Enumeration of Circuits and Minimal Forbidden Sets

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Abstract

In resource-constrained scheduling, it is sometimes important to know all inclusion-minimal subsets of jobs that must not be scheduled simultaneously. These so-called minimal forbidden sets are given implicitly by a linear inequality system, and can be interpreted more generally as the circuits of a particular independence system. We present several complexity results related to computation, enumeration, and counting of the circuits of an independence system. On this account, we also propose a backtracking algorithm that enumerates all minimal forbidden sets for resource constrained scheduling problems.

\textit{Key words: Independence system, Circuit, Enumeration, Minimal forbidden set}

1 Introduction

Given a finite ground set $V$, an independence system is defined as a family $\mathcal{I}$ of subsets of $V$ with two properties. First, $\emptyset \in \mathcal{I}$, and second, any subset of any member of $\mathcal{I}$ also belongs to $\mathcal{I}$. The sets in $\mathcal{I}$ are called independent sets, and the inclusion-maximal independent sets are the bases $\mathcal{B}$ of $\mathcal{I}$. The sets not in $\mathcal{I}$ are called dependent sets, and inclusion-minimal dependent sets are the circuits $\mathcal{C}$ of $\mathcal{I}$; see also [6]. Given a membership oracle for such an independence system $\mathcal{I}$, we are interested in the problem to enumerate all circuits of $\mathcal{I}$. It is obvious that the output size may be exponential in terms of the size of $V$. The complexity is thus measured in terms of the size of both, in- and output. Given that $|V| = n$, and given a (poly($n$)) membership oracle for some collection of subsets $\mathcal{S} \subseteq 2^V$, we use the following definitions; see [4].

\textbf{Definition 1} The enumeration problem for $\mathcal{S}$ is solvable in polynomial total time if there exists an algorithm and a polynomial $p(\cdot, \cdot)$, such that the algorithm correctly outputs all members of $\mathcal{S}$ in $p(n, |\mathcal{S}|)$ time.
Given a sub-collection \( \mathcal{X} \subseteq \mathcal{S} \), the \textit{increments problem} is the problem: either decide that \( \mathcal{X} = \mathcal{S} \), otherwise output a new element in \( \mathcal{S} \setminus \mathcal{X} \).

\textbf{Definition 2} The \textit{enumeration problem} for \( \mathcal{S} \) is solvable in \textit{incremental polynomial time} if there exists an algorithm and a polynomial \( p(\cdot, \cdot) \), such that the algorithm correctly solves the increments problem in \( p(n, |\mathcal{X}|) \) time, for any \( \mathcal{X} \subseteq \mathcal{S} \).

If the enumeration problem for \( \mathcal{S} \) is solvable in incremental polynomial time, it is also solvable in polynomial total time: Starting with \( \mathcal{X} = \emptyset \), the iterative solution of the increments problem yields the complete collection \( \mathcal{S} \), in time polynomial in \( n \) and \( |\mathcal{S}| \). The reverse, however, need not be true.

2 \hspace{1em} The general case

The following theorem is well known; it is proved using a reduction from the \textsc{NP}-complete decision problem \textsc{Satisfiability} [3].

\textbf{Theorem 3} ([5]) Unless \( \text{P=NP} \), there does not exist a polynomial total time algorithm that enumerates the bases \( \mathcal{B} \) of any independence system \( \mathcal{I} \).

Likewise, using a simple duality argument, we can show the following.

\textbf{Theorem 4} Unless \( \text{P=NP} \), there does not exist a polynomial total time algorithm that enumerates the circuits \( \mathcal{C} \) of any independence system \( \mathcal{I} \).

The proof uses the fact that the circuits of \( \mathcal{I} \) are the bases of the following, dual independence system: \( \mathcal{I}^D = \{ W \subseteq V \mid V \setminus W \not\in \mathcal{I} \} \). These two theorems say that, unless \( \text{P=NP} \), an algorithm cannot exist which solves the problem for \textit{any} independence system \( \mathcal{I} \). For particular realizations of the membership oracle of \( \mathcal{I} \) however, efficient enumeration algorithms may well exist.

3 \hspace{1em} Scheduling and linear inequality systems

In resource-constrained scheduling, the input consists of a set of partially ordered jobs \((V, \prec)\) (the partial order representing the precedence constraints) and resource constraints. The latter are given through a number of resource types \( k \) with availabilities \( b_k \), and resource requirements \( a_{kj} \) of these resource types for all jobs \( j \in V \). If a subset \( S \) of jobs consumes more of a resource type than available, the respective jobs in \( S \) may not be processed in parallel. The subsets of jobs that may be processed in parallel define an inde-
pendence system. The circuits of this independence system are either pairs of (precedence-)related jobs, \( \{i, j\} \) with \( i \prec j \), or the so-called minimal forbidden sets (minimal anti-chains of \((V, \prec)\) which may not be processed in parallel). For several algorithmic purposes, e.g., in stochastic scheduling [8], a complete list of the minimal forbidden sets is required. This amounts to the computation of the circuits of the corresponding independence system. The membership oracle of this system is a linear inequality system \( Ax \leq b \), where \( A, b \) contains one row in for each resource type \( k \), and one row for each edge in the comparability graph of the partial order \((V, \prec)\). The circuits are the minimally infeasible \( \{0, 1\} \)-vectors for \( Ax \leq b \); they are the incidence vectors of either pairs of (precedence-)related jobs, or minimal forbidden sets.

Inspired by [2], and using a reduction from the NP-complete decision problem Independent Set in graphs [3], we show:

**Theorem 5** Unless \( P=NP \), there does not exist a polynomial total time algorithm that enumerates the minimally infeasible \( \{0, 1\} \)-vectors of an arbitrary linear inequality system \( Ax \leq b \).

In fact, in [2] it is shown that the decision version of the corresponding increments problem is NP-complete. What is interesting is the fact that the ‘dual’ problem is apparently much easier: It follows from [2] that the increments problem for the maximally feasible \( \{0, 1\} \)-vectors of an arbitrary linear inequality system \( Ax \leq b \) can be solved in quasi-polynomial time, hence there is also a quasi-polynomial total time algorithm for this problem. (Such a result is not likely for the problem considered here, because then all NP-hard problems could be solved in quasi-polynomial time.)

In terms of scheduling, Theorem 5 immediately yields:

**Corollary 6** Unless \( P=NP \), there does not exist a polynomial total time algorithm that enumerates the minimal forbidden sets for any instance of the resource-constrained project scheduling problem.

It turns out that even the computation of the number of minimal forbidden sets is a hard problem, because we can show:

**Theorem 7** The problem to compute the number of minimally infeasible (maximally feasible) \( \{0, 1\} \)-vectors of an arbitrary linear inequality system \( Ax \leq b \) is \#P-complete.

The proof uses a reduction from the problem to compute a maximum cardinality anti-chain of a partial order. While this problem is polynomially solvable, the associated counting problem is known to be \#P-complete [7].

Nevertheless, for practical purposes we implemented a simple backtracking al-
Algorithm that lists the minimal forbidden sets for any instance of the resource-constrained project scheduling problem. In general, this algorithm can have an exponential running time in terms of input and output of the problem. Yet, empirically it improves considerably upon a divide-and-conquer algorithm previously suggested in [5, 1]; see [9]. Moreover, we can show that the algorithm is efficient for an important special case.

Proposition 8 There exists an incremental polynomial time (hence also a polynomial total time algorithm) that enumerates the minimal forbidden sets for any instance of the resource-constrained project scheduling problem, given that the number of resource types is 1.

We refer to [9] for several other results related to enumeration and computation of minimal forbidden sets, as well as detailed computational results.

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References


