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Prioritarian Poverty Comparisons with Cardinal and Ordinal Attributes*

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Abstract

The ethical view of prioritarianism holds that if an extra bundle of attributes is to be allocated to either of two individuals, then priority should be given to the worse off among the two. We consider multidimensional poverty comparisons with cardinal and ordinal attributes and propose three axioms that operationalize the prioritarian view. Each priority axiom, in combination with a handful of standard properties, characterizes a class of poverty measures.

Keywords: Correlation increasing majorization; multidimensional poverty measurement; priority; uniform majorization

JEL classification: *D*31; *D*63; *I*32

I. Introduction

"Benefiting people matters more the worse off these people are." This quote of Parfit (1997, p. 213) summarizes the ethical view of prioritarianism.[1] The view is straightforward to operationalize in the

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[1] Parfit (1997, p. 214) presents prioritarianism as an alternative to egalitarianism. From the prioritarian view, the worse off should be prioritized "but that is only because these people are at a lower *absolute* level. It is irrelevant that these people are worse off *than others*. ... Egalitarians are concerned with *relativities*: with how each person's level compares with the level of other people."

unidimensional setting of income distributions. Standard properties in unidimensional welfare and poverty measurement (with a central role for the Pigou–Dalton transfer principle) do the job (e.g., Fleurbaey, 2001; Tungodden, 2003; Esposito and Lambert, 2011).[2] The implementation of prioritarianism is considerably more challenging in the multidimensional setting. In particular, the absence of a unique well-being indicator (such as income) complicates the identification of the worse off individuals to be prioritized. We consider the setting of multidimensional poverty comparisons and discuss three alternative axioms that operationalize the prioritarian view. The attributes included are either cardinal (e.g., income and life expectancy) or ordinal (e.g., subjective health and physical security). We start with cardinal attributes.

The weakest priority axiom is based on attribute dominance. Suppose that a benefit (an extra bundle of attributes) can be given to either of two poor individuals. If one of the two individuals is worse off in each attribute, then she should receive the extra bundle according to the axiom. If not, then the axiom remains silent. We refer to this axiom as dominance priority. Dominance priority is in the spirit of the Pigou–Dalton bundle dominance principle of Fleurbaey and Trannoy (2003).

The strongest priority axiom is based on the ranking of bundles by the poverty measure itself. As comparing one-person distributions boils down to comparing single bundles, a poverty measure also generates a poverty ranking of individual bundles. Suppose again that an extra bundle of attributes can be given to either of two poor individuals. This version of priority requires that the extra bundle goes to the poorer among the two individuals as judged by the poverty measure itself. We refer to this second axiom as poverty priority. Poverty priority is related to the consistent Pigou–Dalton principle of Bosmans *et al.* (2009). Provided that the poverty measure is monotone in the attributes (an assumption maintained throughout the paper), poverty priority is stronger than (i.e., implies) dominance priority.

Figure 1 illustrates dominance priority and poverty priority. Individual 1's bundle dominates individual 2's bundle. Hence, dominance priority prescribes giving priority to individual 2 over individual 1. Given monotonicity, so does poverty priority. The depicted curve represents

[2] Esposito and Lambert (2011) stress that the distributional concern in unidimensional poverty measurement originates from a prioritarian rather than an egalitarian view. In his pioneering contribution, Watts (1968, p. 326) justifies this concern as follows: "poverty becomes more severe at an increasing rate as successive decrements of income are considered; in other words, … poverty is reduced more by adding $500 to a family's command over goods and services if the family is at 50 percent of the poverty line than if it is at 75 percent." This justification is clearly prioritarian.

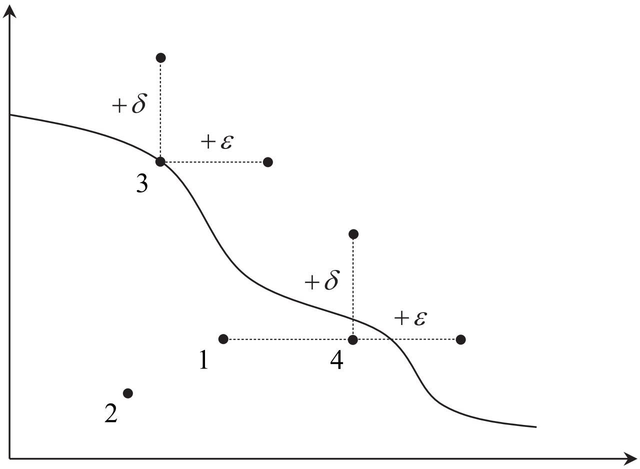

Fig. 1. Who should receive the extra bundle?

a level set of the poverty measure. Clearly, individual 4 is poorer than individual 3. Nonetheless, dominance priority does not recommend giving priority to individual 4 over individual 3, but remains silent. This disregard for the poverty measure's own ranking of the individual bundles could be considered as a shortcoming. In contrast, poverty priority does respect this ranking and prescribes giving priority to individual 4 over individual 3.

However, it could be argued that the implications of the poverty priority axiom are too strong in some cases. Consider again the case in which an extra bundle has to be allocated to either individual 3 or 4. Poverty priority recommends that the extra bundle, δ or ε, is allocated to individual 4. However, individual 4 is already better endowed in the direction of bundle ε than individual 3. If one takes into account possible diminishing returns to well-being (i.e., the possibly greater benefits for individual 3 of obtaining the bundle ε), then it is not clear that individual 4 should receive the bundle. As Parfit (1997, p. 213) puts it, "Benefits to the worse off should be given more weight. This priority is not, however, absolute. … benefits to the worse off could be morally outweighed by sufficiently great benefits to the better off."

This motivates a third priority axiom that strengthens dominance priority, but, contrary to poverty priority, allows for diminishing returns to well-being. If the poorer individual is better endowed in terms of the extra bundle, then this third priority axiom (in contrast to poverty priority) remains silent. If the poorer individual is not better endowed in terms of the extra bundle, then this third axiom follows poverty priority. Applied to individuals 3 and 4 in Figure 1, this axiom recommends that the bundle δ is allocated to individual 4 and remains silent about the allocation of bundle ε. We refer to this third axiom as bundle-dependent priority. Bundle-dependent priority is intermediate in strength between dominance priority and poverty priority.

In the above, we have assumed attributes to be cardinal. Ordinal attributes require special treatment. Priority axioms express the idea that the same increase in attributes is more valuable if the worse off experiences it. However, if, say, the bundle δ in Figure 1 concerns an ordinal attribute, then it is meaningless to state that the potential increase is the same for individuals 3 and 4. This statement is meaningful only if the initial value of the ordinal attribute is equal for the two individuals, such as in the allocation of bundle δ to individual 1 or 4. Hence, in each of the three priority axioms, we will impose this additional condition for bundles containing ordinal attributes.

Our main result characterizes a class of poverty orderings using the bundle-dependent priority axiom in addition to a handful of standard axioms (see Section V, which also discusses two further characterizations based on dominance priority and poverty priority). Let (c^i, o^i) be the attribute bundle of individual i, where c^i is the k-vector listing the values of the cardinal attributes and o^i is the ℓ-vector listing the values of the ordinal attributes. Poverty in a population of size n is measured by the average poverty level

$$\frac{1}{n}\sum_i \pi(c^i, o^i),$$

where the poverty level of individual i is

$$\pi(c^i, o^i) = f\Big[\underbrace{g_1(c_1^i) + g_2(c_2^i) + \cdots + g_k(c_k^i)}_{\text{cardinal}} + \underbrace{h_1(o_1^i) + h_2(o_2^i) + \cdots + h_\ell(o_\ell^i)}_{\text{ordinal}}\Big],$$

where f is a decreasing and convex function, g_j is increasing and concave, and h_j is increasing. The different properties of the functions g_j and h_j reflect the different treatment of cardinal versus ordinal attributes.

Our theorem is the first to characterize a class of multidimensional poverty orderings that deals with cardinal and ordinal attributes jointly. This class encompasses the two main approaches in the body of literature. Atkinson (2003) refers to these approaches as the social welfare approach and the counting approach. The social welfare approach deals exclusively with cardinal attributes and extends concepts of unidimensional social welfare and poverty measurement.[3] The counting approach deals exclusively with ordinal (usually binary) attributes and focuses on counting the number of dimensions in which an individual is deprived.[4]

We highlight two further implications. First, the separability of the individual poverty measure π is neither imposed from the outset, nor implied by strong (and debatable) invariance properties. Rather, it is obtained as a consequence of the bundle-dependent priority axiom in combination with two standard axioms. Second, the properties of the functions f, g_j, and h_j imply that each poverty ordering in the class satisfies uniform majorization (Kolm, 1977), and therefore also the weaker uniform Pigou–Dalton majorization (Weymark, 2006), and correlation increasing majorization (Atkinson and Bourguignon, 1982; Tsui, 1999). Thus, these principles receive a new ethical underpinning using the bundle-dependent priority axiom.

In the next section, we introduce the notation. In Section III, we present the identification criterion and discuss the axioms of representation, focus, and monotonicity. In Section IV, we develop the three priority axioms. In Section V, we present the main result. In Section VI, we discuss the main result.

II. Notation

A population is a finite set of individuals. Each individual is endowed with a bundle of attributes. An attribute bundle is a vector $x = (x_k)_{k \in K}$ of real numbers, where K is a finite set of at least three attributes and x_k is the value of attribute k. The set K of attributes partitions as $C \cup O$, where C is the set of cardinal and O is the set of ordinal attributes. Let $B_C = \mathbb{R}^{|C|}$ and $B_O = \mathbb{R}^{|O|}$. Although attributes are assumed to be continuous, in practice they are approximated by discrete variables. For example, an individual's achievement in education is in essence a continuous concept, but is often measured by the highest degree obtained by the individual. Each bundle x decomposes as (x_C, x_O) with $x_C = (x_k)_{k \in C}$ in B_C and $x_O = (x_k)_{k \in O}$ in B_O. The set $B = B_C \times B_O$ collects all possible bundles. The zero-bundles in B_C, B_O, and B are denoted by 0. For two bundles x and y in B, we write $x \geqslant y$ if $x_k \geqslant y_k$ for each k in K, and $x > y$ if $x \geqslant y$ and $x \neq y$.

[3] See, for example, Tsui (2002), Bourguignon and Chakravarty (2003), Duclos *et al.* (2006), Chakravarty and Silber (2008), and Alkire and Foster (2011). Chakravarty (2009, Chapter 6) provides a survey. Related are studies dedicated to the assessment of poverty over time (e.g., Ligon and Schechter, 2003; Bossert *et al.*, 2012). This framework is also exclusively cardinal because it deals with bundles of incomes, one income per period.

[4] See, for example, Lasso de la Vega (2010), Aaberge and Peluso (2012), and Bossert *et al.* (2013).

Let $x \circ y$ denote the attribute-wise product of two bundles x and y in B; that is, $x \circ y = (x_k y_k)_{k \in K}$.

A set of individuals endowed with a bundle is said to be a distribution. A distribution is fully described by $X = (x^1, x^2, \ldots, x^n)$, where n is the number of individuals in the population, and x^i in B is the bundle of individual $i = 1, 2, \ldots, n$.[5] The domain

$$D = \{(x^1, x^2, \ldots, x^n) | n \in \mathbb{N} \text{ and } x^i \in B \text{ for each } i = 1, 2, \ldots, n\}$$

collects all possible distributions. We do not make a distinction between one-person distributions and bundles; that is, for each bundle x in B, we identify the distribution (x) with x.

A poverty ordering on D is a complete and transitive binary relation in D and is denoted by $\succsim$. We read $X \succsim Y$ as distribution X is at least as good as distribution Y, or equivalently, poverty in X is at most as high as in Y. The asymmetric and symmetric components of $\succsim$ are denoted by $\succ$ and $\sim$.

III. Identification and Three Axioms

The first step in assessing poverty consists of identifying the poor. To determine who is poor and who is not, individual attribute bundles are compared with the poverty thresholds (i.e., the minimally acceptable levels) for the different attributes. For each attribute k in K, let z_k denote the poverty threshold. The vector z in B lists the poverty thresholds for all attributes and is referred to as the poverty bundle. An individual with bundle x in B is said to be deprived in dimension k if $x_k < z_k$.

Typically there exist individuals who are deprived in some dimensions and non-deprived in others. Hence, the identification of the poor depends on the trade-off between the different dimensions. We require only that the trade-off is consistent with how the poverty ordering $\succsim$ on D ranks one-person distributions. That is, an individual with bundle x in B is said to be poor if $z \succ x$ and non-poor if $x \succsim z$. The set P of poor bundles and the set R of non-poor bundles are defined as

$$P = \{x | x \in B \text{ and } z \succ x\} \quad \text{and} \quad R = \{x | x \in B \text{ and } x \succsim z\}.$$

Identification implies that the poverty bundle z extends to a poverty frontier; that is, the set of bundles that are equally good as the poverty bundle z (see Duclos *et al.*, 2006). To sum up, the poverty bundle is exogenous, whereas the poverty frontier through the poverty bundle follows from the poverty ordering.

[5] Below we require the poverty ranking to be anonymous. Therefore, we can use the same labels $1, 2, \ldots$ for individuals across different populations.

We now define the axioms of representation, focus, and monotonicity. The next section develops different versions of the priority axiom.

Representation requires that poverty in a distribution can be judged by its average individual poverty level.

Representation. *There exists a continuous function* $\pi : B \to \mathbb{R}$ *such that, for all distributions* $X = (x^1, x^2, \ldots, x^n)$ *and* $Y = (y^1, y^2, \ldots, y^m)$, *we have*

$$X \succsim Y \quad \text{if and only if} \quad \frac{1}{n}\sum_{i=1}^{n} \pi(x^i) \leqslant \frac{1}{m}\sum_{j=1}^{m} \pi(y^j). \tag{1}$$

The function π can be interpreted as a measure of poverty at the individual level. It is defined up to an affine transformation: for all α and $\beta > 0$, π and $\alpha + \beta\pi$ generate the same poverty ordering $\succsim$. The axiom of representation combines four properties (Tsui, 2002): continuity (small changes in the attribute bundles do not cause large changes in the poverty ranking), anonymity (the names of the individuals do not matter), subgroup consistency (overall poverty increases if poverty increases in a subgroup of the population and remains the same in the complement of this subgroup)[6], and replication invariance (overall poverty does not change if the distribution is replicated).

Focus requires that two distributions are judged as equally poor if the bundles of the poor are the same. Equivalently, replacing a non-poor bundle by the poverty bundle z generates a distribution that is equally good. The focus axiom ensures that the ‘focus’ is solely on the poor individuals.

Focus. *Let* $X = (x^1, \ldots, x^{i-1}, x^i, x^{i+1}, \ldots, x^n)$ *be a distribution. Let* x^i *be a non-poor bundle. Then we have* $X \sim (x^1, \ldots, x^{i-1}, z, x^{i+1}, \ldots, x^n)$.

The focus axiom (as well as several of the axioms discussed next) makes the ordering $\succsim$ dependent on the poverty bundle z. In the notation, we suppress this dependency and write $\succsim$ instead of $\succsim_z$. The imposition of representation and focus implies that we have $\pi(x) = \pi(z)$ for each non-poor bundle x.

Monotonicity demands that poverty decreases if a poor individual receives an additional amount of an attribute.

Monotonicity. *Let* $X = (x^1, \ldots, x^{i-1}, x^i, x^{i+1}, \ldots, x^n)$ *be a distribution. Let* x^i *be a poor bundle. Let* $\varepsilon > 0$ *be a bundle in* B. *Then we have* $(x^1, \ldots, x^{i-1}, x^i + \varepsilon, x^{i+1}, \ldots, x^n) \succ X$.

[6] Many authors regard the separability between individuals inherent in subgroup consistency as being essential to prioritarianism. The same goes for the monotonicity axiom presented below. See, for example, Fleurbaey (2001), Tungodden (2003), and Esposito and Lambert (2011).

An implication of monotonicity is that the poverty ordering is sensitive to both the deprived and the non-deprived attributes of a poor individual. However, monotonicity does not prevent more weight being given to changes in deprived attributes than to changes in non-deprived attributes.[7] The combination of representation, focus, and monotonicity implies that the map π, restricted to the set P of poor bundles, is strictly decreasing in each attribute.

IV. Priority Axioms

Consider a distribution with at least two poor individuals. Assume an indivisible non-negative bundle becomes available. Priority axioms answer the question to which poor individual this extra bundle should be allocated. We distinguish cardinal and ordinal priority axioms, depending on whether the extra bundle (denoted by ε) includes only cardinal or only ordinal attributes.

Let the bundle $\varepsilon = (\varepsilon_C, 0) > 0$ include only cardinal attributes. Let x and y be two bundles such that $x \leqslant y$. Cardinal dominance priority recommends that the extra bundle ε is allocated to the individual endowed with bundle x.

Cardinal dominance priority. *Let $X = (x^1, x^2, \ldots, x^n)$ be a distribution. Let $\varepsilon = (\varepsilon_C, 0) > 0$ be a bundle in $B_C \times B_O$. If $x^i \geqslant x^j$, then*

$$(x^1, \ldots, x^i, \ldots, x^j + \varepsilon, \ldots, x^n) \succsim (x^1, \ldots, x^i + \varepsilon, \ldots, x^j, \ldots, x^n).$$

Cardinal poverty priority relies on the ordering $\succsim$ (restricted to one-person distributions) and recommends that the extra bundle ε is allocated to the poorer individual.

Cardinal poverty priority. *Let $X = (x^1, x^2, \ldots, x^n)$ be a distribution. Let $\varepsilon = (\varepsilon_C, 0) > 0$ be a bundle in $B_C \times B_O$. If $x^i \succsim x^j$, then*

$$(x^1, \ldots, x^i, \ldots, x^j + \varepsilon, \ldots, x^n) \succsim (x^1, \ldots, x^i + \varepsilon, \ldots, x^j, \ldots, x^n).$$

If the poverty ordering $\succsim$ satisfies monotonicity, then $x \geqslant y$ implies $x \succsim y$. As a consequence, monotonicity and cardinal poverty priority entail cardinal dominance priority.

Cardinal poverty priority is a rather demanding ethical requirement. Consider Figure 2, which repeats the example of the introduction.

[7] Tsui (2002) and Bourguignon and Chakravarty (2003) require poverty to be invariant under changes in non-deprived attributes. In contrast, monotonicity interprets an increase in a non-deprived attribute as a (possibly small) improvement.

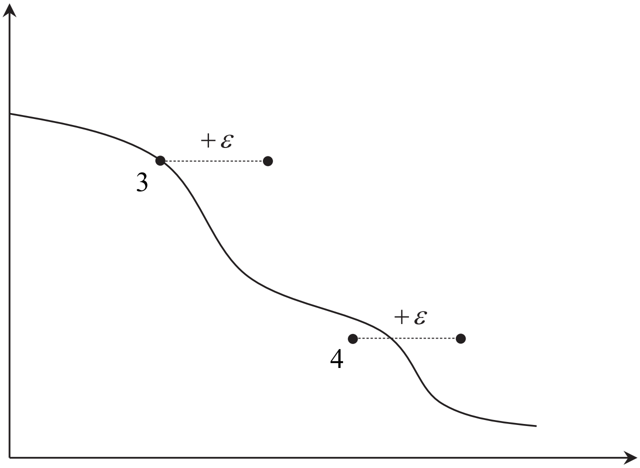

Fig. 2. Who should receive the extra bundle?

According to the poverty ordering (of which an isoline is depicted), individual 4 is poorer than individual 3. By consequence, cardinal poverty priority recommends giving the extra bundle ε to individual 4. However, individual 4 already has more than individual 3 of the attribute in ε. Bundle ε better complements the bundle of 3 than it complements the bundle of 4. Therefore, bundle ε possibly entails greater benefits for individual 3. Cardinal poverty priority disregards these possibly greater benefits for the better off. Although compatible, this is not required by prioritarianism. Indeed, "benefits to the worse off could be morally outweighted by sufficiently great benefits to the better off" (Parfit, 1997, p. 213).

We formulate a third version of priority, cardinal (bundle-dependent) priority. It gives priority to bundle x over bundle y if, in addition to $y \succsim x$, bundle y contains at least as much as bundle x of each attribute for which ε is not zero (i.e., if the attribute-wise product $y \circ \varepsilon$ is at least as great as $x \circ \varepsilon$).

Cardinal priority. *Let* $X = (x^1, x^2, \ldots, x^n)$ *be a distribution. Let* $\varepsilon = (\varepsilon_C, 0) > 0$ *be a bundle in* $B_C \times B_O$. *If* $x^i \succsim x^j$ *and* $x^i \circ \varepsilon \geqslant x^j \circ \varepsilon$, *then*

$$(x^1, \ldots, x^i, \ldots, x^j + \varepsilon, \ldots, x^n) \succsim (x^1, \ldots, x^i + \varepsilon, \ldots, x^j, \ldots, x^n).$$

Given monotonicity, this version is intermediate in strength between cardinal dominance priority and cardinal poverty priority.

Now, let the extra bundle $\varepsilon = (0, \varepsilon_O)$ include only ordinal attributes. The meaning of an increase in an ordinal attribute depends on the amount of the attribute already present. For example, it is not meaningful to say that an increase of an ordinal attribute from 2 to 3 is the same improvement as an increase from 4 to 5. However, if two individuals both have an initial endowment of 2, then an increase to 3 does constitute the same improvement. Therefore, we impose the condition that the two individuals should have the same initial values of the ordinal attributes in bundle ε. We define three versions of ordinal priority.

Ordinal dominance priority. *Let $X = (x^1, x^2, \ldots, x^n)$ be a distribution. Let $\varepsilon = (0, \varepsilon_O) > 0$ be a bundle in $B_C \times B_O$. If $x^i \geqslant x^j$ and $x^i \circ \varepsilon = x^j \circ \varepsilon$, then*

$$(x^1, \ldots, x^i, \ldots, x^j + \varepsilon, \ldots, x^n) \succsim (x^1, \ldots, x^i + \varepsilon, \ldots, x^j, \ldots, x^n).$$

Ordinal poverty priority. *Let $X = (x^1, x^2, \ldots, x^n)$ be a distribution. Let $\varepsilon = (0, \varepsilon_O) > 0$ be a bundle in $B_C \times B_O$. If $x^i \succsim x^j$ and $x^i \circ \varepsilon = x^j \circ \varepsilon$, then*

$$(x^1, \ldots, x^i, \ldots, x^j + \varepsilon, \ldots, x^n) \succsim (x^1, \ldots, x^i + \varepsilon, \ldots, x^j, \ldots, x^n).$$

Due to the restriction on the initial values, ordinal (bundle-dependent) priority coincides with ordinal poverty priority.

Ordinal priority. *This coincides with ordinal poverty priority.*

Table 1 summarizes the different priority axioms. In each of the six cases, the corresponding priority axiom recommends that the extra bundle is allocated to individual j. As ordinal (bundle-dependent) priority and ordinal poverty priority coincide, the corresponding entries in the table also coincide.

Table 1. *Priority axioms*

	Poverty	(Bundle-dependent)	Dominance
Cardinal	$x^i \succsim x^j$	$x^i \succsim x^j$	$x^i \geqslant x^j$
$\varepsilon = (\varepsilon_C, 0) > 0$		$x^i \circ \varepsilon \geqslant x^j \circ \varepsilon$	
Ordinal	$x^i \succsim x^j$	$x^i \succsim x^j$	$x^i \geqslant x^j$
$\varepsilon = (0, \varepsilon_O) > 0$	$x^i \circ \varepsilon = x^j \circ \varepsilon$	$x^i \circ \varepsilon = x^j \circ \varepsilon$	$x^i \circ \varepsilon = x^j \circ \varepsilon$

We now combine the different priority axioms to obtain three final versions of priority. Each version deals with both cardinal and ordinal attributes, but differs in how it assigns priority.

Dominance priority. *Cardinal dominance priority and ordinal dominance priority hold.*

Poverty priority. *Cardinal poverty priority and ordinal poverty priority hold.*

Priority. *Cardinal priority and ordinal priority hold.*

Given monotonicity, the priority axioms are logically connected. A monotonic poverty ordering that satisfies poverty priority also satisfies priority. Furthermore, a monotonic poverty ordering that satisfies priority also satisfies dominance priority.

V. Main Result

Our main result characterizes poverty orderings that satisfy representation, focus, monotonicity, and priority. See the Appendix for the proof.

Theorem 1. *A poverty ordering $\succsim$ on D with poverty bundle z in B satisfies representation, focus, monotonicity, and priority if and only if there exist (a) strictly increasing and concave functions $g_k : \mathbb{R} \to \mathbb{R}$, (b) strictly increasing functions $h_\ell : \mathbb{R} \to \mathbb{R}$, (c) a decreasing and convex continuous function $f : \mathbb{R} \to \mathbb{R}$ with (i) $f(r) = 0$ for each $r \geqslant \zeta = \sum_C g_k(z_k) + \sum_O h_\ell(z_\ell)$, and (ii) f strictly decreasing on the interval $(-\infty, \zeta]$, such that, for all distributions X and Y in D, we have $X \succsim Y$ if and only if*

$$\frac{1}{n}\sum_{i=1}^{n} f\left[\sum\nolimits_C g_k(x_k^i) + \sum\nolimits_O h_\ell(x_\ell^i)\right] \leqslant \frac{1}{m}\sum_{j=1}^{m} f\left[\sum\nolimits_C g_k(y_k^j) + \sum\nolimits_O h_\ell(y_\ell^j)\right]. \tag{2}$$

We indicate how the main result changes if priority is weakened to dominance priority, or is strengthened to poverty priority. The combination of representation, focus, monotonicity, and dominance priority imposes that π has non-increasing increments. That is,

$$\pi(x) - \pi(x+\varepsilon) \geqslant \pi(y) - \pi(y+\varepsilon)$$

for all bundles x and y in P with $x \leqslant y$ and for each bundle ε in B with $x_O \circ \varepsilon_O = y_O \circ \varepsilon_O$. To see this, start from equation (1) and apply dominance priority to the two-person distribution (x, y). Clearly, dominance priority imposes little structure on π.

Strengthening priority to poverty priority implies that the functions g_k in the theorem are linear. The individual poverty measure becomes

$$\pi(x) = f\left[\sum\nolimits_C w_k x_k + \sum\nolimits_O h_\ell(x_\ell)\right]$$

where $w_k > 0$ is a weight for each attribute k in C. To see this, start from equation (2). Consider two bundles x and y in P such that $\pi(x) = \pi(y)$ and a cardinal attribute k. Let ε be a non-negative bundle in B with $\varepsilon_\ell = 0$ for each attribute $\ell \neq k$. Cardinal poverty priority imposes $\pi(x+\varepsilon) = \pi(y+\varepsilon)$. Hence, the gain from x to $x+\varepsilon$ is equal to the gain from y to $y+\varepsilon$; that is,

$$g_k(x_k+\varepsilon_k) - g_k(x_k) = g_k(y_k+\varepsilon_k) - g_k(y_k).$$

As the slope $\left[g_k(x_k+\varepsilon_k) - g_k(x_k)\right]/\varepsilon_k$ does not depend on x_k, linearity follows. This requirement of perfect substitutability of the cardinal attributes shows that poverty priority is a demanding axiom.[8]

VI. Discussion

Our theorem is the first to characterize a class of multidimensional poverty orderings that deals with cardinal and ordinal attributes jointly.[9] Without the focus axiom, the poverty ordering becomes a welfare ordering. Thus, our theorem also applies in the welfare setting.

Together, the four axioms require the individual poverty function π to take the form $f(\sum_C g_k + \sum_O h_\ell)$. We explain the role played by the different components of the individual poverty function and discuss some broader implications.

The sum $\sum_C g_k + \sum_O h_\ell$ is a measure of individual well-being. The functions g_k and h_ℓ determine the trade-offs between the attributes. The difference between cardinal and ordinal priority implies that the maps g_k for the cardinal attributes are strictly increasing and concave, whereas the maps h_ℓ for the ordinal attributes are strictly increasing only. A popular functional form for the cardinal attributes is $g_k : x_k \mapsto w_k x_k^\beta$ with $w_k > 0$ for each k and $0 < \beta < 1$ (e.g., Bourguignon and Chakravarty, 2003). If the ordinal attributes are binary (the typical setting of what Atkinson, 2003, refers to as the counting approach), the ordinal part becomes a weighted count. Indeed, absorbing $\sum_O h_\ell(0)$ in f, the ordinal part can be written as $\sum_O w_\ell x_\ell$, where $w_\ell = h_\ell(1) - h_\ell(0) > 0$ is the weight of attribute ℓ. If we combine the specified cardinal and ordinal parts, then individual well-being is equal to $\sum_C w_k x_k^\beta + \sum_O w_\ell x_\ell$.

The well-being measure $\sum_C g_k + \sum_O h_\ell$ specifies the indifference map for the poor, including the shape of the poverty frontier. The properties of g_k imply that the indifference maps in the space of the cardinal attributes

[8] Perfect substitutability of the cardinal attributes can be avoided by focusing lexicographically on the worst off. However, this lexicographic poverty ordering does not satisfy continuity.

[9] Duclos *et al.* (2007) and Alkire and Foster (2011) also propose poverty orderings that handle cardinal and ordinal attributes jointly. However, they do not provide characterizations.

are convex. This precludes the use of the dual-cutoff approach suggested by Alkire and Foster (2011), except for the union case.

The map f is decreasing and convex and turns individual well-being levels into individual poverty levels. In the above example, $\zeta = \sum_C w_k z_k^\beta + \sum_O w_\ell z_\ell$ measures well-being at the poverty bundle z. A popular functional form for the poverty function is $f: r \mapsto \left[\max\{0, (\zeta - r)/\zeta\}\right]^\alpha$ with $\alpha \geqslant 1$ measuring the curvature of f (e.g., Foster *et al.*, 1984). The larger the curvature of f, the larger the relative weight of the worse off poor in the poverty ordering (Zheng, 2000, Proposition 1, and Bosmans, 2014, Proposition 2). Absolute priority (lexicographically) to the worst off can be approached arbitrarily closely.

We now turn to two broader implications of the theorem. First, well-being must be separable in each attribute. Attribute-separability is a common property of multidimensional poverty orderings.[10] In the body of axiomatic literature, attribute-separability follows typically from a strong (and debatable) ratio-scale invariance axiom that requires poverty to remain the same if the attribute values and the poverty threshold in one dimension are multiplied by the same positive constant (Tsui, 2002; Chakravarty and Silber, 2008; for a critical discussion, see Weymark, 2006). Our theorem (step 1 of the proof) offers an alternative justification of attribute-separability that relies on adding priority to representation and monotonicity.

Second, the properties of the functions f, g_k, and h_ℓ guarantee that the standard equity principles uniform majorization and correlation increasing majorization are satisfied. As a consequence, also the weaker uniform Pigou–Dalton majorization (Weymark, 2006) is satisfied. Therefore, these principles, which are prominent but controversial in the body of literature (see Bosmans *et al.* (2015) for a discussion), receive a new ethical underpinning using priority.

Uniform majorization presupposes a setting where each individual has the same ordinal bundle. The principle demands that post-multiplying the distribution of cardinal bundles by a non-permutation bistochastic matrix does not increase poverty.[11] We decompose a distribution X as (X_C, X_O) where X_C is the matrix $(x_C^1, x_C^2, \ldots, x_C^n)$ and $X_O = (x_O^1, x_O^2, \ldots, x_O^n)$.

Uniform majorization. *Let* $X = (X_C, X_O)$ *be a distribution with* $x_O^1 = x_O^2 = \cdots = x_O^n$. *Let* M *be a non-permutation bistochastic matrix. Then,* $(X_C M, X_O) \succsim (X_C, X_O)$.

[10] See Chakravarty *et al.* (1998), Tsui (2002), Bourguignon and Chakravarty (2003), Chakravarty and Silber (2008), Alkire and Foster (2011), and Bossert *et al.* (2013).

[11] A bistochastic matrix is a non-negative square matrix in which each row and each column sums to one. A permutation matrix is a bistochastic matrix that contains only zeros and ones.

The poverty ordering in the theorem satisfies uniform majorization. Tsui (2002, Proposition 3) shows that convexity of the function π (in the cardinal attributes) is a necessary and sufficient condition for uniform majorization. The concavity of the functions g_k and the decreasingness and convexity of f indeed imply $\pi[\alpha x_C + (1-\alpha)y_C, x_O] \leqslant \alpha\pi(x_C, x_O) + (1-\alpha)\pi(y_C, x_O)$ for all bundles x_C and y_C in B_C, each bundle x_O in B_O, and each α in the interval $[0,1]$.

Correlation increasing majorization requires that switching attributes between two individuals until one individual has more of each attribute than the other does not decrease poverty. Correlation increasing majorization applies to both cardinal and ordinal attributes. Consider two bundles x and y. Let $x \wedge y$ be the bundle $(\min\{x_k, y_k\})_{k\in K}$ and let $x \vee y$ be $(\max\{x_k, y_k\})_{k\in K}$. Note that $x + y = (x \wedge y) + (x \vee y)$.

Correlation increasing majorization. *Let* $X = (x^1, \ldots, x^i, \ldots, x^j, \ldots, x^n)$ *be a distribution. Then,* $X \succsim (x^1, \ldots, x^i \vee x^j, \ldots, x^i \wedge x^j, \ldots, x^n)$.

Dominance priority, and thus also the stronger versions of priority, imply correlation increasing majorization. To see this, consider a distribution X and two individuals i and j. Construct a distribution Y from X such that $y^j = x^i \wedge x^j$ and $y^k = x^k$ for each individual $k \neq j$. We have that $y^i \geqslant y^j$. Define $\varepsilon = x^j - y^j = x^j - (x^i \wedge x^j) \geqslant 0$ and verify that $y^i \circ \varepsilon = y^j \circ \varepsilon$ holds. Dominance priority implies

$$(\ldots, y^i, \ldots, y^j + \varepsilon, \ldots) \succsim (\ldots, y^i + \varepsilon, \ldots, y^j, \ldots),$$

or equivalently,

$$(\ldots, x^i, \ldots, x^j, \ldots) \succsim (\ldots, x^i \vee x^j, \ldots, x^i \wedge x^j, \ldots),$$

as required.

Appendix

Proof of Theorem 1: The representation defined in the theorem satisfies all axioms. The reverse implication is split into four steps. Recall first that a poverty ordering $\succsim$ on D satisfies representation, focus, and monotonicity if and only if there exists a continuous individual poverty function $\pi : B \to \mathbb{R}$, with (a) $\pi(x) = \pi(z)$ for each non-poor bundle x, and (b) π strictly decreasing on the set P of poor bundles, such that equation (1) holds.

Step 1: Representation, monotonicity, and priority imply separability in attributes.

Consider an attribute k in K and two equally poor bundles x

and y with $x_k = y_k$. Let ε be a bundle in B with $\varepsilon_\ell = 0$ for each attribute $\ell \neq k$ and $\varepsilon_k > 0$. As x and y are equally poor, priority implies that also the distributions $(x+\varepsilon, y)$ and $(x, y+\varepsilon)$ are equally poor. Representation implies that $\pi(x+\varepsilon) = \pi(y+\varepsilon)$.

We argue that, starting from the same assumptions, also the equality $\pi(x-\varepsilon) = \pi(y-\varepsilon)$ must hold. The argument is by contradiction and starts from the inequality $\pi(x-\varepsilon) < \pi(y-\varepsilon)$. As π is strictly decreasing in P and $\pi(x) = \pi(y)$, we have $\pi(y) < \pi(x-\varepsilon) < \pi(y-\varepsilon)$. Because π is continuous, there exists a scalar λ with $0 < \lambda < 1$, such that

$$\pi(x-\varepsilon) = \pi(y-\lambda\varepsilon).$$

The distribution $(x-\varepsilon, y-\lambda\varepsilon)$ satisfies $x_k - \varepsilon_k \leqslant y_k - \lambda\varepsilon_k$. Apply cardinal priority and obtain

$$(x-\varepsilon+\varepsilon, y-\lambda\varepsilon) \succsim (x-\varepsilon, y-\lambda\varepsilon+\varepsilon).$$

Consequently,

$$\pi(x) + \pi(y-\lambda\varepsilon) \leqslant \pi(x-\varepsilon) + \pi[y+(1-\lambda)\varepsilon].$$

The equality $\pi(x-\varepsilon) = \pi(y-\lambda\varepsilon)$ implies the inequality $\pi(x) \leqslant \pi[y+(1-\lambda)\varepsilon]$. Recall that x and y are equally poor (i.e., $\pi(x) = \pi(y)$) and $(1-\lambda)\varepsilon > 0$. Hence, this inequality conflicts with monotonicity in P.

Both arguments together imply the following result. Let k in K be an attribute, and let x and y be two bundles in P with $x_k = y_k$. Obtain x' and y' in P from x and y by replacing $x_k = y_k$ by $x'_k = y'_k$. Then, $\pi(x) = \pi(y)$ implies $\pi(x') = \pi(y')$. Now, modify the assumption $\pi(x) = \pi(y)$ to $\pi(x) \geqslant \pi(y)$. Let $c = (\max\{y_k - x_k, 0\})_{k \in K}$ and note that $c_k = 0$. We have $\pi(x) \geqslant \pi(y) \geqslant \pi(x+c)$. As π is continuous, there must exist a scalar λ with $0 \leqslant \lambda \leqslant 1$, such that $\pi(y) = \pi(x+\lambda c)$. As a consequence, $\pi(x'+\lambda c) = \pi(y')$ holds and, using monotonicity, we get $\pi(x') \geqslant \pi(y')$. To sum up, π is separable in attributes. That is, for all bundles x, x', y, and y' in P and for each attribute k in K, if $x_k = y_k$ and $x'_k = y'_k$, and $x_\ell = x'_\ell$ and $y_\ell = y'_\ell$ for each attribute $\ell \neq k$, then $\pi(x) \geqslant \pi(y)$ implies $\pi(x') \geqslant \pi(y')$.

Step 2: Application of Debreu (1960).

Continuity and separability of the poverty function π in P, and the assumption of at least three attributes, allows for an additive representation (Debreu, 1960). More precisely, there exist continuous maps $\bar{f}_k : \mathbb{R} \to \mathbb{R}$ for each k in K, such that, for each x and y in P, we have

$$\pi(x) \leqslant \pi(y) \quad \text{if and only if} \quad \sum_K \bar{f}_k(x_k) \leqslant \sum_K \bar{f}_k(y_k).$$

As π is strictly decreasing in the set P of poor bundles, also the maps $\bar{f}_k$ are strictly decreasing. Let $f_k = \bar{f}_k(0) - \bar{f}_k$ for each k and obtain that $\pi(x) = f\left[\sum_K f_k(x_k)\right]$ for each x in P with (a) f_k continuous, strictly increasing, with $f_k(0) = 0$, and (b) $f : \mathbb{R} \to \mathbb{R}$ continuous and strictly decreasing on the interval $(-\infty, \zeta]$ with $\zeta = \sum_K f_k(z_k)$. We can normalize $f(\zeta) = 0$ without loss of generality because π is defined up to an affine transformation.

Step 3: The map f is convex.

Consider two poor bundles x and y such that $x \leqslant y$ and $x_k = y_k$. Let ε be a bundle that is zero in each dimension except for dimension k ($\varepsilon_k > 0$). Apply priority to the distribution (x, y) and obtain the inequality

$$f\left[f_k(x_k + \varepsilon_k) + \sum_{\ell \neq k} f_\ell(x_\ell)\right] - f\left[\sum_K f_k(x_k)\right]$$
$$\leqslant f\left[f_k(y_k + \varepsilon_k) + \sum_{\ell \neq k} f_\ell(y_\ell)\right] - f\left[\sum_K f_k(y_k)\right].$$

As $x_k = y_k$, the inequality can be rewritten as

$$f(\delta + a) - f(a) \leqslant f(\delta + b) - f(b),$$

where

$$a = \sum_K f_k(x_k) \leqslant b = \sum_K f_k(y_k)$$

and

$$\delta = f_k(x_k + \varepsilon_k) - f_k(x_k) = f_k(y_k + \varepsilon_k) - f_k(y_k).$$

Conclude that the map f is convex.

Step 4: For each cardinal attribute k in C, the map f_k is concave.

Consider two equally poor bundles x and y that satisfy $x_k \leqslant y_k$. Again, let ε be a bundle that is zero in each dimension except for dimension k ($\varepsilon_k > 0$). Apply cardinal priority to the distribution (x, y) and obtain the inequality

$$f\left[f_k(x_k+\varepsilon_k)+\sum_{\ell\neq k} f_\ell(x_\ell)\right]+\pi(y)$$
$$\leqslant f\left[f_k(y_k+\varepsilon_k)+\sum_{\ell\neq k} f_\ell(y_\ell)\right]+\pi(x).$$

As x and y are assumed to be equally poor ($\pi(x)=\pi(y)$) and because f is strictly decreasing, it follows that

$$f_k(x_k+\varepsilon)-f_k(x_k)\geqslant f_k(y_k+\varepsilon)-f_k(y_k),$$

and hence the map f_k is concave for each k in C. Write g_k for f_k with k in C, and h_ℓ for f_ℓ with ℓ in O. Equation (2) follows. □

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