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Mechanisms for division problems with single-dipped preferences

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Abstract

A mechanism allocates one unit of an infinitely divisible commodity among agents reporting a number between zero and one. Nash, Pareto optimal Nash, and strong equilibria are analyzed for the case where the agents have single-dipped preferences. One of the main results is that when the mechanism is anonymous, monotonic, standard, and order preserving, then the Pareto optimal Nash and strong equilibria coincide and assign Pareto optimal allocations that are characterized by so-called maximal coalitions: members of a maximal coalition prefer an equal coalition share over obtaining zero, whereas the outside agents prefer zero over obtaining an equal share from joining the coalition.

Keywords: division problems, single-dipped preferences, mechanisms, Nash equilibrium, strong equilibrium

JEL classification: C72, D71

1 Introduction

We consider the problem of allocating one unit of an infinitely divisible commodity among agents with single-dipped preferences. A single-dipped preference has a worst point, the dip, and preference strictly increases in both directions away from the dip. Such a preference may arise from maximizing a strictly quasiconvex utility function on a (budget) line, and reflects that an agent prefers extremes over combinations – for instance, a university employee may prefer either only teaching or only research over a combination of the two.

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We take a mechanism design approach: each agent reports a number between zero and one, and a mechanism is a function assigning an allocation of the commodity among the agents, which is evaluated by the agents according to their preferences. Under a number of conditions on mechanisms, we analyze the Nash, Pareto optimal Nash, and strong equilibria for each single-dipped preference profile, and the resulting allocations, in the induced game. Mechanisms are related to (social choice) rules: these assign an allocation to each profile of preferences. In particular, a rule which only depends on the dips of the reported preferences, gives rise to a mechanism.

Almost throughout, we assume that a mechanism is anonymous and monotonic. The latter condition means that if an agent reports a higher or lower number, then that agent's share increases or decreases, if possible. The motivation for this monotonicity requirement is that it provides the agents with ample possibilities to influence their shares – thus, making the mechanism sufficiently sensitive to the strategies of the agents.

After preliminaries in Section 2, in Section 3 we discuss Nash equilibria of games induced by a mechanism and single-dipped preference profiles. The main insight here is that in every Nash equilibrium each agent plays 0 or 1, and we characterize all Nash equilibria (Theorem 3.5). If there are two agents then a Nash equilibrium always exists (Proposition 3.7), but this is no longer true for more than two agents.

In Section 4 we consider Pareto optimal Nash equilibria, and we show that an additional condition on a mechanism, namely that when every agent plays 0 or 1, the agents who play 0 receive 0 and the agents who play 1 equally share the commodity, is necessary and sufficient for the existence of a Pareto optimal Nash equilibrium for all games, i.e., all preference profiles. Moreover, in this case the Pareto optimal Nash equilibria are exactly those strategy profiles where agents in a so-called maximal coalition play 1 and the other agents play 0 – 'maximal' means that as many agents as possible (given the restrictions of best reply and Pareto optimality) play 1 and get a positive share. Under the further condition of order preservation on a mechanism – meaning that playing a higher number than another agent results in obtaining a higher share than that agent – these Pareto optimal Nash equilibria are, moreover, strong equilibria (Aumann, 1959): no coalition can profitably deviate. As a consequence, under the mentioned conditions on a mechanism, a selection – denoted by M – of the Pareto social choice correspondence is implemented in strong equilibrium, namely picking the Pareto optimal allocations that are characterized by so-called maximal coalitions: members of a maximal coalition prefer an equal coalition share over obtaining zero, whereas the outside agents prefer zero over obtaining an equal share from joining the coalition.

In Section 5 we provide an axiomatic characterization of this social choice correspondence M. We show that M is the maximal correspondence satisfying minimal envy Pareto optimality, equal division lower bound, and sharing index order preservation. This first condition requires that allocations are Pareto optimal and, within the set of Pareto optimal allocations, only those are selected from at which the number of envious agents – agents who prefer some one else's share over their own – is as small as possible. The second condition requires that each agent (weakly) prefers its share over an equal division of the commodity. The third condition says that an agent who is willing to equally share the commodity with more agents than some other agent, does not receive less than this other agent.

Sprumont (1991) shows that under a few natural conditions, the so-called uniform rule is the unique strategy-proof (Gibbard, 1973; Satterthwaite, 1975) rule for division problems with single-peaked preferences – a preference is single-peaked if there is a unique best point, the peak, and preference decreases in both directions away from this peak. Bochet et al (2021) – combining work of Bochet and Sakai (2009) and Thomson (2010) – show that under similar assumptions as ours, equilibria (Nash, Pareto optimal Nash, strong) end up in the allocation assigned by the uniform rule – see the concluding Section 6. While the uniform rule for single-peaked preferences is strategy-proof, we show in Section 6 that no selection from the implemented correspondence M for single-dipped preferences is strategy-proof. Further, the uniform rule satisfies the first two of the three conditions characterizing M, as described in the preceding paragraph.

Single-dipped and single-peaked preferences were already studied by Inada (1964). For single-dipped preferences in division problems, see Klaus et al (1997), who characterize Pareto optimal allocations (we use their result in Section 4), and study strategy-proofness of rules. For strategy-proofness in problems with indivisible goods and single-dipped preferences see Klaus (2001a,b) and Tamura (2022), and for probabilistic rules see Ehlers (2002). Doghmi (2013) shows that Maskin monotonicity is still a necessary condition for implementation; indeed, it is not difficult to show that the correspondence M is Maskin monotonic.

There is a relatively large literature on single-dipped preferences and public goods (also sometimes called public bads), including Peremans and Storcken (1999), Barberà et al (2012), Bossert and Peters (2014), Öztürk et al (2013, 2014), Manjunath (2014), Ayllón and Caramuta (2016), Tapki (2016), Yamamura (2016), Lahiri et al (2017), and Feigenbaum et al (2020).

2 Preliminaries

In this section we introduce allocations, preferences, mechanisms, rules, and equilibria.

2.1 Allocations, preferences, mechanisms, and equilibria

For $n \in \mathbb{N}$ with $n \geq 2$, let $N = \{1, \ldots, n\}$ be the set of *agents*. Among these agents one unit of a perfectly divisible commodity has to be distributed. The set of all *allocations* is denoted by $\mathcal{A} = \left\{x \in [0,1]^N \mid \sum_{i \in N} x_i = 1\right\}$. A subset of agents is also called a *coalition*.

An agent's *preference* is a transitive and complete binary relation R on the interval $[0,1]$. We denote by P *strict* preference, and by I *indifference*: $\alpha P \beta$ if $\alpha R \beta$ and not $\beta R \alpha$, and $\alpha I \beta$ if $\alpha R \beta$ and $\beta R \alpha$, for $\alpha, \beta \in [0,1]$. By $R_N = (R_i)_{i \in N}$ we denote a *profile* of preferences (for N).

An allocation $x \in \mathcal{A}$ is *Pareto optimal* at a preference profile R_N if there is no $x' \in \mathcal{A}$ such that $x'_i R_i x_i$ for all $i \in N$ and $x'_i P_i x_i$ for at least one $i \in N$.

In this paper we focus on mechanisms in order to select allocations. A *mechanism* is a map $g : [0,1]^N \to \mathcal{A}$. A preference profile R_N and a mechanism g induce a non-cooperative game (R_N, g) as follows. Each agent $i \in N$ has *strategy set* $[0,1]$. A *profile* of strategies $r = (r_i)_{i \in N} \in [0,1]^N$ results in an allocation $g(r) \in \mathcal{A}$, evaluated by each agent i via R_i. A profile r^* is a *Nash equilibrium* of the game (R_N, g) if for all $i \in N$ and $r_i \in [0,1]$,

$$g_i(r^*) R_i g_i(r_i, r^*_{-i}),$$

where $r^*_{-i} = (r^*_j)_{j \in N \setminus \{i\}}$. A Nash equilibrium r^* is a *Pareto optimal Nash equilibrium* of the game (R_N, g) if $g(r^*)$ is Pareto optimal at R_N. A profile r^* is a *strong equilibrium* of the game (R_N, g) if there are no $\emptyset \neq S \subseteq N$ and $r'_S \in [0,1]^S$ such that

$$g_i(r'_S, r^*_{N \setminus S}) R_i g_i(r^*) \text{ for all } i \in S \text{ and } g_i(r'_S, r^*_{N \setminus S}) P_i g_i(r^*) \text{ for some } i \in S,$$

where $r^*_{N \setminus S} = (r^*_i)_{i \in N \setminus S}$.

In most of this paper we focus on single-dipped preferences. A preference R is *single-dipped* if there is a *dip* $d(R) \in [0,1]$ such that for all $\alpha, \beta \in [0,1]$,

$$\alpha < \beta \leq d(R) \Rightarrow \alpha P \beta \quad \text{and} \quad \alpha > \beta \geq d(R) \Rightarrow \alpha P \beta.$$

The set of all single-dipped preferences is denoted by $\mathcal{D}$, and $\mathcal{D}^N$ is the set of all single-dipped preference profiles.

A preference R is *single-peaked* if there is a *peak* $p(R) \in [0,1]$ such that for all $\alpha, \beta \in [0,1]$,

$$p(R) \geq \alpha > \beta \Rightarrow \alpha P \beta \quad \text{and} \quad p(R) \leq \alpha < \beta \Rightarrow \alpha P \beta.$$

The set of all single-peaked preferences is denoted by $\mathcal{P}$, and $\mathcal{P}^N$ is the set of all single-peaked preference profiles.

2.2 Mechanisms versus rules

A mechanism is – indeed – a mechanical device that is used to non-cooperatively determine an allocation, given a preference profile. A (*social choice*) *rule* is a map φ assigning to each preference profile within a given set, an allocation. If such a rule φ on $\mathcal{D}^N$ or on $\mathcal{P}^N$ depends only on the dips or only on the peaks of a preference profile (i.e., is *dips-only* or *peaks-only*), then it can be identified with a mechanism according to our definition. An agent's strategy can then be interpreted as the agent reporting a dip or peak – not necessarily the true dip or peak. In this sense, the peaks-only rules for single-peaked preference profiles studied in for instance Sprumont (1991) or Bochet et al (2021) can be seen as mechanisms. On the other hand, a property like Pareto optimality makes sense for rules (meaning that they assign a Pareto optimal allocation to each preference profile), but not for mechanisms, which are defined independently of preference profiles. In most of what follows, we impose the following additional conditions on a mechanism g:

- *anonymity*: $g_i(r^\pi) = g_{\pi(i)}(r)$ for all $r \in [0,1]^N$ and every permutation π of N, where $r^\pi = (r_{\pi(i)})_{i\in N}$.
- *monotonicity*: for all $r \in [0,1]^N$, $i \in N$ and $r'_i \in [0,1]$,

$$r'_i > r_i \text{ and } g_i(r) < 1 \Rightarrow g_i(r'_i, r_{-i}) > g_i(r),$$

$$r'_i < r_i \text{ and } g_i(r) > 0 \Rightarrow g_i(r'_i, r_{-i}) < g_i(r),$$

  where (r'_i, r_{-i}) is obtained from r by replacing r_i by r'_i.

The set of all anonymous and monotonic mechanisms is denoted by $\mathcal{G}$.

The monotonicity condition is closely related to the condition of 'strict own-peak monotonicity' in Bochet et al (2021) when the latter is applied to rules that are peaks-only. The difference is that the condition in Bochet et al (2021) allows that an agent i receives 0 when that agent's strategy r_i is positive. Under our monotonicity condition this is not possible (see Lemma 3.2).

We conclude this section with two examples of mechanisms in $\mathcal{G}$.

Example 2.1 Let $N = \{1,2\}$ and let $g : [0,1]^N \to \mathcal{A}$ be defined by for each $r \in [0,1]^N$,

$$g(r) = \left(\frac{1 + r_1 - r_2}{2}, \frac{1 - r_1 + r_2}{2}\right).$$

Then g is anonymous and monotonic, and thus $g \in \mathcal{G}$. △

Example 2.2 Let $g : [0,1]^N \to \mathcal{A}$ be defined by for each $r \in [0,1]^N$ and $i \in N$,

$$g_i(r) = \begin{cases} \frac{r_i}{\sum_{j \in N} r_j} & \text{if } \sum_{j \in N} r_j \geq 1 \\ 1 - \frac{(n-1)(1-r_i)}{\sum_{j \in N}(1-r_j)} & \text{if } \sum_{j \in N} r_j \leq 1. \end{cases}$$

This mechanism corresponds to the 'symmetrized proportional rule' in Bochet et al (2021). Again, g is anonymous and monotonic, and therefore $g \in \mathcal{G}$. $\triangle$

In the next two sections we analyze Nash equilibria, Pareto optimal Nash equilibria, and strong equilibria in games induced by single-dipped preference profiles and mechanisms in $\mathcal{G}$.

3 Nash equilibrium

Before stating the main results, we formulate two elementary lemmas concerning single-dipped preferences and mechanisms, respectively. The first lemma recalls the well-known fact (Inada, 1964) that if an agent with a single-dipped preference prefers α to β in $[0,1]$, then this agent prefers α to each γ between α and β. This will be used several times in the sequel.

Lemma 3.1 *Let $R \in \mathcal{D}$ and let $\alpha, \beta \in [0,1]$ with $\alpha R \beta$. Then $\alpha R \gamma$ for all $\gamma \in [0,1]$ with $\min\{\alpha,\beta\} \leq \gamma \leq \max\{\alpha,\beta\}$.*

Proof. If $d(R) \leq \min\{\alpha,\beta\}$, then $d(R) \leq \beta \leq \alpha$, so $\alpha R \gamma$ for all $\beta \leq \gamma \leq \alpha$. If $d(R) \geq \max\{\alpha,\beta\}$, then $d(R) \geq \beta \geq \alpha$, so $\alpha R \gamma$ for all $\alpha \leq \gamma \leq \beta$. If $\min\{\alpha,\beta\} < d(R) < \max\{\alpha,\beta\}$, then we have $\alpha R \gamma$ for all $\min\{\alpha, d(R)\} \leq \gamma \leq \max\{\alpha, d(R)\}$, and $\alpha R \beta R \gamma$ for all $\min\{\beta, d(R)\} \leq \gamma \leq \max\{\beta, d(R)\}$. Therefore, $\alpha R \gamma$ for all $\min\{\alpha,\beta\} \leq \gamma \leq \max\{\alpha,\beta\}$. $\square$

The next lemma shows that a monotonic mechanism assigns 0 to an agent only if its strategy is 0, and assigns 1 to an agent only if its strategy is 1.

Lemma 3.2 *Let g be a monotonic mechanism and let $r \in [0,1]^N$. Then $r_i = 0$ for each $i \in N$ with $g_i(r) = 0$, and $r_i = 1$ for each $i \in N$ with $g_i(r) = 1$.*

Proof. For each $i \in N$ with $g_i(r) = 0$, if $r_i \neq 0$, then $g_i(r'_i, r_{-i}) = 0$ for all $0 \leq r'_i < r_i$, which contradicts monotonicity of g. For each $i \in N$ with $g_i(r) = 1$, if $r_i \neq 1$, then $g_i(r'_i, r_{-i}) = 1$ for all $r_i < r'_i \leq 1$, which again contradicts monotonicity of g. $\square$

The following two lemmas are about properties of Nash equilibria for single-dipped preference profiles. We first show that for a monotonic mechanism and a single-dipped preference profile, no agent receives its dip in a Nash equilibrium.

Lemma 3.3 *Let $R_N \in \mathcal{D}^N$ and let g be a monotonic mechanism. If a strategy profile $r^* \in [0,1]^N$ is a Nash equilibrium of (R_N, g), then $g_i(r^*) \neq d(R_i)$ for all $i \in N$.*

Proof. Let $i \in N$. Assume, to the contrary, that $r^* \in [0,1]^N$ with $g_i(r^*) = d(R_i)$, is a Nash equilibrium of (R_N, g). Then we have $g_i(r_i, r^*_{-i}) = d(R_i)$ for all $r_i \in [0,1]$, which is a contradiction to monotonicity of g. □

Next, we show that, in a Nash equilibrium, an agent's strategy is 0 if that agent receives less than its dip, and is 1 if that agent receives more than its dip.

Lemma 3.4 *Let $R_N \in \mathcal{D}^N$, let g be a monotonic mechanism, and let strategy profile $r^* \in [0,1]^N$ be a Nash equilibrium of (R_N, g). Then $r^*_i = 0$ for all $i \in N$ with $g_i(r^*) < d(R_i)$, and $r^*_i = 1$ for all $i \in N$ with $g_i(r^*) > d(R_i)$.*

Proof. Let $i \in N$ with $g_i(r^*) < d(R_i)$. If $g_i(r^*) = 0$, then $r^*_i = 0$ by Lemma 3.2. If $g_i(r^*) > 0$ with $r^*_i \neq 0$, then from monotonicity, we have $g_i(r_i, r^*_{-i}) < g_i(r^*) < d(R_i)$ for all $0 \leq r_i < r^*_i$. This implies that $g_i(r_i, r^*_{-i}) P_i g_i(r^*)$, which is a contradiction to the assumption that r^* is a Nash equilibrium. Hence, $r^*_i = 0$.

The case $g_i(r^*) > d(R_i)$ is analogous. □

We now introduce some additional notation for a mechanism $g \in \mathcal{G}$. For each $S \subseteq N$, define $e^S \in \mathbb{R}^N$ by $e^S_i = 1$ for all $i \in S$, and $e^S_j = 0$ for all $j \in N \setminus S$. Then, by anonymity we have $g_i(e^\emptyset) = g_i(e^N) = \frac{1}{n}$ for all $i \in N$, and there exist numbers $p^1(g), \ldots, p^{n-1}(g) \in [0,1]$ such that for each $\emptyset \neq S \subsetneq N$ and $i \in S$,

$$g_i(e^S) = p^s(g),$$

where $s = |S|$. It follows that for each $\emptyset \neq S \subsetneq N$ and $j \in N \setminus S$,

$$g_j(e^S) = q^s(g),$$

where $sp^s(g) + (n-s)q^s(g) = 1$ for all $s = 1, \ldots, n-1$. When no confusion arises, the notations $p^s(g)$ and $q^s(g)$ are abbreviated to p^s and q^s, respectively. For convenience, we denote $p^0 = p^n = q^0 = q^n = \frac{1}{n}$. Then, by monotonicity and Lemma 3.2, it holds that for each $i \in N$ and $S \subseteq N \setminus \{i\}$,

$$p^{s+1} = g_i(e^{S \cup \{i\}}) > g_i(e^S) = q^s.$$

The following theorem characterizes the Nash equilibria in games induced by single-dipped preference profiles and mechanisms in $\mathcal{G}$.

Theorem 3.5 *Let $R_N \in \mathcal{D}^N$, $g \in \mathcal{G}$, and $r^* \in [0,1]^N$. Then r^* is a Nash equilibrium of (R_N, g) if and only if $r^* = e^S$ for $S \subseteq N$ such that $p^s R_i q^{s-1}$ for all $i \in S$ and $q^s R_j p^{s+1}$ for all $j \in N \setminus S$.*

Proof. For the if-part, assume that $r^* = e^S$ for $S \subseteq N$ such that $p^s R_i q^{s-1}$ for all $i \in S$ and $q^s R_j p^{s+1}$ for all $j \in N \setminus S$. We prove that r^* is a Nash equilibrium.

For each $i \in S$, we have $r_i^* = 1$ and $p^s R_i q^{s-1}$, which means that $g_i(1, r_{-i}^*) R_i g_i(0, r_{-i}^*)$. With monotonicity, it holds that $g_i(0, r_{-i}^*) \leq g_i(r_i, r_{-i}^*) \leq g_i(1, r_{-i}^*)$ for all $r_i \in [0,1]$. According to Lemma 3.1, we conclude that $g_i(r^*) R_i g_i(r_i, r_{-i}^*)$ for all $r_i \in [0,1]$. For each $j \in N \setminus S$, we have $r_j^* = 0$ and $q^s R_j p^{s+1}$, which means that $g_j(0, r_{-j}^*) R_j g_j(1, r_{-j}^*)$. From monotonicity and Lemma 3.1 again, it holds that $g_j(r^*) R_j g_j(r_j, r_{-j}^*)$ for all $r_j \in [0,1]$. So, $r^* = e^S$ is a Nash equilibrium.

For the only-if part, assume that r^* is a Nash equilibrium. From Lemmas 3.3 and 3.4, we have $r^* = e^S$ for some $S \subseteq N$. In view of $g_i(r^*) R_i g_i(0, r_{-i}^*)$ for all $i \in S$ and $g_j(r^*) R_j g_j(1, r_{-j}^*)$ for all $j \in N \setminus S$, it holds that $p^s R_i q^{s-1}$ for all $i \in S$ and $q^s R_j p^{s+1}$ for all $j \in N \setminus S$. □

Theorem 3.5 can also be used to show that a Nash equilibrium does not have to exist, as for instance in the following example.

Example 3.6 Let $N = \{1,2,3\}$ and let $g \in \mathcal{G}$ satisfy that $p^2 > p^1$. By this assumption and monotonicity, it follows that $q^2 < q^1 < \frac{1}{3} < p^1 < p^2$. Consider $R_N \in \mathcal{D}^N$ such that $q^2 P_1 q^1 P_1 p^2 P_1 p^1 P_1 \frac{1}{3}$, $p^2 P_2 q^2 P_2 q^1 P_2 \frac{1}{3} P_2 p^1$ and $q^1 P_3 p^2$. Then $e^{\emptyset}$ is not a Nash equilibrium in view of $p^1 P_1 \frac{1}{3}$; $e^{\{1\}}$ and $e^{\{3\}}$ are not Nash equilibria in view of $p^2 P_2 q^1$; $e^{\{2\}}$ is not a Nash equilibrium in view of $\frac{1}{3} P_2 p^1$; $e^{\{1,2\}}$ and $e^{\{1,3\}}$ are not Nash equilibria in view of $q^1 P_1 p^2$; $e^{\{2,3\}}$ is not a Nash equilibrium in view of $q^1 P_3 p^2$; and e^N is not a Nash equilibrium in view of $q^2 P_1 \frac{1}{3}$. From Theorem 3.5 it follows that the game (R_N, g) has no Nash equilibrium.

A possible mechanism $g \in \mathcal{G}$ to which this example applies is as follows. For each $r \in [0,1]^N$ and distinct $i, j, k \in N$ let

$$g_i(r) = \frac{8 + 2r_i - r_j - r_k + 2r_i r_j + 2r_i r_k - 4r_j r_k}{24}.$$

Since $g(1,0,0) = (\frac{10}{24}, \frac{7}{24}, \frac{7}{24})$ and $g(1,1,0) = (\frac{11}{24}, \frac{11}{24}, \frac{2}{24})$, we have $q^2 = \frac{2}{24} < q^1 = \frac{7}{24} < \frac{1}{3} < p^1 = \frac{10}{24} < p^2 = \frac{11}{24}$. △

We conclude this section with the result that for two agents a Nash equilibrium always exists. For the proof, see the Appendix.

Proposition 3.7 *Let $N = \{1,2\}$, $R_N \in \mathcal{D}^N$ and $g \in \mathcal{G}$. Then the game (R_N, g) has a Nash equilibrium.*

4 Pareto optimal Nash equilibrium, strong equilibrium, and implementation

In this section, we first consider Pareto optimal Nash equilibria, i.e., Nash equilibria resulting in Pareto optimal allocations. Next, we strengthen this to strong equilibria: no subset of agents can profitably deviate, in the sense that every member is at least as well off, and at least one member is better off. Third, we discuss the related issue of implementation: which social choice correspondence, i.e., multi-valued rule, collects exactly the Pareto optimal Nash equilibria or strong equilibria for a given mechanism?

4.1 Pareto optimal Nash equilibrium

Pareto optimal allocations for single-dipped preference profiles were characterized by Klaus et al (1997). For each $R_N \in \mathcal{D}^N$, we denote by $N_+(R_N) = \{i \in N \mid 1P_i0\}$ the set of agents who strictly prefer 1 to 0, by $N_0(R_N) = \{i \in N \mid 0I_i1\}$ the set of agents who are indifferent between 0 and 1, and by $N_-(R_N) = \{i \in N \mid 0P_i1\}$ the set of agents who strictly prefer 0 to 1. The characterization by Klaus et al (1997) is as follows.

Lemma 4.1 *Let $R_N \in \mathcal{D}^N$. An allocation $x \in \mathcal{A}$ is Pareto optimal at R_N if and only if*

(i) *If $N_+(R_N) \neq \emptyset$, then $x_i = 0$ for every $i \in N \setminus N_+(R_N)$, and for every $i \in N_+(R_N)$ either $x_i = 0$ or x_iP_i0.*

(ii) *If $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$, then $x = e^{\{i\}}$ for some $i \in N_0(R_N)$.*

(iii) *If $N_-(R_N) = N$, then for every $i \in N$ either $x_i = 1$ or x_iP_i1.*

We first introduce so-called maximal coalitions, which are useful to describe Pareto optimal Nash equilibria.

Definition 4.2 Let $R_N \in \mathcal{D}^N$.

(a) The *sharing index* of an agent $i \in N$ at R_N is the number $m_i(R_N)$ defined by

$$m_i(R_N) = \begin{cases} 0 & \text{if } i \notin N_+(R_N) \\ \max\left\{k \in \{1, \ldots, |N_+(R_N)|\} \;\middle|\; \frac{1}{k}P_i0\right\} & \text{if } i \in N_+(R_N). \end{cases}$$

(b) A coalition $S \subseteq N$ is a *maximal coalition* at R_N if the following holds.

(i) If $N_+(R_N) \neq \emptyset$, then $S \subseteq N_+(R_N)$ such that $m_i(R_N) \geq |S|$ for every $i \in S$ and $m_j(R_N) \leq |S|$ for every $j \in N \setminus S$.

(ii) If $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$, then $S = \{i\}$ for some $i \in N_0(R_N)$.

(iii) If $N_-(R_N) = N$, and $\{j \in N \mid 1R_j\frac{1}{n}\} \neq \emptyset$, then $S = \{i\}$ for some $i \in N$ with $1R_i\frac{1}{n}$.

(iv) If $N_-(R_N) = N$, and $\{j \in N \mid 1R_j\frac{1}{n}\} = \emptyset$, then $S = \emptyset$.

The collection of all maximal coalitions at R_N is denoted by $\mathcal{M}(R_N)$.

$\triangle$

The sharing index of an agent $i \in N_+(R_N)$ is the maximal size of a coalition of agents strictly preferring one over zero, including i, such that equally sharing the commodity with the members of this coalition is still preferable over receiving 0. For $i \notin N_+(R_N)$, this is zero. In Case (i) in (b), a maximal coalition consists of agents who strictly prefer 1 over 0 at R_N. Such a coalition is formed by starting with the agent(s) with maximal sharing index, next adding agent(s) with second maximal sharing index, etc., until the size of the coalition exceeds the sharing indices of the remaining agents. See Example 4.3 for an illustration. In a similar spirit, in Case (ii), a maximal coalition consists of any arbitrary single agent indifferent between 0 and 1. In Case (iii), where all agents strictly prefer 0 over 1, a maximal coalition consists of an arbitrary single agent who (weakly) prefers 1 over $\frac{1}{n}$. If there are no such agents, then Case (iv) applies and the only maximal coalition is the empty coalition.

Example 4.3 Let $N = \{1, 2, 3\}$ and let R_N satisfy $\frac{1}{3}P_1 0$, $\frac{1}{2}P_i 0$ and $0R_i\frac{1}{3}$ for $i = 2, 3$. Then $N_+(R_N) = N$, $m_1(R_N) = 3$, and $m_2(R_N) = m_3(R_N) = 2$. To construct a maximal coalition we start with agent 1 and then add either agent 2 or agent 3, to obtain $\{1, 2\}$ and $\{1, 3\}$ as maximal coalitions. Coalition $\{2, 3\}$ is not maximal since $m_1(R_N) = 3 > 2 = |\{2, 3\}|$, and coalition N is not maximal since $m_2(R_N) = 2 < 3 = |N|$. Also singleton coalitions are not maximal: $\{1\}$ is not maximal since $m_2(R_N) = 2 > |\{1\}|$, $\{2\}$ is not maximal since $m_1(R_N) = 3 > |\{2\}|$, and $\{3\}$ is not maximal since $m_1(R_N) = 3 > |\{3\}|$. $\triangle$

The basic reason why maximal coalitions play a role in our analysis, especially in the case where $N_+(R_N) \neq \emptyset$, is that a member of such a coalition prefers receiving an equal share over receiving 0 and therefore would not deviate and leave the coalition; on the other hand, there is no outside agent who would gain by joining the coalition. This will be made precise in Theorem 4.5.

We first formulate an additional property for a mechanism g.

Standardness g is *standard* if $g(e^S) = \frac{1}{|S|}e^S$ for every $\emptyset \neq S \subseteq N$.

If g is standard, then $p^s = \frac{1}{s}$ and $q^s = 0$ for each $s = 1, 2, \ldots, n-1$. The mechanisms in Examples 2.1 and 2.2 are standard, but the mechanism in Example 3.6 is not standard.

We show that standardness of a mechanism is a necessary and sufficient condition for all games based on this mechanism to have a Pareto optimal Nash equilibrium.

Lemma 4.4 *Let $g \in \mathcal{G}$ and suppose that (R_N, g) has a Pareto optimal Nash equilibrium for each $R_N \in \mathcal{D}^N$. Then g is standard.*

Proof. For each $S \in 2^N \setminus \{\emptyset, N\}$, we consider $R_N^S \in \mathcal{D}^N$ such that $d(R_i^S) = 0$ for all $i \in S$ and $d(R_j^S) = 1$ for all $j \in N \setminus S$. Then $N_+(R_N^S) = S$. From Lemmas 3.3 and 3.4, it follows that the only Nash equilibrium in the game (R_N^S, g) is $r^* = e^S$. From Lemma 4.1, we have $g_j(r^*) = 0$ for all $j \in N \setminus N_+(R_N^S) = N \setminus S$. It follows that $g_i(r^*) = \frac{1}{|S|}$ for all $i \in S$. Together with $g(e^N) = \frac{1}{|N|}$, we conclude that $g(e^S) = \frac{1}{|S|} e^S$ for all $S \in 2^N \setminus \{\emptyset\}$. This implies that g is standard. $\square$

Lemma 4.4 says that standardness of the mechanism is a necessary condition for a Pareto optimal Nash equilibrium to exist in every game induced by this mechanism. The sufficiency part follows from the following theorem, which is a main result of this paper.

Theorem 4.5 *Let $R_N \in \mathcal{D}^N$ and let $g \in \mathcal{G}$ be standard. A strategy profile $r^* \in [0,1]^N$ is a Pareto optimal Nash equilibrium of (R_N, g) if and only if $r^* = e^S$ for some $S \in \mathcal{M}(R_N)$.*

Proof. For the if-part, let $S \in \mathcal{M}(R_N)$. We prove that $r^* = e^S$ is a Pareto optimal Nash equilibrium.

Case (i): $N_+(R_N) \neq \emptyset$.

Let $i \in S$. Then $r_i^* = 1$ and $g_i(r^*) = \frac{1}{|S|}$. Since $\frac{1}{|S|} \geq \frac{1}{m_i(R_N)}$ and $\frac{1}{m_i(R_N)} P_i 0$, we have that $\frac{1}{|S|} P_i 0$, which implies that $g_i(r^*) P_i g_i(0, r_{-i}^*)$. Monotonicity then implies $g_i(0, r_{-i}^*) \leq g_i(r_i, r_{-i}^*) \leq g_i(r^*)$ for all $r_i \in [0,1]$, and by Lemma 3.1, $g_i(r^*) R_i g_i(r_i, r_{-i}^*)$ for all $r_i \in [0,1]$.

If $i \in N_+(R_N) \setminus S$, then $r_i^* = 0$ and $g_i(r^*) = 0$. In view of $|S| + 1 > m_i(R_N)$, it holds that $|S| + 1 \geq m_i(R_N) + 1$, i.e., $\frac{1}{|S|+1} \leq \frac{1}{m_i(R_N)+1}$. Together with $0 R_i \frac{1}{m_i(R_N)+1}$, by Lemma 3.1, we have $0 R_i \frac{1}{|S|+1}$, which implies that $g_i(r^*) R_i g_i(1, r_{-i}^*)$. With monotonicity and Lemma 3.1 again, we can similarly verify that $g_i(r^*) R_i g_i(r_i, r_{-i}^*)$ for all $r_i \in [0,1]$.

For $i \in N \setminus N_+(R_N)$, in view of $S \subseteq N_+(R_N)$, we have $i \in N \setminus S$, $r_i^* = 0$ and $g_i(r^*) = 0$. In view of $0 R_i 1$, by Lemma 3.1, we have $g_i(r^*) R_i g_i(r_i, r_{-i}^*)$ for all $r_i \in [0,1]$.

Thus, $g_i(r^*) R_i g_i(r_i, r_{-i}^*)$ for all $i \in N$ and $r_i \in [0,1]$, which implies that $r^* = e^S$ is a Nash equilibrium.

Case (ii): $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

Let $S = \{i\}$ with $i \in N_0(R_N)$. Then $g_i(r^*) = 1$ and $g_j(r^*) = 0$ for all $j \in N \setminus \{i\}$. For agent i, in view of $1 R_i 0$, by Lemma 3.1, it holds that $g_i(r^*) R_i g_i(r_i, r_{-i}^*)$ for all $r_i \in [0,1]$. For each agent $j \in N \setminus \{i\}$, in view of $0 R_j 1$, by Lemma 3.1 again, we have $g_j(r^*) R_j g_j(r_j, r_{-j}^*)$ for all $r_j \in [0,1]$. So, $r^* = e^{\{i\}}$ is a Nash equilibrium.

Case (iii): $N_-(R_N) = N$.

If $S = \{i\}$ for some $i \in N$, then $1R_i\frac{1}{n}$, Then $r_j^* = 0$ and $g_j(r^*) = 0$ for all $j \in N \setminus \{i\}$. In view of $0P_j1$, by Lemma 3.1, we have $g_j(r^*)R_jg_j(r_j, r_{-j}^*)$ for all $r_j \in [0,1]$. With monotonicity, we have $g_i(r_i, r_{-i}^*) \geq \frac{1}{n}$ for all $r_i \in (0,1]$. In view of $1R_i\frac{1}{n}$, by Lemma 3.1 again, we have $g_i(r^*)R_ig_i(r_i, r_{-i}^*)$ for all $r_i \in [0,1]$. So, $r^* = e^{\{i\}}$ is a Nash equilibrium.

If $S = \emptyset$, then $\frac{1}{n}P_i1$ for all $i \in N$. For each $i \in N$, if $r_i > 0$, with monotonicity, we have $g_i(r_i, r_{-i}^*) > g_i(0, r_{-i}^*) = \frac{1}{n}$. Together with $\frac{1}{n}P_i1$, by Lemma 3.1, we have $g_i(r^*)R_ig_i(r_i, r_{-i}^*)$ for all $r_i \in [0,1]$. So, $r^* = e^{\emptyset}$ is a Nash equilibrium.

Combining these three cases, we conclude that for each $S \in \mathcal{M}(R_N)$, $r^* = e^S$ is a Nash equilibrium. Lemma 4.1 implies that $g(r^*)$ is Pareto optimal at R_N.

For the only-if part, assume that r^* is a Pareto optimal Nash equilibrium. From Theorem 3.5, it follows that $r^* = e^S$ for some $S \in 2^N$. We prove that $S \in \mathcal{M}(R_N)$.

Case (i): $N_+(R_N) \neq \emptyset$.

Assume, to the contrary, that $S \notin \mathcal{M}(R_N)$. Let $T \in \mathcal{M}(R_N)$. First, we prove that $|S| = |T|$.

Since $g(e^S)$ and $g(e^T)$ are Pareto optimal at R_N, from Lemma 4.1, we have $\frac{1}{|S|}P_i0$ for all $i \in S$, and $\frac{1}{|T|}P_i0$ for all $i \in T$. Since e^S (by assumption) and e^T (from the if-part) are Nash equilibria of (R_N, g), we have $0R_j\frac{1}{|S|+1}$ for all $j \in N \setminus S$, and $0R_j\frac{1}{|T|+1}$ for all $j \in N \setminus T$. If $|S| < |T|$, then there exists $k \in T \setminus S$ such that $\frac{1}{|T|}P_k0$ and $0R_k\frac{1}{|S|+1}$. However, in view of $|S| < |T|$, we have $|S| + 1 \leq |T|$, i.e., $\frac{1}{|T|} \leq \frac{1}{|S|+1}$. From Lemma 3.1, it follows that $0R_k\frac{1}{|T|}$, which is a contradiction. If $|S| > |T|$, we similarly obtain a contradiction. Thus, $|S| = |T|$.

Then, since $S \notin \mathcal{M}(R_N)$ and $|S| = |T|$, there exist $i \in S$ and $j \in N \setminus S$ such that $m_i(R_N) < m_j(R_N)$. By Lemma 4.1, we have $\frac{1}{|S|}P_i0$. It follows that $|S| \leq m_i(R_N)$. So, $|S| < m_j(R_N)$, i.e., $\frac{1}{|S|+1} \geq \frac{1}{m_j(R_N)}$. In view of $\frac{1}{m_j(R_N)}P_j0$, we have $\frac{1}{|S|+1}P_j0$. This implies that $g_j(1, e_{-j}^S)P_jg_j(e^S)$, which contradicts the assumption that e^S is a Nash equilibrium. Thus, $S \in \mathcal{M}(R_N)$.

Case (ii): $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

From Lemma 4.1, $g(e^T)$ is not Pareto optimal for all $T \in 2^N \setminus \mathcal{M}(R_N)$. Thus, $S \in \mathcal{M}(R_N)$.

Case (iii): $N_-(R_N) = N$.

If there exists $i \in N$ such that $1R_i\frac{1}{n}$, then $e^{\emptyset}$ is not a Pareto optimal Nash equilibrium, hence $S \neq \emptyset$. Since $0P_j1$ for all $j \in N$, it follows that e^T is not a Nash equilibrium for each $T \in 2^N$ with $|T| \geq 2$. Hence, $|S| = 1$. For $j \in N$ such that $\frac{1}{n}P_j1$, it is easily seen that $e^{\{j\}}$ is not a Nash equilibrium. Thus, $S \in \mathcal{M}(R_N)$.

Finally, suppose that $\{i \in N \mid 1R_i\frac{1}{n}\} = \emptyset$, i.e., $\frac{1}{n}P_i1$ for all $i \in N$. If $T \neq \emptyset$, then since $0P_i1$ and $\frac{1}{n}P_i1$ for all $i \in N$, it follows that $g_i(e^{T\setminus\{i\}})P_ig_i(e^T)$ for all $i \in T$, which implies that e^T is not a Nash equilibrium. So, $S = \emptyset \in \mathcal{M}(R_N)$, and the proof of the theorem is complete. $\square$

Theorem 4.5 shows that for a standard mechanism, the Pareto optimal Nash equilibria are those strategy profiles in which all agents in a maximal coalition play 1 and all other agents play 0. Since there exists at least one maximal coalition for every single-dipped preference profile, Lemma 4.4 and Theorem 4.5 imply the result announced earlier.

Corollary 4.6 *Let $g \in \mathcal{G}$. There exists a Pareto optimal Nash equilibrium of (R_N, g) for every $R_N \in \mathcal{D}^N$ if and only if g is standard.*

The next example shows that for a game based on a standard mechanism, besides Pareto optimal Nash equilibria, there may exist Nash equilibria without Pareto optimal outcomes, or Pareto optimal outcomes, not obtained in any Nash equilibrium.

Example 4.7 Let $N = \{1, 2\}$ and let $g \in \mathcal{G}$ be as in Example 2.1.

(a) Consider $R_N \in \mathcal{D}^N$ such that $1P_1 0 P_1 \frac{1}{2}$ and $0P_2 1 P_2 \frac{1}{2}$. Then, we have $g_1(0,1)P_1 g_1(1,1)$ and $g_2(0,1)P_2 g_2(0,0)$. With monotonicity and Lemma 3.1, it follows that $g_1(0,1)R_1 g_1(r_1,1)$ and $g_2(0,1)R_2 g_2(0,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $e^{\{2\}} = (0,1)$ is a Nash equilibrium. However, $g(e^{\{2\}}) = (0,1)$ is not Pareto optimal at R_N. In fact, Theorem 4.5 implies that the unique Pareto optimal Nash equilibrium is $e^{\{1\}} = (1,0)$.

(b) Consider $R_N \in \mathcal{D}^N$ such that $d(R_1) = d(R_2) = 0$. Then $x = g(\frac{1}{2}, \frac{1}{3}) = (\frac{7}{12}, \frac{5}{12})$ is Pareto optimal at R_N, but there is no $S \in 2^N$ such that $g(e^S) = x$. Thus, Theorem 3.5 implies that there is no Nash equilibrium r^* such that $g(r^*) = x$. In fact, $m_1(R_N) = m_2(R_N) = 2$, and hence the unique maximal coalition is N. From Theorem 4.5 (or direct inspection), the unique Pareto optimal Nash equilibrium is $e^N = (1,1)$. $\triangle$

4.2 Strong equilibrium

In this subsection we consider a further strengthening of Pareto optimal Nash equilibrium, namely strong equilibrium (Aumann, 1959): no coalition can profitably deviate. We will show that the Pareto optimal Nash equilibria and strong equilibria coincide if, besides anonymous, monotonic, and standard, the mechanism is order preserving.[1]

Order preservation A mechanism g is *order preserving* if $g_i(r) \geq g_j(r)$ for all $r \in [0,1]^N$ and $i, j \in N$ with $r_i \geq r_j$.

Theorem 4.8 *Let $R_N \in \mathcal{D}^N$ and let $g \in \mathcal{G}$ be standard and order preserving. Then a strategy profile is a Pareto optimal Nash equilibrium of (R_N, g) if and only if it is a strong equilibrium.*

[1] A similar condition also occurs in Bochet et al (2021) under the name 'peak order preservation'.

Proof. We start with the only-if part. Let $S \in \mathcal{M}(R_N)$. By Theorem 4.5, it is sufficient to verify that e^S is a strong equilibrium. Assume, to the contrary, that there exist $T \in 2^N \setminus \{\emptyset\}$ and $r_T \in [0,1]^T$ such that $g_i(r_T, e^S_{N\setminus T}) R_i g_i(e^S)$ for all $i \in T$ and $g_j(r_T, e^S_{N\setminus T}) P_j g_j(e^S)$ for some $j \in T$. We consider three cases.

Case (i): $N_+(R_N) \neq \emptyset$.

If $S \cap T \neq \emptyset$, then for each $i \in S \cap T$, it holds that $g_i(r_T, e^S_{N\setminus T}) \geq \frac{1}{|S|}$ in view of $\frac{1}{|S|} P_i 0$ from Theorem 4.5 and $g_i(r_T, e^S_{N\setminus T}) R_i g_i(e^S)$ by assumption. By order preservation, it follows that $g_j(r_T, e^S_{N\setminus T}) \geq g_i(r_T, e^S_{N\setminus T}) \geq \frac{1}{|S|}$ for all $j \in S \setminus T$. So, we have $g_i(r_T, e^S_{N\setminus T}) = \frac{1}{|S|} = g_i(e^S)$ for all $i \in S$, and $g_j(r_T, e^S_{N\setminus T}) = 0 = g_j(e^S)$ for all $j \in N \setminus S$, i.e., $g_k(r_T, e^S_{N\setminus T}) I_k g_k(e^S)$ for all $k \in T$, contradicting our assumption.

If $S \cap T = \emptyset$, then we claim that $g_i(r_T, e^S_{N\setminus T}) \leq \frac{1}{|S|+1}$ for each $i \in T$. If not, take $i \in T$ with $g_i(r_T, e^S_{N\setminus T}) > \frac{1}{|S|+1}$. Then $g_j(r_T, e^S_{N\setminus T}) \geq g_i(r_T, e^S_{N\setminus T}) > \frac{1}{|S|+1}$ for all $j \in S$. It follows that $\sum_{k \in T\cup S} g_k(r_T, e^S_{N\setminus T}) > 1$, which is not possible. In view of $0 R_i \frac{1}{|S|+1}$ from Theorem 4.5, together with Lemma 3.1, we have $g_i(e^S) R_i g_i(r_T, e^S_{N\setminus T})$ for all $i \in T$, which contradicts our assumption.

Case (ii): $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

In this case, $S = \{i\}$ for some $i \in N_0(R_N)$. Then $g(e^S) = e^{\{i\}}$. Since $1 I_i 0$ and $0 R_j 1$ for all $j \in N \setminus \{i\}$, by Lemma 3.1 we have $g_k(e^S) R_i g_k(r_T, e^S_{N\setminus T})$ for all $k \in T$, which is a contradiction to our assumption.

Case (iii): $N_-(R_N) = N$.

If $S = \{i\}$ for some $i \in N$, then $1 R_i \frac{1}{n}$. It follows that $g_j(e^S) = 0$ for all $j \in N \setminus \{i\}$. Since $e^{\{i\}}$ is a Nash equilibrium, it holds that $T \neq \{i\}$. For each $k \in T \setminus \{i\}$, we have $g_k(e^S) R_k g_k(r_T, e^S_{N\setminus T})$ from $0 P_k 1$ and Lemma 3.1. Together with our assumption, it follows that $g_k(r_T, e^S_{N\setminus T}) = g_k(e^S) = 0$ for all $k \in T \setminus \{i\}$. By order preservation, we have $g_j(r_T, e^S_{N\setminus T}) = 0$ for all $j \in N \setminus \{i\}$. So, $g(r_T, e^S_{N\setminus T}) = g(e^S)$, which is a contradiction.

If $S = \emptyset$, then $\frac{1}{n} P_i 1$ for all $i \in N$. For each $k \in T$, in view of $g_k(r_T, e^S_{N\setminus T}) R_k g_k(e^S)$ and $0 P_k \frac{1}{n} P_k 1$, we have $g_k(r_T, e^S_{N\setminus T}) \leq \frac{1}{n}$. By order preservation, it holds that $g_j(r_T, e^S_{N\setminus T}) \leq g_k(r_T, e^S_{N\setminus T}) \leq \frac{1}{n}$ for all $j \in N \setminus T$ and $k \in T$. So, $g_k(r_T, e^S_{N\setminus T}) = g_k(e^S) = \frac{1}{n}$ for all $k \in T$, which is a contradiction. This concludes the proof of the only-if part.

For the if-part, suppose that r^* is a strong equilibrium of (R_N, g). Obviously, r^* is a Nash equilibrium. By Theorem 3.5, there is a coalition S such that $r^* = e^S$. Since g is standard, we have $g(e^S) = \frac{1}{|S|} e^S$ if $S \neq \emptyset$. If $S = \emptyset$, then $g(e^S) = \frac{1}{n} e^N$.

If $S = \emptyset$, then, since e^S is a Nash equilibrium, we have $\frac{1}{n} R_i 1$ for all $i \in N$, which implies that $g(e^S) = \frac{1}{n} e^N$ is Pareto optimal.

If $|S| \geq 2$, then, again since e^S is a Nash equilibrium, $\frac{1}{|S|} R_i 0$ for all $i \in S$; in this case, if $x_i R_i g_i(e^S)$ for some $x \in \mathcal{A}$ and all $i \in N$, then in particular $x_i \geq \frac{1}{|S|}$ for all $i \in S$, which implies $x = g(e^S)$ and, thus, $g(e^S)$ is Pareto optimal.

Finally, suppose that $|S| = 1$, say $S = \{n\}$.

If $1P_n0$ then clearly $g(e^S) = (0, \dots, 0, 1)$ is Pareto optimal.

If $1I_n0$ and there is some $j \neq n$ with $1P_j0$, then $\{j, n\}$ can profitably deviate by $r_j = 1$ and $r_n = 0$, contradicting that e^S is a strong equilibrium; hence, $0R_j1$ for all $j \neq n$, so that $g(e^S) = (0, \dots, 0, 1)$ is Pareto optimal.

If $0P_n1$ and there is some $j \neq n$ with $1R_j0$, then $\{j, n\}$ can profitably deviate by $r_j = 1$ and $r_n = 0$, contradicting that e^S is a strong equilibrium; hence, $0P_j1$ for all $j \neq n$, so that $g(e^S) = (0, \dots, 0, 1)$ is Pareto optimal. This concludes the proof of the if-part. □

4.3 Implementation

In this subsection we reformulate our main results in terms of implementation. A *social choice correspondence* F is a map assigning to each preference profile $R_N \in \mathcal{D}^N$ a nonempty set of allocations. If this set always consists of exactly one allocation, then F is a rule, as defined earlier in Section 2. We say that a mechanism g *implements* F *in Pareto optimal Nash equilibrium* if

$$F(R_N) = \{g(r) \in \mathcal{A} \mid r \text{ is a Pareto optimal Nash equilibrium of } (R_N, g)\}$$

for every preference profile $R_N \in \mathcal{D}^N$. Mechanism g *implements* F *in strong equilibrium* if

$$F(R_N) = \{g(r) \in \mathcal{A} \mid r \text{ is a strong equilibrium of } (R_N, g)\}$$

for every preference profile $R_N \in \mathcal{D}^N$. For each $S \subseteq N$ define the allocation $\hat{e}^S \in \mathcal{A}$ by

$$\hat{e}^S = \begin{cases} \frac{1}{|S|} e^S & \text{if } S \neq \emptyset \\ (\frac{1}{n}, \dots, \frac{1}{n}) & \text{if } S = \emptyset. \end{cases}$$

Define the social choice correspondence M on $\mathcal{D}^N$ by

$$M(R_N) = \{\hat{e}^S \in \mathcal{A} \mid S \in \mathcal{M}(R_N)\}$$

for every $R_N \in \mathcal{D}^N$. We now have the following corollary from Theorems 4.5 and 4.8.

Corollary 4.9 *Let $g \in \mathcal{G}$. If g is standard, then g implements M in Pareto optimal Nash equilibrium. If g is standard and order preserving, then g implements M in strong equilibrium.*

5 An axiomatic characterization of the correspondence M

In this section we present an axiomatic characterization of the correspondence M, i.e., the correspondence implemented in Pareto optimal Nash or strong equilibrium as in Corollary 4.9.

Unless stated otherwise, F is a social choice correspondence defined on $\mathcal{D}^N$. In order to formulate the first axiom we define the concept of an envious agent. Let $R_N \in \mathcal{D}^N$ and $x \in \mathcal{A}$. An agent $i \in N$ is (an) *envious (agent) at R_N and x* if $x_j P_i x_i$ for some $j \in N$. We denote by $E(R_N, x)$ the set of all envious agents at R_N and x. By $PO(R_N)$ we denote the set of all Pareto optimal allocations at R_N.

The first axiom requires that F assigns Pareto optimal allocations and, among those, only allocations with a minimal number of envious agents.

Minimal envy Pareto optimality $x \in PO(R_N)$ and $|E(R_N, x)| \leq |E(R_N, y)|$ for all $R_N \in \mathcal{D}^N$, $x \in F(R_N)$, and $y \in PO(R_N)$.

The second condition requires that every agent (weakly) prefers an assigned allocation over equal division. Under the same name, this condition occurs in Thomson (2010) for the case of single-peaked preferences; it also occurs already in Pazner (1977) under the name 'per-capita-fairness'.

Equal division lower bound $x_i R_i \frac{1}{n}$ for all $R_N \in \mathcal{D}_N$, $x \in F(R_N)$, and $i \in N$.

The third and final condition requires that a higher sharing index cannot result in a lower share. A higher sharing index expresses more eagerness to receive a nonzero share, and this should not result in a lower share.

Sharing index order preservation $x_i \geq x_j$ for all $R_N \in \mathcal{D}^N$, $x \in F(R_N)$, and $i, j \in N$ with $m_i(R_N) > m_j(R_N)$.

Our characterization result says that M is the maximal correspondence with these three properties.

Theorem 5.1 *A social choice correspondence F satisfies minimal envy Pareto optimality, equal division lower bound, and sharing index order preservation, if and only if $F(R_N) \subseteq M(R_N)$ for all $R_N \in \mathcal{D}^N$.*

The proof of this theorem is based on a number of lemmas. We first show that M satisfies the three axioms in the theorem.

Lemma 5.2 *The social choice correspondence M on $\mathcal{D}^N$ is minimal envy Pareto optimal.*

Proof. Let $R_N \in \mathcal{D}^N$ and $S \in \mathcal{M}(R_N)$. Then, by Corollary 4.9, $\hat{e}^S$ is Pareto optimal. Denote

$$\mu(R_N) = \min_{y \in PO(R_N)} |E(R_N, y)|.$$

We have to show that $|E(R_N, \hat{e}^S)| = \mu(R_N)$. To this end, we distinguish four cases.

Case (i): $N_+(R_N) \neq \emptyset$.

Denote

$$S^* = \{i \in N_+(R_N) \setminus S \mid m_i(R_N) = |S|\}.$$

We claim that $E(R_N, \hat{e}^S) = S^*$.

To prove this claim, first observe that there is no envious agent in S since $\frac{1}{|S|} R_i \frac{1}{m_i(R_N)} P_i 0$ for all $i \in S$, and there is also no envious agent in $N \setminus N_+(R_N)$ since $0 P_i \frac{1}{|S|}$ for all $i \in N \setminus N_+(R_N)$. For the agents in $N_+(R_N) \setminus S$, we consider the following three subcases.

(i.a) $N_+(R_N) \setminus S = \emptyset$.

In this case, $S^* = \emptyset$, and $\hat{e}^S_i = \frac{1}{|S|} = \frac{1}{m_i(R_N)} = \frac{1}{|N_+(R_N)|}$ for all $i \in S$. This, together with $\frac{1}{m_i(R_N)} P_i 0$ for all $i \in S$, implies that there is no envious agent in $N_+(R_N)$. Hence, $E(R_N, \hat{e}^S) = \emptyset = S^*$.

(i.b) $N_+(R_N) \setminus S \neq \emptyset$ and $S^* = \emptyset$.

Let $i \in N_+(R_N) \setminus S$. Then $m_i(R_N) < |S|$, hence $\frac{1}{m_i(R_N)+1} \geq \frac{1}{|S|} > 0$. This and $0 R_i \frac{1}{m_i(R_N)+1}$, imply that $0 R_i \frac{1}{|S|}$, hence there is no envious agent in $N_+(R_N) \setminus S$. Hence, again $E(R_N, \hat{e}^S) = \emptyset = S^*$.

(i.c) $N_+(R_N) \setminus S \neq \emptyset$ and $S^* \neq \emptyset$.

Every agent $i \in S^*$ is an envious agent in view of $\frac{1}{|S|} P_i 0$. If $i \in N_+(R_N) \setminus (S \cup S^*)$, then $m_i(R_N) < |S|$, and similarly as in case (i.b), i is not envious. Hence, also in this case, $E(R_N, \hat{e}^S) = S^*$.

Hence, $E(R_N, \hat{e}^S) = S^*$ in all three subcases.

To complete the proof for Case (i), we show that $\mu(R_N) = |S^*|$.

Assume, to the contrary, that there exists $x \in PO(R_N)$ such that $|E(R_N, x)| < |S^*|$. If there exists $j \in S \cup S^*$ such that $x_j > \frac{1}{|S|}$, then

$$|\{i \in S \cup S^* \mid x_i = \max_{k \in N} x_k\}| \leq |S| - 1.$$

Since $\frac{1}{|S|} P_i 0$ for all $i \in S \cup S^*$, it follows that $(\max_{k \in N} x_k) P_i x_i$ for all $i \in S \cup S^*$ with $x_i \neq \max_{k \in N} x_k$. This means that at least $|S^*| + 1$ agents in $S \cup S^*$ are envious at x, which is a contradiction. Hence, $x_i \leq \frac{1}{|S|}$ for all $i \in S \cup S^*$.

Suppose there exists $j \in N_+(R_N) \setminus (S \cup S^*)$ with $x_j > 0$. Since $m_j(R_N) < |S|$ and $0 R_j \frac{1}{m_j(R_N)+1}$, we have $\frac{1}{m_j(R_N)+1} \geq \frac{1}{|S|}$ and $0 R_j \frac{1}{|S|}$. Hence, since $x \in PO(R_N)$, $x_j > \frac{1}{|S|}$. It follows that

$$|\{i \in S \cup S^* \mid x_i \geq x_j\}| \leq |S| - 1,$$

which implies that at least $|S^*|+1$ agents in $S \cup S^*$ are envious at x, which is a contradiction. Thus, $x_j = 0$ and therefore $\sum_{i \in S \cup S^*} x_i = 1$.

Next, suppose there exists $j \in S^*$ with $0 < x_j < \frac{1}{|S|}$. By Lemma 4.1(i), and since $m_j(R_N) = |S|$, we have $x_j > \frac{1}{|S|+1}$. Hence,

$$|\{i \in S \cup S^* \mid x_i = \max_{k \in N} x_k\}| \leq |S|,$$

and the number of envious agents at x is at least $|S^*|$, contradicting the assumption $|E(R_N, x)| < |S^*|$. Thus, $\mu(R_N) = |S^*|$, and the proof of Case (i) is complete.

Case (ii): $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

In this case, $S = \{i\}$ for some $i \in N_0(R_N)$, and $|E(R_N, \hat{e}^S)| = \mu(R_N) = 0$.

Case (iii): $N_-(R_N) = N$ and $\{j \in N \mid 1 R_j \frac{1}{n}\} \neq \emptyset$.

In this case, $S = \{j\}$ for some $j \in N$ such that $1 R_j \frac{1}{n}$. Since $E(R_N, \hat{e}^S) = \{j\}$, it follows that $\mu(R_N) \leq 1$. We show that $\mu(R_N) = 1$.

Consider any $x \in PO(R_N)$. If $x_i = 1$ for some agent i, then i is an envious agent. Otherwise, from Lemma 4.1(iii), we have $x_j P_j 1$ for all $j \in N$. Since $(\frac{1}{n}, \ldots, \frac{1}{n}) \notin PO(R_N)$, there exist distinct $k, l \in N$ such that $x_k < x_l$. It follows that $x_k P_l x_l P_l 1$, which means that l is an envious agent. Hence, $\mu(R_N) = 1$.

Case (iv): $N_-(R_N) = N$ and $\{j \in N \mid 1 R_j \frac{1}{n}\} = \emptyset$.

Then $S = \emptyset$, and $|E(R_N, \hat{e}^S)| = |E(R_N, (\frac{1}{n}, \ldots, \frac{1}{n}))| = 0 = \mu(R_N)$.

This completes the proof of the lemma. $\square$

Lemma 5.3 *The social choice correspondence M on $\mathcal{D}^N$ satisfies equal division lower bound.*

Proof. Let $R_N \in \mathcal{D}^N$ and $S \in \mathcal{M}(R_N)$. We show that $\hat{e}^S_i R_i \frac{1}{n}$ for all $i \in N$, by considering four cases.

Case (i): $N_+(R_N) \neq \emptyset$.

For each $i \in S$, $\frac{1}{|S|} R_i \frac{1}{n}$ since $\frac{1}{m_i(R_N)} P_i 0$ and $0 < \frac{1}{n} \leq \frac{1}{m_i(R_N)} \leq \frac{1}{|S|}$. For each $i \in N_+(R_N) \setminus S$, $0 R_i \frac{1}{n}$ since $0 R_i \frac{1}{m_i(R_N)+1}$ and $0 < \frac{1}{n} \leq \frac{1}{m_i(R_N)+1}$. For each $i \in N \setminus N_+(R_N)$, $0 P_i \frac{1}{n}$. Thus, $\hat{e}^S_i R_i \frac{1}{n}$ for all $i \in N$.

Case (ii): $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

In this case, $S = \{i\}$ for some $i \in N_0(R_N)$. Clearly, $\hat{e}^S_i R_i \frac{1}{n}$ for all $i \in N$.

Case (iii): $N_-(R_N) = N$ and $\{j \in N \mid 1 R_j \frac{1}{n}\} \neq \emptyset$.

In this case, $S = \{j\}$ for some $j \in N$ such that $1 R_j \frac{1}{n}$. Since $0 R_i 1$ for all $i \in N \setminus \{j\}$, we have $\hat{e}_i^S R_i \frac{1}{n}$ for all $i \in N$.

Case (iv): $N_-(R_N) = N$ and $\{j \in N \mid 1 R_j \frac{1}{n}\} = \emptyset$.

Then $S = \emptyset$, and $\hat{e}_i^S I_i \frac{1}{n}$ for all $i \in N$. □

Lemma 5.4 *The social choice correspondence M on $\mathcal{D}^N$ satisfies sharing index order preservation.*

Proof. Let $R_N \in \mathcal{D}^N$ and $S \in \mathcal{M}(R_N)$. Let $i, j \in N$ with $m_i(R_N) > m_j(R_N)$. Then $i \in N_+(R_N)$. If $i \in S$, then $\hat{e}_i^S = \frac{1}{|S|} \geq \hat{e}_j^S \in \{0, \frac{1}{|S|}\}$. If $i \in N_+(R_N) \setminus S$, then $\hat{e}_i^S = 0 = \hat{e}_j^S$. □

Proof of Theorem 5.1 The if-part of the theorem follows from Lemmas 5.2–5.4 and the observation that these lemmas also hold for any F with $F(R_N) \subseteq M(R_N)$ for all $R_N \in \mathcal{D}^N$.

For the only-if part, assume that F satisfies minimal envy Pareto optimality, equal division lower bound, and sharing index order preservation. Let $R_N \in \mathcal{D}^N$ and $x \in F(R_N)$. We show that $x \in M(R_N)$, by distinguishing four cases.

Case (i): $N_+(R_N) \neq \emptyset$.

Let $S \in \mathcal{M}(R_N)$, and $S^* = \{i \in N_+(R_N) \setminus S \mid m_i(R_N) = |S|\}$. By Lemma 4.1(i), $x_i = 0$ for all $i \in N \setminus N_+(R_N)$. If $x_j > 0$ for some $j \in N_+(R_N) \setminus (S \cup S^*)$, then by Pareto optimality and $m_j(R_N) < |S|$ it follows that $x_j > \frac{1}{|S|}$. By sharing index order preservation, this implies $x_i > \frac{1}{|S|}$ also for all $i \in S \cup S^*$, but then $\sum_{i \in N} x_i > 1$, a contradiction. Therefore, $\sum_{i \in S \cup S^*} x_i = 1$.

Denote $S^+ = \{i \in S \cup S^* \mid x_i > 0\}$. If $|S^+| > |S|$, then sharing index order preservation and $0 R_i \frac{1}{|S|+1}$ for all $i \in N_+(R_N)$ with $m_i(R_N) = |S|$, imply that $\sum_{i \in S \cup S^*} x_i > 1$, which is a contradiction. So, $|S^+| \leq |S|$, and therefore $\max_{i \in S \cup S^*} x_i \geq \frac{1}{|S|}$. Since $\mu(R_N) = |S^*|$ by Lemma 5.2, minimal envy Pareto optimality requires that $|S^+| = |S|$ and $x_i = x_j$ for all $i, j \in S^+$. In turn, this implies that $x \in M(R_N)$.

Case(ii): $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

Since in this case $PO(R_N) = M(R_N)$, we have $x \in M(R_N)$.

Case (iii): $N_-(R_N) = N$ and $\{j \in N \mid 1 R_j \frac{1}{n}\} \neq \emptyset$.

Equal division lower bound implies $x_i \leq \frac{1}{n}$ for each $i \in N$ with $\frac{1}{n} P_i 1$. Equal division lower bound and Pareto optimality imply $x_i \leq \frac{1}{n}$ or $x_i = 1$ for each $i \in N$ with $1 R_i \frac{1}{n}$. Since $(\frac{1}{n}, \ldots, \frac{1}{n}) \notin PO(R_N)$, this implies that $x_j = 1$ for some $j \in N$ such that $1 R_j \frac{1}{n}$. Hence, $x \in M(R_N)$.

Case (iv): $N_-(R_N) = N$ and $\{j \in N \mid 1 R_j \frac{1}{n}\} = \emptyset$.

In this case, $M(R_N) = \{(\frac{1}{n}, \ldots, \frac{1}{n})\}$, and therefore minimal envy Pareto optimality implies that there are no envious agents at x. If $y \in PO(R_N)$ with $y \neq (\frac{1}{n}, \ldots, \frac{1}{n})$, then there are $i, j \in N$ such that $y_i < \frac{1}{n} < y_j$. Since $0 P_j \frac{1}{n} P_j 1$, we have $y_i P_j y_j$, which means that player j is envious at R_N and y, and thus $y \neq x$. Thus, $x = (\frac{1}{n}, \ldots, \frac{1}{n}) \in M(R_N)$. This completes the proof of the theorem. $\square$

The following example shows that the axioms in Theorem 5.1 are logically independent.

Example 5.5

(a) Let $n = 3$ and let $\tilde{R}_N \in \mathcal{D}^N$ be a preference profile with $d(\tilde{R}_i) = \frac{1}{3}$ and $1\tilde{P}_i\frac{1}{2}\tilde{P}_i0$ for all $i \in N$. Define F by $F(\tilde{R}_N) = \{(1,0,0)\}$ and $F(R_N) = M(R_N)$ for all $R_N \in \mathcal{D}^N \setminus \{\tilde{R}_N\}$. Then F satisfies equal division lower bound and, since $m_1(\tilde{R}_N) = m_2(\tilde{R}_N) = m_1(\tilde{R}_N) = 2$, also sharing index order preservation. However, $E(\tilde{R}_N, (1,0,0)) = \{2,3\}$, $E(\tilde{R}_N, (\frac{1}{2}, \frac{1}{2}, 0)) = \{3\}$, and both $(1,0,0)$ and $(\frac{1}{2}, \frac{1}{2}, 0)$ are Pareto optimal at $\tilde{R}_N$, so that F violates minimal envy Pareto optimality. Note that $M(\tilde{R}_N) = \{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}$, so that $F \nsubseteq M$.

(b) Let $n = 2$ and let $\tilde{R}_N \in \mathcal{D}^N$ be a preference profile with $d(\tilde{R}_1) = 1$, $d(\tilde{R}_2) = \frac{1}{2}$, and $0\tilde{P}_21$. Define F by $F(\tilde{R}_N) = \{(1,0)\}$ and $F(R_N) = M(R_N)$ for all $R_N \in \mathcal{D}^N \setminus \{\tilde{R}_N\}$. Then F does not satisfy equal division lower bound. Since $m_1(\tilde{R}_N) = m_2(\tilde{R}_N) = 0$, F satisfies sharing index order preservation. Since $(1,0)$ is Pareto optimal at $\tilde{R}_N$, $E(\tilde{R}_N, (1,0)) = \{1\}$, and at every Pareto optimal allocation at $\tilde{R}_N$ there is exactly one envious player, F satisfies minimal envy Pareto optimality. Note that $M(\tilde{R}_N) = \{(0,1)\}$, so that $F \nsubseteq M$.

(c) Let $n = 4$ and let $\tilde{R}_N \in \mathcal{D}^N$ be a preference profile with $d(\tilde{R}_i) = \frac{1}{4}$ and $1\tilde{P}_i\frac{1}{3}\tilde{P}_i0$ for $i = 1,2$; and $d(\tilde{R}_i) = \frac{1}{4}$, $1\tilde{P}_i\frac{1}{2}\tilde{P}_i0$, and $0\tilde{P}_i\frac{1}{3}$ for $i = 3,4$. Then $m_1(\tilde{R}_N) = m_2(\tilde{R}_N) = 3$ and $m_3(\tilde{R}_N) = m_4(\tilde{R}_N) = 2$. Define F by $F(\tilde{R}_N) = \{(0,0,\frac{1}{2},\frac{1}{2})\}$ and $F(R_N) = M(R_N)$ for all $R_N \in \mathcal{D}^N \setminus \{\tilde{R}_N\}$. Then F is not sharing index order preserving. Note that $M(\tilde{R}_N) = \{(\frac{1}{2}, \frac{1}{2}, 0, 0)\}$. Since $|E(\tilde{R}_N, (\frac{1}{2}, \frac{1}{2}, 0, 0))| = |\{3,4\}| = |\{1,2\}| = |E(\tilde{R}_N, (0,0,\frac{1}{2},\frac{1}{2}))|$, and $(0,0,\frac{1}{2},\frac{1}{2})$ is Pareto optimal at $\tilde{R}_N$, we have that F satisfies minimal envy Pareto optimality. Also, F satisfies equal division lower bound, but $F(\tilde{R}_N) \nsubseteq M(\tilde{R}_N)$. $\triangle$

6 Concluding remarks

We have shown that in division problems with single-dipped preferences, the Pareto optimal Nash and strong equilibria of games induced by a fairly general class of mechanisms, result in Pareto optimal allocations characterized by maximal coalitions.

The obvious counterpart, the case of single-peaked preferences, is extensively studied in Bochet et al (2021). The result that is most closely related to our approach is their Theorem 2, which applies to peaks-only rules – these are equivalent to mechanisms in our sense. Under conditions on rules (mechanisms g), partly similar to ours, they show that the Pareto optimal Nash equilibria and strong equilibria in a game (R_N, g) coincide and result in the uniform allocation, for every $R_N \in \mathcal{P}^N$. An allocation $x \in \mathcal{A}$ is the *uniform allocation at* $R_N \in \mathcal{P}^N$

if there is a $\lambda \in [0,1]$ such that

$$x_i = \begin{cases} \min\{p(R_i), \lambda\} & \text{if } \sum_{i \in N} p(R_i) \geq 1 \\ \max\{p(R_i), \lambda\} & \text{if } \sum_{i \in N} p(R_i) \leq 1. \end{cases}$$

The uniform allocation is the allocation assigned by the uniform rule (single-valued social choice correspondence) U, characterized by Sprumont (1991). At the uniform allocation, either all agents obtain at most their peaks or all agents obtain at least their peaks, or both, and thus the uniform allocation is is indeed Pareto optimal (it is 'same-sided').

Sprumont (1991) shows that the uniform rule is the unique anonymous, Pareto optimal, and strategy-proof rule. Recall that a rule F is strategy-proof if $F_i(R_N) R_i F_i(R'_i, R_{N\setminus\{i\}})$ for every preference profile R_N, agent $i \in N$, and preference R'_i, where preferences are chosen within a specific domain, for instance $\mathcal{P}$ or $\mathcal{D}$. The following example shows that, in the single-dipped case, rules obtained by selecting from M are not strategy-proof. Let $F\colon \mathcal{D}^N \to \mathcal{A}$ such that $F(R_N) \in M(R_N)$ for every $R_N \in \mathcal{D}^N$.

Example 6.1 Let $R_N \in \mathcal{D}^N$ such that $0 P_i 1 R_i \frac{1}{n}$ for all $i \in N$. Then $M(R_N) = \{\{i\} \mid i \in N\}$, and therefore $F(R_N) = e^{\{j\}}$ for some $j \in N$ (cf. Theorem 4.5). Consider $R'_j \in \mathcal{D}$ such that $0 P'_j 1$ and $\frac{1}{n} P'_j 1$. Then $M(R'_j, R_{-j}) = \{\{i\} \mid i \in N\setminus\{j\}\}$, and therefore $F_j(R'_j, R_{-j}) = 0$, so that $F_j(R'_j, R_{-j}) P_j F_j(R_N)$. Hence, F is not strategy-proof. $\triangle$

The uniform rule for single-peaked preference profiles is Pareto optimal and envy-free (at the uniform allocation no agent envies any other agent), thus trivially satisfies minimal envy Pareto optimality formulated for single-peaked preferences. It also satisfies equal division lower bound (Thomson, 2010). It is not hard to see that these conditions do not uniquely characterize the uniform rule, but it is not obvious what the analogue of sharing index order preservation for single-peaked preferences is.

A natural extension of our analysis and the analysis in Bochet et al (2021) is to other domains of preferences, notably if both single-dipped and single-peaked preferences in a profile are allowed.

Appendix: Proof of Proposition 3.7

By $p^{s+1} > q^s$ for all $s = 0, 1, \ldots, n-1$, we have $p^1 > \frac{1}{2} > q^1$. We consider three cases.

(a) Suppose that $p^1 P_1 q^1$. Then, by Lemma 3.1, $p^1 R_1 \frac{1}{2}$.

(a1) First suppose that $q^1 R_2 p^1$. Then, by Lemma 3.1, $q^1 R_2 \frac{1}{2}$. It follows that $g_1(1,0) R_1 g_1(0,0)$ and $g_2(1,0) R_2 g_1(1,1)$. With monotonicity and Lemma 3.1 again, it holds that

$g_1(1,0)P_1g_1(r_1,0)$ and $g_2(1,0)P_2g_2(1,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (1,0)$ is a Nash equilibrium.

(a2) Second, suppose that $p^1R_2q^1$. Then, by Lemma 3.1, $p^1R_2\frac{1}{2}$.

(a2.1) If $\frac{1}{2}P_1q^1$ and $\frac{1}{2}P_2q^1$, then $g_1(1,1)P_1g_1(0,1)$ and $g_2(1,1)P_2g_2(1,0)$. From monotonicity and Lemma 3.1, it holds that $g_1(1,1)R_1g_1(r_1,1)$ and $g_2(1,1)R_2g_2(1,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (1,1)$ is a Nash equilibrium.

(a2.2) If $q^1R_1\frac{1}{2}$, together with $p^1R_2\frac{1}{2}$, then $g_1(1,0)R_1g_1(0,0)$ and $g_2(1,0)R_2g_2(1,1)$. With monotonicity and Lemma 3.1 again, it holds that $g_1(1,0)R_1g_1(r_1,0)$ and $g_2(1,0)R_2g_2(1,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (1,0)$ is a Nash equilibrium.

(a2.3) If $q^1R_2\frac{1}{2}$, then similar to (a2.2), we can prove that $r^* = (0,1)$ is a Nash equilibrium.

(b) Suppose that $p^1I_1q^1$. Then, $p^1R_1\frac{1}{2}$ and $q^1R_1\frac{1}{2}$.

(b1) If $p^1P_2q^1$, then $p^1R_2\frac{1}{2}$. So, $g_1(0,1)P_1g_1(1,1)$ and $g_2(0,1)P_2g_2(0,0)$. With monotonicity and Lemma 3.1, it holds that $g_1(0,1)R_1g_1(r_1,1)$ and $g_2(0,1)R_2g_2(0,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (0,1)$ is a Nash equilibrium.

(b2) If $q^1R_2p^1$, then $q^1R_2\frac{1}{2}$. Similar to (b1), we can prove that $r^* = (1,0)$ is a Nash equilibrium.

(c) Suppose that $q^1P_1p^1$. Then, by Lemma 3.1, $q^1R_1\frac{1}{2}$.

(c1) First, suppose that $p^1R_2q^1$, then similar to (a1), it follows that $r^* = (0,1)$ is a Nash equilibrium.

(c2) Second, suppose that $q^1P_2p^1$. Then, $q^1R_2\frac{1}{2}$.

(c2.1) If $\frac{1}{2}P_1p^1$ and $\frac{1}{2}P_2p^1$, then $g_1(0,0)P_1g_1(1,0)$ and $g_2(0,0)P_2g_1(0,1)$. With monotonicity and Lemma 3.1, it holds that $g_1(0,0)R_1g_1(r_1,0)$ and $g_2(0,0)R_2g_2(0,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (0,0)$ is a Nash equilibrium.

(c2.2) If $p^1R_1\frac{1}{2}$, together with $q^1R_2\frac{1}{2}$, we have $g_1(1,0)R_1g_1(0,0)$ and $g_2(1,0)R_2g_1(1,1)$. With monotonicity and Lemma 3.1, it holds that $g_1(1,0)R_1g_1(r_1,0)$ and $g_2(1,0)P_2g_2(1,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (1,0)$ is a Nash equilibrium.

(c2.3) If, finally, $p^1R_2\frac{1}{2}$, then similar to (c2.2), it can be proved $r^* = (0,1)$ is a Nash equilibrium. $\square$

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