Rational probability measures

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Abstract

In this paper we introduce the concept of rational probability measures. These are probability measures that map every Borel event to a rational number. We show that a rational probability measure has a finite support. As a consequence we prove a new version of Kolmogorov extension theorem. In the second part of the paper we define N-rational probability measures as the set of probability measures that map every Borel event to a rational number with denominator in $N \subseteq \mathbb{N}$. We show that for every finite $N \subseteq \mathbb{N}$, the set of N-rational probability measures is closed in the space of Borel probability measures. The latter is not true when N is infinite.

1. Introduction

A rational probability measure maps every Borel set to a rational number. Rational probability measures are used to model beliefs held by agents who exhibit a specific form of bounded rationality in their reasoning process. Namely, we consider agents whose language does not contain sentences of the form *tomorrow it will rain with probability* $\sqrt{2}/2$. This restriction may be due to their limited understanding of real numbers, as opposed to rationals, and therefore they cannot consciously hold such beliefs.

Or main result (Theorem 3.1) shows that the support of every rational probability measure is necessarily finite. This follows from the fact hat every series of rational numbers converging to a rational limit has infinitely many subseries converging to an irrational number (Badea, 1987).

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A direct consequence of the previous seemingly surprising result is a new version of Kolmogorov extension theorem (Proposition 4.1), stating that a sequence of rational probability measures over separable metrizable spaces is uniquely extended to a unique — nt necessarily rational — probability measure over the product space.

In the second part of the paper, we restrict our attention to measures that map every Borel set to a rational number with denominator belonging to a subset of the natural numbers, $N \subseteq \mathbb{N}$. We call these measures N-rational, and we show that if N is finite then the set of N-rational probability measures is closed in the space of Borel probability measures (Proposition 5.1). The latter is not true when N is infinite. The space of N-rational probability measures captures probabilistic beliefs of agents who exhibit an additional form of bounded rationality, in that their language can only express certain rational numbers, e.g., we have in mind agents whose probabilistic beliefs are outputs of a computer that can only express decimals with at most n digits. A straightforward consequence of the previous result is that for any finite $N \subseteq \mathbb{N}$, a metrizable space is separable (Polish) if and only if the space of N-rational probability measures is separable and metrizable (Poilsih).

2. Preliminaries

In this section we present some definitions and standard results. For more details, we refer to Aliprantis and Border (1994), or Kechris (1995).

A topological space $(X, \mathcal{T})$ is metrizable whenever there is a metric $d : X \times X \to \mathbb{R}$ such that (X, d) is a metric space. A topological space is called separable if it contains a countable dense subset. Recall that a metrizable space is separable if and only if it is second countable. Any subspace of a separable metrizable space is separable and metrizable. The product of countably many separable metrizable spaces is separable and metrizable. A metrizable space is complete whenever every Cauchy sequence converges in X. A topological space is called Polish whenever it is separable and completely metrizable.

Let X be a separable metrizable space, together with the Borel σ-algebra, $\mathcal{B}$. As usual, $\Delta(X)$ denotes the space of probability measures on $(X, \mathcal{B})$, endowed with the topology of weak convergence. Recall that the topology of weak convergence, which is usually denoted by w^*, is the coarsest topology that makes the mapping $\mu \mapsto \int f d\mu$ continuous, for every bounded and continuous real-valued function, f. The space $\Delta(X)$ is separable and metrizable (Polish) if and only if X is separable and metrizable (Polish). For further properties of w^*, we refer to Aliprantis and Border (1994, Ch. 15). For some $\mu \in \Delta(X)$ let $\Gamma(\mu)$ denote its support, i.e., the set of all points $X \in X$ such that every $T \in \mathcal{T}$ with $x \in T$ has positive measure: $\Gamma(\mu) = \{x \in X : x \in T \in \mathcal{T} \Rightarrow \mu(T) > 0\}$. The support is the smallest closed subset of X with measure equal to 1. If X is separable and metrizable, the support is unique (Parthasarathy, 1967, Thm. 2.1). We say that a measure $\mu \in \Delta(X)$ is tight whenever $\mu(B) = \sup\{\mu(K) \mid K \subseteq B,\ K \in \mathcal{B} \text{ and } K \text{ compact}\}$ for all $B \in \mathcal{B}$.

3. Properties of rational probability measures

Unless stated otherwise, we assume that X separable and metrizable.

Definition 3.1. We define the set of ***rational probability measures*** by

$$\Delta^{\mathbb{Q}}(X) := \left\{ \mu \in \Delta(X) \;\middle|\; \mu(B) \in \mathbb{Q}, \forall B \in \mathcal{B} \right\}.$$

Theorem 3.1. *Every $\mu \in \Delta^{\mathbb{Q}}(X)$ has a finite support.*

Proof. Consider an arbitrary $\mu \in \Delta^{\mathbb{Q}}(X)$, and consider the set of singletons with positive measure,

$$\Gamma := \{x \in X : \mu(\{x\}) > 0\}. \tag{3.1}$$

First, we show that Γ is non-empty. Suppose that μ is a non-atomic measure. Then, it follows from Fremlin (2003, p. 46) that for every $\xi \in (0, 1)$ there is some $B \in \mathcal{B}$ such that $\mu(B) = \xi$, which contradicts $\mu \in \Delta^{\mathbb{Q}}(X)$ if we consider some $\xi \in \mathbb{R} \setminus \mathbb{Q}$. Hence, there is at least one atom $A \in \mathcal{B}$. Now, it follows from Aliprantis and Border (1994, Lem. 12.18) that A contains a singleton of positive measure, implying that Γ is non-empty.

Second, we show that Γ is countable. Let $\{\Gamma_n\ ;\ n \geq 1\}$ be a countable partition of Γ, with

$$\Gamma_n := \left\{ x \in \Gamma : \frac{1}{n+1} < \mu(\{x\}) \leq \frac{1}{n} \right\}.$$

If Γ is uncountable, there is some $n \geq 1$ such that Γ_n is uncountable, implying that there is a countably infinite $\{x_1, x_2, ...\} \subseteq \Gamma_n$. Finally, observe that

$$\begin{aligned}
\mu(X) &\geq \mu(\Gamma_n) \\
&\geq \sum_{k=1}^{\infty} \mu(\{x_k\}) \\
&> \sum_{k=1}^{\infty} \frac{1}{n+1} \\
&= \infty,
\end{aligned}$$

which is a contradiction.

Third, we show that $\mu(\Gamma) = 1$. Assume otherwise, implying that $\mu(X \setminus \Gamma) > 0$. Since Γ is countable, it is Borel, implying that $X \setminus \Gamma$ is also Borel. Hence, it follows from Aliprantis and Border

(1994, Lem. 12.18) that there is $x \in X \setminus \Gamma$ with $\mu(\{x\}) > 0$, implying, by Eq. (3.1), that $x \in \Gamma$, which is a contradiction.

Now, suppose that Γ is infinite, and construct a sequence of rational numbers $\{\mu(\{x_k\})\}_{k>0}$ where $\Gamma = \{x_1, x_2, ...\}$. It follows from $\mu(\Gamma) = 1$ that

$$\sum_{k=1}^{\infty} \mu(\{x_k\}) = 1.$$

Then, it follows from Badea (1987, Prop., p. 225) that there is a subsequence $\{x'_k\}_{k>0}$ of $\{x_k\}_{k>0}$ such that

$$\mu\Big(\bigcup_{k=1}^{\infty} \{x'_k\}\Big) = \sum_{k=1}^{\infty} \mu\big(\{x'_k\}\big)$$

is an irrational number, thus contradicting the hypothesis that $\mu \in \Delta^{\mathbb{Q}}(X)$.

Therefore, Γ is necessarily finite. Moreover, it is closed, as the finite union of singletons, implying that $\Gamma(\mu) = \Gamma$, which completes the proof. □

Remark 3.1. The previous quite surprising result rules out probability measures with countably infinite support, even if the probability of each atom is a rational number, as illustrated by the following example. ◁

Example 3.1. Let $X = \{x_1, x_2, ...\}$ and suppose that $\mu \in \Delta(X)$ is such that

$$\mu\big(\{x_k\}\big) = 2^{-k}$$

for every $k > 0$. It is straightforward verifying that

$$\sum_{k=1}^{\infty} \mu\big(\{x_k\}\big) = 1,$$

thus confirming that μ is a probability measure.

Consider the Fibonacci sequence, $\{\alpha_k\}_{k \geq 0}$, and define the increasing sequence $\{y_k\}_{k>0}$ by

$$\begin{aligned} y_k &:= \mu\big(\{x_{\alpha_1}, ..., x_{\alpha_k}\}\big) \\ &= \sum_{n=1}^{k} \mu\big(\{x_{\alpha_n}\}\big) \\ &= \sum_{n=1}^{k} 2^{-\alpha_n} \\ &= \frac{\sum_{n=1}^{k} 2^{\alpha_k - \alpha_n}}{2^{\alpha_k}} \end{aligned} \tag{3.2}$$

and observe that each y_k is a rational number, with nominator

$$p_k = \sum_{n=1}^{k} 2^{\alpha_k - \alpha_n} \tag{3.3}$$

and denominator

$$q_k = 2^{\alpha_k}$$

for each $k > 0$. Recall from Brun's criterion (Brun, 1910) that $\lim_{k\to\infty} y_k$ is an irrational number, if the following three conditions hold for all $k > 0$:

(a) $p_k < p_{k+1}$,

(b) $y_k < y_{k+1}$,

(c) $(p_{k+2} - p_{k+1})/(q_{k+2} - q_{k+1}) < (p_{k+1} - p_k)/(q_{k+1} - q_k)$.

The first two conditions trivially follow from Equations (3.3) and (3.2) respectively. Now, observe that

$$\begin{aligned}
\frac{p_{k+1} - p_k}{q_{k+1} - q_k} &= \frac{\sum_{n=1}^{k+1} 2^{\alpha_{k+1}-\alpha_n} - \sum_{n=1}^{k} 2^{\alpha_k - \alpha_n}}{2^{\alpha_{k+1}} - 2^{\alpha_k}} \\
&= \frac{1 + \left(2^{\alpha_{k+1}} - 2^{\alpha_k}\right) \sum_{n=1}^{k} 2^{-\alpha_n}}{2^{\alpha_{k+1}} - 2^{\alpha_k}} \\
&= \frac{1}{2^{\alpha_{k+1}} - 2^{\alpha_k}} + \sum_{n=1}^{k} 2^{-\alpha_n},
\end{aligned}$$

and consider the difference

$$\begin{aligned}
\frac{p_{k+1} - p_k}{q_{k+1} - q_k} - \frac{p_{k+2} - p_{k+1}}{q_{k+2} - q_{k+1}} &= \frac{1}{2^{\alpha_{k+1}} - 2^{\alpha_k}} + \sum_{n=1}^{k} 2^{-\alpha_n} - \frac{1}{2^{\alpha_{k+2}} - 2^{\alpha_{k+1}}} - \sum_{n=1}^{k+1} 2^{-\alpha_n} \\
&= \frac{1}{2^{\alpha_{k+1}} - 2^{\alpha_k}} - \frac{1}{2^{\alpha_{k+2}} - 2^{\alpha_{k+1}}} - \frac{1}{2^{\alpha_{k+1}}} \\
&= \frac{1}{2^{\alpha_{k+1}} - 2^{\alpha_k}} - \frac{1}{2^{\alpha_{k+1}}\left(2^{\alpha_{k+2}-\alpha_{k+1}} - 1\right)} - \frac{1}{2^{\alpha_{k+1}}} \\
&= \frac{1}{2^{\alpha_{k+1}} - 2^{\alpha_k}} - \frac{2^{\alpha_{k+2}-\alpha_{k+1}}}{2^{\alpha_{k+1}}\left(2^{\alpha_{k+2}-\alpha_{k+1}} - 1\right)} \\
&= \frac{1}{2^{\alpha_{k+1}} - 2^{\alpha_k}} - \frac{2^{\alpha_k}}{2^{\alpha_k} 2^{\alpha_{k+1}-\alpha_k}\left(2^{\alpha_k} - 1\right)} \\
&= \frac{1}{2^{\alpha_{k+1}} - 2^{\alpha_k}} - \frac{1}{2^{\alpha_{k+1}} - 2^{\alpha_{k+1}-\alpha_k}} \\
&= \frac{1}{2^{\alpha_{k+1}} - 2^{\alpha_k}} - \frac{1}{2^{\alpha_{k+1}} - 2^{\alpha_{k-1}}}
\end{aligned}$$

which is strictly positive, since $\alpha_k > \alpha_{k-1}$, thus proving that (c) holds, and therefore

$$\mu\Big(\bigcup_{k=1}^{\infty} \{x_{\alpha_k}\}\Big) = \lim_{k\to\infty} y_k$$

is an irrational number. Hence, $\mu \notin \Delta^{\mathbb{Q}}(X)$. ◁

Corollary 3.1. *Every $\mu \in \Delta^{\mathbb{Q}}(X)$ is tight.*

Proof. Recall from Aliprantis and Border (1994, Lemma 12.6) that μ is tight if and only if for all $\varepsilon > 0$ there is some compact $K \subseteq X$ such that $\mu(K) > \mu(X) - \varepsilon$. It follows from Theorem 3.1 that $\Gamma(\mu)$ is finite, and therefore compact. Combine the latter with the fact that for every $\varepsilon > 0$,

$$\mu\big(\Gamma(\mu)\big) > \mu(X) - \varepsilon$$

and the proof is completed. □

Remark 3.2. The previous result extends Thm. 17.11 from Kechris (1995) to spaces which are not complete, when the probability measure is rational. ◁

4. A version of Kolmogorov extension theorem

Consider a sequence of separable metrizable spaces $\{X_k\}_{k>0}$ with the corresponding Borel σ-algebras, $\mathcal{B}_k$, and let $X := \prod_{k>0} X_k$, together with the product σ-algebra, $\mathcal{B} := \bigotimes_{k>0} \mathcal{B}_k$. For every finite $K \subseteq \mathbb{N}$, consider the sub-σ-algebra of $\mathcal{B}$, defined by

$$\mathcal{B}_K := \Big\{ \big(\prod_{k\in K} B_k\big) \times \big(\prod_{k\notin K} X_k\big) \;\Big|\; B_k \in \mathcal{B}_k,\ \forall k \in K \Big\},$$

and let μ_K be a Borel probability measure on $(X, \mathcal{B}_K)$. Observe that if $K \subseteq L$ then $\mathcal{B}_K \subseteq \mathcal{B}_L$, implying that $\{\mathcal{B}_K\}$ is an increasing net of σ-algebras on X. We say that the collection $\{(\mathcal{B}_K, \mu_K)\}$ is Kolmogorov consistent in X, whenever

$$\mathcal{B}_K \subseteq \mathcal{B}_L \Rightarrow \mu_K(B) = \mu_L(B)$$

for all $B \in \mathcal{B}_K$.

Then, Kolmogorov Extension Theorem provides sufficient conditions for the existence of a unique Borel probability measure $\mu \in \Delta(X, \mathcal{B})$ extending every μ_K (see, Aliprantis and Border, 1994, Ch. 15.6). In other words, there is a unique $\mu \in \Delta(X, \mathcal{B})$ such that for every finite $K \subseteq \mathbb{N}$,

$$\mu_K(B) = \mu(B)$$

for all $B \in \mathcal{B}_K$.

Proposition 4.1. *Let $\{X_k\}_{k>0}$ be a family of separable metrizable spaces, together with the Borel σ-algebras $\mathcal{B}_k$. For each finite $K \subseteq \mathbb{N}$, let μ_K be a rational probability measure on $(X, \mathcal{B}_K)$. Assume that the distributions $\{\mu_K\}$ are Kolmogorov consistent. Then, there is a unique probability measure on $(X, \mathcal{B})$ that extends every μ_K.*

Proof. Since X_k is separable and metrizable, it follows that X_K is Hausdorff for every finite $K \subseteq \mathbb{N}$. Moreover, all μ_K are tight (Corollary 3.1). Then, the result follows directly from Aliprantis and Border (1994, Cor. 15.28). $\square$

Remark 4.1. Observe that the unique measure $\mu \in \Delta(X)$ extending every μ_K is not necessarily rational as illustrated in the following example. $\triangleleft$

Example 4.1. Let $X_k := \{0, 1\}$, endowed with the discrete σ-algebra, $\mathcal{B}_k$. Consider the Cantor set,

$$X := \prod_{k=1}^{\infty} X_k = \{0, 1\}^{\mathbb{N}}.$$

For each $k > 0$, let α_k be the k-digit approximation of $\sqrt{2}/2$, and

$$\beta_k := \alpha_k + 10^{-k}.$$

Now, let $\mu^k \in \Delta(X)$ be such that

$$\begin{aligned}
\mu^k\Big(\underbrace{\{1\} \times \cdots \times \{1\}}_{k \text{ times}} \times \prod_{\ell=k+1}^{\infty} X_\ell\Big) &= 1 - \beta_1 \\
&\ \ \vdots \\
\mu^k\Big(\underbrace{\{0\} \times \cdots \times \{0\}}_{n \text{ times}} \times \underbrace{\{1\} \times \cdots \times \{1\}}_{k-n \text{ times}} \times \prod_{\ell=k+1}^{\infty} X_\ell\Big) &= \beta_n - \beta_{n+1} \\
&\ \ \vdots \\
\mu^k\Big(\underbrace{\{0\} \times \cdots \times \{0\}}_{k \text{ times}} \times \prod_{\ell=k+1}^{\infty} X_\ell\Big) &= \beta_k,
\end{aligned}$$

and for every $K \subseteq \{1, ..., k\}$, define $\mu_K \in \Delta(X, \mathcal{B}_K)$ by

$$\mu_K = \int_{\prod_{\ell \in \{1,\dots,k\} \setminus K} X_\ell} d\mu^k,$$

which is obviously rational for every finite $K \subseteq \mathbb{N}$. Then, it is straightforward that the unique $\mu \in \Delta(X)$ extending every μ_K is not rational as

$$\mu\big(\{0\} \times \{0\} \times \cdots\big) = \lim_{k \to \infty} \beta_k = \frac{\sqrt{2}}{2},$$

is obviously an irrational number. $\triangleleft$

Remark 4.2. The previous result extends shows that the X_k's do need to be complete for Kolmogorov extension theorem (Aliprantis and Border, 1994, Cor. 15.27) to hold, as long as the corresponding probability measures are rational. $\triangleleft$

5. The space of N-rational probability measures

For some given $N \subseteq \mathbb{N}$, we restrict attention to those probability measures in $\Delta(X)$ that map every Borel event to a rational number with the denominator belonging to N. Consider the subset of the rational numbers,

$$\mathbb{Q}_N := \{m/n : m = 0, ..., n \ ; \ n \in N\}.$$

Definition 5.1. For some $N \subseteq \mathbb{N}$, we define the set of ***N-rational probability measures*** by

$$\Delta^N(X) := \left\{ \mu \in \Delta(X) \ \middle| \ \mu(B) \in \mathbb{Q}_N, \forall B \in \mathcal{B} \right\}.$$

Obviously, the space of $\mathbb{N}$-rational probability measures coincides with the space of rational probability measures, i.e.,

$$\Delta^{\mathbb{N}}(X) = \Delta^{\mathbb{Q}}(X).$$

Proposition 5.1. *If N is finite, $\Delta^N(X)$ is a closed subset of $\Delta(X)$.*

Proof. It suffices to show that an arbitrary convergent sequence $\{\mu_k\}$ of elements of $\Delta^N(X)$ has its limit in $\Delta^N(X)$, i.e., if $\mu_k \overset{w^*}{\to} \mu$, then $\mu \in \Delta^N(X)$.

For an arbitrary finite $N \subseteq \mathbb{N}$, let

$$N^* := \max_{n \in N} n.$$

Consider the metric $d : X \times X \to \mathbb{R}$ which is assumed to be compatible with the topology on X. For every $x \in X$, denote the radius $\delta > 0$ open neighborhood of x by

$$B(x, \delta) := \{x' \in X : d(x, x') < \delta\},$$

and the corresponding closed subset by

$$\overline{B(x, \delta)} := \{x' \in X : d(x, x') \leq \delta\}.$$

For an arbitrary $x \in X$, suppose that there is some $\delta > 0$ such that there are at most finitely many $k > 0$ with $\mu_k\big(B(x, \delta)\big) > 0$, implying that

$$\liminf_{k>0} \mu_k\big(B(x, \delta)\big) = 0. \tag{5.1}$$

Since $\mu_k \overset{w^*}{\to} \mu$, it follows from Theorem 15.3 in Aliprantis and Border (1994) that $\mu\big(B(x, \delta)\big) \leq \liminf_{k>0} \mu_k\big(B(x, \delta)\big)$, which together with Eq. (5.1) yields

$$\mu\big(B(x, \delta)\big) = 0,$$

thus implying $x \notin \Gamma(\mu)$.

Now, suppose instead that for every $\delta > 0$ there are infinitely many $k > 0$ with $\mu_k\big(B(x,\delta)\big) > 0$, and therefore with $\mu_k\big(\overline{B(x,\delta)}\big) > 0$. Since $\mu_k \in \Delta^N(X)$, it follows that $\mu_k\big(\overline{B(x,\delta)}\big) > 0$ if and only if $\mu_k\big(\overline{B(x,\delta)}\big) \geq 1/N^*$, implying that

$$\limsup_{k>0} \mu_k\big(\overline{B(x,\delta)}\big) \geq 1/N^*. \tag{5.2}$$

Once again, since $\mu_k \xrightarrow{w^*} \mu$, it follows from Aliprantis and Border (1994, Thm. 15.3) that $\mu\big(\overline{B(x,\delta)}\big) \geq \limsup_{k>0} \mu_k\big(\overline{B(x,\delta)}\big)$, which together with Eq. (5.2) yields

$$\mu\big(\overline{B(x,\delta)}\big) \geq 1/N^*. \tag{5.3}$$

Now, consider a sequence of positive reals $\{\delta_n\}$ with $\delta_n \downarrow 0$, which induces a sequence of Borel events $\{\overline{B(x,\delta_n)}\}$ such that

$$\limsup_{n>0} \overline{B(x,\delta_n)} = \{x\}. \tag{5.4}$$

Recall from Billingsley (1995, Thm. 4.1) that

$$\mu\big(\limsup_{n>0} \overline{B(x,\delta_n)}\big) \geq \limsup_{n>0} \mu\big(\overline{B(x,\delta_n)}\big),$$

which, together with (5.3) and (5.4), yields $\mu(\{x\}) \geq 1/N^*$. Hence, $x \in \Gamma(\mu)$ if and only if $\mu(\{x\}) \geq 1/N^*$, implying that $\Gamma(\mu)$ is finite.

Now, let $x \in \Gamma(\mu)$. It follows from applying Aliprantis and Border (1994, Thm. 15.3) twice (once for (5.5) and once for (5.6)), that for every $\delta > 0$,

$$\begin{aligned}
\mu\big(\overline{B(x,\delta)}\big) &\geq \limsup_{k>0} \mu_k\big(\overline{B(x,\delta)}\big) && (5.5)\\
&\geq \limsup_{k>0} \mu_k\big(B(x,\delta)\big) \\
&\geq \liminf_{k>0} \mu_k\big(B(x,\delta)\big) \\
&\geq \mu\big(B(x,\delta)\big). && (5.6)
\end{aligned}$$

Since $\Gamma(\mu)$ is finite, consider some $\rho > 0$ such that

$$\rho < \min_{x' \in \Gamma(\mu) \setminus \{x\}} d(x,x').$$

Then, obviously $\mu\big(\overline{B(x,\rho)}\big) = \mu\big(B(x,\rho)\big) = \mu(\{x\})$. Hence, it follows from (5.5–5.6) that

$$\mu(\{x\}) = \lim_{k>0} \mu_k\big(B(x,\rho)\big).$$

Finally, since the sequence $\{\mu_k\big(B(x,\rho)\big)\}_{k>0}$ takes values in the finite set $\mathbb{Q}_N$, its limit will also belong to $\mathbb{Q}_N$, which completes the proof. $\square$

Corollary 5.1. *Let N be finite.*

(a) $\Delta^N(X)$ *is separable and metrizable if and only if* X *is separable and metrizable.*

(b) $\Delta^N(X)$ *is Polish if and only if* X *is Polish.*

Proof. It follows directly from Proposition 5.1 and Aliprantis and Border (1994, Thm. 15.12 & Thm. 15.15). □

Remark 5.1. The previous result does not necessarily hold when N is infinite, e.g., let $N = \mathbb{N}$ and $X = \{x_1, x_2\}$, and consider a sequence of measures $\{\mu_k\}_{k>0}$ such that $\mu_k(\{x_1\}) = \alpha_k$, where α_k is the k-digit decimal approximation of $\sqrt{2}/2$. It is straightforward verifying that $\mu_k \in \Delta^{\mathbb{Q}}(X)$ for all $k > 0$. Then, let $\mu \in \Delta(X)$ satisfy $\mu(\{x_1\}) = \sqrt{2}/2$, and observe that $\mu_k \xrightarrow{w^*} \mu$, implying that $\Delta^{\mathbb{Q}}(X)$ is not closed in $\Delta(X)$. ◁

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