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Abstract

This paper introduces the new class of one-bound core games, where the core can be described by either a lower bound or an upper bound on the payoffs of the players, named lower bound core games and upper bound core games, respectively. We study the relation of the class of one-bound core games with several other classes of games and characterize the new class by the structure of the core and in terms of Davis-Maschler reduced games. We also provide explicit expressions and axiomatic characterizations of the nucleolus for one-bound core games, and show that the nucleolus coincides with the Shapley value when these games are convex.

Keywords: one-bound core games, lower bound core games, upper bound core games, core, nucleolus
JEL classification: C71

1 Introduction

In a cooperative game with transferable utility, coalitions of cooperating players are able to attain joint revenues. A characteristic function models this feature by assigning to each possible coalition a real number, called worth, reflecting these joint revenues. Depending on the structure of the characteristic function, different classes of games arise. For a class of games, a main issue in each game is how to allocate the worth of the grand coalition, consisting of all players in the game. Solution concepts assign to each game in a certain class such an allocation. A central benchmark for evaluating solutions is the core, which equals the set of allocations that for each coalition assign in total at least the worth to its members. The nucleolus (cf. Schmeidler 1969) is a particular solution that assigns to each game with a nonempty core the unique core allocation that lexicographically minimizes the maximal excesses over all coalitions.

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In this paper, we introduce a new class of cooperative games, called one-bound core games, where the core can be described by either a lower bound or an upper bound on the payoffs of the players. A game is a lower bound core game if the core can be described by a lower bound on the payoffs of the players, and an upper bound core game if the core can be described by an upper bound on the payoffs of the players. Both lower bound core games and upper bound core games are specific two-bound core games (cf. Gong et al. 2022b), where the core can be described by a lower bound and an upper bound on the payoffs of the players. Moreover, upper bound core games generalize 1-convex games (cf. Driessen 1985), where the worth of the grand coalition is large enough for each nonempty coalition to cover its worth while allocating to all non-members their marginal contributions to the grand coalition. We provide a necessary and sufficient condition for one-bound core games to be convex (cf. Shapley 1971).

We characterize one-bound core games by the structure of the core. A game with nonempty core is a lower bound core game if and only if each player obtains its maximal payoff within the core exactly when the other players obtain their minimal payoffs within the core, or equivalently, in each extreme point of the core precisely one player obtains its maximal payoff within the core and all other players obtain their minimal payoffs within the core. Similarly, a game with nonempty core is an upper bound core game if and only if each player obtains its minimal payoff within the core exactly when the other players obtain their maximal payoffs within the core, or equivalently, in each extreme point of the core precisely one player obtains its minimal payoff within the core and all other players obtain their maximal payoffs within the core. We show that a game with nonempty core is a lower bound core game if and only if all Davis-Maschler reduced games with respect to core allocations have the same lower exact core bound. Similarly, a game with nonempty core is an upper bound core game if and only if all Davis-Maschler reduced games with respect to core allocations have the same upper exact core bound.

We also study the nucleolus for one-bound core games. We show that it is the unique pre-imputation that is a convex combination of the two exact core bounds. We provide axiomatic characterizations based on new properties that require that the difference between the allocation and the minimal payoff or maximal payoff within the core is equal for all players. For convex one-bound core games, the nucleolus coincides with the Shapley value (cf. Shapley 1953), another well-known solution for cooperative games.

The remainder of this paper is organized as follows. Section 2 provides preliminary definitions and notation for cooperative games. Section 3 introduces and studies one-bound core games. Section 4 analyzes the nucleolus for one-bound core games. Section 5 concludes.

2 Preliminaries

Let N be a nonempty and finite set of *players* and let $2^N = \{S \mid S \subseteq N\}$ be the set of all *coalitions*. For all $x \in \mathbb{R}^N$, we denote $x_S = (x_i)_{i \in S}$ for all $S \in 2^N \setminus \{\emptyset\}$. For all $x, y \in \mathbb{R}^N$, we denote $x \leq y$ if $x_i \leq y_i$ for all $i \in N$, $x \geq y$ if $x_i \geq y_i$ for all $i \in N$, and $x + y = (x_i + y_i)_{i \in N}$.

A *(transferable utility) game* is a pair (N, v), where $v : 2^N \to \mathbb{R}$ assigns to each coalition $S \in 2^N$ its *worth* $v(S) \in \mathbb{R}$ such that $v(\emptyset) = 0$. The class of all games with player set N is denoted by Γ^N. For simplicity, we write $v \in \Gamma^N$ rather than $(N, v) \in \Gamma^N$.

For each game $v \in \Gamma^N$, the set of *pre-imputations* $X(v) \subseteq \mathbb{R}^N$ is given by

$$X(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N) \right\},$$

and the *core* $C(v) \subseteq \mathbb{R}^N$ is given by

$$C(v) = \left\{ x \in X(v) \;\middle|\; \forall S \in 2^N : \sum_{i \in S} x_i \geq v(S) \right\}.$$

A game $v \in \Gamma^N$ is *balanced* (cf. Bondareva 1963 and Shapley 1967) if and only if $C(v) \neq \emptyset$. The class of all balanced games with player set N is denoted by Γ_b^N.

For each game $v \in \Gamma_b^N$, the *lower exact core bound* $l^*(v) \in \mathbb{R}^N$ (cf. Bondareva and Driessen 1994) is given by

$$l_i^*(v) = \min_{x \in C(v)} x_i \quad \text{for all } i \in N,$$

and the *upper exact core bound* $u^*(v) \in \mathbb{R}^N$ (cf. Bondareva and Driessen 1994) is given by

$$u_i^*(v) = \max_{x \in C(v)} x_i \quad \text{for all } i \in N.$$

A game $v \in \Gamma_b^N$ is a *two-bound core game* (cf. Gong et al. 2022b) if there exist $l, u \in \mathbb{R}^N$ such that $C(v) = \{x \in X(v) \mid l \leq x \leq u\}$. Gong et al. (2022b) showed that a game $v \in \Gamma_b^N$ is a two-bound core game if and only if

$$C(v) = \{x \in X(v) \mid l^*(v) \leq x \leq u^*(v)\}.$$

The class of all two-bound core games with player set N is denoted by Γ_t^N. Gong et al. (2022b) showed that $\Gamma_t^N = \Gamma_b^N$ if and only if $|N| \leq 3$.

A game $v \in \Gamma_b^N$ is a *1-convex game* (cf. Driessen 1985) if

$$v(S) + \sum_{i \in N \setminus S} (v(N) - v(N \setminus \{i\})) \leq v(N) \quad \text{for all } S \in 2^N \setminus \{\emptyset\}.$$

The class of all 1-convex games with player set N is denoted by Γ_{1c}^N.

A game $v \in \Gamma^N$ is *convex* (cf. Shapley 1971) if and only if

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \quad \text{for all } i \in N \text{ and all } S \subseteq T \subseteq N \setminus \{i\}.$$

The class of all convex games with player set N is denoted by Γ_c^N. Shapley (1971) showed that $\Gamma_c^N \subseteq \Gamma_b^N$, and $\Gamma_c^N = \Gamma_b^N$ if and only if $|N| \leq 2$. Moreover, for each $v \in \Gamma_c^N$, $l_i^*(v) = v(\{i\})$ and $u_i^*(v) = v(N) - v(N \setminus \{i\})$ for all $i \in N$.

A *solution* φ on a domain of games assigns to each game v in this domain an allocation $\varphi(v) \in X(v)$. The *nucleolus* η (cf. Schmeidler 1969) is the solution that assigns to each game $v \in \Gamma_b^N$ the allocation $x \in X(v)$ that lexicographically minimizes the maximal excesses $v(S) - \sum_{i \in S} x_i$ over all $S \in 2^N \setminus \{\emptyset\}$. Clearly, $\eta(v) \in C(v)$ for all $v \in \Gamma_b^N$. Gong et al. (2022b) showed that the nucleolus of a two-bound core game $v \in \Gamma_t^N$ is for each $i \in N$ given by

$$\eta_i(v) = \begin{cases} l_i^*(v) + \min\left\{\frac{1}{2}(u_i^*(v) - l_i^*(v)), \lambda\right\} & \text{if } \frac{1}{2}\sum_{i \in N}(u_i^*(v) + l_i^*(v)) \geq v(N); \\ l_i^*(v) + \max\left\{\frac{1}{2}(u_i^*(v) - l_i^*(v)), u_i^*(v) - l_i^*(v) - \lambda\right\} & \text{if } \frac{1}{2}\sum_{i \in N}(u_i^*(v) + l_i^*(v)) \leq v(N), \end{cases}$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} \eta_i(v) = v(N)$. The *Shapley value* ϕ (cf. Shapley 1953) is the solution that assigns to each game $v \in \Gamma^N$ the allocation given by

$$\phi_i(v) = \sum_{S \in 2^N : i \notin S} \frac{|S|!(|N| - |S| - 1)!}{|N|!} (v(S \cup \{i\}) - v(S)) \quad \text{for all } i \in N.$$

Shapley (1971) showed that $\phi(v) \in C(v)$ for all $v \in \Gamma_c^N$.

3 One-bound core games

In this section, we introduce the new class of one-bound core games, where the core can be described by either a lower bound or an upper bound on the payoffs of the players. A game is a lower bound core game if the core can be described by a lower bound on the payoffs of the players, and an upper bound core game if the core can be described by an upper bound on the payoffs of the players.

Definition 1
A game $v \in \Gamma_b^N$ is a *lower bound core game* if there exists $l \in \mathbb{R}^N$ such that

$$C(v) = \{x \in X(v) \mid l \leq x\}.$$

A game $v \in \Gamma_b^N$ is an *upper bound core game* if there exists $u \in \mathbb{R}^N$ such that

$$C(v) = \{x \in X(v) \mid x \leq u\}.$$

A game is a *one-bound core game* if it is a lower bound core game or an upper bound core game.

The class of all lower bound core games with player set N is denoted by Γ_l^N. The class of all upper bound core games with player set N is denoted by Γ_u^N. It turns out that each lower bound core game and each upper bound core game can be described by the lower exact core bound and the upper exact core bound, respectively.

Lemma 1

(i) A game $v \in \Gamma_b^N$ is a lower bound core game if and only if $C(v) = \{x \in X(v) \mid l^*(v) \leq x\}$.

(ii) A game $v \in \Gamma_b^N$ is an upper bound core game if and only if $C(v) = \{x \in X(v) \mid x \leq u^*(v)\}$.

Proof. (i) The if-part follows directly from the definition of lower bound core games. For the only-if part, assume that $C(v) = \{x \in X(v) \mid l \leq x\}$ for some $l \in \mathbb{R}^N$. Then $l \leq l^*(v)$, which implies that

$$C(v) \subseteq \{x \in X(v) \mid l^*(v) \leq x\} \subseteq \{x \in X(v) \mid l \leq x\} = C(v).$$

Hence, $C(v) = \{x \in X(v) \mid l^*(v) \leq x\}$.

(ii) The proof is analogous to the proof of (i). □

All lower bound core games with at most two players are upper bound core games, and all upper bound core games with at most two players are lower bound core games, but this does not hold for more players.

Theorem 1
$\Gamma_l^N = \Gamma_u^N$ if and only if $|N| \leq 2$.

Proof. Assume that $|N| \leq 2$. Let $v \in \Gamma_b^N$. Then $l_i^*(v) = v(\{i\})$ and $u_i^*(v) = v(N) - v(N \setminus \{i\})$ for all $i \in N$, which implies that $C(v) = \{x \in X(v) \mid l^*(v) \leq x\} = \{x \in X(v) \mid x \leq u^*(v)\}$, so $v \in \Gamma_l^N \cap \Gamma_u^N$. Hence, $\Gamma_l^N = \Gamma_u^N = \Gamma_b^N$.

Let $v \in \Gamma_b^N$ with $|N| \geq 3$ be defined by

$$v(S) = \begin{cases} 1 & \text{if } S = N; \\ 0 & \text{otherwise.} \end{cases}$$

Then $l_i^*(v) = 0$ and $u_i^*(v) = 1$ for all $i \in N$. This implies that $C(v) = \{x \in X(v) \mid l^*(v) \leq x\}$ and $C(v) \neq \{x \in X(v) \mid x \leq u^*(v)\}$. Hence, $v \in \Gamma_l^N \setminus \Gamma_u^N$.

Let $v \in \Gamma_b^N$ with $|N| \geq 3$ be defined by

$$v(S) = \begin{cases} |N| - 1 & \text{if } S = N; \\ |N| - 2 & \text{if } |S| = |N| - 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then $l_i^*(v) = 0$ and $u_i^*(v) = 1$ for all $i \in N$. This implies that $C(v) \neq \{x \in X(v) \mid l^*(v) \leq x\}$ and $C(v) = \{x \in X(v) \mid x \leq u^*(v)\}$. Hence, $v \in \Gamma_u^N \setminus \Gamma_l^N$. □

As the following theorem shows, all 1-convex games are upper bound core games, and all one-bound core games are two-bound core games. Strict inclusion depends on the number of players in the game.

Theorem 2

(i) If $|N| = 1$*, then* $\Gamma^N_{1c} = \Gamma^N_l = \Gamma^N_u = \Gamma^N_t = \Gamma^N_b = \Gamma^N$.

(ii) If $|N| = 2$*, then* $\Gamma^N_{1c} = \Gamma^N_l = \Gamma^N_u = \Gamma^N_t = \Gamma^N_b \subsetneq \Gamma^N$.

(iii) If $|N| = 3$*, then* $\Gamma^N_l \subsetneq \Gamma^N_t = \Gamma^N_b \subsetneq \Gamma^N$ *and* $\Gamma^N_{1c} \subsetneq \Gamma^N_u \subsetneq \Gamma^N_t = \Gamma^N_b \subsetneq \Gamma^N$.

(iv) If $|N| \geq 4$*, then* $\Gamma^N_l \subsetneq \Gamma^N_t \subsetneq \Gamma^N_b \subsetneq \Gamma^N$ *and* $\Gamma^N_{1c} \subsetneq \Gamma^N_u \subsetneq \Gamma^N_t \subsetneq \Gamma^N_b \subsetneq \Gamma^N$.

Proof. First, we show that $\Gamma^N_l \subseteq \Gamma^N_t \subseteq \Gamma^N_b \subseteq \Gamma^N$ and $\Gamma^N_{1c} \subseteq \Gamma^N_u \subseteq \Gamma^N_t \subseteq \Gamma^N_b \subseteq \Gamma^N$.

Let $v \in \Gamma^N_l$. Then $C(v) \subseteq \{x \in X(v) \mid l^*(v) \leq x \leq u^*(v)\} \subseteq \{x \in X(v) \mid l^*(v) \leq x\} = C(v)$, so $C(v) = \{x \in X(v) \mid l^*(v) \leq x \leq u^*(v)\}$. Hence, $v \in \Gamma^N_t$.

Let $v \in \Gamma^N_{1c}$. If $x \in X(v)$ and $x \leq u^*(v)$, then for each $S \in 2^N \setminus \{\emptyset\}$,

$$\sum_{i \in S} x_i = \sum_{i \in N} x_i - \sum_{i \in N \setminus S} x_i \geq v(N) - \sum_{i \in N \setminus S} u^*_i(v) \geq v(N) - \sum_{i \in N \setminus S} (v(N) - v(N \setminus \{i\})) \geq v(S),$$

so $x \in C(v)$. This implies that $C(v) = \{x \in X(v) \mid x \leq u^*(v)\}$. Hence, $v \in \Gamma^N_u$.

Let $v \in \Gamma^N_u$. Then $C(v) \subseteq \{x \in X(v) \mid l^*(v) \leq x \leq u^*(v)\} \subseteq \{x \in X(v) \mid x \leq u^*(v)\} = C(v)$, so $C(v) = \{x \in X(v) \mid l^*(v) \leq x \leq u^*(v)\}$. Hence, $v \in \Gamma^N_t$.

(i) & *(ii)* Assume that $|N| \in \{1, 2\}$. Clearly, $\Gamma^N_b = \Gamma^N$ if and only if $|N| = 1$. Let $v \in \Gamma^N_b$. Then for each $S \in 2^N \setminus \{\emptyset\}$,

$$v(S) + \sum_{i \in N \setminus S} (v(N) - v(N \setminus \{i\})) = v(N).$$

This implies that $v \in \Gamma^N_{1c}$. Hence, $\Gamma^N_{1c} = \Gamma^N_l = \Gamma^N_u = \Gamma^N_t = \Gamma^N_b$.

(iii) & *(iv)* Assume that $|N| \geq 3$. Let $v \in \Gamma^N_u$ be defined by

$$v(S) = \begin{cases} |S| & \text{if } |S| \in \{1, |N|\}; \\ 0 & \text{otherwise.} \end{cases}$$

Then $v(N) - v(N \setminus \{i\}) = |N|$ for all $i \in N$. This implies that for each $S \in 2^N$ with $|S| = 1$,

$$v(S) + \sum_{i \in N \setminus S} (v(N) - v(N \setminus \{i\})) = 1 + (|N| - 1)\,|N| > |N| = v(N),$$

so $v \notin \Gamma^N_{1c}$. Hence, $\Gamma^N_{1c} \subsetneq \Gamma^N_u$.

Let $v \in \Gamma_t^N$ be defined by

$$v(S) = \begin{cases} 3 & \text{if } S = N; \\ 1 & \text{if } |S| = |N| - 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then $l_i^*(v) = 0$ and $u_i^*(v) = 2$ for all $i \in N$. This implies that $C(v) \neq \{x \in X(v) \mid l^*(v) \leq x\}$ and $C(v) \neq \{x \in X(v) \mid x \leq u^*(v)\}$, so $v \notin \Gamma_l^N \cup \Gamma_u^N$. Hence, $\Gamma_l^N \subsetneq \Gamma_t^N$ and $\Gamma_u^N \subsetneq \Gamma_t^N$. □

As the following example shows, one-bound core games are not necessarily convex, and convex games are not necessarily one-bound core games.

Example 1
Let $v \in \Gamma_l^N \cup \Gamma_u^N$ with $|N| \geq 3$ be defined by

$$v(S) = \begin{cases} |S| & \text{if } |S| \in \{1, |N|\}; \\ 0 & \text{otherwise.} \end{cases}$$

Then $v(\{i\}) - v(\emptyset) = 1 > -1 = v(\{i,j\}) - v(\{j\})$ for all distinct $i, j \in N$, so $v \notin \Gamma_c^N$.

Now, let $v \in \Gamma_c^N$ with $|N| \geq 3$ be defined by

$$v(S) = \begin{cases} 3 & \text{if } S = N; \\ 1 & \text{if } |S| = |N| - 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then $l_i^*(v) = 0$ and $u_i^*(v) = 2$ for all $i \in N$, which implies that $C(v) \neq \{x \in X(v) \mid l^*(v) \leq x\}$ and $C(v) \neq \{x \in X(v) \mid x \leq u^*(v)\}$, so $v \notin \Gamma_l^N \cup \Gamma_u^N$. △

We provide a necessary and sufficient condition for one-bound core games to be convex.

Theorem 3

(i) A lower bound core game $v \in \Gamma_l^N$ is convex if and only if

$$\sum_{i \in S} l_i^*(v) = v(S) \quad \textit{for all } S \in 2^N \setminus \{N\}.$$

(ii) An upper bound core game $v \in \Gamma_u^N$ is convex if and only if

$$\sum_{i \in N \setminus S} u_i^*(v) = v(N) - v(S) \quad \textit{for all } S \in 2^N \setminus \{\emptyset\}.$$

Proof. (*i*) Let $v \in \Gamma_l^N$. Assume that $\sum_{i \in S} l_i^*(v) = v(S)$ for all $S \in 2^N \setminus \{N\}$. Let $i \in N$ and let $S \subseteq N \setminus \{i\}$. If $S = N \setminus \{i\}$, then

$$v(S \cup \{i\}) - v(S) = v(N) - v(N \setminus \{i\}) = v(N) - \sum_{j \in N \setminus \{i\}} l_j^*(v) \geq \sum_{j \in N} l_j^*(v) - \sum_{j \in N \setminus \{i\}} l_j^*(v) = l_i^*(v).$$

If $S \neq N \setminus \{i\}$, then $v(S \cup \{i\}) - v(S) = \sum_{j \in S \cup \{i\}} l_j^*(v) - \sum_{j \in S} l_j^*(v) = l_i^*(v)$. This implies that $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $S \subseteq T \subseteq N \setminus \{i\}$. Hence, $v \in \Gamma_c^N$.

Let $v \in \Gamma_l^N$. Then $(v(N) - \sum_{j \in N \setminus \{i\}} l_j^*(v), l_{N \setminus \{i\}}^*(v)) \in C(v)$ for all $i \in N$, which implies that $\sum_{i \in S} l_i^*(v) \geq v(S)$ for all $S \in 2^N \setminus \{N\}$. Assume that $v \in \Gamma_c^N$. Let $S \in 2^N \setminus \{\emptyset, N\}$. Denote $S = \{i_1, \ldots, i_{|S|}\}$. Then

$$\sum_{i \in S} l_i^*(v) = \sum_{i \in S} v(\{i\}) = \sum_{k=1}^{|S|} (v(\{i_k\}) - v(\emptyset)) \leq \sum_{k=1}^{|S|} (v(\{i_1, \ldots, i_k\}) - v(\{i_1, \ldots, i_{k-1}\})) = v(S),$$

where the first equality and the inequality follow from convexity. Hence, $\sum_{i \in S} l_i^*(v) = v(S)$ for all $S \in 2^N \setminus \{N\}$.

(ii) Let $v \in \Gamma_u^N$. Assume that $\sum_{i \in N \setminus S} u_i^*(v) = v(N) - v(S)$ for all $S \in 2^N \setminus \{\emptyset\}$. Let $i \in N$ and let $S \subseteq N \setminus \{i\}$. If $S = \emptyset$, then

$$v(S \cup \{i\}) - v(S) = v(\{i\}) = v(N) - \sum_{j \in N \setminus \{i\}} u_j^*(v) \leq \sum_{j \in N} u_j^*(v) - \sum_{j \in N \setminus \{i\}} u_j^*(v) = u_i^*(v).$$

If $S \neq \emptyset$, then

$$v(S \cup \{i\}) - v(S) = \left(v(N) - \sum_{j \in N \setminus (S \cup \{i\})} u_j^*(v) \right) - \left(v(N) - \sum_{j \in N \setminus S} u_j^*(v) \right) = u_i^*(v).$$

This implies that $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $S \subseteq T \subseteq N \setminus \{i\}$. Hence, $v \in \Gamma_c^N$.

Let $v \in \Gamma_u^N$. Then $(v(N) - \sum_{j \in N \setminus \{i\}} u_j^*(v), u_{N \setminus \{i\}}^*(v)) \in C(v)$ for all $i \in N$, which implies that $\sum_{i \in N \setminus S} u_i^*(v) \leq v(N) - v(S)$ for all $S \in 2^N \setminus \{\emptyset\}$. Assume that $v \in \Gamma_c^N$. Let $S \in 2^N \setminus \{\emptyset, N\}$. Denote $N \setminus S = \{i_1, \ldots, i_{|N \setminus S|}\}$. Then

$$\begin{aligned}
\sum_{i \in N \setminus S} u_i^*(v) &= \sum_{i \in N \setminus S} (v(N) - v(N \setminus \{i\})) \\
&= \sum_{k=1}^{|N \setminus S|} (v(N) - v(N \setminus \{i_k\})) \\
&= \sum_{k=1}^{|N \setminus S|} (v((N \setminus \{i_k\}) \cup \{i_k\}) - v(N \setminus \{i_k\})) \\
&\geq \sum_{k=1}^{|N \setminus S|} (v((N \setminus \{i_1, \ldots, i_k\}) \cup \{i_k\}) - v(N \setminus \{i_1, \ldots, i_k\})) \\
&= v(N) - v(N \setminus (N \setminus S)) \\
&= v(N) - v(S),
\end{aligned}$$

where the first equality and the inequality follow from convexity. Hence, $\sum_{i \in N \setminus S} u_i^*(v) = v(N) - v(S)$ for all $S \in 2^N \setminus \{\emptyset\}$. □

One-bound core games are characterized by the structure of the core. A balanced game is a lower bound core game if and only if each player obtains its maximal payoff within the core exactly when the other players obtain their minimal payoffs within the core, or equivalently, in each extreme point of the core precisely one player obtains its maximal payoff within the core and all other players obtain their minimal payoffs within the core. Similarly, a balanced game is an upper bound core game if and only if each player obtains its minimal payoff within the core exactly when the other players obtain their maximal payoffs within the core, or equivalently, in each extreme point of the core precisely one player obtains its minimal payoff within the core and all other players obtain their maximal payoffs within the core. These observations are captured by the following theorem.

Theorem 4

(i) A game $v \in \Gamma_b^N$ is a lower bound core game if and only if $u_i^*(v) + \sum_{j \in N \setminus \{i\}} l_j^*(v) = v(N)$ for all $i \in N$, or equivalently, $C(v) = conv\{(u_i^*(v), l_{N \setminus \{i\}}^*(v)) \mid i \in N\}$.[1]

(ii) A game $v \in \Gamma_b^N$ is an upper bound core game if and only if $l_i^*(v) + \sum_{j \in N \setminus \{i\}} u_j^*(v) = v(N)$ for all $i \in N$, or equivalently, $C(v) = conv\{(l_i^*(v), u_{N \setminus \{i\}}^*(v)) \mid i \in N\}$.

Proof. (i) Assume that $v \in \Gamma_l^N$. Let $i \in N$. For each $x \in C(v)$ with $x_i = u_i^*(v)$,

$$u_i^*(v) = x_i = v(N) - \sum_{j \in N \setminus \{i\}} x_j \leq v(N) - \sum_{j \in N \setminus \{i\}} l_j^*(v).$$

For $x \in \mathbb{R}^N$ with $x_i = v(N) - \sum_{j \in N \setminus \{i\}} l_j^*(v)$ and $x_j = l_j^*(v)$ for all $j \in N \setminus \{i\}$, we have $x \in X(v)$ and $l^*(v) \leq x$, which implies that $x \in C(v)$, so $u_i^*(v) \geq x_i = v(N) - \sum_{j \in N \setminus \{i\}} l_j^*(v)$. Hence, $u_i^*(v) = v(N) - \sum_{j \in N \setminus \{i\}} l_j^*(v)$ for all $i \in N$.

For each $i \in N$ and each $x \in C(v)$ such that $x_i = u_i^*(v)$, we have $x_j = l_j^*(v)$ for all $j \in N \setminus \{i\}$, which implies that $(u_i^*(v), l_{N \setminus \{i\}}^*(v)) \in C(v)$. Convexity of the core implies that $conv\{(u_i^*(v), l_{N \setminus \{i\}}^*(v)) \mid i \in N\} \subseteq C(v)$. Let $x \in C(v)$. Define $\lambda \in [0,1]^N$ by

$$\lambda_i = \frac{x_i - l_i^*(v)}{v(N) - \sum_{j \in N} l_j^*(v)} \quad \text{for all } i \in N.$$

Then $\sum_{i \in N} \lambda_i = 1$ and $x = \sum_{i \in N} \lambda_i (u_i^*(v), l_{N \setminus \{i\}}^*(v))$, so $x \in conv\{(u_i^*(v), l_{N \setminus \{i\}}^*(v)) \mid i \in N\}$. Hence, $C(v) = conv\{(u_i^*(v), l_{N \setminus \{i\}}^*(v)) \mid i \in N\}$.

Let $v \in \Gamma_b^N$. Assume that $C(v) = conv\{(u_i^*(v), l_{N \setminus \{i\}}^*(v)) \mid i \in N\}$. Then $\sum_{i \in S} l_i^*(v) \geq v(S)$ for all $S \in 2^N \setminus \{N\}$. Let $x \in X(v)$ be such that $l^*(v) \leq x$. For each $S \in 2^N \setminus \{N\}$,

$$\sum_{i \in S} x_i \geq \sum_{i \in S} l_i^*(v) \geq v(S),$$

which implies that $x \in C(v)$, so $C(v) = \{x \in X(v) \mid l^*(v) \leq x\}$. Hence, $v \in \Gamma_l^N$.

[1] The convex hull $conv(Y)$ of a set $Y \subseteq \mathbb{R}^N$ is the smallest convex set containing Y.

(*ii*) Assume that $v \in \Gamma_u^N$. Let $i \in N$. For each $x \in C(v)$ with $x = l_i^*(v)$,

$$l_i^*(v) = x_i = v(N) - \sum_{j \in N \setminus \{i\}} x_j \geq v(N) - \sum_{j \in N \setminus \{i\}} u_j^*(v).$$

For $x \in \mathbb{R}^N$ with $x_i = v(N) - \sum_{j \in N \setminus \{i\}} u_j^*(v)$ and $x_j = u_j^*(v)$ for all $j \in N \setminus \{i\}$, we have $x \in X(v)$ and $x \leq u^*(v)$, which implies that $x \in C(v)$, so $l_i^*(v) \leq x_i = v(N) - \sum_{j \in N \setminus \{i\}} u_j^*(v)$. Hence, $l_i^*(v) = v(N) - \sum_{j \in N \setminus \{i\}} u_j^*(v)$ for all $i \in N$.

For each $i \in N$ and each $x \in C(v)$ such that $x_i = l_i^*(v)$, we have $x_j = u_j^*(v)$ for all $j \in N \setminus \{i\}$, which implies that $(l_i^*(v), u_{N \setminus \{i\}}^*(v)) \in C(v)$. Convexity of the core implies that $conv\{(l_i^*(v), u_{N \setminus \{i\}}^*(v)) \mid i \in N\} \subseteq C(v)$. Let $x \in C(v)$. Define $\lambda \in [0,1]^N$ by

$$\lambda_i = \frac{u_i^*(v) - x_i}{\sum_{j \in N} u_j^*(v) - v(N)} \quad \text{for all } i \in N.$$

Then $\sum_{i \in N} \lambda_i = 1$ and $x = \sum_{i \in N} \lambda_i (l_i^*(v), u_{N \setminus \{i\}}^*(v))$, so $x \in conv\{(l_i^*(v), u_{N \setminus \{i\}}^*(v)) \mid i \in N\}$. Hence, $C(v) = conv\{(l_i^*(v), u_{N \setminus \{i\}}^*(v)) \mid i \in N\}$.

Let $v \in \Gamma_b^N$. Assume that $C(v) = conv\{(l_i^*(v), u_{N \setminus \{i\}}^*(v)) \mid i \in N\}$. Then $v(N) - \sum_{i \in N \setminus S} u_i^*(v) \geq v(S)$ for all $S \in 2^N \setminus \{\emptyset\}$. Let $x \in X(v)$ be such that $x \leq u^*(v)$. For each $S \in 2^N \setminus \{\emptyset\}$,

$$\sum_{i \in S} x_i = \sum_{i \in N} x_i - \sum_{i \in N \setminus S} x_i = v(N) - \sum_{i \in N \setminus S} x_i \geq v(N) - \sum_{i \in N \setminus S} u_i^*(v) \geq v(S),$$

which implies that $x \in C(v)$, so $C(v) = \{x \in X(v) \mid x \leq u^*(v)\}$. Hence, $v \in \Gamma_u^N$. □

Corollary 1
Let $v \in \Gamma_l^N \cup \Gamma_u^N$. Then $|C(v)| = 1$ if and only if $l_i^*(v) = u_i^*(v)$ for some $i \in N$.

The *reduced game* (cf. Davis and Maschler 1965) of $v \in \Gamma_b^N$ on $T \in 2^N \setminus \{\emptyset\}$ with respect to $x \in \mathbb{R}^N$, denoted by $v_T^x \in \Gamma^T$, is defined by

$$v_T^x(S) = \begin{cases} v(N) - \sum_{i \in N \setminus T} x_i & \text{if } S = T; \\ \max_{Q \subseteq N \setminus T} \left\{ v(S \cup Q) - \sum_{i \in Q} x_i \right\} & \text{if } S \in 2^T \setminus \{\emptyset, T\}; \\ 0 & \text{if } S = \emptyset. \end{cases}$$

In other words, the worth of a coalition in a reduced game is defined as the maximal remainder in cooperation with any subgroup of players in the original game that are not present in the reduced game. We show that a balanced game is a lower bound core game if and only if all reduced games with respect to core allocations have the same lower exact core bound. Similarly, a balanced game is an upper bound core game if and only if all reduced games with respect to core allocations have the same upper exact core bound. We use the following lemma, which follows from Peleg (1986) and Hwang and Sudhölter (2001).

Lemma 2
Let $v \in \Gamma_b^N$, let $T \in 2^N \setminus \{\emptyset\}$, and let $x \in C(v)$. Then $C(v_T^x) = \{y \in X(v_T^x) \mid (y, x_{N\setminus T}) \in C(v)\}$.

Theorem 5

(i) A game $v \in \Gamma_b^N$ is a lower bound core game if and only if $l^*(v_T^x) = l_T^*(v)$ for all $T \in 2^N$ with $|T| \geq 2$ and all $x \in C(v)$.

(ii) A game $v \in \Gamma_b^N$ is an upper bound core game if and only if $u^*(v_T^x) = u_T^*(v)$ for all $T \in 2^N$ with $|T| \geq 2$ and all $x \in C(v)$.

Proof. (i) Let $v \in \Gamma_b^N$. For the only-if part, assume that $v \in \Gamma_l^N$. Let $T \in 2^N$ with $|T| \geq 2$ and let $x \in C(v)$. By Lemma 2, $l^*(v_T^x) \geq l_T^*(v)$. For each $i \in T$, define $y^i \in X(v_T^x)$ by $y_i^i = \sum_{j \in T} x_j - \sum_{j \in T\setminus\{i\}} l_j^*(v)$ and $y_j^i = l_j^*(v)$ for all $j \in T \setminus \{i\}$. For each $i \in T$,

$$y_i^i = \sum_{j \in T} x_j - \sum_{j \in T\setminus\{i\}} l_j^*(v) = x_i + \sum_{j \in T\setminus\{i\}} x_j - \sum_{j \in T\setminus\{i\}} l_j^*(v) \geq x_i \geq l_i^*(v),$$

which implies that $l^*(v) \leq (y^i, x_{N\setminus T})$, so $(y^i, x_{N\setminus T}) \in C(v)$. By Lemma 2, $y^i \in C(v_T^x)$ for all $i \in T$, so $l^*(v_T^x) \leq l_T^*(v)$. Hence, $l^*(v_T^x) = l_T^*(v)$.

For the if-part, assume that $l^*(v_T^x) = l_T^*(v)$ for all $T \in 2^N$ with $|T| \geq 2$ and all $x \in C(v)$. If $|N| \leq 2$, then $v \in \Gamma_l^N$ by Theorem 2. Suppose that $|N| \geq 3$. Denote $N = \{1, \ldots, |N|\}$. Let $x^1 \in C(v)$ be such that $x_1^1 = l_1^*(v)$. Then $l^*(v_{N\setminus\{1\}}^{x^1}) = l_{N\setminus\{1\}}^*(v)$. Let $x^2 \in C(v_{N\setminus\{1\}}^{x^1})$ be such that $x_2^2 = l_2^*(v_{N\setminus\{1\}}^{x^1}) = l_2^*(v)$. By Lemma 2, $(x^2, x_1^1) \in C(v)$. Moreover, $l^*(v_{N\setminus\{1,2\}}^{(x^2,x_1^1)}) = l_{N\setminus\{1,2\}}^*(v)$ if $|N| > 3$. If $|N| > 3$, let $x^3 \in C(v_{N\setminus\{1,2\}}^{(x^2,x_1^1)})$ be such that $x_3^3 = l_3^*(v_{N\setminus\{1,2\}}^{(x^2,x_1^1)}) = l_3^*(v)$. By Lemma 2, $(x^3, x_2^2, x_1^1) \in C(v)$. Continuing this reasoning, $(v(N) - \sum_{i=1}^{|N|-1} l_i^*(v), l_{\{1,\ldots,|N|-1\}}^*(v)) \in C(v)$. This holds for all permutations, so $(u_i^*(v), l_{N\setminus\{i\}}^*(v)) \in C(v)$ for all $i \in N$. Convexity of the core implies that $conv\{(u_i^*(v), l_{N\setminus\{i\}}^*(v)) \mid i \in N\} \subseteq C(v)$. Now, let $x \in C(v)$. Define $\lambda \in [0,1]^N$ by

$$\lambda_i = \frac{x_i - l_i^*(v)}{v(N) - \sum_{j \in N} l_j^*(v)} \quad \text{for all } i \in N.$$

Then $\sum_{i \in N} \lambda_i = 1$ and $x = \sum_{i \in N} \lambda_i (u_i^*(v), l_{N\setminus\{i\}}^*(v))$, so $x \in conv\{(u_i^*(v), l_{N\setminus\{i\}}^*(v)) \mid i \in N\}$. This implies that $C(v) = conv\{(u_i^*(v), l_{N\setminus\{i\}}^*(v)) \mid i \in N\}$. Hence, by Theorem 4, $v \in \Gamma_l^N$.

(ii) Let $v \in \Gamma_b^N$. For the only-if part, assume that $v \in \Gamma_u^N$. Let $T \in 2^N$ with $|T| \geq 2$ and let $x \in C(v)$. Then $u^*(v_T^x) \leq u_T^*(v)$. For each $i \in T$, define $y^i \in X(v_T^x)$ by $y_i^i = \sum_{j \in T} x_j - \sum_{j \in T\setminus\{i\}} u_j^*(v)$ and $y_j^i = u_j^*(v)$ for all $j \in T \setminus \{i\}$. For each $i \in T$,

$$y_i^i = \sum_{j \in T} x_j - \sum_{j \in T\setminus\{i\}} u_j^*(v) = x_i + \sum_{j \in T\setminus\{i\}} x_j - \sum_{j \in T\setminus\{i\}} u_j^*(v) \leq x_i \leq u_i^*(v),$$

which implies that $(y^i, x_{N\setminus T}) \leq u^*(v)$, so $(y^i, x_{N\setminus T}) \in C(v)$. By Lemma 2, $y^i \in C(v_T^x)$ for all $i \in T$, so $u^*(v_T^x) \geq u_T^*(v)$. Hence, $u^*(v_T^x) = u_T^*(v)$.

For the if-part, assume that $u^*(v_T^x) = u_T^*(v)$ for all $T \in 2^N$ with $|T| \geq 2$ and all $x \in C(v)$. If $|N| \leq 2$, then $v \in \Gamma_u^N$ by Theorem 2. Suppose that $|N| \geq 3$. Denote $N = \{1, \ldots, |N|\}$. Let $x^1 \in C(v)$ be such that $x_1^1 = u_1^*(v)$. Then $u^*(v_{N\setminus\{1\}}^{x^1}) = u_{N\setminus\{1\}}^*(v)$. Let $x^2 \in C(v_{N\setminus\{1\}}^{x^1})$ be such that $x_2^2 = u_2^*(v_{N\setminus\{1\}}^{x^1}) = u_2^*(v)$. By Lemma 2, $(x^2, x_1^1) \in C(v)$. Moreover, $u^*(v_{N\setminus\{1,2\}}^{(x^2,x_1^1)}) = u_{N\setminus\{1,2\}}^*(v)$ if $|N| > 3$. If $|N| > 3$, let $x^3 \in C(v_{N\setminus\{1,2\}}^{(x^2,x_1^1)})$ be such that $x_3^3 = u_3^*(v_{N\setminus\{1,2\}}^{(x^2,x_1^1)}) = u_3^*(v)$. By Lemma 2, $(x^3, x_2^2, x_1^1) \in C(v)$. Continuing this reasoning, $(v(N) - \sum_{i=1}^{|N|-1} u_i^*(v), u_{\{1,\ldots,|N|-1\}}^*(v)) \in C(v)$. This holds for all permutations, so $(l_i^*(v), u_{N\setminus\{i\}}^*(v)) \in C(v)$ for all $i \in N$. Convexity of the core implies that $conv\{(l_i^*(v), u_{N\setminus\{i\}}^*(v)) \mid i \in N\} \subseteq C(v)$. Now, let $x \in C(v)$. Define $\lambda \in [0,1]^N$ by

$$\lambda_i = \frac{u_i^*(v) - x_i}{\sum_{j\in N} u_j^*(v) - v(N)} \quad \text{for all } i \in N.$$

Then $\sum_{i\in N} \lambda_i = 1$ and $x = \sum_{i\in N} \lambda_i (l_i^*(v), u_{N\setminus\{i\}}^*(v))$, so $x \in conv\{(l_i^*(v), u_{N\setminus\{i\}}^*(v)) \mid i \in N\}$. This implies that $C(v) = conv\{(l_i^*(v), u_{N\setminus\{i\}}^*(v)) \mid i \in N\}$. Hence, by Theorem 4, $v \in \Gamma_u^N$. □

4 Nucleolus

In this section, we analyze the nucleolus for one-bound core games. The nucleolus for one-bound core games is the unique pre-imputation that is a convex combination of the lower exact core bound and the upper exact core bound.

Theorem 6

(i) Let $v \in \Gamma_l^N$ be a lower bound core game. Then

$$\eta(v) = \frac{1}{|N|} u^*(v) + \left(1 - \frac{1}{|N|}\right) l^*(v).$$

(ii) Let $v \in \Gamma_u^N$ be an upper bound core game. Then

$$\eta(v) = \frac{1}{|N|} l^*(v) + \left(1 - \frac{1}{|N|}\right) u^*(v).$$

Proof. (*i*) By Theorem 2, $v \in \Gamma_t^N$. By Theorem 4, $u_i^*(v) - l_i^*(v) = v(N) - \sum_{j\in N} l_j^*(v)$ for all $i \in N$. This implies that $\eta_i(v) - l_i^*(v) = \eta_j(v) - l_j^*(v)$ for all $i, j \in N$, so for each $i \in N$, we have

$$\begin{aligned}
\eta_i(v) &= l_i^*(v) + \frac{1}{|N|} \left(v(N) - \sum_{j\in N} l_j^*(v) \right) \\
&= l_i^*(v) + \frac{1}{|N|} \left(u_i^*(v) - l_i^*(v) \right) \\
&= \frac{1}{|N|} u_i^*(v) + \left(1 - \frac{1}{|N|}\right) l_i^*(v).
\end{aligned}$$

(*ii*) By Theorem 2, $v \in \Gamma_t^N$. By Theorem 4, $u_i^*(v) - l_i^*(v) = \sum_{j \in N} u_j^*(v) - v(N)$ for all $i \in N$. This implies that $\eta_i(v) - l_i^*(v) = \eta_j(v) - l_j^*(v)$ for all $i, j \in N$, so for each $i \in N$, we have

$$\begin{aligned}\eta_i(v) &= l_i^*(v) + \frac{1}{|N|}\left(v(N) - \sum_{j \in N} l_j^*(v)\right)\\ &= l_i^*(v) + \frac{1}{|N|}\left(l_i^*(v) + \sum_{j \in N \setminus \{i\}} u_j^*(v) - \sum_{j \in N} l_j^*(v)\right)\\ &= l_i^*(v) + \frac{1}{|N|}\sum_{j \in N \setminus \{i\}} \left(u_j^*(v) - l_j^*(v)\right)\\ &= l_i^*(v) + \frac{1}{|N|}\left(|N| - 1\right)\left(u_i^*(v) - l_i^*(v)\right)\\ &= \frac{1}{|N|} l_i^*(v) + \left(1 - \frac{1}{|N|}\right) u_i^*(v).\end{aligned}$$

□

Corollary 2
Let $v \in \Gamma_l^N \cup \Gamma_u^N$ *be a one-bound core game. Then*

$$\eta(v) = \lambda l^*(v) + (1 - \lambda)\, u^*(v),$$

where $\lambda \in [0, 1]$ *is such that* $\sum_{i \in N} \eta_i(v) = v(N)$.

The nucleolus for one-bound core games is characterized by the properties that require that the difference between the allocation and the minimal payoff or maximal payoff within the core is equal for all players. We refer to these properties as balanced lower gaps and balanced upper gaps, respectively.

Definition 2
A solution φ on a subdomain of balanced games satisfies *balanced lower gaps* if for each game v in this domain, it holds that $\varphi_i(v) - l_i^*(v) = \varphi_j(v) - l_j^*(v)$ for all $i, j \in N$.

A solution φ on a subdomain of balanced games satisfies *balanced upper gaps* if for each game v in this domain, it holds that $u_i^*(v) - \varphi_i(v) = u_j^*(v) - \varphi_j(v)$ for all $i, j \in N$.

Theorem 7

(i) The nucleolus is the unique solution for one-bound core games satisfying balanced lower gaps.[2]

(ii) The nucleolus is the unique solution for one-bound core games satisfying balanced upper gaps.[3]

Proof. (*i*) Let $v \in \Gamma_l^N \cup \Gamma_u^N$. Let $i, j \in N$. If $v \in \Gamma_l^N$, then Theorem 4 implies that

$$u_i^*(v) - l_i^*(v) = v(N) - \sum_{k \in N} l_k^*(v) = u_j^*(v) - l_j^*(v).$$

If $v \in \Gamma_u^N$, then Theorem 4 implies that

$$u_i^*(v) - l_i^*(v) = \sum_{k \in N} u_k^*(v) - v(N) = u_j^*(v) - l_j^*(v).$$

By Theorem 6,

$$\eta_i(v) - l_i^*(v) = \frac{1}{|N|}\left(u_i^*(v) - l_i^*(v)\right) = \frac{1}{|N|}\left(u_j^*(v) - l_j^*(v)\right) = \eta_j(v) - l_j^*(v).$$

Hence, the nucleolus satisfies balanced lower gaps.

Let φ be a solution on $\Gamma_l^N \cup \Gamma_u^N$ satisfying balanced lower gaps. Let $i \in N$. If $v \in \Gamma_l^N$, then Theorem 4 implies that $v(N) - u_i^*(v) = \sum_{j \in N \setminus \{i\}} l_j^*(v)$, so

$$\begin{aligned}
\varphi_i(v) &= u_i^*(v) + (v(N) - u_i^*(v)) + (\varphi_i(v) - v(N)) \\
&= u_i^*(v) + \sum_{j \in N \setminus \{i\}} l_j^*(v) - \sum_{j \in N \setminus \{i\}} \varphi_j(v) \\
&= u_i^*(v) - \sum_{j \in N \setminus \{i\}} \left(\varphi_j(v) - l_j^*(v)\right) \\
&= u_i^*(v) - \sum_{j \in N \setminus \{i\}} \left(\varphi_i(v) - l_i^*(v)\right) \\
&= u_i^*(v) - (|N| - 1)\left(\varphi_i(v) - l_i^*(v)\right) \\
&= u_i^*(v) + (|N| - 1)\, l_i^*(v) - (|N| - 1)\, \varphi_i(v),
\end{aligned}$$

where the fourth equality follows from balanced lower gaps, and rewriting yields

$$\varphi_i(v) = \frac{1}{|N|} u_i^*(v) + \left(1 - \frac{1}{|N|}\right) l_i^*(v).$$

[2] In fact, the nucleolus is the unique solution satisfying balanced lower gaps on each subdomain of one-bound core games.

[3] In fact, the nucleolus is the unique solution satisfying balanced upper gaps on each subdomain of one-bound core games.

If $v \in \Gamma_u^N$, then Theorem 4 implies that $l_j^*(v) = v(N) - \sum_{k \in N \setminus \{j\}} u_k^*(v)$ for all $j \in N$, so

$$\begin{aligned}
\varphi_i(v) &= l_i^*(v) + \frac{1}{|N|}|N| \left(\varphi_i(v) - l_i^*(v)\right) \\
&= l_i^*(v) + \frac{1}{|N|} \sum_{j \in N} \left(\varphi_j(v) - l_j^*(v)\right) \\
&= l_i^*(v) + \frac{1}{|N|} \left(\sum_{j \in N} \varphi_j(v) - \sum_{j \in N} l_j^*(v) \right) \\
&= l_i^*(v) + \frac{1}{|N|} \left(v(N) - \sum_{j \in N} \left(v(N) - \sum_{k \in N \setminus \{j\}} u_k^*(v) \right) \right) \\
&= l_i^*(v) + \frac{1}{|N|} \left(v(N) - |N| v(N) + \sum_{j \in N} \sum_{k \in N \setminus \{j\}} u_k^*(v) \right) \\
&= l_i^*(v) + \frac{1}{|N|} \left((1 - |N|) v(N) + (|N| - 1) \sum_{j \in N} u_j^*(v) \right) \\
&= l_i^*(v) + \frac{1}{|N|} (|N| - 1) \left(\sum_{j \in N} u_j^*(v) - v(N) \right) \\
&= l_i^*(v) + \frac{1}{|N|} (|N| - 1) \left(u_i^*(v) - l_i^*(v)\right) \\
&= \frac{1}{|N|} l_i^*(v) + \left(1 - \frac{1}{|N|}\right) u_i^*(v),
\end{aligned}$$

where the second equality follows from balanced lower gaps. Hence, by Theorem 6, $\varphi_i(v) = \eta_i(v)$.

(ii) The proof is analogous to the proof of *(i)*. □

The nucleolus assigns to each one-bound core game a specific core allocation. As the following example shows, the Shapley value does not assign to each one-bound core game a core allocation.

Example 2

Let $N = \{1, 2, 3\}$ and let $v \in \Gamma_l^N \cap \Gamma_u^N$ be defined by

$$v(S) = \begin{cases} 6 & \text{if } S \in \{\{1,2\}, \{1,3\}, N\}; \\ 0 & \text{otherwise.} \end{cases}$$

Then $l^*(v) = u^*(v) = (6, 0, 0)$, so $\eta(v) = (6, 0, 0)$ and $\eta(v) \in C(v)$. However, $\phi(v) = (4, 1, 1)$ and $\phi(v) \notin C(v)$. △

However, the Shapley value assigns to each convex one-bound core game a specific core allocation. In fact, as the following theorem states, the nucleolus and the Shapley value coincide for convex one-bound core games.

Theorem 8
Let $v \in \Gamma_l^N \cup \Gamma_u^N$ be a one-bound core game. If v is convex, then the nucleolus coincides with the Shapley value.

Proof. We only prove the case $v \in \Gamma_u^N$; the case $v \in \Gamma_l^N$ follows analogously. Assume that $v \in \Gamma_c^N$. Let $i \in N$. By Theorem 3, for each $S \in 2^N \setminus \{\emptyset\}$ with $i \notin S$,

$$v(S \cup \{i\}) - v(S) = \left(v(N) - \sum_{j \in N \setminus (S \cup \{i\})} u_j^*(v) \right) - \left(v(N) - \sum_{j \in N \setminus S} u_j^*(v) \right) = u_i^*(v).$$

This implies that

$$\begin{aligned}
\phi_i(v) &= \sum_{S \in 2^N : i \notin S} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \left(v(S \cup \{i\}) - v(S) \right) \\
&= \frac{1}{|N|} v(\{i\}) + \sum_{S \in 2^N \setminus \{\emptyset\} : i \notin S} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \left(v(S \cup \{i\}) - v(S) \right) \\
&= \frac{1}{|N|} l_i^*(v) + \sum_{S \in 2^N \setminus \{\emptyset\} : i \notin S} \frac{|S|!(|N| - |S| - 1)!}{|N|!} u_i^*(v) \\
&= \frac{1}{|N|} l_i^*(v) + \sum_{k=1}^{|N|-1} \binom{|N| - 1}{k} \frac{k!(|N| - k - 1)!}{|N|!} u_i^*(v) \\
&= \frac{1}{|N|} l_i^*(v) + \sum_{k=1}^{|N|-1} \frac{(|N| - 1)!}{k!(|N| - k - 1)!} \frac{k!(|N| - k - 1)!}{|N|!} u_i^*(v) \\
&= \frac{1}{|N|} l_i^*(v) + \sum_{k=1}^{|N|-1} \frac{1}{|N|} u_i^*(v) \\
&= \frac{1}{|N|} l_i^*(v) + \frac{|N| - 1}{|N|} u_i^*(v) \\
&= \frac{1}{|N|} l_i^*(v) + \left(1 - \frac{1}{|N|} \right) u_i^*(v).
\end{aligned}$$

Hence, by Theorem 6, $\varphi_i(v) = \eta_i(v)$. □

5 Concluding remarks

In this paper, the new class of one-bound core games is introduced. By Theorem 2, all one-bound core games are two-bound core games. By Lemma 2 and Theorem 5, all reduced games of one-bound core games with respect to core allocations are one-bound core games. This implies that the axiomatic characterizations of the core, the nucleolus, and the egalitarian core (cf. Arin and Iñarra 2001) provided by Gong et al. (2022a) on the class of two-bound core games can be reformulated on the class of one-bound core games. However, the exact core bounds of reduced two-bound core games are not necessarily the same. By Theorem 5, if all reduced games of a balanced game have the same exact core bound, then it is necessarily a one-bound core game.

References

Arin, J. and E. Iñarra (2001). Egalitarian solutions in the core. *International Journal of Game Theory*, **30**(2), 187–193.

Bondareva, O. (1963). Some applications of linear programming methods to the theory of cooperative games. *Problemy Kibernetiki*, **10**, 119–139.

Bondareva, O. and T. Driessen (1994). Extensive coverings and exact core bounds. *Games and Economic Behavior*, **6**(2), 212–219.

Davis, M. and M. Maschler (1965). The kernel of a cooperative game. *Naval Research Logistics Quarterly*, **12**(3), 223–259.

Driessen, T. (1985). Properties of 1-convex n-person games. *OR Spectrum*, **7**(1), 19–26.

Gong, D., B. Dietzenbacher, and H. Peters (2022a). Reduced two-bound core games. *Mathematical Methods of Operations Research*, **96**(3), 447–457.

Gong, D., B. Dietzenbacher, and H. Peters (2022b). Two-bound core games and the nucleolus. *Annals of Operations Research*, doi:10.1007/s10479–022–04949–0.

Hwang, Y. and P. Sudhölter (2001). Axiomatizations of the core on the universal domain and other natural domains. *International Journal of Game Theory*, **29**(4), 597–623.

Peleg, B. (1986). On the reduced game property and its converse. *International Journal of Game Theory*, **15**(3), 187–200.

Schmeidler, D. (1969). The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics*, **17**(6), 1163–1170.

Shapley, L. (1953). A value for n-person games. In: H. Kuhn and A. Tucker (Eds.), *Contributions to the Theory of Games II*, pp. 307–317. Princeton University Press.

Shapley, L. (1967). On balanced sets and cores. *Naval Research Logistics Quarterly*, **14**(4), 453–460.

Shapley, L. (1971). Cores of convex games. *International Journal of Game Theory*, **1**(1), 11–26.