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Doudou Gong, Bas Dietzenbacher,
Hans Peters

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GSBE

Maastricht University School of Business and Economics
Graduate School of Business and Economics

P.O Box 616
NL- 6200 MD Maastricht
The Netherlands

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Doudou Gong[1,2*] Bas Dietzenbacher[2†] Hans Peters[2‡]

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Abstract

This paper introduces and studies a random arrival rule for bankruptcy problems with nontransferable utility. This bankruptcy rule generalizes the random arrival rule for bankruptcy problems with transferable utility which assigns the unique efficient allocation proportional to the sum of marginal vectors. We provide two axiomatic characterizations based on symmetry and monotonicity, respectively.

Keywords: NTU-bankruptcy problems, random arrival rule, axiomatic analysis
JEL classification: C79, D63, D74

1 Introduction

Bankruptcy problems with nontransferable utility, or NTU-bankruptcy problems (cf. Orshan et al. 2003), are an alternative interpretation of bargaining problems with claims (cf. Chun and Thomson 1992) that generalize bankruptcy problems with transferable utility (cf. O'Neill 1982). Multiple claimants with possibly nonlinear utility functions over money have incompatible claims on an exogenous estate. Which allocation of the estate among the claimants should be selected?

Already in the seminal paper on bankruptcy problems with transferable utility, O'Neill (1982) proposed the random arrival rule, also known as the recursive completion rule or the run-to-the-bank rule. Suppose that the claimants sequentially arrive and claim their entitled part of the estate. The estate is allocated on a so called first-come first-served basis, i.e. claimants are in turn allocated their claim, or the remaining part of the estate if not possible. Each possible order in which the claimants arrive results in a specific allocation, also referred to as marginal vector. Assuming that all orders are equally likely, the random arrival rule assigns the expected payoffs to the claimants. This is mathematically equivalent

*E-mail: d.gong@maastrichtuniversity.nl.
†E-mail: b.dietzenbacher@maastrichtuniversity.nl.
‡E-mail: h.peters@maastrichtuniversity.nl.
[1]School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an, 710072, China.
[2]Department of Quantitative Economics, Maastricht University, Maastricht, 6200 MD, The Netherlands.

to the arithmetic average over all marginal vectors, which corresponds to the unique efficient allocation proportional to the sum of marginal vectors. For bankruptcy problems with two claimants, the random arrival rule coincides with the contested garment principle (cf. Aumann and Maschler 1985), also known as the concede-and-divide principle.

The random arrival rule for bankruptcy problems with transferable utility is well-studied from an axiomatic perspective, see e.g. Thomson (2019). Most of the results exploit the fact that the random arrival rule is obtained by applying the Shapley value (cf. Shapley 1953) to the corresponding bankruptcy game with transferable utility (cf. O'Neill 1982). The provided axiomatic characterizations are based on consistency (cf. Hwang and Wang 2009; Albizuri et al. 2010), potentials and balanced payoffs (cf. Hwang 2015), and a weakening of additivity (cf. Morgenstern and Domínguez 2019).

Other studies on the random arrival rule for bankruptcy problems with transferable utility focused on limit behavior (cf. Chun and Lee 2007), computational complexity (cf. Aziz 2013), and sampling estimation (cf. Saavedra-Nieves and Saavedra-Nieves 2020). Extensions of the random arrival rule have been proposed for bankruptcy problems with multiple issues (cf. González-Alcón et al. 2007), bankruptcy problems with multiple references (cf. Borrero et al. 2018), and bankruptcy problems with withdrawal limits (cf. Sanchez-Soriano 2021).

In this paper, we generalize the random arrival rule to bankruptcy problems with nontransferable utility based on a generalized concept of marginal vectors. For bankruptcy problems with two claimants, the new random arrival rule assigns the unique efficient allocation proportional to the sum of marginal vectors. However, for more than two claimants, this definition may lead to an allocation where some claimants get more than their claims. By explicitly bounding the payoffs by the claims, we extend the random arrival rule to bankruptcy problems with an arbitrary number of claimants. We provide two axiomatic characterizations based on symmetry and monotonicity, respectively. In particular, we axiomatically compare the random arrival rule to the well-known proportional rule (cf. Chun and Thomson 1992; Lombardi and Yoshihara 2010).

The remainder of this paper is organized as follows. Section 2 provides preliminary notions and notations for bankruptcy problems with nontransferable utility. Section 3 introduces and studies the random arrival rule.

2 Preliminaries

Let N with $|N| \geq 2$ be a finite set of *claimants*. Denote $2^N = \{S \mid S \subseteq N\}$. An *order* of N is a bijection $\sigma : \{1, \ldots, |N|\} \to N$. The set of all orders of N is denoted by Π^N. For all $x, y \in \mathbb{R}^N_+$, $x \leq y$ denotes $x_i \leq y_i$ for all $i \in N$, and $x < y$ denotes $x_i < y_i$ for all $i \in N$. The notations $\geq$ and $>$ are defined analogously. For all $x \in \mathbb{R}^N$ and all $S \in 2^N$, $x_S \in \mathbb{R}^S$ denotes $x_S = (x_i)_{i \in S}$.

A *bankruptcy problem with nontransferable utility*, or *(NTU-)bankruptcy problem* (cf. Orshan et al. 2003), is a triple (N, E, c), where $E \subseteq \mathbb{R}^N_+$ with $E \cap \mathbb{R}^N_{++} \neq \emptyset$ is the *estate* assumed to be

- nonempty and compact;
- comprehensive, i.e. $E = \{x \in \mathbb{R}^N_+ \mid \exists y \in E : y \geq x\}$;
- nonleveled, i.e. $\{x \in E \mid \neg\exists y \in E : y > x\} = \{x \in E \mid \neg\exists y \in E : y \geq x, y \neq x\}$;

and $c \in \mathbb{R}^N_{++} \setminus E$ is the vector of *claims* of N on E. Note that we do not assume that the estate is convex in order to allow for utility functions that are not necessarily of the Von Neumann-Morgenstern type, but all results do not rely on the admission of nonconvex estates and can be restated on the domain of NTU-bankruptcy problems with convex estate. Let $\mathcal{B}^N$ denote the class of all NTU-bankruptcy problems with set of claimants N. For convenience, an NTU-bankruptcy problem is denoted by $(E, c) \in \mathcal{B}^N$.

Let $(E, c) \in \mathcal{B}^N$. The vector of *utopia values* $u^E \in \mathbb{R}^N_{++} \setminus E$ is defined by

$$u_i^E = \max\{x_i \mid x \in E\} \quad \text{for all } i \in N.$$

The vector of *truncated claims* $\widehat{c}^E \in \mathbb{R}^N_{++} \setminus E$ is defined by

$$\widehat{c}_i^E = \min\{c_i, u_i^E\} \quad \text{for all } i \in N.$$

The *truncated estate* $\widehat{E}^c \subseteq \mathbb{R}^N_+$ is defined by

$$\widehat{E}^c = \{x \in E \mid x \leq c\}.$$

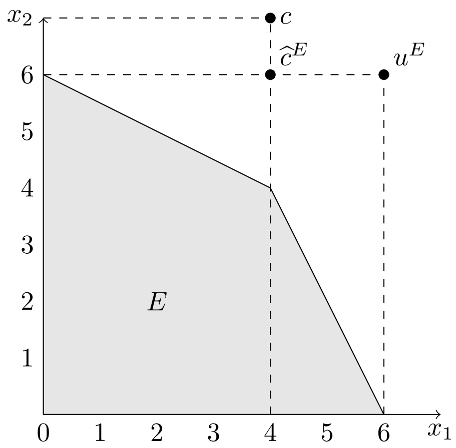
Example 1
Let $N = \{1, 2\}$ and let $(E, c) \in \mathcal{B}^N$ be the bankruptcy problem with estate

$$E = \left\{x \in \mathbb{R}^N_+ \;\middle|\; 2x_2 \leq \min\{-x_1 + 12, -4x_1 + 24\}\right\}$$

and claims $c = (4, 7)$. Then $u^E = (6, 6)$, $\widehat{c}^E = (4, 6)$, and

$$\widehat{E}^c = \left\{x \in \mathbb{R}^N_+ \;\middle|\; x_1 \leq 4 \text{ and } x_1 + 2x_2 \leq 12\right\}.$$

This is illustrated in the following picture.



$\triangle$

A *bankruptcy rule* $f : \mathcal{B}^N \to \mathbb{R}^N_+$ assigns to each bankruptcy problem $(E, c) \in \mathcal{B}^N$ an allocation $f(E, c) \in \widehat{E}^c$. The *proportional rule* (cf. Chun and Thomson 1992; Lombardi and Yoshihara 2010) $Prop : \mathcal{B}^N \to \mathbb{R}^N_+$ assigns to each bankruptcy problem $(E, c) \in \mathcal{B}^N$ the allocation

$$Prop(E, c) = \lambda c,$$

where $\lambda = \max\{t \in (0, 1) \mid tc \in E\}$. Note that $Prop(E, c) \in \mathbb{R}^N_{++}$ for all $(E, c) \in \mathcal{B}^N$ because $E \cap \mathbb{R}^N_{++} \neq \emptyset$ and $c \in \mathbb{R}^N_{++}$.

3 A random arrival rule

This section introduces and studies a random arrival rule for bankruptcy problems with nontransferable utility. Suppose that the claimants sequentially arrive and in turn get their claim, or maximal remaining utility if not possible. This induces a so-called marginal vector (see also Otten et al. 1998 and Dietzenbacher 2018). The random arrival rule assigns to each NTU-bankruptcy problem with two claimants the unique efficient allocation proportional to the sum of marginal vectors.

Let $(E, c) \in \mathcal{B}^N$. For all $\sigma \in \Pi^N$, the *marginal vector* $m^\sigma(E, c) \in \mathbb{R}^N_+$ is defined by

$$m^\sigma_{\sigma(k)}(E, c) = \max\left\{t \in \mathbb{R}_+ \;\middle|\; (m^\sigma_{\sigma(1)}(E, c), \ldots, m^\sigma_{\sigma(k-1)}(E, c), t, 0, \ldots, 0) \in \widehat{E}^c\right\}$$

for all $k \in \{1, \ldots, |N|\}$. Their sum is denoted by

$$M(E, c) = \sum_{\sigma \in \Pi^N} m^\sigma(E, c).$$

Note that $M(E, c) \in \mathbb{R}^N_{++}$ because $m^\sigma_{\sigma(1)}(E, c) = \widehat{c}^E_{\sigma(1)}$ for all $\sigma \in \Pi^N$ and $\widehat{c}^E \in \mathbb{R}^N_{++}$. Moreover, $m^\sigma(E, \widehat{c}^E) = m^\sigma(E, c)$ for all $\sigma \in \Pi^N$, and consequently $M(E, \widehat{c}^E) = M(E, c)$.

Definition 1
For $|N| = 2$, the *random arrival rule* $RA : \mathcal{B}^N \to \mathbb{R}^N_+$ assigns to each bankruptcy problem $(E, c) \in \mathcal{B}^N$ the allocation

$$RA(E, c) = \lambda M(E, c),$$

where $\lambda = \max\{t \in (0, 1) \mid tM(E, c) \in E\}$.

Remark
The random arrival rule is well-defined for bankruptcy problems with two claimants, i.e. $RA(E, c) \in \widehat{E}^c$ *for all* $(E, c) \in \mathcal{B}^N$ *with* $|N| = 2$.

Proof. Let $(E, c) \in \mathcal{B}^N$ with $|N| = 2$. Then

$$M_i(E, c) = \widehat{c}^E_i + \max\left\{t \in \mathbb{R}_+ \;\middle|\; (t, \widehat{c}^E_{N\setminus\{i\}}) \in E\right\} \quad \text{for all } i \in N.$$

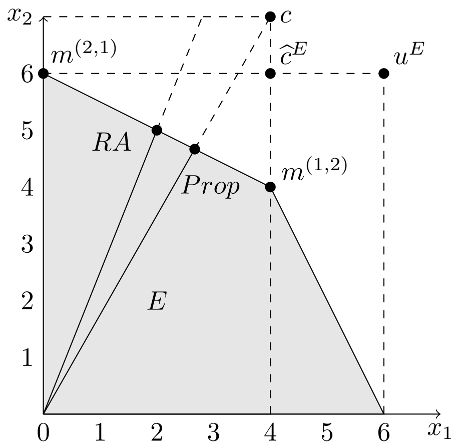
Since $\widehat{c}^E \notin E$, we have

$$2\max\left\{t \in \mathbb{R}_+ \;\middle|\; (t, \widehat{c}^E_{N\setminus\{i\}}) \in E\right\} < M_i(E, c) < 2\widehat{c}^E_i \quad \text{for all } i \in N.$$

Suppose that there exists $i \in N$ such that $RA_i(E, c) = \lambda M_i(E, c) \geq \widehat{c}^E_i$. Then $\lambda > \frac{1}{2}$. However, $RA_{N\setminus\{i\}}(E, c) = \lambda M_{N\setminus\{i\}}(E, c) \leq \max\{t \in \mathbb{R}_+ \mid (t, \widehat{c}^E_i) \in E\}$, which implies that $\lambda < \frac{1}{2}$. This is a contradiction, so $RA(E, c) < \widehat{c}^E$. Hence, $RA(E, c) \in \widehat{E}^c$. $\square$

Note that $RA(E, c) \in \mathbb{R}^N_{++}$ for all $(E, c) \in \mathcal{B}^N$ because $E \cap \mathbb{R}^N_{++} \neq \emptyset$ and $M(E, c) \in \mathbb{R}^N_{++}$.

Example 2
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathcal{B}^N$ from Example 1 with $E = \{x \in \mathbb{R}^N_+ \mid 2x_2 \leq \min\{-x_1 + 12, -4x_1 + 24\}\}$ and $c = (4, 7)$. Then $Prop(E, c) = (\frac{8}{3}, \frac{14}{3})$, $m^{(1,2)}(E, c) = (4, 4)$, $m^{(2,1)}(E, c) = (0, 6)$, $M(E, c) = (4, 10)$, and $RA(E, c) = (2, 5)$. This is illustrated in the following picture.



$\triangle$

The random arrival rule assigns to each NTU-bankruptcy problem with two claimants the unique efficient allocation proportional to the sum of marginal vectors. This generalizes the contested garment principle (cf. Aumann and Maschler 1985), also known as the concede-and-divide principle. Note that this generalization is an alternative for the relative adjustment principle introduced by Dietzenbacher et al. (2020). To axiomatically compare the random arrival rule with the proportional rule, we introduce the following properties.

Pareto optimality:
for all $(E, c) \in \mathcal{B}^N$ and all $x \in E$, we have $x \not> f(E, c)$.

Scale covariance:
for all $(E, c) \in \mathcal{B}^N$ and all $a \in \mathbb{R}^N_{++}$, we have $f(aE, ac) = af(E, c)$.[1]

Symmetry:
for all $(E, c) \in \mathcal{B}^N$ and all $i, j \in N$ with $E = \{x \in \mathbb{R}^N_+ \mid \exists y \in E : y_i = x_j, y_j = x_i, y_{N\setminus\{i,j\}} = x_{N\setminus\{i,j\}}\}$ and $c_i = c_j$, we have $f_i(E, c) = f_j(E, c)$.

c-symmetry:
for all $(E, c) \in \mathcal{B}^N$ and all $i, j \in N$ with $c_i = c_j$, we have $f_i(E, c) = f_j(E, c)$.

M-symmetry:
for all $(E, c) \in \mathcal{B}^N$ and all $i, j \in N$ with $M_i(E, c) = M_j(E, c)$, we have $f_i(E, c) = f_j(E, c)$.

Pareto optimality requires that no other estate allocation is better for all claimants. Scale covariance requires covariance under individual rescaling of utility. Symmetry requires that symmetric claimants (w.r.t. the estate) with equal claims get equal payoffs. c-symmetry requires that claimants with equal claims get equal payoffs. M-symmetry requires that claimants with equal accumulated marginal payoffs get equal payoffs. Note that c-symmetry and M-symmetry both imply symmetry. Both the proportional rule and the random arrival rule satisfy Pareto optimality, scale covariance, and symmetry. If we strengthen symmetry to c-symmetry or M-symmetry, we obtain an axiomatic characterization of the proportional rule or the random arrival rule, respectively, for bankruptcy problems with two claimants.

Theorem 1
For bankruptcy problems with two claimants, the proportional rule is the unique bankruptcy rule satisfying Pareto optimality, scale covariance, and c-symmetry.[2]

Proof. Clearly, the proportional rule for bankruptcy problems with two claimants satisfies Pareto optimality, scale covariance, and c-symmetry. Let $f : \mathcal{B}^N \to \mathbb{R}^N_+$ with $|N| = 2$ be a bankruptcy rule satisfying Pareto optimality, scale covariance, and c-symmetry. Let

[1] Here, $aE = \{ax \in \mathbb{R}^N_+ \mid x \in E\}$ and $ac = (a_i c_i)_{i \in N}$.

[2] Also for bankruptcy problems with an arbitrary number of claimants, the proportional rule is characterized by these properties.

$(E,c) \in \mathcal{B}^N$. By scale covariance, assume without loss of generality that $c_i = c_j$ for all $i,j \in N$. By c-symmetry, $f_i(E,c) = f_j(E,c)$ for all $i,j \in N$. By Pareto optimality, $f(E,c) = Prop(E,c)$. Hence, $f = Prop$. □

Theorem 2
For bankruptcy problems with two claimants, the random arrival rule is the unique bankruptcy rule satisfying Pareto optimality, scale covariance, and M-symmetry.

Proof. Clearly, the random arrival rule for bankruptcy problems with two claimants satisfies Pareto optimality, scale covariance, and M-symmetry. Let $f : \mathcal{B}^N \to \mathbb{R}^N_+$ with $|N| = 2$ be a bankruptcy rule satisfying Pareto optimality, scale covariance, and M-symmetry. Let $(E,c) \in \mathcal{B}^N$. By scale covariance, assume without loss of generality that $M_i(E,c) = M_j(E,c)$ for all $i,j \in N$. By M-symmetry, $f_i(E,c) = f_j(E,c)$ for all $i,j \in N$. By Pareto optimality, $f(E,c) = RA(E,c)$. Hence, $f = RA$. □

Let $N = \{1,2\}$. The bankruptcy rule $f^0 : \mathcal{B}^N \to \mathbb{R}^N_+$ assigning to each $(E,c) \in \mathcal{B}^N$ the allocation $f^0(E,c) = (0,0)$ satisfies scale covariance, c-symmetry, and M-symmetry, but not Pareto optimality. The bankruptcy rule $f^{PR} : \mathcal{B}^N \to \mathbb{R}^N_+$ assigning to each $(E,c) \in \mathcal{B}^N$ the allocation

$$f^{PR}(E,c) = \begin{cases} Prop(E,c) & \text{if } c_1 = c_2; \\ RA(E,c) & \text{otherwise}; \end{cases}$$

satisfies Pareto optimality and c-symmetry, but not scale covariance nor M-symmetry. The bankruptcy rule $f^{RP} : \mathcal{B}^N \to \mathbb{R}^N_+$ assigning to each $(E,c) \in \mathcal{B}^N$ the allocation

$$f^{RP}(E,c) = \begin{cases} RA(E,c) & \text{if } M_1(E,c) = M_2(E,c); \\ Prop(E,c) & \text{otherwise}; \end{cases}$$

satisfies Pareto optimality and M-symmetry, but not scale covariance nor c-symmetry. Hence, the properties in Theorems 1 and 2 are independent. An overview is presented in the following table.[3]

$N = \{1,2\}$	$Prop$	RA	f^0	f^{PR}	f^{RP}
Pareto optimality	$+^*$	$+^*$	$-$	$+$	$+$
scale covariance	$+^*$	$+^*$	$+$	$-$	$-$
symmetry	$+$	$+$	$+$	$+$	$+$
c-symmetry	$+^*$	$-$	$+$	$+$	$-$
M-symmetry	$-$	$+^*$	$+$	$-$	$+$

[3]Here, $+$ indicates that the rule satisfies the property, $-$ indicates that the rule does not satisfy the property, and $*$ indicates the axiomatic characterizations.

For bankruptcy problems with an arbitrary number of claimants, the unique efficient allocation proportional to the sum of marginal vectors does not necessarily belong to the truncated estate. This is shown by the following example.

Example 3
Let $N = \{1, 2, 3\}$ and consider the bankruptcy problem $(E, c) \in \mathcal{B}^N$ with

$$E = \text{c.c.h.}\left\{(\tfrac{72}{35}, 0, 0), (0, \tfrac{72}{35}, 0), (0, 0, 6), (\tfrac{18}{10}, \tfrac{18}{10}, 0), (2, 0, 4), (0, 2, 4), (\tfrac{17}{10}, \tfrac{17}{10}, 4)\right\}$$

and $c = (2, 2, 4)$.[4] The marginal vectors are presented in following table.

σ	m_1^σ	m_2^σ	m_3^σ
$(1,2,3)$	2	$\frac{2}{5}$	0
$(1,3,2)$	2	0	4
$(2,1,3)$	$\frac{2}{5}$	2	0
$(2,3,1)$	0	2	4
$(3,1,2)$	2	0	4
$(3,2,1)$	0	2	4

Then $M(E, c) = (\frac{32}{5}, \frac{32}{5}, \frac{80}{5})$ and $\lambda M(E, c) = (\frac{204}{125}, \frac{204}{125}, \frac{510}{125})$, where $\lambda = \max\{t \in (0, 1) \mid tM(E, c) \in E\} = \frac{51}{200}$. This means that $\lambda M_3(E, c) = \frac{510}{125} > 4 = c_3$. $\triangle$

Nevertheless, the accumulated marginal payoffs could still be considered as benchmarks on which the random arrival rule is based. By explicitly bounding the payoffs by the claims, we define the random arrival rule for NTU-bankruptcy problems as the unique efficient allocation as proportionally as possible to the sum of marginal vectors. This extends the random arrival rule for bankruptcy problems with two claimants to an arbitrary number of claimants.

Definition 2
The *random arrival rule* $RA : \mathcal{B}^N \to \mathbb{R}_+^N$ assigns to each bankruptcy problem $(E, c) \in \mathcal{B}^N$ the allocation

$$RA_i(E, c) = \min\{\lambda M_i(E, c), c_i\} \quad \text{for all } i \in N,$$

where $\lambda = \max\{t \in (0, 1) \mid (\min\{\lambda M_i(E, c), c_i\})_{i \in N} \in E\}$.

Note that $RA(E, c) \in \widehat{E}^c$ for all $(E, c) \in \mathcal{B}^N$ because $RA(E, c) \in E$ and $RA(E, c) \leq c$, so the random arrival is well-defined. Moreover, $RA(E, c) \in \mathbb{R}_{++}^N$ for all $(E, c) \in \mathcal{B}^N$ because $E \cap \mathbb{R}_{++}^N \neq \emptyset$, $M(E, c) \in \mathbb{R}_{++}^N$, and $c \in \mathbb{R}_{++}^N$. Definitions 1 and 2 are equivalent for NTU-bankruptcy problems with two claimants. Definition 2 generalizes the random arrival rule for TU-bankruptcy problems with an arbitrary number of claimants (cf. O'Neill 1982).

[4]Here, c.c.h. denotes the comprehensive convex hull, i.e. the smallest comprehensive and convex set containing it.

Example 4
Let $N = \{1,2,3\}$ and consider the bankruptcy problem $(E,c) \in \mathcal{B}^N$ from Example 3 with $E = \text{c.c.h.}\{(\frac{72}{35},0,0),(0,\frac{72}{35},0),(0,0,6),(\frac{18}{10},\frac{18}{10},0),(2,0,4),(0,2,4),(\frac{17}{10},\frac{17}{10},4)\}$ and $c = (2,2,4)$. Then $Prop(E,c) = (\frac{12}{7},\frac{12}{7},\frac{24}{7})$ and $RA(E,c) = (\frac{17}{10},\frac{17}{10},4)$. △

The proportional rule and the random arrival rule for bankruptcy problems with an arbitrary number of claimants both satisfy Pareto optimality, scale covariance, and symmetry. For a further axiomatic comparison, we introduce the following monotonicity properties.

Monotonicity:
for all $(E,c),(E',c) \in \mathcal{B}^N$ with $E \subseteq E'$, we have $f(E,c) \leq f(E',c)$.

Conditional monotonicity:
for all $(E,c),(E',c) \in \mathcal{B}^N$ with $E \subseteq E'$, and $f_i(E,c) = f_j(E,c)$ and $RA_i(E',c) = RA_j(E',c)$ for all $i,j \in N$, we have $f(E,c) \leq f(E',c)$.

Monotonicity requires that no claimant is worse off when the estate enlarges. Conditional monotonicity requires that no claimant is worse off when the estate enlarges, provided that all claimants got equal payoffs before the enlargement and equal random arrival rule payoffs after the enlargement. Note that monotonicity implies conditional monotonicity. If monotonicity is imposed in addition to Pareto optimality, scale covariance, and symmetry, then we obtain an axiomatic characterization of the proportional rule.

Theorem 3 (cf. Chun and Thomson 1992; Albizuri et al. 2020)
The proportional rule is the unique bankruptcy rule satisfying Pareto optimality, scale covariance, symmetry, and monotonicity.

The random arrival rule does not satisfy monotonicity, but trivially satisfies conditional monotonicity. Needless to say, the proportional rule also satisfies Pareto optimality, scale covariance, symmetry, and conditional monotonicity. However, if we additionally impose truncation invariance, which requires invariance under truncating the claims by the utopia values, then we obtain an axiomatic characterization of the random arrival rule.

Truncation invariance:
for all $(E,c) \in \mathcal{B}^N$, we have $f(E,c) = f(E,\hat{c}^E)$.

Theorem 4
The random arrival rule is the unique bankruptcy rule satisfying Pareto optimality, scale covariance, symmetry, conditional monotonicity, and truncation invariance.

Proof. Clearly, the random arrival rule satisfies Pareto optimality, scale covariance, symmetry, conditional monotonicity, and truncation invariance. Let $f : \mathcal{B}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying Pareto optimality, scale covariance, symmetry, conditional monotonicity, and truncation invariance. Let $(E, c) \in \mathcal{B}^N$. By scale covariance, assume without loss of generality that $RA_i(E, c) = RA_j(E, c)$ for all $i, j \in N$. Let $\varepsilon > 0$ be small enough. Define $E^\varepsilon \subseteq \mathbb{R}^N_+$ by

$$E^\varepsilon = \text{c.c.h.} \left\{ \left(\left(RA_i(E,c) - \frac{|S|-1}{|N|-1}\varepsilon \right)_{i \in S}, (0)_{i \in N \setminus S} \right) \,\middle|\, S \in 2^N \setminus \{\emptyset\} \right\}.$$

Then $E^\varepsilon \subseteq E$, and $u_i^{E^\varepsilon} = RA_i(E, c)$ and $\widehat{c}_i^{E^\varepsilon} = RA_i(E, c)$ for all $i \in N$. By symmetry, $f_i(E^\varepsilon, \widehat{c}^{E^\varepsilon}) = f_j(E^\varepsilon, \widehat{c}^{E^\varepsilon})$ for all $i, j \in N$. By Pareto optimality, $f_i(E^\varepsilon, \widehat{c}^{E^\varepsilon}) = RA_i(E, c) - \varepsilon$ for all $i \in N$. By truncation invariance, $f_i(E^\varepsilon, c) = RA_i(E, c) - \varepsilon$ for all $i \in N$. This means that, $f_i(E^\varepsilon, c) = f_j(E^\varepsilon, c)$ for all $i, j \in N$. By conditional monotonicity, for all small $\varepsilon > 0$, $f_i(E, c) \geq f_i(E^\varepsilon, c) = RA_i(E, c) - \varepsilon$ for all $i \in N$. This implies that $f_i(E, c) = RA_i(E, c)$ for all $i \in N$. Hence, $f = RA$. □

The bankruptcy rule $f^0 : \mathcal{B}^N \to \mathbb{R}^N_+$ satisfies scale covariance, symmetry, monotonicity, and truncation invariance, but not Pareto optimality. The *constrained equal awards rule* $CEA : \mathcal{B}^N \to \mathbb{R}^N_+$ assigning to each $(E, c) \in \mathcal{B}^N$ the allocation

$$CEA(E, c) = (\min\{\lambda, c_i\})_{i \in N},$$

where $\lambda = \max\{t \in \mathbb{R}_{++} \mid (\min\{t, c_i\})_{i \in N} \in E\}$, satisfies Pareto optimality, symmetry, monotonicity, and truncation invariance, but not scale covariance. For each $\sigma \in \Pi^N$, the bankruptcy rule $m^\sigma : \mathcal{B}^N \to \mathbb{R}^N_+$ assigning to each $(E, c) \in \mathcal{B}^N$ the allocation $m^\sigma(E, c)$ satisfies Pareto optimality, scale covariance, monotonicity, and truncation invariance, but not symmetry. The *truncated proportional rule* $TProp : \mathcal{B}^N \to \mathbb{R}^N_+$ assigning to each $(E, c) \in \mathcal{B}^N$ the allocation

$$TProp(E, c) = \lambda \widehat{c}^E,$$

where $\lambda = \max\{t \in (0, 1) \mid t\widehat{c}^E \in E\}$, satisfies Pareto optimality, scale covariance, symmetry, and truncation invariance, but not conditional monotonicity. Hence, the properties in Theorems 3 and 4 are independent. An overview is presented in the following table.

	$Prop$	RA	f^0	CEA	m^σ	$TProp$
Pareto optimality	$+^*$	$+^*$	$-$	$+$	$+$	$+$
scale covariance	$+^*$	$+^*$	$+$	$-$	$+$	$+$
symmetry	$+^*$	$+^*$	$+$	$+$	$-$	$+$
monotonicity	$+^*$	$-$	$+$	$+$	$+$	$-$
conditional monotonicity	$+$	$+^*$	$+$	$+$	$+$	$-$
truncation invariance	$-$	$+^*$	$+$	$+$	$+$	$+$

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