Monte Carlo pricing in the Schobel-Zhu model and its extensions

Journal of Computational Finance **17**(3), 1–29

Monte Carlo pricing in the Schöbel–Zhu model and its extensions

Alexander van Haastrecht
VU University Amsterdam, Department of Finance, De Boelelaan 1105,
1081 HV Amsterdam, The Netherlands; email: a.van.haastrecht@vu.nl

Roger Lord
Cardano, Quantitative Analytics, Weena 690, 21st floor, 3012 CN Rotterdam,
The Netherlands; email: r.lord@cardano.com

Antoon Pelsser
Maastricht University, Department of Finance, Department of Quantitative
Economics, PO Box 616, 6200 MD Maastricht, The Netherlands;
email: a.pelsser@maastrichtuniversity.nl

(Received March 7, 2011; revised October 22, 2011; accepted January 6, 2012)

We propose a simulation algorithm for the Schöbel–Zhu model and its extension to include stochastic interest rates. Both schemes are derived by analyzing the lessons learned from the Broadie and Kaya scheme on how to avoid the so-called leaking correlation phenomenon in the simulation of the Heston model. All introduced schemes are exponentially affine in expectation, which greatly facilitates the derivation of a martingale correction. In addition we study the regularity of each scheme. The numerical results indicate that our scheme consistently outperforms the Euler scheme. For a special case of the Schöbel–Zhu model which coincides with the Heston model, our scheme performs similarly to the QE-M scheme of Andersen. The results reaffirm that when simulating stochastic volatility models it is of the utmost importance to match the correlation between the asset price and the stochastic volatility process.

1 INTRODUCTION

Stochastic volatility models have become the de facto standard to price and hedge complex financial products; in derivative models the behavior of financial derivatives

The authors would like to thank Jeroen Decuypere, Peter Jäckel, participants of the London Cass Business School seminar series, and an anonymous referee, for their valuable comments and suggestions.

is usually modeled by stochastic differential equations that jointly describe the movements of the underlying financial assets, such as stock prices, stock variances and interest rates. Though certain models yield closed-form solutions for some products, the vast majority cannot be priced in closed-form. Nonetheless, Monte Carlo methods provide a popular and flexible pricing alternative to value such exotic derivatives. Due to technical advances such as multi-processor programming, increasing computational power and variance reduction techniques, Monte Carlo techniques are expected to become even more widely applicable in the near future. Taking these advances into account, Monte Carlo techniques are still computationally relatively expensive, hence much attention focuses on efficient simulation schemes aiming to minimize computational effort while retaining a high degree of accuracy.

In the last decade the literature on efficient simulation schemes for stochastic volatility has evolved. Approaches to price derivatives with stochastic volatility models were described in Hull and White (1987), Stein and Stein (1991), Heston (1993) and the Schöbel and Zhu (1999) model. The latter two models stand out for allowing the stochastic volatility to be correlated with the underlying asset, while still allowing for closed-form formulas for most vanilla options used in the model's calibration. Discretization schemes for models have been described by several authors, for example Jäckel (2002), Glasserman (2003), Kahl and Jäckel (2006), Andersen (2008), Lord *et al* (2008), Smith (2008) and van Haastrecht and Pelsser (2010). Most of these papers focus on efficient discretization methods for the Heston (1993) model, paying particular attention to the discretization of the underlying square-root variance process. Andersen (2008) was the first to make the key observation that for any discretization scheme of the Heston (1993) model it is crucially important to match the correlation between the underlying and the variance process as close as possible. Simple Euler schemes which do not take this into account suffer from the so-called leaking correlation phenomenon.

In this paper simulation schemes are presented for the Schöbel and Zhu (1999) (SZ) stochastic volatility model and its extensions. Instead of only focusing on the simulation of the volatility process, which in the case of the SZ model is normally distributed and hence can easily be simulated exactly, like Andersen we also pay particular attention to the aforementioned "leaking correlation" issue. It appears that this issue is a general problem in the simulation of stochastic volatility models. As we aim for our analysis to be as broadly applicable as possible, we also consider an extension of the SZ model which incorporates stochastic interest rates: the Schöbel–Zhu–Hull–White (SZHW) model, as considered in van Haastrecht *et al* (2009). This extension combines the SZ model with the one-factor Gaussian interest rate model of Hull and White (1993), allowing for a general correlation structure between all processes. This is closely related to the recent advances in the development of a market for long-maturity European options in equity and exchange rate derivatives, showing liquid quotes for European options ranging up to fifteen years, for which maturities we feel a model including stochastic interest rates is more suitable. Finally, we note that the methods presented here also facilitate the pricing of interest rate derivatives in the context of stochastic volatility London Interbank Offered Rate (LIBOR) market models (see, for example, Zhu 2007).

The remainder of the paper is organized as follows. First, the SZ model is described in Section 2. Section 3 analyzes the problem of leaking correlations in the Schöbel and Zhu (1999) and Heston (1993) stochastic volatility models. In Section 4 discretization schemes are presented for the SZ model. These results are extended with stochastic interest rates in Section 5. In Section 6 numerical examples are worked out, showing the impact of leaking correlations in Monte Carlo methods for stochastic volatility models. Conclusions are given in Section 7.

2 THE SCHÖBEL–ZHU MODEL

The risk-neutral log-asset price dynamics of the Schöbel and Zhu (1999) model read

$$\mathrm{d}\ln x(t) = -\tfrac{1}{2}\nu^2(t)\,\mathrm{d}t + \nu(t)\,\mathrm{d}W_x(t), \qquad \ln x(0) = \ln(x_0), \tag{2.1}$$

$$\mathrm{d}\nu(t) = \kappa(\psi - \nu(t))\,\mathrm{d}t + \tau\,\mathrm{d}W_\nu(t), \qquad \nu(0) = \nu_0, \tag{2.2}$$

where κ, ψ, τ are positive parameters corresponding to the mean reversion, the long-term volatility and the volatility of the volatility process, and with $W_x(t)$ and $W_\nu(t)$ being two Brownian motions under some probability measure $\mathcal{Q}$ with linear correlation coefficient $\rho_{x\nu}$. The variance process is defined as $\nu^2(t)$, whose dynamics can be obtained from (2.2) using Ito's lemma and are given by

$$\mathrm{d}\nu^2(t) = 2\kappa\left(\frac{\tau^2}{2\kappa} + \psi\nu(t) - \nu^2(t)\right)\mathrm{d}t + 2\tau\sqrt{\nu^2(t)}\,\mathrm{d}W_\nu(t), \quad \nu^2(0) = \nu_0^2, \tag{2.3}$$

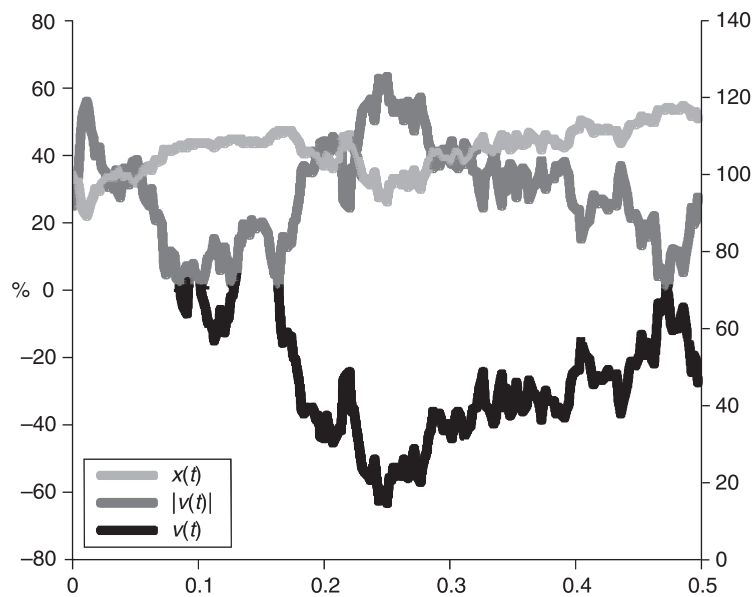
which will be used in Section 4 for the model's simulation.

Here, $x(t)$ represents the asset price process (eg, a stock, a foreign exchange rate or a LIBOR rate), and is assumed to be a martingale in the chosen probability measure. Since it is straightforward to allow for a deterministic drift, we omit this for ease of exposure. In Section 5 we elaborate more on the case of stochastic interest rates. At first sight, one curious property of the Schöbel and Zhu (1999) model is that the volatility process $\nu(t)$ affects the sign of the instantaneous correlation between $\nu(t)$ and $\ln x(t)$. Indeed, we can show that

$$\mathrm{Corr}(\mathrm{d}\ln x(t), \mathrm{d}\nu(t)) = \frac{\rho_{x\nu}\nu(t)\tau}{\sqrt{\nu^2(t)\tau^2}} = \rho_{x\nu}\,\mathrm{sgn}(\nu(t)). \tag{2.4}$$

This effect is visualized in Figure 1 on the next page, where we have plotted a sample path of $x(t)$, $\nu(t)$ and $|v(t)|$.

FIGURE 1 Sample path of $x(t)$, $\nu(t)$ and $|\nu(t)|$.



SZ parameters are $\kappa = \tau = 1$, $\nu(0) = \psi = 25\%$, $\rho_{x\nu} = -1$ and $x(0) = 100$.

When $\nu(t)$ is negative and decreasing, the asset price is increasing, contrary to what we would expect from the parameter configuration. The key lies therein that $\nu(t)$ should not be interpreted as the volatility of the underlying asset.[1] It is merely a latent variable which drives the true volatility of the asset, the true volatility being defined as the square root of the variance.

If one applies the Ito–Tanaka theorem (see, for example, Revuz and Yor 1999) to derive the dynamics of $|\nu(t)|$, we can indeed show that

$$\mathrm{Corr}(\mathrm{d}\ln x(t), \mathrm{d}|\nu(t)|) = \frac{\rho_{x\nu}|\nu(t)|\tau}{\sqrt{\nu^2(t)\tau^2}} = \rho_{x\nu}, \tag{2.5}$$

exactly as we would like it to be. We can intuitively also see this by considering the case where $\nu(t)$ is far away from the origin, and considering the case where $\nu(t) > 0$ and $\nu(t) < 0$. This will lead to the same result.

[1] It would be preferable to write "volatility", but we will refrain from doing this unless it causes confusion.

3 LEAKING CORRELATION IN STOCHASTIC VOLATILITY MODELS

One of the major problems Andersen (2008) signalled with Euler schemes of the Heston (1993) model (see Lord *et al* (2008) for an overview) is their inability to generate a correlation between the increments of the asset and the stochastic volatility processes which resembles that of the true process. As the correlation parameter in stochastic volatility models is an important determinant of the skew in implied volatilities, we can imagine that not being able to match this parameter can lead to a significant mispricing of options with strikes that are further away from the at-the-money level.

Such problems in the Heston model are partially caused by the fact that an Euler discretization tries to approximate a Feller process, which is guaranteed to be positive, by a Gaussian process. While stochastic volatility in the Schöbel and Zhu (1999) (SZ) model is itself Gaussian, we will show that "leaking correlation" as this phenomenon has been dubbed, is still an issue. Before we can design an effective simulation scheme for the SZ model and its extensions, we will take an excursion to the Heston model and pinpoint exactly why Andersen's simulation schemes are successful in reproducing the right correlation.

For completeness, we first state the dynamics of the Heston model. In this model, the dynamics of the log asset are described by the following set of SDEs:

$$\mathrm{d}\ln x(t) = -\tfrac{1}{2}\nu^2(t)\,\mathrm{d}t + \sqrt{\nu^2(t)}\,\mathrm{d}W_x(t), \qquad \ln x(0) = \ln(x_0), \quad (3.1)$$

$$\mathrm{d}\nu^2(t) = \kappa_H(\psi_H - \nu^2(t))\,\mathrm{d}t + \tau_H\sqrt{\nu^2(t)}\,\mathrm{d}W_\nu(t), \qquad \nu(0) = \nu_0, \quad (3.2)$$

where again $\nu^2(t)$ is the variance of the log-asset price.

In this section we will focus on a special case of the SZ model where the long-term level of mean reversion for the volatility $\nu(t)$, ψ, equals zero. This special case also happens to be a special case of the Heston model, which can be seen from the dynamics of $\nu^2(t)$:

$$\mathrm{d}\nu^2(t) = (\tau^2 - 2\kappa\nu^2(t))\,\mathrm{d}t + 2\tau\sqrt{\nu^2(t)}\,\mathrm{d}W_\nu(t). \quad (3.3)$$

In this case the Heston and SZ parameters are related as follows

$$\kappa_H \mapsto 2\kappa, \quad \psi_H \mapsto \frac{\tau^2}{2\kappa}, \quad \tau_H \mapsto 2\tau. \quad (3.4)$$

Recall that from (2.2) we can easily see that the volatility process follows a standard Gaussian distribution. When $\psi = 0$, we can write

$$\nu(t+\Delta) = K_1\nu(t) + K_2 Z_\nu, \quad (3.5)$$

with

$$K_1 = \mathrm{e}^{-\kappa\Delta} \quad \text{and} \quad K_2 = \tau\sqrt{\frac{1-\mathrm{e}^{-2\kappa\Delta}}{2\kappa}}. \quad (3.6)$$

Turning to the log-asset price, integrating the SDE in (2.1) and (3.1) yields

$$\ln x(t+\Delta) = \ln x(t) - \frac{1}{2}\int_t^{t+\Delta} \nu^2(u)\,\mathrm{d}u + \rho_{x\nu}\int_t^{t+\Delta} \nu(u)\,\mathrm{d}W_\nu(u) + \hat{\rho}_{x\nu}\int_t^{t+\Delta} \upsilon(u)\,\mathrm{d}\tilde{W}_x(u), \quad (3.7)$$

where W_ν and $\tilde{W}_x$ are independent Brownian motions and $\hat{\rho}_{x\nu} := \sqrt{1-\rho_{x\nu}^2}$. Using an Euler discretization, this would become

$$\ln x(t+\Delta) = \ln x(t) - \tfrac{1}{2}\nu^2(t)\Delta + \nu(t)\sqrt{\Delta}(\rho_{x\nu}Z_\nu + \sqrt{1-\rho_{x\nu}^2}Z_x), \quad (3.8)$$

with Z_ν, Z_x standard normally distributed random variables. Conditional upon $x(t)$ and $\nu(t)$, the correlation between $\ln x(t+\Delta)$ and $\nu(t+\Delta)$ equals

$$\begin{aligned}\mathrm{Corr}_t[\ln x(t+\Delta), \nu(t+\Delta)] &= \frac{\mathrm{Cov}_t[\ln x(t+\Delta), \nu(t+\Delta)]}{K_2|\nu(t)|\sqrt{\Delta}} \\ &= \frac{\mathrm{Cov}_t[\ln x(t+\Delta), Z_\nu]}{|\nu(t)|\sqrt{\Delta}} \\ &= \rho_{x\nu}\,\mathrm{sgn}(\nu(t)). \quad (3.9)\end{aligned}$$

So with a naive Euler discretization it seems there is no "leaking correlation", as reported for the Heston model in Andersen (2008), as this perfectly matches the instantaneous correlation between $\mathrm{d}\ln x(t)$ and $\mathrm{d}\nu(t)$ in (2.5). Let us turn to $\nu^2(t)$, however, which is a quadratic Gaussian. Note that the asset price dynamics, which are correlated with the underlying, are of quadratic order. That is, denoting $O(\nu^2)$ as the order in its mean square sense, we have that

$$\int_t^{t+\Delta} \nu(u)\,\mathrm{d}W_\nu(u) = \frac{\nu^2(t+\Delta) - \nu^2(t)}{2\tau} - \tfrac{1}{2}\tau\Delta + \frac{\kappa}{\tau}\int_t^{t+\Delta} \nu^2(u)\,\mathrm{d}u \sim O(\nu^2) \quad (3.10)$$

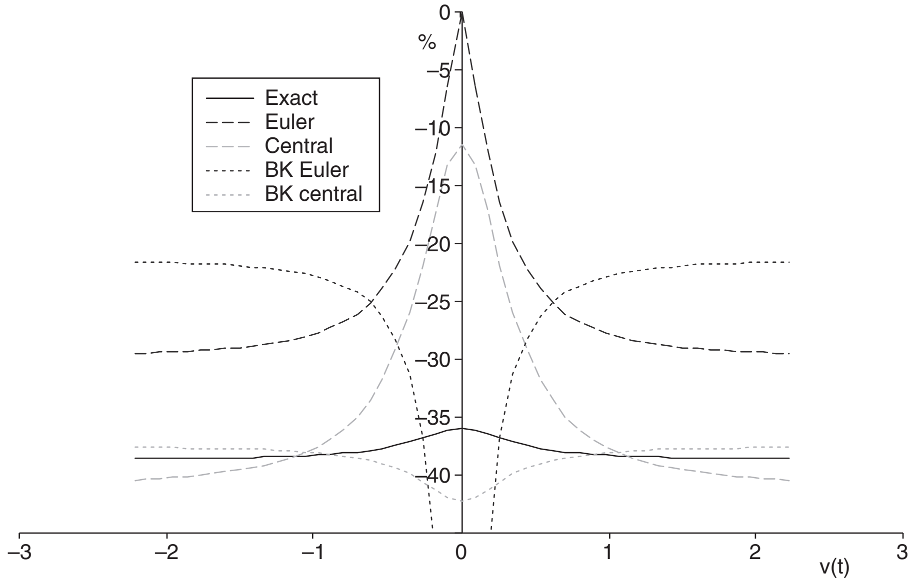
(see, for example, Andersen (2008) and (3.14)).

Hence, in simulating the asset price, it is of particular importance that the correlation between $\mathrm{d}\ln x(t)$ and $\mathrm{d}\nu^2(t)$ is preserved. After some calculations, we can show that in the Euler scheme

$$\mathrm{Corr}_t[\ln x(t+\Delta), \nu^2(t+\Delta)] = \rho_{x\nu}\frac{K_1|\nu(t)|}{\sqrt{K_1^2\nu^2(t) + \frac{1}{2}K_2^2}}, \quad (3.11)$$

which tends to $\rho_{x\nu}$ as $\nu(t)$ tends to plus or minus infinity, but can differ substantially when $\nu(t)$ is close to zero and even equals zero when $\nu(t)$ does. In this sense an Euler

FIGURE 2 Correlation between $\ln x(t+\Delta)$ and $\nu^2(t+\Delta)$ for various values of the volatility $\nu(t)$.



Here we have used the parameters $\kappa = \tau = 1$, $\rho = -0.3$ and $\Delta = \frac{1}{4}$. The central schemes uses $\delta_1 = \delta_2 = 0.5$ for the drift interpolations.

discretization in the SZ model also suffers from leaking correlation in the same way as in the Heston model. This behavior is visualized in Figure 2.

In the above schemes we simulate $\nu(t)$ exactly. All the discretizations therefore refer to how (3.1) is discretized. Euler therefore means an Euler discretization of (3.1), central implies we use (3.13) with $\delta_1 = \delta_2 = \frac{1}{2}$. BK Euler implies we use (3.15) for the log asset, with an Euler discretization of the integrated variance, and BK central also implies we use (3.15) for the log asset, but with a central discretization for (3.12).

Clearly the correlation from an Euler scheme is far from the exact correlation (see Andersen 2008, Appendix A). A typical range for the volatility, for the Heston parameter setting of the above table, is between and around the positive unit interval. For instance if $\nu(t) = 30\%$, we have that more than 99% of the probability mass of $\nu(t+\Delta)$ lies between -0.91 and 1.38. Note that this interval corresponds exactly to the region where the correlation of the Euler scheme is misaligned the most with the true correlation. The question is what the best way is to improve upon the Euler scheme. When we simulate $x(t+\Delta)$, we would already have computed $\nu(t+\Delta)$.

One possibility is therefore to approximate the integrated variance in (3.7) using a second-order approximation to compute

$$\int_t^{t+\Delta} \nu^2(u)\,\mathrm{d}u \approx \delta_1 \nu^2(t) + \delta_2 \nu^2(t+\Delta)\Delta, \tag{3.12}$$

leading to

$$\begin{aligned}\ln x(t+\Delta) = \ln x(t) &- \tfrac{1}{2}[\delta_1 \nu^2(t) + \delta_2 \nu^2(t+\Delta)]\Delta \\ &+ \rho_{x\nu}\nu(t)t(W_\nu(t+\Delta) - W_\nu(t)) \\ &+ \hat{\rho}_{x\nu}\sqrt{\delta_1 \nu^2(t) + \delta_2 \nu^2(t+\Delta)}(\tilde{W}_x(t+\Delta) - \tilde{W}_x(t)).\end{aligned} \tag{3.13}$$

A special case is the central discretization, where $\delta_1 = \delta_2 = \frac{1}{2}$. As can be seen from Figure 2 on the preceding page, using a central discretization does improve the correlation behavior somewhat, although it is still quite far from the true correlation. Andersen's discretization for the log-asset price uses an insight of Broadie and Kaya (2006) which, as can be seen by integrating the SDE of (3.3) directly, relates the first stochastic integral in (3.7) in terms of already simulated quantities and the integrated variance

$$\int_t^{t+\Delta} \nu(u)\,\mathrm{d}W_\nu(u) = \frac{\nu^2(t+\Delta) - \nu^2(t)}{2\tau} - \tfrac{1}{2}\tau\Delta + \frac{\kappa}{\tau}\int_t^{t+\Delta} \nu^2(u)\,\mathrm{d}u. \tag{3.14}$$

Substituting this in (3.7) yields

$$\begin{aligned}\ln x(t+\Delta) = \ln x(t) + \left(\frac{\rho_{x\nu}\kappa}{\tau} - \frac{1}{2}\right)\int_t^{t+\Delta} \nu^2(u)\,\mathrm{d}u + \frac{\rho_{x\nu}}{2\tau}[\nu^2(t+\Delta) - \nu^2(t)] \\ + \sqrt{1-\rho_{x\nu}^2}\int_t^{t+\Delta} \nu(u)\,\mathrm{d}\tilde{W}_x(u).\end{aligned} \tag{3.15}$$

Once again we can choose to approximate the integrated variance with an Euler discretization or, as Andersen does in the Heston model, with a central discretization. The correlation for both schemes is analyzed in Figure 2 on the preceding page. As we can see, it is the combination of the insight of Broadie and Kaya (2006) and the central discretization which brings the correlation much more in line with the true correlation. For this reason all simulation schemes we consider in the remainder of this paper will use Broadie and Kaya's insight, as well as a central discretization for the integrated variance.

4 SIMULATION IN THE SCHÖBEL–ZHU MODEL

Having demonstrated in the previous section how to best preserve the correlation structure between the asset and stochastic volatility processes in a special case of the SZ model, we will now formulate our simulation scheme for the full SZ model. In addition, we will demonstrate how to apply a martingale correction such that the forward price of the asset is matched exactly.

For those readers wondering whether an exact simulation of the SZ model is feasible à la Broadie and Kaya (2006), it should be mentioned that, contrary to the Heston model, the increment of the log asset price process is not normally distributed conditional upon the old and new realizations of the volatility process, and the integrated variance process. In addition to the mentioned realizations, we also need to condition on the integrated volatility process, which complicates matters considerably. Nevertheless, as we have seen in the case of the Heston model, schemes based on a simple drift interpolation method are computationally much more efficient than exact transform-based methods (see Lord *et al* (2008), Andersen (2008) or van Haastrecht and Pelsser (2010)). From a practical point of view it is therefore not a disadvantage that an exact simulation is not feasible.

4.1 Simulation scheme for the SZ model

As the volatility process ν in (3.2) follows an Ornstein–Uhlenbeck process, we have the following explicit solution for $\nu(t+\Delta)$ (conditional on the time-t filtration):

$$\nu(t+\Delta) = \nu(t)\mathrm{e}^{-\kappa\Delta} + \int_t^{t+\Delta} \kappa\psi \mathrm{e}^{-\kappa(t+\Delta-u)}\,\mathrm{d}u + \int_t^{t+\Delta} \tau \mathrm{e}^{-\kappa(t+\Delta-u)}\,\mathrm{d}W_\nu(u).$$

As follows from Ito's isometry that $(\nu(t+\Delta) \mid \nu(t))$ is normally distributed with mean $\mu_\nu := K_1\nu(t) + K_2$ and standard deviation $\sigma_\nu := K_3$, a sample of $\nu(t+\Delta) \mid \nu(t)$ can be obtained by setting

$$\nu(t+\Delta) = K_1\nu(t) + K_2 + K_3 Z_\nu, \tag{4.1}$$

where

$$K_1 := \mathrm{e}^{-\kappa\Delta}, \quad K_2 := \psi(1-\mathrm{e}^{-\kappa\Delta}), \tag{4.2}$$

$$K_3 := \tau\sqrt{\frac{1}{2\kappa}(1-\mathrm{e}^{-2\kappa\Delta})}, \tag{4.3}$$

and Z_ν is a sample from the standard normal distribution. This can be generated directly and efficiently by "inverting" the standard normal distribution (see, for example, Acklam 2003). Note that the above sampling of the volatility process, immediately also gives us realizations for the variance process.

As the previous section demonstrated, it is beneficial to apply the Broadie and Kaya (2006) insight and replace $\int_t^{t+\Delta} \nu(u)\,\mathrm{d}\tilde{W}_x(u)$ in (3.7) by expressing it in other model quantities. This can be achieved by integrating (2.3), leading to

$$\int_t^{t+\Delta} \nu(u)\,\mathrm{d}W_\nu(u) = \frac{1}{2\tau}\left(\nu^2(t+\Delta) - \nu^2(t) - \tau^2\Delta + 2\kappa\int_t^{t+\Delta} \nu^2(u)\,\mathrm{d}u - 2\kappa\int_t^{t+\Delta} \psi\nu(u)\,\mathrm{d}u\right). \tag{4.4}$$

Substituting (4.4) in (3.7) yields

$$\ln x(t+\Delta) = \ln x(t) - \frac{1}{2}\int_t^{t+\Delta} \nu(u)\,\mathrm{d}u + \frac{\rho_{x\nu}\kappa}{\tau}\int_t^{t+\Delta} (\nu^2(u) - \psi\nu(u))\,\mathrm{d}u + \frac{\rho_{x\nu}}{2\tau}(\nu^2(t+\Delta) - \nu^2(t) - \tau^2\Delta) + \sqrt{1-\rho_{x\nu}^2}\int_t^{t+\Delta} \nu(u)\tilde{W}_x(u). \tag{4.5}$$

As in the previous section, we replace the integrals over the variance and volatility by linear combinations of their realizations at t and $t+\Delta$

$$\int_t^{t+\Delta} \nu^p(u)\,\mathrm{d}u \mid \nu(t), \nu(t+\Delta) \approx \delta_1\nu^p(t) + \delta_2\nu^p(t+\Delta), \tag{4.6}$$

for $p \in \{1,2\}$ and some constants δ_1, δ_2. These constants can be set in several ways: an Euler-like setting would read $\delta_1 = 1, \delta_2 = 0$, while a central/mid-point/predictor–corrector method uses $\delta_1 = \delta_2 = \frac{1}{2}$. By applying the above drift interpolation method in (4.5), one obtains the following discretization scheme:

$$\ln x(t+\Delta) = \ln x(t) + C_0 + C_1\nu(t) + C_2\nu(t+\Delta) + C_3\nu^2(t) + C_4\nu^2(t+\Delta) + \sqrt{\delta_1\nu^2(t) + \delta_2\nu^2(t+\Delta)}\,C_5 Z_x \tag{4.7}$$

with

$$C_0 = -\tfrac{1}{2}\rho_{x\nu}\tau\Delta, \qquad C_1 = -\delta_1\rho_{x\nu}\frac{\psi\kappa\Delta}{\tau},$$

$$C_2 = -\delta_2\rho_{x\nu}\frac{\psi\kappa\Delta}{\tau}, \qquad C_3 = -\tfrac{1}{2}\delta_1\Delta + \frac{\rho_{x\nu}}{\tau}(\delta_1\kappa\Delta - \tfrac{1}{2}),$$

$$C_4 = -\tfrac{1}{2}\delta_2\Delta + \frac{\rho_{x\nu}}{\tau}(\delta_2\kappa\Delta + \tfrac{1}{2}), \qquad C_5 = \sqrt{1-\rho_{x\nu}^2}\sqrt{\Delta}.$$

Despite the fact the scheme is based on the exact solution of the asset and volatility processes, the discretization for the asset is in general not a martingale, and its net drift away from a martingale can be significant for certain parameter choices. In the following section we show how to enforce this martingale condition. As (4.7) is exponentially affine after we exponentiate and take expectations with respect to the Gaussian random variates, we will refer to this scheme as an exponentially affine in expectation (EAE) scheme. This property will prove to be very convenient in enforcing the exact martingale condition.

4.2 Martingale correction, regularity

As discussed in Andersen and Piterbarg (2007), the continuous-time asset price process $x(t)$ might not have finite higher moments, but it will always be a martingale under the chosen measure

$$\mathbb{E}^{\mathcal{Q}}[x(t+\Delta) \mid \mathcal{F}_t] = x(t) < \infty. \tag{4.8}$$

If we replace $x(t+\Delta)$ by its discretization $\tilde{x}(t+\Delta)$, the martingale condition is no longer satisfied. Though the net drift away from the martingale is controllable by reducing the size of the time step, its size, as mentioned, can be significant depending on the parameters of the model. Following Glasserman and Zhao (2000) and Andersen (2008), we investigate whether it is possible to exactly satisfy this martingale property. Additionally, we look at the regularity of the discretization scheme: that is, we look at whether there might be parameter values where the discretization $\tilde{x}$ could blow up (see also Andersen (2008)).

First of all note that, by the tower law of conditional expectations, we have

$$\mathbb{E}^{\mathcal{Q}}[\tilde{x}(t+\Delta) \mid \mathcal{F}_t] = \mathbb{E}\{\mathbb{E}^{\mathcal{Q}}[\tilde{x}(t+\Delta) \mid \mathcal{F}_t, \nu(t+\Delta)] \mid \mathcal{F}_t\}, \tag{4.9}$$

hence for the martingale condition (4.8) to hold, we need the latter expectation to equal $\tilde{x}(t)$; using the moment-generating function of the normal distribution, we have the following for the discretized asset price $\tilde{x}(t+\Delta)$:

$$\begin{aligned}
&\tilde{x}(t+\Delta) \\
&\quad = \tilde{x}(t)\exp[C_0^* + C_1 v(t) + C_3 v^2(t)]\mathbb{E}\{\exp[C_2 v(t+\Delta) + C_4 v^2(t+\Delta)] \\
&\qquad \times \mathbb{E}^{\mathcal{Q}}(\exp[\sqrt{\delta_1 \nu^2(t) + \delta_2 \nu^2(t+\Delta)}\, C_5 Z_x] \mid \mathcal{F}_t, v(t+\Delta)) \\
&\quad = \tilde{x}(t)\exp[C_0^* + C_1 v(t) + C_3 v^2(t)] \\
&\qquad \times \mathbb{E}[\exp[C_2 v(t+\Delta) + C_4 v^2(t+\Delta) + \tfrac{1}{2} C_5^2(\delta_1 \nu^2(t) + \delta_2 \nu^2(t+\Delta))]].
\end{aligned} \tag{4.10}$$

As mentioned earlier, this is where the EAE property of the scheme becomes apparent. We are left with evaluating the expectation of an exponentially affine form. Taking the $\mathcal{F}_t$-measurable terms out of the expectation, and dividing by $\tilde{x}(t)$, we thus find that the following expectation has to be satisfied for the martingale condition,

$$\begin{aligned}
1 &= \exp[C_0^* + D_1 \nu(t) + D_3 \nu^2(t)]\mathbb{E}^{\mathcal{Q}}\{\exp[D_2 \nu(t+\Delta) + D_4 \nu^2(t+\Delta)] \mid \mathcal{F}_t\} \\
&= \exp[C_0^* + D_1 \nu(t) + D_3 \nu^2(t)]\Psi_H(1),
\end{aligned} \tag{4.11}$$

where $\Psi_H(t)$ denotes the moment-generating function of the (discretized) process

$$H := D_2\nu(t+\Delta) + D_4\nu^2(t+\Delta), \tag{4.12}$$

evaluated in the point t, with

$$\begin{aligned} D_1 &:= C_1, \\ D_2 &:= C_2, \\ D_3 &:= C_3 + \tfrac{1}{2}(1-\rho_{x\nu}^2)\delta_1\Delta, \\ D_4 &:= C_4 + \tfrac{1}{2}(1-\rho_{x\nu}^2)\delta_2\Delta. \end{aligned} \tag{4.13}$$

If the regularity condition $\Psi_H(1) < \infty$ is satisfied, the martingale condition (ie, (4.11)) can be satisfied by setting

$$C_0^* := -D_1\nu(t) - D_3\nu^2(t) - \ln(\Psi_H(1)). \tag{4.14}$$

It now remains to determine the moment-generating function of the random variable H and investigate its existence. To this end, we will use the following corollary.

COROLLARY 4.1 *Let X be a normally distributed random variable with mean μ and variance σ^2; furthermore, let p and q be two constants. Then provided that the regularity condition $uq\sigma^2 < 1$ is satisfied, the moment-generating function of $Y := pX + \frac{1}{2}qX^2$ is given by*

$$\mathbb{E}\exp(uY) = \exp\left(-\frac{p^2}{2q}\right)\left(\frac{1}{\sqrt{1-uq\sigma^2}}\right)\exp\left(\frac{\frac{1}{2}\lambda uq\sigma^2}{1-uq\sigma^2}\right), \tag{4.15}$$

with

$$\lambda = \left(\frac{\mu + p/q}{\sigma}\right)^2. \tag{4.16}$$

PROOF For instance see Johnson *et al* (1994). □

Since the volatility process, conditional upon $\mathcal{F}_t$, is normally distributed, we can immediately use Corollary 4.1 with $p = D_2$ and $q = 2D_4$. Provided that $2D_4\sigma_\nu^2 < 1$, we find that $\Psi_H(1)$ is given by

$$\Psi_H(1) = \exp\left(-\frac{D_2^2}{4D_4}\right)\left(\frac{1}{\sqrt{1-2D_4K_3^2}}\right)\exp\left(\frac{\lambda_\nu D_4 K_3^2}{1-2D_4K_3^2}\right) \tag{4.17}$$

with

$$\lambda_\nu := \left(\frac{\mu_\nu + (D_2/2D_4)}{K_3}\right)^2 = \left(\frac{\nu(t)K_1 + K_2 + (D_2/2D_4)}{K_3}\right)^2, \tag{4.18}$$

with K_1, K_2 as defined in (4.2) and (4.3).

The following proposition applies the above result to the martingale correction in (4.7) and the corresponding regularity condition.

PROPOSITION 4.2 *The regularity of the simulation scheme (4.7) holds if and only if the following regularity condition is satisfied:*

$$\frac{\tau^2}{\kappa}(1-\mathrm{e}^{-2\kappa\Delta})\bigg[-\tfrac{1}{2}\rho_{xv}^2\delta_2\Delta+\frac{\rho_{xv}}{\tau}(\delta_2\kappa\Delta+\tfrac{1}{2})\bigg]<1, \tag{4.19}$$

Given that this condition is satisfied, we can ensure the martingale property in the SZ-scheme of (4.7) by replacing the constant C_0 by

$$C_0^* = E_0 + E_1 v(t) + E_2 v^2(t), \tag{4.20}$$

with

$$\begin{aligned} E_0 &:= \tfrac{1}{2}\ln(1-2D_4K_3^2) - \frac{D_4(K_2+(D_2/2D_4))^2}{1-2D_4K_3^2}, \\ E_1 &:= -D_1 - \frac{2D_4K_1(K_2+(D_2/2D_4))}{1-2D_4K_3^2}, \\ E_2 &:= -D_3 - \frac{D_4K_1^2}{1-2D_4K_3^2} \end{aligned} \tag{4.21}$$

and where K_1, K_2, K_3 are as defined in (4.2), (4.3) and $D_1, \dots, D_4$ are as in (4.13).

PROOF Follows immediately from the results above. □

REMARK 4.3 We note that (4.19) is not restrictive; for negative ρ_{xv} (which is more often the case than in option markets), the condition is automatically satisfied. However, for (strictly) positive ρ_{xv} the condition (4.19) imposes a limit on the size of the time step. Nonetheless, for practical sizes of the time step (eg, $\Delta = \frac{1}{4}$), it is unlikely that the regularity condition (4.19) will be violated. For example, with $\kappa = 1$, $\tau = \frac{1}{2}$, $\delta_1 = \delta_2 = \frac{1}{2}$ and $\rho_{xv} = 1$, this condition is satisfied as long as $\Delta < 6.18$.

5 MONTE CARLO PRICING UNDER STOCHASTIC INTEREST RATES

The SZHW model as introduced in van Haastrecht *et al* (2009) extends the Schöbel and Zhu (1999) model for stochastic volatility with Hull and White (1993) stochastic interest rates. It can for example be used to value equity options or equity-interest rate hybrids. It models three key variables which are allowed to be correlated with each other: the asset price $x(t)$, the Hull and White (1993) interest rate process $r(t)$ and the stochastic volatility of the asset, which follows an Ornstein–Uhlenbeck process in accordance with (Schöbel and Zhu 1999). The risk-neutral asset price dynamics of the SZHW model hence read:

$$dx(t) = x(t)r(t)\,dt + x(t)\nu(t)\,dW_x(t), \qquad x(0) = x_0, \tag{5.1}$$

$$dr(t) = (\theta(t) - ar(t))\,dt + \sigma\,dW_r(t), \qquad r(0) = r_0, \tag{5.2}$$

$$d\nu(t) = \kappa(\psi - \nu(t))\,dt + \tau\,dW_\nu(t), \qquad \nu(0) = \nu_0, \tag{5.3}$$

where $a, \sigma, \kappa, \nu_0, \psi, \tau$ are positive parameters which can be inferred from market data and correspond to the mean reversion and volatility of the short rate process, and the mean reversion, long-term volatility and volatility of the volatility process, respectively. The quantity r_0 and the deterministic function $\theta(t)$ are used to match the currently observed term structure of interest rates (see, for example, Hull and White 1993). Finally, $\tilde{W}(t) = (W_x(t), W_r(t), W_\nu(t))$ denotes a Brownian motion under the risk-neutral measure $\mathbb{Q}$ with covariance matrix

$$\operatorname{Var}(\tilde{W}(t)) = \begin{pmatrix} 1 & \rho_{xr} & \rho_{x\nu} \\ \rho_{xr} & 1 & \rho_{r\nu} \\ \rho_{x\nu} & \rho_{r\nu} & 1 \end{pmatrix} t. \tag{5.4}$$

Though we can price vanilla options in an SZHW model by transforming the characteristic function of the log-asset price (see, for example, van Haastrecht *et al* 2009), sometimes there is a need to price more complex securities, such as path-dependent or multi-asset securities, for which we have to resort to, for example, Monte Carlo simulation. In this section we will present a simulation scheme for the SZHW model, based on the insights of the previous section.

First of all, instead of looking at these dynamics under the risk-neutral bank account measure, we change the underlying probability measure to evaluate this expectation under the T-forward probability measure $\mathbb{Q}^T$ (see, for example, Geman *et al* 1995). Effectively this reduces the dimension of the Monte Carlo simulation as we can eliminate the path dependency of the stochastic interest rates in discounting future cash flows as we discount using the zero-coupon bond maturing at time T, rather than using the money market account. We note that it may not always be a good idea to simulate under the T-forward probability measure, due to the fact that dividing by a numeraire that is a discount bond can lead to a higher sampling error than when we divide by a money market account that accumulates through time (see, for example, Glasserman and Zhao 2000; Andersen and Piterbarg 2010, Section 14.6.1.3). Since the issues behind the simulation and their corresponding solutions are the same, regardless of whether we choose a risk-neutral or a T-forward measure, we choose to work with the T-forward measure and ease our notation.

To this end we define $y(t,T)$, the logarithm of the forward asset price $F(t,T)$, as

$$y(t,T) := \ln\left(\frac{x(t)}{P(t,T)}\right) = \ln F(t,T). \tag{5.5}$$

An application of Ito's lemma yields the following asset price dynamics where $P(t,T)$ is the time-t value of a zero-coupon bond maturing at time T:

$$\mathrm{d}y(t,T) = -\tfrac{1}{2}\nu_F^2(t)\,\mathrm{d}t + \nu(t)\,\mathrm{d}W_x^T(t) + \sigma B_r(t,T)\,\mathrm{d}W_r^T(t), \tag{5.6}$$

$$\nu_F^2(t) := \nu^2(t) + 2\rho_{xr}\nu(t)\sigma B_r(t,T) + \sigma^2 B_r^2(t,T), \tag{5.7}$$

with $B_r(u,T) := (1/a)[1-\mathrm{e}^{-a(T-u)}]$ and where the volatility and variance dynamics read

$$\mathrm{d}\nu(t) = \kappa\bigg(\bigg(\psi - \frac{\rho_{r\nu}\sigma\tau}{\kappa}B_r(t,T)\bigg) - \nu(t)\bigg)\,\mathrm{d}t + \tau\,\mathrm{d}W_\nu^T(t), \tag{5.8}$$

$$\mathrm{d}\nu^2(t) = 2\kappa\bigg(\frac{\tau^2}{2\kappa} + \bigg[\psi - \frac{\rho_{r\nu}\sigma\tau}{\kappa}B_r(t,T)\bigg]\nu(t) - \nu^2(t)\bigg)\,\mathrm{d}t + 2\tau\nu(t)\,\mathrm{d}W_\nu^T(t). \tag{5.9}$$

All of the above SDEs are specified in the T-forward measure, induced by using $P(\cdot,T)$ as the numeraire asset. To this end, W_x^T, W_r^T and W_ν^T are all Brownian motions under the T-forward measure. Before turning to the simulation of the asset price dynamics, we first consider the simulation of the Gaussian rate and volatility process which is common in both of the schemes we will consider.

5.1 Interest rate and variance simulation

For the Ornstein–Uhlenbeck stochastic volatility process, we have the following solution under the T-forward measure $\mathbb{Q}^T$:

$$\nu(t+\Delta) = \nu(t)\mathrm{e}^{-\kappa\Delta} + \int_t^{t+\Delta}\kappa\bigg(\psi - \frac{\rho_{r\nu}\sigma\tau}{\kappa}B_r(u,T)\bigg)\mathrm{e}^{-\kappa(t+\Delta-u)}\,\mathrm{d}u + \int_t^{t+\Delta}\tau\mathrm{e}^{-\kappa(t+\Delta-u)}\mathrm{d}W_\nu^T(u). \tag{5.10}$$

From Ito's isometry we therefore have that $(\nu(t+\Delta)\mid\nu(t))$ is normally distributed with mean $\mu_\nu = K_1\nu(t) + K_2$ and variance $\sigma_\nu^2 = K_3^2$. A sample of $\nu(t+\Delta)\mid\nu(t)$ can be obtained by

$$\nu(t+\Delta) = K_1\nu(t) + K_2 + K_3 Z_\nu, \tag{5.11}$$

with Z_ν a standard normal distributed random variable and where

$$K_1 := \mathrm{e}^{-\kappa\Delta},$$

$$K_2 := \bigg(\psi - \frac{\rho_{r\nu}\sigma\tau}{a\kappa}\bigg)(1-\mathrm{e}^{-\kappa\Delta}) - \frac{\rho_{r\nu}\sigma\tau}{a(\kappa+a)}(\mathrm{e}^{-a(T-t)-\kappa\Delta} - \mathrm{e}^{-a(T-t-\Delta)}), \tag{5.12}$$

$$K_3 := \tau\sqrt{\frac{1}{2\kappa}(1-\mathrm{e}^{-2\kappa\Delta})}. \tag{5.13}$$

Though the volatility and the directly related variance process can be simulated from their exact distributions, we need to resort to discretization methods for a (joint) asset price sampling. We will deal with this in the following sections.

5.2 Asset price sampling scheme

Recall that we have the following solution for the SZHW log-asset price solution under the T-forward measure $\mathbb{Q}^T$:

$$y(t+\Delta,T) = y(t,T) - \frac{1}{2}\int_t^{t+\Delta} \nu_F^2(u)\,\mathrm{d}u + \sigma\int_t^{t+\Delta} B_r(u,T)\,\mathrm{d}W_r^T(u) + \int_t^{t+\Delta} \nu(u)\,\mathrm{d}W_x^T(u) \quad (5.14)$$

with

$$\nu_F^2(u) = \nu^2(u) + 2\rho_{xr}\nu(u)\sigma B_r(u,T) + \sigma^2 B_r^2(u,T), \quad (5.15)$$

and where $W_x^T(u)$ and $W_r^T(u)$ are correlated Brownian motions. In a Monte Carlo simulation it is often convenient to express these correlated Brownian motions in terms of three orthogonal components $\tilde{W}_\nu^T$, $\tilde{W}_x^T$ and $\tilde{W}_r^T$, eg, by using a Cholesky decomposition; the asset dynamics of (5.14) hence become

$$\begin{aligned} y(t+\Delta,T) &= y(t,T) - \frac{1}{2}\int_t^{t+\Delta} \nu_F^2(u)\,\mathrm{d}u \\ &\quad + \int_t^{t+\Delta} \nu(u)\,\mathrm{d}(\rho_{x\nu}\tilde{W}_\nu^T(u) + \sqrt{1-\rho_{x\nu}^2}\,\tilde{W}_x^T(u)) \\ &\quad + \sigma\int_t^{t+\Delta} B_r(u,T)\,\mathrm{d}(\rho_{r\nu}\tilde{W}_\nu^T(u) + \omega_{xr}\tilde{W}_x^T(u) + \sqrt{1-\rho_{r\nu}^2-\omega_{xr}^2}\,\tilde{W}_r^T(u)), \end{aligned} \quad (5.16)$$

with

$$\omega_{xr} = \frac{\rho_{xr} - \rho_{x\nu}\rho_{r\nu}}{\sqrt{1-\rho_{x\nu}^2}}. \quad (5.17)$$

As Section 4 demonstrated, it is beneficial to apply the Broadie and Kaya (2006) insight and replace $\int_t^{t+\Delta} \nu(u)\,\mathrm{d}\tilde{W}_x^T(u)$ in (5.16) by expressing it in other model quantities. This can be achieved by integrating (5.9), leading to

$$\int_t^{t+\Delta} \nu(u)\,\mathrm{d}\tilde{W}_\nu^T(u) = \frac{1}{2\tau}\bigg[\nu^2(t+\Delta) - \nu^2(t) - \tau^2\Delta + 2\kappa\int_t^{t+\Delta}\nu^2(u)\,\mathrm{d}u - 2\kappa\int_t^{t+\Delta}\Big(\psi - \frac{\rho_{r\nu}\sigma\tau}{\kappa}B_r(u,T)\Big)\nu(u)\,\mathrm{d}u\bigg]. \quad (5.18)$$

Substituting (5.18) in (5.16) yields

$$\begin{aligned}
y(t+\Delta,T) = y(t,T) &- \frac{1}{2}\int_t^{t+\Delta} \nu_F^2(u)\,\mathrm{d}u \\
&+ \frac{\rho_{x\nu}\kappa}{\tau}\int_t^{t+\Delta}\biggl[\nu^2(u) - \biggl(\psi - \frac{\rho_{r\nu}\sigma\tau}{\kappa}B_r(u,T)\biggr)\nu(u)\biggr]\mathrm{d}u \\
&+ \frac{\rho_{x\nu}}{2\tau}(\nu^2(t+\Delta) - \nu^2(t) - \tau^2\Delta) \\
&+ \int_t^{t+\Delta}(\sqrt{1-\rho_{x\nu}^2}\,\nu(u) + \omega_{xr}\sigma B_r(u,T))\,\mathrm{d}\tilde{W}_x^T(u) \\
&+ \rho_{r\nu}\int_t^{t+\Delta}\sigma B_r(u,T)\,\mathrm{d}\tilde{W}_\nu^T(u) \\
&+ \sqrt{1-\rho_{r\nu}^2-\omega_{xr}^2}\int_t^{t+\Delta}\sigma B_r(u,T)\,\mathrm{d}\tilde{W}_r^T(u).
\end{aligned} \tag{5.19}$$

This leaves us with three stochastic integrals, which we tackle in order of complexity. We start with the last, which follows directly from Ito's isometry

$$\int_t^{t+\Delta}\sigma B_r(u,T)\,\mathrm{d}\tilde{W}_r^T(u) \sim \sqrt{\int_t^{t+\Delta}\sigma^2 B_r^2(u,T)\,\mathrm{d}u}\,Z_r, \tag{5.20}$$

with Z_r an independent standard normal random variable. The first stochastic integral in (5.19) follows similarly as

$$\begin{aligned}
&\int_t^{t+\Delta}\biggl(\sqrt{1-\rho_{x\nu}^2}\,\nu(u) + \omega_{xr}\sigma B_r(u,T)\biggr)\mathrm{d}\tilde{W}_x^T(u) \\
&\quad\sim \biggl(\int_t^{t+\Delta}((1-\rho_{x\nu}^2)\nu^2(u) + 2\sqrt{1-\rho_{x\nu}^2}\,\omega_{xr}\sigma B_r(u,T)\nu(u) \\
&\qquad\qquad + \omega_{xr}^2\sigma^2 B_r^2(u,T))\,\mathrm{d}u\biggr)^{1/2} Z_x,
\end{aligned} \tag{5.21}$$

with Z_x an independent standard normal distributed random variable. Finally, the second integral follows from the fact that the pair

$$\biggl(\int_t^{t+\Delta}\sigma B_r(u,T)\,\mathrm{d}\tilde{W}_\nu^T(u), \int_t^{t+\Delta}\mathrm{d}\tilde{W}_\nu^T(u)\biggr)$$

follows a bivariate normal distribution with correlation $\rho_{\nu\nu 2}(t, t+\Delta)$ and a conditioning argument

$$\int_t^{t+\Delta} \sigma B_r(u,T)\,\mathrm{d}\tilde{W}_\nu^T(u) \,\bigg|\, \int_t^{t+\Delta} \mathrm{d}\tilde{W}_\nu^T(u) \\ \sim \sqrt{G(t,t+\Delta)}(\rho_{\nu\nu 2}(t,t+\Delta)Z_\nu + \sqrt{1-\rho_{\nu\nu 2}^2(t,t+\Delta)}Z_{\nu 2}), \tag{5.22}$$

$$\rho_{\nu\nu 2}(t,t+\Delta) := \frac{\int_t^{t+\Delta} \sigma B_r(u,T)\,\mathrm{d}u}{\sqrt{\Delta G(t,t+\Delta)}}, \tag{5.23}$$

with $Z_{\nu 2}$ an independent standard normal random variable and with

$$G(t,t+\Delta) \\ := \int_t^{t+\Delta} \sigma^2 B_r^2(u,T)\,\mathrm{d}u \\ = \frac{\sigma^2}{a^2}\bigg[\Delta + \frac{1}{2a}\mathrm{e}^{-2a(T-t-\Delta)} - \frac{2}{a}\mathrm{e}^{-a(T-t-\Delta)} - \frac{1}{2a}\mathrm{e}^{-2a(T-t)} + \frac{2}{a}\mathrm{e}^{-a(T-t)}\bigg]. \tag{5.24}$$

Having eliminated all stochastic integrals, we are left with deterministic integrals over $\sigma B_r(u,T)$, $\nu(u)$ and powers thereof. For the deterministic integrals over $\sigma B_r(u,T)$ we use the following explicit solutions:

$$H(t,t+\Delta) := \int_t^{t+\Delta} \sigma B_r(u,T)\,\mathrm{d}u = \frac{\sigma}{a}\bigg[\Delta - \frac{1}{a}\mathrm{e}^{-a(T-t-\Delta)} + \frac{1}{a}\mathrm{e}^{-a(T-t)}\bigg]. \tag{5.25}$$

whereas we will approximate all integrals over $\nu(u)$ by using the predictor–corrector method:

$$\int_t^{t+\Delta} \nu^p(u)\,\mathrm{d}u \approx (\delta_1 \nu^p(t) + \delta_2 \nu^p(t+\Delta))\Delta, \tag{5.26}$$

for $p \in \{1,2\}$ and some constants δ_1, δ_2.

Collecting all terms once again yields an EAE scheme for the SZHW model:

$$y(t+\Delta,T) = y(t,T) + C_0 + C_1\nu(t) + C_2\nu(t+\Delta) + C_3\nu^2(t) \\ + C_4\nu^2(t+\Delta) + C_5 Z_x + C_6 Z_\nu + C_7 Z_{\nu 2} + C_8 Z_r, \tag{5.27}$$

where

$$C_0 = -\tfrac{1}{2}[G(t,t+\Delta) + \rho_{x\nu}\tau\Delta],$$

$$C_1 = -\delta_1\bigg(\rho_{xr}H(t,t+\Delta) + \rho_{x\nu}\bigg[\frac{\psi\kappa\Delta}{\tau} - \rho_{r\nu}H(t,t+\Delta)\bigg]\bigg),$$

$$
\begin{aligned}
C_2 &= -\delta_2\bigg(\rho_{xr}H(t,t+\Delta)+\rho_{x\nu}\bigg[\frac{\psi\kappa\Delta}{\tau}-\rho_{r\nu}H(t,t+\Delta)\bigg]\bigg),\\
C_3 &= -\tfrac{1}{2}\delta_1\Delta+\frac{\rho_{x\nu}}{\tau}(\delta_1\kappa\Delta-\tfrac{1}{2}),\\
C_4 &= -\tfrac{1}{2}\delta_2\Delta+\frac{\rho_{x\nu}}{\tau}(\delta_2\kappa\Delta+\tfrac{1}{2}),\\
C_5 &= \sqrt{C_{50}+C_{51}\nu(t)+C_{52}\nu(t+\Delta)+C_{53}\nu^2(t)+C_{54}\nu^2(t+\Delta)},\\
C_{50} &= \omega_{xr}^2 G(t,t+\Delta),\\
C_{51} &= 2\delta_1\omega_{xr}\sqrt{1-\rho_{x\nu}}H(t,t+\Delta),\\
C_{52} &= 2\delta_2\omega_{xr}\sqrt{1-\rho_{x\nu}^2}H(t,t+\Delta),\\
C_{53} &= \delta_1\Delta(1-\rho_{x\nu}^2),\\
C_{54} &= \delta_2\Delta(1-\rho_{x\nu}^2),\\
C_6 &= \rho_{r\nu}\sqrt{G(t,t+\Delta)}\rho_{vv2}(t,t+\Delta),\\
C_7 &= \rho_{r\nu}\sqrt{G(t,t+\Delta)}\sqrt{1-\rho_{vv2}^2(t,t+\Delta)},\\
C_8 &= \sqrt{1-\rho_{r\nu}^2-\omega_{xr}^2}\sqrt{G(t,t+\Delta)}.
\end{aligned}
$$

Similar to the SZ scheme (4.7), the above simulation scheme might have a net drift away from the martingale and violate the (no-arbitrage) martingale property. Nonetheless, in the following section, we show that we can easily enforce this martingale condition by replacing the constant C_0 with C_0^* of (4.14).

5.3 Martingale correction, regularity

In this section, using similar techniques to those used in Section 4.2, we will investigate how to enforce the martingale property of the discretized asset price process $\tilde{x}$ in predictor–corrector scheme (5.27). Furthermore, we investigate the regularity of the proposed discretization scheme, ie, we investigate whether there are parameter values where the $\tilde{x}$-process blows up.

By the tower law of conditional expectations, we have that the discretization $\tilde{F}$ of the forward asset price F (see (5.5)) satisfies:

$$
\mathbb{E}^{\mathbb{Q}^T}[\tilde{F}(t+\Delta)\mid\mathcal{F}_t]=\mathbb{E}^{\mathbb{Q}^T}\{\mathbb{E}^{\mathbb{Q}^T}[\tilde{F}(t+\Delta)\mid\mathcal{F}_t,\nu(t+\Delta)]\mid\mathcal{F}_t\}. \tag{5.28}
$$

If we want the martingale condition (4.8) to hold, we need $\tilde{F}(t)$ to equal the latter expectation. We express the inner expectation completely in terms of $\nu(t)$ and $\nu(t+\Delta)$ by exponentiating (5.27), taking the expectation over the independent normal distributions Z_x, Z_{v2} and Z_r, and noting from (5.12) that

$$Z_\nu := \frac{\nu(t+\Delta) - \nu(t)K_1 - K_2}{K_3}. \tag{5.29}$$

We obtain the following expression:

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}^T}[\tilde{F}(t+\Delta) \mid \mathcal{F}_t, \nu(T)] \\ &\quad = \tilde{F}(t)\exp[D_0 + D_1\nu(t) + D_2\nu(t+\Delta) + D_3\nu^2(t) + D_4\nu^2(t+\Delta)],\end{aligned} \tag{5.30}$$

where

$$\left.\begin{aligned} &D_0 := C_0^* + \tfrac{1}{2}C_8^2 + \tfrac{1}{2}C_7^2 + \tfrac{1}{2}C_{50} - C_6\frac{K_2}{K_3}, \qquad D_1 := C_1 + \tfrac{1}{2}C_{51} - \frac{K_1}{K_3}C_6, \\ &D_2 := C_2 + \tfrac{1}{2}C_{52} + \frac{1}{K_3}C_6, \qquad D_3 := C_3 + \tfrac{1}{2}C_{53}, \qquad D_4 := C_4 + \tfrac{1}{2}C_{54}. \end{aligned}\right\} \tag{5.31}$$

Once again, due to the EAE property of our scheme, this term is exponentially affine. By substituting (5.30) in (5.28), we find that the following condition has to be satisfied for the martingale condition to hold:

$$1 = \mathbb{E}^{\mathbb{Q}^T}\{\exp[D_0 + D_1\nu(t) + D_2\nu(t+\Delta) + D_3\nu^2(t) + D_4\nu^2(t+\Delta)] \mid \mathcal{F}_t\}. \tag{5.32}$$

Taking the $\mathcal{F}_t$-measurable terms out of the expectation and collecting terms, we obtain

$$\begin{aligned} 1 &= \exp[D_0 + D_1\nu(t) + D_3\nu^2(t)]\mathbb{E}^{\mathbb{Q}^T}\{\exp[D_2\nu(t+\Delta) + D_4\nu^2(t+\Delta)] \mid \mathcal{F}_t\} \\ &= \exp[D_0 + D_1\nu(t) + D_3\nu^2(t)]\Psi_H(1), \end{aligned} \tag{5.33}$$

where $\Psi_H(t)$ denotes the moment-generating function of the (discretized) process

$$H := D_2\nu(t+\Delta) + D_4\nu^2(t+\Delta), \tag{5.34}$$

evaluated in the point t. Hence expanding D_0, we have that for the martingale condition to hold we need

$$1 = \exp\left[C_0^* + \tfrac{1}{2}C_8^2 + \tfrac{1}{2}C_7^2 + \tfrac{1}{2}C_{50} - C_6\frac{K_2}{K_3} + D_1\nu(t) + D_3\nu^2(t)\right]\Psi_H(1), \tag{5.35}$$

which (assuming the regularity condition $\Psi_H(1) < \infty$ is satisfied) can be established by setting

$$C_0^* := -\tfrac{1}{2}C_8^2 - \tfrac{1}{2}C_7^2 - \tfrac{1}{2}C_{50} + C_6\frac{K_2}{K_3} - D_1\nu(t) - D_3\nu^2(t) - \ln(\Psi_H(1)). \tag{5.36}$$

As $\nu(t+\Delta)$ is still Gaussian under $\mathcal{Q}^T$, $\Psi_H(1)$ and its regularity can be determined in a similar fashion to Section 4.2. The following proposition applies the above result to the martingale correction and the regularity of the simulation scheme (5.27).

PROPOSITION 5.1 *The regularity of the simulation scheme (5.27) holds if and only if the regularity condition (4.19) is satisfied. Given that this condition is satisfied, we can ensure the martingale property in the SZHW-scheme of (5.27) by replacing the constant C_0 by*

$$C_0^* = E_0 + E_1\nu(t) + E_2\nu^2(t), \tag{5.37}$$

where

$$E_0 := \tfrac{1}{2}\ln(1-2D_4K_3^2) - \frac{D_4(K_2 + (D_2/2D_4))^2}{1-2D_4K_3^2} - \tfrac{1}{2}C_{50} + C_6\frac{K_2}{K_3} - \tfrac{1}{2}C_7^8 - \tfrac{1}{2}C_8^2 + \frac{D_2^2}{4D_4}, \tag{5.38}$$

$$E_1 := -D_1 - \frac{2D_4K_1(K_2 + (D_2/2D_4))}{1-2D_4K_3^2},$$

$$E_2 := -D_3 - \frac{D_4K_1^2}{1-2D_4K_3^2}, \tag{5.39}$$

with K_1, K_2, K_3 as defined in (5.12), (5.13) and $D_1, \dots, D_4$ as in (5.31).

PROOF Follows directly from the above results. □

6 NUMERICAL RESULTS

Any simulation scheme has to be tested. As they say, the proof of the pudding is in the eating. In this section our goal is to test our proposed simulation schemes and compare them with alternate schemes. In our comparisons we focus on the bias of European call prices, where by "bias" we mean $\mathbb{E}[\hat{\alpha}] - \alpha$, where α is the true price of the European call and $\hat{\alpha}$ is its Monte Carlo estimator. It is of high importance for practitioners to have a bias as small as possible for reasonable sizes of the time step. Ideally we would like to be able to simulate the relevant quantities only at those points in time that are relevant to the option contract that is being priced. Unfortunately, that is not always possible as has certainly become clear from several papers on the simulation of the Heston model (see, for example, Lord *et al* 2008; van Haastrecht and Pelsser 2010).

Table 1 on the next page contains the parameter configurations for our test cases. Section 6.1 deals with the simulation scheme for the special case of the SZ model that collapses to a Heston model (see Section 3). Volatilities for Case I are similar to those in the equity market at the time of writing. In this test case we not only compare our scheme to an Euler scheme, but also to the best-performing scheme considered in

TABLE 1 Test cases for the Schöbel–Zhu/Heston, Schöbel–Zhu and Schöbel–Zhu–Hull–White simulation schemes; in all cases, $S(0) = 100$.

Example	Type	κ	τ	$v(0)$	θ	ρ_{xv}	r	a	σ	ρ_{xr}	ρ_{rv}
Case I	Call-5Y	0.1	0.3	0.0	0.0	−0.6	0.00	—	—	—	—
Case II	Call-10Y	0.4	0.4	0.2	0.2	−0.9	0.04	—	—	—	—
Case III	Call-15Y	0.4	0.4	0.2	0.2	−0.7	0.04	0.03	0.01	0.2	0.15

Andersen (2008), the QE-M scheme. Indeed, for some of the test cases in Andersen (2008) the martingale correction was not necessary, but for some it certainly was. Since when pricing derivatives, matching the forward is a sine qua non, we feel that it should therefore always be included in any real-life application of such a simulation scheme. Moreover, since in the EAE scheme the martingale correction can in some test cases be significant, we feel it would not be fair to compare it to the QE scheme without a martingale correction.

Finally, we also undertake a comparison with a scheme for the Heston model recently proposed in Zhu (2011). In Case II we consider a setting of the SZ model which does not collapse to the Heston model. Here, the volatilities correspond to levels seen at the end of 2008 and beginning of 2009. Finally, we also test the scheme which was proposed in Section 5 for the SZHW model: Case III deals with normal, perhaps slightly excited, long-term market volatilities.

All numerical examples are based on a million simulation paths, where we used the asset price as a control variate and the Mersenne twister to generate pseudorandom uniform numbers.

6.1 Results for the Heston/Schöbel–Zhu model

In Case I we consider a special case of the SZ model which corresponds to the Heston model. First we will address the question of to what extent the Broadie–Kaya insight, the central discretization of the integrated variance, or the combination of both affects the size of the bias of our scheme. To this end we will compare the following schemes.

- A simple Euler scheme, where we sample the volatility from its exact (normal) distribution, and discretize the log asset according to (3.7).
- A scheme we refer to as Euler–central, which is the same as above, except that we discretize the log asset according to (3.13), using $\delta_1 = \delta_2 = \frac{1}{2}$.
- A scheme we refer to as Euler–BK, where we apply the Broadie–Kaya insight, but discretize the integrated variance using an Euler discretization. In formulas, this means we discretize the log asset according to (3.15), where the integrated

TABLE 2 Estimated call option price biases for Case I.

(a) $K = 100$

Δ	Euler		Euler–central		BK–Euler		EAE	
1	−1.914	(±0.062)	−0.009*	(±0.065)	−2.626	(±0.055)	−0.130	(±0.057)
1/2	−0.781	(±0.060)	0.037*	(±0.061)	−1.272	(±0.056)	−0.020*	(±0.057)
1/4	−0.348	(±0.058)	0.052*	(±0.059)	−0.672	(±0.057)	−0.039*	(±0.057)
1/8	−0.136	(±0.058)	0.033*	(±0.058)	−0.325	(±0.057)	−0.003*	(±0.057)
1/16	0.002*	(±0.058)	0.032*	(±0.058)	−0.150	(±0.057)	0.013*	(±0.057)
1/32	−0.015*	(±0.057)	0.011*	(±0.057)	−0.089	(±0.057)	−0.010*	(±0.057)

(b) $K = 140$

Δ	Euler		Euler–central		BK–Euler		EAE	
1	0.058*	(±0.074)	1.859	(±0.079)	−2.442	(±0.064)	−0.354	(±0.066)
1/2	0.226	(±0.071)	1.035	(±0.072)	−1.123	(±0.065)	−0.080	(±0.066)
1/4	0.160	(±0.068)	0.588	(±0.070)	−0.580	(±0.066)	−0.051*	(±0.066)
1/8	0.137	(±0.067)	0.281	(±0.068)	−0.287	(±0.066)	−0.012*	(±0.066)
1/16	0.138	(±0.067)	0.172	(±0.067)	−0.134	(±0.066)	0.003*	(±0.066)
1/32	0.053*	(±0.066)	0.080	(±0.066)	−0.084	(±0.065)	−0.015*	(±0.066)

(c) $K = 60$

Δ	Euler		Euler–central		BK–Euler		EAE	
1	−1.707	(±0.036)	−0.576	(±0.039)	−1.680	(±0.034)	−0.005*	(±0.036)
1/2	−0.793	(±0.037)	−0.300	(±0.038)	−0.865	(±0.035)	0.004*	(±0.036)
1/4	−0.381	(±0.037)	−0.133	(±0.037)	−0.461	(±0.036)	−0.017*	(±0.036)
1/8	−0.170	(±0.037)	−0.055	(±0.037)	−0.223	(±0.036)	0.003*	(±0.037)
1/16	−0.044	(±0.037)	−0.025*	(±0.037)	−0.111	(±0.036)	0.004*	(±0.036)
1/32	−0.024*	(±0.037)	−0.013*	(±0.037)	−0.058	(±0.036)	−0.001*	(±0.036)

Numbers in parentheses are the widths of the confidence interval at a 99% confidence level. Asterisks denote the biases that were not significantly different from zero. EThe exact price for strikes $K = 100$, 140 and 60 are, respectively, 27.90, 14.23 and 50.34.

variance as well as the stochastic integral are approximated using an Euler discretization; for this scheme we have included a martingale correction.

- Our EAE scheme, which combines the central discretization with Broadie–Kaya's insight, together with a martingale correction.

In Table 2, we have displayed estimated call option price biases for Case I, as a function of the strike level ($K = 60$, 100 or 140) and the time step Δ (1 through 1/32).

Numbers in parentheses are the widths of the confidence interval at a 99% confidence level. Asterisks denote the biases that were not significantly different from zero. We note a couple of interesting things. First of all, we noticed that the Broadie–Kaya insight by itself does not lead to an improvement in the correlation structure, as witnessed in Figure 2 on page 7. This is confirmed here. Secondly, the central discretization does seem to improve the bias of the simulation scheme, except in the case of high strikes. Overall, we can conclude that adding the Broadie–Kaya insight and the central discretization together yields the best result, where the central discretization appears to have the highest impact.

We now turn to a comparison with other schemes. Since we are considering a special case of the Heston model, we can also compare the QE-M discretization of Andersen (2008). Whereas our EAE scheme samples from the exact distributions of $\nu(t)$ and $\nu^2(t)$, Andersen's QE scheme uses

- a quadratic Gaussian distribution when $\mathrm{Var}[\nu^2(t+\Delta)]/\mathbb{E}[\nu^2(t+\Delta)] \leqslant 1.5$,
- a mixture of zero and an exponential function, otherwise.

Our scheme and the QE-M scheme therefore differ for low values of $\nu(t)$, to be precise when

$$\nu^2(t) \in \left[0, \frac{(\mathrm{e}^{\kappa\Delta}-1)\tau^2}{4\kappa}\right]. \tag{6.1}$$

Finally, we also compare our scheme with the Zhu (2011) scheme. In this scheme, the SDE for the square root of the stochastic variance is derived. As the square root is not differentiable in zero, Ito's lemma is incorrectly applied here (see, for example, Kahl and Jäckel 2006; Lord *et al* 2008). Luckily, Zhu's best-performing method, a moment-matching method, does not depend too much on this premise. The numerical examples in his paper suggest that this method is comparable to Andersen's QE scheme for low values of the volatility of variance parameter, but is outperformed for realistic levels of the volatility of variance parameter.

From the results it is clear that – at least for this parameter configuration – Zhu's method is better than a simple Euler discretization for lower strikes, though for higher strikes the Euler scheme wins. However, the QE-M and EAE methods are much better in terms of bias. Both methods are too close to be able to distinguish between them.

As we eventually want to judge a scheme based on its efficiency, we should look at its accuracy in combination with the computational effort of the methods. To this end we also report the computational times for the four simulation methods that are provided in Table 3 on the facing page.

From Table 3 on the facing page we can see the Euler scheme takes the least time to compute, followed by Zhu's method, the EAE scheme and the QE-M scheme. Still, the efficiency of the QE-M scheme and the EAE method by far outperforms those of

TABLE 3 Computational times in seconds for Case I for the Euler, Zhu, QE-M and EAE schemes, all with 1 000 000 sample paths.

Δ	Euler	Zhu	QE-M	EAE
1	0.9	1.0	1.2	1.1
1/2	1.8	2.0	2.4	2.2
1/4	3.6	4.0	5.0	4.3
1/8	7.0	7.9	10.2	8.6
1/16	13.9	15.7	20.8	17.1
1/32	28.1	31.3	41.9	33.5

the Euler and Zhu's method, as can be seen if we take a look at the accuracy of the methods in Table 2 on page 23. From that table, we can see that the EAE and QE-M only need two or four time steps a year to produce a scheme with no significant bias, whereas the Euler and Zhu schemes in most cases need at least sixteen time steps a year to produce a scheme negligible bias. Though the QE-M scheme and the EAE method produce a similar accuracy, the EAE method is more efficient. This can be explained by the fact that the exact Gaussian volatility distribution of the Schöbel and Zhu (1999) model is explicitly utilized in the EAE method, whereas the variance simulation of the QE-M method is tailored for the Heston (1993) model.

6.2 Results for the Schöbel–Zhu and Schöbel–Zhu–Hull–White models

We move on to Cases II and III, which are slightly more benign due to a nonzero value of ψ^2 in Case II and the inclusion of stochastic interest rates for Case III. The numerical results for these cases can be found in Table 4 on the next page and Table 5 on page 27. Computational times for both cases are very similar to those reported in Table 3 and are hence omitted.[3] We only consider the EAE scheme here and compare it to the simplest scheme, an Euler scheme.

In Case II, a non-Heston SZ model, the differences between the Euler and EAE methods are indeed closer, though still noticeably in favor of the EAE method. From Table 5 on page 27 we can see that especially for in- and out-of-the-money options, the EAE scheme significantly outperforms the Euler scheme.

Finally, we take a look at the performance of the simulation schemes for the SZHW model, where in addition to the SZ model we have stochastic interest rates that are correlated with both the underlying and the stochastic volatility process. While the

[2] This makes the distribution less fat tailed.
[3] These are available from the authors upon request.

TABLE 4 Estimated call option price biases for Case II.

(a) $K = 100$

Δ	Euler		EAE	
1	−0.828	(±0.059)	−0.389	(±0.050)
1/2	−0.314	(±0.055)	−0.165	(±0.050)
1/4	−0.145	(±0.052)	−0.034*	(±0.050)
1/8	−0.068	(±0.051)	0.014*	(±0.050)
1/16	−0.043*	(±0.051)	0.005*	(±0.050)
1/32	0.017*	(±0.050)	0.003*	(±0.050)

(b) $K = 140$

Δ	Euler		EAE	
1	−0.495	(±0.080)	−0.457	(±0.063)
1/2	−0.110	(±0.072)	−0.204	(±0.064)
1/4	−0.032*	(±0.068)	−0.041*	(±0.064)
1/8	−0.003*	(±0.066)	0.013*	(±0.064)
1/16	0.002*	(±0.065)	0.005*	(±0.064)
1/32	0.034*	(±0.065)	0.005*	(±0.064)

(c) $K = 60$

Δ	Euler		EAE	
1	−0.699	(±0.035)	−0.278	(±0.031)
1/2	−0.298	(±0.033)	−0.110	(±0.032)
1/4	−0.151	(±0.033)	−0.023*	(±0.032)
1/8	−0.070	(±0.032)	0.007*	(±0.032)
1/16	−0.046	(±0.032)	0.004*	(±0.032)
1/32	0.005*	(±0.032)	0.004*	(±0.032)

Results for Case II, exact prices: 56.77, 45.34 and 70.89. Numbers in parentheses are the widths of the confidence interval at a 99% confidence level. Asterisks denote the biases that were not significantly different from zero.

addition of stochastic interest rates complicates the scheme slightly, the picture is similar to before, as can be seen from Table 5 on the facing page.

Again the EAE method produces a much smaller discretization error than the Euler scheme, allowing the user to utilize bigger time steps, instead of the smaller ones we would be confined to when using the Euler method. For example, for the strikes considered we could safely use a time step of a quarter of a year for the EAE method,

TABLE 5 Estimated call option price biases for Case III.

(a) $K = 100$

Δ	Euler		EAE	
1	−0.323	(±0.032)	−0.069	(±0.032)
1/2	−0.151	(±0.032)	−0.036	(±0.032)
1/4	−0.087	(±0.032)	0.008*	(±0.032)
1/8	−0.029*	(±0.032)	0.005*	(±0.032)
1/16	−0.006*	(±0.032)	−0.022*	(±0.032)
1/32	0.001*	(±0.032)	0.001*	(±0.032)

(b) $K = 140$

Δ	Euler		EAE	
1	−0.299	(±0.047)	−0.085	(±0.046)
1/2	−0.127	(±0.046)	−0.046*	(±0.046)
1/4	−0.077	(±0.046)	0.012*	(±0.046)
1/8	−0.014*	(±0.046)	0.006*	(±0.046)
1/16	−0.002*	(±0.046)	−0.022*	(±0.046)
1/32	0.014*	(±0.046)	−0.008*	(±0.046)

(c) $K = 60$

Δ	Euler		EAE	
1	−0.190	(±0.016)	−0.041	(±0.017)
1/2	−0.095	(±0.017)	−0.017	(±0.017)
1/4	−0.052	(±0.017)	0.005*	(±0.017)
1/8	−0.019	(±0.017)	0.005*	(±0.017)
1/16	0.005*	(±0.017)	−0.007*	(±0.017)
1/32	−0.001*	(±0.017)	0.004*	(±0.017)

Results for Case III, exact prices: 53.75, 40.69 and 69.97. Numbers in parentheses are the widths of the confidence interval at a 99% confidence level. Asterisks denote the biases that were not significantly different from zero.

and have a bias which is not significantly different from zero. In the Euler method, this is only achieved with a time step equal to 1/16.

7 CONCLUSION

A major problem signaled with Euler schemes in the simulation of stochastic volatility models is their inability to generate the proper correlation between the increments of the asset and the stochastic volatility processes. As the correlation parameter in the stochastic volatility models is an important determinant of the skew in implied volatilities, not being able to match this parameter leads to a significant mispricing of options with strikes far away from the at-the-money level. In the Heston (1993) model this so-called leaking correlation problem is partially caused by the fact that an Euler discretization tries to approximate a square-root process using a Gaussian process. However, even when the stochastic volatility itself is Gaussian, such as in Schöbel and Zhu (1999)-like models, we have shown that the problem of leaking correlation is still an issue.

In this paper we have proposed simulation algorithms for the SZ model and its extensions. By analyzing the lessons learned from the Andersen (2008) quadratic exponential scheme for the simulation of the Heston model, we conclude that the central discretization of the integrated variance terms affects the bias most. In addition, the insight from Broadie and Kaya (2006) helps to improve the correlation structure of the resulting discretization. Based on these insights, we have formulated a simulation scheme for the SZ model which is tailored to match the correlation between the increments of the asset price and the variance processes of the continuous-time dynamics. A simulation scheme for the SZHW model considered in van Haastrecht *et al* (2009), which incorporates the need for stochastic interest rates, was derived as well. This is closely related to the recent advances in the development of markets for long-term derivatives, for which maturities the inclusion of stochastic interest rates in a derivatives pricing model is more appropriate.

All introduced schemes have been carefully chosen to be EAE, which greatly facilitates the derivation of a martingale correction. Finally, we numerically compared the new simulation schemes with other recent schemes in the literature. For a special case of the SZ model which coincides with Heston, our proposed scheme has a similar performance to the QE-M scheme of Andersen (2008), while being slightly more efficient in terms of computational time required. For the general SZ and SZHW model, it has been demonstrated that our scheme consistently outperforms the Euler scheme. These results affirm that Andersen's result is more widely applicable than to the Heston model alone; we conclude that for the simulation of stochastic volatility models, it is of great importance to match the correlation between the asset price and its stochastic volatility (and variance) process.

REFERENCES

Acklam, P. (2003). An algorithm for computing the inverse normal cumulative distribution function. URL: http://home.online.no/pjacklam/notes/invnorm/index.html.

Andersen, L. B. G. (2008). Simple and efficient simulation of the Heston stochastic volatility model. *The Journal of Computational Finance* **11**(3), 1–42.

Andersen, L. B. G., and Piterbarg, V. (2007). Moment explosions in stochastic volatility models. *Finance and Stochastics* **11**(1), 29–50.

Andersen, L. B. G., and Piterbarg, V. V. (2010). *Interest Rate Modeling, Volume II: Term Structure Models*, 1st edn. Atlantic Financial Press.

Broadie, M., and Kaya, Ö. (2006). Exact simulation of stochastic volatility and other affine jump diffusion models. *Operations Research* **54**(2), 217–231.

Geman, H., Karoui, E., and Rochet, J. C. (1995). Changes of numeraire, changes of probability measures and pricing of options. *Journal of Applied Probability* **32**, 443–548.

Glasserman, P. (2003). *Monte Carlo Methods in Financial Engineering*. Springer.

Glasserman, P., and Zhao, Z. (2000). Arbitrage-free discretization of lognormal forward Libor and swap rate models. *Finance and Stochastics* **4**(1), 35–68.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* **6**, 327–343.

Hull, J., and White, A. (1987). The pricing of options on assets with stochastic volatilities. *Journal of Finance* **42**(2), 281–300.

Hull, J., and White, A. (1993). One factor interest rate models and the valuation of interest rate derivative securities. *Journal of Financial and Quantitative Analysis* **28**(2), 235–254.

Jäckel, P. (2002). *Monte Carlo Methods in Finance*. Wiley Finance.

Johnson, N. L., Kotz, S., and Balakrishnan, N. (1994). *Continuous Univariate Distributions*, 2nd edn, Vol. 2. Wiley.

Kahl, C., and Jäckel, P. (2006). Fast strong approximation Monte-Carlo schemes for stochastic volatility models. *Quantitative Finance* **6**(6), 513–536.

Lord, R., Koekkoek, R., and van Dijk, D. (2008). A comparison of biased simulation schemes for stochastic volatility models. *Quantitative Finance* **10**(2), 177–194.

Revuz, D., and Yor, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd edn. Springer.

Schöbel, R., and Zhu, J. (1999). Stochastic volatility with an Ornstein–Uhlenbeck process: an extension. *European Finance Review* **4**, 23–46.

Smith, R. D. (2008). An almost exact simulation method for the Heston model. *The Journal of Computational Finance* **11**(1), 115–125.

Stein, E. M., and Stein, J. C. (1991). Stock-price distributions with stochastic volatility: an analytic approach. *Review of Financial Studies* **4**, 727–752.

van Haastrecht, A., and Pelsser, A. A. J. (2010). Efficient, almost exact simulation of the Heston stochastic volatility model. *International Journal of Theoretical and Applied Finance* **13**(1), 1–43.

van Haastrecht, A., Lord, R., Pelsser, A. A. J., and Schrager, D. (2009). Pricing long-dated insurance contracts with stochastic volatility and stochastic interest rates. *Insurance: Mathematics and Economics* **45**(3), 436–448.

Zhu, J. (2007). An extended Libor market model with nested stochastic volatility dynamics. URL: http://ssrn.com/abstract=955352.

Zhu, J. (2011). A simple and exact simulation approach to Heston model. *Journal of Derivatives* **18**(4), 26–36.