# Resource location games

Operations Research Letters 47 (2019) 300–304

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

# Resource location games

Loe Schlicher [a,*], Marieke Musegaas [b], Evelot Westerink-Duijzer [b]

[a] Faculty of Military Sciences, Netherlands Defense Academy, P.O. Box 90002, 4800 PA, Breda, The Netherlands
[b] Econometric Institute, Erasmus School of Economics, Erasmus University Rotterdam, P.O. Box 1738, 3000 DR, Rotterdam, The Netherlands

A R T I C L E I N F O

*Article history:*
Received 25 January 2019
Received in revised form 29 March 2019
Accepted 1 April 2019
Available online 17 April 2019

*Keywords:*
Cooperative game
Core
Resource-Profit allocations
Reallocation of resources

A B S T R A C T

In this paper, we introduce and analyze resource location games. We show core non-emptiness by providing a set of intuitive core allocations, called Resource-Profit allocations. In addition, we present a sufficient condition for which the core and the set of Resource-Profit allocations coincide. Finally, we provide an example showing that when the sufficient condition is not satisfied, the coincidence is not guaranteed.



## 1. Introduction

Consider a setting with several regions (e.g., villages, municipalities, or small districts), each inhabited by several residents. All these residents are interested in the realization of the same type of task (e.g., mowing the lawn, cleaning a rain gutter, or pruning the hedge). Such a task can be executed with a single resource, and each resident may or may not own such a single resource (e.g., a lawn mower, a gutter ladder, or a hedge trimmer). If a resident holds (and so has access to) such a resource, it generates a resident-specific profit (e.g., the profit or utility realized by mowing the lawn, cleaning the rain gutter, or pruning the hedge).

Residents amongst, but also within, regions can decide to collaborate. In such a collaboration, the participants decide in which regions to locate their resources. Each resource is then shared, and used, amongst all participants in the region where the resource is located, a so-called covered region. Such type of situations, in which a resource is shared and used amongst all participants in a covered region, is reasonable when, for instance, demand per participant is low (e.g., a hedge trimmer is only used a couple of hours, per year) or capacity of the resource is high. The aim of the collaborating residents is to (re)allocate the resources in such a way that total profit (i.e., the sum of the profits of the participating residents that belong to a covered region), is maximized. Typically, this results in additional profit (compared to the situation without any collaboration amongst the players) and thus the question arises about how to allocate this additional profit in a fair and efficient way amongst the collaborating participants. In this paper, we investigate this joint profit allocation aspect in a resource location (RL) situation. To tackle this aspect, we introduce an RL game wherein residents are represented by players that each may or may not own a single resource and each have an associated profit, indicating the worth of having access to a resource.

For these RL games, we study properties of the core (i.e., the set of all possible allocations for which no group of players has an incentive not to collaborate). We distinguish between the case with more resources than regions (i.e., oversupply) and the case with not more resources than regions (i.e., no oversupply). For both cases, we show that the core is non-empty. For the oversupply case, we provide a complete description of the core. For the no oversupply case, we provide a subset of the core. We do so by providing a set of intuitive core allocations, called Resource-Profit (RP) allocations. These RP allocations are based on a uniform price of owning a resource and the player-specific profit. In addition, for the no oversupply case, we present a sufficient condition for which the core and the set of RP allocations coincide. As a side result, we are able to identify how these RP allocations can be constructed via any core allocation. Finally, we provide an example showing that when the sufficient condition is not satisfied, the coincidence is not guaranteed, i.e., the set of RP allocations is a proper subset of the core.

RL games belong to the class of resource pooling games, in which resources are reallocated, or shared, amongst players to realize additional profit. In the last couple of years, there is an increasing interest in these games. Some examples are the pooling of technicians in the service industry [1], pooling of capacity in a production environment [2,8], pooling of emergency vehicles in health care [6], reallocation of inventory in a retail setting [13], pooling of spare parts in the capital intensive goods

* Corresponding author.
*E-mail addresses:* lpj.schlicher@mindef.nl (L. Schlicher), musegaas@ese.eur.nl (M. Musegaas), duijzer@ese.eur.nl (E. Westerink-Duijzer).

https://doi.org/10.1016/j.orl.2019.04.002
0167-6377/

industry [3–5], and reallocation of spare parts and repair vans in a railway setting [10,11]. To the best of our knowledge, there are no resource pooling games in literature that consider our specific situation —the one in which players can share resources within a region and reallocate them amongst the regions. The only exception is the classical Böhm-Bawerk horse market (BBHM) game, which has been studied extensively in literature (see, e.g., [7,12,14,15]). In BBHM games, there are sellers that each have one horse for sale and buyers that each wish to buy one such horse. These horses are all alike, while the sellers and buyers may have different valuations for such a horse. When collaborating, horses are sold towards those buyers that value horses most. Shapley and Shubik [12] showed that the core of these games coincides with a particular set of market allocations, which, per player, depends on its valuation and a uniform market price. Clearly, we study a generalization of BBHM games: when each region (of an RL game) inhabits exactly one player (with or without a resource), the players with a resource can be seen as potential sellers and the players without a resource can be seen as potential buyers. Hence, in the spirit of BBHM games, we contribute to the literature by generalizing this classical game and some of its corresponding results.

The outline of this paper is as follows. In Section 2, we introduce RL situations and describe the associated RL games. Then, in Section 3, we formally define the set of RP allocations and we analyze its relation with the core of RL games. We conclude this paper with a final remark about this relation. We want to emphasize that proofs of lemmas and theorems are relegated to Appendix A. For the main results, which are presented in the form of theorems, we also give a sketch of proof in the main text.

## 2. RL situations and associated RL games

### 2.1. RL situation

An RL situation can be summarized by a tuple $\theta = (N, r, w, \mathscr{D}, D)$, where $N \subseteq \mathbb{N}$ is a finite set of players (e.g., residents). The parameter $r_i \in \{0, 1\}$ indicates whether player $i \in N$ owns a resource ($r_i = 1$) or not ($r_i = 0$). The vector $r \in \{0, 1\}^N$ summarizes these parameters. It is assumed that there is at least one player who owns a resource, i.e., $\sum_{i \in N} r_i \geq 1$. The player-specific profit $w_i \geq 0$ specifies the profit player $i \in N$ realizes in case he has access to a resource. A player has access to a resource if he owns a resource. When players collaborate, there is also another way of having access to a resource, which will be discussed later on in this section. The vector $w \in \mathbb{R}^N_+$ summarizes the player-specific profits. The set $\mathscr{D} \subseteq \mathbb{N}$ is a finite set of regions. Furthermore, $D_j \subseteq N$ represents the set of players that belong to region $j \in \mathscr{D}$. Every player belongs to exactly one region and thus the family of sets of players $D = \{D_j \mid j \in \mathscr{D}\}$ is a partition of $N$. The set of all RL situations is denoted by $\Theta$.

It is assumed that the nature of the resources is such that the players within the same region can share resources with each other, i.e., all players within the same region can benefit from a single resource. As a consequence, if a region contains at least one player who owns a resource and all players in this region decide to cooperate, then this resource can be donated to this region and so every player in this region has access to a resource and thus receives its player-specific profit. Note that, due to the nature of the resources, it does not make a difference for a region whether it has one, or multiple players with a resource. Indeed, the remaining resources (if any) could be allocated to other regions —and this calls for collaboration amongst the regions as well. Doing this in an optimal way boils down to allocating the $\sum_{i \in N} r_i$ resources to the regions for which the regional profit, i.e., the sum of its player-specific profits, is the highest. These regions are called the covered regions and the remaining ones are called non-covered regions. We denote $\mathscr{D}^c_N \subseteq \mathscr{D}$ as the set of covered regions. Then, by assuming that initially there is no reallocation of resources and moreover resources are not shared amongst players in the same region (i.e., no collaboration within and amongst the regions), the maximal joint profit increase due to cooperation equals

$$\sum_{j \in \mathscr{D}^c_N} \sum_{i \in D_j} w_i - \sum_{i \in N} r_i w_i.$$

Here, the first part equals the sum of the regional profits of the covered regions, i.e., the total profit when there is full collaboration. The second part equals the sum of the player-specific profits of the players who initially own a resource, i.e., the total profit when there is no collaboration at all.

### 2.2. RL games

A cooperative game is a pair $(N, v)$ where $N$ denotes a non-empty, finite set of players and $v : 2^N \to \mathbb{R}$ assigns a monetary payoff to each coalition $S \subseteq N$, where $2^N$ denotes the collection of all subsets of $N$. The coalitional value $v(S)$ denotes the highest payoff the coalition $S$ can jointly generate by means of optimal cooperation without help of players in $N \backslash S$. Coalition $N$ is called the grand coalition. Furthermore, by convention, $v(\emptyset) = 0$.

In order to define a cooperative game associated with RL situations, we first need to introduce some notions and definitions. For each coalition $S \subseteq N$, $R(S)$ indicates the total number of resources in coalition $S$, i.e., $R(S) = \sum_{i \in S} r_i$. Additionally, for each region $j \in \mathscr{D}$, $D_j(S)$ identifies the players of coalition $S$ that belong to region $j$, i.e., $D_j(S) = D_j \cap S$. The set $\mathscr{D}_S \subseteq \mathscr{D}$ contains the regions for which there exists a player of coalition $S$ that belongs to this region, i.e., $\mathscr{D}_S = \{j \in \mathscr{D} \mid D_j(S) \neq \emptyset\}$. Moreover, we denote the sum of the player-specific profits of all players in coalition $S$ that belong to region $j$ by $W_j(S)$ and thus $W_j(S) = \sum_{i \in D_j(S)} w_i$. We call $W_j(S)$ the regional profit of region $j$ for coalition $S$.

To tackle the allocation problem of the maximal joint profit increase in an RL situation $\theta = (N, r, w, \mathscr{D}, D)$, one can analyze an associated cooperative game $(N, v^\theta)$. Here, for a coalition $S \subseteq N \backslash \{\emptyset\}$, $v^\theta(S)$ reflects the maximal joint profit this coalition can make. For this, we assume that the players in $S$ can only reallocate their own resources. Moreover, a player in coalition $S$ cannot benefit from the resource of a player in the same region if he does not belong to coalition $S$. As a consequence, it is optimal for coalition $S$ to allocate his $R(S)$ resources to the $R(S)$ regions in $\mathscr{D}_S$ for which the regional profits for coalition $S$ are the highest. In order to define $v^\theta(S)$ formally, we first introduce the bijection $\sigma_S : \{1, 2, \ldots, |\mathscr{D}_S|\} \to \mathscr{D}_S$. This bijection is uniquely defined and orders the regions in $\mathscr{D}_S$ in such a way that they are in non-increasing order with respect to regional profits for coalition $S$. Moreover, if there is a tie, then the region with the smallest index is chosen first. Formally,

$$\begin{aligned}
&\sigma_S(1) = \min\{j \in \mathscr{D}_S \mid W_j(S) \geq W_k(S) \ \text{ for all } \ k \in \mathscr{D}_S\},\\
&\sigma_S(i) = \min\{j \in \mathscr{D}_S \backslash \{\sigma_S(1), \ldots, \sigma_S(i-1)\} \mid \\
&\qquad\qquad W_j(S) \geq W_k(S) \ \text{ for all } \ k \in \mathscr{D}_S \backslash \{\sigma_S(1), \ldots, \sigma_S(i-1)\}\big\},
\end{aligned}$$

for every $i \in \{2, 3, \ldots, |\mathscr{D}_S|\}$. As a result, coalition $S$ allocates a resource to every region $j \in \mathscr{D}_S$ with $\sigma_S^{-1}(j) \leq R(S)$. We denote the set of covered regions for coalition $S$ by $\mathscr{D}^c_S = \{j \in \mathscr{D}_S \mid \sigma_S^{-1}(j) \leq R(S)\}$ and the set of non-covered regions by $\mathscr{D}^{nc}_S = \{j \in \mathscr{D}_S \mid \sigma_S^{-1}(j) > R(S)\}$. The following definition provides the formal definition of an RL game.

**Definition 1.** For every RL situation $\theta \in \Theta$, the associated RL game $(N, v^\theta)$ is defined by

$$v^\theta(S) = \sum_{j \in \mathscr{D}_S^c} W_j(S),$$

for all $S \subseteq N \backslash \{\emptyset\}$ and $v^\theta(\emptyset) = 0$.

We conclude this section with an illustrative example.

**Example 1.** Let $\theta \in \Theta$ with $N = \{1, 2, 3\}$, $r = (0, 0, 1)$, $w = (4, 6, 2)$, $\mathscr{D} = \{4, 5\}$, $D_4 = \{1\}$, and $D_5 = \{2, 3\}$. In Table 1, we present the coalitional values of $(N, v^\theta)$.

**Table 1**
The RL game $(N, v^\theta)$ of Example 1.

| $S$ | $\emptyset$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1, 2\}$ | $\{1, 3\}$ | $\{2, 3\}$ | $\{1, 2, 3\}$ |
|---|---|---|---|---|---|---|---|---|
| $v^\theta(S)$ | 0 | 0 | 0 | 2 | 0 | 4 | 8 | 8 |

Player 3 is the only player with a resource. When he cooperates with others, he can either keep it in his own region, or give it to another region. Since $w_1 > w_3$, player 3 donates his resource to region 4 when cooperating with player 1, but keeps it in region 5 when cooperating with player 2. When all the players cooperate, it is best to allocate the resource to region 5. Then, both player 2 and player 3 use it, which results in a profit of $w_2 + w_3 = 6 + 2 = 8$. $\diamond$

## 3. The core of RL games

The core $\mathscr{C}(N, v)$ of a cooperative game $(N, v)$ is formally defined as the set of all allocations $x \in \mathbb{R}^N$ that are efficient $\left(\sum_{i \in N} x_i = v(N)\right)$ and stable $(\sum_{i \in S} x_i \geq v(S)$ for all $S \subset N)$. Convexity (see [9]) is a desirable property for cooperative games, since the core of a convex game is always non-empty and its structure is known (namely, the core of a convex game is the convex hull of the marginal vectors). RL games are in general not convex (see for instance Example 1). So, for studying the core of RL games, we have to investigate those games in more detail.

In Lemma 1 we present a result for core allocations that is frequently used throughout this paper. This lemma resembles that a coalition cannot claim too much from the value of the grand coalition, because this would not leave enough for the players outside the coalition. Recall that all proofs are relegated to Appendix A.

**Lemma 1.** *Let $(N, v)$ be a cooperative game and let $S \subset N$. For every $x \in \mathscr{C}(N, v)$ it holds that*

$$\sum_{i \in S} x_i \leq v(N) - v(N \backslash S).$$

In Lemma 2 we show that any coalition in an RL game can realize a coalitional value at most equal to the sum of the player-specific profits of all the players in that coalition. Moreover, in case there are enough resources for all regions of this coalition (i.e., no undersupply of resources for coalition $S$), all player-specific profits can be realized.

**Lemma 2.** *Let $\theta \in \Theta$ be an RL situation and let $(N, v^\theta)$ be the associated RL game. For any coalition $S \subseteq N$, the following holds:*

*(i)* $v^\theta(S) \leq \sum_{i \in S} w_i$ *if* $R(S) < |\mathscr{D}_S|$,
*(ii)* $v^\theta(S) = \sum_{i \in S} w_i$ *if* $R(S) \geq |\mathscr{D}_S|$.

Note that in case of oversupply of resources for the grand coalition, it is also possible that there is no oversupply of resources for some coalitions. In other words, even though we consider in Section 3.1 the case $R(N) > |\mathscr{D}|$, it is still possible that there exists a coalition $S \subset N$ with $R(S) \leq |\mathscr{D}_S|$.

### 3.1. Oversupply of resources: $R(N) > |\mathscr{D}|$

The following theorem shows that in case of oversupply of resources, the core coincides with the vector of player-specific profits, which implies that the value of a resource reduces to zero.

**Theorem 1.** *Let $\theta \in \Theta$ be an RL situation with $R(N) > |\mathscr{D}|$ and let $(N, v^\theta)$ be the associated RL game. It holds that*

$$\mathscr{C}(N, v^\theta) = \{w\}.$$

We now give a sketch of proof for this result. The proof starts with showing that each player cannot claim more than its own profit, which follows by exploiting the results of Lemma 1 and Lemma 2. Subsequently, by exploiting the efficiency property, it follows that each player exactly claims its own profit. Finally, we prove that the vector of player-specific profits is a core element, which follows by exploiting the results of Lemma 2.

**Remark 1.** If the condition of Theorem 1 is not satisfied, the coincidence is not guaranteed. To see this, consider for instance an adjusted version of Example 1 with $r = (1, 1, 0)$ and thus $R(N) = 2 \not> 2 = |\mathscr{D}|$. Then, $\{w\} = \{(4, 6, 2)\} \subset \text{Conv}\,(\{(4, 6, 2), (4, 8, 0)\}) = \mathscr{C}(N, v^\theta)$ and thus the coincidence does not hold. Moreover, note that the condition of Theorem 1 is only a sufficient condition and not a necessary condition. To see this, consider for instance another adjusted version of Example 1 with $r = (1, 1, 1)$ and $\mathscr{D} = \{4, 5, 6\}$, where $D_4 = \{1\}$, $D_5 = \{2\}$ and $D_6 = \{3\}$. Then, the core coincides with the vector of player-specific profits, i.e., $\mathscr{C}(N, v^\theta) = \{(4, 6, 2)\} = \{w\}$, but $R(N) = 3 \not> 3 = |\mathscr{D}|$.

### 3.2. No oversupply of resources: $R(N) \leq |\mathscr{D}|$

We start with introducing the intuitive core allocations, which per player $i \in N$, consists of two components. The first component is the resource component $\gamma \cdot r_i$ that compensates for owning a resource. The second component is the profit component $\alpha_i$ that compensates for the profit realized by a player. The allocation, which we call a Resource-Profit (RP) allocation, is then formulated as

$$\gamma \cdot r_i + \alpha_i \text{ for all } i \in N.$$

We continue by formally defining these two components. First, we introduce the resource component, which depends on $\gamma$. This parameter is defined as follows:

$$\gamma \in \begin{cases} \left[W_{\sigma_N(R(N)+1)}(N), W_{\sigma_N(R(N))}(N)\right] & \text{if } R(N) < |\mathscr{D}|, \\ \left[0, W_{\sigma_N(R(N))}(N)\right] & \text{if } R(N) = |\mathscr{D}|. \end{cases} \tag{1}$$

The parameter $\gamma$ resembles the principle of a market price. Firstly, because $\gamma$ is at least equal to the regional profit of a region that has highest regional profit amongst all non-covered regions. Secondly, because $\gamma$ is at most equal to the regional profit of a region that has lowest regional profit amongst all covered regions. Hence, any other price (than $\gamma$) would always give (some) players incentives to sell (or buy) a resource for a lower (or higher) price. The profit component is defined as follows:

$$\alpha_i \in \begin{cases} [0, w_i] & \text{for all } i \in D_j(N) \text{ with } j \in \mathscr{D}_N^c, \\ \{0\} & \text{for all } i \in D_j(N) \text{ with } j \in \mathscr{D}_N^{nc}, \end{cases} \tag{2}$$

with the additional condition that

$$\sum_{i \in D_j(N)} \alpha_i = W_j(N) - \gamma \text{ for all } j \in \mathscr{D}_N^c. \tag{3}$$

So, players that belong to a covered region can divide the regional profit, minus the price of the resource (that covers the region), freely, with the restriction that no one can demand more than their player-specific profit.

Next, for every RL situation $\theta \in \Theta$, we denote the set of RP allocations by

$$\Omega^{\theta} = \left\{ x \in \mathbb{R}^N \middle| x_i = \gamma \cdot r_i + \alpha_i \text{ for all } i \in N, \ (1), (2), (3) \right\}.$$

We are now ready to give a partial description of the core of RL games, i.e., ready to show that RP allocations are core allocations.

**Theorem 2.** *Let $\theta \in \Theta$ be an RL situation for which $R(N) \leq |\mathscr{D}|$ and let $(N, v^{\theta})$ be the associated RL game. It holds that*

$$\Omega^{\theta} \subseteq \mathscr{C}(N, v^{\theta}).$$

We now give a sketch of proof for this second main result. Efficiency of RP allocations follows by the construction of the resource and profit components in combination with the fact that there is no oversupply of resources. For stability, we use that the sum of resource and profit components of the players in a region exceeds the regional profit.

An interesting follow-up question is under which conditions (if any) every core allocation can be described in terms of an RP allocation. In Theorem 3, we present a sufficient condition under which this is true, i.e., a sufficient condition under which the core coincides with the set of RP allocations. First, we introduce three relevant lemmas that illustrate properties of core allocations in RL games.

The following lemma shows that players who do not own a resource themselves, can claim only a limited share of the total profit.

**Lemma 3.** *Let $\theta \in \Theta$ be an RL situation and let $(N, v^{\theta})$ be the associated RL game. Let $i \in N$ with $r_i = 0$. For any $x \in \mathscr{C}(N, v^{\theta})$ it holds that*

$$x_i \in \begin{cases} \{0\} & \text{if } i \in D_j(N) \text{ for some } j \in \mathscr{D}_N^{nc}, \\ [0, w_i] & \text{if } i \in D_j(N) \text{ for some } j \in \mathscr{D}_N^{c}. \end{cases}$$

For a cooperative game $(N, v)$, we define a coalition $S \subseteq N$ to be self-dual valued if

$$v(N) = v(S) + v(N \backslash S).$$

By Lemma 1, self-dual valued coalitions cannot claim more than their own coalitional value. Thus, by stability, they receive exactly their own coalitional value in every core allocation, i.e., $\sum_{i \in S} x_i = v(S)$ for every $x \in \mathscr{C}(N, v)$ and every self-dual valued coalition $S \subseteq N$. In the following two lemmas we provide two examples of self-dual valued coalitions in RL games.

**Lemma 4.** *Let $\theta \in \Theta$ be an RL situation and let $(N, v^{\theta})$ be the associated RL game. Let $J \subseteq \mathscr{D}_N^c$ with $\sum_{j \in J} R(D_j(N)) = |J|$. Then, coalition $\cup_{j \in J} D_j(N)$ is a self-dual valued coalition. As a consequence, for any $x \in \mathscr{C}(N, v^{\theta})$, it holds that*

$$\sum_{j \in J} \sum_{i \in D_j(N)} x_i = \sum_{j \in J} W_j(N).$$

**Lemma 5.** *Let $\theta \in \Theta$ be an RL situation and let $(N, v^{\theta})$ be the associated RL game. Let $j \in \mathscr{D}_N^c$ with $R(D_j(N)) = 0$. Moreover, let $i \in D_l(N)$ for some $l \in \mathscr{D}_N^{nc}$ with $r_i = 1$. Then, coalition $D_j(N) \cup \{i\}$ is a self-dual valued coalition. As a consequence, for any $x \in \mathscr{C}(N, v^{\theta})$, it holds that*

$$x_i + \sum_{k \in D_j(N)} x_k = W_j(N).$$

We are now ready to present a sufficient condition for which the core and the set of RP allocations coincide, namely the condition that each covered region has no more than two players who initially have a resource.

**Theorem 3.** *Let $\theta \in \Theta$ be an RL situation with $R(N) \leq |\mathscr{D}|$, $R(D_j(N)) \leq 2$ for all $j \in \mathscr{D}_N^c$ and let $(N, v^{\theta})$ be the associated RL game. It holds that*

$$\Omega^{\theta} = \mathscr{C}(N, v^{\theta}).$$

We now provide a sketch of proof for this last main result. We start the proof of this theorem by observing that, based on Theorem 2, it suffices to show that the core is a subset of the set of RP allocations. In particular, we do so by showing that every core allocation can be written as an RP allocation. For that, we distinguish between two cases: the situation in which each covered region has exactly one resource and the situation in which this is not the case. Then, per case, we construct a resource component $(\gamma)$ and a vector of profit components $((\alpha_i)_{i \in N})$ such that they form a core allocation. Finally, we show that these components do satisfy the properties of an RP allocation, i.e., the conditions in (1)–(3).

**Remark 2.** The condition of Theorem 3 resembles the idea that a covered region should have limited bargaining power. If this condition is not satisfied, the coincidence is not guaranteed (see Example 2). Moreover, note that the condition of Theorem 3 is only a sufficient condition and not a necessary condition. To see this, consider for instance an adjusted version of Example 2 with $w = (0, 0, 0, 1, 1)$, then $\Omega^{\theta} = \{(0, 0, 0, 1, 1)\} = \mathscr{C}(N, v^{\theta})$, but $R(D_6(N)) = 3 \not\leq 2$.

**Example 2.** Let $\theta \in \Theta$ be an RL situation with $N = \{1, 2, 3, 4, 5\}$, $w = (1, 2, 3, 4, 5)$, $r = (1, 1, 1, 0, 0)$, $D = \{6, 7, 8\}$, $D_6 = \{1, 2, 3\}$, $D_7 = \{4\}$, and $D_8 = \{5\}$. In Table 2, we present the coalitional values of RL game $(N, v^{\theta})$.

**Table 2**
The RL game $(N, v^{\theta})$ of Example 2.

| $S$ | $v^{\theta}(S)$ | $S$ | $v^{\theta}(S)$ | $S$ | $v^{\theta}(S)$ | $S$ | $v^{\theta}(S)$ |
|---|---|---|---|---|---|---|---|
| $\emptyset$ | 0 | {1, 4} | 4 | {1, 2, 3} | 6 | {2, 4, 5} | 5 |
| {1} | 1 | {1, 5} | 5 | {1, 2, 4} | 7 | {3, 4, 5} | 5 |
| {2} | 2 | {2, 3} | 5 | {1, 2, 5} | 8 | {1, 2, 3, 4} | 10 |
| {3} | 3 | {2, 4} | 4 | {1, 3, 4} | 8 | {1, 2, 3, 5} | 11 |
| {4} | 0 | {2, 5} | 5 | {1, 3, 5} | 9 | {1, 2, 4, 5} | 9 |
| {5} | 0 | {3, 4} | 4 | {1, 4, 5} | 5 | {1, 3, 4, 5} | 9 |
| {1, 2} | 3 | {3, 5} | 5 | {2, 3, 4} | 9 | {2, 3, 4, 5} | 10 |
| {1, 3} | 4 | {4, 5} | 0 | {2, 3, 5} | 10 | {1, 2, 3, 4, 5} | 15 |

It can be checked that $x = (5, 5, 5, 0, 0) \in \mathscr{C}(N, v^{\theta})$. Now, suppose that $x \in \Omega^{\theta}$. So, for each $i \in N$, we can write $x_i = \gamma \cdot r_i + \alpha_i$. For $i \in \{4, 5\}$ this boils down to $\alpha_4 = x_4 = 0$ and $\alpha_5 = x_5 = 0$, because $r_4 = r_5 = 0$. Moreover, since $x \in \Omega^{\theta}$, it holds that $\gamma + \sum_{i \in D_j(N)} \alpha_i = W_j(N)$ for all $j \in \mathscr{D}_N^c$. So, for $j = 7$, this boils down to $\gamma + \alpha_4 = W_7(N) = 4$ and thus $\gamma = 4$. Now, observe that $\gamma + \alpha_5 = 4 \neq 5 = W_8(N)$, which is a contradiction. Hence, $x \notin \Omega^{\theta}$. ◇

## Acknowledgments

The authors would like to thank the review team for their useful suggestions and comments.

## Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.orl.2019.04.002.

## References


[1] S. Anily, M. Haviv, Cooperation in service systems, Oper. Res. 58 (3) (2010) 660–673.

[2] S. Anily, M. Haviv, Line balancing in parallel M/M/1 lines and loss systems as cooperative games, Prod. Oper. Manage. 26 (8) (2017) 1568–1584.

[3] M. Guajardo, M. Rönnqvist, Cost allocation in inventory pools of spare parts with service-differentiated demand classes, Int. J. Prod. Res. 53 (1) (2015) 220–237.

[4] F. Karsten, R. Basten, Pooling of spare parts between multiple users: How to share the benefits? European J. Oper. Res. 233 (1) (2014) 94–104.

[5] F. Karsten, M. Slikker, G. Van Houtum, Inventory pooling games for expensive, low-demand spare parts, Nav. Res. Logist. 59 (5) (2012) 311–324.

[6] F. Karsten, M. Slikker, G. Van Houtum, Resource pooling and cost allocation among independent service providers, Oper. Res. 63 (2) (2015) 476–488.

[7] M. Núñez, C. Rafels, The Böhm–Bawerk horse market: a cooperative analysis, Internat. J. Game Theory 33 (3) (2005) 421–430.

[8] U. Özen, M. Reiman, Q. Wang, On the core of cooperative queueing games, Oper. Res. Lett. 39 (5) (2011) 385–389.

[9] B. Peleg, P. Sudhölter, Introduction to the Theory of Cooperative Games, Vol. 34, Springer Science & Business Media, 2007.

[10] L. Schlicher, M. Slikker, G. Van Houtum, A note on maximal covering location games, Oper. Res. Lett. 45 (1) (2017) 98–103.

[11] L. Schlicher, M. Slikker, G. Van Houtum, Probabilistic resource pooling games, Nav. Res. Logist. 64 (7) (2017) 531–546.

[12] L. Shapley, M. Shubik, The assignment game I: The core, Internat. J. Game Theory 1 (1) (1971) 111–130.

[13] G. Sošić, Transshipment of inventories among retailers: Myopic vs. farsighted stability, Manage. Sci. 52 (10) (2006) 1493–1508.

[14] O. Tejada, Analysis of the core of multi-sided Böhm–Bawerk assignment markets, Top 21 (1) (2013) 189–205.

[15] O. Tejada, M. Núñez, The nucleolus and the core-center of multi-sided Böhm–Bawerk assignment markets, Math. Methods Oper. Res. 75 (2) (2012) 199–220.