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Fair and efficient allocations when preferences are single-dipped

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Abstract

One unit of an infinitely divisible and non-disposable commodity has to be allocated among a group of agents with single-dipped preferences. We combine Pareto optimality with equal treatment of equals, the equal division lower bound, the equal division core, envy-freeness, and group envy-freeness. For each of these fairness requirements, we provide a necessary and sufficient condition for compatibility with Pareto optimality and we characterize all corresponding allocations for each preference profile.

Keywords: resource allocation, single-dipped preferences, Pareto optimality, fairness
JEL classification: D63, D71

1 Introduction

One unit of an infinitely divisible and non-disposable commodity has to be allocated among a group of agents with single-dipped preferences, i.e. for each agent there exists a unique least preferred share (the dip), and preferences increase in both directions away from the dip. This type of preferences arise in situations where extremes are preferred over combinations, such as teaching activities and management tasks at a university department, two-goods exchange economies with fixed prices and strictly quasiconvex utility functions, and common-pool resource allocation problems under increasing returns to scale.

Within this context, Klaus et al. (1997) characterized the *Pareto optimal* allocations, i.e. allocations for which no other allocation is weakly preferred by each agent and strictly preferred by some agent. On top of that, they imposed strategy-proofness, i.e. no agent has an incentive to misrepresent preferences, and other robustness properties. Klaus (2001a) and Klaus (2001b) continued this study by combining Pareto optimality with coalitional

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strategy-proofness, or with strategy-proofness and another robustness or solidarity property. All the rules obtained in these studies assign the entire commodity to one of the agents, and consequently violate various fairness requirements. Kranich (2019) showed that even the divide-and-choose method is unfair in this context. To overcome the fairness issue, Ehlers (2002) introduced probabilistic allocation rules. Doghmi (2013a), Doghmi (2013b), Gong et al. (2022), and Inoue and Yamamura (2023) took a mechanism design approach and focused on Nash implementation. Other studies focused on indivisible commodities: see for instance the recent works of Fujinaka and Wakayama (2023) and Tamura (2023).[1]

Klaus (2001a) already observed that Pareto optimality is not always compatible with the weak fairness requirement of *equal treatment of equals*, i.e. each agent is indifferent between its own share and the share of any other agent with the same preferences. In this paper, we provide a necessary and sufficient condition for compatibility of Pareto optimality and equal treatment of equals. Moreover, we characterize for each preference profile all Pareto optimal allocations that satisfy equal treatment of equals. We also study the compatibility of Pareto optimality with other fairness requirements: the *equal division lower bound*, i.e. each agent weakly prefers the allocation to an equal share, the *equal division core*, i.e. for each group of agents, no redistribution of equal shares is weakly preferred by each group member and strictly preferred by some group member, *envy-freeness*, i.e. each agent weakly prefers its own share to the share of any other agent, and *group envy-freeness*, i.e. for each group of agents, no redistribution of the share of any group of agents with the same size is weakly preferred by each group member and strictly preferred by some group member. If an allocation is group envy-free, then it is envy-free and belongs to the equal division core. If an allocation is envy-free, then it satisfies equal treatment of equals and the equal division lower bound. If an allocation belongs to the equal division core, then it satisfies the equal division lower bound. For each of these fairness requirements, we provide a necessary and sufficient condition for compatibility with Pareto optimality and we characterize all corresponding allocations for each preference profile. Among the fairness requirements we consider, only the equal division lower bound is compatible with Pareto optimality for each preference profile.

This paper is organized as follows. Section 2 introduces the model and defines Pareto optimality. Section 3 characterizes the Pareto optimal allocations that satisfy equal treatment of equals. Section 4 characterizes the Pareto optimal allocations that satisfy the equal division lower bound or belong to the equal division core. Section 5 characterizes Pareto optimal allocations that are envy-free or group envy-free. Section 6 concludes.

[1] A majority of the literature has focused on single-dipped preferences in situations with public commodities. These problems arise when for instance a public facility with strongly negative externalities needs to be located along a road. For more details on those problems, we refer to Barberà et al. (2012), Öztürk et al. (2013), Öztürk et al. (2014), Manjunath (2014), Tapki (2016), Ayllón and Caramuta (2016), Yamamura (2016), Lahiri et al. (2017), Han et al. (2018), Alcalde-Unzu and Vorsatz (2018), Lahiri and Storcken (2019), Feigenbaum et al. (2020), and Thomson (2023).

2 Model

Let N be a finite set of at least two **agents**. One unit of an infinitely divisible and non-disposable commodity has to be allocated among the agents. Each agent $i \in N$ has complete, transitive, continuous, and single-dipped preferences represented by the binary relation R_i over $[0, 1]$, i.e. there exists $d(R_i) \in [0, 1]$ such that $x_i \; P_i \; y_i$ for all $x_i, y_i \in [0, 1]$ with $x_i < y_i \leq d(R_i)$ or $d(R_i) \leq y_i < x_i$. Here, $x_i \; P_i \; y_i$ denotes $x_i \; R_i \; y_i$ and $\neg y_i \; R_i \; x_i$. Moreover, $x_i \; I_i \; y_i$ denotes $x_i \; R_i \; y_i$ and $y_i \; R_i \; x_i$. As usual, the interpretation is that agent $i \in N$ weakly prefers x_i to y_i if $x_i \; R_i \; y_i$, strictly prefers x_i to y_i if $x_i \; P_i \; y_i$, and is indifferent between x_i and y_i if $x_i \; I_i \; y_i$. The set of such preferences is denoted by $\mathcal{R}$, and $\mathcal{R}^N = \mathcal{R} \times \cdots \times \mathcal{R}$ is the set of **preference profiles**.

Remark
Let $i \in N$, $R_i \in \mathcal{R}$, and $x_i, y_i \in [0, 1]$. The following statements hold:

- *If $x_i \; R_i \; y_i$, then $x_i \; P_i \; z_i$ for each $z_i \in [0, 1]$ with $\min\{x_i, y_i\} < z_i < \max\{x_i, y_i\}$.*
- *If $x_i \; R_i \; y_i$ and $x_i < y_i$, then $z_i \; P_i \; x_i$ for each $z_i \in [0, x_i)$.*
- *If $x_i \; R_i \; y_i$ and $x_i > y_i$, then $z_i \; P_i \; x_i$ for each $z_i \in (x_i, 1]$.*

Let $R \in \mathcal{R}^N$ be a preference profile. The set of agents that strictly prefer 0 to 1 is denoted by $N_0(R) = \{i \in N \mid 0 \; P_i \; 1\}$, the set of agents that are indifferent between 0 and 1 is denoted by $N_{0,1}(R) = \{i \in N \mid 0 \; I_i \; 1\}$, and the set of agents that strictly prefer 1 to 0 is denoted by $N_1(R) = \{i \in N \mid 1 \; P_i \; 0\}$. Note that $N_0(R) \cup N_{0,1}(R) \cup N_1(R) = N$.

An **allocation** is an $x \in [0, 1]^N$ such that $\sum_{i \in N} x_i = 1$. The set of allocations is denoted by X. Throughout this paper, we focus on Pareto optimal allocations, i.e. allocations for which no other allocation is weakly preferred by each agent and strictly preferred by some agent. Formally, for preference profile $R \in \mathcal{R}^N$, an allocation $x \in X$ is **Pareto optimal** if there does not exist $y \in X$ such that $y_i \; R_i \; x_i$ for each $i \in N$ and $y_i \; P_i \; x_i$ for some $i \in N$. The set of Pareto optimal allocations is denoted by $P(R)$. Klaus et al. (1997) characterized the Pareto optimal allocations. The proof is provided by Klaus (2001a).

Theorem 1 (Klaus et al. 1997 and Klaus 2001a)
Let $R \in \mathcal{R}^N$ be a preference profile and let $x \in X$ be an allocation. Then x is Pareto optimal, i.e. $x \in P(R)$, if and only if the following statements hold:

- *If $N_1(R) \neq \emptyset$, then $x_i = 0$ or $x_i \; P_i \; 0$ for each $i \in N$.*
- *If $N_1(R) = \emptyset$ and $N_{0,1}(R) \neq \emptyset$, then $x_i = 1$ for some $i \in N_{0,1}(R)$.*
- *If $N_0(R) = N$, then $x_i = 1$ or $x_i \; P_i \; 1$ for each $i \in N$.*

Corollary 1
Let $R \in \mathcal{R}^N$ be a preference profile. Then a Pareto optimal allocation exists, i.e. $P(R) \neq \emptyset$.

3 Equal treatment of equals

This section studies Pareto optimal allocations that satisfy equal treatment of equals, i.e. allocations for which each agent is indifferent between its own share and the share of any other agent with the same preferences. Formally, for preference profile $R \in \mathcal{R}^N$, an allocation $x \in X$ satisfies **equal treatment of equals** if $x_i \; I_i \; x_j$ for all $i, j \in N$ with $R_i = R_j$. The set of allocations satisfying equal treatment of equals is denoted by $E(R)$.

The following example shows that Pareto optimal allocations satisfying equal treatment of equals do not necessarily exist.

Example 1
Let $R \in \mathcal{R}^N$ be a preference profile with $R_i = R_j$ for all $i, j \in N$ such that $1 \; P_i \; 0 \; P_i \; \frac{1}{2}$ for each $i \in N$. Then $P(R) \cap E(R) = \emptyset$. △

For an agent $i \in N$ and a preference profile $R \in \mathcal{R}^N$, the maximum share that 0 is weakly preferred to is denoted by $k_i^1(R) = \max\{x_i \in [0,1] \mid 0 \; R_i \; x_i\}$, and the minimum share that 1 is weakly preferred to is denoted by $k_i^0(R) = \min\{x_i \in [0,1] \mid 1 \; R_i \; x_i\}$. Note that $k_i^1(R) = 1$ if and only if $i \notin N_1(R)$, and $k_i^0(R) = 0$ if and only if $i \notin N_0(R)$.

We characterize the Pareto optimal allocations that satisfy equal treatment of equals.

Theorem 2
Let $R \in \mathcal{R}^N$ be a preference profile and let $x \in P(R)$ be a Pareto optimal allocation. Then x satisfies equal treatment of equals, i.e. $x \in E(R)$, if and only if the following statements hold:

- *If $N_1(R) \neq \emptyset$, then*
  - *$x_i = 0$ or $x_i > k_i^1(R)$ for each $i \in N$;*
  - *$x_i = x_j$ for all $i, j \in N$ with $R_i = R_j$.*
- *If $N_1(R) = \emptyset$ and $N_{0,1}(R) \neq \emptyset$, then $x_i = 1$ for some $i \in N_{0,1}(R)$.*
- *If $N_0(R) = N$, then*
  - *$x_i = 1$ or $x_i < k_i^0(R)$ for each $i \in N$;*
  - *$x_i = x_j$ for all $i, j \in N$ with $R_i = R_j$.*

Proof. Clearly, if the statements hold, then $x \in E(R)$. We prove the only-if part. By Theorem 1, it suffices to show that if $x \in E(R)$, then the following statements hold:

(i) If $N_1(R) \neq \emptyset$, then $x_i = 0$ or $x_i > k_i^1(R)$ for each $i \in N$, and $x_i = x_j$ for all $i, j \in N$ with $R_i = R_j$.

(ii) If $N_0(R) = N$, then $x_i = 1$ or $x_i < k_i^0(R)$ for each $i \in N$, and $x_i = x_j$ for all $i, j \in N$ with $R_i = R_j$.

(i) Assume that $N_1(R) \neq \emptyset$. By Theorem 1, $x_i = 0$ or $x_i \; P_i \; 0$ for each $i \in N$. For the sake of contradiction, assume that there exists $i \in N$ such that $0 < x_i \leq k_i^1(R)$. Then $0 \; R_i \; k_i^1(R)$ implies that $0 \; R_i \; x_i$. This contradicts that $x_i \; P_i \; 0$. Hence, $x_i = 0$ or $x_i > k_i^1(R)$ for each $i \in N$. For the sake of contraposition, assume that there exist $i, j \in N$ with $R_i = R_j$ such that $x_i \neq x_j$. Assume without loss of generality that $x_i > x_j$. Then $x_i \; P_i \; 0$ implies that $x_i \; P_i \; x_j$, so $x \notin E(R)$. Hence, $x_i = x_j$ for all $i, j \in N$ with $R_i = R_j$.

(ii) Assume that $N_0(R) = N$. By Theorem 1, $x_i = 1$ or $x_i \; P_i \; 1$ for each $i \in N$. For the sake of contradiction, assume that there exists $i \in N$ such that $k_i^0(R) \leq x_i < 1$. Then $1 \; R_i \; k_i^1(R)$ implies that $1 \; R_i \; x_i$. This contradicts that $x_i \; P_i \; 1$. Hence, $x_i = 1$ or $x_i > k_i^1(R)$ for each $i \in N$. For the sake of contraposition, assume that there exist $i, j \in N$ with $R_i = R_j$ such that $x_i \neq x_j$. Assume without loss of generality that $x_i < x_j$. Then $x_i \; P_i \; 1$ implies that $x_i \; P_i \; x_j$, so $x \notin E(R)$. Hence, $x_i = x_j$ for all $i, j \in N$ with $R_i = R_j$. □

Theorem 2 provides a necessary and sufficient condition for compatibility of Pareto optimality with equal treatment of equals.

Corollary 2
Let $R \in \mathcal{R}^N$ be a preference profile. Then a Pareto optimal allocation satisfying equal treatment of equals exists, i.e. $P(R) \cap E(R) \neq \emptyset$, if and only if one of the following statements hold:

- $N_1(R) \neq \emptyset$ *and* $\sum_{j \in N : R_i = R_j} k_j^1(R) < 1$ *for some* $i \in N_1(R)$.
- $N_1(R) = \emptyset$ *and* $N_{0,1}(R) \neq \emptyset$.
- $N_0(R) = N$ *and there exists* $i \in N$ *such that* $R_i \neq R_j$ *for each* $j \in N \setminus \{i\}$.
- $N_0(R) = N$ *and* $\sum_{i \in N} k_i^0(R) > 1$.

4 Equal division lower bound

This section studies Pareto optimal allocations that satisfy the equal division lower bound, i.e. allocations weakly preferred to an equal share allocation by each agent. Formally, for preference profile $R \in \mathcal{R}^N$, an allocation $x \in X$ satisfies the **equal division lower bound** if $x_i \; R_i \; \frac{1}{|N|}$ for each $i \in N$. The set of allocations satisfying the equal division lower bound is denoted by $B_{ed}(R)$.

The following example shows that Pareto optimal allocations satisfying equal treatment of equals do not necessarily satisfy the equal division lower bound.

Example 2
Let $R \in \mathcal{R}^N$ be a preference profile with $R_i \neq R_j$ for all $i, j \in N$ such that $\frac{1}{|N|} \; P_i \; 0$ for each $i \in N$. For each $x \in X$ with $x_i = 1$ for some $i \in N$, it holds that $x \in (P(R) \cap E(R)) \setminus B_{ed}(R)$. △

For preference profile $R \in \mathcal{R}^N$, the set of agents that strictly prefer $\frac{1}{|N|}$ to 0 is denoted by $M_1(R) = \{i \in N \mid \frac{1}{|N|} \ P_i \ 0\}$, and the set of agents that strictly prefer $\frac{1}{|N|}$ to 1 is denoted by $M_0(R) = \{i \in N \mid \frac{1}{|N|} \ P_i \ 1\}$. Note that $M_1(R) \subseteq N_1(R)$ and $M_0(R) \subseteq N_0(R)$.

We characterize the Pareto optimal allocations that satisfy the equal division lower bound.

Theorem 3

Let $R \in \mathcal{R}^N$ be a preference profile and let $x \in P(R)$ be a Pareto optimal allocation. Then x satisfies the equal division lower bound, i.e. $x \in B_{ed}(R)$, if and only if the following statements hold:

- *If $N_1(R) \neq \emptyset$, then*
  - *$x_i \geq \frac{1}{|N|}$ for each $i \in M_1(R)$;*
  - *$x_i = 0$ or $x_i \ P_i \ 0$ for each $i \in N \setminus M_1(R)$.*
- *If $N_1(R) = \emptyset$ and $N_{0,1}(R) \neq \emptyset$, then $x_i = 1$ for some $i \in N_{0,1}(R)$.*
- *If $N_0(R) = N$ and $M_0(R) \neq N$, then $x_i = 1$ for some $i \in N \setminus M_0(R)$.*
- *If $M_0(R) = N$, then $x_i = \frac{1}{|N|}$ for each $i \in N$.*

Proof. Clearly, if the statements hold, then $x \in B_{ed}(R)$. We prove the only-if part. By Theorem 1, it suffices to show that if $x \in B_{ed}(R)$, then the following statements hold:

(i) If $N_1(R) \neq \emptyset$, then $x_i \geq \frac{1}{|N|}$ for each $i \in M_1(R)$.

(ii) If $N_0(R) = N$ and $M_0(R) \neq N$, then $x_i = 1$ for some $i \in N \setminus M_0(R)$.

(iii) If $M_0(R) = N$, then $x_i = \frac{1}{|N|}$ for each $i \in N$.

(i) Assume that $N_1(R) \neq \emptyset$. For the sake of contraposition, assume that there exists $i \in M_1(R)$ such that $x_i < \frac{1}{|N|}$. Then $\frac{1}{|N|} \ P_i \ 0$ implies that $\frac{1}{|N|} \ P_i \ x_i$. Hence, $x \notin B_{ed}(R)$.

(ii) Assume that $N_0(R) = N$ and $M_0(R) \neq N$. Let $x \in B_{ed}(R)$. For each $i \in M_0(R)$, $\frac{1}{|N|} \ P_i \ 1$ implies that $x_i \leq \frac{1}{|N|}$. For each $i \in N \setminus M_0(R)$, $0 \ P_i \ 1 \ R_i \ \frac{1}{|N|}$ implies that $x_i = 1$ or $x_i < \frac{1}{|N|}$. Hence, $x_i = 1$ for some $i \in N \setminus M_0(R)$.

(iii) Assume that $M_0(R) = N$. Let $x \in B_{ed}(R)$. For each $i \in N$, $\frac{1}{|N|} \ P_i \ 1$ implies that $x_i \leq \frac{1}{|N|}$. Hence, $x_i = \frac{1}{|N|}$ for each $i \in N$. □

Theorem 3 implies that Pareto optimal allocations satisfying the equal division lower bound always exist.

Corollary 3

Let $R \in \mathcal{R}^N$ be a preference profile. Then a Pareto optimal allocation satisfying the equal division lower bound exists, i.e. $P(R) \cap B_{ed}(R) \neq \emptyset$.

A stronger requirement than the equal division lower bound is the equal division core. An allocation belongs to the equal division core if for each group of agents, no redistribution of equal shares is weakly preferred by each group member and strictly preferred by some group member. The equal division core can be understood as a group version of the equal division lower bound. Formally, for preference profile $R \in \mathcal{R}^N$, an allocation $x \in X$ belongs to the **equal division core** if for each $S \subseteq N$ there does not exist $y \in \mathbb{R}^S_+$ with $\sum_{i \in S} y_i = \frac{|S|}{|N|}$ such that $y_i \; R_i \; x_i$ for each $i \in S$ and $y_i \; P_i \; x_i$ for some $i \in S$. The equal division core is denoted by $C_{ed}(R)$. Note that $C_{ed}(R) \subseteq P(R) \cap B_{ed}(R)$.

The following example shows that equal division core allocations do not necessarily satisfy equal treatment of equals.

Example 3

Let $R \in \mathcal{R}^N$ be a preference profile with $R_i = R_j$ for all $i, j \in N$ such that $0 \; P_i \; 1 \; P_i \; \frac{1}{|N|}$ for each $i \in N$. For each $x \in X$ with $x_i = 1$ for some $i \in N$, it holds that $x \in C_{ed}(R) \setminus E(R)$. $\triangle$

We characterize the equal division core allocations.

Theorem 4

Let $R \in \mathcal{R}^N$ be a preference profile and let $x \in P(R)$ be a Pareto optimal allocation. Then x belongs to the equal division core, i.e. $x \in C_{ed}(R)$, if and only if the following statements hold:

- *If $M_1(R) = N$, then $x_i = \frac{1}{|N|}$ for each $i \in N$.*
- *If $M_1(R) \neq N$ and $N_1(R) \neq \emptyset$, then there exists $i \in N_1(R)$ such that $x_i = 1$ and $0 \; R_j \; \frac{|N|-1}{|N|}$ for each $j \in N \setminus \{i\}$.*
- *If $N_1(R) = \emptyset$ and $N_{0,1}(R) \neq \emptyset$, then $x_i = 1$ for some $i \in N_{0,1}(R)$.*
- *If $N_0(R) = N$ and $M_0(R) \neq N$, then $x_i = 1$ for some $i \in N \setminus M_0(R)$.*
- *If $M_0(R) = N$, then $x_i = \frac{1}{|N|}$ for each $i \in N$.*

Proof. Clearly, if the statements hold, then $x \in C_{ed}(R)$. We prove the only-if part. Let $x \in C_{ed}(R)$. By Theorem 3, it suffices to show that if $M_1(R) \neq N$ and $N_1(R) \neq \emptyset$, then there exists $i \in N_1(R)$ such that $x_i = 1$ and $0 \; R_j \; \frac{|N|-1}{|N|}$ for each $j \in N \setminus \{i\}$. Assume that $M_1(R) \neq N$ and $N_1(R) \neq \emptyset$. Define $N^+ = \{i \in N \mid x_i > 0\}$. By Theorem 3, $M_1(R) \subseteq N^+$ and $x_i \; P_i \; 0$ for each $i \in N^+$.

Moreover, it holds that

$$\begin{aligned}\frac{|N|-|N^+|+1}{|N|} &= \frac{|N|-|N^+|+1}{|N|} - \frac{1}{|N^+|} + \frac{1}{|N^+|}\\ &= \frac{|N||N^+|-|N^+||N^+|+|N^+|-|N|}{|N||N^+|} + \frac{1}{|N^+|}\\ &= \frac{(|N|-|N^+|)(|N^+|-1)}{|N||N^+|} + \frac{1}{|N^+|}\\ &\geq \frac{1}{|N^+|},\end{aligned}$$

with strict inequality if and only if $1 < |N^+| < |N|$.

For the sake of contradiction, suppose that $x_i \neq \frac{1}{|N^+|}$ for some $i \in N^+$. Then there exists $i \in N^+$ such that $x_i < \frac{1}{|N^+|}$. Define $y \in \mathbb{R}_+^{N\setminus(N^+\setminus\{i\})}$ by $y_i = \frac{|N\setminus(N^+\setminus\{i\})|}{|N|}$ and $y_j = 0$ for each $j \in N \setminus N^+$. Then $x_i \; P_i \; 0$ and $y_i = \frac{|N\setminus(N^+\setminus\{i\})|}{|N|} \geq \frac{1}{|N^+|} > x_i$ imply that $y_i \; R_i \; \frac{1}{|N^+|} \; P_i \; x_i$. Moreover, $y_j \; R_j \; x_j$ for each $j \in N \setminus N^+$. This contradicts that $x \in C_{ed}(R)$. Hence, $x_i = \frac{1}{|N^+|}$ for each $i \in N^+$.

For the sake of contradiction, suppose that $|N^+| > 1$. If $N^+ = N$, then $x_i = \frac{1}{|N|}$ and $x_i \; P_i \; 0$ contradict that $0 \; R_i \; \frac{1}{|N|}$ for each $i \in N_1(R) \setminus M_1(R)$. Hence, $1 < |N^+| < |N|$. Let $i \in N^+$ and define $y \in \mathbb{R}_+^{N\setminus(N^+\setminus\{i\})}$ by $y_i = \frac{|N\setminus(N^+\setminus\{i\})|}{|N|}$ and $y_j = 0$ for each $j \in N \setminus N^+$. Then $x_i \; P_i \; 0$ and $y_i = \frac{|N\setminus(N^+\setminus\{i\})|}{|N|} > \frac{1}{|N^+|} = x_i$ imply that $y_i \; P_i \; x_i$. Moreover, $y_j \; R_j \; x_j$ for each $j \in N \setminus N^+$. This contradicts that $x \in C_{ed}(R)$. Hence, $|N^+| = 1$.

For the sake of contradiction, suppose that there exists $i \in N \setminus N^+$ such that $\frac{|N|-1}{|N|} \; P_i \; 0$. Define $y \in \mathbb{R}_+^{N\setminus N^+}$ by $y_i = \frac{|N|-1}{|N|}$ and $y_j = 0$ for each $j \in N \setminus (N^+ \cup \{i\})$. Then $\frac{|N|-1}{|N|} \; P_i \; 0$ and $x_i = 0$ imply that $y_i \; P_i \; x_i$. Moreover, $y_j \; R_j \; x_j$ for each $j \in N \setminus (N^+ \cup \{i\})$. This contradicts that $x \in C_{ed}(R)$. Hence, $0 \; R_i \; \frac{|N|-1}{|N|}$ for each $i \in N \setminus N^+$. □

Theorem 4 provides a necessary and sufficient condition for nonemptiness of the equal division core.

Corollary 4

Let $R \in \mathcal{R}^N$ be a preference profile. Then an equal division core allocation exists, i.e. $C_{ed}(R) \neq \emptyset$, if and only if one of the following statements hold:

- $M_1(R) = N$.
- *There exists $i \in N_1(R)$ such that $0 \; R_j \; \frac{|N|-1}{|N|}$ for each $j \in N \setminus \{i\}$.*
- $N_1(R) = \emptyset$.

5 Envy-freeness

This section studies Pareto optimal allocations that are envy-free, i.e. allocations for which each agent weakly prefers its own share to the share of any other agent. Formally, for preference profile $R \in \mathcal{R}^N$, an allocation $x \in X$ is **envy-free** if $x_i \ R_i \ x_j$ for all $i, j \in N$. The set of envy-free allocations is denoted by $F(R)$. We show that each envy-free allocation satisfies both equal treatment of equals and the equal division lower bound.

Lemma 1
Let $R \in \mathcal{R}^N$ be a preference profile. If an allocation is envy-free, then it satisfies equal treatment of equals and the equal division lower bound, i.e. $F(R) \subseteq E(R) \cap B_{ed}(R)$.

Proof. For the sake of contraposition, let $x \in X \setminus (E(R) \cap B_{ed}(R))$. If $x \notin E(R)$, then there exist $i, j \in N$ with $R_i = R_j$ such that $x_j \ P_i \ x_i$, which implies that $x \notin F(R)$. Suppose that $x \notin B_{ed}(R)$. Then there exists $i \in N$ such that $\frac{1}{|N|} \ P_i \ x_i$. Assume without loss of generality that $x_i < \frac{1}{|N|}$. Then there exists $j \in N$ such that $x_j > \frac{1}{|N|}$. This implies that $x_j \ P_i \ \frac{1}{|N|} \ P_i \ x_i$. Hence, $x \notin F(R)$. □

By Example 1, Pareto optimal and envy-free allocations do not necessarily exist. By Example 3, allocations that belong to the equal division core are not necessarily envy-free. For two agents, Lemma 1 implies that each Pareto optimal and envy-free allocation belongs to the equal division core. However, the following example shows that this implication does not hold for more than two agents.

Example 4
Let $N = \{1, \ldots, |N|\}$ with $|N| > 2$ and let $R \in \mathcal{R}^N$ be a preference profile such that $N_1(R) = M_1(R) = \{1, 2\}$. Then $P(R) \cap F(R) = \{(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)\}$ and $C_{ed}(R) = \emptyset$. △

Following Gong et al. (2022), for an agent $i \in N$ and a preference profile $R \in \mathcal{R}^N$, the **sharing index** (with respect to 0) $s_i(R) \in \{0, 1, \ldots, |N|\}$ is the maximum size of the group of agents with whom equal sharing is preferred to 0, or zero otherwise, i.e.

$$s_i(R) = \begin{cases} \max\{k \in \{1, \ldots, |N|\} \mid \frac{1}{k} \ P_i \ 0\} & \text{if } i \in N_1(R); \\ 0 & \text{if } i \notin N_1(R). \end{cases}$$

Note that $s_i(R) = |N|$ if and only if $i \in M_1(R)$.

We characterize the Pareto optimal and envy-free allocations.

Theorem 5
Let $R \in \mathcal{R}^N$ be a preference profile and let $x \in P(R)$ be a Pareto optimal allocation. Then x is envy-free, i.e. $x \in F(R)$, if and only if the following statements hold:

- *If* $N_1(R) \neq \emptyset$*, then there exists* $S \subseteq N$ *such that*

  - $x_i = \frac{1}{|S|}$ *and* $s_i(R) \geq |S|$ *for each* $i \in S$*;*
  - $x_i = 0$ *and* $s_i(R) < |S|$ *for each* $i \in N \setminus S$*.*

- *If* $N_1(R) = \emptyset$ *and* $N_{0,1}(R) \neq \emptyset$*, then* $x_i = 1$ *for some* $i \in N_{0,1}(R)$*.*

- *If* $N_0(R) = N$*, then* $M_0(R) = N$ *and* $x_i = \frac{1}{|N|}$ *for each* $i \in N$*.*

Proof. Clearly, if the statements hold, then $x \in F(R)$. We prove the only-if part. By Lemma 1 and Theorem 3, it suffices to show that if $x \in F(R)$, then the following statements hold:

(i) If $N_1(R) \neq \emptyset$, then there exists $S \subseteq N$ such that $x_i = \frac{1}{|S|}$ and $s_i(R) \geq |S|$ for each $i \in S$, and $x_i = 0$ and $s_i(R) < |S|$ for each $i \in N \setminus S$.

(ii) If $N_0(R) = N$, then $M_0(R) = N$.

(i) Assume that $N_1(R) \neq \emptyset$. Let $x \in F(R)$. Define $N^+ = \{i \in N \mid x_i > 0\}$. By Theorem 3, $M_1(R) \subseteq N^+$ and $x_i \ P_i \ 0$ for each $i \in N^+$. If there exist $i, j \in N^+$ such that $x_i < x_j$, then $x_i \ P_i \ 0$ implies that $x_j \ P_i \ x_i$, which contradicts that $x \in F(R)$, so $x_i = \frac{1}{|N^+|}$ for each $i \in N^+$. Then $x_i \ P_i \ 0$ implies that $\frac{1}{|N^+|} \ P_i \ 0$ for each $i \in N^+$, and $x \in F(R)$ implies that $0 \ R_i \ \frac{1}{|N^+|}$ for each $i \in N \setminus N^+$. This implies that $s_i(R) \geq |N^+|$ for each $i \in N^+$ and $s_i(R) < |N^+|$ for each $i \in N \setminus N^+$. Hence, there exists $S \subseteq N$ such that $x_i = \frac{1}{|S|}$ and $s_i(R) \geq |S|$ for each $i \in S$, and $x_i = 0$ and $s_i(R) < |S|$ for each $i \in N \setminus S$.

(ii) Assume that $N_0(R) = N$. For the sake of contraposition, assume that $M_0(R) \neq N$. By Theorem 3, there exists $i \in N \setminus M_0(R)$ such that $x_i = 1$ and $x_j = 0$ for each $j \in N \setminus \{i\}$. Then $i \in N_0(R)$ implies that $x_j \ P_i \ x_i$ for each $j \in N \setminus \{i\}$. Hence, $x \notin F(R)$. □

Theorem 5 provides a necessary and sufficient condition for compatibility of Pareto optimality with envy-freeness.

Corollary 5
Let $R \in \mathcal{R}^N$ *be a preference profile. Then a Pareto optimal and envy-free allocation exists, i.e.* $P(R) \cap F(R) \neq \emptyset$*, if and only if one of the following statements hold:*

- *There exists* $S \subseteq N$ *such that*

  - $s_i(R) \geq |S|$ *for each* $i \in S$*;*
  - $s_i(R) < |S|$ *for each* $i \in N \setminus S$*.*

- $N_1(R) = \emptyset$ *and* $N_{0,1}(R) \neq \emptyset$*.*

- $M_0(R) = N$*.*

A stronger requirement than envy-freeness is group envy-freeness. An allocation is group envy-free if for each group of agents, no redistribution of the share of any group of agents with the same size is weakly preferred by each group member and strictly preferred by some group member. Formally, for preference profile $R \in \mathcal{R}^N$, an allocation $x \in X$ is **group envy-free** if for each $S \subseteq N$ there does not exist $y \in \mathbb{R}^S_+$ with $\sum_{i \in S} y_i = \sum_{i \in S'} x_i$ for some $S' \subseteq N$ with $|S'| = |S|$ such that $y_i \; R_i \; x_i$ for each $i \in S$ and $y_i \; P_i \; x_i$ for some $i \in S$. The set of group envy-free allocations is denoted by $G(R)$. Note that $G(R) \subseteq P(R) \cap F(R)$. By Lemma 1, this implies that each group envy-free allocation satisfies both equal treatment of equals and the equal division lower bound. We show that each group envy-free allocation even belongs to the equal division core.

Lemma 2

Let $R \in \mathcal{R}^N$ be a preference profile. If an allocation is group envy-free, then it belongs to the equal division core, i.e. $G(R) \subseteq C_{ed}(R)$.

Proof. For the sake of contraposition, let $x \in X \setminus C_{ed}(R)$. Then there exist $S \subseteq N$ and $y \in \mathbb{R}^S_+$ with $\sum_{i \in S} y_i = \frac{|S|}{|N|}$ such that $y_i \; R_i \; x_i$ for each $i \in S$ and $y_i \; P_i \; x_i$ for some $i \in S$. If $\sum_{i \in S'} x_i = \frac{|S|}{|N|}$ for some $S' \subseteq N$ with $|S'| = |S|$, then $x \notin G(R)$. Suppose that $\sum_{i \in S'} x_i \neq \frac{|S|}{|N|}$ for each $S' \subseteq N$ with $|S'| = |S|$. Assume without loss of generality that $\sum_{i \in S} x_i < \frac{|S|}{|N|}$. Then there exists $S' \subseteq N$ with $|S'| = |S|$ such that $\sum_{i \in S'} x_i > \frac{|S|}{|N|}$. Define $z \in \mathbb{R}^S_+$ with $\sum_{i \in S} z_i = \sum_{i \in S'} x_i$ such that $z_i = y_i$ for each $i \in S$ with $y_i \leq x_i$, and $z_i > y_i$ for each $i \in S$ with $y_i > x_i$. Then $z_i \; R_i \; y_i \; R_i \; x_i$ for each $i \in S$ with $y_i \leq x_i$, and $z_i \; P_i \; y_i \; R_i \; x_i$ for each $i \in S$ with $y_i > x_i$. Hence, $x \notin G(R)$. □

We characterize the Pareto optimal and group envy-free allocations.

Theorem 6

Let $R \in \mathcal{R}^N$ be a preference profile and let $x \in P(R)$ be a Pareto optimal allocation. Then x is group envy-free, i.e. $x \in G(R)$, if and only if the following statements hold:

- *If $M_1(R) = N$, then $x_i = \frac{1}{|N|}$ for each $i \in N$.*
- *If $M_1(R) \neq N$ and $N_1(R) \neq \emptyset$, then $x_i = 1$ or $i \notin N_1(R)$ for each $i \in N$.*
- *If $N_1(R) = \emptyset$ and $N_{0,1}(R) \neq \emptyset$, then $x_i = 1$ for some $i \in N_{0,1}(R)$.*
- *If $N_0(R) = N$, then $M_0(R) = N$ and $x_i = \frac{1}{|N|}$ for each $i \in N$.*

Proof. Clearly, if the statements hold, then $x \in G(R)$. We prove the only-if part. Let $x \in G(R)$. By Lemma 2, Theorem 4, and Theorem 5, it suffices to show that if $M_1(R) \neq N$ and $N_1(R) \neq \emptyset$, then $x_i = 1$ or $i \notin N_1(R)$ for each $i \in N$. Assume that $M_1(R) \neq N$ and $N_1(R) \neq \emptyset$. By Theorem 4, there exists $i \in N_1(R)$ such that $x_i = 1$. By Theorem 5, this implies that $s_j(R) = 0$ for each $j \in N \setminus \{i\}$, so $j \notin N_1(R)$ for each $j \in N \setminus \{i\}$. Hence, $x_i = 1$ or $i \notin N_1(R)$ for each $i \in N$. □

Theorem 6 provides a necessary and sufficient condition for existence of group envy-free allocations.

Corollary 6

Let $R \in \mathcal{R}^N$ be a preference profile. Then a group envy-free allocation exists, i.e. $G(R) \neq \emptyset$, if and only if one of the following statements hold:

- $M_1(R) = N$.
- $|N_1(R)| = 1$.
- $N_1(R) = \emptyset$ *and* $N_{0,1}(R) \neq \emptyset$.
- $M_0(R) = N$.

6 Concluding remarks

This paper studied the compatibility of Pareto optimality with various fairness requirements in the context of allocation problems where agents have single-dipped preferences. These fairness requirements included equal treatment of equals, the equal division lower bound, the equal division core, envy-freeness, and group envy-freeness. All logical relations between them are summarized in Figure 1.

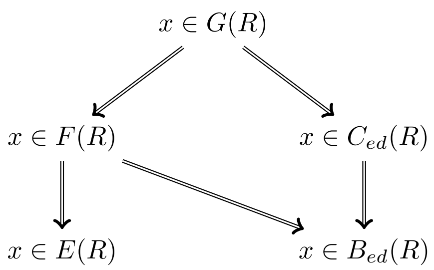

Figure 1: Let $R \in \mathcal{R}^N$ be a preference profile and let $x \in X$ be an allocation. All relations between equal treatment of equals (E), the equal division lower bound (B_{ed}), the equal division core (C_{ed}), envy-freeness (F), and group envy-freeness (G) are presented.

For each of these fairness requirements, we provided a necessary and sufficient condition for compatibility with Pareto optimality and we characterized all corresponding allocations for each preference profile. Among them, only the equal division lower bound is compatible with Pareto optimality for each preference profile.

We conclude this paper by introducing two procedures for selecting fair and Pareto optimal allocations for each preference profile.

The first procedure selects a Pareto optimal allocation that is group envy-free whenever possible, is envy-free whenever possible, and satisfies the equal division lower bound. For preference profile $R \in \mathcal{R}^N$, select $x \in P(R)$ in the following way:

- If $N_1(R) \neq \emptyset$, then for some $S \subseteq N$,
  - $x_i = \frac{1}{|S|}$ and $s_i(R) \geq |S|$ for each $i \in S$;
  - $x_i = 0$ and $s_i(R) \leq |S|$ for each $i \in N \setminus S$.[2]
- If $N_1(R) = \emptyset$ and $N_{0,1}(R) \neq \emptyset$, then $x_i = 1$ for some $i \in N_{0,1}(R)$.
- If $N_0(R) = N$ and $M_0(R) \neq N$, then $x_i = 1$ for some $i \in N \setminus M_0(R)$.
- If $M_0(R) = N$, then $x_i = \frac{1}{|N|}$ for each $i \in N$.

Then $x \in G(R)$ if $G(R) \neq \emptyset$, $x \in F(R)$ if $P(R) \cap F(R) \neq \emptyset$, and $x \in B_{ed}(R)$.

The second procedure selects a Pareto optimal allocation that is group envy-free whenever possible, belongs to the equal division core whenever possible, and satisfies the equal division lower bound. For preference profile $R \in \mathcal{R}^N$, select $x \in P(R)$ in the following way:

- If $M_1(R) \neq \emptyset$, then $x_i = \frac{1}{|M_1(R)|}$ for each $i \in M_1(R)$, and $x_i = 0$ for each $i \in N \setminus M_1(R)$.
- If $M_1(R) = \emptyset$ and $N_1(R) \neq \emptyset$, then $x_i = 1$ for some $i \in \text{argmin}\{k_j^1(R) \mid j \in N\}$.
- If $N_1(R) = \emptyset$ and $N_{0,1}(R) \neq \emptyset$, then $x_i = 1$ for some $i \in N_{0,1}(R)$.
- If $N_0(R) = N$ and $M_0(R) \neq N$, then $x_i = 1$ for some $i \in N \setminus M_0(R)$.
- If $M_0(R) = N$, then $x_i = \frac{1}{|N|}$ for each $i \in N$.

Then $x \in G(R)$ if $G(R) \neq \emptyset$, $x \in C_{ed}(R)$ if $C_{ed}(R) \neq \emptyset$, and $x \in B_{ed}(R)$.

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[2]Such a coalition can be constructed by selecting agents in an order of non-increasing sharing indices.

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