varphi(f(x))$. Since $d_\infty(\varphi, \beta) < \eta\delta$, there is a $z \in \beta(f(x))$ with $\|f(y) - z\|_\infty < \eta\delta$. So a $z' \in \Delta'$ can be found such that $\|y - z'\|_\infty < \delta$ and $f(z') = z$. Then $z' \in \beta(x)$ by Lemma 1. Hence, $\psi(x) \subset B_\delta(\beta(x))$. In an analogous way, one proves that $\beta(x) \subset B_\delta(\psi(x))$.

The foregoing implies that $\text{fix}(\psi) \cap f^{-1}(V)$ is not empty. So we can take an $x^* \in \text{fix}(\psi) \cap f^{-1}(V)$. Then $f(x^*) \in \text{fix}(\varphi)$ and $f(x^*) \in V$. Hence $\text{fix}(\varphi) \cap V$ is not empty. $\lhd$

Finally, we come to the second stability concept introduced by Hillas.

Let, for a game Γ and for each player i, $\varepsilon_i : \mathscr{P}(A_i) \to [0, 1]$ be a mapping, where $\mathscr{P}(A_i)$ is the class of all non-empty proper subsets of A_i. We call $\varepsilon := (\varepsilon_i)_{i \in N}$ a *perturbation system* for Γ. For such a perturbation system ε, $\Gamma[\varepsilon]$ is the game where, for all i, the strategy space of player i is restricted to $\Delta_i(\varepsilon) := \{x_i \in \Delta_i \mid x_i(T) \geq \varepsilon_i(T) \text{ for all } T \in \mathscr{P}(A_i)\}$ with $x_i(T) := \sum_{a \in T} x_{ia}$. Note that the strategy spaces are full-dimensional if $\|\varepsilon\|_\infty := \max\{\varepsilon_i(T) \mid i \in N, T \in \mathscr{P}(A_i)\}$ is small.

Following Hillas (1990), a closed subset S of Δ is called a *Q-set* if for any neighborhood V of S there is a number $\delta > 0$ such that $E(\Gamma[\varepsilon]) \cap V$ is not empty, for all perturbation systems ε with $\|\varepsilon\|_\infty < \delta$.

Since the solution σ_Q that assigns to a game the collection of Q-sets is not invariant, the following theorem is important.

Theorem 6. *The solution σ_Q that assigns to a game the collection of Q-sets is regular.*

Proof: We only show that σ_Q has the projection property. Let f be a reduction map from a game Γ' to the game Γ. Let S be a Q-set of Γ'. In order to prove that $f(S)$ is a Q-set of the game Γ, we first note that $f(S)$ is a closed, non-empty set.

Now take a neighborhood V of $f(S)$. We have to show that there exists a number $\delta > 0$ such that $\Gamma[\varepsilon] \cap V$ is not empty whenever $\|\varepsilon\|_\infty < \delta$. To this end, note that there is a number $\delta > 0$ such that $E(\Gamma'[\xi]) \cap f^{-1}(V)$ is not empty for every perturbation system ξ for Γ' with $\|\xi\|_\infty < \delta$, since S is a Q-set of Γ' and $f^{-1}(V)$ is a neighborhood of S.

Now take a perturbation system ε for Γ with $\|\varepsilon\|_\infty < \delta$. We will show that $E(\Gamma[\varepsilon])$ and V have a non-empty intersection. Define the perturbation system $\tilde{\varepsilon}$ for the game Γ' as follows. Write $\Gamma = \langle A, u\rangle$ and $\Gamma' = \langle B, v\rangle$. By Lemma 1 of Vermeulen and Jansen (1997a) we know that each A_i is a subset of $f_i(B_i)$. Thus we can choose a subset C_i of B_i such that f_i is one-to-one and onto from C_i to A_i. Now define for a player i and proper subset T of B_i

$$\tilde{\varepsilon}_i(T) := \begin{cases} \varepsilon_i(f_i(T)) & \text{if } T \subset C_i \text{ and } \phi \neq T \neq C_i \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, since $\|\varepsilon\|_\infty < \delta$, $\|\tilde{\varepsilon}\|_\infty < \delta$. Hence, $E(\Gamma'[\tilde{\varepsilon}]) \cap f^{-1}(V)$ is not empty, and we can take a strategy $x \in E(\Gamma'[\tilde{\varepsilon}]) \cap f^{-1}(V)$. Then $f(x) \in V$, and the proof is complete if we can show that $f(x) \in E(\Gamma[\varepsilon])$. Since for all $i \in N$, and all proper subsets T of C_i

$$f_i(x)(f_i(T)) \geq x_i(T) \geq \tilde{\varepsilon}_i(T) = \varepsilon_i(f_i(T)),$$

and every proper subset of A_i is equal to a set of the form $f_i(T)$ for exactly one proper subset T of C_i, we know that $f(x)$ is an element of $\Delta(\varepsilon)$. Now let $y_i \in \Delta_i(\varepsilon)$. Define $z_i \in \Delta_i'$ by

$$z_{ib} := \begin{cases} y_{if_i(b)} & \text{if } b \in C_i \\ 0 & \text{else.} \end{cases}$$

It is straightforward to show that z_i is an element of $\Delta_i(\tilde{\varepsilon})$, and $f_i(z_i) = y_i$. Hence,

$$u_i(f(x)_{-i} \mid y_i) = u_i(f(x)_{-i} \mid f_i(z_i)) = v_i(x_{-i}|z_i) \leq v_i(x_{-i}|x_i) = u_i(f(x)).$$

Since y_i was arbitrarily chosen, $f(x) \in E(\Gamma[\varepsilon])$. $\triangleleft$

7. On properties of modified regular solutions

In this section we will investigate which properties of a regular solution σ are inherited by its modification σ_{mod}.

7.1. Admissible best reply invariance

The first property we will study is abr-invariance. Mertens (1987) showed that an abr-invariant solution that also satisfies a strong version of invariance is ordinal. In this section we will show that our modification method preserves abr-invariance. As a consequence, any solution that will satisfy the stronger version of invariance after modification will be ordinal. In particular this holds for homotopy stability.

Two games $\Gamma = \langle A, u\rangle$ and $\Gamma^* = \langle A, u^*\rangle$ are called *admissible-best-reply-equivalent* (abr-equivalent) if the admissible best replies in the game Γ coincide with those in the game Γ^*. A solution τ is called *abr-invariant* (cf. Mertens (1987)) if $\tau(\Gamma)$ equals $\tau(\Gamma^*)$ for any pair of abr-equivalent games Γ and Γ^*.

In order to show that the solution σ_{mod} is abr-invariant if σ is, we need the following result that can easily be proved with the help of Lemma 1.

Lemma 3. *If two games Γ and Γ^* are abr-equivalent, then Γ_f and Γ_f^* are abr-equivalent for any reduction map f.*

Obviously, given that σ is abr-invariant, this lemma together with Proposition 1 implies that a closed subset of Δ_A is e-stable for the game Γ if and only if it is e-stable for the game Γ^*. Hence, σ_{mod} is abr-invariant if σ has this property.

7.2. Connectedness, admissibility and backward induction

Theorem 7 (Connectedness). *Let σ be a regular solution. If, for a game Γ, every minimal element of $\sigma(\Gamma)$ is connected, then all elements of $\sigma_{mod}(\Gamma)$ are connected.*

Proof: The proof consists of two parts.

(a) First we will show that every minimal e-stable set for Γ is connected. So take a minimal e-stable set T for Γ and suppose that T is not connected. Then there are two disjoint non-empty closed sets T_1 and T_2 in Δ with $T_1 \cup T_2 = T$. Furthermore, T_1 and T_2 are no e-stable sets for Γ, since T is minimal. Then there are pairs (Γ', f) and (Γ'', g) with $\Gamma' \rightarrow_g \Gamma$ and $\Gamma'' \rightarrow_g \Gamma$ such that $f^{-1}(T_1)$ is not an element of $\sigma(\Gamma')$ and $g^{-1}(T_2)$ is not an element of $\sigma(\Gamma'')$. Let $\tilde{\Gamma}$, h and k be as in Lemma 2. Then, by the projection property of σ, $h^{-1}(f^{-1}(T_1)) = (f \circ h)^{-1}(T_1)$ is not an element of $\sigma(\tilde{\Gamma})$ and $k^{-1}(g^{-1}(T_1)) = (g \circ k)^{-1}(T_1)$ is not an element of $\sigma(\tilde{\Gamma})$.

However, by Proposition 1 we know that $(f \circ h)^{-1}(T)$ is an element of $\sigma(\tilde{\Gamma})$, since T is an e-stable set for Γ. Consequently

$$\begin{aligned}(f \circ h)^{-1}(T) &= (f \circ h)^{-1}(T_1) \cup (f \circ h)^{-1}(T_2) \\ &= (f \circ h)^{-1}(T_1) \cup (g \circ k)^{-1}(T_2)\end{aligned}$$

contains a minimal, hence connected, element of $\sigma(\tilde{\Gamma})$. Since $(f \circ h)^{-1}(T_1)$ and $(g \circ k)^{-1}(T_2)$ are non-empty, closed and disjoint sets, that connected element of $\sigma(\tilde{\Gamma})$ must be contained in one of these sets. Therefore, one of these sets is an element of $\sigma(\tilde{\Gamma})$ by the monotonicity property of σ, which yields a contradiction.

(b) Let S be an element of $\sigma_{mod}(\Gamma)$. Then there is a pair (Γ', f) with $\Gamma' \rightarrow_f \Gamma$ and a minimal e-stable set T for Γ' with $f(T) = S$. From part (a) it follows that T is connected. Hence, S must be connected since f is continuous. $\triangleleft$

Definition 5. A solution σ is called *perfect-valued* if, for every game Γ, each **minimal** element of $\sigma(\Gamma)$ is contained in $PE(\Gamma)$.

In the proof of the next theorem we use the following result of Vermeulen and Jansen (1997a).

Lemma 4. *Let f be a reduction map from a game Γ' to Γ. Then the set of perfect equilibria of the game Γ' equals the inverse image under f of the set of perfect equilibria of Γ.*

Theorem 8 (Admissibility). *If σ is a perfect-valued, regular solution, then all elements of $\sigma_{mod}(\Gamma)$ are contained in $PE(\Gamma)$.*

Proof: Again the proof is in two parts.

(a) First we show that every minimal e-stable set for Γ is contained in $PE(\Gamma)$. Let S be an e-stable set for Γ. Take a pair (Γ', f) with $\Gamma' \to_f \Gamma$. Then, $f^{-1}(S)$ is an element of $\sigma(\Gamma')$. By the assumption in the theorem and the monotonicity of σ, this implies that $f^{-1}(S) \cap PE(\Gamma')$ is an element of $\sigma(\Gamma')$. Since by Lemma 4, $f^{-1}(S) \cap PE(\Gamma') = f^{-1}(S) \cap f^{-1}(PE(\Gamma))$, $f^{-1}(S \cap PE(\Gamma))$ is an element of $\sigma(\Gamma')$. Hence, $S \cap PE(\Gamma)$ is an e-stable set for Γ. Therefore, any minimal e-stable set for Γ is contained in $PE(\Gamma)$.

(b) Secondly we show that every element of $\sigma_{mod}(\Gamma)$ must be contained in $PE(\Gamma)$. Suppose that S is an element of $\sigma_{mod}(\Gamma)$. Then there is a pair (Γ', f) with $\Gamma' \to_f \Gamma$ and a minimal e-stable set T for Γ' with $f(T) = S$. From part (a) it follows that T is a subset of $PE(\Gamma')$. So, by Lemma 4, $S = f(T) \subset f(PE(\Gamma')) = PE(\Gamma)$. ◁

Since every element of $\sigma_{mod}(\Gamma)$ is also an element of $\sigma(\Gamma)$, we have a proof of

Theorem 9 (Backward Induction). *Let σ be a regular solution. If, for every game Γ, every element of $\sigma(\Gamma)$ contains a proper equilibrium, then all elements of $\sigma_{mod}(\Gamma)$ contain a proper equilibrium.*

7.3. Deletion of a strategy

In this section the preliminary work for the next section dealing with the Independence of inadmissible strategies will be done.

Let $\Gamma = \langle A, u \rangle$ be a fixed game. We deal with the situation that one of the pure strategies of player j, say b, is deleted. The game induced by the deletion of b is the game $\Gamma^* = \langle A^*, u^* \rangle$ with

$$A_i^* = \begin{cases} A_i & \text{if } i \neq j \\ A_j \backslash \{b\} & \text{if } i = j \end{cases}$$

and u_i^* is the restriction of u_i to $\prod_i A_i^*$.

For an $x_j \in \Delta(A_j)_0 := \{y_j \in \Delta(A_j) \mid y_{jb} = 0\}$, $v_j(x_j)$ is the strategy in $\Delta(A_j^*)$ obtained by the deletion of the b-coordinate of the strategy x_j. In order to analyze the interaction between reductions and the deletion of a strategy, let f be a reduction map from a game $(\Gamma^*)' = \langle C, v \rangle$ to the game Γ^*. Now we will construct a reduction map g from a game Γ' to the game Γ. This construction will be used frequently in the next sections. To this end let, for each $c \in C_j$, $\zeta_j(c)$ in $\Delta(A_j)$ be the strategy obtained from $f_j(c) \in \Delta(A_j^*)$ by inserting a zero as the b-coordinate. Define $D_j := C_j \cup \{b\}$ and let $g_j : \Delta(D_j) \to \Delta(A_j)$ be the linear extension of the map defined by

$$g_j(d) := \begin{cases} b & \text{if } d = b \\ \zeta_j(d) & \text{if } d \in C_j. \end{cases}$$

If $\mu : \Delta(D_j)_0 \to \Delta(C_j)$ is the map associated with the deletion of b, the following lemma implies that the diagram

$$\begin{array}{ccc} \Delta(D_j)_0 & \xrightarrow{\mu_j} & \Delta(C_j) \\ \Big\downarrow g_j & & \Big\downarrow f_j \\ \Delta(A_j)_0 & \xrightarrow{\nu_j} & \Delta(A_j^b) \end{array}$$

commutes.

Lemma 5. $\nu_j \circ g_j = f_j \circ \mu_j$.

Proof: Note that $g_j(x_j) \in \Delta(A_j)_0$ for all $x_j \in \Delta(D_j)_0$. We only have to prove that $(\nu_j \circ g_j)(d) = (f_j \circ \mu_j)(d)$ for all $d \in D_j \backslash \{b\}$, since all maps involved are linear. However,

$$\nu_j(g_j(d)) = \nu_j(\zeta_j(d)) = f_j(d) = f_j(\mu_j(d)). \qquad \triangleleft$$

Now let, for all $i \neq j$, g_i from $\Delta(D_i) := \Delta(C_i)$ to $\Delta(A_i) = \Delta(A_i^*)$ be equal to f_i. Then, for $g := (g_i)_{i \in N}$ the game Γ_g is well defined.

Lemma 6. $(\Gamma^*)' = (\Gamma_g)^*$.

Proof: Note that both games have Δ_C as strategy space. So, we have to show that $v_i = (u_i \circ g)^*$. Take an x in Δ_C. Then there is a strategy profile $z \in \Delta_D$ with $z_{jb} = 0$ and $\mu(z) = x$. Hence,

$$v_i(x) = v_i(\mu(z)) = (u_i^* \circ f)(\mu(z)) = (u_i^* \circ \nu \circ g)(z) = (u_i \circ g)(z)$$

$$= (u_i \circ g)^*(\mu(z)) = (u_i \circ g)^*(x)$$

where the third equality follows from Lemma 5. $\triangleleft$

7.4. Independence of inadmissible strategies

In this section Γ will again be a fixed game. First we suppose that S is an e-stable set for Γ.

Lemma 7. *Suppose that the pure strategy b of player j is not an admissible best reply to S in the game Γ. Then it is not an admissible best reply to $g^{-1}(S)$ in the game Γ_g.*

Proof: Suppose that the pure strategy b of player j in the game Γ_g is an admissible best reply to $g^{-1}(S)$. Then a strategy profile $x \in g^{-1}(S)$ can be found such that $b \in B_j^a(x_{-j})$. Hence there is a completely mixed sequence $(x^k)_{k \in \mathbb{N}}$ in Δ_D converging to x such that $b \in \beta_j(x_{-j}^k)$ for all k. Then however, $(g(x^k))_{k \in \mathbb{N}}$ is a completely mixed sequence in Δ_A converging to $g(x) \in S$ and, for all k,

$$b = g_j(b) \in \beta_j(g(x^k)_{-j}).$$

This leads to $b \in B_j^a(S)$. Contradiction. $\triangleleft$

Definition 6. A perfect-valued solution τ is *independent of inadmissible strategies* if for any game Γ the following holds: if $S \in \tau(\Gamma)$ and $b \notin B_j^a(S)$, then $S' := \nu(S \cap \Delta_0)$ contains an element of $\tau(\Gamma^*)$. Here Γ^* is the game induced by the deletion of the pure strategy b of player j in the game Γ.

Theorem 10 (Independence of inadmissible strategies). *If σ is a perfect-valued, regular solution that is independent of inadmissible strategies, then σ_{mod} is independent of inadmissible strategies.*

Proof: Let $S \in \sigma_{mod}(\Gamma)$ and $b \notin B_j^a(S)$. To show: the set S' contains an element of $\sigma_{mod}(\Gamma^*)$.

(a) Since S is e-stable, we can choose a minimal e-stable subset T of S by Theorem 2 (b). Since each e-stable set of Γ^* contains a minimal e-stable set of Γ^*, and hence an element of $\sigma_{mod}(\Gamma^*)$ by Theorem 2 (b), it is sufficient to show that $T' \subset S'$ is an e-stable set for Γ^*.

(b) In order to show that T' is an e-stable set for the game Γ^*, we take a reduction map f from a game $(\Gamma^*)'$ to the game Γ^*. Let g be the reduction map from Γ_g to Γ as constructed in the previous section. Then, since T is minimal e-stable, and hence an element of $\sigma_{mod}(\Gamma)$, $g^{-1}(T)$ is an element of $\sigma(\Gamma_g)$. Moreover, since $T \in \sigma(\Gamma)$ and σ is perfect-valued,

$$g^{-1}(T) \subset g^{-1}(PE(\Gamma)) = PE(\Gamma_g) \subset \{x \in \Delta_D \mid x_{jb} = 0\}.$$

This implies that $\mu(g^{-1}(T)) = g^{-1}(T)'$. Therefore, since σ is monotonous and independent of inadmissible strategies, $\mu(g^{-1}(T))$ is an element of $\sigma((\Gamma_g)^*)$. So, $\mu(g^{-1}(T))$ is an element of $\sigma((\Gamma^*)')$ by Lemma 6. Furthermore, in view of Lemma 5,

$$f(\mu(g^{-1}(T))) = (f \circ \mu)(g^{-1}(T)) = (\nu \circ g)(g^{-1}(T)) = \nu(T) = T',$$

which implies that $\mu(g^{-1}(T))$ is a subset of $f^{-1}(T')$. Hence, by monotonicity, $f^{-1}(T')$ is an element of $\sigma((\Gamma^*)')$. Therefore, T' is an e-stable set of the game Γ^*. $\triangleleft$

8. Final remarks

Let us finally stress that it is possible to find other ways to modify a solution in order to obtain an invariant one. However, if we insist on preservation of the requirements of the Kohlberg-Mertens properties, this does require a careful selection procedure. To our knowledge, the modification presented here is the first one to preserve all requirements Kohlberg and Mertens (1986) originally mentioned.

To give an example of an alternative modification, let σ be a given solution. We call a closed set S in the strategy space of a game Γ d-stable if $\pi(S) \in \sigma(\Gamma^*)$, where Γ^* is the reduced normal form of Γ and π is the associated reduction map. Consider the solution σ' that assigns the collection $\sigma'(\Gamma)$ of minimal d-stable sets to Γ. Then one can show that σ' is invariant and not empty whenever the original solution σ is not empty and satisfies the closedness property. In the next example we show that by modifying the solution σ_Q in this way the backward induction property is lost.

Example 3. Reducing the game

$$\Gamma = \begin{bmatrix} (6,0) & (6,0) \\ (8,0) & (0,8) \\ (0,8) & (8,0) \\ (3,4) & (7,0) \end{bmatrix},$$

leads to

$$\Gamma^* = \begin{bmatrix} (6,0) & (6,0) \\ (8,0) & (0,8) \\ (0,8) & (8,0) \end{bmatrix}.$$

Furthermore, it is straightforward to show that $\sigma_Q(\Gamma^*) = \{(e_1, q) \in \Delta_3 \times \Delta_2 \mid q_1 \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}\}$. Now, for any d-stable set S for the game Γ, we know that $(e_1, q) \in S$ for $q_1 \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$. This implies that

$$\sigma'_Q(\Gamma) = \{T\},$$

where $T = \{(e_1, q) \in \Delta_4 \times \Delta_2 \mid q_1 \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}\}$. The set T however does not contain the only proper equilibrium $(e_1, (\frac{7}{12}, \frac{5}{12}))$ of the game Γ. Note that all quasi-stable sets contain a proper equilibrium and that $(\sigma_Q)_{mod}(\Gamma) = \{E(\Gamma)\}$. $\triangleleft$

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