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Optimal dividends and ALM under unhedgeable risk☆

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HIGHLIGHTS

- We derive optimal investment decisions for insurance companies under unhedgeable risk.
- We study the trade off between the optimal hedge and the fully diversified portfolios.
- We show how to price unhedgeable risk.
- We derive the distribution of the time of bankruptcy.

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ABSTRACT

In this paper we develop a framework for optimal investment decisions for insurance companies in the presence of (partially) unhedgeable risk. The perspective that we choose is from an insurance company that maximises the stream of dividends paid to its shareholders. The policy instruments that the company has are the dividend policy and the investment policy. Using stochastic control theory, we derive simultaneously the optimal investment policy and the optimal dividend policy, taking the insurance risks to be given. We study the trade off between investing in the optimal hedge portfolio and the fully diversified portfolio. We show next how the pricing of unhedgeable risk can also be embedded in our framework. Finally, we derive the distribution of the time of bankruptcy and demonstrate its usefulness in calibrating the model.

1. Introduction

Insurance companies are faced with risks of many types. These include financial risks such as risks inherent in the investment process, but also non-financial risks such as the insurance claims that are at the core of the insurance operation. While most financial risks are generally assumed to be hedgeable, which means that such risks can be replicated in the financial markets, insurance claims are generally considered to be unhedgeable as no replicating portfolio exists for most "insurance events".

In this paper we aim to develop a framework for optimal investment decisions for insurance companies in the presence of (partially) unhedgeable risks. The perspective that we choose is

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from an insurance company that tries to maximise the stream of dividends paid to its shareholders. The policy instruments that the company has to this end are the dividend policy and the investment policy. The insurance company can continue to pay dividends until bankruptcy, and hence the time of bankruptcy is also endogenously controlled by the dividend and investment policies.

The problem of optimising dividends payout schemes has a long history in actuarial mathematics; see, for example, the early contributions by De Finetti (1957), Borch (1967, 1969), Bühlmann (1970) and Gerber (1972, 1979). More recently the study of the problem has received an important impulse by the application of controlled diffusion techniques; see, for example, Paulsen and Gjessing (1997) and the overview paper by Taksar (2000).

The starting points for this paper are the results obtained in the papers by Asmussen and Taksar (1997, AT hereafter), Højgaard and Taksar (1999, HT99 hereafter) and Højgaard and Taksar (2004, HT04 hereafter). Especially the results of HT99 and HT04 are quite interesting. In HT99, they analyse the case where an insurance company searches both an optimal dividend policy and an optimal level of reinsurance. In HT04, they consider the case where also the investment risk can be controlled.

This paper asks a different, yet basic question which appears to have been overlooked in the optimal dividends literature: how does the presence of partially hedgeable and partially unhedgeable risk (which is the usual case in insurance) impact the optimal investment and optimal dividend policies? To answer this question we distinguish carefully between hedgeable risks (i.e., risks that are traded on the financial markets) and unhedgeable risks (i.e., risks that are not traded on the financial markets). This distinction allows us to study the fundamental trade off between investing in the optimal hedge portfolio (reducing risk exposure) and investing in the fully diversified portfolio (increasing expected asset returns). This trade off is at the core of Asset–Liability Management (ALM).

Given our setup, we can use our results to infer what price should be charged for accepting additional unhedgeable risks such that the value of the insurance company remains unchanged. This provides a novel mechanism for the valuation of unhedgeable risks which can be viewed as the marriage of equivalent utility valuation on the one hand, and value and dividend optimisation in ruin theory on the other. We also derive the non-trivial probability distribution of the time of bankruptcy, and we illustrate how this information can be used to calibrate the model such that the implied default probabilities are consistent with observed default probabilities for insurance companies.

The outline of this paper is as follows. In Section 2 we introduce our framework. In Section 3 we derive the optimal policies and we illustrate the derived solution by means of an example. Section 4 discusses the pricing of insurance and Section 5 studies the time of bankruptcy. Section 6 analyzes the optimisation problem under general utility functions and, finally, Section 7 contains some concluding remarks.

2. Stylised insurance company

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$, which we assume to satisfy *the usual assumptions* (completed and right-continuous). This filtration represents the flow of information on which decisions are based. All Brownian motions that we consider below are defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to its filtration.

The surplus S_t of the insurance company is defined as the difference in value between assets and liabilities. The insurance company remains solvent as long as $S_t > 0$. We propose to model the surplus process by

$$dS_t = (\boldsymbol{\alpha}'\boldsymbol{\mu}_A + m)dt + \begin{pmatrix}\boldsymbol{\alpha}\\ -1\end{pmatrix}' \begin{pmatrix}\boldsymbol{\Sigma}_A & \boldsymbol{\sigma}_{AM}\\ \boldsymbol{\sigma}_{AM}' & \sigma_M^2\end{pmatrix}^{1/2} \times \begin{pmatrix}d\boldsymbol{W}_A\\ dW_M\end{pmatrix} - \sigma_I dW_I. \tag{2.1}$$

To explain this elaborate model for the surplus process we focus first on its liability component, and then on its asset component.

We assume that the liability component is driven by two sources of risk: the diffusion term $\sigma_I dW_I$ which represents the insurance risks, and the diffusion term $\sigma_M dW_M$ which represents the financial market risk component of the liabilities. Many types of insurance liabilities, for example unit-linked or participating contracts, have exposure to financial market risk. We will assume (without loss of generality) that the standard Brownian motions W_M and W_I are independent. The drift term of the surplus process contains a (positive) margin m that the insurance company has built into its liability process to cover the insurance risks and management fees. We assume that there is competition in the insurance market and that m is exogenously given and not a control variable for the insurance company. Please note that the constants m, σ_M and σ_I are absolute quantities and not "percentages".

We assume that the assets of the insurance company can only be invested in financial markets. However, the insurance company can choose from a universe of N investment categories. The $(N\times 1)$-vector $\boldsymbol{\mu}_A$ denotes the vector of expected investment returns, the $(N \times N)$-matrix $\boldsymbol{\Sigma}_A$ denotes the covariance matrix of the investment returns (which means that $\boldsymbol{\Sigma}_A$ is symmetric) and $\boldsymbol{W}_A$ is an N-dimensional standard Brownian motion. We assume that $\boldsymbol{\Sigma}_A$ is a positive definite matrix so that it is non-singular. The vector $\boldsymbol{\alpha}$ captures the exposure in absolute terms to each of the N investment categories. Please note that the constants $\boldsymbol{\mu}_A$ and $\boldsymbol{\Sigma}_A$ denote a vector of return percentages and a matrix of return variances, respectively.

In Eq. (2.1) we have stacked the $N+1$ sources of financial market risk together in an $(N + 1)$-vector, with an $(N + 1) \times (N + 1)$ covariance matrix. The $(N \times 1)$-vector $\boldsymbol{\sigma}_{AM}$ denotes the covariance of each asset category with the insurance liability portfolio. When the financial risk of the insurance liabilities is spanned by the N investment opportunities, then the vector $\boldsymbol{\sigma}_{AM}$ is collinear with the matrix $\boldsymbol{\Sigma}_A$ and as a consequence the $(N + 1) \times (N + 1)$ covariance matrix is rank-deficient. In this case it will be possible to choose a vector $\boldsymbol{\alpha}$ such that all financial risk drivers are eliminated. This is known as the *replicating portfolio*. In this case the surplus process reduces to $dS_t = mdt - \sigma_I dW_I$. This means that when the insurance company decides to invest in the replicating portfolio, the surplus process is driven by pure insurance risks only. The optimal dividend policy for this special case is investigated in the AT paper.

If the insurance company decides to deviate from the replicating portfolio then the surplus process may benefit from additional excess returns, but at the cost of increased risk. It is this risk/return trade off which is the subject of so-called ALM (Asset–Liability Management) models.

To lighten the notation for the analysis of the surplus process, we replace the $N + 2$ Brownian motions by a single diffusion term which has the same law:

$$dS_t = \left(\boldsymbol{\alpha}'\boldsymbol{\mu}_A + m\right) dt + \left(\begin{pmatrix}\boldsymbol{\alpha}\\ -1\end{pmatrix}' \begin{pmatrix}\boldsymbol{\Sigma}_A & \boldsymbol{\sigma}_{AM}\\ \boldsymbol{\sigma}_{AM}' & \sigma_M^2\end{pmatrix} \times \begin{pmatrix}\boldsymbol{\alpha}\\ -1\end{pmatrix} + \sigma_I^2\right)^{1/2} dW. \tag{2.2}$$

For a typical insurance company the surplus S is a factor 10–20 smaller than the total asset portfolio A (or the liability portfolio

L). Therefore an arithmetic specification of the surplus process S_t seems a reasonable approximation. It can moreover be viewed as the diffusion limit of the classical Cramér–Lundberg risk model (see also AT, HT99 and HT04, and the references therein), and of an analogous jump model for gains and losses on assets, with a risk premium as drift term. Considering alternative more realistic surplus processes, with e.g., asymmetric and heavy tails, means that one has to resort to numerical methods to compute the optimal policies in the (already complex) dynamic programming problem specified below.

3. Optimal policies

After HT we seek the optimal solution for the following dynamic programming problem:

$$\sup_{\boldsymbol{\alpha},D} \mathbb{E}\int_0^{\tau} e^{-ct}dD_t, \tag{3.1}$$

$$\text{s.t. } dS_t = (\boldsymbol{\alpha}'\boldsymbol{\mu}_A + m)dt + \left(\begin{pmatrix}\boldsymbol{\alpha}\\ -1\end{pmatrix}'\begin{pmatrix}\boldsymbol{\Sigma}_A & \boldsymbol{\sigma}_{AM}\\ \boldsymbol{\sigma}_{AM}' & \sigma_M^2\end{pmatrix}\times\begin{pmatrix}\boldsymbol{\alpha}\\ -1\end{pmatrix} + \sigma_I^2\right)^{1/2} dW - dD_t,$$

$$S_0 = x - D_0,$$

where D_t denotes the cumulative dividend payout process, and τ denotes the time of bankruptcy defined as $\tau := \inf\{t : S_t = 0\}$, x denotes the initial surplus of the insurance company, and c denotes the (subjective) discount rate that shareholders use to discount future dividends. In our search for the optimal dividend and optimal investment policies we restrict ourselves to cumulative dividend payout processes that are adapted to $(\mathcal{F}_t)_{t\geq 0}$, that are non-decreasing and right-continuous and satisfy $D_{0-} = 0$.

Remark 3.1. When deriving the optimal dividend and optimal investment policies it is assumed that the management of the insurance company acts in the shareholders' interests. We thus refrain from possible agency problems between shareholders and management.

Remark 3.2. In (3.1), the expected discounted dividend stream paid to the shareholders is maximised, implicitly assuming that shareholders admit a linear utility function. In particular, the risk discount rate c of the shareholders does not react to changes in riskiness of the balance sheet of the insurance company. The dynamic programming problem becomes much more complex in case one would consider non-linear utility functions; see Hubalek and Schachermayer (2004), Thonhauser and Albrecher (2007) or Grandits et al. (2007) for (partial) extensions in this direction of the simpler 1-dimensional case without investment risk control, and also Section 6 below.

Remark 3.3. As is usual in ruin models, it is assumed that bankruptcy takes place when $S_t = 0$ for the first time, even though in reality the insurance company may decide to raise external funds at (or prior to) such occasion. The decision whether or not to raise external funds would be based on a trade off between incurring high costs of external financing while realising future profits on the one hand and not incurring high costs of external financing and not realising future profits on the other hand. We refrain from making such trade off and assume that bankruptcy takes place with certainty as soon as $S_t = 0$ for the first time.

Following HT, we define a value function $V(x) := \mathbb{E}\int_0^{\tau} e^{-ct}dD_t$, which is the expected value of the discounted dividends given the initial level of surplus x. Note that from this definition it follows that $V(0) = 0$, because when $x = 0$ the insurance company immediately goes bankrupt and no dividends will ever be paid to the shareholders.

Using similar arguments as HT we find that $V(x)$ satisfies the following HJB equation:

$$\max\left\{\max_{\boldsymbol{\alpha}}\frac{1}{2}\left(\begin{pmatrix}\boldsymbol{\alpha}\\ -1\end{pmatrix}'\begin{pmatrix}\boldsymbol{\Sigma}_A & \boldsymbol{\sigma}_{AM}\\ \boldsymbol{\sigma}_{AM}' & \sigma_M^2\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\ -1\end{pmatrix} + \sigma_I^2\right)\right.$$

$$\left.\times V''(x) + (\boldsymbol{\alpha}'\boldsymbol{\mu}_A + m)V'(x) - cV(x),\ 1 - V'(x)\right\} = 0. \tag{3.2}$$

Let us start in the region where $V'(x) > 1$, so $V(x)$ must satisfy

$$\max_{\boldsymbol{\alpha}}\frac{1}{2}\left(\begin{pmatrix}\boldsymbol{\alpha}\\ -1\end{pmatrix}'\begin{pmatrix}\boldsymbol{\Sigma}_A & \boldsymbol{\sigma}_{AM}\\ \boldsymbol{\sigma}_{AM}' & \sigma_M^2\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\ -1\end{pmatrix} + \sigma_I^2\right)V''(x)$$

$$+ (\boldsymbol{\alpha}'\boldsymbol{\mu}_A + m)V'(x) - cV(x) = 0. \tag{3.3}$$

The expression on the left-hand side is maximised for

$$\boldsymbol{\alpha}^*(x) = \left(-\frac{V'(x)}{V''(x)}\right)\boldsymbol{\Sigma}_A^{-1}\boldsymbol{\mu}_A + \boldsymbol{\Sigma}_A^{-1}\boldsymbol{\sigma}_{AM}. \tag{3.4}$$

We can interpret the optimal portfolio $\boldsymbol{\alpha}^*$ as follows: the optimal portfolio consists of two parts. The term $\boldsymbol{\Sigma}_A^{-1}\boldsymbol{\sigma}_{AM}$ is the hedge portfolio that replicates as much of the (financial) liability risks as possible. Note that this term does not depend on the level of the surplus x. The term $\boldsymbol{\Sigma}_A^{-1}\boldsymbol{\mu}_A$ is the mean–variance optimal "Merton portfolio" (see Merton, 1969, 1971). The exposure to the Merton portfolio depends only on the level of the surplus x through the function $-V'(x)/V''(x)$. Hence, we find that, like in the CAPM, we get a two-fund separation solution for the optimal portfolio, and therefore the optimal choice of N assets can be reduced to a 1-dimensional problem.

The result we have found has important consequences for the ALM process of an insurance company: In a first stage, the insurance company can determine the optimal hedge portfolio $\boldsymbol{\Sigma}_A^{-1}\boldsymbol{\sigma}_{AM}$. This is a fixed portfolio that does not depend on the surplus position of the insurance company, but is determined by the nature of the liability portfolio. In a second stage, the insurance company can determine the mean–variance optimal portfolio $\boldsymbol{\Sigma}_A^{-1}\boldsymbol{\mu}_A$. This is the "speculative" portfolio that the insurance company uses to optimise its expected asset returns. The composition of this portfolio is given exogenously, only the amount invested in this portfolio depends on the surplus x. More specifically, the dependence on the surplus x is via a "risk tolerance" term $-V'(x)/V''(x)$ which is the reciprocal of the absolute risk aversion measure $-V''(x)/V'(x)$ introduced by Pratt (1964) in the context of utility functions.

The variance σ_U^2 of the unhedgeable risk consists of two components:

$$\sigma_U^2 := \left(\sigma_M^2 - \boldsymbol{\sigma}_{AM}'\boldsymbol{\Sigma}_A^{-1}\boldsymbol{\sigma}_{AM}\right) + \sigma_I^2;$$

the first term (between brackets) is the market risk of the portfolio of insurance liabilities that is not hedged by the optimal hedge portfolio while the second term is the variance σ_I^2 of the (non-traded) insurance risks. It is the variance σ_U^2 that determines the mean–variance trade off that the insurance company has to make in the ALM process.

If we define the risk tolerance $\beta(x) := -V'(x)/V''(x)$ in expression (3.4) and substitute into (3.3) we obtain

$$\frac{1}{2}\left(\beta(x)^2\sigma_A^2 + \sigma_U^2\right)V''(x)$$

$$+ \left(\beta(x)\sigma_A^2 + \mu_U\right)V'(x) - cV(x) = 0; \tag{3.5}$$

where $\sigma_A^2 := \boldsymbol{\mu}_A'\boldsymbol{\Sigma}_A^{-1}\boldsymbol{\mu}_A$,

$$\sigma_U^2 := \left(\sigma_M^2 - \boldsymbol{\sigma}_{AM}' \boldsymbol{\Sigma}_A^{-1} \boldsymbol{\sigma}_{AM}\right) + \sigma_I^2,$$

$$\mu_U := \boldsymbol{\mu}_A' \boldsymbol{\Sigma}_A^{-1} \boldsymbol{\sigma}_{AM} + m.$$

Before we proceed, we will make the additional assumption that (due to risk management or regulatory restrictions) there is an upper bound on the investment position $\boldsymbol{\alpha}^*(x)$, which is equivalent to imposing an upper bound on the risk tolerance $\beta(x)$. Hence, we assume $\beta(x) \leq M$ for an exogenously given upper bound M.

To construct a solution to (3.5), we assume there are three regions:

1. $0 < x < u_0$, where we follow a "dynamic" ALM policy with $\beta(x) < M$, and no dividends are paid out;
2. $u_0 < x < u_1$, where we follow a "maximum risk" strategy with $\beta(x) \equiv M$, and no dividends are paid out;
3. $u_1 < x$, where we pay out immediately the excess surplus $x - u_1$ as dividends to the shareholders. Such a dividend strategy is called a *barrier strategy*.

Remark 3.4 (*Using Barrier Strategy by Insurance Companies in Reality*)**.** It is our observation that insurance companies do implement such barrier strategies in reality, albeit in a modified form. In the real world, insurance companies make a distinction between "tied surplus" and "free surplus". The tied surplus is needed as a buffer capital against the risky insurance and investment positions, but the free surplus (i.e. the surplus $x > u_1$) is not immediately paid out to the shareholders, but is held "on behalf of" the shareholders to do strategic acquisitions. In a perfectly transparent world, holding the free surplus $x > u_1$ will increase the value function in exactly the same linear fashion as the value function we derive in Section 3.3.

3.1. Dynamic region $0 < x < u_0$

To construct a solution for the "dynamic" region $0 < x < u_0$, we could substitute the definition for $\beta(x)$ back into Eq. (3.5) and obtain a differential equation for $V(x)$. Unfortunately, the resulting non-linear differential equation is very difficult to solve directly. We therefore proceed along a different path and solve for the risk tolerance $\beta(x)$.

Substituting $V''(x) = -V'(x)/\beta(x)$ into (3.5) leads to

$$\frac{1}{2}\left(\sigma_A^2 \beta(x) + 2\mu_U - \frac{\sigma_U^2}{\beta(x)}\right) V'(x) - cV(x) = 0. \tag{3.6}$$

Taking the derivative with respect to x of this equation leads to

$$\frac{1}{2}\left(\sigma_A^2 + \frac{\sigma_U^2}{\beta(x)^2}\right) \beta'(x) V'(x) + \frac{1}{2}\left(\sigma_A^2 \beta(x) + 2\mu_U - \frac{\sigma_U^2}{\beta(x)}\right) V''(x) - cV'(x) = 0. \tag{3.7}$$

Substituting once more $V''(x) = -V'(x)/\beta(x)$ into (3.7) leads to

$$\frac{1}{2}\left(\sigma_A^2 + \frac{\sigma_U^2}{\beta(x)^2}\right) \beta'(x) V'(x) - \frac{1}{2}\left(\sigma_A^2 \beta(x) + 2\mu_U - \frac{\sigma_U^2}{\beta(x)}\right) \frac{V'(x)}{\beta(x)} - cV'(x) = 0. \tag{3.8}$$

As the value function V is an increasing function, we have that V' is strictly positive for all x. Hence, we are allowed to divide (3.8) by V' and we obtain a differential equation for $\beta(x)$:

$$\beta'(x) = \frac{(\sigma_A^2 + 2c)\beta(x)^2 + 2\mu_U \beta(x) - \sigma_U^2}{\sigma_A^2 \beta(x)^2 + \sigma_U^2}. \tag{3.9}$$

This is a first order ordinary differential equation (ODE) of the form

$$\frac{d\beta}{dx} = \frac{A\beta^2 + B\beta - C}{\beta^2 + C}; \tag{3.10}$$

$$\text{with } A := \frac{\sigma_A^2 + 2c}{\sigma_A^2}, \qquad B := \frac{2\mu_U}{\sigma_A^2}, \qquad C := \frac{\sigma_U^2}{\sigma_A^2}.$$

We can express the solution to (3.10) in the form:

$$\int \frac{\beta^2 + C}{A\beta^2 + B\beta - C} d\beta - C_\beta = \int dx. \tag{3.11}$$

The integral on the right-hand side of (3.11) is trivial and is equal to x. The integral on the left-hand side of (3.11) is a rational function in β which can be integrated analytically. We find the following expression for $\beta(x)$:

$$\frac{B^2 + 2A(1+A)C}{2A^2\sqrt{B^2 + 4AC}} \ln\left(\frac{2A\beta + B - \sqrt{B^2 + 4AC}}{2A\beta + B + \sqrt{B^2 + 4AC}}\right) - \frac{B}{2A^2} \ln\left(A\beta^2 + B\beta - C\right) + \frac{\beta}{A} - C_\beta = x. \tag{3.12}$$

If we set $x = 0$ in Eq. (3.6), use $V(0) = 0$ and divide by $V'(0)$, we obtain

$$\sigma_A^2 \beta(0) + 2\mu_U - \frac{\sigma_U^2}{\beta(0)} = 0. \tag{3.13}$$

If we multiply by $\beta(0)$ we obtain a quadratic equation. Selecting the positive root gives the following expression for $\beta(0)$:

$$\beta(0) = -\frac{\mu_U}{\sigma_A^2} + \sqrt{\left(\frac{\mu_U}{\sigma_A^2}\right)^2 + \left(\frac{\sigma_U}{\sigma_A}\right)^2}. \tag{3.14}$$

If we substitute this expression for $\beta(0)$ into (3.12) for $x = 0$ we can solve for C_β, but the resulting expression is omitted here for brevity.

Notice that (3.12) is the expression for the *inverse* function of $\beta(x)$. Let us denote this inverse function by $x(\beta)$. Although we do not obtain an explicit expression for $\beta(x)$, the implicit Eq. (3.12) is still quite useful: the inverse function $x(\beta)$ is strictly increasing. Hence, $\beta(x)$ itself is also strictly increasing in x. So for increasing levels of the surplus x, the optimal investment policy for the insurance company is to hold an ever increasing amount of risky assets until the maximum risk tolerance level M is reached. The surplus level u_0 is defined as the first point where the risk tolerance β reaches the maximum level M. If we substitute $\beta = M$ into (3.12) we obtain directly an analytical expression for u_0.

Let us now construct the expression for $V(x)$ on $0 < x < u_0$. The definition $\beta(x) = -V'(x)/V''(x)$ gives a differential equation for $V(x)$. If we take the reciprocal on both sides and integrate we obtain

$$C_0 - \int \frac{1}{\beta(x)} dx = \int \frac{V''(x)}{V'(x)} dx. \tag{3.15}$$

The integral on the right-hand side easily evaluates to $\ln V'(x)$. The left-hand integral is slightly more complicated since an explicit expression for $\beta(x)$ is not available. We can evaluate the integral if we perform a change of variable from dx to $d\beta$. Using the Change of Variables Theorem $dx = (dx/d\beta)d\beta = \left(1/(d\beta/dx)\right)d\beta$, we can substitute the expression for $\beta'(x)$ given in Eq. (3.11) into the left-hand side of Eq. (3.15):

$$C_0 - \int \frac{1}{\beta}\left(\frac{\beta^2 + C}{A\beta^2 + B\beta - C}\right) d\beta = \ln V'(x). \tag{3.16}$$

The left-hand side is a rational function in β that can be integrated explicitly. After taking the exponential we obtain:

$$V'(x) = e^{C_0}\beta(x)\frac{\left(\frac{2A\beta(x)+B+\sqrt{B^2+4AC}}{2A\beta(x)+B-\sqrt{B^2+4AC}}\right)^{\frac{B(A-1)}{2A\sqrt{B^2+4AC}}}}{\left(A\beta(x)^2+B\beta(x)-C\right)^{\frac{A+1}{2A}}}$$
$$= e^{C_0}\beta(x) \times \frac{\left(\frac{\left(\sigma_A^2+2c\right)\beta(x)+\mu_U+\sqrt{\mu_U^2+\left(\sigma_A^2+2c\right)\sigma_U^2}}{\left(\sigma_A^2+2c\right)\beta(x)+\mu_U-\sqrt{\mu_U^2+\left(\sigma_A^2+2c\right)\sigma_U^2}}\right)^{\frac{c\mu_U}{\left(\sigma_A^2+2c\right)\sqrt{\mu_U^2+\left(\sigma_A^2+2c\right)\sigma_U^2}}}}{\left(\frac{\left(\sigma_A^2+2c\right)\beta(x)^2+2\mu_U\beta(x)-\sigma_U^2}{\sigma_A^2}\right)^{\frac{\sigma_A^2+c}{\sigma_A^2+2c}}}. \quad (3.17)$$

If we substitute this result for $V'(x)$ into Eq. (3.6) we obtain the following expression for $V(x)$:

$$V(x) = C_0^*\left(\sigma_A^2\beta(x)^2+2\mu_U\beta(x)-\sigma_U^2\right) \times \frac{\left(\frac{\left(\sigma_A^2+2c\right)\beta(x)+\mu_U+\sqrt{\mu_U^2+\left(\sigma_A^2+2c\right)\sigma_U^2}}{\left(\sigma_A^2+2c\right)\beta(x)+\mu_U-\sqrt{\mu_U^2+\left(\sigma_A^2+2c\right)\sigma_U^2}}\right)^{\frac{c\mu_U}{\left(\sigma_A^2+2c\right)\sqrt{\mu_U^2+\left(\sigma_A^2+2c\right)\sigma_U^2}}}}{\left(\left(\sigma_A^2+2c\right)\beta(x)^2+2\mu_U\beta(x)-\sigma_U^2\right)^{\frac{\sigma_A^2+c}{\sigma_A^2+2c}}}, \quad (3.18)$$

where C_0^* denotes an arbitrary constant that will be solved later in Eq. (3.24) and with μ_U, σ_A and σ_U as defined below (3.5).

3.2. Maximum risk region $u_0 < x < u_1$

On the interval $u_0 < x < u_1$, the insurance company will follow the "maximum risk" strategy by holding the risk tolerance $\beta(x) \equiv M$. The value function on the interval $u_0 < x < u_1$, denoted by $V_1(x)$, is therefore a solution of the equation

$$\frac{1}{2}\sigma_{MM}^2V''(x)+\mu_{MM}V'(x)-cV(x)=0; \quad (3.19)$$

with $\mu_{MM} := (M\sigma_A^2+\mu_U), \qquad \sigma_{MM}^2 := \left(M^2\sigma_A^2+\sigma_U^2\right).$

The solution to this second order ODE is given by

$$V_1(x) = C_1e^{\gamma_{M+}(x-u_0)}+C_2e^{\gamma_{M-}(x-u_0)}; \quad (3.20)$$

with $\gamma_{M\pm} := \frac{1}{\sigma_{MM}^2}\left(-\mu_{MM} \pm \sqrt{\mu_{MM}^2+2c\sigma_{MM}^2}\right).$

At the point $x = u_0$, we know $\beta(u_0) = M = -V'(u_0)/V''(u_0)$. We also know that $V(u_0) = V_1(u_0)$. These two pieces of information are sufficient to determine C_1 and C_2:

$$\begin{cases} V(u_0) = C_1 + C_2 \\ 0 = C_1\left(\gamma_{M+}+M\gamma_{M+}^2\right)+C_2\left(\gamma_{M-}+M\gamma_{M-}^2\right). \end{cases} \quad (3.21)$$

The function $V_1(x)$ is therefore given by

$$V_1(x) = V(u_0)\frac{\left(\gamma_{M+}+M\gamma_{M+}^2\right)e^{\gamma_{M-}(x-u_0)}-\left(\gamma_{M-}+M\gamma_{M-}^2\right)e^{\gamma_{M+}(x-u_0)}}{(\gamma_{M+}-\gamma_{M-})+M\left(\gamma_{M+}^2-\gamma_{M-}^2\right)}. \quad (3.22)$$

Please note that at this point the function value $V(u_0)$ still depends on the (unresolved) constant C_0^*. This final constant will be determined in the next section.

3.3. Dividend region $u_1 < x$

We can now solve for the upper limit u_1. From Eq. (3.2) it follows that for $x > u_1$, the insurer immediately pays out the excess surplus $x - u_1$ as dividends. Hence, the function $V(x)$ satisfies the equation $V'(x) = 1$. The solution is given by $V_2(x) = C_3 + x$. The point u_1 is the point where the function $V_1(x)$ makes a "smooth" contact with the function V_2. This means that the first and second derivatives should match at the point u_1.

Since $V_2(x)$ is a straight line, its second derivative is 0. We can solve for u_1 from the equation $V_1''(u_1) = 0$. This leads to:

$$u_1 = u_0+\frac{1}{\gamma_{M+}-\gamma_{M-}}\ln\left(\frac{\left(\gamma_{M+}+M\gamma_{M+}^2\right)\gamma_{M-}^2}{\left(\gamma_{M-}+M\gamma_{M-}^2\right)\gamma_{M+}^2}\right). \quad (3.23)$$

Given this value for u_1, we can now solve for the remaining constant C_0^* in $V(u_0)$ from the condition $V_1'(u_1) = 1$. This leads to

$$V(u_0) = \frac{(\gamma_{M+}-\gamma_{M-})+M\left(\gamma_{M+}^2-\gamma_{M-}^2\right)}{\left(\gamma_{M+}+M\gamma_{M+}^2\right)\gamma_{M-}e^{\gamma_{M-}(u_1-u_0)}-\left(\gamma_{M-}+M\gamma_{M-}^2\right)\gamma_{M+}e^{\gamma_{M+}(u_1-u_0)}}. \quad (3.24)$$

Finally, we solve for C_3 from $V_1(u_1) - u_1 = C_3$.

3.4. Example

Let us illustrate the derived solution with an example. Although our derivation so far has been fully N-dimensional, we see from Eq. (3.5) that the problem of determining the value function $V(x)$ is essentially a one-dimensional problem. Hence, for ease of exposition, we will use a one-dimensional setup for our numerical illustration. We therefore set $N = 1$, and take $\Sigma_A =: s_A^2$ (and hence $\sigma_{AM} = s_A\sigma_M$). The parameter specification is set as in Table 1. Fig. 1 displays the optimal investment in the risky asset in the "speculative" portfolio $\beta(x)\mu_A/s_A^2$ as a function of the initial surplus x.

Table 1
Parameter specification.

Parameter	Value
σ_M	2.47 G€
σ_I	4.57 G€
s_A	15.0%
m	1.26 G€
μ_A	3.00%
M	15 G€
c	5.00%

This table reports the parameter values chosen when generating Figs. 1 and 2. The values are adapted from the annual report of a large European insurance company. The "absolute" quantities are obtained by multiplying returns and volatilities with the asset and liability values reported by the insurance company. All "absolute" quantities are reported in billions of euro (G€).

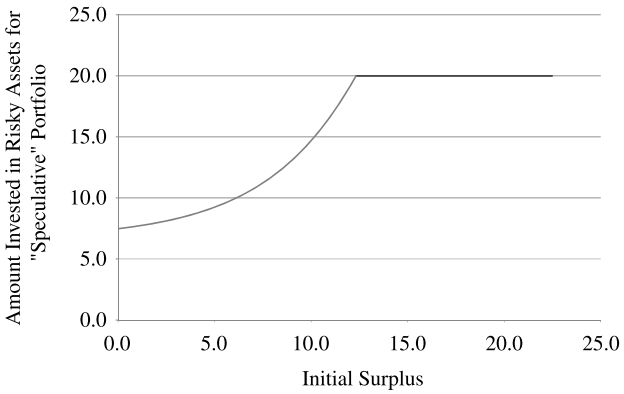

Fig. 1. Optimal investment in "speculative" portfolio as a function of the initial surplus. This figure displays the optimal investment in the "speculative" portfolio $\beta(x)\mu_A/s_A^2$ derived in Section 3 as a function of the initial surplus x on the interval $0 < x < u_0$ and $u_0 < x < u_1$. Recall that when $x > u_0$ a "maximum risk" strategy $M\mu_A/s_A^2$ is adopted. The units of the initial surplus and optimal investments are billion Euros. The parameters used to generate this graph are reported in Table 1.

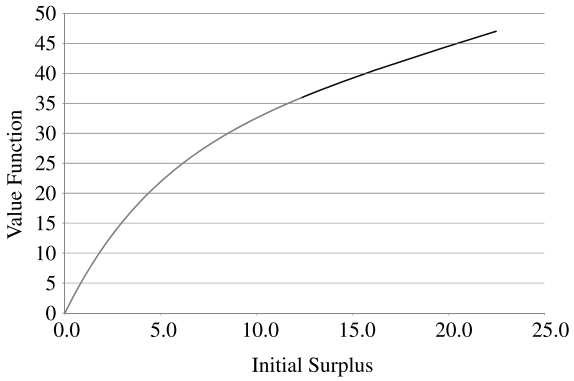

Fig. 2. Value function ($V(x)$). This figure displays the value function derived in Section 3 as a function of the initial surplus x. The grey part corresponds to $V(x)$ on the interval $0 < x < u_0$ while the black part corresponds to $V_1(x)$ on the interval $u_0 < x < u_1$. Recall that on the interval $x > u_1$, $V_2(x)$ is a straight line. The units of the initial surplus and value function are billion Euros. The parameters used to generate this graph are reported in Table 1.

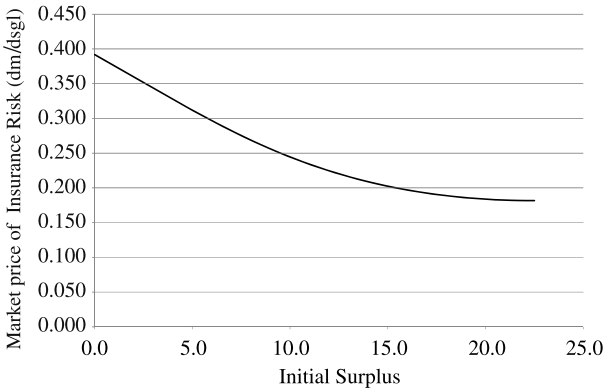

Fig. 3. Market price of insurance risk $dm/d\sigma_I$. This figure illustrates the sensitivity of the market price of insurance risk $dm/d\sigma_I$; see Section 4. The units of the initial surplus are billion Euros. The parameters used to generate this graph are reported in Table 1.

Next, Fig. 2 displays the expected value of the discounted dividends under the optimal investment and optimal dividend policy as a function of the initial surplus, i.e., the value function. For our parameters, we find the following values $u_0 = 12.3$ and $u_1 = 22.5$ (rounded to three significant digits).

4. The pricing of insurance

The value function turns out to be decreasing in σ_I. The sensitivity of the value function with respect to σ_I can be interpreted as the *marginal price of insurance risk*. It provides a novel mechanism for the valuation of unhedgeable risks which can be viewed as the marriage of equivalent utility valuation on the one hand, and value and dividend optimisation in ruin theory on the other. The marginal price of insurance risk is such that the shareholders are indifferent between bearing an additional unit of insurance risk (as measured by σ_I) while receiving an immediate dividend payout equal to the marginal price of insurance risk, and not bearing the additional unit of insurance risk.

Alternatively (but essentially equivalently), to price insurance risk one may determine the increase of the margin m that offsets the decrease of the value function when an additional unit of insurance risk is borne. Let us denote this quantity by $dm/d\sigma_I$. Suppose we consider the value function $V(x; m, \sigma_I)$ as a function of the parameters m and σ_I. Then if we insist that the "total derivative" $dV = 0$, we obtain (with slight abuse of notation)

$$\frac{\partial V}{\partial \sigma_I} + \frac{\partial V}{\partial m}\frac{dm}{d\sigma_I} = 0$$

and from this it follows that we can express $dm/d\sigma_I$ as

$$\frac{dm}{d\sigma_I} = -\frac{\partial V}{\partial \sigma_I}\Big/\frac{\partial V}{\partial m}. \tag{4.1}$$

Both derivatives in the numerator and the denominator can be calculated analytically given the analytical expressions for $V(x)$ we have derived.

The quantity $dm/d\sigma_I$ can be interpreted as the increase in the margin m that the shareholders require to accept one additional unit of insurance risk σ_I. This quantity could also be interpreted as the "market (or shareholders) price of insurance risk". It is the compensating rate for which shareholders, under the optimal investment and dividend policies maximising shareholders' value, are indifferent between accepting and not accepting to bear additional insurance risk.

Let us consider again the example of Section 3.4. Fig. 3 displays the market price of insurance risk $dm/d\sigma_I$ for different values of initial surplus x. Note that the market price of insurance risk is not a constant in our model, but depends on the initial surplus x. For low levels of initial surplus, an increase in insurance risk σ_I leads to a relatively large increase in the probability of bankruptcy. Therefore, the shareholders require a higher compensation at low levels of surplus, compared to higher levels of surplus.

5. The time of bankruptcy

In this section we study the distribution of the time of bankruptcy τ. We denote by $\varphi(\lambda, x) = \mathbb{E}[e^{-\lambda\tau}]$, $\lambda > 0$, the *Laplace transform* of (the distribution function of) τ.

Because the optimal dividend policy is a barrier strategy, the (modified) surplus process x is a Brownian motion with a *reflecting* barrier (at the level u_1 where the excess surplus is paid out as dividends to the shareholders) and an *absorbing* barrier at the level 0 (at which bankruptcy takes place).

In the region $0 < x < u_0$ the surplus process x follows the stochastic differential equation $dx = \left(\beta(x)\sigma_A^2 + \mu_U\right)dt + \sqrt{\beta(x)^2\sigma_A^2 + \sigma_U^2}\,dW$, therefore, the Laplace transform $\varphi(\lambda, x)$ is a solution of the ODE

$$\frac{1}{2}\left(\beta(x)^2\sigma_A^2 + \sigma_U^2\right)\varphi''(x) + \left(\beta(x)\sigma_A^2 + \mu_U\right)\varphi'(x) - \lambda\varphi(x) = 0 \tag{5.1}$$

in the region $0 < x < u_0$.

In the region $u_0 < x < u_1$ the surplus process x follows the stochastic differential equation $dx = \left(M\sigma_A^2 + \mu_U\right)dt + \sqrt{M^2\sigma_A^2 + \sigma_U^2}\,dW$, and the function $\varphi(\lambda, x)$ is a solution of the ODE

$$\frac{1}{2}\left(M^2\sigma_A^2 + \sigma_U^2\right)\varphi''(x) + \left(M\sigma_A^2 + \mu_U\right)\varphi'(x) - \lambda\varphi(x) = 0 \tag{5.2}$$

in the region $u_0 < x < u_1$.

Due to the reflecting boundary at u_1 and the absorbing boundary at 0, the Laplace transform $\varphi(\lambda, x)$ satisfies the following boundary conditions (see, for example, Cox and Miller, 1965, Chapter 5.7):

$$\varphi(\lambda, 0) = 1, \tag{5.3}$$

$$\frac{d}{dx}\varphi(\lambda, u_1) = 0. \tag{5.4}$$

5.1. Solution for φ in the region $0 < x < u_0$

Due to the complexity of the function $\beta(x)$, we cannot provide a fully analytical solution to (5.1). However, since we are interested in finding an expression for $\mathbb{E}[\tau] = -\frac{d}{d\lambda}\varphi(\lambda = 0, x)$ we only need to find an expression for $\frac{d}{d\lambda}\varphi(\lambda = 0, x)$.

If we take the derivative with respect to λ in (5.1) we find:

$$\frac{1}{2}\big(\beta(x)^2\sigma_A^2 + \sigma_U^2\big)\varphi_\lambda''(x) + \big(\beta(x)\sigma_A^2 + \mu_U\big)\varphi_\lambda'(x) - \varphi(x) - \lambda\varphi_\lambda(x) = 0, \tag{5.5}$$

where $\varphi_\lambda(x)$ is shorthand notation for $\frac{d}{d\lambda}\varphi(\lambda, x)$. If we now evaluate (5.5) at our point of interest $\lambda = 0$, then the ODE simplifies to

$$\frac{1}{2}\big(\beta(x)^2\sigma_A^2 + \sigma_U^2\big)\varphi_\lambda''(x) + \big(\beta(x)\sigma_A^2 + \mu_U\big)\varphi_\lambda'(x) = \varphi(0, x) = 1. \tag{5.6}$$

The substitution $\varphi(0, x) = 1$ follows immediately from $\varphi(0, x) = \mathbb{E}[e^{-0\tau}] = 1$. The ODE we have now obtained is a non-homogeneous first-order ODE, which we can solve.

Let us proceed to construct the solution. First, we make the substitution $\xi(x) := \varphi_\lambda'(x)$. We then obtain the equation

$$\frac{1}{2}\big(\beta(x)^2\sigma_A^2 + \sigma_U^2\big)\xi'(x) + \big(\beta(x)\sigma_A^2 + \mu_U\big)\xi(x) = 1. \tag{5.7}$$

We can now do a change of variables from x to $\beta(x)$ since $\beta(x)$ is monotonically increasing in x. This leads to

$$\frac{1}{2}\big(\beta(x)^2\sigma_A^2 + \sigma_U^2\big)\xi'(\beta(x))\beta'(x) + \big(\beta(x)\sigma_A^2 + \mu_U\big)\xi(\beta(x)) = 1. \tag{5.8}$$

If we substitute the expression for $\beta'(x)$ given in (3.9), then (5.8) simplifies to

$$\frac{1}{2}\big((\sigma_A^2 + 2c)\beta^2 + 2\mu_U\beta - \sigma_U^2\big)\xi'(\beta) + \big(\beta\sigma_A^2 + \mu_U\big)\xi(\beta) = 1, \tag{5.9}$$

which is a first-order ODE that depends on β only.

The next step is to find a solution for the homogeneous equation

$$\frac{1}{2}\big((\sigma_A^2 + 2c)\beta^2 + 2\mu_U\beta - \sigma_U^2\big)\xi'(\beta) + \big(\beta\sigma_A^2 + \mu_U\big)\xi(\beta) = 0. \tag{5.10}$$

The solution to the homogeneous equation (5.10) can be represented as

$$\xi_H(\beta) = \exp\left(-\int \frac{\beta\sigma_A^2 + \mu_U}{\frac{1}{2}\big((\sigma_A^2 + 2c)\beta^2 + 2\mu_U\beta - \sigma_U^2\big)}d\beta\right). \tag{5.11}$$

This expression can be evaluated as

$$\xi_H(\beta) = \frac{\left(\frac{(\sigma_A^2+2c)\beta+\mu_U+\sqrt{\mu_U^2+(\sigma_A^2+2c)\sigma_U^2}}{(\sigma_A^2+2c)\beta+\mu_U-\sqrt{\mu_U^2+(\sigma_A^2+2c)\sigma_U^2}}\right)^{\frac{2c\mu_U}{(\sigma_A^2+2c)\sqrt{\mu_U^2+(\sigma_A^2+2c)\sigma_U^2}}}}{\big((\sigma_A^2 + 2c)\beta^2 + 2\mu_U\beta - \sigma_U^2\big)^{\frac{\sigma_A^2}{\sigma_A^2+2c}}}. \tag{5.12}$$

The solution to the non-homogeneous equation (5.8) can be found by "variation of constants". If we use the *ansatz* $\xi(\beta) = C_\xi(\beta)\xi_H(\beta)$, and substitute this ansatz into (5.8), we obtain the following differential equation for C_ξ:

$$\frac{1}{2}\big((\sigma_A^2 + 2c)\beta^2 + 2\mu_U\beta - \sigma_U^2\big)C_\xi'(\beta)\xi_H(\beta) = 1. \tag{5.13}$$

Hence, $C_\xi(\beta)$ can be represented in integral form as

$$C_\xi(\beta) = \int \frac{1}{\frac{1}{2}\big((\sigma_A^2 + 2c)\beta^2 + 2\mu_U\beta - \sigma_U^2\big)\xi_H(\beta)}d\beta. \tag{5.14}$$

Summarising, we can represent the solution to (5.6) as

$$\varphi_\lambda'(x) = \left(C_\varphi + \int \frac{1}{\frac{1}{2}\Big((\sigma_A^2 + 2c)\beta(x)^2 + 2\mu_U\beta(x) - \sigma_U^2\Big)\xi_H(\beta(x))}d\beta(x)\right) \times \xi_H(\beta(x)), \tag{5.15}$$

where C_φ denotes an arbitrary integration constant.

Finally, the function $\varphi_\lambda(x)$ can be found by integrating the previous expression

$$\begin{aligned}\varphi_\lambda(x) &= \int_0^x \varphi_\lambda'(y)dy \\ &= \int_0^x \varphi_\lambda'\big(\beta(y)\big)\frac{1}{\beta'(y)}d\beta(y) \\ &= \int_{\beta(0)}^{\beta(x)} \varphi_\lambda'(\beta)\left(\frac{\sigma_A^2\beta^2 + \sigma_U^2}{(\sigma_A^2 + 2c)\beta^2 + 2\mu_U\beta - \sigma_U^2}\right)d\beta.\end{aligned} \tag{5.16}$$

Note that we have chosen the boundaries of the integration in such a way that the boundary condition (5.3) at $\varphi_\lambda(0) = 0$ is already satisfied. Unfortunately, we cannot simplify the integral in (5.16) any further, and we have to resort to numerical integration. We can however still track the dependence on the constant C_φ by rewriting (5.16) as the sum of two terms:

$$\varphi_\lambda(x) = \Phi_0(x) + C_\varphi\Phi_1(x) \tag{5.17}$$

with

$$\begin{aligned}\Phi_0(x) &:= \int_{\beta(0)}^{\beta(x)}\left(\int_{\beta(0)}^{\beta}\frac{d\gamma}{\frac{1}{2}\big((\sigma_A^2 + 2c)\gamma^2 + 2\mu_U\gamma - \sigma_U^2\big)\xi_H(\gamma)}\right) \times \xi_H(\beta)\frac{\sigma_A^2\beta^2 + \sigma_U^2}{(\sigma_A^2 + 2c)\beta^2 + 2\mu_U\beta - \sigma_U^2}\,d\beta \\ \Phi_1(x) &:= \int_{\beta(0)}^{\beta(x)}\xi_H(\beta)\frac{\sigma_A^2\beta^2 + \sigma_U^2}{(\sigma_A^2 + 2c)\beta^2 + 2\mu_U\beta - \sigma_U^2}\,d\beta.\end{aligned}$$

The constant C_φ will be resolved in the next section when we will solve for the upper boundary condition (5.4).

5.2. Solution for φ in the region $u_0 < x < u_1$

In the region $u_0 < x < u_1$ the function $\varphi_1(\lambda, x)$ is given by the ODE (5.2). As noted in the previous section, we are ultimately interested in $\varphi_{\lambda 1}(x) := \frac{d}{d\lambda}\varphi_1(0, x)$. Using a similar argument as in the previous section, we find the $\varphi_{\lambda 1}(x)$ satisfies the ODE

$$\frac{1}{2}\sigma_{MM}^2\varphi_{\lambda 1}''(x) + \mu_{MM}\varphi_{\lambda 1}'(x) = 1, \tag{5.18}$$

with σ_{MM}^2 and μ_{MM} as defined in (3.19). The solution to this ODE is

$$\varphi_{\lambda 1}(x) = G_1 + \frac{x - u_0}{\mu_{MM}} + G_2\exp\left(-\frac{\mu_{MM}}{\frac{1}{2}\sigma_{MM}^2}(x - u_0)\right). \tag{5.19}$$

At the point $x = u_0$, we know $\beta(u_0) = M$. Furthermore, at $x = u_0$, the function φ must satisfy a "smooth pasting" condition. This implies that $\varphi_\lambda(u_0) = \varphi_{\lambda 1}(u_0)$ and that $\varphi_\lambda'(u_0) = \varphi_{\lambda 1}'(u_0)$. Furthermore, we have the upper boundary condition $\varphi_{\lambda 1}'(u_1) = 0$. These

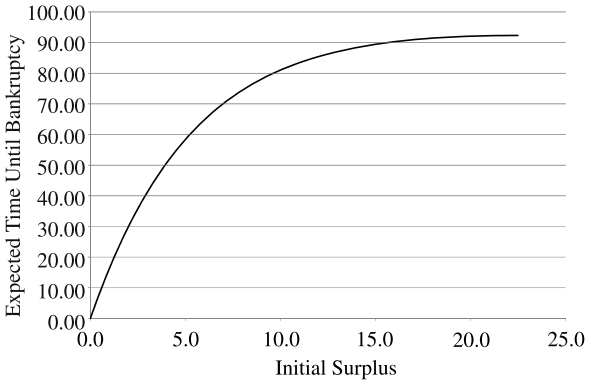

Fig. 4. Expected time until bankruptcy as a function of the initial surplus. This figure displays the expected time until bankruptcy τ derived in Section 5 as a function of the initial surplus x. The units of the initial surplus are billion Euros, the time is measured in years. The parameters used to generate this graph are reported in Table 1.

three pieces of information are sufficient to determine the integration constants G_1, G_2 and C_φ:

$$\begin{cases} \Phi_0(u_0) + C_\varphi \Phi_1(u_0) = G_1 + G_2, \\ \left(C_\varphi + \int_{\beta(0)}^{M} \frac{d\beta}{\frac{1}{2}\left((\sigma_A^2 + 2c)\beta^2 + 2\mu_U\beta - \sigma_U^2\right)\xi_H(\beta)} \right) \\ \quad \times \xi_H(M) = \frac{1}{\mu_{MM}} - G_2 \frac{\mu_{MM}}{\frac{1}{2}\sigma_{MM}^2}. \\ \frac{1}{\mu_{MM}} - G_2 \frac{\mu_{MM}}{\frac{1}{2}\sigma_{MM}^2} \exp\left(-\frac{\mu_{MM}}{\frac{1}{2}\sigma_{MM}^2}(u_1 - u_0) \right) = 0. \end{cases} \tag{5.20}$$

These are three linear equations that are straightforward to solve, and the resulting expressions are omitted here for brevity.

5.3. Results for τ

The expectation of the time until bankruptcy τ for a given surplus x can be calculated as $\mathbb{E}[\tau|x] = -\varphi_\lambda(x)$, where we can use the closed-form expressions we have found for both the intervals $0 < x < u_0$ and $u_0 < x < u_1$.

The expected time until bankruptcy can be useful in calibrating the model. In particular, it may be used to infer the dividend discount rate c. Suppose that the insurance company aims for a given expected survival time (which is closely related to the probability of bankruptcy over a 1-year horizon). Then, using the expression for $\mathbb{E}[\tau]$ (which implicitly depends on c), one can solve for the value of c for which, under the corresponding optimal investment and optimal dividend policy, the survival time aimed at is achieved.

Let us consider again the example of Section 3.4. Fig. 4 displays the expected time of bankruptcy under the optimal investment and optimal dividend policy as a function of the initial surplus x. With a dividend discount rate of 5.00% and an initial surplus of 17 billion Euros, the expected time of bankruptcy is 93.7 years. With a dividend discount rate of 4.00% (3.50%) and the same initial surplus of 17 billion Euros, the expected time of bankruptcy would be 153 (206) years. Under Solvency II, the European Commission proposes a Solvency Capital Requirement which is such that the annual probability of insurer bankruptcy (i.e., the event that the value of liabilities exceeds the value of assets) is $5 \cdot 10^{-3}$, which corresponds, under serial independence, to an expected time of bankruptcy of 200 years.

6. General utility functions

In this section, we study the optimal investment policy and the optimal dividend policy under a general utility specification. That is, we replace the linear utility function implicitly assumed in our dynamic programming problem (3.1) by a general utility function $U : \mathbb{R}_+ \to \mathbb{R}_+$ and maximise the expected value of the discounted utility of dividend payments until ruin; see also Remark 3.2.

We start by making fairly general assumptions on U. In particular, we assume that U is non-decreasing and furthermore that U satisfies the (selection of) Inada conditions that U is concave, $U(0) = 0$ and $\lim_{x\uparrow+\infty} U'(x) = 0$. Special cases of interest, satisfying our assumptions, are that of (the bounded from below version of) power utility:

$$U(x) = \frac{x^\xi}{\xi}, \quad \xi \in (0, 1),$$

and exponential utility:

$$U(x) = 1 - e^{-x/\theta}, \quad \theta > 0.$$

Under these assumptions on U we may suppose without loss of generality that the dividend process D is absolutely continuous with respect to the Lebesgue measure, because a singular part of D does not contribute to the objective function. We therefore suppose that the process $(D_t)_{t\geq 0}$ admits a density process $(d_t)_{t\geq 0}$:

$$D_t = \int_0^t d_s ds, \quad \text{a.s., } t \geq 0.$$

Using similar arguments as in HT (who restrict attention to linear utility functions but whose arguments can be adapted to apply also in this general setting) we then find that, on the "dynamic" region where the upper bound on the investment position is strictly satisfied, the value function $V(x) \equiv \mathbb{E}\int_0^\tau e^{-ct}U(d_t)dt$ satisfies the HJB equation

$$\sup_{\boldsymbol{\alpha},d} \left(\frac{1}{2} \left(\begin{pmatrix} \boldsymbol{\alpha} \\ -1 \end{pmatrix}' \begin{pmatrix} \boldsymbol{\Sigma}_A & \boldsymbol{\sigma}_{AM} \\ \boldsymbol{\sigma}_{AM}' & \sigma_M^2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ -1 \end{pmatrix} + \sigma_I^2 \right) V''(x) \right.$$
$$\left. + (\boldsymbol{\alpha}'\boldsymbol{\mu}_A + m - d)V'(x) - cV(x) + U(d) \right) = 0. \tag{6.1}$$

The supremum is attained at

$$d^*(x) = (U')^{-1}(V'(x)); \tag{6.2}$$

$$\boldsymbol{\alpha}^*(x) = \left(-\frac{V'(x)}{V''(x)} \right) (\boldsymbol{\Sigma}_A)^{-1} \boldsymbol{\mu}_A + (\boldsymbol{\Sigma}_A)^{-1} \boldsymbol{\sigma}_{AM}. \tag{6.3}$$

Substituting the solutions (6.2) and (6.3) back into Eq. (6.1) yields a non-linear ODE for $V(x)$ that has to be solved numerically. Also, the corresponding ODE for $\beta(x) = -\frac{V'(x)}{V''(x)}$ (or its inverse) can no longer be solved analytically (as opposed to the linear utility case).

Nevertheless, (6.2) and (6.3) allow us to characterise the full structure of the solution. Similar to the linear utility case, on the active region, the optimal investment portfolio consists of two parts: the optimal hedge portfolio and the mean–variance optimal "Merton portfolio" (two-fund separation). Again, a two-stage procedure applies, as in Section 3: In a first stage, the insurer can determine analytically the optimal hedge portfolio, ignoring the surplus position of the company. In a second stage the insurer then determines the mean–variance optimal portfolio, the composition of which can be calculated analytically while the exposure to which should now be determined numerically.

Notice that the optimal dividend strategy is no longer a barrier strategy. In the power utility case it converges to a barrier strategy when $\xi \uparrow 1$ ($\xi = 1$: linear utility case). We thus find that risk-averse shareholders prefer some "early" dividend payouts rather than increasing the surplus of the company at the maximum rate which would allow for more aggressive risk-taking.

7. Conclusions

In this paper we have developed results for the optimal dividend payout and investment decisions for insurance companies in the presence of (partially) unhedgeable risks. While the search for optimal dividend and optimal investment strategies has been quite an active area of research in recent years, we ask in our paper a different, yet important, question: how does the presence of partially hedgeable and unhedgeable risk – which is the usual case in insurance – affect the optimal dividend and investment policies?

Our results can be summarised as follows. First, we are able to characterise analytically the optimal dividend strategy, which under linear preferences is a barrier strategy. It is our observation that such behaviour is consistent with the way in which insurance companies make a distinction between "tied surplus" and "free surplus".

Second, we are able to give an analytical characterisation of the optimal investment strategy that the insurance company should follow in a general N-asset setting. We find that the optimal strategy consists of two parts. First, we identify a hedge portfolio that replicates as much as possible of the liability risks with traded assets. This hedge portfolio does not depend on the level of surplus. Second, we identify a mean–variance optimal Merton portfolio. The exposure to this portfolio depends in a non-linear fashion on the surplus of the insurance company. These results have direct and important implications for the optimal ALM strategies that insurance companies should follow.

Next, we have presented a novel mechanism for the valuation of unhedgeable risks. Our results allow for an analytical treatment of this valuation problem. Finally, we also derive the probability distribution of the time of bankruptcy in closed form, and we illustrate how this information can be used to calibrate our model such that the implied default probabilities are consistent with observed default probabilities.

To be able to derive these analytical results, we have to make some rather strong assumptions on the utility function of the shareholders and the price processes that the assets and liabilities follow. We have partially explored what the consequences are of relaxing these assumptions, and we believe this is also an interesting direction for further research.

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