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On the time-optimal 2D-trajectories in non-uniform mediums

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On the Time-Optimal 2D-Trajectories in Non-Uniform Mediums

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Abstract. This paper addresses two-dimensional trajectory optimization problems, where the mover's speed monotonically decreases/increases in one of the space's coordinates. For instance, it is well-known that the absolute value of the helicopter speed decreases in altitude because an air pressure drops. Then, it is not trivial to find a time-minimal helicopter trajectory from one given point to another.
We address such problems in different settings for the medium (atmosphere) and in the presence of obstacles. First, we consider the basic problem without any obstacles, where the mover's speed decreases linearly in altitude. We show that the problem is reducible to the L'Hopital's problem, one of the well-studied problems in geometrical optics. In this case, the time-optimal trajectory is a circular segment, and therefore can be expressed in a closed analytic form.
Next, we address the problem with linear speed decrease in presence of rectilinear obstacles. We show that this problem can be solved in polynomial time. Finally, we consider the case without obstacles, where the medium is non-uniform and the mover's velocity is a piece-wise linear monotonically decreasing function. This is a widely accepted model in our motivating application of helicopter flights. In this case, we reduce the problem to the solution of a system of polynomial equations of fixed degree. Here, if the number of breakpoints of the velocity function is a constant, the algebraic elimination theory allows us to solve the problem in constant time with any given level of precision.

1 Introduction

The classical trajectory optimization problems in computational geometry usually assume a constant speed of the moving objects. Moreover, many fundamental geometric problems assume that a moving object, a *mover*, is a single point in two- or three-dimensional space. Under these two assumptions, length-optimal and time-optimal trajectories are the same. Therefore, dealing with a time-optimization problem, we can apply, for instance, the shortest-paths algorithms; see, e.g., [1, 2]. The situation is totally different when we assume that the speed of the mover depends on the position in the space. In this case, length-optimal and time-optimal trajectories can deviate from each other significantly. One example for such a problem is finding shortest paths in weighted regions [12, 13], in which the plane is divided into a finite number of regions in each of which the speed is constant. In this paper we investigate such a shortest path problem where the speed changes continuously in one of the space's coordinates.

Given a mover in three-dimensional Euclidean space, assume that the absolute value of the mover's speed decreases or increases in one of the space coordinates. This is the real-life setting in any helicopter flight. The greater the altitude of the helicopter, the less the atmospheric pressure and, consequently, the less the air density. In turn, the reduction in air density will reduce the power available, and then, the maximum speed of the helicopter decreases with altitude. Though the (concave) function of the helicopter's maximum velocity in altitude is quite complex and hardly admits a closed analytical form, it is widely accepted to approximate it by linear or piece-wise linear functions with just few breakpoints, see, e.g. [3]. Therefore, we can formulate the *basic helicopter problem* as follows. Given is a source point A and a destination point B in three-dimensional Euclidean space, where, traditionally, the vertical y-axis represents points altitudes while the x- and z-axes represent the surface coordinates. No obstacles (also no ground level) are present. The

absolute value of the mover's speed decreases linearly in altitude, i.e., $v(y) = max\{v_0 - qy, 0\}$, where $v_0 > 0$ is some speed intercept (for instance, the helicopter's maximum velocity at the ground level) and $q \geq 0$ is the velocity degradation rate. One has to find a time-optimal trajectory to fly the mover (*helicopter*) from A to B. Notice that this three-dimensional problem can be easily reduced to the two-dimensional problem. This is because the time-optimal trajectory clearly belongs to the plane orthogonal to the surface and containing points A and B. The *general helicopter problem* is the natural generalization of the basic helicopter problem, where the velocity is represented by the piece-wise linear function with finite number of breakpoints.

The *helicopter problem with obstacles* is formulated as follows: given points A and B in three-dimensional Euclidean space, along with a set of polyhedral obstacles, find a time-optimal trajectory to fly the helicopter from A to B without hitting the interior of any of the obstacles. It is noticeable that the helicopter problem with obstacles is a generalization of the classical three-dimensional Euclidean shortest-path problem: if all the velocity degradation rates are equal to zero, the problems are equivalent. It is well known that the two-dimensional Euclidean shortest-path problem is polynomially solvable [18], the three-dimensional problem is NP-hard [7], but admits polynomial time approximation schemes [1]. Notice that one can derive a polynomial-time approximation scheme to the three-dimensional general helicopter problem with obstacles by discretizing/scaling the space and constructing a weighted complete graph, where the edge weight is the time to travel between two vertices using the straight line trajectory. The presence of obstacles can be taken into account setting the edge weight to positive infinity if the straight line between two vertices hits interior of an obstacle. Now we can search for the shortest path in the obtained graph.

In the time-optimization setting, to tackle the general problem with obstacles, one might need a complete characterization of the set of optimal solutions to the basic problem without obstacles, the set of, so-called, *motion primitives*. A typical illustration of lifting from motion primitives to solutions to the general problem can be found, for example, in robotics, see [8–10]. In this paper we derive motion primitives for the basic helicopter problem. We show that the basic helicopter problem is reducible to the L'Hopital's problem [4], one of the most studied problems in geometrical optics and mechanics. As a consequence we have that the trajectory following a certain circle segment with endpoints A and B is the time-optimal trajectory in the class of continuously differentiable functions. We use Euler-Lagrange equations from the calculus of variations. One can see this also as a variant of the Pontryagin Maximum (Minimum) Principle [20]. This type of techniques is quite common in optimal control theory in general, and in robotics in particular; see, e.g., [5, 6].

Then, we generalize results for the basic helicopter problem in two ways. First, we show that the basic helicopter problem with rectilinear obstacles is solvable in polynomial time by a dynamic programming algorithm. Second, we reduce the general helicopter problem without obstacles to the system of fixed degree multivariable polynomial equations. If the number of breakpoints in the piece-wise linear velocity function is a constant, the resulting system has a fixed number of variables and it can be solved in polynomial time by the algebraic elimination theory; see, e.g., [11].

2 Solution to the basic helicopter problem

First, we notice that if the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ lay on the same vertical line (i.e., they share the same $x-$coordinate: $x_1 = x_2$), then the time-optimal trajectory L_{AB} is just a piece of the straight line $x = x_1$ between A and B. We omit the proof as it is straightforward. We also assume that $q > 0$, for otherwise we are in the well-studied setting of the classical Euclidean shortest-path problem. Thus, from now on we consider only the case where $x_1 \neq x_2$ and $q > 0$. We show that in this case the circle segment is the unique time-optimal trajectory. The first observation is about the convexity of any time-optimal trajectory.

Lemma 1. *For any two points A and B in $\mathbb{R}^2 \cap \{(x, y) : v_o - qy > 0\}$, a time-optimal trajectory between A and B is a convex function of x.*

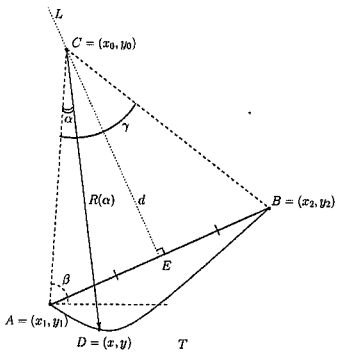


Fig. 1. Convex trajectory T in the polar system of coordinates

By Lemma 1, we can restrict our search for time-optimal trajectories to the class of convex functions in x. Let T be an arbitrary continuously differentiable convex trajectory between A and B. Let L be the line equidistant from A and B. Since T is convex in x, it is possible to observe every point of T from any point C on L that lies above the line segment AB. We consider the trajectory T in the polar system of coordinates with observation point $C = (x_0, y_0)$, where $C \in L$ is chosen such that the velocity in C is 0, i.e., $y_0 = \frac{v_0}{q}$. For illustration see Figure 1. We refer to the angle between CA and the x-axis as β and to the angle between CA and CB as γ. Consider an arbitrary point $D = (x, y)$ of the trajectory T. In the chosen polar system of coordinates, the point D is completely determined by α and $R(\alpha)$, where α is the angle between CA and CD, and $R(\alpha)$ is the length of the interval CD:

$$x = x_0 - R(\alpha)\cos(\alpha + \beta), \tag{1}$$

$$y = y_0 - R(\alpha)\sin(\alpha + \beta). \tag{2}$$

First, we write the integral which represents the time needed to travel along T. For a sufficiently small piece of the trajectory, we may assume that the velocity v remains constant within the piece. Let the length of the piece be denoted by Δl, then the time to travel along the piece is $\Delta t \approx \frac{\Delta l}{v}$. The velocity v is completely determined by the altitude of $D = (x, y)$. Therefore, by Equation (2), we have $v = v_0 - qy = qR(\alpha)\sin(\alpha+\beta)$, as $v_0 - qy_0 = 0$ by the choice of C. We know that for continuously differentiable trajectories the length of the piece Δl is determined by $\Delta l = \sqrt{R'^2(\alpha) + R^2(\alpha)}\Delta\alpha$, see, e.g. [16]. Therefore, the time needed to travel along T is determined by the following integral:

$$t_T = \int_0^\gamma \frac{\sqrt{R'^2(\alpha) + R^2(\alpha)}\, d\alpha}{qR(\alpha)\sin(\alpha+\beta)} = \frac{1}{q}\int_0^\gamma \sqrt{\left(\frac{R'(\alpha)}{R(\alpha)}\right)^2 + 1}\frac{d\alpha}{\sin(\alpha+\beta)}. \tag{3}$$

Equation (3) allows us to proof the following theorem about the unique time-optimal trajectory between A and B. We defer the proof to the journal version of the paper, see also Appendix A.

Theorem 1. *Let C be the intersection point of the line $y = \frac{v_0}{q}$ and the line equidistant from A and B. The segment T_{AB} of the circle with center C and radius $R = |CA| = |CB|$ is a time-optimal trajectory between A and B. The time needed to travel along T_{AB} is*

$$t_{opt} = t_{T_{AB}} = \frac{1}{q}\int_0^\gamma \frac{d\alpha}{\sin(\alpha+\beta)} = \frac{1}{q}\ln\left|\frac{\tan\frac{\beta}{2} + \tan\frac{\gamma}{2}}{\tan\frac{\beta}{2} - \tan^2\frac{\beta}{2}\tan\frac{\gamma}{2}}\right|. \tag{4}$$

Here, $\tan\frac{\gamma}{2}$ and $\tan\frac{\beta}{2}$ can be determined in terms of coordinates $A=(x_1,y_1)$ and $B=(x_2,y_2)$. Point $C=(x_0,y_0)$ has an altitude $y_0=\frac{v_0}{q}$. Since C belongs to the line L equidistant from A and B, we have following set of values:

$$x_0=-\frac{y_2-y_1}{x_2-x_1}y_0-\frac{x_1^2+y_1^2-x_2^2-y_2^2}{2(x_2-x_1)}, \tag{5}$$

$$R=|CA|=\sqrt{(x_0-x_1)^2+(y_0-y_1)^2}=\sqrt{(x_0-x_2)^2+(y_0-y_2)^2}, \tag{6}$$

Now we determine the tangents of β and γ in the following way: $\tan\frac{\gamma}{2}=\frac{\sqrt{R^2-d^2}}{d}$. Since $\sin\beta=\frac{y_0-y_1}{R}$ and $\cos\beta=\frac{x_0-x_1}{R}$, then $\tan\frac{\beta}{2}=\frac{\sin\beta}{1+\cos\beta}=\frac{y_0-y_1}{R+x_0-x_1}$, where $d=\sqrt{(x_0-\frac{x_1+x_2}{2})^2+(y_0-\frac{y_1+y_2}{2})^2}$.

To demonstrate the reducibility of the problem to the L'Hopital's problem, we present the time needed to travel along the trajectory T in the following way:

$$t_T=\int_{y_1}^{y_2}\frac{\sqrt{1+\dot{x}^2(y)}}{v_0-qy}dy,$$

as we have to go from the altitude y_1 to the altitude y_2 along the trajectory $x(y)$, when speed depends linearly at altitude, so $v(y)=v_0-qy$, and the length of the trajectory piece is $\Delta l=\sqrt{1+\dot{x}^2(y)}$, see [4]. This integral can be seen as a solution to the L'Hopital's problem, see [4], Chapter 3.

3 The basic helicopter problem with rectilinear obstacles

In this section we consider the case with rectilinear obstacles in 2D. The obstacles are given as the sequence of points $S=\{(x_1,y_1),\ldots,(x_n,y_n)\}$, so that every pair of consecutive points have exactly one same coordinate (either x_i or y_i) and these same coordinates alternate, see Figure 2. We assume that the helicopter goes from the left to the right without any loops, furthermore, the start point $A=(x_1,y_1)$ and the end point $B=(x_2,y_2)$ belong to the set S, otherwise we "draw" an additional step on the left of the terrain with the corner A, for example, and add A to S. Here and below we refer to the straight line segment between any two points K and M as L_{KM} and to the time-optimal circle segment between K and M from Theorem 1 as T_{KM}, the center of this circle is $C_{KM}=(x_0,y_0)$, where $y_0=\frac{V_0}{q}$ and x_0 and radius R_{KM} are determined by Equations (5)–(6) respectively.

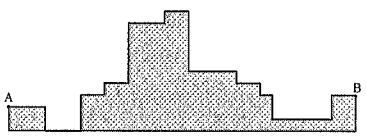


Fig. 2. The helicopter problem with rectilinear obstacles

3.1 Dynamic programming algorithm

The idea for the algorithm computing a time-optimal two-dimensional helicopter trajectory in presence of rectilinear obstacles is to find all potential breakpoints such that the trajectory changes the motion primitives in these points. If the amount of these points is polynomial then a simple

dynamic program can solve the problem in polynomial time. Obviously, the points of the set S are already potential breakpoints. Are there any other points? The answer is "yes" and it is given in the following lemma. It considers the time-optimal trajectory with respect to the one step obstacle, see Figure 3. The proof is also deferred to the journal version of the paper, see also Appendix B.

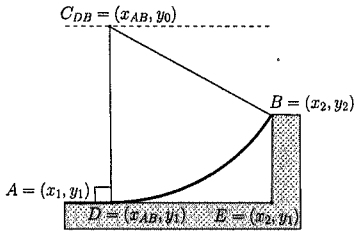


Fig. 3. The optimal trajectory for the one step case, T_{AB} hits L_{AE}

Lemma 2. *For three consecutive points $A = (x_1, y_1)$, $E = (x_2, y_1)$ and $B = (x_2, y_2)$ from S, such that $y_1 < y_2$, the time-optimal trajectory from A to B with obstacles is defined in the following way:*

a) if C_{AB} is to the left of A then T_{AB} is a time-optimal trajectory,
b) if C_{AB} is to the right of A then time-optimal trajectory is the combination of the straight horizontal line segment L_{AD} and the circle segment T_{DB}, where D and C_{DB} determine a vertical line. In particular,

$$D = (x_2 - \sqrt{(y_2 - y_1)(2y_0 - y_1 - y_2)}, y_1). \tag{7}$$

We observe that by Lemma 2 for any pair of points $A = (x_1, y_1), B = (x_2, y_2)$ from S, it is sufficient to introduce at most one potential breakpoint $D = (x_{AB}, y_1)$ on the horizontal line $y = y_1$, if it is impossible to go along T_{AB} and it is possible to go along the straight line L_{AD} and circle segment T_{DB}. If there are other obstacles between A and B and the new trajectory hits some obstacle, then the time-optimal trajectory with rectilinear obstacles from A to B passes some other potential breakpoint E between them. This fact is stated in Lemma 3. Hence, all necessary potential breakpoints are determined by Lemma 2. The main ingredients to prove Lemma 3 are the following observations, the proofs are sketched in Appendix C:

Observation 1 *For any three points $A = (x_1, y_1), B = (x_2, y_2)$ and $M = (x_3, y_3)$ such that $x_1 < x_3 < x_2$ and $y_3 < y_1$, $y_3 < y_2$, T_{AB} lies above T_{AM} and T_{MB} on the interval $[x_1, x_2]$.*

Observation 2 *For any three consecutive points $A = (x_1, y_1), C = (x_2, y_1)$ and $B = (x_2, y_2)$ such that $x_1 < x_2$ and T_{AB} hits L_{AC}, the time-optimal trajectory lies above T_{AB} on the interval $[x_1, x_2]$.*

We conclude the set of observations with the so-called "ground level case", when the unique obstacle between A and B is a horizontal line.

Observation 3 *Given four consecutive points $A = (x_1, y_1), C = (x_1, y_2), D = (x_2, y_2)$ and $B = (x_2, y_3)$ from S such that $x_1 < x_2$, $y_2 < y_1$, $y_2 < y_3$ and T_{AB} hits L_{CD}, the time-optimal trajectory with obstacles is the combination of the circle segment T_{AK}, straight line L_{KL} and circle segment T_{LB}, where K and L are the points of intersection of T_{AD} and T_{CB} with L_{CD} respectively.*

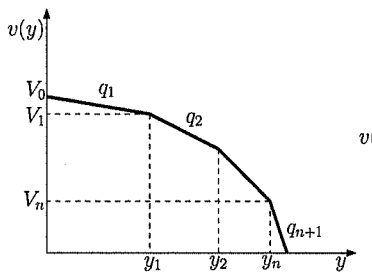
Finally, we present Lemma 3. The proof is deferred to the journal version of the paper, see also Appendix C. It is the extensive case study for the obstacle patterns between two points A and B.

Lemma 3. *If the time-optimal trajectory T_{AB} between any two potential breakpoints $A = (x_1, y_1), B = (x_2, y_2)$, $x_1 < x_2$, hits the obstacles, then there exists a potential breakpoint C between A and B such that the time-optimal trajectory with obstacles from A to B passes C.*

Now the problem is reduced to the problem of finding a shortest path in a weighted graph. All the potential breakpoints are the vertices of the graph, the number of vertices is not more than $n + n^2 = O(n^2)$. There is an edge between two vertices if the time-optimal trajectory between corresponding potential breakpoints, given by Lemma 2, does not hit any obstacle. The time needed to go along this trajectory denotes the weight of the edge. Lemma 3 guarantees that the graph is connected and the time-optimal trajectory corresponds to the shortest path in the weighted graph. This problem is solvable in time $O(|V|^2)$, see, e.g. [19]. Hence, we have $O(n^4)$-time algorithm for 2D basic helicopter problem with rectilinear obstacles.

4 General helicopter problem

In this section we consider the general helicopter problem where the function of velocity is a piece-wise linear concave function in altitude with finite number of breakpoints $y_1, \ldots, y_n$, see Figure 4. Let $V_i, i \in I$ denote the velocity at the altitude y_i. By concavity of the function we have that $q_1 \leq q_2 \leq \ldots \leq q_{n+1}$. This is a reasonable assumption as the helicopter looses the speed quicker on high altitudes. Let us define the altitude layers L_i, $i \in I = \{0, \ldots n\}$, in the following natural way: $L_i = \{y \mid y_i \leq y \leq y_{i+1}\}, i \in I \setminus \{0, n\}$, $L_0 = \{y \mid y \leq y_1\}$ and $L_n = \{y \mid y \geq y_n\}$. Therefore, the velocity function is given as follows:



$$v(y) = \begin{cases} V_0 - q_1 y & : y \in L_0, \\ V_1 - q_2 y & : y \in L_1, \\ \ldots & \ldots \\ \max\{V_n - q_{n+1} y, 0\} & : y \in L_n. \end{cases}$$

Fig. 4. Piece-wise linear velocity function

Consider the general helicopter problem where the helicopter must fly from $A = (x_A, y_0)$ to $B = (x_B, y_{n+1})$. Without loss of generality assume, that $x_A < x_B$ and $y_0 \in L_0$, $y_{n+1} \in L_n$. This setting yields several natural restrictions. First, we observe that the time-optimal trajectory from A to B is the set of circle and linear segments, as within one layer L_i, $i \in \{0, \ldots, n\}$, it is either the circle segment or the linear segment, see Section 2. Second, for any $y \geq y_1$ the time-optimal trajectory is the monotonically increasing function. The reason is as follows. Assume that a non-increasing trajectory T_{KB} is determined as the time-optimal trajectory within the layer L_1, see Figure 5. Then the piece T_{KL} of this trajectory, which belongs to the layer L_0, is not optimal anymore. It is better to follow T_{AL} within the layer L_0 and then continue with T_{LB}, a contradiction.

As the time-optimal trajectory is the set of circle and straight line segments, we restrict ourselves first to the case when only circle segments are allowed. The reason is to make the proofs less complicated. Later on we show how to generalize the result to the case of linear and circle segments.

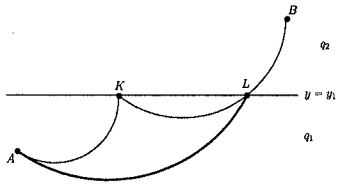


Fig. 5. Monotonically increasing circle segments when $y \geq y_1$. The trajectory T_{AL} plus T_{LB} is better than T_{AK} plus T_{KB}

4.1 Only circle segments

In this subsection we consider the setting when all the pieces of the time-optimal trajectory from A to B are circle segments, hence the time needed to go along this trajectory can be calculated as the sum of times to go along the circle segments in layers L_i's, $i \in \{0, \ldots, n\}$.

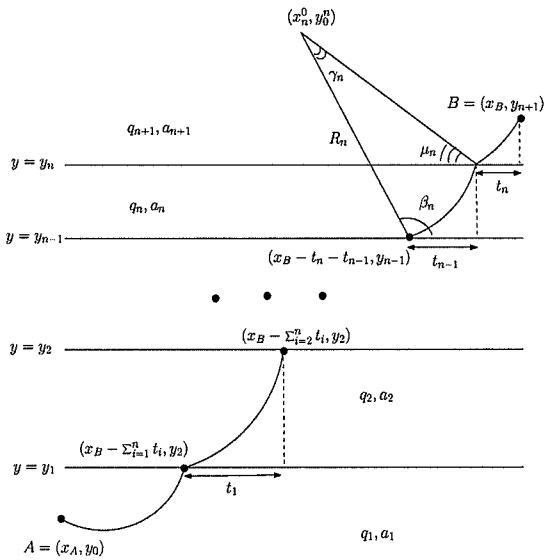


Fig. 6. Time-optimal trajectory is the set of circle segments, $x_0^i, \beta_i, \mu_i, \gamma_i$ and R_i are functions at $t_1, \ldots, t_n$

For every layer L_i, $i \in \{0, \ldots, n\}$ we introduce the set of parameters, see Figure 6:

- $y_0^{i+1} = \frac{V_i}{q_{i+1}}$ is the altitude of the time-optimal circle segment center on this layer,
- $a_{i+1} = \sqrt{(2y_0^{i+1} - y_{i+1} - y_i)(y_{i+1} - y_i)}$, as determined in Lemma 2. It is the maximal horizontal distance between two points on lines $y = y_i$ and $y = y_{i+1}$ such that the time-optimal circle segment is a monotonically increasing function,

and a set of variables and functions:

- t_i, such that for any i greater than zero t_i is the horizontal distance between the points of intersection the lines $y = y_i$ and $y = y_{i+1}$ by the time-optimal trajectory from A to B, so that the coordinates of intersection are $(x_B - \sum_{i=k}^n t_i, y_k), k \in \{1, \ldots, n\}$. The values of the t_i's

are strictly positive, as otherwise we have the vertical line segment, and $t_i \leq a_{i+1}, i > 0$, as the circle segment should be a monotonically increasing function for $i > 0$. We introduce the redundant variable t_0: $x_B - \sum_{i=0}^{n} t_i = x_A$, where t_0 is the horizontal distance between point A and the point of intersection of the time-optimal trajectory from A to B with the line $y = y_1$. It is strictly positive as otherwise we go further to the left than point A, which is not optimal. It can be bigger than a_1 as within the layer L_0 the time-optimal trajectory is not necessary the monotonically increasing function,

- $x_0^{i+1}(t_1, \ldots, t_n)$ is the x-coordinate of the time-optimal circle segment center on this layer,
- $R_{i+1}(t_1, \ldots, t_n)$ is the radius of the time-optimal circle segment on this layer,
- $\beta_{i+1}(t_1, \ldots, t_n)$ and $\gamma_{i+1}(t_1, \ldots, t_n)$ are the angles defined in Theorem 1, and $\mu_{i+1}(t_1, \ldots, t_n)$ is the angle between CB and x-axis in terms of Figure 1, in particular, $\mu_{i+1} = \pi - \gamma_{i+1} - \beta_{i+1}$.

Now, we can write down the time needed to go alone this trajectory as the sum of the times to go alone the time-optimal trajectories within one layer L_i, see Section 2:

$$t(t_1, \ldots, t_n) = \sum_{k=0}^{n} \frac{1}{q_{k+1}} \int_0^{\gamma_{k+1}(t_1, \ldots, t_n)} \frac{d\alpha}{sin(\alpha + \beta_{k+1}(t_1, \ldots, t_n))}.$$

To minimize $t(t_1, \ldots, t_n)$, let us first find the stationary points of the function, so we have to find the partial derivatives of $t(t_1, \ldots, t_n)$ with respect to $t_i, i \in \{1, \ldots, n\}$ and equate them with zero. Direct differentiation leads to the following results, see Appendix B for the details:

$$\frac{\partial t(t_1, \ldots, t_n)}{\partial t_i} = -\sum_{k=1}^{n+1} \left[\frac{\partial \beta_k(t_1, \ldots, t_n)}{\partial t_i} \frac{1}{sin\ \beta_k(t_1, \ldots, t_n)} + \frac{\partial \mu_k(t_1, \ldots, t_n)}{\partial t_i} \frac{1}{sin\ \mu_k(t_1, \ldots, t_n)} \right] = 0.$$

We find $\beta_k(t_1, \ldots, t_n)$ and $\mu_k(t_1, \ldots, t_n)$, for $k \geq 2$, using the formulas from Section 2, as all the derivatives in this case can be obtained in the similar way. The case $k = 1$ differs in the number of variables and it is considered later. Within the layer $L_k, k \geq 2$, the time-optimal trajectory goes from the point $(x_B - \sum_{i=k-1}^{n} t_i, y_{k-1})$ to the point $(x_B - \sum_{i=k}^{n} t_i, y_k)$, so from the Equations (5)–(6) and the following formulae for the trigonometrical functions:

$$x_0^k(t_1, \ldots, t_n) = -\frac{a_k^2}{2t_{k-1}} + x_B - \sum_{i=k}^{n} t_i - \frac{t_{k-1}}{2}, \text{ then}$$

$$R_k(t_1, \ldots, t_n) = \sqrt{\left(\frac{a_k^2}{2t_{k-1}} + \frac{t_{k-1}}{2} \right)^2 + (y_0^k - y_k)^2} = R_k(t_{k-1}), \text{ and}$$

$$\sin\ \beta_k(t_1, \ldots, t_n) = \frac{y_0^k - y_{k-1}}{R_k(t_{k-1})} = \sin\ \beta_k(t_{k-1}),$$

$$\cos\ \beta_k(t_1, \ldots, t_n) = \frac{x_0^k(t_1, \ldots, t_n) - (x_B - \sum_{i=k-1}^{n} t_i)}{R_k(t_{k-1})} = -\frac{a_k^2 - t_{k-1}^2}{2t_{k-1}R_k(t_{k-1})} = \cos\ \beta_k(t_{k-1}),$$

$$\sin\ \mu_k(t_1, \ldots, t_n) = \frac{y_0^k - y_k}{R_k(t_{k-1})} = \sin\ \mu_k(t_{k-1}),$$

$$\cos\ \mu_k(t_1, \ldots, t_n) = \frac{x_B - \sum_{i=k}^{n} t_i - x_0^k(t_1, \ldots, t_n)}{R_k(t_{k-1})} = \frac{a_k^2 + t_{k-1}^2}{2t_{k-1}R_k(t_{k-1})} = \cos\ \mu_k(t_{k-1}).$$

Thus, we see that R_k, β_k and μ_k are functions of t_{k-1} only, so for any other $i, i \neq (k-1)$, the partial derivatives of β_k and μ_k with respect to t_i are equal to zero. After differentiation and some

simplifications we conclude that

$$\frac{\beta_k'(t_{k-1})}{sin\ \beta_k(t_{k-1})} + \frac{\mu_k'(t_{k-1})}{sin\ \mu_k(t_{k-1})} = -\frac{1}{R_k(t_{k-1})},\ k = 2, \ldots, n+1.$$

We find $\beta_1(t_1, \ldots, t_n)$ and $\mu_1(t_1, \ldots, t_n)$ in the similar way. The start point of the time-optimal trajectory is $A = (x_A, y_0)$ and the end point is $(x_B - \sum_{k=1}^{n} t_i, y_1)$. The only difference is that these functions depend on all n variables $t_1, \ldots, t_n$. We find by differentiation that:

$$\frac{\partial \mu_1}{\partial t_i}\frac{1}{sin\ \mu_1(t_1, \ldots, t_n)} + \frac{\partial \beta_1}{\partial t_i}\frac{1}{sin\ \beta_1(t_1, \ldots, t_n)} = \frac{1}{R_1(t_1, \ldots, t_n)},\ i = 1, \ldots, n.$$

Hence, the stationary point of the function $t(t_1, \ldots, t_n)$ satisfies the following set of conditions:

$$\frac{\partial t(t_1, \ldots, t_n)}{\partial t_k} = -\left(\frac{1}{q_1 R_1(t_1, \ldots, t_n)}\right) - \left(-\frac{1}{q_{k+1} R_{k+1}(t_k)}\right) = 0,\ k = 1, \ldots, n, \text{or equivalently}$$

$$q_1 R_1(t_1, \ldots, t_n) = q_2 R_2(t_1) = \ldots = q_{n+1} R_{n+1}(t_n), \tag{8}$$

where $t_i \in (0, a_{i+1}]$ and $x_B - x_A - \sum_{i=1}^{n} t_i = t_0 > 0$.

It is possible to simplify the set of Equations (8) further: consider every consecutive pair of expressions k and $k+1$, $k = 1, \ldots, n$, and use the fact that the point $(x_B - \sum_{i=k}^{n} t_i, y_k)$ belongs at the same time to the circles with radii $R_k(t_{k-1})$ and $R_{k+1}(t_k)$ as the time-optimal trajectory has to be continuous. We use the redundant variable t_0 to write down the system in the uniform way:

$$q_{k+1}(a_{k+1}^2 - t_k^2)t_{k-1} = q_k(a_k^2 + t_{k-1}^2)t_k,\ k = 1, \ldots, n, \tag{9}$$

$$\sum_{i=0}^{n} t_i = x_B - x_A, \tag{10}$$

$$t_i \in (0, a_{i+1}],\ i > 0, \quad t_0 > 0. \tag{11}$$

Now, let us show that not more than one solution of the system of Equations (8) on the domain $\Pi_{i=1}^{n}(0, a_{i+1}]$ exists, hence system (9)—(11) also has not more than one solution. Suppose there exists a solution $(t_1^*, \ldots, t_n^*)$ of the system (8) with the corresponding trajectory T, see Figure 7. Assume that there is another solution $(t_1', \ldots, t_n')$ with the corresponding trajectory T'. The circle segments of both trajectories satisfy the system of Equations (8), so $q_1 R_1(t_1^*, \ldots, t_n^*) = q_2 R_2(t_1^*) = \ldots = q_{n+1} R_{n+1}(t_n^*)$ and $q_1 R_1(t_1', \ldots, t_n') = q_2 R_2(t_1') = \ldots = q_{n+1} R_{n+1}(t_n')$. We denote $R_i(t_1^*, \ldots, t_n^*)$ as R_i^* and $R(t_1', \ldots, t_n')$ as R_i'.

Assume that T' is completely above T and they intersect only in the end points A and B. K^* and L^* are the points where T intersects $y = y_1$ and $y = y_n$ respectively, K' and L' are the points where T' intersects $y = y_1$ and $y = y_n$ respectively. K' is to the left of K^* and the left end A of the trajectories T and T' is fixed, so $R_1^* < R_1'$. L' is to the left of L^* and the right end B of T and T' is fixed, so $R_{n+1}^* > R_{n+1}'$. Both T and T' represent the solution of the system (8) so

$$q_{n+1}R_{n+1}' \ < \ q_{n+1}R_{n+1}^* = \ldots = q_1 R_1^* \ < \ q_1 R_1' = \ldots = q_{n+1}R_{n+1}',$$

a contradiction. All other cases, like T' is completely below T or they intersect in some intermediate point $M \neq B$, are proven in the same way. Therefore, not more than one solution of the system (8) exists.

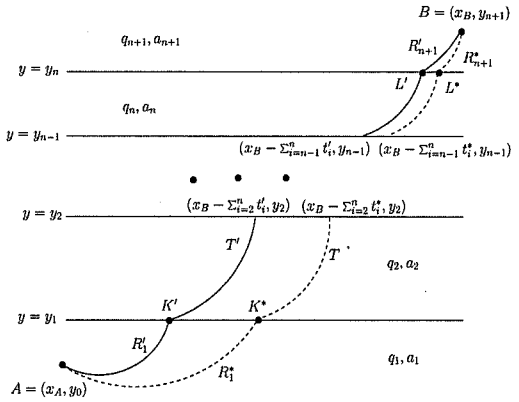


Fig. 7. Uniqueness of the solution

Furthermore, it is easy to check that if $t_k \in (0, a_{k+1}]$ is known then there exists not more than one value of $t_{k-1} \in (0, a_k]$, because two possible values of t_{k-1} from Equation (9) are:

$$t_{k-1}^{1,2} = \frac{q_{k+1}(a_{k+1}^2 - t_k^2) \pm \sqrt{q_{k+1}^2(a_{k+1}^2 - t_k^2)^2 - 4q_k^2 a_k^2 t_k^2}}{2q_k t_k}, \tag{12}$$

if the discriminant $q_{k+1}^2(a_{k+1}^2 - t_k^2)^2 - 4q_k^2 a_k^2 t_k^2$ is nonnegative. $t_{k-1} \leq a_k$, so only the solution with *minus* reminds, as otherwise the condition is broken. Hence, if the discriminant is nonnegative, there exists unique solution $t_{k-1} = t_{k-1}(t_k)$. So, given t_n we can find $t_k, k = n-1, \ldots, 1$, using the next equation from the set, and either there exists the unique solution of the system or there is no solution, so the time-optimal trajectory which crosses $(x_B - t_n, y_n)$ and which is the set of circle segments does not exist.

Now, we have to solve the system of polynomial equations (9)—(11). We use the tools of computational algebraic geometry and commutative algebra to solve the system, see, e.g., [11]. The brief introduction into the theory is given in Appendix C. In particular, the Elimination Theorem guaranties that we can find a polynomial consequence of the system (9)—(11) in finitely many steps, such that it contains only t_n as a variable. There are numerical methods to find the roots of such an equation with high precision (for example, the Jenkins-Traub algorithm, see [21]). If there exists a root $t_n^* \in (0, a_{n+1}]$ then we can find all other values of t_k^* from Equation (12). Otherwise there is no time-optimal trajectory which is a set of circle segments.

There are some practical problems with the method suggested above, see [17, 22]. Computing the roots of the univariate polynomial can be ill-conditioned for polynomials of degree greater than 14, 15, while Groebner basis can contain as a consequence a polynomial at one variable of high degree. In fact, for our case the polynomial consequence, which contains only t_n, is a crucial one. All other values of t_i, $i < n$, can be found directly from the quadratic Equation (12). As an example, if the velocity function has only one breakpoint then the highest degree of the polynomial at t_n is 3 (hence, it is solvable analytically), for two breakpoints the highest degree is 7. So, if the number of breakpoints is small then the method is applicable.

4.2 Circle and line segments

In this subsection we briefly sketch the search of the potential time-optimal solution which contains circle and straight line segments. In fact, we reduce this case to the previous one, which contains

only circle segments. The straight line segment appears when t_k is equal to zero, so we have $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ combinations of possible set of zero t_k's. We assume that the number of breakpoints of the velocity function n is constant, so 2^n is a constant factor.

Assume that only one t_k is equal to zero, then the time needed to go alone the trajectory which contains circle segments and one straight line segment is:

$$t(t_1, \ldots, t_n) = \sum_{i=1,\ldots,k,k+2,\ldots,n} \int_0^{\gamma_i(t_1,\ldots,t_n)} \frac{d\alpha}{\sin(\alpha + \beta_i(t_1, \ldots, t_n))} + t_{line}, \tag{13}$$

where t_{line} is the time to go alone the vertical line segment within the layer L_k from the altitude $y = y_k$ to the altitude $y = y_{k+1}$. It is a constant with respect to the variables $t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n$.

We find the stationary point conditions for Equation (13) in the similar way as in Subsection 4.1. The partial derivative of the term t_{line} with respect to any $t_i, i \neq k$, is equal to zero, as it is a constant. The functions $\beta_i(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)$ and $\mu_i(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)$, $i \neq (k+1)$, are still the functions at t_{k-1} only, as we do the same steps, replacing t_k with zero. β_{k+1} and μ_{k+1} are not determined any more. So we end up with the similar system of equations:

$$q_1 R_1(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n) = q_i R_i(t_{i-1}),\ i = 1, \ldots, k, k+2, \ldots, n. \tag{14}$$

If more than one of the t_k's are equal to zero, then Equation (13) contains more constant factors, but the rest of the analysis remains the same. Finally, we choose the best trajectory with respect to time (determined by Equation (13)) among the trajectories generated by some set of t_i's which are equal to zero. The case when the trajectory contains only circle segments is also included, as it is generated by the empty set of zero t_i's.

5 Conclusion

In this paper we have addressed the problem of determining a time-optimal helicopter trajectory between two points in two-dimensional space, where the speed of the helicopter depends on the altitude. We have characterized the time-optimal trajectories for the basic problem, where the speed decreases linearly in altitude. We used the motions primitives to solve the basic helicopter problem with rectilinear obstacles in 2D in polynomial time and to reduce the general helicopter problem with piece-wise linear velocity function to the set of multivariate polynomial equations. If the number of velocity function breakpoints is constant then it enables us to solve the general helicopter problem in polynomial time using the algebraic elimination theory.

There are several interesting research directions that can be followed. First, it is an open question whether the general helicopter problem with polygonal obstacles in 2D admits a polynomial time algorithm. Second, are there efficient polynomial time approximation schemes for the general helicopter problem with obstacles in 3D? Notice, the three-dimensional problem with obstacles is NP-hard as it generalizes the Euclidean shortest-path problem with obstacles in 3D.

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A Solution to the basic helicopter problem

In this section we derive motion primitives to the basic helicopter problem.

Theorem 2. *Let C be the intersection point of the line $y = \frac{v_0}{q}$ and the line equidistant from A and B. The segment T_{AB} of the circle with center C and radius $R = |CA| = |CB|$ is a time-optimal trajectory between A and B. The time needed to travel along T_{AB} is*

$$t_{opt} = t_{T_{AB}} = \frac{1}{q}\int_0^\gamma \frac{d\alpha}{\sin(\alpha+\beta)} = \frac{1}{q}\ln\left|\frac{\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}}{\tan\frac{\beta}{2}-\tan^2\frac{\beta}{2}\tan\frac{\gamma}{2}}\right|,$$

where β is the angle between CA and the x-axis, and γ is the angle between CA and CB.

Proof. The beginning of the proof is given in Section 2. The following integral is derived to calculate the time needed to go alone the arbitrary convex trajectory T:

$$t_T = \int_0^\gamma \frac{\sqrt{R'^2(\alpha)+R^2(\alpha)}\,d\alpha}{qR(\alpha)\sin(\alpha+\beta)} = \frac{1}{q}\int_0^\gamma \sqrt{\left(\frac{R'(\alpha)}{R(\alpha)}\right)^2+1}\,\frac{d\alpha}{\sin(\alpha+\beta)}.$$

Notice, this integral is well defined as $0 < \alpha+\beta < \pi$ for any $\alpha \in [0,\gamma]$ (β and γ are two of the three angles of the triangle). Furthermore, for the trajectory T we have that $R(0) = R(\gamma) = R = |CA| = |CB|$, as the point C is equidistant from A and B.

Now, we replace $R(\alpha)$ with the function $f(\alpha) = \ln(R(\alpha))$. Since $R(\alpha)$ is continuously differentiable, the function $f(\alpha)$ is continuously differentiable as well. Furthermore, $f'(\alpha) = \frac{R'(\alpha)}{R(\alpha)}$, and the following boundary conditions hold: $f(0) = f(\gamma) = \ln R$, where $R = |CA| = |CB|$. For the function $f(\alpha)$ the travel time is calculated as follows:

$$t_T = \frac{1}{q}\int_0^\gamma \sqrt{1+f'^2(\alpha)}\,\frac{d\alpha}{\sin(\alpha+\beta)}. \tag{15}$$

Now, we have to find all minimizers of the functional (15). The following theorem of the calculus of variation is used:

Theorem 3 ([15], p. 21; see also [14]). *Let $S(y) = \int_a^b F(t,y(t),y'(t))dt$ be a functional, where $y : [a,b] \subset \mathbb{R} \to X$ is differentiable and $y(a) = y_a, y(b) = y_b$, y' is the continuous derivative of y and F is a real-valued function with continuous first partial derivatives. If the function $y(t)$ is a stationary point of the functional $S(y)$, then it satisfies the equation:*

$$F'_y(t,y(t),y'(t)) - \frac{d}{dt}F'_{y'}(t,y(t),y'(t)) = 0. \tag{16}$$

Notice that the trajectory is a circle segment if and only if the radius-vector $R(\alpha)$ is constant, which is possible if and only if the function $f(\alpha) = \ln(R(\alpha))$ is also a constant. Therefore, we can concentrate on Equation (15) proving that $f(\alpha) = const$ is the unique minimizer of the functional (15).

1) We prove first that the circle segment with the center C and radius $R = |CA|$ is a time-optimal trajectory.
   Consider an arbitrary trajectory T. $f'^2(\alpha) \geq 0$ for any T, therefore, the following lower bound exists:
$$t_T \geq \frac{1}{q}\int_0^\gamma \frac{d\alpha}{\sin(\alpha+\beta)}.$$

On the other hand, the last integral represents the time needed to travel along the circle segment T_{AB} with the center C and the constant radius $R = |CA|$. Therefore, the derived circle segment is at least as good as any other trajectory.

2) Now, we show that T_{AB} is the unique time-optimal trajectory in the class of continuously differentiable functions. Consider the functional (15) with the integrand $F(\alpha, f, f') = \frac{\sqrt{1+f'^2(\alpha)}}{\sin(\alpha+\beta)}$. Suppose $f(\alpha)$ is the minimizer of the functional. Therefore, the Euler-Lagrange Equation (16) holds for $f(\alpha)$, because it is a necessary condition for the stationary point of any functional. We obtain from Equation (16) that $\frac{d}{d\alpha}(F'_{f'}) = 0$ as $F'_f = 0$. Hence,

$$F'_{f'} = \frac{f'(\alpha)}{\sqrt{1+f'^2(\alpha)}\sin(\alpha+\beta)} = const. \tag{17}$$

Since $\sqrt{1+f'^2(\alpha)} \geq 1$ and $\sin(\alpha+\beta) > 0$, $(0 < \alpha+\beta < \pi)$, the denominator in Equation (17) is always positive. So the sign of $f'(\alpha)$ depends only from the sign of the constant. If $const = 0$ then $f'(\alpha) = 0$ and $f(\alpha)$ is a constant function, so as $R(\alpha)$. This minimizer is the circle segment described above. If $const > 0 (< 0)$ then $f(\alpha)$ is strictly increasing (decreasing), but this is the contradiction to the boundary condition $f(0) = f(\gamma) = \ln R$.
Hence, there are no other time-optimal trajectories but the circle segment T_{AB} with the center C and the radius $R = |CA| = |CB|$.

The time needed to travel along the circle segment T_{AB} can be computed as the value of the integral:

$$t_{opt} = \frac{1}{q}\int_0^{\gamma} \frac{d\alpha}{\sin(\alpha+\beta)} = \frac{1}{q}\int_{\beta}^{\beta+\gamma} \frac{d\tau}{\sin\tau} = \frac{1}{q}\int_{\tan\frac{\beta}{2}}^{\tan\frac{\beta+\gamma}{2}} \frac{dt}{t} =$$
$$= \frac{1}{q}\ln\left|\frac{\tan\frac{\beta+\gamma}{2}}{\tan\frac{\beta}{2}}\right| = \frac{1}{q}\ln\left|\frac{\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}}{\tan\frac{\beta}{2}-\tan^2\frac{\beta}{2}\tan\frac{\gamma}{2}}\right|.$$

□

B Potential breakpoints, one step obstacle case

Lemma 4. *For three consecutive points $A = (x_1, y_1)$, $E = (x_2, y_1)$ and $B = (x_2, y_2)$ from S, such that $y_1 < y_2$, the time-optimal trajectory from A to B with obstacles is:*

a) if C_{AB} is to the left of A then T_{AB} is a time-optimal trajectory,

b) if C_{AB} is to the right of A then time-optimal trajectory is the combination of the straight horizontal line segment L_{AD} and the circle segment T_{DB}, where D and C_{DB} determine a vertical line. In particular,

$$D = (x_2 - \sqrt{(y_2-y_1)(2y_0-y_1-y_2)}, y_1).$$

Proof. a) Follows directly from Theorem 1, as T_{AB} does not hit the horizontal and vertical obstacles L_{AE} and L_{EB}.

b) T_{AB} hits L_{AE} as C_{AB} is to the right of A. Let us find the first point $D = (x, y_1)$ to the right of A such that T_{DB} does not hit the obstacle L_{AE}. The circle does not hit L_{AE} for the first time when L_{AE} is the tangent line of T_{DB}. So the x-coordinate of the circle center C_{DB} and D are the same. Hence, $R_{DB} = d(C_{DB}, D) = d(C_{DB}, B)$ and C_{DB} has coordinates (x, y_0). We have the following equation:

$$(x-x)^2+(y_0-y_1)^2=(x_2-x)^2+(y_0-y_2)^2,$$
$$(x-x_2)^2=(y_2-y_1)(2y_0-y_1-y_2)\geq 0,\ \text{ as } y_0\geq y_2>y_1,$$
$$x=x_2-\sqrt{(y_2-y_1)(2y_0-y_1-y_2)},\ \text{ as } x\leq x_2.$$

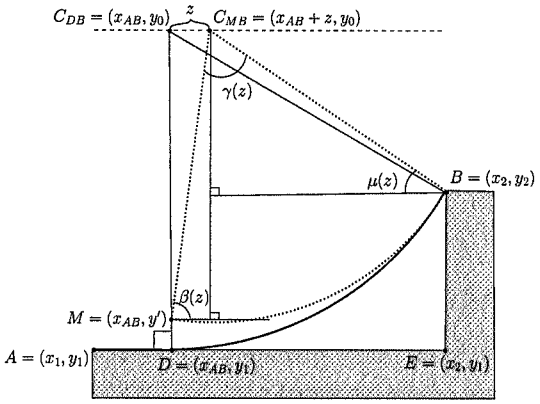

Fig. 8. The optimal trajectory for the one step case, T_{AB} hits L_{AE}

Here and below we denote the found value of x for A and B as x_{AB}. We show now that the constructed trajectory L_{AD} plus T_{DB} is time-optimal with obstacles. Suppose there exists another trajectory T such that $t_T < t_{L_{AD}} + t_{T_{DB}}$. T has to cross the line $x = x_{AB}$ at some altitude above y_1 to avoid the obstacle L_{AE}, for example at point $M = (x_{AB}, y')$, $y' \in [y_1, y_0]$, see Figure 8. M splits the trajectory T into two parts:

- AM such that $t_{AM} \geq t_{L_{AD}}$, the proof is the same as the proof of Lemma 1,
- MB, the trajectory MB is unknown, but the best we can do is to go from M to B along T_{MB}. Let us show that $t_{T_{MB}} \geq t_{T_{DB}}$.
  The position of the point C_{MB} is a one-to-one function of the position of the point M. For the simplicity we consider z as a parameter, where $C_{MB} = (x_{AB} + z, y_0)$. By Theorem 1 the time needed to go along T_{MB} is given by the following integral:

$$t_{T_{MB}}(z) = \frac{1}{q}\int_0^{\gamma(z)} \frac{d\alpha}{sin(\alpha+\beta(z))}.$$

We show that the derivative of $t_{T_{MB}}(z)$ is positive and thus the function $t_{T_{MB}}(z)$ is monotonically increasing:

$$\begin{aligned}\frac{d}{dz}(qt_{T_{MB}}(z)) &= \int_0^{\gamma(z)} -\frac{cos(\alpha+\beta(z))\beta'(z)d\alpha}{sin^2(\alpha+\beta(z))} + \frac{\gamma'(z)}{sin(\gamma(z)+\beta(z))} = \\ &= -\beta'(z)\int_{sin\,\beta(z)}^{\sin(\beta(z)+\gamma(z))} \frac{d\tau}{\tau^2} + \frac{\gamma'(z)}{sin(\gamma(z)+\beta(z))} = \\ &= \frac{\beta'(z)+\gamma'(z)}{sin(\gamma(z)+\beta(z))} - \frac{\beta'(z)}{sin\,\beta(z)} = -\left[\frac{\beta'(z)}{sin\,\beta(z)} + \frac{\mu'(z)}{sin\,\mu(z)}\right],\end{aligned}$$

where $\mu(z) = \pi - \gamma(z) - \beta(z)$ is the angle between $C_{MB}B$ and the horizontal axis. $R_{MB} = R(z) = \sqrt{(x_{AB} + z - x_2)^2 + (y_0 - y_2)^2}$, then $cos\,\beta(z) = \frac{z}{R(z)}$, $sin\,\beta(z) = \frac{\sqrt{R^2(z) - z^2}}{R(z)}$, $cos\,\mu(z) = \frac{x_2 - x_{AB} - z}{R(z)}$, $sin\,\mu(z) = \frac{y_0 - y_2}{R(z)}$. Hence we obtain,

$$\frac{d}{dz}(qt_{T_{MB}}(z)) = -\left[\frac{\beta'(z)}{sin\,\beta(z)} + \frac{\mu'(z)}{sin\,\mu(z)}\right] = \frac{z(x_2 - x_{AB})}{R(z)(R^2(z) - z^2)} \geq 0,$$

as $x_2 > x_{AB}$ and the denominator is positive, so the time increases with the growth of z. $\frac{d}{dz}(qt_{T_{MB}}(z)) = 0$ if and only if $z = 0$, which means $C_{MB} = C_{DB}$. We can conclude that $t_{T_{DB}}$ is smaller than t_{MB} for any trajectory MB.

So, the constructed trajectory L_{AD} plus T_{DB} is the time-optimal for one step as

$$t_{L_{AD}} + t_{T_{DB}} \leq t_{AM} + t_{T_{MB}} \leq t_T.$$

□

C Dynamic programming to the basic helicopter problem with rectilinear obstacles

In this appendix we prove the set of observations used in Lemma 3 and the lemma itself.

Observation 4 *For any three points $A = (x_1, y_1), B = (x_2, y_2)$ and $M = (x_3, y_3)$ such that $x_1 < x_3 < x_2$ and $y_3 < y_1$, $y_3 < y_2$, T_{AB} lies above T_{AM} and T_{MB} on the interval $[x_1, x_2]$.*

Proof. Denote the x-coordinates of the centers of the circle segments T_{AM}, T_{MB} and T_{AB} as x_0^{AM}, x_0^{MB} and x_0^{AB} respectively. Then $x_0^{MB} \leq x_0^{AB} \leq x_0^{AM}$ as $x_0^{AB} = \lambda x_0^{MB} + (1 - \lambda)x_0^{AM}$ for $\lambda = \frac{x_2 - x_3}{x_2 - x_1} \in [0, 1]$.

The circle segment T_{AB} is given by the equation $(x - x_0^{AB})^2 + (y - y_0)^2 = R_{AB}^2$, where $R_{AB}^2 = (x_0^{AB} - x_1)^2 + (y_0 - y_1)^2$, so $y = y_0 - \sqrt{R_{AB}^2 - (x - x_0^{AB})^2}$. Similarly, the circle segment T_{AM} is given by the equation $y = y_0 - \sqrt{R_{AM}^2 - (x - x_0^{AM})^2}$, where $R_{AM}^2 = (x_0^{AM} - x_1)^2 + (y_0 - y_1)^2$. The circle segment T_{AB} is above the circle segment T_{AM} if $y_0 - \sqrt{R_{AB}^2 - (x - x_0^{AB})^2}$ is bigger than $y_0 - \sqrt{R_{AM}^2 - (x - x_0^{AM})^2}$ for any $x \in [x_1, x_2]$. It is enough to compare the expressions under the radical to verify this fact:

$$\begin{aligned}
&R_{AB}^2 - (x - x_0^{AB})^2 - (R_{AM}^2 - (x - x_0^{AM})^2) = \\
&(x_0^{AB} - x_1)^2 + (y_0 - y_1)^2 - (x - x_0^{AB})^2 - (x_0^{AM} - x_1)^2 - (y_0 - y_1)^2 + (x - x_0^{AM})^2 = \\
&2(x_0^{AB} - x_0^{AM})(x - x_1) \leq 0.
\end{aligned}$$

So $\sqrt{R_{AB}^2 - (x - x_0^{AB})^2} \leq \sqrt{R_{AM}^2 - (x - x_0^{AM})^2}$ for any $x \in [x_1, x_2]$, hence, $y_0 - \sqrt{R_{AB}^2 - (x - x_0^{AB})^2} \geq y_0 - \sqrt{R_{AM}^2 - (x - x_0^{AM})^2}$. Therefore, T_{AB} is higher than T_{AM}. The proof that T_{AB} is higher than T_{MB} is similar.

□

Observation 5 *For any three consecutive points $A = (x_1, y_1), C = (x_2, y_1)$ and $B = (x_2, y_2)$ such that $x_1 < x_2$ and T_{AB} hits L_{AC}, the time-optimal trajectory lies above T_{AB} on the interval $[x_1, x_2]$.*

Proof. We consider $x_1 = 0$ and $D = (\Delta x, y_1)$ without loss of generality, where Δx is determined by Lemma 2b) as in this case L_{AD} plus T_{DB} is the time-optimal trajectory with one step obstacle. Furthermore, $\Delta x = x_0^{DB}$. Then by Equation (5)

$$x_0^{AB} = \frac{-2(y_2 - y_1)y_0 - y_1^2 + x_2^2 + y_2^2}{2x_2},$$
$$x_0^{DB} = \Delta x = \frac{-2(y_2 - y_1)y_0 - y_1^2 + x_2^2 + y_2^2}{2(x_2 - \Delta x)} + \frac{(\Delta x)^2}{2(\Delta x - x_2)},$$

so $2x_0^{DB}(x_2 - \Delta x) + (\Delta x)^2 = 2x_2 x_0^{AB}$, hence $x_0^{DB} - x_0^{AB} = \Delta x - x_0^{AB} = \frac{(\Delta x)^2}{2x_2} \geq 0$. Hence, $x_0^{DB} \geq x_0^{AB}$, so the circle segment T_{DB} is above T_{AB} for any $x \in [x_1, x_2]$, the proof of this fact is similar to the proof of Observation 1.

□

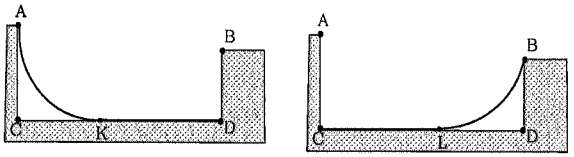

Fig. 9. The time-optimal trajectory for the unique horizontal obstacle between A and B

Observation 6 *Given four consecutive points $A = (x_1, y_1), C = (x_1, y_2), D = (x_2, y_2)$ and $B = (x_2, y_3)$ from S such that $x_1 < x_2$, $y_2 < y_1$, $y_2 < y_3$ and T_{AB} hits L_{CD}, the time-optimal trajectory with obstacles is the combination of the circle segment T_{AK}, straight line L_{KL} and circle segment T_{LB}, where K and L are the points of intersection of T_{AD} and T_{CB} with L_{CD} respectively.*

Proof. We consider the triplets A, C, D and C, D, B as the input to Lemma 2 (see Figure 9). T_{AK} does not intersect T_{LB} by Observation 1, as T_{AB} hits L_{CD}.

□

Lemma 5. *If the time-optimal trajectory T_{AB} between any two potential breakpoints $A = (x_1, y_1), B = (x_2, y_2)$, $x_1 < x_2$ hits the obstacles, then there exists a potential breakpoint C between A and B such that the time-optimal trajectory with obstacles from A to B passes C.*

Proof. Assume the statement is not correct, hence the trajectory hits some obstacle and there exists a time-optimal trajectory T with obstacles from A to B such that it does not hit any obstacle and does not pass any potential breakpoint.

We split the proof into five parts depending on the type of "terrain" between A and B.

1) *The obstacles between A and B are monotonically increasing (or decreasing), see Figure 10.*
First we notice that T is convex, otherwise we can always locally shortcut it with the straight line in such a way that it still does not pass any potential breakpoint, but the straight line is always better than any trajectory above it by Lemma 1, so T is not time-optimal.
We assume without loss of generality that $y_1 < y_2$. Let us consider the last obstacle $C = (x_3, y_3)$ on the way along T, T does not pass C by assumption. By Lemma 2 the time-optimal trajectory from C to B has one of two shapes:

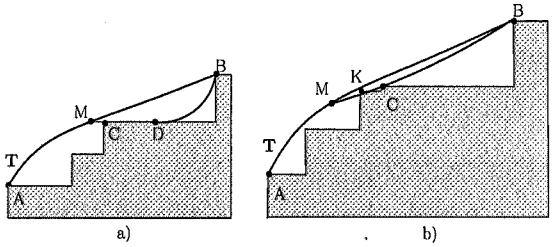

Fig. 10. Monotonic obstacle case

- it is the combination of the straight line L_{CD} and T_{DB}, see Figure 10a). The intersection of the line $y = y_3$ and the trajectory T is point M. The time-optimal trajectory from M to B is the combination of the straight line L_{MD} and the circle segment T_{DB} by Lemma 2b). Then the following combined trajectory is better than T: T before the point M, straight line L_{MD} and T_{DB}, and it passes the potential breakpoint C, a contradiction.
- it is T_{CB}, see Figure 10b). M is the point of intersection of T and T_{CB}. If $T_{CB} = T_{MB}$ does not hit any obstacle before crossing T or it just crosses the obstacle in the potential breakpoint K then the trajectory T between A and M plus T_{MB} is better than T and it passes the potential breakpoint C, a contradiction.
  If T_{MB} hits the obstacle which is determined by the potential breakpoint K then we consider T_{KB} (or L_{KD_K} and T_{D_KB}) by Lemma 2a) (or b)) and so on. We have only a finite number of obstacles so finally we can find the "last" obstacle L such that the time-optimal trajectory from Lemma 2 does not hit this obstacle. Any new circle segment or combination of the line and circle segment lies higher than the previous one by induction, so it does not hit any obstacle to the right of the considered potential breakpoint. We use the time-optimal trajectory which is determined by the potential breakpoint L and B to shortcut T as in the previous case, a contradiction.

2) *There is an obstacle between A and B which is higher than A and B, see Figure 11.* By Lemma 1 we can always shortcut T with the line L_{KM} which passes 2 potential breakpoints, a contradiction.

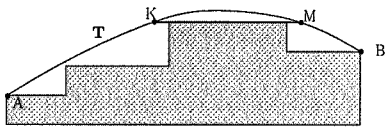

Fig. 11. Concave obstacle case

3) *The obstacles between A and B is a convex rectilinear function, see Figure 12.* Similar to case 1) the trajectory T is convex. Denote the lowest point of the trajectory T as point $M = (x_M, y_M)$. We consider two following cases:
   - *All the obstacles which are hit by T_{AB} are lower than M, see Figure 12a).* We introduce a new instance: the horizonal line $y = y'$ at the altitude of the highest obstacle $K = (x', y')$, hit by T_{AB}. T is a valid trajectory for the new instance. The time-optimal trajectory T' for the new instance by Observation 3 is the circle segment T_{AK_1}, line segment $L_{K_1K_2}$ and circle

segment T_{K_2B}, where K_1, K_2 belongs to the line $y = y'$. By Observation 2 this trajectory does not hit any obstacle of the original instance, so it is a valid trajectory for the original instance. T' is time-optimal for the new instance, hence it is better than T. So, T' is also better than T for the original instance, a contradiction.

- *There is at least one obstacle $K = (x', y')$, which is hit by T_{AB}, and it is higher than M, see Figure 12b).* We introduce a new instance: the horizontal line $y = y_M$. If either T_{AM} or T_{MB} hits the obstacles then we are in the settings of the case 1) for one of the segments T_{AM} or T_{MB}, hence there is a trajectory T' which is better than T and it passes one of the potential breakpoints.
  So we only consider the case when T_{AM} and T_{MB} do not hit any obstacle. It means that T is just a combination of T_{AM} and T_{MB}. By Observation 1 T_{AB} is above T, so T_{AB} also does not hit any obstacle, a contradiction.

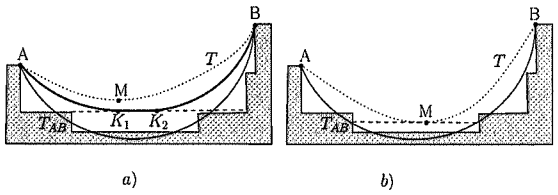

Fig. 12. Convex obstacle case

4) *There is an obstacle between A and B which is higher than A, lower than B and it is the point of non-monotonicity, see Figure 13.* If the trajectory T "goes down" after this obstacle, see Figure 13a), we can shortcut it as in case 2), a contradiction. If it stays higher, see Figure 13b), then we can introduce a new equivalent monotonic instance as in case 1).

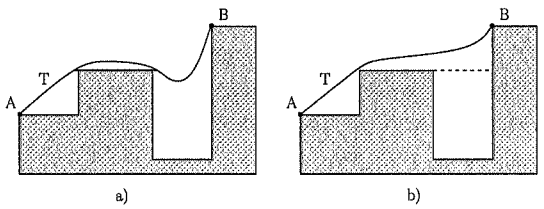

Fig. 13. Non-monotonic obstacle case, the obstacle is higher than A

5) *The obstacles between A and B are not presented by a convex rectilinear function, and all the obstacles are lower than A and B, see Figure 14.* The obstacle function is not convex, so there exists an obstacle which is concave. If T "goes down" on both sides of this obstacle we can shortcut is as in case 2), see Figure 14a), a contradiction. If T stays higher at least on one side, see Figure 14b), then we can introduce a new convex instance as in case 4), so we are in the settings of the case 3).

Finally we conclude that if the time-optimal trajectory between two potential breakpoints A and B hits an obstacle then there is a potential breakpoint C between A and B such that the time-optimal trajectory with obstacles from A to B passes C.

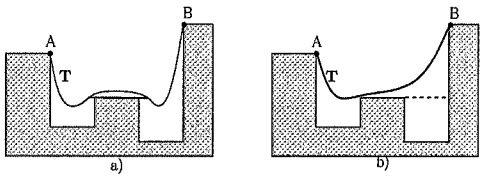

Fig. 14. Non-monotonic obstacle case, the obstacle is lower than A and B

□

D Elimination theory

In this appendix we briefly introduce some notions and facts of computational algebraic geometry and commutative algebra about polynomials of n variables $x_1, \ldots, x_n$ (see [11]):

- The set of all polynomials at $x_1, \ldots, x_n$ with coefficients in the field k is denoted $k[x_1, \ldots, x_n]$.
- The product $cx_1^{\alpha_1} \cdots x_n^{\alpha_n}, c \in k$, is called a *leading term* of the polynomial if $\sum_{i=1}^{n} \alpha_i$ is the biggest among all the sums of the powers.
- A subset $I \subseteq k[x_1, \ldots, x_n]$ is an *ideal* if it satisfies:
  - $0 \in I$,
  - if $f, g \in I$ then $f + g \in I$,
  - if $f \in I$ and $h \in k[x_1, \ldots, x_n]$ then $hf \in I$.
- If $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ then $\langle f_1, \ldots, f_s \rangle = \{\sum_{i=1}^{s} h_i f_i \mid h_1, \ldots, h_s \in k[x_1, \ldots, x_n]\}$ is an ideal of $k[x_1, \ldots, x_n]$.
- We can think of $\langle f_1, \ldots, f_s \rangle$ as consisting of all "polynomial consequences" of the equations $f_1 = \ldots = f_s = 0$.
- An ideal I is *finitely generated* if there exist $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ such that $I = \langle f_1, \ldots, f_s, \rangle$. $f_1, \ldots, f_s$ are a *basis* of I.
- The lexicographical order *lex* is determined as follows: $x_1^{\alpha_1} \cdots x_n^{\alpha_n} >_{lex} x_1^{\beta_1} \cdots x_n^{\beta_n}$ if in the product $x_1^{\alpha_1 - \beta_1} \cdots x_n^{\alpha_n - \beta_n}$ the leftmost nonzero entry among $\alpha_i - \beta_i$ is positive.
- The set $\{g_1, \ldots, g_s\} \subset I$ is called a *Groebner basis* of I iff the leading term of every element of I is divisible into the leading term of at least one g_i.
- The Groebner basis of any $\emptyset \neq I \subset k[x_1, \ldots, x_n]$ can be found in finite number of steps by *Buchberger's algorithm.*
- Given $I = \langle f_1, \ldots, f_s \rangle \subset k[x_1, \ldots, x_n]$ the *l-th elimination ideal* I_l, $l \leq 1$, is the ideal of $k[x_{l+1}, \ldots, x_n]$ defined by $I_l = I \cap k[x_{l+1}, \ldots, x_n]$. Thus, I_l consists of all consequences of $f_1 = \ldots = f_s = 0$ which eliminate the variables $x_1, \ldots, x_l$.
- *The Elimination Theorem*: Let $I \subset k[x_1, \ldots, x_n]$ be an ideal and let G be a Groebner basis of I with respect to lex order $x_1 > x_2 > \ldots > x_n$. Then, for every $1 \leq l \leq n$, the set $G_l = G \cap k[x_{l+1}, \ldots, x_n]$ is a Groebner basis of the l-th elimination ideal I_l.