

# Stability in Matching with Couples having Non-Responsive Preferences

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# Stability in Matching with Couples having Non-Responsive Preferences \*

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## Abstract

The paper studies matching markets where institutions are matched with possibly more than one individual. The matching market contains some couples who view the pair of jobs as complements. We specify that the couples have a “weak” preference to be matched together. We first assume that the institutions have common preference over all the individuals. We then characterize under which weak preferences of couples a stable matching exist. We then impose further conditions on the common preference of institutions over the individuals and prove existence of stable matching for unrestricted couple preferences. Finally, we establish a result on stability by relaxing the condition on common preference of institutions over individuals and assuming different preferences for different institutions.

KEYWORDS. two-sided matching, stability, weak responsiveness

JEL CLASSIFICATION CODES. C78, D47

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# 1 Introduction

In many different contexts, there is a centralized matching procedure by which individuals on one side of the market are matched with institutions on the other side of the market. These include the market for lawyers in Canada, children in schools in the USA, doctors and senior-level health-care professionals in several countries, etc. There is a huge literature which has been developed on various market designs to find out an “optimal” matching procedure to produce stable matchings. A matching is stable when there does not exist any institution-individual pairs which can block the original matching by getting matched together, such that both of them are better off compared to their original matching. It was shown by Roth (1984) [9] that it is possible to have mechanisms which induced only one side of the market to correctly reveal their preferences. However, the results on stability have been more encouraging as the received doctrine is that stable matchings do exist under appropriate domain restrictions. But to achieve that, institutions must view individuals as substitutes and individuals also must only care about the institution to which they are matched. It was first pointed out by Roth (1984) [8] that the presence of couples in the labour market may lead to an impossibility result where no stable matching may exist. This can happen because couples may view pairs of jobs as complements. Thus, the assumptions which consider the choices of individuals to be independent of each other might not apply. Klaus and Klijn (2005) [5] identify the maximal domain of preferences of couples under which stable matchings exist. The maximal domain satisfies responsiveness, meaning that a couple is better off when any member of the couple is matched with a more preferred institution. However, Kojima, Pathak, and Roth (2010) [6] point out that Responsiveness is not satisfied in their data sets because couples show strong preference to be matched in institutions situated in the same geographical area.

In this paper, we consider a set of doctors including some couples. We focus on the issue of existence of stable matchings with couples. Furthermore, we first look at the scenario when all the institutions have common prefer-

ences over singleton doctors. This can be easily justified if hospitals rank doctors according to grades of some common examination. The institutions have a linear order over the set of doctors. However, the preference over the set of doctors might vary from one institution to the other even when the preference on singleton doctors remains the same. The starting point of our analysis is how to model the common preferences of the institutions and how to model the preference ordering of any couple over pairs of positions, given the individual preferences of each member of the couple. We look into a setting where the set of institutions is a finite set and there is no information about the “distance” between any pair of institutions. But when a couple is matched with the same institution, then the distance trivially becomes zero. Thus as assumed by Dutta and Massó (1997) [3], we have an option to assume that a couple prefers to be matched at the same institution rather than being matched with different institutions.

We assume that the preferences of institutions satisfy responsiveness, i.e., for two allocations of an institution, which differ in exactly one doctor, the institution prefers the allocation with the better doctor. We then analyse the situation where couples’ preferences violate responsiveness as long as they can be together in any possible institution. We show that under common preferences of institutions we will have stable matchings if, and only if, the couples’ preferences do not violate responsiveness with respect to the more preferred institution of the couple.

We then restrict the condition of common institution preferences such that the couples are ordered consecutively. We finally show that under lexicographic preferences of institutions, stable matchings exist for unrestricted couples’ preferences if, and only if, the couples are ordered consecutively under the common institutions preference over individuals.

Finally, we find out the consequences of relaxing the condition of common preferences of the institutions over individuals. We establish a result for the existence of stable matchings when institutions’ do not have common preference over individuals. It turns out that the results proved earlier are

not sufficient to prove the existence of stable matchings in this scenario. Thus we need to impose further restrictions on couples' preferences in order to get a stable matching.

In section 2, we formally introduce the model, providing all the necessary definitions, notations and algorithms which are used throughout this paper. In section 3, we give a necessary and sufficient condition on couples' preferences which always guarantees a stable matching when institutions have common preference over individual doctors. In section 4, we further restrict the common preference over individuals, and look for the conditions which guarantee stable matchings for unrestricted couples' preferences. Finally, in section 5, we consider the case where institutions may not have common preference over individual doctors and give conditions for stable matchings.

## 2 The Framework

We consider many-to-one matching between doctors and hospitals. We denote by  $H$  the set of hospitals. We use the notation  $\bar{H}$  to denote  $H \cup \{\emptyset\}$ . The interpretation of  $\emptyset$  is that if some doctor is matched with  $\emptyset$ , then that doctor is unmatched. Each hospital  $h \in H$  has a finite capacity  $\kappa_h \geq 1$ . We denote by  $D$  the set of doctors.  $F, M, S$  is a collection of pairwise disjoint subsets of  $D$  such that  $|M| = |F|$ . Here,  $S$  is the set of single doctors which are not a part of any couple and  $F$  and  $M$  together form couples. We further denote the set of couples by  $C = \{\{f_1, m_1\}, \dots, \{f_k, m_k\}\}$ . We denote a couple by  $c = \{f, m\}$ . Throughout this paper, we assume  $|H| \geq 2$ ,  $|D| \geq 4$  and  $|C| \geq 1$ . We also assume that the total number of vacancies in all hospitals in  $H$  is equal to the total number of doctors available, i.e.,  $\sum_{h \in H} \kappa_h = |D|$ .

Consider a couple  $c = \{f, m\}$ . Then, an allocation of the couple  $c$  is an element  $(h, h')$  of  $\bar{H}^2$  where hospital  $h$  is matched with doctor  $f$  and hospital  $h'$  is matched with doctor  $m$ .

For notational convenience, we do not use braces for singleton sets.

## 2.1 Matching

DEFINITION 1 *A matching is a mapping  $\mu : H \cup D \rightarrow \bar{H} \cup 2^D$  such that*

- (i) *for all  $h \in H$ ,  $\mu(h) \subseteq D$  with  $|\mu(h)| \leq \kappa_h$ ,*
- (ii) *for all  $d \in D$ ,  $\mu(d) \in \bar{H}$ , and*
- (iii) *for all  $d \in D$  and  $h \in H$ ,  $\mu(d) = h$  if and only if  $d \in \mu(h)$ .*

The first condition of the definition says that every hospital  $h \in H$  can be matched with at most  $\kappa_h$  many doctors. The second condition says that every doctor can be either matched with exactly one hospital in  $H$  or be unmatched. The third condition captures the basic idea of matching that if a doctor is matched with a hospital, then that hospital is also matched with that doctor.

## 2.2 Preferences

In this section, we introduce the notion of preferences of doctors and hospitals, and present the restrictions on them.

For a set  $X$ , we denote by  $\mathbb{L}(X)$  the set of linear orders, i.e., complete, reflexive, transitive, and antisymmetric binary relations over  $X$ . An element of  $\mathbb{L}(X)$  is called a preference over  $X$ . For  $P \in \mathbb{L}(X)$  and  $k \leq |X|$ , we define the rank  $r_k(P) = x$  if and only if  $|\{y \in X : yPx\}| = k$ . This means that the rank of  $x$  in  $P$  is  $k$  if and only if there are  $k - 1$  alternatives  $y_1, \dots, y_{k-1}$  such that  $y_i \neq x$  and  $y_iPx$  for all  $i \in \{1, \dots, k - 1\}$ .

### 2.2.1 Preferences of Hospitals

A preference of a hospital  $h \in H$ , denoted by  $P_h$ , is a linear order over the feasible sets of doctors  $\{D' \subseteq D : |D'| \leq \kappa_h\}$ , i.e.,  $P_h$  is an element of  $\mathbb{L}(\{D' \subseteq D : |D'| \leq \kappa_h\})$ . We assume hospitals' preferences to be responsive which we define below.

**DEFINITION 2** A preference  $P_h$  of a hospital  $h \in H$  with capacity  $\kappa_h$  satisfies Responsiveness if for any  $D' \subseteq D$  with  $|D'| \leq \kappa_h$  the following hold:

- (i) for any  $d' \in D'$  and any  $d \in D \setminus D'$ ,  $((D' \cup d) \setminus d')P_h D'$  if and only if  $dP_h d'$ , and
- (ii) for any  $D'' \subsetneq D'$ ,  $D'P_h D''$ .

Here the first condition says that there are no complementaries across doctors, and the second condition says that all the doctors are acceptable for hospital  $h$ . Throughout this paper, we assume that hospitals' preferences satisfy responsiveness.

In the following, we define the notion of common preference over individual doctors. We call this preference - Common Preference over Individuals (CPI)

**DEFINITION 3** A collection of preferences  $(P_h)_{h \in H}$  of hospitals in  $H$  is common over individual doctors if for all pairs of hospitals  $h, h' \in H$  and for all  $d, d' \in D$ ,  $dP_h d'$  if and only if  $dP_{h'} d'$ .

Note that, *CPI* implies that all the hospitals have common preference over individual doctors. However, it does not impose any restriction on the preferences of the hospitals over larger subsets of doctors.

Unless mentioned otherwise, we *CPI* at every collection of preferences of the hospital. Under *CPI*, the common restriction of  $(P_h)_{h \in H}$  over individual doctors is defined as  $P^0 \in \mathbb{L}(D)$  such that for all  $d, d' \in D$ ,  $dP^0 d'$  if  $dP_h d'$  for all  $h \in H$ . Throughout this paper, we use  $P^0$  to denote *CPI*. Whenever we consider a *CPI*  $P^0$ , we assume for ease of presentation that the indexation of the doctors in couples is such that  $fP^0 m$  for every couple  $c = \{f, m\} \in C$ , and  $m_i P^0 m_j$  for all  $i < j \in \{1, \dots, k\}$ . This is without loss of generality as we consider only one *CPI* at every given context.

### 2.2.2 Preferences of Doctors

A preference of a doctor  $d \in D$ , denoted by  $P_d$ , is a linear order over  $\bar{H}$ , i.e.,  $P_d$  is an element of  $\mathbb{L}(\bar{H})$ . We assume  $hP_d\emptyset$  for all  $h \in H$  and all  $d \in D$ . Having defined the preferences of the doctors (as singles), now we proceed to define the preferences of the couples.

#### Preferences of Couples

A preference of a couple  $c = \{f, m\} \in C$ , denoted by  $P_c$ , is a linear order over  $\bar{H}^2$ , i.e.,  $P_c$  is an element of  $\mathbb{L}(\bar{H}^2)$ .

In this paper, we intend to deviate from responsiveness in a ‘minimal’ way and study its consequences on stability. We assume that a preference of a couple is almost responsive except in the situations where both the members of the couple get to stay together in some hospital. The usual definition of responsive couple preference means that for two allocations of a couple that differ in the allocation for only one couple member, the couple prefers the allocation where that member is assigned to his/her more preferred hospital. However, here we allow for a couple to violate responsiveness only if the couple gets allocated to the same hospital. We call this ‘preference for togetherness’.

In the following, we define responsiveness for couples’ preferences. The notion of responsiveness is in principle the same as that for a preference of a hospital. However, for the sake of clarity, we present the formal definition below.

**DEFINITION 4** *Let  $c = \{f, m\} \in C$  be any couple. Let  $P_f$  be a preference of  $f$  and  $P_m$  be a preference of  $m$ . Then, a preference  $P_c \in \mathbb{L}(\bar{H}^2)$  of the couple  $c$  satisfies Responsiveness if the following holds. For all  $h, h_1, h_2 \in \bar{H}$ , we have*

(i)  $(h, h_1)P_c(h, h_2)$  if and only if  $h_1P_mh_2$ , and

(ii)  $(h_1, h)P_c(h_2, h)$  if and only if  $h_1P_fh_2$

For a couple  $c$ , by  $\mathcal{D}_c^R$  we denote the set of responsive preferences of  $c$ .

**DEFINITION 5** Let  $c = \{f, m\}$  be any couple. Then, a preference ordering  $\bar{P}_c \in \mathbb{L}(\bar{H}^2)$  of the couple  $c$  satisfies Responsiveness Violated for Togetherness (RVT) if there is a responsive preference  $P_c \in \mathcal{D}_c^R$  of the couple  $c$  such that

- (i) for all  $h \in H$  and all  $(h_1, h_2) \in \bar{H}^2$ ,  $(h, h)P_c(h_1, h_2)$  implies  $(h, h)\bar{P}_c(h_1, h_2)$ ,  
and
- (ii) for all  $(h, h'), (h_1, h_2) \in \bar{H}^2$  such that  $h \neq h'$  and  $h_1 \neq h_2$ ,  $(h, h')P_c(h_1, h_2)$   
if and only if  $(h, h')\bar{P}_c(h_1, h_2)$ .

For a couple  $c$ , by  $\mathcal{D}_c^{RVT}$  we denote the set of RVT preferences of  $c$ .

Note that, RVT implies that couples' preferences can violate responsiveness only in order to be together in some hospital. Further note that, by taking  $h_1 = h_2$  in Condition (i) of Definition 5, it follows that the relative ordering among the allocations where both the doctors of a couple  $c$  are in the same hospital does not change from  $P_c$  to  $\bar{P}_c$ . Thus,  $\bar{P}_c$  is obtained from  $P_c$  by shifting hospital pairs  $(h, h)$  to higher preference positions (lower ranks).

**REMARK 1** In the rest of the paper, we assume that for all hospitals  $h \in H$ , we have  $\kappa_h \geq 2$ .

### 2.2.3 Preference Profiles

In this section, we define the notion of a preference profile.

**DEFINITION 6** A preference profile  $\underline{P}$  for hospitals in  $H$  and doctors in  $D$  with couples  $C$  is defined as a collection of preferences  $(\{\underline{P}_d\}_{d \in D}, \{\underline{P}_c\}_{c \in C}, \{\underline{P}_h\}_{h \in H})$  where for all  $d \in D$ ,  $\underline{P}_d$  is a preference of doctor  $d$ , for all  $c \in C$ ,  $\underline{P}_c$  is a preference of couple  $c$ , and for all  $h \in H$ ,  $\underline{P}_h$  is a preference of hospital  $h$ .

By a matching problem, we mean the set of hospitals with corresponding capacities, the set of doctors with its partition into the set of  $F$  and  $M$ , and a preference profile.

## 2.3 Stability

Our model is formally equivalent to a many-to-many matching market as a couple looks for two positions and hospitals have at least two positions. Thus, one can have different notions of stability based on different types of permissible blocking coalitions.<sup>1</sup>

Blocking pairs can be a hospital and a single doctor, or a pair of hospitals and a couple.

**DEFINITION 7** *Let  $s \in S$  be a single doctor and  $h \in H$  be a hospital. Then, for two matchings  $\mu, \mu'$ , we write  $\mu \rightarrow_{\{h,s\}} \mu'$  if*

- (i)  $\mu'(h) = (\mu(h) \setminus D') \cup s$  for some (possibly empty)  $D' \subseteq \mu(h)$ , and
- (ii)  $\mu'(h') = \mu(h') \setminus s$  for all  $h' \neq h$ .

The statement  $\mu \rightarrow_{\{h,s\}} \mu'$  captures the idea that the hospital  $h$  and the single doctor  $s$  can change their allocations under  $\mu$  to that under  $\mu'$  because  $h$  can release some doctors and hire  $s$ . Moreover, hospitals other than  $h$  continue to retain their allocations unless they were matched with  $s$ .

**DEFINITION 8** *Let  $c = \{f, m\} \in C$  be a couple. Let  $(h_1, h_2) \in H^2$ . Then, for two matchings  $\mu, \mu'$ , we write  $\mu \rightarrow_{\{(h_1, h_2), c\}} \mu'$  if*

- (i)  $\mu'(h_1) = (\mu(h_1) \setminus D') \cup f$ , for some (possibly empty)  $D' \subseteq \mu(h_1)$ ,
- (ii)  $\mu'(h_2) = (\mu(h_2) \setminus D'') \cup m$ , for some (possibly empty)  $D'' \subseteq \mu(h_2)$ , and
- (iii)  $\mu'(h) = \mu(h) \setminus \{f, m\}$  for all  $h \in H \setminus \{h_1, h_2\}$ .

The statement  $\mu \rightarrow_{\{(h_1, h_2), c\}} \mu'$  captures the idea that the pair of hospitals  $\{h_1, h_2\}$  and the couples  $c$  can change their allocations under  $\mu$  to that under  $\mu'$  because each of the hospitals can release some doctors and hire one or

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<sup>1</sup>See Roth (1984) [8], Roth (1984) [9], Konishi and Ünver (2006) [7], Echenique and Oviedo (2006) [4] for some alternative notions of stability in many-to-many matchings.

both doctors of the couple. Moreover, hospitals other than the hospitals in  $\{h_1, h_2\}$  continue to retain their allocations unless they were matched with any member of the couple  $c$ .

In the following, we introduce the notion of a blocking.

**DEFINITION 9** *Let  $h \in H$  be a hospital and  $s \in S$  be a single doctor. Then,  $(h, s)$  blocks  $\mu$  through  $\mu'$  if  $\mu \rightarrow_{\{h,s\}} \mu'$ ,  $\mu'(h)P_h\mu(h)$ , and  $hP_s\mu(s)$ .*

**DEFINITION 10** *Let  $c = \{f, m\}$  be a couple. Let  $(h_1, h_2) \in H^2$ . Then,  $((h_1, h_2), c)$  blocks  $\mu$  through  $\mu'$  if  $\mu \rightarrow_{\{(h_1, h_2), c\}} \mu'$ ,  $\mu'(c)P_c\mu(c)$  and  $\mu'(h)P_h\mu(h)$  for all  $h \in \{h_1, h_2\}$  such that  $\mu'(h) \not\subseteq \mu(h)$ .*

Thus, a hospital  $h$  and a single doctor  $s$  block  $\mu$  if there exists a matching  $\mu'$  with  $\mu \rightarrow_{\{h,s\}} \mu'$  such that both  $h$  and  $s$  are better off in  $\mu'$  as compared to  $\mu$ . Similarly, hospitals  $(h_1, h_2)$  and couple  $c$  block  $\mu$  if there exists a matching  $\mu'$  with  $\mu \rightarrow_{\{(h_1, h_2), c\}} \mu'$  such that (i)  $c$  as a couple is better off in  $\mu'$  as compared to  $\mu$ , and (ii) every hospital in  $\{h_1, h_2\}$  which is receiving some new doctor from the couple  $c$  in  $\mu'$  is better off in  $\mu'$  as compared to  $\mu$ .

Note that, we do not require that both hospitals  $h_1$  and  $h_2$  must be better off in  $\mu'$ . In particular, if a hospital is not receiving any new doctor from the couple  $c$  in the matching  $\mu'$ , then that hospital has no control over the block, and hence we do not require it to be better off by the blocking.

**DEFINITION 11** *A matching  $\mu$  is stable if it is not blocked by any pair  $(h, s)$  where  $h \in H$  and  $s \in S$ , or by any pair  $((h_1, h_2), c)$  where  $(h_1, h_2) \in H^2$  and  $c \in C$ .*

Note that, members of a couple move according to their couple preferences, in particular, a member of a couple does not block according to his/her individual preference.

**REMARK 2** *Note that, by our assumptions on the preferences of hospitals and doctors, hospitals find all doctors acceptable and doctors find all hospitals acceptable. Therefore, all matchings are individually rational.*

## 2.4 Algorithm

In this section we present a well-known doctor proposing deferred acceptance algorithm (DPDA) that we will use throughout the paper to match hospitals with doctors. Our existence proof uses a modification of the Gale-Shapley deferred acceptance algorithm with the doctors making the proposals (DPDA). We give a very short description of DPDA.

*DPDA:* In stage 1 of the algorithm, all doctors simultaneously propose to their most preferred hospitals. Each hospital  $h$  provisionally accepts up to  $\kappa_h$  most preferred doctors. If a hospital has received more than  $\kappa_h$  proposals, then it rejects all the doctors after its  $\kappa_h$  most preferred doctors. In any step  $k$ , the unmatched doctors propose to their most preferred hospital from the remaining set of hospitals who have not rejected them in any of the earlier steps. In any stage of DPDA, since each hospital accepts  $\kappa_h$  most preferred doctors, it may reject some doctors that it had provisionally accepted earlier. Hospitals whose provisional list of accepted doctors is less than their maximum capacity can still add to their accepted list if they have received fresh proposals. Thus the algorithm finally terminates when each doctor is matched or has been rejected by all hospitals.

**REMARK 3** *Note that in DPDA, each individual doctor proposes according to his/her individual preference. Thus, couples do not play any role in the DPDA.*

Now we present another well-known algorithm called Serial Dictatorship Algorithm (SDA). In the SDA, the highest-ranked doctor according to CPI chooses his/her most-preferred hospital, and in general the  $k$ -highest ranked doctor chooses his/her most preferred hospital among the hospitals with available vacancy after all the higher ranked doctors have made their choices.

**REMARK 4** *DPDA and SDA produce the same matching under CPI.*

## 2.5 Conditions for Stability under RVT

In this section, we provide conditions on couples' preferences satisfying RVT that guarantee the existence of stable matchings.

Let  $P_C^0 = (\{P_d^0\}_{d \in D \setminus S}, \{P_c^0\}_{c \in C})$  be a given collection of preferences of the doctors that are in some couple, and of the couples in  $C$ . Let  $P^0$  be an CPI. Recall that, we assume the indexation of the doctors in couples to be such that  $f_i P^0 m_i$  for all  $i \leq k$ . Then, by  $\mathcal{D}(P_C^0, P^0)$  we denote the set of preference profiles where doctors  $d \in D \setminus S$  and couples  $c \in C$  have preferences as in  $P_C^0$  and the CPI is  $P^0$ , i.e.,  $\mathcal{D}(P_C^0, P^0) = \{\underline{P} : \underline{P}_d = P_d^0 \text{ for all } d \in D \setminus S, \underline{P}_c = P_c^0 \text{ for all } c \in C, \text{ and } P^0 \text{ is the CPI of } \underline{P}\}$ .

**CONDITION 1** *Suppose  $(P_C^0, P^0)$  is such that  $P_c^0 \in \mathcal{D}_c^{RVT}$  for all  $c \in C$  and  $P^0$  is the CPI. Then, for each couple  $c = \{f, m\} \in C$  and for all  $h, h' \in H$ ,  $(h, h)P_c^0(h', h)$  implies  $hP_f^0 h'$ .*

Condition 1 says the following. Consider a couple  $c = \{f, m\}$ . Suppose that a RVT preference of  $c$  prefers an allocation where both the doctors of  $c$  are in  $h$  to another allocation where  $f$  is in  $h'$  and  $m$  is in  $h$ . Then, it must be the case that  $f$  prefers hospital  $h$  over  $h'$  according to its individual preference. Thus, Condition 1 implies that couples' preferences are always responsive with respect to  $f$ , i.e., to be together, compromise is always made by  $m$ .

Now we state our Theorem, which says that a stable matching exists at every preference profile, if and only if the couples' preferences satisfy Condition 1. Thus, Condition 1 always gives a stable matching for every preference profile. However, if a couple violates Condition 1, then we can always find a preference profile with no stable matching.

**THEOREM 1** *Suppose  $P_C^0$  is such that  $P_c^0 \in \mathcal{D}_c^{RVT}$  for all  $c \in C$ . Then, a stable matching exists at every preference profile in  $\mathcal{D}(P_C^0, P^0)$  if and only if  $(P_C^0, P^0)$  satisfies Condition 1.*

*Proof:* [Necessity] Suppose  $P_C^0$  is such that  $P_c^0 \in \mathcal{D}_c^{RVT}$  for all  $c \in C$ . Suppose further that  $(P_C^0, P^0)$  does not satisfy Condition 1. We show that there is

a preference profile in  $\underline{\mathcal{D}}(P_C^0, P^0)$  with no stable matching. Since  $(P_C^0, P^0)$  does not satisfy Condition 1, there exists a couple  $c = \{f, m\}$ , two hospitals  $h_1, h_2 \in H$ , such that  $(h_1, h_1)P_c^0(h_2, h_1)$  and  $h_2P_f^0h_1$ .

Since  $h_2P_f^0h_1$ , it follows from the definition of RVT that  $(h_2, h_2)P_c^0(h_1, h_2)$ . Consider a preference profile  $\underline{P}$  in  $\underline{\mathcal{D}}(P_C^0, P^0)$  such that the CPI  $P^0$  satisfies  $fP^0d_1P^0d_2P^0m$ , and  $r_1(\underline{P}_{d_1}) = h_1$ ,  $r_1(\underline{P}_{d_2}) = h_2$ . Suppose  $|\{d : dP^0f \text{ and } r_1(\underline{P}_d) = h_2\}| = \kappa_{h_2} - 2$ ,  $|\{d : dP^0f \text{ and } r_1(\underline{P}_d) = h_1\}| = \kappa_{h_1} - 2$ , and  $|\{d : dP^0f \text{ and } r_1(\underline{P}_d) = h\}| = \kappa_h$  for all  $h \neq h_1, h_2$ . Suppose further that the preferences of all couples other than  $c$  satisfy responsiveness. We show that there is no stable matching at  $\underline{P}$ . Suppose  $\mu$  is a stable matching at  $\underline{P}$ . Since  $\mu$  is stable, it must be that  $\mu(d) = r_1(\underline{P}_d)$  for all  $dP^0f$ . Moreover, since  $|\{d : dP^0f \text{ and } r_1(\underline{P}_d) = h_2\}| = \kappa_{h_2} - 2$  and  $|\{d : dP^0f \text{ and } r_1(\underline{P}_d) = h_1\}| = \kappa_{h_1} - 2$ , there are exactly 2 positions left in  $h_2$  and exactly  $n$  positions left in  $h_1$  and  $h_2$  after matching all the doctors  $d$  such that  $dP^0f$ . Now, we distinguish the following cases for the allocation of the couple  $c$ .

- Suppose  $\mu(c) = (h_2, h_2)$ . Then,  $(h_2, d_2)$  blocks  $\mu$  as  $r_1(\underline{P}_{d_2}) = h_2$  and  $d_2P^0m$ .
- Suppose  $\mu(c) = (h_1, h_2)$ . Then,  $((h_2, h_2), c)$  blocks  $\mu$  as  $fP^0d_1P^0d_2$  and by RVT,  $(h_2, h_2)P_c^0(h_1, h_2)$ .
- Suppose  $\mu(c) = (h_1, h_1)$ . Then,  $(h_1, d_1)$  blocks  $\mu$  as  $r_1(\underline{P}_{d_1}) = h_1$  and  $d_1P^0m$ .
- Suppose  $\mu(c) = (h_2, h_1)$ . Then,  $((h_1, h_1), c)$  blocks  $\mu$  as  $fP^0d_1P^0d_2$  and by RVT,  $(h_1, h_1)P_c^0(h_2, h_1)$ .

This completes the proof of the necessity part.

[Sufficiency] The proof of this part is constructive. Suppose  $P_C^0$  is such that  $P_c^0 \in \mathcal{D}_c^{RVT}$  for all  $c \in C$ . Suppose further that  $(P_C^0, P^0)$  satisfies Condition 1. Take  $\underline{P} \in \underline{\mathcal{D}}(P_C^0, P^0)$ . We construct an algorithm that produces a stable matching in  $\underline{P}$ . For each couple  $c = \{f, m\}$  and each hospital  $h$ , define

the conditional preference of  $m$  given  $h$ ,  $P_{m|h}^0 \in \mathbb{L}(H)$ , in the following way:  $h' P_{m|h}^0 h''$  if and only if  $(h, h') P_c^0(h, h'')$ . Recall that, by our initial assumption on CPI,  $m_i P^0 m_j$  for all  $i < j \in \{1, \dots, k\}$ . In the following, we present our algorithm.

*Algorithm 1:* Algorithm 1 involves  $k + 1$  steps. We present the 1<sup>st</sup> step, and a general step of the algorithm.

*Step 1:* Use SDA to match all doctors ranked above  $m_1$  according to  $P^0$ . Let  $f_1$  is matched to some hospital, say  $h$ . Then, match  $m_1$  using SDA, where  $m_1$  bids according to  $P_{m_1|h}^0$ .

⋮

*Step  $j$ :* Use SDA to match all doctors ranked below  $m_{j-1}$  and above  $m_j$  according to  $P^0$ . Let  $f_j$  is matched with some hospital, say  $h$ , then match  $m_j$  by SDA where  $m_j$  bids according to  $P_{m_j|h}^0$ .

⋮

Continue this process till Step  $k$  and then match the remaining doctors by SDA at step  $k + 1$ .

We show that Algorithm 1 produces a stable matching at  $\mathcal{P}$ . Let  $\mu$  be the outcome of Algorithm 1. We distinguish the following cases.

*Case 1:* Suppose  $(h, s)$  blocks  $\mu$  through  $\mu'$ . Note that, by the nature of Algorithm 1, all doctors that propose before  $s$  are more preferred to  $s$  according to the CPI. Since  $s \notin \mu(h)$ , by the nature of Algorithm 1, this means either  $d P^0 s$  for all  $d \in \mu(h)$  and  $|\mu(h)| = \kappa_h$ , or  $\mu(s) \mathcal{P}_s h$ . If  $d P^0 s$  for all  $d \in \mu(h)$  and  $|\mu(h)| = \kappa_h$ , then by responsiveness of hospitals' preferences, we have  $\mu(h) \mathcal{P}_h \mu'(h)$ . Thus hospital  $h$  does not block with  $s$ . On the other hand, if  $\mu(s) \mathcal{P}_s h$ , then clearly  $s$  does not block with hospital  $h$ . This contradicts that  $(h, s)$  blocks  $\mu$ .

*Case 2:* Suppose  $((h_1, h_2), c)$  blocks  $\mu$  where  $c = \{f, m\}$ .

We first show  $((\mu(f), h_2), c)$  blocks  $\mu$ . Note that, if  $\mu(f) = h_1$ , then there is nothing to show.

First, we claim that  $\mu(f)P_f^0h_1$ . Assume for contradiction that  $h_1P_f^0\mu(f)$ . Note that, by the nature of Algorithm 1,  $f$  proposes according to preference  $P_f^0$ , and all the doctors that propose before  $f$  are preferred to  $f$  according to the CPI  $P^0$ . Since  $f \notin \mu(h_1)$ , it must be that  $dP^0f$  for all  $d \in \mu(h_1)$  and  $|\mu(h_1)| = \kappa_{h_1}$ . By responsiveness of hospitals' preferences, this means  $\mu(h_1)\underline{P}_{h_1}\mu'(h_1)$ . This is a contradiction to the fact that  $((h_1, h_2), c)$  blocks  $\mu$  through  $\mu'$ . Therefore,  $\mu(f)P_f^0h_1$ .

Now, we show that  $(\mu(f), h_2)P_c^0(h_1, h_2)$ . Suppose  $h_1 \neq h_2$ . Since  $\mu(f)P_f^0h_1$ , by RVT it follows that  $(\mu(f), m)P_c^0(h_1, h_2)$ . Now suppose  $h_1 = h_2 = h$ . Since  $\mu(f)P_f^0h_1$ , by Condition 1 we have  $(\mu(f), h_2)P_c^0(h, h)$ .

Since  $((h_1, h_2), c)$  blocks  $\mu$ ,  $(\mu(f), h_2)P_c^0(h_1, h_2)$  implies  $((\mu(f), h_2), c)$  blocks  $\mu$ . Now we proceed to show that  $((\mu(f), h_2), c)$  cannot block  $\mu$ .

Suppose  $\mu(f) = h$  for some  $h \in H$ . Then, by Algorithm 1 and the definition of  $P_{m|h}^0$ , we have  $h_2P_{m|h}^0\mu(m)$ . By the nature of Algorithm 1, all doctors that propose before  $m$  are preferred to  $m$  according to the CPI  $P^0$ . Since  $m \notin \mu(h_2)$ , it must be that  $dP^0m$  for all  $d \in \mu(h_2)$  and  $|\mu(h_2)| = \kappa_{h_2}$ . By responsiveness of hospitals' preferences  $\mu(h_2)\underline{P}_{h_2}\mu'(h_2)$ . This contradicts that  $((\mu(f), h_2), c)$  is a block.

This completes the proof of the sufficiency part. ■

## 2.6 Existence of Stable Matching with Adjacent CPI

In this section, we consider restrictions on the Identical Hospital Preference and investigate the existence of stable matching under those restrictions. We relax the RVT condition on the preferences of the couples, by assuming that a couple can have any preference over the sets of hospitals irrespective of the preferences of the individual doctors in that couple over individual hospitals. Such preferences of couples are called unrestricted preferences. More formally, the set of unrestricted preferences of a couple  $c \in C$  is  $\mathbb{L}(\bar{H}^2)$ .

However, we still assume that any couple prefers an allocation where each member of the couple is matched to some hospital over an allocation where at least one member of the couple is unmatched.

In the following, we define lexicographic preferences of the hospitals. Let  $P_h$  be a preference of a hospital  $h$  and  $D' \subseteq D$ . Then, define  $r_k(P_h, D') = d$  if and only if  $|\{d' \in D' : d' P_h d\}| = k$ .

**DEFINITION 12** *A preference  $P_h$  of a hospital  $h$  with capacity  $\kappa_h$  is called lexicographic if for all  $D', D'' \subseteq D$  with  $|D''| = |D'| \leq \kappa_h$ ,  $D' P_h D''$  if and only if there exists  $l \in \{1, \dots, |D'|\}$  such that  $r_l(P_h, D') P_h r_l(P_h, D'')$  and  $r_m(P_h, D') = r_m(P_h, D'')$  for all  $m < l$ . The set of lexicographic preferences of a hospital  $h$  is denoted by  $\mathcal{D}_h^L$ .*

**DEFINITION 13** *A preference profile  $\underline{P}$  with lexicographic hospitals' preferences and RVT couples' preferences is defined as  $(\{\underline{P}_d\}_{d \in D}, \{\underline{P}_c\}_{c \in C}, \{\underline{P}_h\}_{h \in H})$  where  $\underline{P}_d \in \mathbb{L}(\bar{H})$  for all  $d \in D$ ,  $\underline{P}_c \in \mathcal{D}_c^{RVT}$  for all  $c \in C$ ,  $\underline{P}_h \in \mathcal{D}_h^L$  for all  $h \in H$ , and hospitals' preferences satisfy CPI. The set of preference profiles with lexicographic hospitals' preferences and RVT couples' preferences is denoted by  $\mathcal{D}^{LR}$ .*

**DEFINITION 14** *A preference profile  $\underline{P}$  with lexicographic hospitals' preferences and unrestricted couples' preferences is defined as  $(\{\underline{P}_d\}_{d \in D}, \{\underline{P}_c\}_{c \in C}, \{\underline{P}_h\}_{h \in H})$  where  $\underline{P}_d \in \mathbb{L}(\bar{H})$  for all  $d \in D$ ,  $\underline{P}_c \in \mathbb{L}(\bar{H}^2)$  for all  $c \in C$ ,  $\underline{P}_h \in \mathcal{D}_h^L$  for all  $h \in H$ , and hospitals' preferences satisfy CPI. The set of all preference profiles with lexicographic hospitals' preferences and unrestricted couples' preferences is denoted by  $\mathcal{D}^{LU}$ .*

In the following, we introduce the notion of Adjacent CPI (ACPI). ACPI implies that for any couple  $c = \{f, m\}$ , there cannot be a doctor other than the members of the couple that lies in-between  $f$  and  $m$  in the CPI. Recall that, whenever we consider an CPI  $P^0$ , we assume for any couple  $c = \{f, m\}$  that  $f P^0 m$ . Below, we provide a formal definition of ACPI.

**DEFINITION 15** Let  $P^0$  be an CPI. Then,  $P^0$  satisfies *Adjacent CPI (ACPI)* if for any couple  $c = \{f, m\} \in C$  there does not exist any  $d \in D \setminus \{f, m\}$ , such that  $fP^0dP^0m$ .

**DEFINITION 16** Let  $P^0$  be an CPI. Then, the collection of preference profiles where

- hospitals in  $H$  have lexicographic preferences and couples' preferences satisfy RVT, denoted by  $\mathcal{D}^{LR}(P^0)$ , is defined as  $\mathcal{D}^{LR}(P^0) = \{\underline{P} \in \mathcal{D}^{LR} : P^0 \text{ is the CPI of } \underline{P}\}$ ,
- hospitals in  $H$  have lexicographic preferences and couples' preferences are unrestricted, denoted by  $\mathcal{D}^{LU}(P^0)$ , is defined as  $\mathcal{D}^{LU}(P^0) = \{\underline{P} \in \mathcal{D}^{LU} : P^0 \text{ is the CPI of } \underline{P}\}$ .

Note that, for any CPI  $P^0 \in \mathbb{L}(D)$ ,  $\mathcal{D}^{LR}(P^0) \subseteq \mathcal{D}^{LU}(P^0)$ .

Our next theorem says that ACPI is a necessary condition for the existence of stable matching at every preference profile where hospitals have lexicographic preferences and couples' preferences satisfy RVT.

**THEOREM 2** Let  $P^0$  be an CPI. Suppose a stable matching exists at every preference profile in  $\mathcal{D}^{LR}(P^0)$ . Then,  $P^0$  satisfies ACPI.

*Proof:* Consider an CPI  $P^0$ . Suppose  $P^0$  does not satisfy ACPI. We show that there exists a preference profile in  $\mathcal{D}^{LR}(P^0)$  with no stable matching. Since  $P^0$  does not satisfy ACPI, there exists a couple  $c = \{f, m\}$  and doctor  $d_1 \notin c$  such that  $fP^0d_1P^0m$ . Take two hospitals  $h_1, h_2 \in H$  and  $d_2 \in D$  such that  $mP^0d_2$ . Consider a preference profile  $\underline{P}$  in  $\mathcal{D}^{LR}(P^0)$  such that  $r_1(\underline{P}_{d_1}) = h_1$  and  $h_2\underline{P}_{d_1}h$  for all  $h \in H \setminus \{h_1\}$ . Further,  $r_1(\underline{P}_{d_2}) = h_2$  and  $h_1\underline{P}_{d_2}h$  for all  $h \in H \setminus \{h_2\}$ . Let the preference of couple  $c$  be such that  $r_1(\underline{P}_m) = h_1$  and  $h_2\underline{P}_mh$  for all  $h \in H \setminus \{h_1\}$ . Also,  $r_1(\underline{P}_f) = h_2$  and  $h_1\underline{P}_fh$  for all  $h \in H \setminus \{h_2\}$ . We further assume that  $(h_1, h_1)\underline{P}_c(h_2, h_1)$  and  $(h_1, h_1)\underline{P}_c(h_2, h_2)$ . An allocation of the couple  $c$ , where at least one doctor in  $c$  is matched to a hospital  $h \notin \{h_1, h_2\}$  is assumed to be responsive and is ranked

below all the allocations, where both the members of the couple are either matched to  $h_1$  or  $h_2$ . Let  $|\{d : r_1(\underline{P}_d) = h_2\}| = \kappa_{h_2} - 2$ ,  $|\{d : r_1(\underline{P}_d) = h_1\}| = \kappa_{h_1} - 2$ , and  $|\{d : r_1(\underline{P}_d) = h\}| = \kappa_h$  for all  $h \neq h_1, h_2$ . Finally, we assume that the preferences of all couples other than  $c$  satisfy responsiveness. We show that there is no stable matching at this preference profile. Let  $\mu$  be a stable matching at this preference profile. Since  $\sum_{h \in H} \kappa_h = |D|$ , by the preferences of  $d_1, d_2$  and  $c$  we must have  $\mu(d) \in \{h_1, h_2\}$  for all  $d \in \{f, m, d_1, d_2\}$ . Also, since  $\mu$  is stable, by the given preferences of the doctors in  $D$ , it must be that  $\mu(d) = r_1(\underline{P}_d)$  for all  $d \in D \setminus \{f, m, d_1, d_2\}$ . Now we distinguish the following cases for the allocation of the couple  $c$ .

- Suppose  $\mu(c) = (h_1, h_1)$ . Then,  $(h_1, d_1)$  blocks  $\mu$  as  $h_1 \underline{P}_{d_1} h_2$  and  $d_1 P^0 m$ .
- Suppose  $\mu(c) = (h_2, h_1)$ . Then,  $((h_1, h_1), c)$  blocks  $\mu$  as  $f P^0 d_1 P^0 d_2$  and  $(h_1, h_1) \underline{P}_c (h_2, h_1)$ .
- Suppose  $\mu(c) = (h_1, h_2)$ . Note that since  $h_2 \underline{P}_f h_1$ , by RVT  $(h_2, h_2) \underline{P}_c (h_1, h_2)$ . This, together with the fact that  $f P^0 d_1 P^0 d_2$ , means  $\mu$  is blocked by  $((h_2, h_2), c)$ .
- Suppose  $\mu(c) = (h_2, h_2)$ . Since  $(h_1, h_1) \underline{P}_c (h_2, h_2)$ ,  $f P^0 d_1 P^0 d_2$  and hospitals' preferences are lexicographic,  $\mu$  is blocked by  $((h_1, h_1), c)$ .

This completes the proof of Theorem 2. ■

Now we prove the converse of Theorem 2 which states that if the hospitals' preferences satisfy ACPI, then a stable matching exists at every preference profile where hospitals' preferences are lexicographic and couples' preferences satisfy RVT. However, we prove a stronger version of this, where we show that if the hospitals' preferences satisfy ACPI, then stable matching exists at every preference profile where hospitals' preferences are lexicographic and couples' preferences are unrestricted.

**THEOREM 3** *Let  $P^0$  be an CPI. Suppose  $P^0$  satisfies ACPI. Then, a stable matching exists at every preference profile in  $\mathcal{D}^{LU}(P^0)$ .*

*Proof:* The proof of Theorem 3 is constructive. Suppose  $P^0$  satisfies ACPI. We construct an algorithm that produces a stable matching at every preference profile in  $\mathcal{D}^{LU}(P^0)$ . Take  $\underline{P} \in \mathcal{D}^{LU}(P^0)$ . Recall that, by our initial assumption on CPI,  $m_i P^0 m_j$  for all  $i < j \in \{1, \dots, k\}$ . Since  $P^0$  satisfies ACPI, this means  $f_i P^0 f_j$  for all  $i < j \in \{1, \dots, k\}$ . Now, we present our algorithm.

*Algorithm 2:* Algorithm 2 involves  $k + 1$  steps. We present the 1<sup>st</sup> step, and a general step of the algorithm. At every step, a couple proposes to a set of hospitals. Whenever a hospital receives a set of proposals at some step, it accepts all proposals if it has adequate vacancies, otherwise it rejects all the proposals.

*Step 1:* Use SDA to match all the doctors ranked above  $f_1$  according to  $P^0$ . Let  $c_1 = \{f_1, m_1\}$  propose on  $r_1(\underline{P}_{c_1})$ . If at least one doctor of the couple is rejected, then let  $c_1$  propose to  $r_2(\underline{P}_{c_1})$ , and so on. Continue this process till some  $l$  such that all the members of the couple  $c_1$  are accepted by corresponding hospital in  $r_l(\underline{P}_{c_1})$ .

⋮

*Step  $j$ :* Use SDA to match all the doctors that ranked below  $c_{j-1} = \{f_{j+1}, m_{j+1}\}$  and above  $c_j = \{f_j, m_j\}$  according to  $P^0$ . Let  $c_j$  propose on  $r_1(\underline{P}_{c_j})$ . If at least one doctor of the couple is rejected, then let  $c_j$  propose to  $r_2(\underline{P}_{c_j})$ , and so on. Continue this process till some  $l$  such that all the members of the couple  $c_j$  are accepted by corresponding hospital in  $r_l(\underline{P}_{c_j})$ .

⋮

Continue this process till Step  $k$  and then match the remaining doctors by SDA at the step  $k + 1$ .

We show that Algorithm 2 produces a stable matching at  $\underline{P}$ . Let  $\mu$  be the outcome of Algorithm 2. We distinguish the following cases.

*Case 1:* Suppose  $(h, s)$  blocks  $\mu$  through  $\mu'$ . Note that, by the nature of Algorithm 2, all the doctors that propose before  $s$  are preferred to  $s$  according

to the ACPI  $P^0$ . Moreover, for any  $c = \{f, m\} \in C$ , if  $fP^0s$ , then by ACPI,  $mP^0s$ . Since  $s \notin \mu(h)$ , by the nature of Algorithm 2, we have either  $dP^0s$  for all  $d \in \mu(h)$  and  $|\mu(h)| = \kappa_h$ , or  $\mu(s)\underline{P}_sh$ . If  $dP^0s$  for all  $d \in \mu(h)$  and  $|\mu(h)| = \kappa_h$ , then by responsiveness of hospitals' preferences, we have  $\mu(h)\underline{P}_h\mu'(h)$ . Therefore hospital  $h$  does not block with  $s$ . On the other hand, if  $\mu(s)\underline{P}_sh$ , then  $s$  does not block with hospital  $h$ . This contradicts that  $(h, s)$  blocks  $\mu$ .

*Case 2:* Suppose  $((h_1, h_2), c)$  blocks  $\mu$  through  $\mu'$  where  $c = \{f, m\}$ . Then, it must be that  $(h_1, h_2)\underline{P}_c(\mu(f), \mu(m))$ . By the nature of Algorithm 2, this means couple  $c$  proposes to  $(h_1, h_2)$  before proposing to  $(\mu(f), \mu(m))$ , and some hospital, say  $h_i \in \{h_1, h_2\}$ , rejects at least one member of the couple  $c$ . We look at the following cases

*Case 2.1:* Suppose  $h_1 \neq h_2$ . Since  $h_i$  rejects a doctor from couple  $c$ , it must be that  $h_i$  has no vacancies when  $c$  proposes to  $(h_1, h_2)$ . This means that if  $((h_1, h_2), c)$  blocks  $\mu$  through  $\mu'$  then  $\mu(h_i)\underline{P}_{h_i}\mu'(h_i)$ . This together with the fact that  $\mu'(h_i) \not\subseteq \mu(h_i)$ , is a contradiction to the definition of a block.

*Case 2.2:* Suppose  $h_1 = h_2$ . Because  $h_1$  rejects at least one member of  $c$ , it must be that  $h_1$  has less than 2 vacancies when  $c$  proposes to  $(h_1, h_1)$ . Let  $D'$  be the set of doctors that are present in  $h_1$  at the time when  $c$  makes a proposal to  $(h_1, h_1)$ . By ACPI and the nature Algorithm 2, this means each doctor in  $D'$  is preferred to both the doctors of the couple  $c$ . Again, by the nature of Algorithm 2, it follows that  $D' \subseteq \mu(h_1)$ . This means  $h_1$  must release some doctors from  $D'$  for the block  $((h_1, h_1), c)$  to block  $\mu$  through  $\mu'$ . Since  $d\underline{P}_{h_1}f\underline{P}_{h_1}m$  and  $\underline{P}_{h_1}$  is lexicographic, we have  $\mu(h_1)\underline{P}_{h_1}\mu'(h_1)$ . This together with the fact that  $\mu'(h_1) \not\subseteq \mu(h_1)$ , is a contradiction to the definition of a block. This completes the proof of Theorem 3. ■

In what follows, we show by means of an example that the lexicographic assumption on the hospitals' preferences is necessary for Theorem 3. In other words, we show that if hospitals' preferences are not lexicographic, then

existence of a stable matching is not guaranteed even if hospitals' preferences follow ACPI. In fact, we show a stronger version where the existence of a stable matching is not guaranteed under the additional assumption that couples' preferences satisfy RVT.

**EXAMPLE 1** Consider a matching problem where  $H = \{h_1, h_2\}$  with  $\kappa_{h_1} = \kappa_{h_2} = 2$ ,  $D = \{f, m, s_1, s_2\}$  and there is exactly one couple  $c = \{f, m\}$  in  $C$ . The preferences of individual doctors, preference of the couple and ACPI of hospitals on the set of individual doctors is given in Table 1. The couple's preferences over pairs where one member is matched with a hospital  $h \in H$  and the other one is unmatched is not shown in the table, but assumed to be responsive and ranked below the shown pairs. Finally, we assume,  $\{f, m\}P_{h_1}\{s_1, s_2\}$ . Note that, the preference of the hospital  $h_1$  is not lexicographic. Further note that, the preference of the couple  $c$  satisfies RVT.

$P^0$	$P_{s_1}$	$P_{s_2}$	$P_f$	$P_m$	$P_c$
$s_1$	$h_1$	$h_2$	$h_1$	$h_2$	$(h_1, h_1)$
$f$	$h_2$	$h_1$	$h_2$	$h_1$	$(h_2, h_2)$
$m$					$(h_1, h_2)$
$s_2$					$(h_2, h_1)$

Table 1: Preferences

Clearly, for a stable matching  $\mu$ , each hospital should get exactly 2 doctors. We consider all 4 possible cases of couple matching.

- Suppose  $\mu(c) = (h_1, h_1)$ . Since  $s_1P^0m$  and  $h_1P_{s_1}h_2$ ,  $\mu$  is blocked by  $(h_1, s_1)$ .
- Suppose  $\mu(c) = (h_2, h_2)$ . Since  $\{f, m\}P_{h_1}\{s_1, s_2\}$  and  $(h_1, h_1)P_c(h_2, h_2)$ ,  $\mu$  is blocked by  $((h_1, h_1), c)$ .
- Suppose  $\mu(c) = (h_1, h_2)$  or  $(h_2, h_1)$ . We show  $\mu$  is blocked when  $\mu(c) = (h_1, h_2)$ , the proof of the same for  $\mu(c) = (h_2, h_1)$  can be obtained

by changing the roles of  $f$  and  $m$ . Suppose  $\mu(c) = (h_1, h_2)$ . Since  $h_1 P_{s_1} h_2$  and  $s_1 P^0 s_2$ , if  $\mu(h_1) = s_2$ , then  $\mu$  is blocked by  $(h_1, s_1)$ . Now suppose  $\mu(s_1) = h_1$ . Since  $\mu(f) = h_1$ , this means  $\mu(s_2) = h_2$ . Then,  $(h_2, h_2) P_c (h_1, h_2)$  and  $f P^0 s_2$  imply  $\mu$  is blocked by  $((h_2, h_2), c)$ .

This shows that there is no stable matching for the given matching problem.

## 2.7 Matching Market with Non-Identical Hospital Preferences and Couples

In this section, we investigate what happens if the CPI condition is slightly relaxed. However, we assume that for all couples  $c = \{f, m\} \in C$  and for all hospitals  $h \in H$ ,  $f P_h m$ .

In what follows, we show by the means of an example that the CPI assumption on hospitals' preferences is necessary for Theorem 1. In other words, we show that if hospitals' preferences do not satisfy CPI, then the existence of a stable matching is not guaranteed even if couples' preferences satisfy Condition 1.

**EXAMPLE 2** Consider a matching problem where  $H = \{h_1, h_2, h_3\}$  with  $\kappa_{h_1} = \kappa_{h_2} = \kappa_{h_3} = 2$ ,  $D = \{f, m, s_1, s_2, s_3, s_4\}$ , and there is exactly one couple  $c = \{f, m\}$  in  $C$ . The preferences of individual doctors, preference of the couple, and preferences of hospitals on the set of individual doctors is given in Table 2. The couple's preference over pairs where one member is matched with a hospital and the other one is unmatched is not shown in the table, but assumed to be responsive and ranked below the shown pairs. Note that, the preference of the couple  $c$  satisfies Condition 1.

Let  $\mu$  be a stable matching at the preference profile given in Table 2. Since  $\mu$  is a stable matching and  $r_1(P_{h_1}) = s_3$  and  $r_1(P_{s_3}) = h_1$ , it must be that  $\mu(s_3) = h_1$ . Similarly,  $r_1(P_{h_2}) = s_4$  and  $r_1(P_{s_4}) = h_2$ , it must be that  $\mu(s_4) = h_2$ . Thus we can just consider a reduced Matching problem, where  $\kappa_{h_1} = \kappa_{h_2} = 1$  and  $\kappa_{h_3} = 2$  and  $s_1, s_2, f, m$  are the only doctors to be matched.

$P_{h_1}$	$P_{h_2}$	$P_{h_3}$	$P_{s_1}$	$P_{s_2}$	$P_{s_3}$	$P_{s_4}$	$P_f$	$P_m$	$P_c$
$s_3$	$s_4$	$s_3$	$h_2$	$h_3$	$h_1$	$h_2$	$h_1$	$h_2$	$(h_1, h_2)$
$s_4$	$s_3$	$s_4$	$h_1$	$h_1$	$h_2$	$h_1$	$h_3$	$h_1$	$(h_1, h_1)$
$s_1$	$f$	$f$	$h_3$	$h_2$	$h_3$	$h_3$	$h_2$	$h_3$	$(h_1, h_3)$
$f$	$m$	$m$							$(h_3, h_3)$
$m$	$s_1$	$s_1$							$(h_3, h_2)$
$s_2$	$s_2$	$s_2$							$(h_3, h_1)$
									$(h_2, h_2)$
									$(h_2, h_1)$
									$(h_2, h_3)$

Table 2: Preferences

Finally, since  $\mu$  is stable and  $s_1 P_h s_2$  for all  $h \in H$ , it must be that either  $\mu(s_1) = \mu(s_2)$  or  $\mu(s_1) P_{s_1} \mu(s_2)$ . Now we look at different cases for possible allocations of the couple. Note that since it can not be the case that  $\mu(c) = (h_1, h_1)$  or  $\mu(c) = (h_2, h_2)$ , we do not consider these two cases.

- Suppose  $\mu(c) = (h_1, h_2)$ . Since  $h_1 P_{s_1} h_3$  and  $s_1 P_{h_1} f$ ,  $\mu$  is blocked by  $(h_1, s_1)$ .
- Suppose  $\mu(c) = (h_1, h_3)$ . Since  $(h_1, h_2) P_c (h_1, h_3)$  and  $m P_{h_2} s_1$ ,  $\mu$  is blocked by  $(h_2, m)$ .
- Suppose  $\mu(c) = (h_3, h_3)$ . Since  $(h_1, h_2) P_c (h_3, h_3)$ ,  $m P_{h_2} s_1$ , and  $f P_{h_1} s_2$ ,  $\mu$  is blocked by  $((h_1, h_2), c)$ .
- Suppose  $\mu(c) = (h_3, h_2)$ . Since  $(h_3, h_3) P_c (h_3, h_2)$  and  $m P_{h_3} s_2$ ,  $\mu$  is blocked by  $(h_3, m)$ .
- Suppose  $\mu(c) = (h_3, h_1)$ . Since  $(h_3, h_3) P_c (h_3, h_1)$  and  $m P_{h_3} s_2$ ,  $\mu$  is blocked by  $(h_3, m)$ .
- Suppose  $\mu(c) = (h_2, h_1)$ . Since  $h_1 P_{s_1} h_3$  and  $s_1 P_{h_1} m$ ,  $\mu$  is blocked by  $(h_1, s_1)$ .

- Suppose  $\mu(c) = (h_2, h_3)$ . Since  $(h_3, h_3)P_c(h_2, h_3)$  and  $fP_{h_3s_2}$ ,  $\mu$  is blocked by  $(h_3, f)$ .

Thus, there is no stable matching at this preference profile.

In view of the above example, we look for a condition on couples' preferences that is sufficient to ensure the existence of a stable matching when hospitals' preferences do not follow CPI. Recall that, we have a condition on hospitals' preferences that  $f$  is preferred to  $m$  for all the hospitals  $h$ .

Let  $P_C^0 = (\{P_d^0\}_{d \in D \setminus S}, \{P_c^0\}_{c \in C})$  be a given collection of preferences of the doctors that are in some couple, and of the couples in  $C$ . Then, by  $\mathcal{D}(P_C^0)$  we denote the set of preference profiles where doctors  $d \in D \setminus S$  and couples in  $c \in C$  have preferences as in  $P_C^0$ , i.e.,  $\mathcal{D}(P_C^0) = \{\underline{P} : \underline{P}_d = P_d^0 \text{ for all } d \in D \setminus S \text{ and } \underline{P}_c = P_c^0 \text{ for all } c \in C\}$ .

**CONDITION 2** *Suppose  $P_C^0$  is such that  $P_c^0 \in \mathcal{D}_c^{RVT}$  for all  $c \in C$ . Then, there exists a responsive preference  $P_c \in \mathcal{D}_c^R$  for all  $c \in C$  such that for all  $c = \{f, m\} \in C$  and all  $(h_1, h_2), (h_3, h_4) \in (\bar{H} \times \bar{H}) \setminus (h_f, h_f)$  where  $r_1(P_f^0) = h_f$ , we have  $(h_1, h_2)P_c(h_3, h_4)$  if and only if  $(h_1, h_2)P_c^0(h_3, h_4)$ .*

Condition 2 implies that for a couple  $c = \{f, m\}$ ,  $P_c^0$  satisfies responsiveness over all pairs of hospitals except  $(h_f, h_f)$  where  $r_1(P_f^0) = h_f$ . Furthermore,  $P_c^0$  violates responsiveness for togetherness only when both members of the couple get a position at  $h_f$ . In other words, couples' preferences always satisfy responsiveness with respect to  $f$ , and  $m$  is ready to violate responsiveness only if  $f$  gets its highest preferred hospital.

In the following theorem, we show that existence of a stable matching is guaranteed at a preference profile if the couples' preferences satisfy Condition 2.

**THEOREM 4** *Suppose  $P_C^0$  is such that  $P_c^0 \in \mathcal{D}_c^{RVT}$  for all  $c \in C$ . Then, a stable matching exists at every preference profile in  $\mathcal{D}(P_C^0)$  if  $P_C^0$  satisfies Condition 2.*

*Proof:* The proof of Theorem 4 is constructive. Suppose  $P_C^0$  is such that  $P_c^0 \in \mathcal{D}_c^{RVT}$  for all  $c \in C$ . Suppose further that  $P_C^0$  satisfies Condition 2. Take  $\underline{P} \in \underline{\mathcal{D}}(P_C^0)$ . We construct an algorithm that produces a stable matching in  $\underline{P}$ . For each  $c = \{f, m\} \in C$  there exists some  $P_c \in \mathcal{D}_c^R$  such that for all  $(h_1, h_2), (h_3, h_4) \in (\bar{H} \times \bar{H}) \setminus (h_f, h_f)$  where  $r_1(P_f^0) = h_f$ , we have  $(h_1, h_2)P_c(h_3, h_4)$  if and only if  $(h_1, h_2)P_c^0(h_3, h_4)$ . For each couple  $c = \{f, m\}$  and each  $h \in H$ , define a conditional preference of  $f$  given  $h$ ,  $P_{m|h}^0 \in \mathbb{L}(H)$ , in the following way:  $h'P_{m|h}^0 h''$  if and only if  $(h, h')P_c^0(h, h'')$ . In the following lemma, we establish a connection between  $P_m^0$  and  $P_{m|h_f}^0$ .

**LEMMA 1** *Suppose  $c = \{f, m\}$  is a couple and  $h_1, h_2, h_f$  are all distinct hospitals. Then,  $h_1P_m^0 h_2$  implies  $h_1P_{m|h_f}^0 h_2$ .*

*Proof:* Assume for contradiction that  $h_1P_m^0 h_2$  and  $h_2P_{m|h_f}^0 h_1$ . Since  $h_2P_{m|h_f}^0 h_1$ , we have  $(h_f, h_2)P_c^0(h_f, h_1)$ . As  $h_1, h_2, h_f$  are all distinct, by Condition 2 we have  $(h_f, h_2)P_c(h_f, h_1)$ . Because  $P_c \in \mathcal{D}_c^R$ , this means  $h_2P_m^0 h_1$ , which is a contradiction. This completes the proof of the lemma.  $\square$

Now we present our algorithm that produces a stable matching at  $\underline{P}$ .

*Algorithm 3:* Use DPDA where every doctor bids as a single doctor. For all  $c = \{f, m\}$ ,  $f$  proposes according to  $P_f^0$  and  $m$  proposes according to  $P_{m|h_f}^0$  where  $r_1(P_f^0) = h_f$ . Also, for all  $s \in S$ ,  $s$  proposes according to  $\underline{P}_s$ .

The following lemma establishes an important property of DPDA. The proof of the lemma is elementary, however we present it for the sake of completeness. Let  $\mu$  be the outcome of Algorithm 3.

**LEMMA 2** *Suppose a doctor  $d$  is rejected by hospital  $h$  at some stage of Algorithm 3. Then  $(h, d)$  can not block  $\mu$  through  $\mu'$ .*

*Proof:* Since  $h$  has rejected  $d$  during some stage of Algorithm 3, it must be that hospital  $h$  had  $\kappa_h$  many proposals from doctors that are better than  $d$  according to  $\underline{P}_h$  at the time when  $h$  rejected  $d$ . Therefore, by the nature of DPDA, all the doctors that are matched with  $h$  at the end of Algorithm 3

must be better than  $d$  according to  $\underline{P}_h$ . So,  $h$  will not block with  $d$ . This completes the proof of the lemma.  $\square$

Now, we show that Algorithm 3 produces a stable matching at  $\underline{P}$ . We distinguish the following cases.

*Case 1:* Suppose  $(h, s)$  blocks  $\mu$  through  $\mu'$ . Since  $s$  blocks with  $h$ , we have  $h\underline{P}_s\mu(s)$ . Therefore, it must be that  $s$  proposed to  $h$  and was rejected by  $h$  earlier in Algorithm 3. Hence, by Lemma 2,  $(h, s)$  cannot block  $\mu$ .

*Case 2:* Suppose  $((h_1, h_2), c)$  blocks  $\mu$  through  $\mu'$  where  $c = \{f, m\}$ . Here,  $h_1$  and  $h_2$  are not necessarily different. Since  $c$  blocks with  $(h_1, h_2)$ , we have  $(h_1, h_2)P_c^0(\mu(f), \mu(m))$ .

*Case 2.1:* Suppose  $\mu(f) = h_f$ . By the definition of  $P_c^0$ , this means  $(h_f, h_2)P_c^0(h_1, h_2)$ . Since  $(h_1, h_2)P_c^0(h_f, \mu(m))$ , this implies  $(h_f, h_2)P_c^0(h_f, \mu(m))$ . Because  $((h_1, h_2), c)$  blocks  $\mu$  through  $\mu'$  and  $\mu(f) = h_f$ , it follows that  $((h_f, h_2), c)$  also blocks  $\mu$ . Note that,  $(h_f, h_2)P_c^0(h_f, \mu(m))$  implies  $h_2P_{m|h_f}^0\mu(m)$ . Therefore, by the definition of Algorithm 3, it must be that  $m$  proposed to  $h_2$  and got rejected at an earlier stage of Algorithm 3. Hence, by Lemma 2,  $((h_f, h_2), c)$  cannot block  $\mu$ .

*Case 2.2:* Suppose  $\mu(f) \neq h_f$ . Since  $f$  bids according to  $P_f^0$ , using similar logic as before, it follows that either  $h_1 = \mu(f)$  or  $\mu(f)P_f^0h_1$ . This, together with the facts that  $(h_1, h_2)P_c^0(\mu(f), \mu(m))$  and  $\mu(f) \neq h_f$ , implies  $h_2P_m^0\mu(m)$ . Because  $\mu(f) \neq h_f$ , it must be that  $|\mu(h_f)| = \kappa_{h_f}$  and  $d\underline{P}_{h_f}f$  for all  $d \in \mu(h_f)$ . Since  $f\underline{P}_hm$  for all  $h \in H$ , this means  $d\underline{P}_{h_f}m$  for all  $d \in \mu(h_f)$ . Therefore,  $\mu(m) \neq h_f$ . Moreover, since  $((h_1, h_2), c)$  blocks  $\mu$  through  $\mu'$ , it follows that  $h_2 \neq h_f$ . Because  $h_2P_m^0\mu(m)$ , it must be that  $h_2 \neq \mu(m)$ . As  $\mu(m), h_2, h_f$  are all distinct and  $h_2P_m^0\mu(m)$ , by Lemma 1 we have  $h_2P_{m|h_f}^0\mu(m)$ . Therefore, by the definition of Algorithm 3, it must be that  $m$  proposed to  $h_2$  and got rejected at an earlier stage of Algorithm 3. Hence, by Lemma 2,  $((h_1, h_2), c)$  cannot block  $\mu$ .

This completes the proof of Theorem 4.  $\blacksquare$

### 3 Conclusion

We considered many-to-one matching problems between doctors and hospitals, where doctors consisted of some couples. We saw that under common preference of hospitals over individual doctors, stable matchings exist only when the lesser preferred member is ready to violate responsiveness to be together with the more preferred member. However, in the final section, we saw that a stable matching may not exist when hospitals do not have common preferences over singleton doctors. Thus, under arbitrary hospitals' preferences, it becomes harder to obtain stable matching when couples' preferences violate responsiveness.

We also saw that even when the hospitals have common preference over individuals, it is only possible for couples to have arbitrary violation of responsiveness only if the members of the couple are ordered consecutively and the hospitals' preferences are lexicographic. Thus, we need more restrictions on hospitals preferences to obtain stable matchings when couples' preferences are arbitrary

However, under common preference over individuals, it might still be possible to obtain a stable matching by further weakening RVT. It might be an interesting problem to characterize the preferences of couples which always guarantee a stable matching hospitals' preferences are common over individual doctors.

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