

# Generic Pricing of FX, inflation and stock options under stochastic interest rates and stochastic volatility

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# Generic pricing of FX, inflation and stock options under stochastic interest rates and stochastic volatility

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We consider the pricing of FX, inflation and stock options under stochastic interest rates and stochastic volatility, for which we use a generic multi-currency framework. We allow for a general correlation structure between the drivers of the volatility, the inflation index, the domestic (nominal) and the foreign (real) rates. Having the flexibility to correlate the underlying FX/inflation/stock index with both stochastic volatility and stochastic interest rates yields a realistic model that is of practical importance for the pricing and hedging of options with a long-term exposure. We derive explicit valuation formulas for various securities, such as vanilla call/put options, forward starting options, inflation-indexed swaps and inflation caps/floors. These vanilla derivatives can be valued in closed form under Schöbel and Zhu [*Eur. Finance Rev.*, 1999, 4, 23–46] stochastic volatility, whereas we devise an (Monte Carlo) approximation in the form of a very effective control variate for the general Heston [*Rev. Financial Stud.*, 1993, 6, 327–343] model. Finally, we investigate the quality of this approximation numerically and consider a calibration example to FX and inflation market data.

*Keywords:* Foreign Exchange; Inflation; Equity; Stochastic volatility; Stochastic interest rates; Hybrids

## 1. Introduction

Markets for long maturity and hybrid derivatives are developing continuously. Not only are increasingly exotic structures being created, but also the markets for plain vanilla derivatives are growing. One recent advance is the development of long maturity option markets across various asset classes. During the last few years, long maturity securities, such as Target Auto Redemption Notes (TARN) equity-interest rate options (see, e.g., Caps 2007), Power-Reverse Dual-Currency (PRDC) Foreign Exchange (FX) swaps (see, e.g., Piterbarg 2005) and inflation-indexed Limited Price Indices (LPI) structures (see, e.g., Mercurio 2005 and Mercurio and Moreni 2006a), have become increasingly popular. Whereas for FX,

inflation and hybrid structures, which explicitly depend on the evolution of future interest rates, it is immediately clear that the use of stochastic interest rates is crucial in a derivative pricing model, the addition of stochastic rates is also important for the pricing and, particularly, the hedging of long maturity equity derivatives (see, e.g., Bakshi *et al.* 2000). First, the option's rho, which measures/hedges the interest rate risk of the derivative, is increasing with time to maturity. Secondly, stochastic interest rates are important for exotic option pricing since the numeraire is the discount bond associated with the maturity of the option. Because long-term interest rates are, to a reasonable degree, correlated with FX/inflation/equity indices, the rates directly influence the pricing kernel used in exotic option pricing.

Most investment banks have now standardized a three-factor modeling framework to price cross-currency

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(i.e. FX and inflation) options (Sippel and Ohkoshi 2002, Jarrow and Yildirim 2003, Piterbarg 2005), where the index follows a log-normal process and the interest rates of the two currencies are driven by one-factor Gaussian (see, e.g., Hull and White 1993) models. The choice of Gaussian assumptions for the interest rates and log-normality for the index has allowed for a very efficient, essentially closed-form, calibration to at-the-money options on the index, i.e. on the FX rate or stock price. The assumption of log-normality for an index, although technically very convenient, does not find justification in financial equity markets (see, e.g., Bakshi *et al.* 1997), FX markets (see, e.g., Piterbarg 2005 or Caps 2007) or in inflation markets (see, e.g., Mercurio and Moreni 2006a, Kenyon 2008 or Mercurio and Moreni 2009). In fact, the markets for these products exhibit a strong volatility skew or smile, implying log index returns deviating from normality and suggesting the use of skewed and heavier tailed distributions. Moreover, many multi-currency structures (such as LPIs or PRDCs) are particularly sensitive to volatility skews/smiles as they often incorporate multiple strikes as well as callable/knockout components. Hence, appropriate exotic option pricing models, which need to quantify the volatility exposure in such structures, should at least be able to incorporate the smiles/skews in vanilla markets. While various methods exist to incorporate volatility smiles (i.e. local volatility, stochastic volatility and/or jumps), the calibration of such models is by no means trivial. A skew mechanism is normally applied to the forward index price (i.e. the FX rate, CPI/Equity index), but to price multi-currency options, a term structure involving various time points of the forward index is also required. The incorporation of stochastic interest rates makes the connection between the two particularly non-trivial (see, e.g., Piterbarg 2005 or Antonov *et al.* 2008). Although the issue is important, Piterbarg (2005) even dubs it “perhaps even the most important current outstanding problems for quantitative research departments worldwide”, there is remarkably little literature available on the subject, even though the problem has attracted the attention of both practitioners and academia (see, e.g., van der Ploeg 2007).

A few approaches have recently been suggested. A local volatility approach is used by Piterbarg (2005), who derives approximating formulas for calibration. Andreasen (2006) combines Heston (1993) stochastic volatility with independent stochastic interest rate drivers and derives closed-form Fourier expressions for vanilla options. To correlate the independent rate drivers with the FX rate, Andreasen (2006) uses an indirect approach in the form of a volatility displacement parameter, which has some disadvantages as it can lead to extreme model parameters (see, e.g., Antonov *et al.* 2008). The latter framework is generalized by Kainth and Saravanamuttu (2007), who consider the pricing of double no-touch options in a model with stochastic correlation and double

Heston dynamics for the stochastic volatility. The calibration of FX option stochastic interest rate Heston (1993) stochastic volatility under a full correlation structure is undertaken by Antonov *et al.* (2008), who use Markovian projection to derive approximation formulas. Although their projection technique is elegant, the quality of their approximation deteriorates for larger maturities or more extreme model parameters. The exact pricing of FX options under Schöbel and Zhu (1999) stochastic volatility, single-factor Gaussian rates and a full correlation structure was recently considered by van Haastrecht *et al.* (2008). The modeling of the inflation smile has been considered by several authors (see, e.g., Belgrade *et al.* 2004, Kenyon 2008 and Mercurio and Moreni 2009).

In this paper, building on the results of Piterbarg (2005), Andreasen (2006), Antonov *et al.* (2008) and van Haastrecht *et al.* (2008), we consider the pricing of foreign exchange, inflation and stock options under Heston (1993) and Schöbel and Zhu (1999) stochastic volatility and under multi-factor Gaussian interest rates with a full correlation structure. Since stock and FX options are a special (nested) cases of inflation-indexed caps/floors,<sup>†</sup> we will mainly focus on the pricing of inflation index derivatives. Hence, the stock and FX model option pricing formulas follow directly from our generalization of the foreign exchange inflation framework of Jarrow and Yildirim (2003). The paper is organized as follows. Section 2 introduces our new model. Section 3 considers the basic vanilla derivatives and section 3.1 considers the pricing methodology. In section 4 we derive the characteristic functions (CF) required for the Fourier-based pricing methods. Under Schöbel and Zhu (1999) stochastic volatility we can derive the CF of our model in closed form, but under Heston (1993) stochastic volatility it is extremely challenging to derive the CF of the general model in closed form, nonetheless we demonstrate how the CF of the special (uncorrelated) case can be used as a simple and efficient control variate for the general model. Finally, section 6 concludes.

## 2. The model

Before introducing the general model, we first consider the Jarrow and Yildirim (2003) model, which can be seen as a special (degenerate) case of our model. The Jarrow and Yildirim (2003) framework for modeling inflation and real rates is based on a foreign-exchange analogy between the real and nominal economy. That is, the real rates are seen as interest rates in the real (foreign) economy, whereas the nominal rates represent the interest rates in the nominal (domestic) economy. The inflation index then represents the exchange rate between the nominal (domestic) and real (foreign) currency. There are several assumptions that can be made with respect to the evolution of these dynamics. We first discuss the classical

<sup>†</sup>In our framework an inflation option can be seen as a forward-starting FX option, hence the pricing of an FX option follows from the pricing of an inflation option by setting the forward starting date equal to the current date. A stock option can be seen as an FX option in which (possibly deterministic) foreign interest rates represent the continuous dividend yield.

Jarrow and Yildirim (2003) model, before turning to generalized model setups. For clarity, we will use constant model parameters in both frameworks, however this can clearly be extended to time-dependent model parameters.

**2.1. Special case: The Jarrow–Yildirim (2003) model**

Jarrow and Yildirim (2003) assume that the real-world evolution of the nominal and real instantaneous forward rates is given by HJM dynamics, whereas the inflation index is log-normal distributed. Although several choices can be made with respect to the volatility structure within an HJM model, Jarrow and Yildirim (2003) assume that the forward rate volatilities are given by  $\sigma e^{a(T-t)}$ . Using the equivalent formulation of the HJM model in terms of instantaneous short rates results in the following dynamics under the risk-neutral measure  $\mathcal{Q}^n$  (Jarrow and Yildirim 2003).

**Proposition 2.1:** *The  $\mathcal{Q}_n$  dynamics of the instantaneous nominal rate  $n(t)$ , the real rate  $r(t)$  and the inflation index  $I(t)$  are given by*

$$dn(t) = [\vartheta_n(t) - a_n n(t)]dt + \sigma_n dW_n(t), \tag{1}$$

$$dr(t) = [\vartheta_r(t) - \rho_{r,I} \sigma_I \sigma_r - a_r r(t)]dt + \sigma_r(t) dW_r(t), \tag{2}$$

$$dI(t) = I(t)[n(t) - r(t)]dt + \sigma_I I(t) dW_I(t), \tag{3}$$

with  $a_n, a_r, \sigma_n, \sigma_r$  and  $\sigma_I$  positive parameters (possibly time-dependent) and where  $(W_n, W_r, W_I)$  is a Brownian motion under  $\mathcal{Q}^n$  (i.e. with the nominal bank account as numeraire) with correlations  $\rho_{n,r}, \rho_{n,I}$  and  $\rho_{r,I}$ , and with  $\vartheta_n(T)$  and  $\vartheta_r(t)$  deterministic functions that are used to exactly fit the term structure of the nominal and real interest rates.

Note that the covariance in (2) between the inflation and real rate term  $\rho_{r,I} \sigma_I \sigma_r$  arises due to a change of the real to the nominal risk-neutral measure (see, e.g., Geman *et al.* 1996). With this particular volatility structure, Jarrow and Yildirim (2003) thus assumed that both the nominal and real (instantaneous) rates followed Hull and White (1993) processes under their own risk-neutral measure. Moreover, they showed that the real rate still follows an Ornstein–Uhlenbeck process under the nominal risk-neutral measure  $\mathcal{Q}_n$  and that the inflation index  $I(T)$  for each  $t < T$  is log-normal distributed under  $\mathcal{Q}_n$ . In particular, one can write

$$I(T) = I(t) \exp\left(\int_t^T \left[n(u) - r(u) - \frac{1}{2} \sigma_I^2\right] du + \int_t^T \sigma_I dW_I(u)\right). \tag{4}$$

The main advantage of the Jarrow and Yildirim (2003) model is its tractability; one, for example, has analytical formulas for the prices of year-on-year inflation-indexed swaps (Brigo and Mercurio 2006, p. 653, formula (16.15)) and closed-form Black-like formulas for the prices of inflation-indexed caplets (Brigo and Mercurio 2006, p. 663, formula (17.4)). Although one can challenge the one-factor rate models, the greatest disadvantage of the

Jarrow and Yildirim (2003) model for the pricing of inflation derivatives is most often the log-normal assumption of the inflation index, which does not find justification in markets (see, e.g., Mercurio and Moreni 2006b, Kruse 2007 or Kenyon 2008).

**2.2. General model**

In this section we present a general model that can be seen as an extension of the models of Jarrow and Yildirim (2003) and van Haastrecht *et al.* (2008). The first extension is that instead of one-factor Hull and White (1993) models for the instantaneous nominal and real rates, we let the short rate be driven by multiple (correlated) factors. We use an equivalent additive formulation for Hull–White interest rates in terms of a sum of correlated Gaussian factors plus a deterministic function, i.e. we write the model into an affine factor formulation (e.g., Duffie *et al.* 2000, 2003). The deterministic factor can be chosen so as to exactly fit the term structure of the nominal or real interest rates (e.g., Pelsser 2000 or Brigo and Mercurio 2006). The nominal short interest rate is driven by  $K$  correlated Gaussian factors and the real short rate by  $M$  factors. The multi-factor Gaussian interest can hence be represented as

$$n(t) = \varphi_n(t) + \bar{\mathbf{1}} \cdot X_n(t), \quad r(t) = \varphi_r(t) + \bar{\mathbf{1}} \cdot X_r(t), \tag{5}$$

with  $\bar{\mathbf{1}}$  a vector of ones and where  $\varphi_n(t)$  and  $\varphi_r(t)$  are the deterministic functions to fit the nominal and real term structure (in particular,  $\varphi_n(0) = n(0)$  and  $\varphi_r(0) = r(0)$ ) and with  $X_n(t)$  and  $X_r$  Gaussian rate vectors that drive, respectively, the nominal and real rates, i.e. with typical elements the Gaussian factors  $x_n^i(t)$  and  $x_r^j(t)$ .

The second extension in our model is that we make the volatility  $\sigma_I$  stochastic. Moreover, we let this stochastic volatility factor, which from now on we denote by  $\nu(t)$ , be correlated with the instantaneous interest rates and the inflation index. Two popular choices within the stochastic volatility literature are the models of Heston (1993) and Schöbel and Zhu (1999). In the latter, the volatility is modeled as an Ornstein–Uhlenbeck process,

$$d\nu(t) = \kappa[\psi - \nu(t)]dt + \tau dW_\nu(t), \quad \nu(0) = \nu_0, \tag{6}$$

with  $\kappa, \psi$  and  $\sigma_\nu$  positive parameters and where  $W_\nu(t)$  is a Brownian motion that is correlated with the other driving factors, especially the asset price. Note that we have a positive probability that  $\nu(t)$  in (6) can become negative, which will cause the correlation between  $\nu(t)$  and the other driving factors to (temporarily) change sign.

The most popular stochastic volatility model, however, is the Heston (1993) model, which mainly owes its popularity to its analytical tractability. In the Heston model, the variance is modeled by the following Feller/CIR/square-root process:

$$d\nu^2(t) = \kappa[\theta - \nu^2(t)]dt + \xi \nu(t) dW_\nu(t), \quad \nu^2(0) = \nu_0^2, \tag{7}$$

with  $\kappa, \theta$  and  $\xi$  positive parameters and where  $W_\nu$  again represents a Brownian motion that is correlated with the other model factors.

With the multi-factor Gaussian rates and with stochastic volatility *à la* Schöbel–Zhu or Heston, we come to the following proposition for the dynamics of our model.

**Proposition 2.2:** *The  $\mathcal{Q}_n$  dynamics of the  $K$ -factor instantaneous nominal rate  $n(t)$ , the  $M$ -factor real rate  $r(t)$  and the inflation index  $I(t)$  are given by*

$$dx_n^i(t) = -a_n^i x_n^i(t)dt + \sigma_n^i dW_{n_i}(t), \quad i = 1, \dots, K, \quad (8)$$

$$dx_r^j(t) = [-a_r^j x_r^j(t) - \rho_{I, x_r^j} v(t) \sigma_r^j]dt + \sigma_r^j dW_{r_j}(t), \quad j = 1, \dots, M, \quad (9)$$

$$dI(t) = I(t)[n(t) - r(t)]dt + v(t)I(t)dW_I(t), \quad (10)$$

with  $a_n^i(t)$ ,  $a_r^j$ ,  $\sigma_n^i(t)$  and  $\sigma_r^j$  positive parameters,  $v(t)$  the stochastic volatility factor with dynamics given by (6) or (7), and where  $W_*(t) := (W_{n_1}, \dots, W_{n_K}, W_{r_1}, \dots, W_{r_M}, W_v)$  is a Brownian motion under  $\mathcal{Q}^n$  with a (possibly) full correlation structure.

The multi-factor Gaussian model is still very tractable. For example, one has the following analytical formulas for zero-coupon bond prices:

$$P_n(t, T) = \mathbb{E}_n \left\{ e^{-\int_t^T n(u)du} \right\} = A_n(t, T)e^{-B_n(t, T)X_n(t)}, \quad (11)$$

$$P_r(t, T) = \mathbb{E}_r \left\{ e^{-\int_t^T r(u)du} \right\} = A_r(t, T)e^{-B_r(t, T)X_r(t)}, \quad (12)$$

with  $B_n(t, T)$  and  $B_r(t, T)$  vectors with typical elements  $B_n^i(t, T)$  and  $B_r^i(t, T)$ , and where  $A_n(t, T)$ ,  $A_r(t, T)$ ,  $B_n^i(t, T)$  and  $B_r^i(t, T)$  are affine functions (see, e.g., appendix B.1). A useful quantity for the pricing of inflation-indexed options will be the forward inflation index  $I_F(t)$  under the nominal  $T$ -forward measure for a general maturity  $T$ , i.e.

$$I_F(t) = I(t) \frac{P_r(t, T)}{P_n(t, T)}. \quad (13)$$

Hence, since  $I_F(T) = I(T)$ , we can directly substitute the forward inflation index dynamics for the inflation index to price European time- $T$  options. In the following subsection we derive the dynamics of  $I_F(t)$  under the nominal  $T$ -forward measure.

### 2.3. Dynamics under the $T$ -forward measure

Using the change of numeraire technique of Geman *et al.* (1996), we now derive the dynamics of our model under the  $T$ -forward measure for a general maturity  $T$ . Note that, under their risk-neutral measures, the nominal and real discount bond prices follow the processes

$$\begin{aligned} \frac{dP_n(t, T)}{P_n(t, T)} &= n(t)dt + \Sigma_n(t, T)dW_n(t), \\ \frac{dP_r(t, T)}{P_r(t, T)} &= r(t)dt + \Sigma_r(t, T)dW_r(t), \end{aligned} \quad (14)$$

where  $\Sigma_i(t, T)$ ,  $i \in \{n, r\}$ , denotes the vector of zero bond volatilities, with typical element  $\sigma_i^k B_i^k(t, T)$ , and with  $W_i$  a

vector Brownian motion. Hence, by an application of Itô’s lemma, we come to the following proposition.

**Proposition 2.3:** *The  $\mathcal{Q}_n^T$  dynamics for the  $T$ -forward asset price, under the  $T$ -forward Brownian motion  $W^*(t)$ , has the following SDEs:*

$$dx_n^k(t) = [-a_n^k x_n^k - R_{x_n^k, X_n} \Sigma_n(t, T)]dt + \sigma_n^k dW_T^*(t), \quad k = 1, \dots, K, \quad (15)$$

$$dx_r^j(t) = [-a_r^j x_r^j - \sigma_r^j \rho_{I, x_r^j} v(t) - \sigma_r^j R_{x_r^j, X_n} \Sigma_n(t, T)]dt + \sigma_r^j dW_T^*(t), \quad j = 1, \dots, M, \quad (16)$$

$$\frac{dI_F(t)}{I_F(t)} = (v(t) + \Sigma_n(t, T) - \Sigma_r(t, T))dW_T^*(t), \quad (17)$$

where  $R_{x_n^i, X_n}$  and  $R_{x_r^j, X_n}$  denote the correlation vectors between, respectively,  $x_n^i(t)$  and  $x_r^j(t)$  and the vector of nominal interest rate drivers  $X_n(t)$ . The stochastic volatility SDE in the Schöbel–Zhu case is given by

$$dv(t) = \kappa[\xi(t) - v(t)]dt + \tau dW_T^*(t), \quad (18)$$

$$\xi(t) = \psi - \tau \frac{R_{v, X_n}}{\kappa} \Sigma_n(t), \quad (19)$$

while the Heston dynamics become

$$dv^2(t) = \kappa[\zeta(t) - v^2(t)]dt + \xi v(t)dW_T^*(t), \quad (20)$$

$$\zeta(t) = \theta - v(t)\xi \frac{R_{v, X_n}}{\kappa} \Sigma_n(t). \quad (21)$$

Note that correlations in the above model are introduced via vector volatilities.

We can simplify (17) further by switching to logarithmic coordinates: defining

$$z(t) := \log I_F(t) = \log \left( I(t) \frac{P_r(t, T)}{P_n(t, T)} \right), \quad (22)$$

the application of Itô’s lemma yields

$$dz(t) = -\frac{1}{2}v_F^2(t)dt + v_F(t)dW^*(t), \quad (23)$$

with  $v_F(t) := [v(t) + \Sigma_n(t, T) - \Sigma_r(t, T)]$  the instantaneous variance of the forward inflation index (explicitly defined in (50)). Note that we have now transformed the system of proposition 2.1 of the variables  $x_n^1(t), \dots, x_n^K(t)$ ,  $x_r^1(t), \dots, x_r^M(t)$ ,  $I(t)$  and  $v(t)$  under the nominal risk-neutral measure, into the system (18)–(23) of variables  $z(t)$  and  $v_F(t)$  under the  $T$ -forward measure. This latter system will be used to determine the characteristic function of the log inflation rate in our model (see section 4).

### 3. Pricing and applications

In this section we briefly discuss the main vanilla inflation, FX and equity derivatives and discuss how these securities can be priced in closed form by our model.

Before turning to the market-specific structures, we first consider the general pricing methodology.

### 3.1. Pricing

We now discuss the general option pricing framework for inflation, FX and stock options. That is, we briefly review the framework of Carr and Madan (1999) for the pricing of European option prices using Fourier inversion. We then show how this framework can be applied to value inflation, FX and stock derivatives. Under the risk-neutral measure  $\mathcal{Q}$  (i.e. with the bank account as numeraire), we can write the following for the price  $C_T(k)$  of a European option ( $\omega = 1$  for a call,  $\omega = -1$  for a put) maturing at time  $T$ , with strike  $K = \exp(k)$ , on asset  $I$ :

$$C_T(k, \omega) = \mathbb{E}_n \left\{ e^{-\int_t^T r(u)du} [\omega(I(T) - K)]^+ \mid \mathcal{F}_t \right\}, \quad (24)$$

and hence note that to price European options we only need the probability distribution of the  $T$ -forward stock price at time  $T$ . Therefore, instead of evaluating the expected discounted payoff under the risk-neutral bank account measure, we can also change the underlying probability measure to evaluate this expectation under the  $T$ -forward probability measure  $\mathcal{Q}^T$  (see, e.g., Geman *et al.* 1996). This is equivalent to choosing the  $T$ -discount bond as numeraire. Hence, conditional on time  $t$ , we can evaluate the price of a European option ( $\omega = 1$  for a call,  $\omega = -1$  for a put) with strike  $K = \exp(k)$  as

$$C_T(k, \omega) = P_n(t, T) \mathbb{E}_n^{\mathcal{Q}^T} \{ [\omega(I_F^T(T) - K)]^+ \mid \mathcal{F}_t \}, \quad (25)$$

where  $P_n(t, T)$  denotes the price of a (pure) discount bond and  $I_F^T(t) := I(t)/P_n(t, T)$  denotes the  $T$ -forward index price. The above expression can be evaluated numerically by means of a Fourier inversion of the log-asset price characteristic function. Following Carr and Madan (1999), Lewis (2001) and Lord and Kahl (2007), we can then write the call option price (24) with log strike  $k$  in terms of the ( $T$ -forward) characteristic function  $\phi_T$  of the  $T$ -forward log index price  $z(T) := \log I_F^T(t)$ . Provided that the regularity conditions for the Fourier transformations are satisfied, i.e.  $\alpha > 0$  for a call ( $\omega = 1$ ) and  $\alpha > 1$  for a put ( $\omega = -1$ ), one can write the following for the corresponding European option price:

$$C_T(k, \omega, \alpha) = \frac{P_n(t, T)}{\pi} \int_0^\infty \text{Re}(e^{-(\omega\alpha + iv)k} \psi_T(v, \omega, \alpha)) dv, \quad (26)$$

with

$$\psi_T(v, \omega, \alpha) := \frac{\phi_T(v - (\omega\alpha + 1)i)}{(\omega\alpha + iv)(\omega\alpha + 1 + iv)}, \quad (27)$$

where  $\phi_T(u) := \mathbb{E}^{\mathcal{Q}^T} [\exp(iuz(T)) \mid \mathcal{F}_t]$  denotes the  $T$ -forward characteristic function of the log index price. Note that (26) can be efficiently evaluated, either by direct integration or fast Fourier transformation (Carr and Madan 1999, Lee 2004, Lord and Kahl 2007). Thus, for the pricing of call and put options, it suffices to have knowledge of the characteristic function of the log price

process. The characteristic and forward characteristic function under Schöbel and Zhu (1999) volatility can be found in propositions 4.1 and 4.2, respectively, whereas under Heston (1993) volatility these can be found in propositions 4.5 and 4.8.

### 3.2. Inflation derivatives

Before dealing with the pricing of inflation-indexed derivatives within the general model (proposition 2.2), we first discuss the main (vanilla) inflation-indexed securities. We adopt the notation used by Mercurio (2005) and Brigo and Mercurio (2006), who we also refer to for an excellent overview of interest rate and inflation-indexed derivatives and models.

**3.2.1. Inflation-indexed swaps.** Given a set of payment dates  $T_1, \dots, T_M$ , an inflation-indexed swap (IIS) is a swap where, on each date, party A pays party B the inflation rate over a predefined period, while party B pays party A a fixed rate. This inflation rate is calculated as the percentage return of the inflation index (e.g., HICP ex Tobacco in the Eurozone) over the time interval it applies to. The two main IIS contracts that are traded in markets are the zero-coupon inflation-indexed swap (ZCIIS) and the year-on-year inflation-indexed swap (YYIIS). For the ZCIIS, for the payoff at time  $T_M$ , assuming  $T_M = M$  years, party B pays party A the fixed amount

$$N[(1 + K)^M - 1], \quad (28)$$

where  $K$  is the strike (e.g., the break-even inflation rate) and  $N$  the nominal value of the contract. In exchange, party A pays party B, at the final time  $T_M$ , the floating amount of

$$N \left[ \frac{I(T_M)}{I_0} - 1 \right], \quad (29)$$

with  $I(T_M)$ ,  $I_0$  the inflation/CPI index, respectively, at time  $T_M$  and  $T_0$ . For the YYIIS, at each time  $T_i$ , party B pays party A the fixed amount

$$N\phi_i K, \quad (30)$$

where  $\phi_i$  denotes the fixed-leg year fraction for the interval  $[T_{i-1}, T_i]$  and  $N$  the nominal value of the YYIIS. In exchange, at each time  $T_i$ , party A pays party B the floating amount

$$N\psi_i \left[ \frac{I(T_i)}{I(T_{i-1})} - 1 \right], \quad (31)$$

where  $\psi_i$  denotes the fixed-leg year fraction for the interval  $[T_{i-1}, T_i]$  ( $T_0 := 0$ ).

Let  $P_n$  and  $P_r$  denote, respectively, the (zero-coupon) discount bond prices of the real and nominal economy. Then standard no-arbitrage theory and some straightforward rewriting show that the price of a ZCIIS

(zero-coupon inflation-indexed swap) can be expressed as

$$\mathbf{ZCIIS}(t, T_M, I_0, N, K) = N \left[ \frac{I(t)}{I_0} P_r(t, T_M) - (1 + K)^M \right], \tag{32}$$

which is model-independent. That is, the above price is not based on any specific assumptions concerning the evolution of the (real and nominal) interest rates, but simply follows from the absence of arbitrage. This is an important fact, since it allows us to strip, without ambiguity, real zero-coupon bond prices from the quotes prices of ZCIIS. More specifically, given a set of market quotes  $K=K(T_M)$  at time  $t=0$ , we can use equation (32) together with the net present value (28) to determine the discount bonds of the real economy, i.e.

$$P_r(0, T_M) = P_n(0, T_M)(1 + K(T_M))^M. \tag{33}$$

A completely different story applies to the valuation of a YYIIS (year-on-year inflation-indexed swap), which, in fact, depends on the evolution of the underlying quantities and hence its price is model-dependent. Note that the value at time  $t < T_{i-1}$  of the payoff (31) at time  $T_i$  is

$$\begin{aligned} \mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) &= N\psi_i \mathbb{E}_n \left\{ e^{-\int_t^{T_i} n(u)du} \left[ \frac{I(T_i)}{I(T_{i-1})} - 1 \right] \middle| \mathcal{F}_t \right\} \\ &= N\psi_i \mathbb{E}_n \left\{ e^{-\int_t^{T_i} n(u)du} \frac{I(T_i)}{I(T_{i-1})} \middle| \mathcal{F}_t \right\} - N\psi_i P_n(t, T_i), \end{aligned} \tag{34}$$

where  $\mathbb{E}_n$  denotes the expectation under the nominal risk-neutral measure. We briefly comment on why the latter expectation is model-dependent. First note that

$$\begin{aligned} \mathbb{E}_n \left\{ e^{-\int_t^{T_i} n(u)du} \frac{I(T_i)}{I(T_{i-1})} \middle| \mathcal{F}_t \right\} &= \mathbb{E}_n \left\{ e^{-\int_t^{T_{i-1}} n(u)du} P_r(T_{i-1}, T_i) \middle| \mathcal{F}_t \right\}, \end{aligned} \tag{35}$$

hence we can interpret the expectation from (34) as the nominal price of a derivative that pays off (in nominal units) the real zero-coupon bond price  $P_r(T_{i-1}, T_i)$  at time  $T_i$ . Alternatively, we can also evaluate the latter expectation under a different measure (see, e.g., Geman *et al.* 1996). Denote by  $\mathcal{Q}_n^T$  the nominal  $T$ -forward measure for some maturity  $T$  and let  $\mathbb{E}_n^T$  represent the expectation taken under this measure. Then we can write (35) as

$$\begin{aligned} \mathbb{E}_n \left\{ e^{-\int_t^{T_{i-1}} n(u)du} P_r(T_{i-1}, T_i) \middle| \mathcal{F}_t \right\} &= P_n(t, T_{i-1}) \mathbb{E}_n^{T_{i-1}} \{ P_r(T_{i-1}, T_i) \mid \mathcal{F}_t \}. \end{aligned} \tag{36}$$

If the nominal or real rates are deterministic, then this expectation would reduce to the present value (in nominal units) of the forward price of the real zero-coupon bond, i.e. we would then have

$$P_n(t, T_{i-1}) \mathbb{E}_n^{T_{i-1}} \{ P_r(T_{i-1}, T_i) \mid \mathcal{F}_t \} = P_r(T_{i-1}, T_i) P_n(t, T_{i-1}). \tag{37}$$

However, for inflation-linked derivative pricing purposes it is usually desirable (if not necessary) that real rates are

stochastic and the expectation of (34) is model-dependent. In fact, if the nominal and real rates are correlated (and hence stochastic), the change of measure will change the drift of the real rate  $r(t)$  and hence also the expectation of (36). In interest rate terms, this effect is known under the term convexity adjustment (see, e.g., Pelsser 2000 or Brigo and Mercurio 2006). For example, if one assumes one-factor Gaussian rates (as in the JY model), one will see this convexity effect for any non-zero correlation coefficient between the nominal and real rates.

Finally, note that we can also evaluate the expectation of (34) under the  $T_i$ -forward measure, i.e.

$$\begin{aligned} \mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) &= N\psi_i P(t, T_i) \mathbb{E}_n^{T_i} \left\{ \frac{I(T_i)}{I(T_{i-1})} \middle| \mathcal{F}_t \right\} - N\psi_i P_n(t, T_i). \end{aligned} \tag{38}$$

This latter interpretation, which expresses the YYIIS (year-on-year inflation-indexed swap) as the  $T_i$ -forward expectation of the return on the inflation index, is very useful for our pricing methodology (see section 3.1), because it expresses the price of a YYIIS in terms of the distribution of  $I(T_i)/I(T_{i-1})$  under the  $T_{i-1}$ -forward measure.

### 3.3. Inflation-indexed caplets/floorlets

An inflation-indexed caplet can be seen as a call option on the inflation rate implied by the inflation (e.g., CPI) index. Analogously, an inflation-indexed floorlet can be seen as a put option on the same inflation rate. We can write the following for the payoff of an IICplt (inflation-indexed caplet/floorlet) at time  $T_i$ :

$$N\psi_i \left[ \omega \left( \frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa \right) \right]^+, \tag{39}$$

where  $N$  denotes the nominal value of the contract,  $\kappa$  the strike,  $\psi_i$  the year fraction for the interval  $[T_{i-1}, T_i]$  and  $\omega=1$  for a caplet and  $\omega=-1$  for a floorlet. Setting  $K:=1+\kappa$ , standard no-arbitrage theory implies that the value of the payoff (39) at time  $t \leq T_{i-1}$  is

$$\begin{aligned} \mathbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) &= N\psi_i \mathbb{E}_n \left\{ e^{-\int_t^{T_i} n(u)du} \left[ \omega \left( \frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \middle| \mathcal{F}_t \right\} \\ &= N\psi_i P_n(t, T_i) \mathbb{E}_n^{T_i} \left\{ \left[ \omega \left( \frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \middle| \mathcal{F}_t \right\}. \end{aligned} \tag{40}$$

Since (40) is equivalent to a call option on the forward return of the inflation index, the pricing of an inflation-indexed caplet/floorlet is thus very similar to that of a forward starting (clicket) option.

**3.3.1. Pricing.** The crucial quantity for the pricing of the inflation-indexed derivatives in our model (proposition 2.2) is the log-return  $y(T_{i-1}, T_i)$  of the inflation index over the interval  $[T_{i-1}, T_i]$  under the  $T_i$ -forward



measure  $\mathcal{Q}_n^{T_i}$ , i.e.

$$y(T_{i-1}, T_i) = \log\left(\frac{I(T_i)}{I(T_{i-1})}\right), \quad (41)$$

and henceforth we assume that we explicitly know the characteristic function  $\phi_{T_{i-1}, T_i}$  of  $y(T_{i-1}, T_i)$ ,

$$\phi_{T_{i-1}, T_i}(u) := \mathbb{E}_n^{T_i}[\exp(iuy(T_{i-1}, T_i)) \mid \mathcal{F}_t]. \quad (42)$$

The derivations and explicit formulas for the characteristic function(s) are discussed in section 4.

**3.3.2. Pricing of inflation-indexed swaps.** The main two inflation-indexed swaps are the ZCIIS and YYIIS. Recall that the zero-coupon swap is model-independent and is simply given by no-arbitrage arguments, i.e. by (32). Given the characteristic function  $\phi_{T_{i-1}, T_i}(u)$  from (42) of the log-inflation return under the  $T_i$ -forward measure, the pricing of a YYIIS is extremely simple. In fact, recall from (38) that we have the following expression for the price of a YYIIS:

$$\begin{aligned} & \text{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) \\ &= N\psi_i P(t, T_i) \mathbb{E}_n^{T_i} \left\{ \frac{I(T_i)}{I(T_{i-1})} \mid \mathcal{F}_t \right\} - N\psi_i P_n(t, T_i), \end{aligned} \quad (43)$$

and then note that the expectation in the above expression is nothing more than the characteristic function of the log-return evaluated in the complex-valued point  $-i$ ,

$$\begin{aligned} \mathbb{E}_n^{T_i} \left\{ \frac{I(T_i)}{I(T_{i-1})} \mid \mathcal{F}_t \right\} &= \mathbb{E}_n^{T_i} \left\{ \exp \left[ i(-i) \log \left( \frac{I(T_i)}{I(T_{i-1})} \right) \right] \mid \mathcal{F}_t \right\} \\ &= \phi_{T_{i-1}, T_i}(-i). \end{aligned} \quad (44)$$

Hence, the price of a YYIIS is given by the following simple expression:

$$\begin{aligned} & \text{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) \\ &= N\psi_i P(t, T_i) \phi_{T_{i-1}, T_i}(-i) - N\psi_i P_n(t, T_i). \end{aligned} \quad (45)$$

### 3.4. Pricing of inflation-indexed caplets/floorlets

The pricing of forward starting options such as cliquets has recently attracted the attention of both practitioners and academia (see, e.g., Lucić 2003, Hong 2004 and Brigo and Mercurio 2006). In this section we show how one can price inflation call options in the framework of Carr and Madan (1999). Working under the  $T_i$ -forward measure, we are particularly interested in the  $T_i$ -forward log return  $y(T_{i-1}, T_i)$  on the inflation index between the times  $T_{i-1}$  and  $T_i$ , i.e. as defined in 41. From (40) we know that we can express an inflation caplet as a call option on the forward return of the index. We can then place this directly in the Carr and Madan (1999) methodology of section 3.1. That is, using (26) and provided that the regularity conditions are satisfied, i.e.  $\alpha > 0$  for a caplet ( $\omega = 1$ ) and  $\alpha > 1$  for a floorlet ( $\omega = -1$ ), we can write the following for the price of an IICplt

(inflation-indexed caplet):

$$\begin{aligned} & \text{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) \\ &= N\psi_i P_n(t, T_i) \mathbb{E}_n^{T_i} \left\{ \left[ \omega \left( \frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \mid \mathcal{F}_t \right\} \\ &= N\psi_i P_n(t, T_i) \frac{1}{\pi} \int_0^\infty \text{Re} [e^{-(\omega\alpha + iv) \log K} \Psi_{T_{i-1}, T_i}(v, \omega, \alpha)] dv, \end{aligned} \quad (46)$$

with  $\Psi_{T_{i-1}, T_i}(v, \omega, \alpha)$  in (27) a function of the characteristic function  $\phi_{T_{i-1}, T_i}(u)$  of (42). Alternatively, the price of a floorlet can be expressed in terms of the corresponding caplet price (and *vice versa*) by means of put–call parity (see, e.g., Mercurio 2005). Given that we know the characteristic function, formula (46) provides an efficient and accurate way to determine the prices of inflation-indexed caps/floors. What remains is the derivation of this forward characteristic function, which we will discuss in section 4. The corresponding characteristic functions can be found in propositions 4.2 (for Schöbel and Zhu 1999 volatility) and 4.8 (for Heston 1993 volatility).

### 3.5. FX and stock derivatives

The pricing of FX and stock derivatives within the general model (proposition 2.2) can be done using similar techniques as in the previous section with inflation-indexed derivatives. The main difference is that inflation-indexed derivatives are usually forward-starting options, whereas the vanilla FX and stock options do not share this feature. In a way, one can therefore treat FX and stock options within the FX setup of our (proposition 2.2) as nested (degenerate) cases of inflation derivatives by choosing the forward-starting date equal to the current date and normalizing the stock/index price by  $I(0)$ , i.e. in accordance with (41). In a similar spirit, one can envisage a stock option as a FX option in which the foreign instantaneous interest rate represents the stochastic (or deterministic) continuous dividend rate of the stock.

For clarity, we provide the pricing formulas for FX and stock options. Working under the  $T$ -forward measure, the pricing formulas require the characteristic function

$$\phi_T(u) := \mathbb{E}^{\mathcal{Q}^T} [\exp(iuz(T)) \mid \mathcal{F}_t] \quad (47)$$

of the log index/FX rate/stock price  $z(T) := \log I(T)$ . Equipped with this characteristic function, the time- $T$  forward FX-rate  $FFX(T)$  (i.e. with convexity adjustment when the foreign interest rates are stochastic) is given by

$$FFX(T) = \mathbb{E}^{\mathcal{Q}^T} [I(T)] = \phi_T(-i). \quad (48)$$

Provided with the log-asset price characteristic function, one can immediately price a call/put option on the stock or FX rate within the ‘Fourier-inversion’ framework of section 3.1. More specifically, one can directly substitute the characteristic function for  $\phi_T$  into the pricing formulas (26) and (27). Completely analogously to inflation-indexed options, one can price forward-starting (cliquet) options on the forward return of the FX rate/stock index

by substituting the characteristic function  $\phi_{T_{i-1}, T_i}(u)$  of the forward log return (41) into the pricing equations (26) and (27). We will discuss the derivation of both of these characteristic functions in the next section.

**4. Characteristic function of the model**

In this section we turn to the derivation of the characteristic function of the log inflation return under the nominal  $T$ -forward measure  $\mathcal{Q}_n^T$ . For an inflation index that is driven by a Schöbel–Zhu stochastic volatility process, we are able to derive a closed-form expression, whereas for the Heston stochastic volatility case we are able to approximate this characteristic function. Before turning to these derivations, we first turn to the volatility aspect of the inflation index and to the Gaussian interest rates, the treatment of which is common for both volatility choices.

**4.1. Volatility driver and multi-factor Gaussian rates**

To ease notation we introduce some matrix notation. Let  $\Sigma(t, T)$  denote the  $1 \times (1 + K + M)$  column vector of the ‘volatilities’ driving the Brownian motion of the  $T$ -forward inflation index, with corresponding  $(1 + K + M) \times (1 + K + M)$  correlation matrix  $R$ , i.e.

$$\Sigma(t, T) = \begin{bmatrix} \nu(t) \\ \sigma_n^1(t) B_n^1(t, T) \\ \vdots \\ \sigma_n^K(t) B_n^K(t, T) \\ -\sigma_r^1(t) B_r^1(t, T) \\ \vdots \\ -\sigma_r^M(t) B_r^M(t, T) \end{bmatrix},$$

$$R = \begin{pmatrix} 1 & \rho_{x_n^1, \nu} & \cdots & \rho_{x_r^K, \nu} & \rho_{x_r^1, \nu} & \cdots & \rho_{x_r^M, \nu} \\ \rho_{x_n^1, \nu} & 1 & \cdots & \rho_{x_n^1, x_n^K} & \rho_{x_n^1, x_r^1} & \cdots & \rho_{x_n^1, x_r^M} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \rho_{x_n^K, \nu} & \rho_{x_n^1, x_n^K} & \cdots & 1 & \rho_{x_n^K, x_r^1} & \cdots & \rho_{x_n^K, x_r^M} \\ \rho_{x_r^1, \nu} & \rho_{x_n^1, x_r^1} & \cdots & \rho_{x_n^K, x_r^1} & 1 & \cdots & \rho_{x_r^1, x_r^M} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{\nu, x_r^M} & \rho_{x_n^1, x_r^M} & \cdots & \rho_{x_n^K, x_r^M} & \rho_{x_r^1, x_r^M} & \cdots & 1 \end{pmatrix}. \tag{49}$$

Hence we can write the following for the instantaneous variance  $\nu_F(t)$  of the inflation index under the  $T$ -forward measure:

$$\nu_F^2(t) = \Sigma'(t, T) R \Sigma(t, T). \tag{50}$$

Another useful expression is the integrated variance of the multi-factor Gaussian rate process. We can write the following for the instantaneous variance  $\nu_{K,M}(t)$  of the sum of the rate processes:

$$\nu_{K,M}^2(t, T) = \sum_{i=2}^{K+M+1} (\Sigma^{(i)}(t, T))^2 + 2 \sum_{i=2}^{K+M+1} \sum_{j=i+1}^{K+M+1} R^{(i,j)} \Sigma^{(i)}(t, T) \Sigma^{(j)}(t, T), \tag{51}$$

where  $\Sigma^{(i)}$  is the  $i$ th element of the vector  $\Sigma(t, T)$  and  $R^{(i,i)}$  denotes the element in row  $i$  and column  $j$  of matrix  $R$ . Note the shift in index, which is due to the presence of the volatility driver  $\nu(t)$  in (49). For the integrated rate variance  $V_{K+M}(t, T)$ , one has the following expression:

$$V_{K,M}(t, T) := \int_t^T \nu_{K,M}^2(u, T) du = \sum_{i=2}^{K+M+1} C^{(i,i)} + 2 \sum_{i=2}^{K+M+1} \sum_{j=i+1}^{K+M+1} R^{(i,j)} C^{(i,j)}, \tag{52}$$

where  $C^{(i,j)}$  denotes the integrated covariance between the  $i$ th and  $j$ th elements of the vector of rate volatilities  $\Sigma(t)$ .

For the covariance  $C^{(2, K+M+1)}$  between the first and  $K + M$ th element, one, for example, has†

$$C^{(2, K+M+1)} := -\frac{\sigma_n^1 \sigma_r^M}{a_n^1 a_r^M} \left( (T-t) + \frac{e^{-a_n^1(T-t)} - 1}{a_n^1} + \frac{e^{-a_r^M(T-t)} - 1}{a_r^M} - \frac{e^{-(a_n^1 + a_r^M)(T-t)} - 1}{a_n^1 + a_r^M} \right). \tag{54}$$

**4.2. Schöbel–Zhu stochastic volatility**

In this section we determine the characteristic function (under the  $T$ -forward measure) of the forward log-inflation return  $y(T_{i-1}, T_i)$  between times  $T_{i-1}$  and  $T_i$ . For this we first need to determine the characteristic function of the  $T$ -forward log-inflation rate  $z(T)$  for a general maturity  $T$ . Building on the results of van Haastrecht *et al.* (2008), who derive the characteristic function for the one-factor Schöbel–Zhu–Hull–White model, we derive its multi-factor generalization in the following subsection.

†It is indeed possible to consider time-dependent parameters, in which case the covariance  $C^{(2, K+M+1)}$  is given by the time-dependent integral expression

$$C^{(2, K+M+1)} := \int_t^T (\sigma_n^1(u) B_n^1(u, T)) (-\sigma_r^M(u) B_r^M(u, T)) du. \tag{53}$$

We can do this for all formulas in the paper. However, as the resulting integral expressions become obscure, whilst the generalization is obvious, we use constant parameters for clarity of exposition.

**4.2.1. Characteristic function of the log-index price.** We now determine the characteristic function of the reduced system (23), for which we shall use a partial differential approach. Recall from (22) that  $z(t) := \log I_F(t)$  is defined as the  $T$ -forward log-asset price. Subject to the terminal boundary condition

$$f(T, z, v) = \exp(iuz(T)), \tag{55}$$

the Feynman–Kac theorem implies that the expected value of  $\exp(iuz(T))$  equals the solution of the Kolmogorov backward partial differential equation for the joint probability distribution function  $f(t, z, v)$ , i.e.

$$f := f(t, z, v) = \mathbb{E}^{\mathcal{Q}^T}[\exp(iuz(T)) | \mathcal{F}_t]. \tag{56}$$

Thus, the solution for  $f$  equals the characteristic function of the forward asset price dynamics (starting from  $z$  at time  $t$ ). To obtain the Kolmogorov backward partial differential equation for the joint probability distribution function  $f=f(t, y, v)$ , we need to take into account the covariance term  $dz(t) dv(t)$ , which equals

$$\begin{aligned} dz(t)dv(t) &= (v(t) + \Sigma_n(t, T) - \Sigma_r(t, T))dW^*(t)(\tau dW_v^T(t)) \\ &= (\rho_{Iv}\tau v(t) + \tau R_{v,X_n}\Sigma_n(t, T) - \tau\rho_{v,X_r}\Sigma_r(t, T))dt. \end{aligned} \tag{57}$$

The model we are considering is not an affine model in  $z(t)$  and  $v(t)$ , but it is if we enlarge the state space to include  $v^2(t)$ :

$$dz(t) = -\frac{1}{2}v_F^2(t)dt + v_F(t)dW^*(t), \tag{58}$$

$$dv(t) = \kappa[\xi(t) - v(t)]dt + \tau dW_v^T(t), \tag{59}$$

$$\begin{aligned} dv^2(t) &= 2v(t)dv(t) + \tau^2 dt \\ &= 2\kappa\left(\frac{\tau^2}{2\kappa} + \xi(t)v(t) - v^2(t)\right)dt + 2\tau v(t)dW_v(t). \end{aligned} \tag{60}$$

Using (57) and (58) we obtain the following partial differential equation for  $f(t, z, v)$ :

$$\begin{aligned} 0 = & f_t - \frac{1}{2}v_F^2(t)f_z + \kappa(\xi(t) - v(t))f_v + \frac{1}{2}v_F^2(t)f_{zz} \\ & + (\rho_{Iv}\tau v(t) + \tau R_{v,X_n}\Sigma_n(t, T) - \tau\rho_{v,X_r}\Sigma_r(t, T))f_{zv} + \frac{1}{2}\tau^2 f_{vv}. \end{aligned} \tag{61}$$

Solving this partial differential equation, subject to the terminal boundary condition (55), provides us with the characteristic function of the forward asset price dynamics (starting from  $z$  at time  $t$ ). Due to the affine structure of our model, we come to the following proposition.

**Proposition 4.1:** *The characteristic function of the domestic  $T$ -forward log inflation rate of the model with Schöbel and Zhu (1999) stochastic volatility is given by the following closed-form solution:*

$$\begin{aligned} f(t, z, v) &= \exp[A(u, t, T) + B(u, t, T)z(t) \\ &+ C(u, t, T)v(t) + \frac{1}{2}D(u, t, T)v^2(t)], \end{aligned} \tag{62}$$

where

$$\begin{aligned} A(u, t, T) &= -\frac{1}{2}u(i+u)V_{K,M}(t, T) \\ &+ \int_t^T \left\{ \left[ \kappa\psi + (iu-1) \sum_{i=1}^K \rho_{x_n^i} \tau \sigma_n^i B_n^i(t, T) \right. \right. \\ &\left. \left. - iu \sum_{j=1}^M \rho_{x_r^j} \tau \sigma_r^j B_r^j(t, T) \right] C(u, s, T) \right. \\ &\left. + \frac{1}{2}\tau^2 (C^2(u, s, T) + D(u, s, T)) \right\} ds, \end{aligned} \tag{63}$$

$$B(u, t, T) = B := iu, \tag{64}$$

$$\begin{aligned} C(u, t, T) &= -\frac{u(i+u)}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}} \left\{ \gamma_0 (1 + e^{-2\gamma(T-t)}) \right. \\ &+ \sum_{i=1}^K [(\gamma_3^i - \gamma_4^i e^{-2\gamma(T-t)}) - (\gamma_5^i e^{-a_n^i(T-t)} \\ &- \gamma_6^i e^{-(2\gamma+a_n^i)(T-t)}) - \gamma_7^i e^{-\gamma(T-t)}] \\ &- \sum_{j=1}^M [(\gamma_8^j - \gamma_9^j e^{-2\gamma(T-t)}) \\ &\left. - (\gamma_{10}^j e^{-a_r^j(T-t)} - \gamma_{11}^j e^{-(2\gamma+a_r^j)(T-t)}) - \gamma_{12}^j e^{-\gamma(T-t)}] \right\}, \end{aligned} \tag{65}$$

$$D(u, t, T) = -u(i+u) \frac{1 - e^{-2\gamma(T-t)}}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}}, \tag{66}$$

where  $V_{K,M}(t, T)$ , as defined in (52) represents the integrated variance of the Gaussian rate processes with

$$\begin{aligned} \gamma &= \sqrt{(\kappa - \rho_{Iv}\tau B)^2 - \tau^2(B^2 - B)}, \\ \gamma_0 &= \frac{\kappa\psi}{\gamma}, \\ \gamma_1 &= \gamma + (\kappa - \rho_{Iv}\tau B), \\ \gamma_2 &= \gamma - (\kappa - \rho_{Iv}\tau B), \\ \gamma_3^i &= \frac{\rho_{I,x_n^i}\sigma_n^i\gamma_1 + \rho_{x_n^i,v}\sigma_n^i\tau(iu-1)}{a_n^i\gamma}, \\ \gamma_4^i &= \frac{\rho_{I,x_n^i}\sigma_n^i\gamma_2 - \rho_{x_n^i,v}\sigma_n^i\tau(iu-1)}{a_n^i\gamma}, \\ \gamma_5^i &= \frac{\rho_{I,x_n^i}\sigma_n^i\gamma_1 + \rho_{x_n^i,v}\sigma_n^i\tau(iu-1)}{a_n^i(\gamma - a_n^i)}, \\ \gamma_6^i &= \frac{\rho_{I,x_n^i}\sigma_n^i\gamma_2 - \rho_{x_n^i,v}\sigma_n^i\tau(iu-1)}{a_n^i(\gamma + a_n^i)}, \\ \gamma_8^j &= \frac{\rho_{I,x_r^j}\sigma_r^j\gamma_1 + \rho_{x_r^j,v}\sigma_r^j\tau B}{a_r^j\gamma}, \\ \gamma_9^j &= \frac{\rho_{I,x_r^j}\sigma_r^j\gamma_2 - \rho_{x_r^j,v}\sigma_r^j\tau B}{a_r^j\gamma}, \\ \gamma_{10}^j &= \frac{\rho_{I,x_r^j}\sigma_r^j\gamma_1 + \rho_{x_r^j,v}\sigma_r^j\tau B}{a_r^j(\gamma - a_r^j)}, \end{aligned} \tag{67}$$

$$\begin{aligned} \gamma_{11} &= \frac{\rho_{L,x_r^j} \sigma_r^j \gamma_2 - \rho_{x_r^j, v} \sigma_r^j \tau B}{a_r^j (\gamma + a_r^j)}, \\ \gamma_7^j &= (\gamma_3^j - \gamma_4^j) - (\gamma_5^j - \gamma_6^j), \\ \gamma_{12}^j &= (\gamma_8^j - \gamma_9^j) - (\gamma_{10}^j - \gamma_{11}^j). \end{aligned}$$

**Proof:** See appendix A.  $\square$

Using the above characteristic function of the log-inflation index under the  $T$ -forward measure, in the following section we are able to derive the forward starting characteristic of the log-inflation index return.

**4.2.2. Characteristic function of the log index return.** Recently, the pricing of forward starting options has attracted the attention of both practitioners and academia (see, e.g., Lucić 2003, Hong 2004 and van Haastrecht *et al.* 2008, and in an inflation context, Mercurio and Moreni 2006a and Kruse 2007). In this section we consider the pricing of forward starting options such as inflation caplets within the general model setup combined with Schöbel–Zhu volatility. In particular, using the framework of Carr and Madan (1999), as described in section 3.1, it suffices to know the characteristic function of the following log-inflation index return under the  $T_T$ -forward measure:

$$y(T_{i-1}, T_i) := \log\left(\frac{I(T_i)}{I(T_{i-1})}\right) = \log I(T_i) - \log I(T_{i-1}). \tag{68}$$

Since  $I(t) := I_F(t) (P_n(t, T_i)/P_r(t, T_i))$ , we can also express this return in terms of the  $T_T$ -forward log inflation rate  $z(t) := \log(I_F(t))$ , i.e.

$$y(T_{i-1}, T_i) = z(T_i) - z(T_{i-1}) - \log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i). \tag{69}$$

We are then interested in the characteristic function  $\phi_{T_{i-1}, T_i}(u)$  of the log-inflation index return  $y(T_{i-1}, T_i)$  under the  $T_T$ -forward measure, i.e.

$$\phi_{T_{i-1}, T_i}(u) := \mathbb{E}^{Q^T}[\exp(iu(y(T_{i-1}, T_i))) \mid \mathcal{F}_t]. \tag{70}$$

First define

$$\Lambda := \exp(iu[z(T_i) - z(T_{i-1}) - \log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i)]). \tag{71}$$

Hence using the tower law for conditional expectations and the (conditional) characteristic function of our model (62), we obtain the following expression for the characteristic function of the (forward) log-return:

$$\begin{aligned} \phi_{T_{i-1}, T_i}(u) &= \mathbb{E}_n^{T_i}\{\Lambda \mid \mathcal{F}_t\} = \mathbb{E}_n^{T_i}\{\mathbb{E}_n^{T_i}[\Lambda \mid \mathcal{F}_{T_{i-1}}] \mid \mathcal{F}_t\} \\ &= \mathbb{E}_n^{T_i}\{\exp(iu[-z(T_{i-1}) - \log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i)]) \\ &\quad \cdot \mathbb{E}_n^{T_i}[\exp[iuz(T_i)] \mid \mathcal{F}_{T_{i-1}}] \mid \mathcal{F}_t\} \end{aligned}$$

$$\begin{aligned} &= \exp(iu[A_r(T_{i-1}, T_i) - A_n(T_{i-1}, T_i)] \\ &\quad + A(u, T_{i-1}, T_i)) \\ &\quad \cdot \mathbb{E}_n^{T_i} \left\{ \exp \left( iu[B_n(T_{i-1}, T_i)X_n(T_{i-1}) - B_r(T_{i-1}, T_i)X_r(T_{i-1})] + C(u, T_{i-1}, T_i)v(T_{i-1}) + \frac{1}{2}D(u, T_{i-1}, T_i)v^2(T_{i-1}) \right) \middle| \mathcal{F}_t \right\}. \end{aligned} \tag{72}$$

Although the latter expectation depends only on the (correlated) Gaussian variates  $x_n^k(T_{i-1})$ ,  $x_r^j(T_{i-1})$  and  $v(T_{i-1})$ , we also have that the integrated volatility process  $\int_t^{T_{i-1}} v(u)du$  arises in the real rate processes  $x_r^j(T_{i-1})$  (see, e.g., proposition 2.3). To this end, we decompose  $x_r^j(T_{i-1})$  into

$$x_r^j(T_{i-1}) = V^j(T_{i-1}) + \tilde{x}_r^j(T_{i-1}), \tag{73}$$

$$\begin{aligned} V^j(T_{i-1}) &:= \rho_{L,x_r^j} \sigma_r^j \int_t^{T_{i-1}} e^{-a_r^j(T_{i-1}-u)} v(u) du \\ &\sim N(\mu_r^j(t, T_{i-1}), \sigma_r^j(t, T_{i-1})), \end{aligned} \tag{74}$$

$$\begin{aligned} \tilde{x}_r^j(T_{i-1}) &= \mu_r^j(t, T_{i-1}) + \sigma_r^j \int_t^{T_{i-1}} e^{-a_r^j(T_{i-1}-u)} dW_{r_j}^{T_i}(u) \\ &\sim N(\mu_r^j(t, T_{i-1}), \sigma_r^j(t, T_{i-1})), \end{aligned} \tag{75}$$

where  $\mu_r^j(t, T_{i-1})$ ,  $\sigma_r^j(t, T_{i-1})$ ,  $\mu_V^j(t, T_{i-1})$  and  $\sigma_V^j(t, T_{i-1})$  are as defined in (B15), (B16), (B24) and (B25) (see appendix B.2).

Hence, we find that the characteristic function (72) is of the following Gaussian-quadratic form:

$$\begin{aligned} &\exp(iu[A_r(T_{i-1}, T_i) - A_n(T_{i-1}, T_i)] + A(u, T_{i-1}, T_i)) \\ &\quad \times \mathbb{E}_n^{T_i} \left\{ \exp \left( iu[B_n(T_{i-1}, T_i)X_n(T_{i-1}) - B_r(T_{i-1}, T_i)(\bar{V}(T_{i-1}) + \tilde{X}_r(T_{i-1}))] + C(u, T_{i-1}, T_i)v(T_{i-1}) + \frac{1}{2}D(u, T_{i-1}, T_i)v^2(T_{i-1}) \right) \middle| \mathcal{F}_t \right\} \\ &=: \mathbb{E}_n^{T_i} \{\exp[a_0 + a'Z + Z'BZ]\}, \end{aligned} \tag{76}$$

with  $a_0$  a constant,  $a'$  a row vector,  $B$  a matrix and where  $Z$  follows a multivariate standard normal distribution with correlation matrix  $S$ . Thus the random vector  $Z$  consists of the  $1 + K + 2M$  driving elements  $v$ ,  $x_n^1, \dots, x_n^K$ ,  $x_r^1, \dots, x_r^M$  and  $V^1, \dots, V^M$ . Note that since we are only dealing with one quadratic term (i.e.  $v^2(T_{i-1})$ ), we can reduce the quadratic form (76) of the random vector  $X$  to

$$\mathbb{E}_n^{T_i} \{\exp[a_0 + a'Z + b_0Z^{(1)2}]\}, \tag{77}$$

where the constants  $a_0$  and  $b_0$ , the column vector  $a$  and the correlation matrix  $S$  of the standard Gaussian vector  $Z$  can easily be deduced from (76) and are explicitly defined in appendix B.4.

Using the standard theory on Gaussian-quadratic forms (see, e.g., Feuerverger and Wong 2000 or

Glasserman 2003) we can now easily find an explicit solution for (76). Recalling that (76) is equivalent to the characteristic function (72) of the forward return on the log inflation index, we come to the following proposition.

**Proposition 4.2:** *Let  $C$  be a matrix (with typical element  $c_{i,j}$ ) satisfying  $C'C = S$  (e.g., by a Cholesky decomposition). Define*

$$p_j := \sum_{i=1}^{1+K+2M} c_{i,j} a^{(i)}, \tag{78}$$

$$q_1 := \sum_{i=1}^{1+K+2M} c_{i,1}^2 b_0, \tag{79}$$

with correlation matrix  $S$ , column vector  $a$  and constant  $b_0$  as defined in appendix B.4. Provided that  $q_1 < 1/2$ , the characteristic function of the forward log return  $y(T_{i-1}, T_i)$  (68) under the  $T_i$ -forward measure is given by the following closed-form solution:

$$\phi_{T_{i-1}, T_i}(u) = \frac{\left\{ \begin{array}{l} \exp(a_0 + [p_1^2/2q_1(1 - 2q_1)]) \\ -(p_1^2/4q_1) + \sum_{j=2}^{1+K+2M} (p_j^2/2) \end{array} \right\}}{\sqrt{1 - 2q_1}}. \tag{80}$$

**Proof:** Since (76) is equivalent to (72), the characteristic function of the forward return on the log inflation index is given by an explicit solution of the Gaussian-quadratic form (76), which is given by standard theory on quadratic forms (see, e.g., Feuerverger and Wong 2000 or Glasserman 2003).  $\square$

Equipped with the characteristic function of the log-inflation index return, the prices of year-on-year inflation-indexed swaps and inflation-indexed caps/floors are directly obtained by formulas (45) and (46).

### 4.3. Heston stochastic volatility

The characteristic function-based pricing method in our model with Heston (1993) stochastic volatility turns out to be somewhat more involved than under Schöbel and Zhu (1999) stochastic volatility. In fact, for the general model Heston (1993) stochastic volatility we need to resort to approximate methods for the pricing of inflation-indexed options.

Recall from (17) and (20) that the general model dynamics with Heston (1993) volatility under the  $T$ -forward measure  $\mathcal{Q}_n^T$  are given by

$$\frac{dI_F(t)}{I_F(t)} = (v(t) + \Sigma_n(t, T) - \Sigma_r(t, T))dW_*^T(t), \tag{81}$$

$$dv^2(t) = \kappa[\zeta(t) - v^2(t)]dt + \xi v(t)dW_v^T(t). \tag{82}$$

To derive the characteristic function of the log-inflation rate, one can in principle then pursue the same steps as in the model with Schöbel and Zhu (1999) volatility, that is solving the Kolmogorov backward equation for the log-inflation rate with a certain boundary condition. However, due to the square-root volatility process, the

Heston partial differential equation in combination with correlated Gaussian rates is, unfortunately, no longer affine. Hence, contrary to the previous model, there is (as far as we know) no exact closed-form expression for the characteristic function for this model. Nevertheless, in the case where we make the simplifying assumption that the rate processes are perpendicular to the stochastic volatility and the asset price processes, one can easily find a closed-form solution for its characteristic function. For the general case, we consider two alternative pricing methods.

- (1) A projection of the characteristic function of the general model onto the uncorrelated case.
- (2) A control variate-based pricing technique that uses an exact result from the uncorrelated model.

The setup of the following section is therefore as follows. We first discuss the pricing for the log-inflation rate and the log-inflation index return in the model with uncorrelated Heston (1993) stochastic volatility. We then describe a projection technique of the general case onto the uncorrelated model. Finally, although the projection is already found to work quite well, we also discuss the use of the approximate model as control variate in a Monte Carlo pricing procedure of the exact model.

#### 4.3.1. Characteristic function of the log-index price:

**Uncorrelated case.** For the derivation of the characteristic function of the uncorrelated model (i.e. with rate processes perpendicular to the variance and asset price process), we will use two propositions. First, let  $z(t) = \log[I(t)P_r(t, T)/P_n(t, T)]$  denote the  $T$ -forward log-asset price, with dynamics

$$dz(t) = -\frac{1}{2}v^2(t) + v(t)dW_I^T(t), \tag{83}$$

$$dv^2(t) = \kappa[\theta - v^2(t)]dt + \xi v(t)dW_v^T(t), \tag{84}$$

i.e. with stochastic interest rate dynamics. One then has the following proposition regarding the characteristic function of  $z(t)$ .

**Proposition 4.3:** *Conditional on time  $t$ , the characteristic function  $\phi_{HE}(u)$  of the  $T$ -forward log-asset price  $z(T)$  of the classical Heston (1993) model is given by*

$$\phi_{HE}(u) := \exp[iuz(t) + A_{HE}(u, t, T) + B_{HE}(u, t, T)v^2(t)], \tag{85}$$

where

$$A_{HE}(u, t, T) := \theta \kappa \xi^{-2} \left( (\kappa - \rho \xi iu - d)T - 2 \log \left( \frac{1 - g_2 e^{-dT}}{1 - g_2} \right) \right), \tag{86}$$

$$B_{HE}(u, t, T) := \xi^{-2} (\kappa - \rho \xi iu - d) \frac{1 - e^{-dT}}{1 - g_2 e^{-dT}}, \tag{87}$$

and with

$$d := \sqrt{(\rho\xi iu - \kappa)^2 + \xi^2(iu + u^2)}, \tag{88}$$

$$g_2 := \frac{\kappa - \rho\xi iu - d}{\kappa - \rho\xi iu + d}. \tag{89}$$

**Proof:** For the proof we refer to Heston (1993) or Gatheral (2005).  $\square$

Note that, in the literature, one can find two (mathematically) equivalent formulations for the Heston characteristic function: the one presented above can, for example, be found in Albrecher *et al.* (2005) and Gatheral (2005) and is free of a numerical difficulty called branch cutting, while another representation can be found in the original Heston (1993) paper or Kahl and Jäckel (2005), which may cause some numerical difficulties for certain model parameters (Albrecher *et al.* 2005).

The second proposition concerns the interest rates part of the inflation dynamics. To this end, define

$$R_{K,M}(t, T) := -\frac{1}{2}V_{K,M}(t, T) + \int_t^T [\Sigma_n(u, T)dW_n^T(u) - \Sigma_r(u, T)dW_r^T(u)]du. \tag{90}$$

We then come to the following proposition of the characteristic function of  $R_{K,M}(t, T)$ .

**Proposition 4.4:** *The characteristic function  $\phi_R(u)$  of  $R_{K,M}(t, T)$  (90) is given by*

$$\phi_{K,M}(u) := \exp\left[-\frac{1}{2}u(i+u)V_{K,M}(t, T)\right]. \tag{91}$$

**Proof:** As  $\int_t^T \Sigma_i(u, T)du$ ,  $i \in n, r$ , follows a Gaussian distribution with mean 0,  $R_{K,M}(t, T)$  as a sum of Gaussian variates is also Gaussian with mean  $-\frac{1}{2}V_{K,M}(t, T)$ . Using Fubini and Itô's isometry it then follows that  $R_{K,M}(t, T)$  is normally distributed with mean  $-\frac{1}{2}V_{K,M}(t, T)$  and variance  $V_{K,M}(t, T)$ , as explicitly given by (52). Moreover, the characteristic function  $\phi_{K,M}(u)$  of  $R_{K,M}(t, T)$  follows directly as a consequence of this normality.  $\square$

With the results from propositions 4.3 and 4.4, we can now easily determine the characteristic function of the log-inflation index in the uncorrelated model, which results in the following proposition.

**Proposition 4.5:** *The characteristic function  $\phi_F(u)$  for the log-inflation index  $\log I_F(t)$  of the uncorrelated Heston dynamics (81) is given by the following closed-form expression:*

$$\begin{aligned} \phi_F(u) &= \phi_{HE}(u) \cdot \phi_{K,M}(u) \\ &= \exp\left[iuz(t) + A_{HE}(u, t, T) + B_{HE}(u, t, T)v^2(t) - \frac{1}{2}u(i+u)V_{K,M}(t, T)\right]. \end{aligned} \tag{92}$$

**Proof:** Since the Brownian motions driving the Heston dynamics  $z(t)$ , i.e.  $W_I^T(t)$  and  $W_v^T(t)$ , are uncorrelated with the Brownian motions that drive the rate process  $R_{K,M}(t, T)$ , i.e.  $W_{n_i}^T(u)$  and  $W_r^T(u)$ , we can write the log-inflation index dynamics  $\log I_F(t)$  of the dynamics of (23) (or, equivalently, of (81)) as the sum of the above two processes, i.e.

$$\log I_F(t) = z(t) + R_{K,M}(t, T).$$

Since the driving Brownian motions are uncorrelated, we then have that  $z(t)$  is independent of  $R_{K,M}(t, T)$  and, furthermore, that the characteristic function  $\phi_F(u)$  of  $\log I_F(t)$  is given by the product of the characteristic functions of  $z(t)$  and  $R_{K,M}(t, T)$ .  $\square$

**4.3.2. Characteristic function of the log index return: Uncorrelated case.**

We now derive the (forward-starting) characteristic function of the log-inflation index return. As in our model from section 4.2.2, we follow Hong (2004) and van Haastrecht *et al.* (2008), that is we consider the characteristic function  $\phi_{T_{i-1}, T_i}(u)$  of the log-inflation index return

$$\begin{aligned} y(T_{i-1}, T_i) &:= \log\left(\frac{I(T_i)}{I(T_{i-1})}\right) = z(T_i) - z(T_{i-1}) \\ &\quad - \log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i). \end{aligned} \tag{93}$$

In particular, we want to resolve the characteristic function  $\phi_{T_{i-1}, T_i}(u)$  of  $y(T_{i-1}, T_i)$  under the  $T_i$ -forward measure. Using similar arguments (e.g., the tower law for conditional expectations) as in (72), we can obtain the following expression for the forward-starting characteristic function in our (uncorrelated) model:

$$\begin{aligned} \phi_{T_{i-1}, T_i}(u) &= \mathbb{E}_n^{T_i}\{\exp(iu[-z(T_{i-1}) - \log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i)]) \\ &\quad \cdot \mathbb{E}_n^{T_i}\{\exp[iuz(T_i)] \mid \mathcal{F}_{T_{i-1}}\} \mid \mathcal{F}_t\} \\ &= \exp\left(A_{HE}(u, T_{i-1}, T_i) - \frac{1}{2}u(i+u)V_{K,M}(T_{i-1}, T_i)\right) \\ &\quad \cdot \mathbb{E}_n^{T_i}\{\exp(iu[-\log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i)] + B_{HE}(u, T_{i-1}, T_i)v^2(T_{i-1})) \mid \mathcal{F}_t\}. \end{aligned} \tag{94}$$

Hence since the rate processes  $x_n^k(T_{i-1})$  and  $x_r^j(T_{i-1})$  are independent of the variance process  $v^2(T_{i-1})$ , we have

$$\begin{aligned} \phi_{T_{i-1}, T_i}(u) &= \exp\left(A_{HE}(u, T_{i-1}, T_i) - \frac{1}{2}u(i+u)V_{K,M}(T_{i-1}, T_i)\right) \\ &\quad \cdot \mathbb{E}_n^{T_i}\{\exp(iu[-\log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i)]) \mid \mathcal{F}_t\} \\ &\quad \cdot \mathbb{E}_n^{T_i}\{\exp[B_{HE}(u, T_{i-1}, T_i)v^2(T_{i-1})] \mid \mathcal{F}_t\}. \end{aligned} \tag{95}$$

Hence it remains to evaluate the expectations in the latter expression. Since the first expectation can be seen as the characteristic function of the log-bond prices, we have the following proposition.

**Proposition 4.6:** *The characteristic function  $\phi_{K,M}(u)$  of the log-bond prices in (95) under the  $T_i$ -forward measure is*

given by

$$\phi_{K,M}(u) = \exp\left[ iuh_0 - \frac{u^2}{2} h' S_R h \right], \quad (96)$$

with the constant  $h_0$ , column vector  $h$  and correlation matrix  $S_R$  respectively as defined in (B38), (B39) and (B40).

**Proof:** Note that one can write

$$-\log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i) =: h_0 + h' Z_R, \quad (97)$$

with  $Z_R$  the random Gaussian vector consisting of the normalized stochastic parts of the Gaussian factors  $x_n^1, \dots, x_n^K$  and  $x_r^1, \dots, x_r^M$ . Therefore, (97) is nothing more than the characteristic function of the normal distribution  $h_0 + h' Z_R$ , which is given by expression (96). Alternatively, one can see this expectation as a special case of the Gaussian-quadratic form (76) of the model in proposition 4.2, i.e. without the volatility components  $v(t)$  and  $V^j(t)$ .  $\square$

For the calculation of the second expectation of (95) we will use the following property of the square-root process  $v^2(T_{i-1})$ .

**Proposition 4.7:** *Provided that  $2cy < 1$ , the moment-generating function  $\phi_{v^2}(y)$  of  $v^2(T_{i-1})$  is given by*

$$\phi_{v^2}(y) = \mathbb{E}[\exp(yv^2(T_{i-1}))] = \frac{\exp([cy\lambda/(1-2cy)])}{(1-2cy)^{2\kappa\theta/\xi^2}}, \quad (98)$$

where

$$c := \frac{\xi^2(1 - e^{-\kappa(T_{i-1}-t)})}{4\kappa}, \quad (99)$$

$$\lambda := \frac{4\kappa e^{-\kappa(T_{i-1}-t)} v(s)}{\xi^2(1 - e^{-\kappa(T_{i-1}-t)})}. \quad (100)$$

**Proof:** The proposition follows directly from the fact that variance process  $v^2(T_{i-1})$  is distributed as a constant  $c$  times a non-central chi-squared distribution with  $4\kappa\theta/\xi^2$  degrees of freedom and non-centrality parameter  $\lambda$  (see, e.g., Cox *et al.* 1985).  $\square$

Hence we come to the following proposition for the characteristic function  $\phi_{T_{i-1}, T_i}(u)$  as in expression (95).

**Proposition 4.8:** *The forward-starting characteristic function  $\phi_{T_{i-1}, T_i}(u)$  of the model (proposition 2.2) with uncorrelated Heston (1993) stochastic volatility is given by the following closed-form expression:*

$$\phi_{T_{i-1}, T_i}(u) = \exp\left( A_{HE}(u, T_{i-1}, T_i) - \frac{1}{2} u(i+u) V_{K,M}(T_{i-1}, T_i) \right) \times \phi_{K,M}(u) \phi_{v^2}(B_{HE}(u, T_{i-1}, T_i)), \quad (101)$$

with  $A_{HE}(u, t, T_{i-1})$  and  $B_{HE}(u, t, T_{i-1})$  as defined in equations (86) and (87), and with  $\phi_{K,M}(u)$  and  $\phi_{v^2}(u)$  as in propositions 4.6 and 4.7.

**Proof:** The characteristic function (101) of the forward log-inflation index return follows directly by evaluating the two expectations of (95). The first expectation of (95)

equals the characteristic-generating function  $\phi_{K,M}(u)$  of the log-bond prices (97). The second expectation equals the moment-generating function  $\phi_{v^2}$  of the shifted non-central chi-squared distributed random variable  $v^2(T_{i-1})$ , evaluated at the point  $B(u, t, T_{i-1})$ .  $\square$

**4.3.3. Projection of the general case onto the uncorrelated model.**

Since in the general Heston model setup (i.e. with a full correlation structure) the affine structure is destroyed, it is challenging to find the characteristic function of the log-inflation index. We are not aware of a closed-form expression for the characteristic function in the Heston model with correlated Gaussian rates. Nevertheless, one can try to approximate the general dynamics by a simpler process for which a closed-form pricing expression does exist. Where a heuristic approach based on moment-matching techniques was suggested by van Haastrecht (2007), a more rigorous projection method was recently suggested by Antonov *et al.* (2008) that uses a Markovian projection technique of the general model onto the (affine) uncorrelated model. After the projected parameters are determined, one can just use the uncorrelated model and corresponding pricing formulas to price stock, foreign exchange and inflation derivatives. Although the Markovian projection technique is fast and works well for mild parameter settings and short maturities (i.e. when the ‘distance’ between the models is relatively small), the projection is rather involved and deteriorates for longer maturities and more extreme model parameters (i.e. when the ‘distance’ is relatively large), in particular for a large index-rate correlation in combination with a high volatility of the rates. For details on the Markovian projection and numerical results of the approximation, we refer the reader to Antonov *et al.* (2008).

**4.3.4. Monte Carlo pricing method for the general model.**

Instead of approximating the prices of vanilla options in the general Heston setup, e.g. by a projection technique as touched upon in subsection 4.3.3, one can also use a Monte Carlo procedure to price these options. Where the approximation formulas can be rather biased for certain model settings (e.g., see the discussion in subsection 4.3.3), a Monte Carlo estimate has the advantage that it converges to the true option price in the limit for the number of sample paths. Moreover, a Monte Carlo procedure is generic and is straightforward to implement (if not already implemented for exotic option pricing). The main practical disadvantage of a Monte Carlo calibration procedure is the speed with which vanilla options can be calculated within a certain error measure. Since one repeatedly needs to update an error function of the ‘distance’ between model and market vanilla prices, the speed of calculating these model option prices is rather important. Even though one can price multiple options (e.g., on different times) with one Monte Carlo run, the use of closed-form option pricing formulas is often much faster. Nevertheless, with the use of modern-day variance reduction techniques and the

ever-growing computational power (particularly the fact that the Monte Carlo procedure can easily be parallelized over multiple processors), we expect Monte Carlo techniques to become even more popular in the near future.

In this section we present a very effective control variate estimator for the pricing of vanilla options using the general Heston dynamics. To demonstrate its efficiency, we take the pricing of a vanilla call option as an example. To benchmark the numerical results against the Markovian projection, we consider the same hybrid equity-interest rate (stock) example as in Antonov *et al.* (2008). The setup of this section is as follows. We first discuss the control variate technique for the general model, after which we demonstrate which variance reductions can be expected and discuss its numerical efficiency.

**4.3.5. Uncorrelated price as control variate estimator.** As discussed in section 4.3.4, Monte Carlo pricing procedures might be easy to implement and quite generic, but often lack speed and are hence sometimes considered as ‘brute-force’ procedures. Nowadays, however, a whole variety of variance reduction techniques are available to boost the computational efficiency of the Monte Carlo run (see, e.g., Jäckel 2002 or Glasserman 2003 for an overview of such methods). One of these variance reduction techniques is the control variate estimator. The key idea behind this technique is that we can use the error in estimating a similar quantity (from which we know the theoretical value) as a control to correct for the Monte Carlo error for the unknown quantity (Glasserman 2003). The effectiveness of such a control variate depends explicitly on the correlation between the control and the (to be estimated) price. Thus if the control contains much information on the estimated price, it can correct quite a lot of Monte Carlo noise in the resulting estimator (and *vice versa*). Mathematically, it can be shown that, if the correlation between the control and the standard Monte Carlo estimator is correlated with correlation coefficient  $\rho$  in combination with an optimal control parameter, one obtains (on average) a variance reduction of

$$VR(\rho) = \frac{1}{1 - \rho^2}, \tag{102}$$

which can be enormous as  $\rho \rightarrow 1$  (see, e.g., Glasserman 2003).

Before turning to the control variate estimator, we first introduce some notation. Let  $\bar{C}^0$ ,  $\bar{C}^\rho$  and  $C_i^0$ ,  $C_i^\rho$ , respectively, denote the expected (European) call option price and the simulated call option prices for the general (superscript  $\rho$ ) and the uncorrelated (superscript 0) dynamics. Since we know the call option price  $C^0$  of the uncorrelated price in closed form by inverting (101), and this price is usually largely correlated with the call option price  $C^\rho$  of the general model, we propose to use  $C^0$  as a control for  $C^\rho$ . Since the prices are highly correlated, we expect to see large variance reductions of the control variate estimator  $\tilde{C}^\rho(b)$  over the ordinary estimator  $C^\rho$ ,

i.e. from formula (102). The resulting control variate estimator  $\tilde{C}^\rho(b)$  is given by

$$\tilde{C}^\rho(b) = \frac{1}{n} \sum_{i=1}^n (C_i^\rho - b(C_i^0 - \mathbb{E}[C^0])), \tag{103}$$

where  $b$  is a real coefficient. The optimal coefficient  $b^*$  that minimizes the variance of (103) can easily be calculated and is explicitly given by

$$b^* = \frac{\sigma_{C^\rho}}{\sigma_{C^0}} \rho_{C^0, C^\rho} = \frac{\text{Cov}[C^0, C^\rho]}{\text{Var}[C^0]}. \tag{104}$$

Note that one often also needs to estimate  $b^*$  from the simulations and this might induce some bias in the effectiveness (102) of the control variate. However, as discussed by Glasserman (2003), this bias is often very small. In the case where  $\rho_{C^0, C^\rho}$  is close to one and  $\sigma_{C^\rho} \approx \sigma_{C^0}$  (which is often the case), it may even be more efficient to just set  $b^*$  equal to one (since one does not have to estimate  $b^*$  (Glasserman 2003)). The quality of the control variate estimator is investigated in section 5.1.

**5. Applications and numerical results**

In this section, we look at two applications of the model. First, for an equity example and with Heston (1993) stochastic volatility, we test the quality of the control variate estimator  $\tilde{C}^\rho$  of (103), compare it with the Markovian projection technique of Antonov *et al.* (2008) and discuss its practical applicability in a Monte Carlo calibration and/or pricing procedure. Secondly, we consider two applications (one with Heston (1993) and one with Schöbel and Zhu (1999) stochastic volatility) in which we calibrate our model to FX (option) market data. The example explicitly takes into account the pronounced long-term FX implied volatility skew/smile that is present in markets. Finally, the results are compared and analysed.

**5.1. Quality of the control variate estimator**

To test the numerical quality of the control variate estimator  $\tilde{C}^\rho$  of (103), we turn to the pricing of (European) call options under the general hybrid Heston dynamics. To this end, we consider two different parameter settings, listed in table 1. Both test cases roughly correspond to parameter settings that are likely to be encountered in medium- to long-maturity equity markets. The first test case is prevalent in the existing literature. Similar Heston parameter settings, in a pure equity context, are considered by Broadie and Kaya (2006), Andersen (2007) and Lord *et al.* (2008). The second test case is taken from Antonov *et al.* (2008), where it serves to test their Markovian projection approximation, i.e. as touched upon in section 4.3.3. Using these test cases, we first look at the quality of the control as a function of the equity rate correlation coefficient and, secondly, we investigate the efficiency of the control variate estimator (103) as a function of the



Table 1. Numerical test cases for the control variate estimator (103).  $y_r$  denotes the continuous (constant) interest rate yield and  $y_q$  the continuous (constant) dividend yield.

Example	$\kappa$	$\xi$	$\rho_{I,v^2}$	$v(0)$	$\theta$	$y_r$	$y_q$	$a_n$	$\sigma_n$	$\rho_{I,x_n^1}$	$\rho_{v^2,x_n^1}$
Case I	2.0	1.0	-0.3	0.09	0.09	0.04	0.0	0.03	0.007	*	0.0
Case II	0.25	0.625	-0.4	0.0625	0.0625	0.05	0.02	0.05	0.01	0.30	0.15

\*This parameter was varied during the experiments and in all cases  $I(0) = 100$ .

Table 2. Expected variance reductions when using the control variate estimator of (103) instead of the standard Monte Carlo estimator. For various values of  $\rho_{I,x_n^1}$  the expected reduction for a three-year call option with an at-the-money strike is calculated using the estimates  $\hat{b}$  and  $\hat{\rho}_{C^0,C^p}$ , respectively, for the optimal control coefficient and the correlation between the control and the estimated quantity. Parameter settings from case I of table 1. Results based on 50,000 pseudo-random paths.

$\rho_{I,x_n^1}$	$\hat{\rho}_{C^0,C^p}$ (%)	$\hat{b}$	Var. red.	$\rho_{I,x_n^1}$	$\hat{\rho}_{C^0,C^p}$ (%)	$\hat{b}$	Var. red.
-0.9	99.859	0.960	356	0.9	99.864	1.034	367
-0.8	99.911	0.965	562	0.8	99.913	1.031	574
-0.7	99.940	0.970	839	0.7	99.941	1.027	852
-0.6	99.960	0.974	1254	0.6	99.961	1.024	1268
-0.5	99.974	0.979	1937	0.5	99.974	1.020	1950
-0.4	99.984	0.983	3188	0.4	99.984	1.016	3202
-0.3	99.992	0.987	5888	0.3	99.992	1.012	5902
-0.2	99.996	0.992	13,597	0.2	99.996	1.008	13,614
-0.1	99.999	0.996	55,209	0.1	99.999	1.004	55,252
0	100	1	$\infty$				

option maturity and compare it with the Markovian projection technique of Antonov *et al.* (2008).

**5.1.1. Results for case I.** Although the uncorrelated price is often highly correlated with the price of the general model, the efficiency is dependent on the specific model parameters. For example, note that, for  $\rho_{I,x_n^1} = \rho_{v^2,x_n^1} = 0\%$ , the control variate estimator is exact, because in that case the uncorrelated price equals the required estimate. Although the effectiveness depends on both correlation parameters, the impact of the correlation rate-vol is usually much smaller than the impact of the rate-stock correlation (see, e.g., Antonov *et al.* 2008 or van Haastrecht *et al.* 2008). Moreover, from a practical point of view, the rate-stock parameter is most important for the pricing and hedging of hybrid equity-interest rate securities. We therefore restrict ourselves to investigate the impact of the rate-stock parameter on the quality of the control variate estimator. We look at the (empirical) variance reductions for a three-year call option with an ATMF (at-the-money-forward) strike level of 100% for different  $\rho_{I,x_n^1}$ . The results can be found in table 2.

From the table (the case  $\rho_{I,x_n^1} = 1$  does not fit in a valid correlation matrix and is hence omitted), we can see that the control is very effective in all cases, i.e. resulting in large to huge variance reductions. As expected, the variance reductions become larger for smaller absolute values of  $\rho_{I,x_n^1}$ . For the case  $\rho_{I,x_n^1} = 0$ , the control is perfect and results in a zero variance control variate estimator, whereas for larger values of  $|\rho_{I,x_n^1}|$  the correlation between the ‘uncorrelated’ and ‘correlated’ option prices is smaller and therefore reduces the effectiveness of the control, as is

theoretically underpinned by formula (102). Thus from table 2 we can see that the effectiveness of the control, i.e. the resulting variance reduction, depends to a large extent on the absolute value of the correlation ‘between’ interest rates and equity underlying. Finally, it is worth mentioning that because of  $\hat{\rho}_{C^0,C^p}$  and  $\sigma_{C^p}/\sigma_{C^0}$ , the (estimated) optimal coefficients  $\hat{b}$  are also close to one. In such a situation it might be more efficient to just set  $b^* = 1$  and consequently save the computational effort in estimating  $\hat{\rho}_{C^0,C^p}$  (Glasserman 2003).

**5.1.2. Results for case II.** The second test case of table 1 consists of an experiment where we investigate the variance reductions of (103) over the standard Monte Carlo estimator for European call options of different maturities and strikes. Furthermore, since the same parameters were used by Antonov *et al.* (2008), we can use these results to draw a comparison between the Monte Carlo control variate estimator and the Markovian projection technique. The numerical results are given in table 3. From the table, we can see that the control variate estimator by far outperforms the ordinary Monte Carlo estimator. For short- to moderate-maturity options the control variate shows large to huge variance reduction factors, varying from 629 to 7938. For middle- to long-term options, the variance reductions are smaller, but still quite reasonable with reductions from 54 to 371. If we look at the variance reductions over different strike levels, the differences are somewhat smaller. It is worth noting that, for a fixed maturity, the control variate is most effective for out-of-the-money options, which are usually the hardest options to value by (plain) Monte Carlo.

Table 3. Variance reductions for case I of table 1 using 50,000 pseudo-random paths. Reported is the variance reduction factor ('Var. red. '), i.e. the fraction between the variance of the control variate (103) and the standard Monte Carlo estimator.

Maturity (years)	Strike	Sim. vol. (std. dev.)* (ordinary MC)	MP error*	Var. red. (CV variate MC)
1	86.07	24.45 (0.06)	0.04	6381
1	92.77	22.25 (0.05)	0.02	5884
1	100.00	20.36 (0.05)	-0.04	5717
1	107.79	19.42 (0.05)	-0.08	6549
1	116.18	19.67 (0.06)	-0.03	7938
3	77.12	22.61 (0.08)	0.03	661
3	87.82	20.05 (0.08)	0.01	622
3	100.00	17.95 (0.09)	-0.04	629
3	113.87	17.23 (0.13)	-0.09	763
3	129.67	18.02 (0.18)	-0.09	985
5	71.50	21.89 (0.06)	0.06	250
5	84.56	19.43 (0.05)	0.02	240
5	100.00	17.49 (0.06)	-0.05	246
5	118.26	16.83 (0.08)	-0.11	295
5	139.85	17.55 (0.12)	-0.13	371
10	62.23	21.55 (0.07)	0.06	98
10	78.89	19.52 (0.07)	0.00	100
10	100.00	18.01 (0.08)	-0.10	106
10	126.77	17.41 (0.11)	-0.19	124
10	160.70	17.75 (0.16)	-0.24	152
20	51.13	22.28 (0.06)	0.03	54
20	71.50	20.91 (0.06)	-0.05	55
20	100.00	19.94 (0.06)	-0.17	57
20	139.85	19.44 (0.09)	-0.27	63
20	195.58	19.40 (0.13)	0.35	72

\*Results taken from Antonov *et al.* (2008).

We conclude the analysis by comparing the Monte Carlo control variate estimator (103) with the Markovian projection technique. The result of the best projection technique of Antonov *et al.* (2008) is denoted by Heston DV (displaced volatility) and can be found in the fourth column of table 3. The most crucial difference between the methods is that the Markovian projection technique is, in principle, a biased approximation, whereas the control variate is unbiased and converges to the true price. However, in practice, one often only has a limited computational budget available and one will also note bias in the Monte estimates as a consequence of the limited number of simulations; the bias could be larger than the error in the approximation. Essentially, the choice between the methods constitutes a balance between speed and accuracy, which might differ across applications and products. Nevertheless, let us consider one concrete example. Consider the pricing of a 10-year option ATMF call option and, for arguments sake, assume that the Monte Carlo volatility of 18.01 is in fact the true volatility and hence the Markovian projection error is 0.10. We can then ask ourselves how many simulations are needed to improve the error of this approximation in at least 90% of cases. By definition, 90% of all the spanned confidence intervals should contain the 'true' price of 18.01, hence to improve the MP error we should aim to obtain a standard deviation of the Monte Carlo estimated volatility smaller than  $0.10/\Psi^{-1}(95\%)=0.061$  (or, equivalently, a variance smaller than  $0.061^2$ ). Using the fact that the Black and Scholes (1973) ATMF price is close to linear as a function of the

volatility, taking the standard deviation 0.08 of the simulated volatility and the variance reduction factor 108 of the above table and assuming a convergence rate of the Monte Carlo of one over the square root of the number of simulations  $N$ , one finds that one needs

$$M = \frac{\text{Var}_N}{\text{Var}_{\text{REQ}}} \frac{N}{\text{VR}} = \frac{0.08^2}{0.061^2} \frac{50,000}{108} = 802$$

simulations to improve upon the MP error in 90% of cases, with VR the variance reduction factor and where  $\text{Var}_{\text{REQ}}$  represents the required variance corresponding to a confidence level of  $1 - \alpha = 90\%$ . If we, for example, take  $\alpha = 50\%$ , one finds that, on average, one only has to use 134 simulations to perform 'equally as well' as the MP projection. Hence due to the large variance reductions, only a very moderate number of simulations is needed to arrive at a good estimate. Although the above analysis is too small to draw very strong conclusions concerning the comparison between the MP projection technique and the control variate, the main conclusion we would like to draw is that only a moderate number of simulations is required to obtain reliable price/volatility estimates for the above call options. In most situations, a couple of thousand paths will suffice to obtain prices that lie within typical bid-ask spreads.

Finally, we would also like to point out that the MP projection might also be used in conjunction with the control variate estimator (103) in a model calibration procedure. The first point (in future research) could be to investigate the quality of the MP projection as a control

for the exact dynamics. Secondly, in a practical implementation, one might first use the MP approximation to calibrate the model (which consists of most of the iterations) and consecutively use the control variate to refine the (near) optimal parameters found in the previous minimization. Note that (for each new parameter guess), one only needs a single Monte Carlo run to price all options simultaneously. In this way (assuming one uses a sufficiently large number of paths in the last few optimization steps using Monte Carlo) one can obtain the best of both worlds, i.e. the speed of an approximating formula combined with the accuracy of the control variate estimator.

### 5.2. Calibration to the FX market

We test our model by calibrating it to FX option market data. To this end, we consider the same vanilla FX data (see appendix C) as considered by Piterbarg (2005), who uses this set for the calibration of his local volatility model. In an elegant paper, Piterbarg (2005) concludes that, for the pricing and managing of exotic FX derivatives (i.e. PRDCs), it is essential to take the FX implied volatility skew/smile into account. Hence, although FX model setups may differ, i.e. local volatility in Heston (1993) and Piterbarg (2005) stochastic volatility with independent stochastic interest rate drivers in Andreasen (2006) and our stochastic volatility model with multi-factor Gaussian rates and Heston (1993) or Schöbel and Zhu (1999) volatility under a full correlation structure, all these models share the essential feature of explicitly accounting for the FX skew/smile.

For the calibration results of our model we consider the same interest rate and correlation parameters as in Piterbarg (2005), that is the interest curves in the domestic (Japanese yen) and foreign (US dollar) economies are given by

$$\begin{aligned} P_n(0, T) &= \exp(-0.02 \cdot T), \\ P_r(0, T) &= \exp(-0.05 \cdot T), \end{aligned}$$

and the one-factor Hull and White (1993) interest rate parameters for the interest rate evolutions in both currencies are given by

$$\begin{aligned} a_n(t) &:= 0.0\%, & \sigma_n(t) &:= 0.70\%, \\ a_r(t) &:= 5.0\%, & \sigma_r(t) &:= 1.2\%. \end{aligned}$$

The correlation parameters are given by

$$\begin{aligned} \rho_{n,r} &= 25.00\%, & \rho_{I,n} &= \rho_{I,r} = -15.00\%, \\ \rho_{n,v} &= \rho_{r,v} &= 0.00\%. \end{aligned}$$

Note that our stochastic volatility model has the additional flexibility of correlating the domestic or foreign exchanges with the volatility drivers (i.e. through  $\rho_{n,v}$  or  $\rho_{r,v}$ ). However, for simplicity we fix them to zero here. The initial spot FX rate (yen per dollar) is set at 105.00. The 10 expiry dates considered in the calibration and the seven strikes considered per date are given in table C1

of appendix C. For each maturity  $T_n$ , the strikes  $K_i(T_n)$  are computed using the formula

$$\begin{aligned} K_i(T_n) &= F(0, T_n) \exp(0.1 \cdot \delta_i \sqrt{T_n}), \\ \delta_i &\in \{-1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5\}. \end{aligned} \quad (105)$$

In particular, note that the fourth strike level corresponds to the forward FX rate for that date. The implied volatilities corresponding to the above strikes and maturities can be found in table 5 of appendix C. With the above setup, we consider in the next section how well the models (proposition 2.2), i.e. with Heston (1993) and Schöbel and Zhu (1999) stochastic volatility, fit the market implied volatilities of table C2.

### 5.3. Calibration results

We calibrate the models (proposition 2.2) with Schöbel and Zhu (1999) and Heston (1993) stochastic volatility to the various maturities by minimizing the differences between model and market implied volatilities using a local optimization method. The differences are reported in tables C3 and C4 of appendix C. For visual comparison, we present the calibration results for a few maturities in figure 1.

We first consider the model (proposition 2.2) with Schöbel and Zhu (1999) stochastic volatility. The model produces a good fit to the market, as can be seen from table C3 and figure 1, with differences smaller than 0.50% at most points and with a good fit around the at-the-money-forward volatilities and the slope of the volatility skews for each maturity. The model produces calibration results similar to the models of Piterbarg (2005) and Andreasen (2006). The low-strike (in-the-money call) options are underestimated by the model, which seems to have a slight difficulty in fitting the tails of the implied volatility structure, suggesting the addition of an extra factor, e.g. a trivial extension including Poisson-type jumps. Nonetheless, the smiles produced by the model are much closer to the market than a log-normal model would indicate, in particular for in- and out-of-the-money options.

Secondly, we consider the model (proposition 2.2) with Heston (1993) stochastic volatility. For simplicity, we consider uncorrelated stochastic volatility, as we can then directly price the required FX options in closed form. Nonetheless, the results of calibration to call option prices should be very similar to Antonov *et al.* (2008), where the parameters of the general model can often be projected onto parameters of the uncorrelated model, while to a large extent preserving option prices and model characteristics. The calibration results are shown in figure 1 and table C4 of appendix C. We can see that the model again produces a very good fit to the market, with differences now even smaller than 0.30% at most points and with excellent fits across moneyness and maturities. It seems that the Heston (1993) model is slightly better in fitting the extreme/convex FX skew that we calibrated against. In a way it is able to capture both the volatility part of the at-the-money prices, as well as the extremes of the in- and

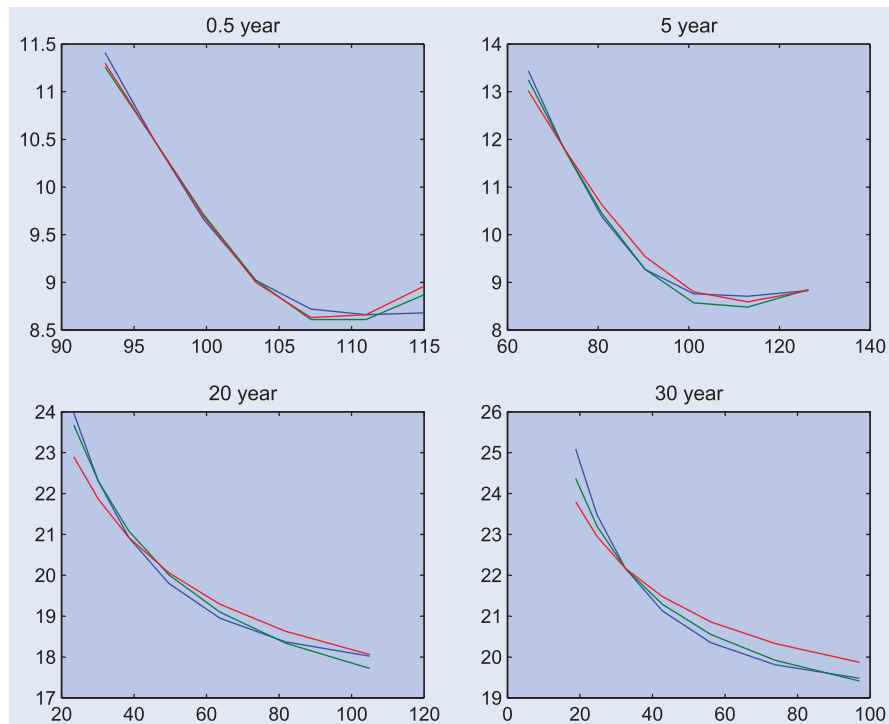


Figure 1. Calibration results for the model with Schöbel–Zhu and Heston stochastic volatility. For maturities 0.5, 5, 20 and 30 the implied volatilities (vertical axis) are plotted versus the corresponding strikes (horizontal axis). The market data are in blue, the model (proposition 2.2) with Schöbel–Zhu volatility in red and the model (proposition 2.2) with Heston volatility in green.

out-of-the-money prices. Alternatively, one can argue that the addition of an extra factor is still needed for the pricing of certain exotic options (see, e.g. Fouque *et al.* 2000 and van der Ploeg 2006), a discussion of which is, however, beyond the scope of this article.

As shown by Piterbarg (2005) and Andreasen (2006), it is of crucial importance to take the FX skew into account for the pricing and managing of exotic FX structures such as PRDCs (power reverse dual contracts) or cliquets. Therefore, since the skews/smiles generated by our stochastic volatility models are much closer to the market than produced by a log-normal model, we can conclude that our stochastic volatility model(s) (proposition 2.2) is better suited to price and manage these exotic FX structures. Finally, although the models of Piterbarg (2005) and Andreasen (2006) account for the FX skew, our model stands out as we model stochastic volatility (versus local volatility used by Piterbarg 2005) and stochastic interest rates, while allowing all driving model factors to be instantaneously correlated with each other (versus independent Gaussian rates used by Andreasen (2006)). Having this flexibility yields a realistic model that is of practical importance for the pricing and hedging of options with long-term FX exposure.

Given data on FX prices, our model can also be used to examine the pricing and hedging performance of products that explicitly depend on future volatility smiles. For FX it is not only the terminal volatility that is important, but also other liquid vol-sensitive instruments may need to be taken into account, depending on the product to be priced. More exotic options such as barrier or double-no-touch options are standard calibration instruments in

the FX space. The calibration to other vol-sensitive instruments and an empirical study of the relative performance of the stochastic volatility models discussed here versus other models for FX (see, e.g., Kainth and Saravanamuttu 2007) is left for future research.

#### 5.4. Calibration to inflation markets

In a recent paper, Mercurio and Moreni (2009) considered the pricing of inflation-indexed year-on-year and zero-coupon caps/floors using a market model with SABR (Hagan *et al.* 2002) stochastic volatility dynamics for year-on-year inflation rates and a log-normal Libor market model for nominal interest rates. Other market model approaches for inflation can, for instance, be found in Belgrade *et al.* (2004), Brigo and Mercurio (2006) and Kenyon (2008). Compared with the latter models, the approach considered by Mercurio and Moreni (2009) stands out by reconciling both zero-coupon and year-on-year quotes. Similar to the framework considered in this paper, these authors consider a full correlation structure between the stochastic quantities underlying the model, whilst preserving closed-form and flexible calibration methods for calibration to market option data.

The differences between market models and low-dimensional Markov models, as considered in this paper, have been described by several authors (see, e.g., Pelsser 2000 and Brigo and Mercurio 2006). Market models explicitly model observable quantities (e.g., year-on-year inflation rates) and due to their dimensionality provide greater calibration flexibility than low-dimensional

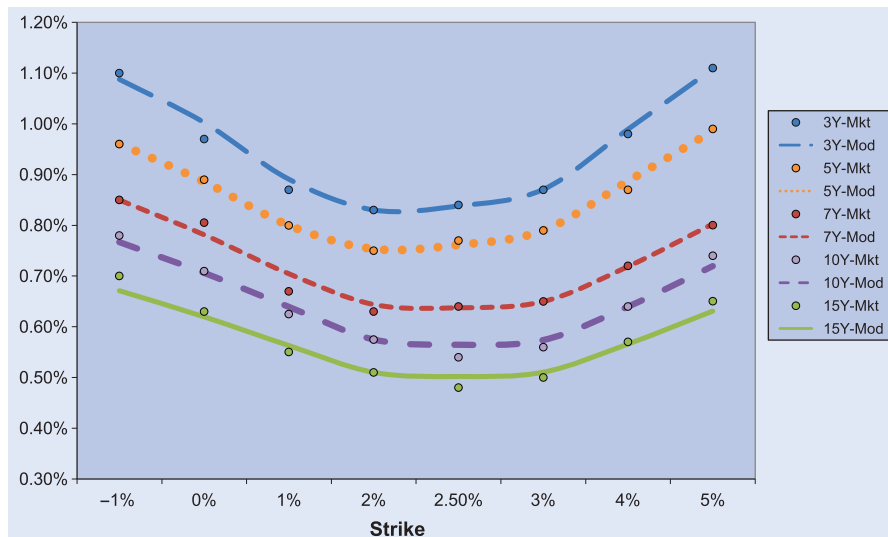


Figure 2. Calibration results. Market and model implied volatilities for caplets/floorlets maturing in 3, 5, 7, 10 and 15 years. Calibration results for the model (proposition 2.2) with Schöbel–Zhu volatility. Data for 4 September 2008, corresponding to the uncorrelated case of figure 1 of Mercurio and Moreni (2009).

Markov models. On the other hand, the dimensionality of such models can also be disadvantageous. For instance, due to a lack of calibration instruments in less liquid markets (such as inflation options), hedges and calibrations may become unstable when using market models (see, e.g., Jäckel and Bonneton 2010). These models also tend to be relatively slow compared with low-dimensional Markov models (see, e.g., Glasserman 2003). In this sense, both market and low-dimensional market models show (dis)advantages and the model choice ultimately depends on the exotic product one wants to price.

It is, however, insightful to compare models based on their calibration capabilities. To this end, we compare calibrations of the model (proposition 2.2) with Schöbel and Zhu (1999) stochastic volatility and one-factor interest rates with the SABR model of Mercurio and Moreni (2009). To test the calibration of this model, we use the same market data and assumptions as in the first case of Mercurio and Moreni (2009) and hence calibrate the model to a set of caplets and floorlets. Adopting this setup has the additional advantage that it enables us to make a proper comparison between these methods. For a further description of the market data we refer the reader to that paper. Calibration results are shown in figure 2. We can see from this figure, in which market and model implied volatilities are reported, that the fit is accurate. We note that the market data display small non-smooth behavior where cap and floor quotes meet (strikes 2–2.5%) or on single strikes (e.g., the 7 year 0% floorlet). Similar to Mercurio and Moreni (2009), we consider these discrepancies as being essentially bound to liquidity reasons and stress that parameterized models also provide useful smoothing tools for such market data. The calibration results of the models with Schöbel–Zhu and SABR stochastic volatility are very similar. Both models are well able to qualitatively fit the shape of the implied volatility, whilst they are also capable of detecting small

market anomalies. Because the liquidity of inflation options is not that large, as can be seen from the wide bid/ask spreads for inflation caps and floors, such smoothing of market data might be very useful, as indicated by Mercurio and Moreni (2009) and Jäckel and Bonneton (2010). We therefore conclude that whilst the Schöbel–Zhu stochastic volatility model has all the advantages of a low-dimensional Markov model, it is also sufficiently flexible to fit prices of vanilla inflation-indexed options in an accurate way.

## 6. Conclusion

We have introduced a generic model incorporating stochastic interest rates and stochastic volatility under a full correlation structure of all driving model factors, with closed-form pricing formulas for vanilla options and which is able to incorporate the market's implied volatility structures. Having the flexibility to correlate the underlying FX/inflation/stock index with both the stochastic volatility and the stochastic interest rates yields a realistic model that is of practical importance for the pricing and hedging of options with long-term exposure. Furthermore, closed-form pricing of vanilla FX, inflation and stock options is a big advantage for the calibration (and sensitivity analysis) of the model. Using Fourier methods, we have shown how vanilla call/put options, forward starting options, year-on-year inflation-indexed swaps and inflation-indexed caps/floors can be valued in closed form. It should be noted that our model can cover Poisson-type jumps with a trivial extension. Under Schöbel and Zhu (1999) stochastic volatility, using its affine properties, we were able to derive the corresponding characteristic functions in closed form. Under Heston (1993) stochastic volatility, the characteristic functions can only be derived explicitly under special zero

correlation assumptions. Nonetheless, we have demonstrated that one can still use these pricing formulas, either by using a projection of the general model onto the uncorrelated case, or by using it as a control variate for the general model. The latter method even results in such large variance reductions that its incorporation in the calibration becomes more than a viable option. Our model can be used for multi-asset purposes (e.g., interest rates, FX, inflation, equity, commodities) and is fast enough for the real-life risk management of large portfolios of such products. We believe that it is particularly suitable for the pricing and hedging of long-dated multi-currency structures (e.g., hybrid TARN options, variable annuities, inflation LPI options and PRDC FX swaps), which are sensitive to both future interest rate evolution as well as movements from the underlying index and/or corresponding volatility smiles.

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**Appendix A: Deriving the characteristic function of the log ‘Schöbel–Zhu’ inflation rate**

In this appendix we prove that the partial differential equation (61), i.e.

$$0 = f_t - \frac{1}{2} v_F^2(t) f_z + \kappa(\xi(t) - v(t)) f_v + \frac{1}{2} v_F^2(t) f_{zz} + \left( \rho_{Iv} \tau v(t) + \sum_{i=1}^K \rho_{x_n^i} \tau \sigma_n^i B_n^i(t, T) - \sum_{j=1}^M \rho_{x_r^j} \tau \sigma_r^j B_r^j(t, T) \right) f_{zv} + \frac{1}{2} \tau^2 f_{vv}, \quad (A1)$$

subject to the terminal boundary condition  $f(T, y, \sigma) = \exp(iuy(T))$ , has a solution given by (62)–(67). To solve this differential equation, we use the ansatz (62), find the corresponding partial derivatives and substitute them into (61). We then obtain a system of ordinary differential equations that is similar to the one-factor model of van Haastrecht *et al.* (2008) and which can be solved using similar methods.

Expanding  $v_F^2(t)$  according to (50) and collecting the terms for  $y(t)$ ,  $v(t)$  and  $\frac{1}{2} v^2(t)$  yields the following system of ordinary differential equations for the functions  $A(u, t, T), \dots, D(u, t, T)$ :

$$0 = \frac{\partial B(u, t, T)}{\partial t} \Rightarrow B(u, t, T) := B, \quad (A2)$$

$$0 = \frac{\partial D(u, t, T)}{\partial t} - 2(\kappa - \rho_{xv} \tau B) D(u, t, T) + \tau^2 D^2(t) + (B^2 - B), \quad (A3)$$

$$0 = \frac{\partial C(u, t, T)}{\partial t} + (\rho_{xv} \tau B - \kappa + \tau^2 D) C(u, t, T) + \left\{ \sum_{i=1}^K [\rho_{I x_n^i} \sigma_n^i B_n^i(t, T)] - \sum_{j=1}^M [\rho_{I x_r^j} \sigma_r^j B_r^j(t, T)] \right\} (B^2 - B) + \left\{ \kappa \xi(t) + \left( \sum_{i=1}^K [\rho_{x_n^i} \tau \sigma_n^i B_n^i(t, T)] - \sum_{j=1}^M [\rho_{x_r^j} \tau \sigma_r^j B_r^j(t, T)] \right) B \right\} D(u, t, T), \quad (A4)$$

$$0 = \frac{\partial A(u, t, T)}{\partial t} + \left[ \kappa \xi(t) + \sum_{i=1}^K \rho_{x_n^i} \tau \sigma_n^i B_n^i(t, T) B - \sum_{j=1}^M \rho_{x_r^j} \tau \sigma_r^j B_r^j(t, T) B \right] C(u, t, T) + \frac{1}{2} \tau^2 (C^2(u, t, T) + D(u, t, T)) + \frac{1}{2} (B^2 - B) v_{K,M}^2(t, T), \quad (A5)$$

with  $v_{K,M}^2(t, T)$  the instantaneous variance of the Gaussian rate processes (see (51)). As already hinted at in equation (A2), it immediately follows that  $B(u, t, T) = B$  equals a constant since its derivative is zero. Subject to the boundary condition (55) we then find

$$B = iu. \quad (A6)$$

The second equation (A3) yields a Riccati equation with constant coefficients and boundary condition  $D(u, T, T) = 0$ , which is equivalent to the PDE for the  $D$  term in the SZHW model (van Haastrecht *et al.* 2008) and has the following solution:

$$D(u, t, T) = -u(i + u) \frac{1 - e^{-2\gamma(T-t)}}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}}, \quad (A7)$$

with

$$\gamma = \sqrt{(\kappa - \rho_{xv} \tau B)^2 - \tau^2 (B^2 - B)}, \quad (A8)$$

$$\gamma_1 = \gamma + \frac{1}{2} q_1 = \gamma + (\kappa - \rho_{xv} \tau B), \quad (A9)$$

$$\gamma_2 = \gamma - \frac{1}{2} q_1 = \gamma - (\kappa - \rho_{xv} \tau B). \quad (A10)$$

The third equation (A4) for  $C(u, t, T)$  looks pretty daunting, but is merely a first-order linear differential equation of the form  $[\partial C(u, t, T)/\partial t] + g(t)C(u, t, T) + h(t) = 0$ , with associated boundary condition  $C(u, T, T) = 0$ . Hence we can represent a solution for  $C(u, t, T)$  as

$$C(u, t, T) = \int_t^T h(s) \exp \left[ \int_t^s g(v) dv \right] ds, \quad (A11)$$

with

$$g(v) = -(\kappa - \rho_{xv} \tau B) + \tau^2 D(u, v, T), \quad (A12)$$

$$\begin{aligned}
 h(s) &= \left( \kappa \xi(u) + \sum_{i=1}^K \left( \rho_{x_i^i, v} \tau \sigma_n^i B_n^i(s, T) \right. \right. \\
 &\quad \left. \left. - \sum_{j=1}^M \rho_{x_i^j, v} \tau \sigma_r^j B_r^j(s, T) \right) B \right) D(u, s, T) \\
 &\quad + \left( \sum_{i=1}^K \rho_{I_{x_i}^i} \sigma_n^i B_n^i(s, T) - \sum_{j=1}^M \rho_{I_{x_r}^j} \sigma_r^j B_r^j(s, T) \right) (B^2 - B) \\
 &= \kappa \psi D(u, s, T) \\
 &\quad + \sum_{i=1}^K \{ \rho_{I_{x_i}^i} \sigma_n^i B_n^i(s, T) (B^2 - B) \\
 &\quad + [\rho_{x_i^i, v} (B - 1) \tau \sigma_n^i B_n^i(s, T)] D(u, s, T) \} \\
 &\quad - \sum_{j=1}^M \{ \rho_{I_{x_r}^j} \sigma_r^j B_r^j(s, T) (B^2 - B) \\
 &\quad + [\rho_{x_r^j, v} B \tau \sigma_r^j B_r^j(s, T)] D(u, s, T) \}. \tag{A13}
 \end{aligned}$$

We first consider the integral over  $g$ . Dividing equation (A12) by  $D(u, t, T)$ , rearranging terms and integrating we find the surprisingly simple solution

$$\begin{aligned}
 \int g(v) dv &= \int -(\kappa - \rho_{xv} \tau B) + \tau^2 D(u, v, T) dv \\
 &= \int (\kappa - \rho_{xv} \tau B) - \frac{(B^2 - B)}{D(u, v, T)} \\
 &\quad - \frac{\partial D(u, v, T)}{\partial v} \frac{1}{D(u, v, T)} dv \\
 &= \log(\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}) + c, \tag{A14}
 \end{aligned}$$

where  $\gamma$ ,  $\gamma_1$  and  $\gamma_2$  are defined in (67) and with  $c$  the integration constant. Hence taking the exponent and filling in the required integration boundaries yields

$$\exp \left[ \int_t^s g(v) dv \right] = \frac{\gamma_1 e^{\gamma(T-s)} + \gamma_2 e^{-\gamma(T-s)}}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}}. \tag{A15}$$

Hence substituting this expression into (A11) we find (after a long but straightforward calculation) for  $C(u, t, T)$

$$\begin{aligned}
 C(u, t, T) &= -\frac{u(i+u)}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}} \left\{ \gamma_0 (1 + e^{-2\gamma(T-t)}) \right. \\
 &\quad + \sum_{i=1}^K [(\gamma_3^i - \gamma_4^i e^{-2\gamma(T-t)}) - (\gamma_5^i e^{-a_n^i(T-t)}) \\
 &\quad - \gamma_6^i e^{-(2\gamma+a_n^i)(T-t)} - \gamma_7^i e^{-\gamma(T-t)}] \\
 &\quad - \sum_{j=1}^M [(\gamma_8^j - \gamma_9^j e^{-2\gamma(T-t)}) - (\gamma_{10}^j e^{-a_r^j(T-t)}) \\
 &\quad \left. - \gamma_{11}^j e^{-(2\gamma+a_r^j)(T-t)} - \gamma_{12}^j e^{-\gamma(T-t)}] \right\}, \tag{A16}
 \end{aligned}$$

with the constants  $\gamma, \gamma_0, \dots, \gamma_{12}^j$  as defined in (67).

Finally, by integration equation (A5) we find the following expression for  $A(u, t, T)$ :

$$\begin{aligned}
 A(u, t, T) &= \int_t^T \left[ \frac{1}{2} (B^2 - B) v_{K,M}^2(t, T) + \kappa \xi(s) C(u, s, T) \right. \\
 &\quad \left. + \frac{1}{2} \tau^2 (C^2(u, s, T) + D(u, s, T)) \right] ds
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{2} u(i+u) V_{K,M}(t, T) \\
 &\quad + \int_t^T \left\{ \left[ \kappa \psi + (iu - 1) \sum_{i=1}^K \rho_{x_i^i, v} \tau \sigma_n^i B_n^i(t, T) \right. \right. \\
 &\quad \left. \left. - iu \sum_{j=1}^M \rho_{x_r^j, v} \tau \sigma_r^j B_r^j(t, T) \right] C(u, s, T) \right. \\
 &\quad \left. + \frac{1}{2} \tau^2 (C^2(u, s, T) + D(u, s, T)) \right\} ds, \tag{A17}
 \end{aligned}$$

where  $V_{K,M}(t, T)$  is the integrated variance of the multi-factor Gaussian rates which can be found by simple integration (see (52)). It is possible to write a closed-form expression for the remaining integral in (A17). As the ordinary differential equation for  $D(u, s, T)$  is exactly the same as in the Heston (1993) or Schöbel and Zhu (1999) model, it will involve a complex logarithm and should therefore be evaluated as outlined by Lord and Kahl (2008) in order to avoid any discontinuities. The main problem, however, lies in the integrals over  $C(u, s, T)$  and  $C^2(u, s, T)$ , which will involve the Gaussian hypergeometric  ${}_2F_1(a, b, c; z)$ . The most efficient way to evaluate this hypergeometric function (according to *Numerical Recipes* Press and Flannery 1992) is to integrate the defining differential equation. Since all of the terms involved in  $D(u, s, T)$  are also required in  $C(u, s, T)$ , numerical integration of the second part of (A17) seems to be the most efficient method for evaluating  $A(u, t, T)$ . Therefore, we conveniently avoid any issues regarding complex discontinuities altogether.

### Appendix B: Analytical properties of the Gaussian factors driving the asset price process

In this section we discuss some properties of the processes that drive the asset price dynamics. That is, we discuss the pricing of bonds under multi-factor Gaussian interest rates (section B.1) and the moments of the Gaussian interest rate processes and the Ornstein–Uhlenbeck distributed volatility process under the  $T$ -forward measure (section B.2).

#### B.1. Zero-coupon bond prices under multi-factor Gaussian rates

In this section we briefly review zero-coupon bond prices of the Gaussian multi-factor rate model, i.e. one has the following analytical formulas for the zero-coupon bond prices (see, e.g., Brigo and Mercurio 2006 for the two-factor model, which can readily be extended to the multi-factor case):

$$\begin{aligned}
 P_n(t, T) &= \mathbb{E}_n \left\{ e^{-\int_t^T n(u) du} \right\} = A_n(t, T) e^{-B_n(t, T) X_n(t)} \\
 &= A_n(t, T) e^{-\sum_{i=0}^K B_n^i(t, T) X_n^i(t)}, \tag{B1}
 \end{aligned}$$

$$A_n(t, T) = \frac{P_n^M(0, T)}{P_n^M(0, t)} \exp \left\{ \frac{1}{2} [V_n(t, T) - V_n(0, T) + V_n(0, t)] \right\}, \tag{B2}$$



where  $B_n^i(t, T) = (1 - e^{-a_n^i(T-t)})/a_n^i$ . It is straightforward to generalize this to the case of time-dependent model parameters, i.e. in that case,  $B_n^i(t, T) := \int_t^T e^{-a(u)(T-u)} du$ . Expressions for the real bond prices  $P_r(t, T)$  and affine terms  $A_r(t, T)$ ,  $B_r(t, T)$  are completely analogous.

For the integrated rate variances  $V_i(t, T)$ , one also has closed-form expressions. To this end we let (as in section 4.1)  $C^{(i,j)}$  and  $R^{(i,j)}$ , respectively, denote the integrated covariance and correlation between the  $i$ th and  $j$ th elements of the vector of rate volatilities  $\Sigma(t)$  of (49). One can then express the integrated rate variances as

$$V_n(t, T) = \sum_{i=2}^{K+1} C^{(i,i)} + 2 \sum_{i=2}^{K+1} \sum_{j=i+1}^{K+1} R^{(i,j)} C^{(i,j)}, \quad (B3)$$

$$V_r(t, T) = \sum_{i=K+2}^{K+M+1} C^{(i,i)} + 2 \sum_{i=K+2}^{K+M+1} \sum_{j=i+1}^{K+M+1} R^{(i,j)} C^{(i,j)}. \quad (B4)$$

Expressions for these covariances are provided in section 4.1.

### B.2. Moments of the interest rate and volatility processes

In this section, we derive the moments of the stochastic factors that drive the nominal, real and volatility rates. Since all factors follow Ornstein–Uhlenbeck processes, the moments can be found relatively easy.

**B.2.1. Moments of the volatility process.** By integrating the  $T_i$ -forward dynamics of (18) conditional on  $\nu(t)$ , we obtain

$$\begin{aligned} \nu(T_{i-1}) &= \nu(t)e^{-\kappa(T_{i-1}-t)} + \int_t^{T_{i-1}} \kappa \xi(u) e^{-\kappa(T_{i-1}-u)} du \\ &\quad + \tau \int_t^{T_{i-1}} e^{-\kappa(T_{i-1}-u)} dW_\nu^T(u), \end{aligned} \quad (B5)$$

where  $\xi(u) := \psi - \sum_{i=1}^K (\rho_{x_n^i, \nu} \sigma_n^i \tau / a_n^i \kappa) [1 - e^{-a_n^i(T_i-u)}]$ . From Itô's isometry, we then have that the mean and variance of  $\nu$  under the  $T_i$ -forward measure are given by

$$\begin{aligned} \mu_\nu &= \nu(t)e^{-\kappa(T_{i-1}-t)} + \left( \psi - \sum_{i=1}^K \frac{\rho_{x_n^i, \nu} \sigma_n^i \tau}{a_n^i \kappa} \right) (1 - e^{-\kappa(T_i-t)}) \\ &\quad - \sum_{i=1}^K \frac{\rho_{x_n^i, \nu} \sigma_n^i \tau}{a_n^i (\kappa + a_n^i)} [e^{-a_n^i(T_i-t) - \kappa(T_{i-1}-t)} - e^{-a_n^i(T_i-T_{i-1})}], \end{aligned} \quad (B6)$$

$$\sigma_\nu^2 = \frac{\tau^2}{2\kappa} (1 - e^{-2\kappa(T_{i-1}-t)}). \quad (B7)$$

**B.2.2. Moments of the rate processes.** Conditional on time  $t$ , one can integrate the rate dynamics of  $x_n^i(T_{i-1})$  and  $x_r^j(T_{i-1})$ , from time  $t$  to  $T_{i-1}$ , to obtain the following explicit solutions (see also Pelsser 2000 or Brigo and

Mercurio 2006):

$$\begin{aligned} x_n^k(T_{i-1}) &= x_n^k(t) e^{-a_n^k(T_{i-1}-t)} - M_{n_k}^{T_i}(t, T_{i-1}) \\ &\quad + \sigma_n^k \int_t^{T_{i-1}} e^{-a_n^k(T_{i-1}-u)} dW_{n_k}^{T_i}(u), \end{aligned} \quad (B8)$$

$$\begin{aligned} x_r^j(T_{i-1}) &= x_r^j(t) e^{-a_r^j(T_{i-1}-t)} - M_{r_j}^{T_i}(t, T_{i-1}) \\ &\quad + \sigma_r^j \int_t^{T_{i-1}} e^{-a_r^j(T_{i-1}-u)} dW_{r_j}^{T_i}(u), \end{aligned} \quad (B9)$$

where

$$\begin{aligned} M_{n_k}^{T_i}(t, T_{i-1}) &= \int_t^{T_{i-1}} \left[ \sigma_n^k \sum_{i=1}^K \rho_{x_n^i, x_n^k} \sigma_n^i B_n^i(u, T_i) \right] e^{-a_n^k(T_{i-1}-u)} du \\ &= \sigma_n^k \frac{1 - e^{-a_n^k(T_{i-1}-t)}}{a_n^k} \sum_{i=1}^K \rho_{x_n^i, x_n^k} \frac{\sigma_n^i}{a_n^i} \\ &\quad - \sigma_n^k \sum_{i=1}^K \rho_{x_n^i, x_n^k} \frac{\sigma_n^i}{a_n^i (a_n^k + a_n^i)} \\ &\quad \times [e^{-a_n^i(T_i-T_{i-1})} - e^{-a_n^i(T_i-t) - a_n^k(T_{i-1}-t)}], \end{aligned} \quad (B10)$$

$$\begin{aligned} M_{r_j}^{T_i}(t, T_{i-1}) &= \int_t^{T_{i-1}} \left[ \rho_{L, x_r^j} \nu(u) \sigma_r^j + \sigma_r^j \sum_{i=1}^K \rho_{x_n^i, x_r^j} \sigma_n^i B_n^i(u, T_i) \right] \\ &\quad \times e^{-a_r^j(T_{i-1}-u)} du \\ &= \tilde{M}_\nu^{T_i}(t, T_{i-1}) + \tilde{M}_{r_j}^{T_i}(t, T_{i-1}). \end{aligned} \quad (B11)$$

In the last step we decompose  $M_{r_j}^{T_i}(t, T_{i-1})$  into a deterministic part, denoted by  $\tilde{M}_{r_j}^{T_i}(t, T_{i-1})$ , and a stochastic part depending on  $\nu(u)$ , denoted by  $\tilde{M}_\nu^{T_i}(t, T_{i-1})$ . The calculation of the  $\tilde{M}_{r_j}^{T_i}(t, T_{i-1})$  term is similar to the nominal interest rate case and results in the following expression:

$$\begin{aligned} \tilde{M}_{r_j}^{T_i}(t, T_{i-1}) &= \sigma_r^j \frac{1 - e^{-a_r^j(T_{i-1}-t)}}{a_r^j} \sum_{i=1}^K \rho_{x_n^i, x_r^j} \frac{\sigma_n^i}{a_n^i} \\ &\quad - \sigma_r^j \sum_{i=1}^K \rho_{x_n^i, x_r^j} \frac{\sigma_n^i}{a_n^i (a_r^j + a_n^i)} \\ &\quad \times [e^{-a_n^i(T_i-T_{i-1})} - e^{-a_n^i(T_i-t) - a_r^j(T_{i-1}-t)}]. \end{aligned} \quad (B12)$$

Hence from Itô's isometry we then have that the mean and variance of  $x_n^k(T_{i-1})$  and  $x_r^j(T_{i-1})$  (conditional on time  $t$ ) are given, respectively, by

$$\mu_n^k(t, T_{i-1}) = x_n^k(t) e^{-a_n^k(T_{i-1}-t)} - M_{n_k}^{T_i}(t, T_{i-1}), \quad (B13)$$

$$(\sigma_n^k(t, T_{i-1}))^2 = \frac{(\sigma_n^k)^2}{2a_n^k} (1 - e^{-2a_n^k(T_{i-1}-t)}), \quad (B14)$$

$$\mu_r^j(t, T_{i-1}) = x_r^j(t) e^{-a_r^j(T_{i-1}-t)} - \tilde{M}_{r_j}^{T_i}(t, T_{i-1}), \quad (B15)$$

$$(\sigma_r^j(t, T_{i-1}))^2 = \frac{(\sigma_r^j)^2}{2a_r^j} (1 - e^{-2a_r^j(T_{i-1}-t)}). \quad (B16)$$

Hence it remains to determine the moments of  $\tilde{M}_\nu^{T_i}(t, T_{i-1})$ , i.e. of

$$\tilde{M}_\nu^{T_i}(t, T_{i-1}) = \sigma_r^j \rho_{L, x_r^j} \int_t^{T_{i-1}} \nu(u) e^{-a_r^j(T_{i-1}-u)} du. \quad (B17)$$

By substituting the explicit solution (B5) for  $v(u)$ , one obtains the following three integrals:

$$\sigma_r^j \rho_{I,x_r^j} v(t) \int_t^{T_{i-1}} e^{-\kappa(u-t)} e^{-a_r^j(T_{i-1}-u)} du, \tag{B18}$$

$$\sigma_r^j \rho_{I,x_r^j} \int_t^{T_{i-1}} \left[ \int_t^u \xi(s) e^{-\kappa(u-s)} ds \right] e^{-a_r^j(T_{i-1}-u)} du, \tag{B19}$$

$$\sigma_r^j \rho_{I,x_r^j} \tau \int_t^{T_{i-1}} \left[ \int_t^u e^{\kappa(u-s)} dW_v^{T_i}(s) \right] e^{-a_r^j(T_{i-1}-u)} du. \tag{B20}$$

The integral of (B18) resolves into

$$v(t) \frac{\sigma_r^j \rho_{I,x_r^j}}{(a_r^j - \kappa)} [e^{-\kappa(T_{i-1}-t)} - e^{-a_r^j(T_{i-1}-t)}]. \tag{B21}$$

Using Fubini's theorem to interchange the order of integration, one finds that the integral of (B19) resolves into

$$\begin{aligned} & \sigma_r^j \rho_{I,x_r^j} \frac{\kappa e^{-a_r^j(T_{i-1}-t)} + (a_r^j - \kappa) - a_r^j e^{-\kappa(T_{i-1}-t)}}{(a_r^j - \kappa) \kappa a_r^j} \\ & \times \left[ \psi - \sum_{i=1}^K \frac{\rho_{x_n^i, v} \sigma_n^i \tau}{a_n^i \kappa} \right] \\ & - \frac{\sigma_r^j \rho_{I,x_r^j}}{\kappa(a_r^j - \kappa)} \sum_{i=1}^K \frac{\rho_{x_n^i, v} \sigma_n^i \tau}{(\kappa + a_n^i) a_n^i (a_r^j + a_n^i)} \\ & \times \{ (a_r^j + a_n^i) e^{-\kappa(T_{i-1}-1) - a_n^i(T_{i-1})} \\ & - (\kappa + a_n^i) e^{-a_r^j(T_{i-1}-t) - a_n^i(T_{i-1})} - (a_r^j - \kappa) e^{-a_n^i(T_{i-1}-t)} \}. \end{aligned} \tag{B22}$$

Again by changing the order of integration, we find that the following expression holds for the stochastic integral of (B20):

$$\frac{\sigma_r^j \rho_{I,x_r^j} \tau}{(a_r^j - \kappa)} \int_t^{T_{i-1}} [e^{-\kappa(T_{i-1}-s)} - e^{-a_r^j(T_{i-1}-s)}] dW_v^{T_i}(s). \tag{B23}$$

Hence from Itô's isometry, we have that  $\tilde{M}_v^{T_i}(t, T_{i-1})$  of (B17) is normally distributed with mean  $\mu_V^j(t, T_{i-1})$  and variance  $(\sigma_V^j(t, T_{i-1}))^2$ , given by

$$\begin{aligned} \mu_V^j(t, T_{i-1}) &= \sigma_r^j \rho_{I,x_r^j} v(t) [e^{-\kappa(T_{i-1}-t)} - e^{-a_r^j(T_{i-1}-t)}] \\ &+ \sigma_r^j \rho_{I,x_r^j} \frac{\kappa e^{-a_r^j(T_{i-1}-t)} + (a_r^j - \kappa) - a_r^j e^{-\kappa(T_{i-1}-t)}}{(a_r^j - \kappa) \kappa a_r^j} \\ &\times \left[ \psi - \sum_{i=1}^K \frac{\rho_{x_n^i, v} \sigma_n^i \tau}{a_n^i \kappa} \right] \\ &- \frac{\sigma_r^j \rho_{I,x_r^j}}{\kappa(a_r^j - \kappa)} \sum_{i=1}^K \frac{\rho_{x_n^i, v} \sigma_n^i \tau}{(\kappa + a_n^i) a_n^i (a_r^j + a_n^i)} \\ &\times \{ (a_r^j + a_n^i) e^{-\kappa(T_{i-1}-1) - a_n^i(T_{i-1})} \\ &- (\kappa + a_n^i) e^{-a_r^j(T_{i-1}-t) - a_n^i(T_{i-1})} \\ &- (a_r^j - \kappa) e^{-a_n^i(T_{i-1}-t)} \}, \end{aligned} \tag{B24}$$

$$\begin{aligned} (\sigma_V^j(t, T_{i-1}))^2 &= \left( \frac{\sigma_r^j \rho_{I,x_r^j} \tau}{(a_r^j - \kappa)} \right)^2 \left\{ \frac{1}{2\kappa} + \frac{1}{2a_r^j} - \frac{2}{(\kappa + a_r^j)} \right. \\ &\left. - \frac{e^{-2\kappa(T_{i-1}-t)}}{2\kappa} - \frac{e^{-2a_r^j(T_{i-1}-t)}}{2a_r^j} + \frac{2e^{-(\kappa+a_r^j)(T_{i-1}-t)}}{(\kappa + a_r^j)} \right\}. \end{aligned} \tag{B25}$$

### B.3. Terminal correlations between the driving factors

In this section we provide simple analytical expressions for the (terminal) correlations between the driving model factors,  $v, x_n^1, \dots, x_n^K, x_r^1, \dots, x_r^K$  and  $V^1, \dots, V^M$ , from the current time  $t$  to time  $T_{i-1}$ . To this end, we consider the following explicit solutions for these Gaussian processes:

$$v(T_{i-1}) = O(dt) + \tau \int_t^{T_{i-1}} e^{-\kappa(T_{i-1}-u)} dW_v^{T_i}(u), \tag{B26}$$

$$x_n^k(T_{i-1}) = O(dt) + \sigma_n^k \int_t^{T_{i-1}} e^{-a_n^k(T_{i-1}-u)} dW_{n_k}^{T_i}(u), \tag{B27}$$

$$x_r^j(T_{i-1}) = O(dt) + \sigma_r^j \int_t^{T_{i-1}} e^{-a_r^j(T_{i-1}-u)} dW_{r_j}^{T_i}(u), \tag{B28}$$

$$\begin{aligned} V^j(T_{i-1}) &= O(dt) + \frac{\sigma_r^j \rho_{I,v} \tau}{(a_r^j - \kappa)} \\ &\times \int_t^{T_{i-1}} [e^{-\kappa(T_{i-1}-u)} - e^{-a_r^j(T_{i-1}-u)}] dW_v^{T_i}(u). \end{aligned} \tag{B29}$$

All of the above processes can be written in the form

$$y_m(T_{i-1}) = O(dt) + c_m \int_t^{T_{i-1}} a_m(u) dW_m(u),$$

hence by Itô's isometry the correlation can easily be calculated. In general, we have that the correlation between, say  $y_1(T_{i-1})$  and  $y_2(T_{i-1})$ , is given by

$$\begin{aligned} \rho_{y_1, y_2}(t, T_{i-1}) &= \frac{\text{Cov}(y_1(T_{i-1}), y_2(T_{i-1}))}{\sqrt{\text{Var}(y_1(T_{i-1})) \times \text{Var}(y_2(T_{i-1}))}} \\ &= \rho_{y_1, y_2} \int_t^{T_{i-1}} c_1 a_1(u) c_2 a_2(u) du \\ &\quad / \sqrt{\int_t^{T_{i-1}} [c_1 a_1(u)]^2 du \times \int_t^{T_{i-1}} [c_2 a_2(u)]^2 du}. \end{aligned} \tag{B30}$$

After identification in (B26)–(B29), one has that  $a_m(u)$  takes two particular forms,

$$a_m(u) = \begin{cases} e^{-b_m(T_{i-1}-u)}, & \text{for } v, x_n^1, \dots, x_n^K, x_r^1, \dots, x_r^K, \\ & b_m \in \{\kappa, a_n^1, \dots, a_n^K, a_r^1, \dots, a_r^M\}, \\ e^{-\kappa(T_{i-1}-u)} - e^{-b_m(T_{i-1}-u)}, & \text{for } V^1, \dots, V^M, \\ & b_m \in \{a_r^1, \dots, a_r^M\}. \end{cases}$$

Hence by combining the above two forms and using formula (B30), one has that the resulting correlations take one of the three forms given below. To ease notation, we first define the following two integral expressions:

$$\begin{aligned}
 I_1(b_m) &= \int_t^{T_{i-1}} c_m^2 [e^{-b_m(T_{i-1}-u)}]^2 du \\
 &= c_m^2 \left\{ \frac{1 - e^{-2b_m(T_{i-1}-t)}}{2b_m} \right\}, \\
 I_2(b_m) &= \int_t^{T_{i-1}} c_m^2 [e^{-\kappa(T_{i-1}-u)} - e^{-b_m(T_{i-1}-u)}]^2 du \\
 &= c_m^2 \left\{ \frac{1}{2\kappa} + \frac{1}{2b_m} - \frac{2}{(\kappa + b_m)} - \frac{e^{-2\kappa(T_{i-1}-t)}}{2\kappa} \right. \\
 &\quad \left. - \frac{e^{-2b_m(T_{i-1}-t)}}{2b_m} + \frac{2e^{-(\kappa+b_m)(T_{i-1}-t)}}{(\kappa + b_m)} \right\}.
 \end{aligned}$$

If  $a_1(u)$  and  $a_2(u)$  are both of the first form, then the correlation between  $y_1(T_{i-1})$  and  $y_2(T_{i-1})$  is given by

$$\rho_{y_1, y_2} = \frac{c_1 c_2}{\sqrt{I_1(b_1)I_1(b_2)}} \frac{1 - e^{-(b_1+b_2)(T_{i-1}-t)}}{(b_1 + b_2)}. \tag{B31}$$

If  $a_1(u)$  is of the first form and  $a_2(u)$  of the second, then the correlation between  $y_1(T_{i-1})$  and  $y_2(T_{i-1})$  is given by

$$\rho_{y_1, y_2} = \frac{c_1 c_2}{\sqrt{I_1(b_1)I_2(b_2)}} \left[ \frac{1 - e^{-(b_1+\kappa)(T_{i-1}-t)}}{(b_1 + \kappa)} - \frac{1 - e^{-(b_1+b_2)(T_{i-1}-t)}}{(b_1 + b_2)} \right]. \tag{B32}$$

Finally, if  $a_1(u)$  and  $a_2(u)$  are both of the second form, then the correlation between  $y_1(T_{i-1})$  and  $y_2(T_{i-1})$  is given by

$$\begin{aligned}
 \rho_{y_1, y_2} &= \frac{c_1 c_2}{\sqrt{I_2(b_1)I_2(b_2)}} \left[ \frac{1 - e^{-2\kappa(T_{i-1}-t)}}{2\kappa} + \frac{1 - e^{-(b_1+b_2)(T_{i-1}-t)}}{(b_1 + b_2)} \right. \\
 &\quad \left. - \frac{1 - e^{-(b_1+\kappa)(T_{i-1}-t)}}{(b_1 + \kappa)} - \frac{1 - e^{-(b_2+\kappa)(T_{i-1}-t)}}{(b_2 + \kappa)} \right]. \tag{B33}
 \end{aligned}$$

**B.4. Constants in the quadratic form (77)**

The constants  $a_0$  and  $b_0$  and the vector  $a$  of the quadratic form (77) can be extracted directly from equation (76) and are given by

$$\begin{aligned}
 a_0 &:= iu[A_r(T_{i-1}, T_i) - A_n(T_{i-1}, T_i)] \\
 &\quad + A(u, T_{i-1}, T_i) + C(T_{i-1})\mu_v(t, T_{i-1}) \\
 &\quad + \frac{1}{2}D(T_{i-1})\mu_v^2(t, T_{i-1}) + iu \sum_{k=1}^K B_n^k(T_{i-1}, T_i)\mu_n^k(t, T_{i-1}) \\
 &\quad - iu \sum_{j=1}^M B_r^j(T_{i-1}, T_i)[\mu_r^j(t, T_{i-1}) + \mu_r^j(t, T_{i-1})], \tag{B34}
 \end{aligned}$$

$$b_0 := \frac{1}{2}D(u, T_{i-1}, T_i)\sigma_v^2(t, T_{i-1}), \tag{B35}$$

$$a := iu \begin{bmatrix} \sigma_v(t, T_{i-1})[C(T_{i-1}) + D(T_{i-1})\mu_v(t, T_{i-1})] \\ \sigma_n^1(t, T_{i-1})B_n^1(T_{i-1}, T_i) \\ \vdots \\ \sigma_n^K(t, T_{i-1})B_n^K(T_{i-1}, T_i) \\ -\sigma_r^1(t, T_{i-1})B_r^1(T_{i-1}, T_i) \\ \vdots \\ -\sigma_r^M(t, T_{i-1})B_r^M(T_{i-1}, T_i) \\ \sigma_r^1(t, T_{i-1})B_r^1(T_{i-1}, T_i) \\ \vdots \\ \sigma_r^M(t, T_{i-1})B_r^M(T_{i-1}, T_i) \end{bmatrix}, \tag{B36}$$

with the  $(1 + K + 2M) \times (1 + K + 2M)$  correlation matrix  $S$  given by

$$S := \begin{pmatrix} 1 & \rho_{x_n^1, v}(t, T_{i-1}) & \dots & \rho_{V^M, v}(t, T_{i-1}) \\ \rho_{x_n^1, v}(t, T_{i-1}) & 1 & \dots & \rho_{x_n^1, V^M}(t, T_{i-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{V^M, v}(t, T_{i-1}) & \rho_{x_n^1, V^M}(t, T_{i-1}) & \dots & 1 \end{pmatrix}. \tag{B37}$$

The moments of the Gaussian factors  $v$ ,  $x_n^k$ ,  $x_r^j$  and  $V^j$  are given by simple analytical expressions (see appendix B2). Where the correlations between all instantaneous quantities are fixed input parameters, the (terminal) correlations  $\rho(t, T_{i-1})$  between the driving processes are model/parameter-dependent. However, these are also given by simple analytical expressions (see appendix B3).

**B.5. Constants in proposition 4.6**

The constant  $h_0$ , vector  $h$  and correlation matrix  $S_R$  can be extracted from equation (97) and are given by

$$\begin{aligned}
 h_0 &:= [A_r(T_{i-1}, T_i) - A_n(T_{i-1}, T_i)] \\
 &\quad + \sum_{k=1}^K B_n^k(T_{i-1}, T_i)\mu_n^k(t, T_{i-1}) \\
 &\quad - \sum_{j=1}^M B_r^j(T_{i-1}, T_i)\mu_r^j(t, T_{i-1}), \tag{B38}
 \end{aligned}$$

$$h := \begin{bmatrix} \sigma_n^1(t, T_{i-1})B_n^1(T_{i-1}, T_i) \\ \dots \\ \sigma_n^K(t, T_{i-1})B_n^K(T_{i-1}, T_i) \\ -\sigma_r^1(t, T_{i-1})B_r^1(T_{i-1}, T_i) \\ \dots \\ -\sigma_r^M(t, T_{i-1})B_r^M(T_{i-1}, T_i) \end{bmatrix}, \tag{B39}$$

with the  $(K + M) \times (K + M)$  correlation matrix  $S_R$  given by

$$S_R := \begin{pmatrix} 1 & \dots & \rho_{x_n^1, x_n^K}(t, T_{i-1}) & \rho_{x_n^1, x_r^1}(t, T_{i-1}) & \dots & \rho_{x_n^1, x_r^M}(t, T_{i-1}) \\ \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ \rho_{x_n^1, x_n^K}(t, T_{i-1}) & \dots & 1 & \rho_{x_n^K, x_r^1}(t, T_{i-1}) & \dots & \rho_{x_n^K, x_r^M}(t, T_{i-1}) \\ \rho_{x_n^1, x_r^1}(t, T_{i-1}) & \dots & \rho_{x_n^K, x_r^1}(t, T_{i-1}) & 1 & \dots & \rho_{x_r^1, x_r^M}(t, T_{i-1}) \\ \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ \rho_{x_n^1, x_r^M}(t, T_{i-1}) & \dots & \rho_{x_n^K, x_r^M}(t, T_{i-1}) & \rho_{x_r^1, x_r^M}(t, T_{i-1}) & \dots & 1 \end{pmatrix}, \tag{B40}$$

and where the above moments and correlations of the Gaussian factors  $x_n^k$  and  $x_r^j$  can be found in appendices B2 and B3.

**Appendix C: FX calibration**

We briefly describe the FX market data set used, which can be found in Piterbarg (2005): the set consists of 10 maturities, each with seven strikes. The strikes are computed according to formula (105). These strikes and corresponding Black and Scholes (1973) implied

volatilities can be found in tables C1 and C2. Note from table C2 that the increasing term structure of the implied volatility and the pronounced implied volatility skew/smile do not die out for long maturities. We then report the detailed calibration results of the model (proposition 2.2) with the above market data. In tables C3 and C4 we report the calibration differences, in implied volatilities for the model (proposition 2.2), with, respectively, Schöbel and Zhu (1999) and Heston (1993) stochastic volatility. For an analysis of these results, see section 5.2.

Table C1. Strikes.

	Strike						
	1	2	3	4	5	6	7
0.5	93.03	96.38	99.84	103.44	107.16	111.02	115.01
1	87.70	92.20	96.93	101.90	107.12	112.61	118.39
3	74.01	80.70	88.00	95.96	104.64	114.11	124.43
5	64.62	72.27	80.81	90.37	101.06	113.02	126.39
7	57.23	65.33	74.57	85.11	97.15	110.89	126.57
10	48.41	56.70	66.41	77.79	91.11	106.72	125.00
15	37.45	45.45	55.16	66.95	81.26	98.62	119.69
20	29.46	36.85	46.08	57.63	72.06	90.12	112.71
25	23.43	30.08	38.63	49.60	63.69	81.77	105.00
30	18.77	24.69	32.46	42.69	56.14	73.82	97.08

Table C2. Market-implied volatility (%).

	Strike						
	1	2	3	4	5	6	7
0.5	11.41	10.49	9.66	9.02	8.72	8.66	8.68
1	12.23	10.98	9.82	8.95	8.59	8.59	8.65
3	12.94	11.35	9.89	8.78	8.34	8.36	8.46
5	13.44	11.84	10.38	9.27	8.76	8.71	8.83
7	14.29	12.68	11.23	10.12	9.52	9.37	9.43
10	16.43	14.79	13.34	12.18	11.43	11.07	10.99
15	20.93	19.13	17.56	16.27	15.29	14.65	14.29
20	22.96	21.19	19.68	18.44	17.50	16.84	16.46
25	23.97	22.31	20.92	19.80	18.95	18.37	18.02
30	25.09	23.48	22.17	21.13	20.35	19.81	19.48

Table C3. Differences, in implied Black volatilities, between market and model values using Schöbel–Zhu stochastic volatility (%).

	Strike						
	1	2	3	4	5	6	7
0.5	-0.11	0.00	0.03	-0.02	-0.09	0.00	0.28
1	-0.18	0.00	0.08	0.00	-0.18	-0.14	0.22
3	-0.47	0.00	0.29	0.30	0.00	-0.16	0.02
5	-0.42	0.00	0.25	0.27	0.04	-0.12	0.00
7	-0.74	0.00	0.57	0.80	0.56	-0.07	-0.81
10	-0.64	0.00	0.44	0.62	0.46	0.00	-0.61
15	-0.45	0.00	0.22	0.22	0.02	-0.35	-0.82
20	-0.83	-0.27	0.07	0.22	0.18	0.00	-0.33
25	-1.07	-0.44	0.00	0.26	0.34	0.26	0.04
30	-1.29	-0.53	0.00	0.35	0.51	0.52	0.39

Table C4. Differences, in implied Black volatilities, between market and model values using Heston stochastic volatility (%).

	Strike						
	1	2	3	4	5	6	7
0.5	-0.15	0.00	0.05	0.00	-0.11	-0.05	0.19
1	-0.20	0.00	0.09	0.00	-0.13	0.00	0.41
3	-0.25	0.00	0.09	0.00	-0.18	-0.06	0.43
5	-0.19	0.00	0.07	0.00	-0.19	-0.23	0.02
7	-0.11	0.00	0.03	0.00	-0.09	-0.14	0.05
10	-0.04	0.01	0.01	0.01	-0.02	-0.05	0.00
15	-0.01	0.01	0.00	-0.01	0.00	-0.01	0.00
20	-0.30	0.00	0.13	0.13	0.00	-0.24	-0.60
25	-0.29	0.00	0.16	0.21	0.15	-0.03	-0.30
30	-0.72	-0.28	0.00	0.16	0.20	0.11	-0.07