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Stationary quasi-perfect equilibrium partitions constitute the recursive core*

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Abstract

We present sufficient conditions for the implementation of the (pessimistic) recursive core (Kóczy, 2007) in discrete partition function form games using a modified version of the sequential coalition formation game by Bloch (1996) extending the results of Kóczy (2008) and –in a slightly different setup– Huang and Sjöström (2006) to games with empty residual cores (respectively, to games that are not r-balanced).

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1 Introduction

Solving cooperative games with externalities remains a difficult problem in game theory despite the numerous attempts that have been made. One school tries to generalise solutions of TU games, the other seeks equilibria of noncooperative coalition formation games. Our aim is to bridge the gap between the two schools and implement the core also for games with externalities. The present paper extends the results by Kóczy (2008).

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Studying the relation of noncooperative models and the core is not new (Chatterjee et al., 1993; Lagunoff, 1994; Perry and Reny, 1994). Most of the work has, however, focussed on TU-games and therefore ignore externalities. Huang and Sjöström (2006) and Kóczy (2008) study games with externalities, but their results are limited to games with non-empty residual cores, or, in terms of sequential coalition formation games: to games with stationary perfect equilibria. It turns out that perfectness is a very demanding condition and the implementation might fail even for games without externalities if we insist on it. We therefore introduce a generalisation, stationary quasi-perfectness, such that the resulting equilibria coincide with the recursive core.

The structure of the paper is as follows. After a basic setup and the introduction of the notation we present the sequential coalition formation game and define the concept stationary quasi perfect equilibrium. Then we define the recursive core and finally we present our results.

2 Preliminaries

Let N denote the set of players. Subsets are called *coalitions*. A partition π_S of S is a splitting of S into disjoint coalitions. Π_S is the set of partitions of S. The game (N, v) is given by the player set N and a *discrete partition function* (DPF, Lucas and Macelli, 1978) $v : \Pi(N) \longmapsto \mathbb{R}^N$, where $v_i(\pi)$ denotes the payoff for player i in case partition π formed. For vectors $x, y \in \mathbb{R}^N$ we write $x >_S y$ if $x_i \geq y_i$ for all $i \in S \subset N$ and there exists $j \in S$ such that $x_j > y_j$.

A *rule of order* ρ is a strict ordering of the players. Let $\rho(S)$ denote the player ranked first in the set S and ρ_S the restriction of the ordering to S.

In the following we present the two approaches. A noncooperative bargaining game that is a slight modification of the sequential coalition formation game of Bloch (1996) and the recursive core (or its adaptation to discrete partition function form games; Kóczy, 2007, 2008), a cooperative solution concept, itself a generalisation of the core.

2.1 Sequential coalition formation

First we present the noncooperative bargaining game. A *game of sequential coalition formation* (SCF) (v, ρ) (Bloch, 1996) is defined by a DPF v and the rule of order ρ. It is played as follows.

1. Start at the highest ranked player.
2. The current player proposes a coalition S it is a member of also specifying the partition of the coalition.
3. If all members of S have approved the proposal, the coalition forms and these players exit. Otherwise following player in S gets the word. He can reject the proposal, become the next proposer and the game continues at step 2. Alternatively, he can accept the proposal and the step is repeated.
4. If all players have left, the game terminates, otherwise return to Step 1.

In the original model the partition of S was trivial: consisting of the single coalition S. For a detailed discussion of the benefits of this modification see Kóczy (2006).

Should the algorithm not terminate we must make special provisions to determine payoffs. Let K be the set of players who have formed partition π_K. Then, for the strategy profile σ the payoffs are given by the following:

$$v_i(\pi(\sigma)) = \begin{cases} 0 & i \in N \setminus K \\ \min_{\pi \supset \pi_K} v_i(\pi) & i \in K \end{cases} \tag{2.1}$$

Note that our definition is different from Bloch's who considers optimistic players, that is, $v_i(\pi(\sigma)) = \min_{\pi \supset \pi_K} v_i(\pi)$ for $i \in K$. The implications of this difference will be clear later, but let us provide a motivation for this change in terms of deviations, a concept we formalise later. A deviation is profitable if it is weakly profitable to all players. Suppose this deviation creates a subgame where the sequential coalition formation game continues indefinitely. In absence of a stable partition, any of the partitions might form. Optimistic players expect the best: a partition beneficial to the deviation will form. Bloch's players' optimism goes further: they individually hope the best, so that a deviation may take place if it is profitable for some but creates losses for others. Pessimism is consistent in this sense: A player will not deviate if any of the possible partitions will create a loss to him, in other words a deviation is profitable for K if and only if it is profitable for each player in K and for each possible partition.

In the following we formalise the model using Bloch's notation.

A *history* $h^t = (\hat{K}(h^t), \pi_{\hat{K}(h^t)}, \hat{T}(h^t), S, i)$ at date t is a list of offers, acceptances and rejections up to period t, where $\hat{K}(h^t) \subset N$ is the set of players who have already left the game, $\pi_{\hat{K}(h^t)} \in \Pi(\hat{K}(h^t))$ is the set of coalitions they have formed, $\hat{T}(h^t)$ is the ongoing proposal, $S \subset N$ who have already accepted the proposal, finally $i \in N$ is the player active at time t. The collection of histories at which i is active is denoted H_i.

Strategy σ_i of player i is a mapping from H_i to his set of actions:

$$\sigma_i(h^t) \in \begin{cases} \{\text{Yes}, \text{No}\} & \text{if } \hat{\tau}(h^t) \neq \emptyset \\ \mathrm{T}(i, \hat{K}(h^t)) & \text{if } \hat{\tau}(h^t) = \emptyset \end{cases} \tag{2.2}$$

where $\mathrm{T}(i, \hat{K}(h^t)) = \Big\{\tau \in \Pi(T), T \subseteq N \setminus \hat{K}(h^t), i \in T\Big\}$, the set of partitions that i can form with the remaining set of players.

We are interested in stationary strategies:

Definition 1. A strategy is *stationary* if it does not depend on history, but only on the current state $s = (\pi_K, \tau)$.

Definition 2. A *stationary perfect equilibrium* σ^* is a strategy profile such that for all players $i \in N$, for all states s and for all strategies σ_i of player i we have

$$v_i(\pi(\sigma^*(s))) \geq v_i(\pi(\sigma_i(s), \sigma^*_{-i}(s))). \tag{2.3}$$

2.2 Recursive core

The second model is a cooperative solution concept, a generalisation of the core to games in partition function form.

Our definition of the recursive core (Kóczy, 2007) is adapted from the pessimistic version to DPF games just as it has been used in Kóczy (2008). First we introduce the notion of *residual game*:

Definition 3 (Residual Game)**.** Let R be a subset of N and π_S a partition of its complement S. The residual game (R, v_{π_S}) is the DPF form game over the player set R and with the DPF $v_{\pi_S} : \Pi(R) \to \mathbb{R}^R$, where $v_{\pi_S}(\pi_R) = v(\pi_R \cup \pi_S)$.

The residual game is a discrete partition function form game and in the recursive core the same solution is used to solve this game as the original one. Before defining the core, please note that as the partition uniquely determines payoffs, instead of imputations or payoffs, the core consists of *partitions*.

Definition 4 (Recursive core)**.** The definition consists of four steps.

1. *Trivial game.* $C(\{1\}, v) = \{\{1\}\}$.

2. *Inductive assumption.* Given the definition of the core $C(R, v)$ for every game with $|R| < k$ players we define dominance for a game of k players. Let $A(R, v)$ denote the *assumption about the game* (R, v). If $C(R, v) \neq \varnothing$ then $A(R, v) = C(R, v)$, otherwise $A(R, v) = \Pi(R)$, the set of partitions.

3. *Dominance.* The partition π is *dominated via the coalition S forming partition π_S* if for all assumptions $\pi_R \in A(N \setminus S, v_{\pi_s})$ we have $v(\pi_S \cup \pi_R) >_S v(\pi)$.

   The partition π is *dominated* if it is dominated via a coalition.

4. *Core.* The *core* of a game of k players is the set of undominated partitions and we denote it by $C(N, v)$.

For a discussion of its properties see Kóczy (2007).

3 Order-independent equilibria and the core

We show that the set of partitions produced by certain order independent equilibria (OIE, Moldovanu and Winter, 1995, p.27) of the SCF game coincide with the recursive core. While previously we have studied strategies that are stationary perfect by our previous result (Kóczy, 2008) OIE under this provision only exist for games with nonempty residual cores. This condition seems rather limiting and we therefore augment the set of interesting strategy profiles with stationary *quasi-perfect* ones.

3.1 Stationary quasi-perfectness

For our implementation we need to modify not only these definitions, but even, slightly, the way the game is played. Bloch's original coalition formation game specifies a characteristic function and a rule of order. Kóczy (2008) has used the concept of order independent equilibrium (Moldovanu and Winter, 1995), where an equilibrium strategy profile must work for all possible orders of players and result in the same payoff vector. This means that a strategy profile is an equilibrium if, for whatever rule of order it neutralises deviations. Here we go a step further and only consider deviations that are profitable for all possible rules of orders. In essence the players choose strategies, possibly announce deviations and then, when all is known, the order is given. For games with non-empty residual cores, the two models yield the same equilibria.

The equivalence result (Kóczy, 2008) predicts that games with empty residual cores do not have stationary perfect equilibria. On the other hand, just as the recursive core may be non-empty even if the game has empty residual cores, with an appropriately defined concept, we may retain some equilibrium-like behaviour in the corresponding sequential coalition formation games, too.

Bloch (1996) presents an example of a game without stationary-prefect strategies. It appears, a slight imperfection, such as the lack of stationary-perfect strategies in the smallest of subgames spoils the existence of stationary perfect equilibria in the main game, even if the subgames in question are simply never played. If we can ignore such "bad" subgames, put them in quarantine (in essence lying that they are good) can save the whole game from poisoning. The good news is that there are indeed irrelevant subgames, that we *can* put into quarantine and worry only about the rest. In the following we first clarify which subgames are relevant and then we define stationary quasi perfect equilibria applying the perfectness condition only to relevant subgames.

Definition 5. For a strategy profile σ a subgame $s = (\pi_{N \setminus S}, \tau)$ is *relevant* if

- s is the original game ($\pi_{N \setminus S} = \tau = \varnothing$),
- there exists a modification σ', such that
  - $\pi_{N \setminus S} \subseteq \pi(\sigma')$,
  - σ and σ' differ in a single action outside subgame s, resulting in $\pi_D = \pi_{N \setminus S} \setminus (\pi(\sigma) \cap \pi(\sigma'))$ leaving the game, and
  - there exists a ρ_S such that $v(\pi(\sigma)) <_D v(\pi(\sigma'))$, or

- it is a relevant subgame of a relevant subgame.

We look at the second case first. As strategy profile σ is played, a deviation occurs. We are of course interested in deviations that actually change the resulting partition, and if so, consider the players that first leave the game. The fact that they all accepted the deviation indicates that they (weakly) benefit from it. If they do not, irrespective of the strategies of the remaining players in the game, there is no need to further specify the strategies for this hypothetical subgame, hence such a subgame is irrelevant. The same holds, for subgames that make a deviation always profitable, but then we want to know what happens after the deviation and hence the subgame is relevant. Finally we must deal with subgames that are more than a single deviation away: here we consider a sequence of subgames with 1-action differences and if these are respectively relevant, the smallest subgame is one, too.

Definition 6. The strategy profile σ is a stationary quasi-perfect equilibrium (SQPE) if

- σ is stationary
- restrictions to subgames that are relevant for σ are also SQPE profiles and
- for all states $s_K = (\pi_K, \tau)$ and for all strategy profiles $\sigma_i'(s_K)$ with $i \notin K$ there exists an order $\rho_{N\setminus K}$ such that

$$v_i(\pi_K \cup \pi(\sigma_i'(s_K), \sigma_{-i}(s_K))) \leq v_i(\pi_K \cup \pi(\sigma(s_K))).$$

We denote the set of stationary quasi perfect equilibria by $\mathrm{SQPE}(N, v)$ and partitions resulting from playing such equilibrium strategies by $\mathrm{SQPP}(N, v)$.

Quasi-perfectness is motivated by the difference between concave and quasi-concave functions: There may be local deviations, but for the global picture stationary perfectness must hold.

Observe that stationary perfect equilibria are also stationary quasi-perfect.

3.2 Results

Now that the concepts have been defined, we can present our main result.

Theorem 1. *Let (N, v) be a DPF form game. Then $C(N, v) = \mathrm{SQPP}(N, v)$.*

The rest of this section is devoted to the inductive proof of this theorem. As the proof is long, we break it into a number of propositions and finally present a summary of these results.

The following proposition requires no proof:

Proposition 2. *Let* $(\{1\}, v)$ *be a trivial, single-player DPF form game. Then* $C(\{1\}, v) = \mathrm{SQPP}(\{1\}, v)$.

Now *assume* that Theorem 1 holds for all games with less than n players. In the following we extend it to games with n players. In order to show $\mathrm{SQPP}(N, v) = C(N, v)$, first we show $\mathrm{SQPP}(N, v) \subseteq C(N, v)$ then $\mathrm{SQPP}(N, v) \supseteq C(N, v)$

Lemma 3. *If Theorem 1 holds for all games with up to* $k-1$ *players,* $\mathrm{SQPP}(N, v) \subseteq C(N, v)$ *for all* k*-player games.*

Proof. If $\mathrm{SQPP}(N, v) = \varnothing$ the result is trivial, so in the following we assume that there exists a SQPE σ that results in the SQPP π. In particular, we assume that $\pi \notin C(N, v)$ and prove contradiction.

Our assumption is, by definition, equivalent to the existence of a profitable deviation π_D. The resulting subgame has fewer players so our inductive assumption applies to it. We discuss three cases.

Case 1. The resulting subgame $(\pi_D, \varnothing)$ is irrelevant. Then for all $\pi_{N \setminus D} \in \Pi(N \setminus D)$ there exists $i \in D$ such that $v_i(\pi_D, \pi_{N \setminus D}) < v_i(\pi)$ – clearly the deviation in the DPF game cannot be profitable; contradiction.

Case 2. The resulting subgame is relevant, the core of the corresponding residual subgame is empty. Then $v(\pi_D, \pi_{N \setminus D}) >_D v(\pi)$ for all $\pi_{N \setminus D}$. Then the following deviation is clearly profitable in the SCF game: when a player in D has it turn, it rejects pending offers and proposes π_D, clearly all in D will accept. Hence π is not a SQPP. Contradiction.

Case 3. The resulting subgame is relevant and the core of the corresponding residual subgame is not empty. By assumption $\sigma(\pi_D, \varnothing)$ is a SQPE such that the deviation from σ to form π_D is not profitable, and by the inductive assumption $\pi(\sigma(\pi_D, \varnothing)) \in C(N \setminus D, v_{\pi_D})$. This, however implies that in the DPF game the deviation π_D is not profitable. Contradiction.

We have discussed all cases, and found the assumptions contradicting. Therefore $\pi \in C(N, v)$. $\square$

Lemma 4. *If Theorem 1 holds for all games with less than k players, then* $\mathrm{SQPP}(N, v) \supseteq C(N, v)$ *for all k-player games with nonempty residual cores.*

Proof. The proof is inspired by that of Bloch (1996, Proposition 3.2) in part, and is by construction. We show that if $\tilde{\pi} \in C(N, v)$ there exists a stationary quasi-perfect strategy profile $\tilde{\sigma} = \tilde{\sigma}(K, \pi_K, \tau)$ such that $\pi(\tilde{\sigma}) = \tilde{\pi}$.

Let $\pi(\tau)$ denote the partition that the acceptance of a proposal τ ultimately produces. In the DPF form game π_K, as a deviation defines a residual game $(N \setminus K, v_{\pi_K})$. We discuss two cases based on the emptiness of the core of this residual game.

If the residual core is *not empty* a "harsh response" to π_K is an element of the residual core $C(N \setminus K, v_{\pi_K})$ ensuring that the deviation π_K is not profitable. That is, $\tilde{\pi}_{N\setminus K}$ satisfies

$$\exists i \in S: \quad v_i(\pi_K, \tilde{\pi}_{N\setminus K}) < v_i(\tilde{\pi}), \text{ or} \tag{3.1}$$

$$\forall i \in S: \quad v_i(\pi_K, \tilde{\pi}_{N\setminus K}) = v_i(\tilde{\pi}). \tag{3.2}$$

Since $\tilde{\pi} \in C(N, v)$ such a $\tilde{\pi}_{N\setminus K}$ exists for all deviations π_K.

If the residual core is *empty* we observe that in order for a deviation to be profitable it must be profitable for *all* residual partitions. Since $\tilde{\pi} \in C(N, v)$, the deviation is not profitable and so there exists a residual partition $\tilde{\pi}_{N\setminus K} \in \Pi(N \setminus K)$ satisfying Condition 3.1 or Condition 3.2.

The stationary strategy $\tilde{\sigma}_i$ for player i is then constructed as follows:

$$\text{If } \pi_K = \varnothing, \quad \tilde{\sigma}_i(K, \pi_K, \varnothing) = \tilde{\pi} \tag{3.3}$$

$$\tilde{\sigma}_i(K, \pi_K, \tau) = \begin{cases} \text{Yes} & \text{if } v_i(\pi(\tau)) \geq v_i(\tilde{\pi}) \\ \text{No} & \text{otherwise.} \end{cases}$$

$$\text{If } \pi_K \neq \varnothing, \quad \tilde{\sigma}_i(K, \pi_K, \varnothing) = \tilde{\pi}_{N\setminus K} \tag{3.4}$$

$$\tilde{\sigma}_i(K, \pi_K, \tau) = \begin{cases} \text{Yes} & \text{if } v_i(\pi(\tau)) \geq v_i(\pi_K, \tilde{\pi}_{N\setminus K}) \\ \text{No} & \text{otherwise.} \end{cases}$$

In equilibrium $\pi(\tilde{\sigma}) = \tilde{\pi}$ and the strategy is stationary by construction so we only need to verify quasi subgame-perfection. We show this by induction. As quasi subgame-perfection holds for a trivial game we may assume that it holds for all games of size less than $|N|$.

Now consider game (N, v) and observe the following. If a set of players K have left the game to form π_K the subgame is simply a coalition formation game with less players. We discuss two cases based on the emptiness of the residual core.

1. If the residual core is not empty, the proposed strategy exhibits the same similarity property: in equilibrium the core partition is proposed and accepted, while residual cores form off-equilibrium.

The original assumption about smaller games then ensures that the off-equilibrium path is quasi subgame perfect so we only need to check whether a deviation τ is ever accepted. This deviation corresponds to a deviation in the DPF game. Since $\tilde{\pi} \in C(N, v)$, by the construction of $\tilde{\pi}_{N \setminus K}$ we know that there exists a player in S for whom the deviation τ is not profitable.

2. If the residual core is empty, the deviation is not profitable irrespective of the residual partition that forms, the subgame is not relevant, and therefore the second condition for quasi-subgame perfectness is satisfied.

The emptiness of the residual core, by our assumption, also implies that the set of order-independent partitions is also empty, thus there are no stationary quasi subgame-perfect equilibrium strategy profiles either. In the absence of such strategy profiles the players in K cannot predict the partition of $\pi_{N \setminus K}$ – in this case, by Expression 2.1, they individually expect the worst. As π_K only forms if it is profitable, it will only if it is profitable for all partitions $\pi_{N \setminus K}$. Since $\tilde{\pi} \in C(N, v)$ this is not the case. This, on the other hand implies that the formation of $\tilde{\pi}$ is unaffected by possible deviations in this subgame, meeting the third condition of stationary quasi-perfectness.

□

Proof of Theorem 1. The proof is by induction. The result holds for trivial, single-player games. Assuming that the result holds for all $k - 1$ player games, the result for k-player games is a corollary of Lemmata 3 & 4. □

4 Conclusion

Theorem 1 holds for arbitrary games in discrete partition function form, but of course it is most interesting for games where some of the residual cores are empty. When a proposal is made in a game without externalities the invited players do not even (need to) consider the residual game and therefore the emptiness of a residual core is not addressed. Huang and Sjöström (2006) and Kóczy (2008) simply restrict their attention to games where the residual cores are non-empty, in fact the r-core (Huang and Sjöström, 2003) is not even defined for games with empty residual cores. As already pointed out by Kóczy (2007) this is not only an enormous limitation given the number of conditions such games must satisfy (one for each residual game), but the definitions/results do not apply to some games without externalities and so they are not generalisations of the well-known results for TU-games. The present paper heals this deficiency.

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