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Pricing long-dated insurance contracts with stochastic interest rates and stochastic volatility☆

Alexander van Haastrecht[a,b,*], Roger Lord[c], Antoon Pelsser[d], David Schrager[e]

[a] Netspar/University of Amsterdam, Department of Quantitative Economics, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands
[b] Delta Lloyd Insurance, Risk Management, Spaklerweg 4, PO Box 1000, 1000 BA Amsterdam, The Netherlands
[c] Cardano, Financial Engineering, 27-28 Clements Lane, London EC4N 7AE, UK
[d] Maastricht University, Department of Finance, Department of Quantitative Economics, PO Box 616, 6200 MD Maastricht, The Netherlands
[e] ING Life Japan, Variable Annuity Market Risk Management, Japan

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ABSTRACT

We consider the pricing of long-dated insurance contracts under stochastic interest rates and stochastic volatility. In particular, we focus on the valuation of insurance options with long-term equity or foreign exchange exposures. Our modeling framework extends the stochastic volatility model of Schöbel and Zhu (1999) by including stochastic interest rates. Moreover, we allow all driving model factors to be instantaneously correlated with each other, i.e. we allow for a general correlation structure between the instantaneous interest rates, the volatilities and the underlying stock returns. As insurance products often incorporate long-term exposures, they are typically more sensitive to changes in the interest rates, volatility and currencies. Therefore, having the flexibility to correlate the underlying asset price with both the stochastic volatility and the stochastic interest rates, yields a realistic model which is of practical importance for the pricing and hedging of such long-term contracts. We show that European options, typically used for the calibration of the model to market prices, and forward starting options can be priced efficiently and in closed-form by means of Fourier inversion techniques. We extensively discuss the numerical implementation of these pricing formulas, allowing for a fast and accurate valuation of European and forward starting options. The model will be especially useful for the pricing and risk management of insurance contracts and other exotic derivatives involving long-term maturities.

1. Introduction

The derivative markets are maturing more and more. Not only are increasingly exotic structures created, the markets for plain vanilla derivatives are also growing. One of the recent advances in equity derivatives and exchange rate derivatives is the development of a market for long-maturity European options.[1] In this paper we develop a stochastic volatility model aimed at pricing and risk managing long-maturity insurance contracts involving equity, interest rate and exchange rate risks.

We extend the models by Stein and Stein (1991) and Schöbel and Zhu (1999) to allow for Hull and White (1993) stochastic interest rates as well as correlation between the stock price process, its stochastic volatility and interest rates. We call it the Schöbel–Zhu Hull–White (SZHW) model. Our model enables to take into account two important factors in the pricing of long-maturity equity or exchange rate derivatives: stochastic volatility and stochastic interest rates, whilst also taking into account the correlation between those processes explicitly. It is hardly necessary to motivate the inclusion of stochastic volatility in a derivative pricing model. The addition of interest rates as a stochastic factor is important when considering long-maturity equity derivatives and has been the subject of empirical investigations most notably by Bakshi et al. (2000). These authors show that the hedging performance of delta hedging strategies of long-maturity options improves when taking stochastic interest rates into account. Interest rate risk is not as much a factor for short maturity options. This result is also intuitively appealing since the interest rate risk of equity derivatives, the option's rho, is increasing with time to maturity. The SZHW

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* Corresponding author at: Netspar/University of Amsterdam, Department of Quantitative Economics, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands.
E-mail addresses: a.vanhaastrecht@uva.nl (A. van Haastrecht), r.lord@cardano.com (R. Lord), a.pelsser@maastrichtuniversity.nl (A. Pelsser), 02037_schrager@ing-life.co.jp (D. Schrager).
[1] The implied volatility service of MarkIT, a financial data provider, shows regular quotes on a large number of major equity indices for option maturities up to 10–15 years.

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model can be used in the pricing and risk management for a range of insurance and exotic derivatives contracts. One can for example think of pension products, variable and guaranteed annuities (e.g. see Ballotta and Haberman (2003)), long-maturity PRDC FX contracts (e.g. see Piterbarg (2005)), rate of return guarantees in Unit-Linked contracts (e.g. see Schrager and Pelsser (2004)) and many other structures which have a long-term nature.

Our paper can be placed in the derivative pricing literature on stochastic volatility models as it adds to or extends work by Stein and Stein (1991), Heston (1993), Schöbel and Zhu (1999) or, since our model can be placed in the affine class, in the more general context of Duffie et al. (2000), Duffie et al. (2003) and van der Ploeg (2006). The SZHW model benefits greatly from the analytical tractability typical for this class of models. Our work can also be viewed as an extension of the work by Amin and Jarrow (1992) on stochastic volatility. In a related paper Ahlip (2008) considers an extension of the Schöbel–Zhu model to Gaussian stochastic interest rates for the pricing of exchange rate options. Upon a closer look however the correlation structure considered by this paper is limited to perfect correlation between the stochastic processes.[2] The affine stochastic volatility models fall in the broader literature on stochastic volatility which covers both volatility modelling for the purpose of derivative pricing as well as real world volatility modeling. Previous papers that covered both stochastic volatility and stochastic interest rates in derivative pricing include: Scott (1997), Bakshi et al. (1997), Amin and Ng (1993) and Andreasen (2006). The SZHW model distinguishes itself from these models by a closed form call pricing formula and/or explicit, rather than implicit, incorporation of the correlation between underlying and the term structure of interest rates.

Our contribution to the existing literature is fourfold:

- First, we derive the conditional characteristic function of the SZHW model in closed form and analyse pricing vanilla equity calls and puts using transform inversion. We also derive a closed form expression for the conditional characteristic function.
- Second, since the practical relevance of any model is limited without a numerical implementation, we extensively consider the efficient implementation of the transform inversion (see Lord and Kahl (2007)) required to price European options. In particular we derive a theoretical result on the limiting behaviour of the conditional characteristic function of the SZHW model which allows us to calculate of the inversion integral much more accurately.
- Third, we consider the pricing of forward starting options.
- Fourth, we generalise the SZHW model to be able to value FX options in a framework where both domestic and foreign interest rate processes are stochastic.

The outline for the remainder of the paper is as follows. First, we introduce the model and focus on the analytical properties. Second, we consider the effect of stochastic interest rates and correlation on the implied volatility term structure. Third, we consider the numerical implementation of the transform inversion integral. Fourth, we consider the pricing of forward starting options. Fifth, we present the extension of the model for FX options involving two interest rate processes. Finally we conclude.

2. The Schöbel–Zhu–Hull–White model

The model we will derive here is a combination of the famous Hull and White (1993) model for the stochastic interest rates and the Schöbel and Zhu (1999) model for stochastic volatility. The model has three key variables, which we allow to be correlated with each other: the stock price $x(t)$, the Hull–White interest rate process $r(t)$ and the stochastic stock volatility which follows an Ornstein–Uhlenbeck process cf. Schöbel and Zhu (1999). The risk-neutral asset price dynamics of the Schöbel–Zhu–Hull–White (SZHW) read:

$$\mathrm{d}x(t) = x(t)r(t)\mathrm{d}t + x(t)v(t)\mathrm{d}W_x(t), \quad x(0) = x_0, \tag{1}$$

$$\mathrm{d}r(t) = \big(\theta(t) - ar(t)\big)\mathrm{d}t + \sigma\mathrm{d}W_r(t), \quad r(0) = r_0, \tag{2}$$

$$\mathrm{d}v(t) = \kappa\big(\psi - v(t)\big)\mathrm{d}t + \tau\mathrm{d}W_v(t), \quad v(0) = v_0, \tag{3}$$

where $a, \sigma, \kappa, \psi, \tau$ are positive parameters which can be inferred from market data and correspond to the mean reversion and volatility of the short rate process, and the mean reversion, long-term volatility and volatility of the volatility process respectively. The quantity r_0 and the deterministic function $\theta(t)$ are used to match the currently observed term structure of interest rates, e.g. see Hull and White (1993). The hidden parameter $v_0 > 0$, corresponds to the current instantaneous volatility and hence should be determined directly from market (e.g. just as the non-observable short interest rate), but is in practice often (mis-)used as extra parameter for calibration. Finally, $\widetilde{W}(t) = \big(W_x(t), W_r(t), W_v(t)\big)$ denotes a Brownian motion under the risk-neutral measure $\mathcal{Q}$ with covariance matrix:

$$\mathrm{Var}\big(\widetilde{W}(t)\big) = \begin{pmatrix} 1 & \rho_{xr} & \rho_{xv} \\ \rho_{xr} & 1 & \rho_{rv} \\ \rho_{xv} & \rho_{rv} & 1 \end{pmatrix} t. \tag{4}$$

Note that as $v(t)$ follows an Ornstein–Uhlenbeck process, there is a slight change that $v(t)$ becomes negative; effectively this implies that the sign of instantaneous correlation between $\ln x(t)$ and $v(t)$ changes as $v(t)$ goes through zero:

$$\begin{aligned}\mathrm{Corr}\Big(\mathrm{d}\ln x(t), \mathrm{d}v(t)\Big) &= \mathrm{Corr}\Big(v(t)\mathrm{d}W_x(t), \tau\mathrm{d}W_v(t)\Big) \\ &= \frac{\rho_{xv}v(t)\tau}{\sqrt{v^2(t)\tau^2}}\mathrm{d}t = \rho_{xv}\,\mathrm{sgn}\Big(v(t)\Big)\mathrm{d}t,\end{aligned} \tag{5}$$

but the actual volatility is $|v(t)|$, which would not have this feature.

2.1. European option pricing

We will now show that general payoffs which are a function of the stock price at maturity T can be priced using the corresponding characteristic function of the log-asset price. Therefore we evaluate the probability distribution of the T-forward stock price at time T. Instead of evaluating expected discounted payoff under the risk-neutral bank account measure, we can also change the underlying probability measure to evaluate this expectation under the T-forward probability measure $\mathcal{Q}^T$ (e.g. see Geman et al. (1996)). This is equivalent to choosing the T-discount bond as numeraire. Hence conditional on time t, we can evaluate the price of a European stock option ($w = 1$ for a call option, $w = -1$ for a put option) with strike $K = \exp(k)$ as

$$\begin{aligned}&\mathbb{E}^{\mathcal{Q}}\left[\exp\left[-\int_t^T r(u)\mathrm{d}u\right]\big(w(S(T) - K)\big)^+|\mathcal{F}_t\right] \\ &= P(t,T)\mathbb{E}^{\mathcal{Q}^T}\Big[\big(w(F^T(T) - K)\big)^+|\mathcal{F}_t\Big],\end{aligned} \tag{6}$$

where $P(t,T)$ denotes the price of a (pure) discount bond and $F^T(t) := \frac{S(t)}{P(t,T)}$ denotes the T-forward stock price. The above expression can be numerically evaluated by means of a Fourier inversion of the log-asset price characteristic function.

Following Carr and Madan (1999), Lewis (2001) and Lord and Kahl (2007), we can then write the call option (6) with log strike

[2] We thank Vladimir Piterbarg for pointing out this paper to us.

k, in terms of the (T-forward) characteristic function ϕ_T of the log asset price $z(T)$, i.e.

$$C_T(k) = P(t,T)\frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big(\mathrm{e}^{-(\alpha+\mathrm{i}v)k}\psi_T(v)\Big)\mathrm{d}v + R\Big(F^T(t),K,\alpha(k)\Big), \tag{7}$$

where the residue term R equals

$$R\Big(F,K,\alpha\Big) := F\cdot 1_{\{\alpha\leq 0\}} - K\cdot 1_{\{\alpha\leq -1\}} - \frac{1}{2}\Big(F\cdot 1_{\{\alpha=0\}} - K\cdot 1_{\{\alpha=-1\}}\Big), \tag{8}$$

with

$$\psi_T(v) := \frac{\phi_T\Big(v-(\alpha+1)\mathrm{i}\Big)}{(\alpha+\mathrm{i}v)(\alpha+1+\mathrm{i}v)}, \tag{9}$$

and where $\phi_T(v) := \mathbb{E}^{\mathcal{Q}^T}\Big[\exp\Big(\mathrm{i}uz(T)\Big)|\mathcal{F}_t\Big]$ denotes the T-forward characteristic function of the log asset price. Thus for the pricing of call options in the SZHW model, it suffices to know the characteristic function of the log-asset price process. We will derive this characteristic function in the following subsection. Section 4 is concerned with the numerical implementation of Eq. (7) and presents an alternative pricing equation which transforms the integration domain to the unit interval and hence avoids truncation errors, see also Lord and Kahl (2007).

2.2. The T-forward dynamics

For the Hull–White model we have the following analytical expression for the discount bond price:

$$P(t,T) = \exp\Big[A_r(t,T) - B_r(t,T)r(t)\Big], \tag{10}$$

where $A_r(t,T)$ is used to calibrate to the interest rate term structure, and with:

$$B_r(t,T) := \frac{1-\mathrm{e}^{-a(T-t)}}{a}. \tag{11}$$

Hence the forward stock price can be expressed as

$$F^T(t) = \frac{S(t)}{\exp\Big[A_r(t,T) - B_r(t,T)r(t)\Big]}. \tag{12}$$

Under the risk-neutral measure $\mathcal{Q}$ (where we use the money market bank account as numeraire) the discount bond price follows the process $\mathrm{d}P(t,T) = r(t)P(t,T)\mathrm{d}t - \sigma B_r(t,T)P(t,T)\mathrm{d}W_r(t)$. Hence, by an application of Ito's lemma, we find the following T-forward stock price process:

$$\mathrm{d}F^T(t) = \Big(\sigma^2 B_r^2(t,T) + \rho_{xr}\nu(t)\sigma B_r(t,T)\Big)F^T(t)\mathrm{d}t + \nu(t)F^T(t)\mathrm{d}W_x(t) + \sigma B_r(t,T)F^T(t)\mathrm{d}W_r(t). \tag{13}$$

By definition the forward stock price will be a martingale under the T-forward measure. This is achieved by defining the following transformations of the Brownian motions:

$$\begin{aligned}\mathrm{d}W_r(t) &\mapsto \mathrm{d}W_r^T(t) - \sigma B_r(t,T)\mathrm{d}t,\\ \mathrm{d}W_x(t) &\mapsto \mathrm{d}W_x^T(t) - \rho_{xr}\sigma B_r(t,T)\mathrm{d}t,\\ \mathrm{d}W_\nu(t) &\mapsto \mathrm{d}W_\nu^T(t) - \rho_{r\nu}\sigma B_r(t,T)\mathrm{d}t.\end{aligned} \tag{14}$$

Hence under the T-forward measure the processes for $F^T(t)$ and $\nu(t)$ are given by

$$\mathrm{d}F^T(t) = \nu(t)F^T(t)\mathrm{d}W_x^T(t) + \sigma B_r(t,T)F^T(t)\mathrm{d}W_r^T(t), \tag{15}$$

$$\mathrm{d}\nu(t) = \kappa\Big(\Big(\psi - \frac{\rho_{r\nu}\sigma\tau}{\kappa}B_r(t,T)\Big) - \nu(t)\Big)\mathrm{d}t + \tau\mathrm{d}W_\nu^T(t), \tag{16}$$

where $W_x^T(t)$, $W_r^T(t)$, $W_\nu^T(t)$ are now Brownian motions under the T-forward $\mathcal{Q}^T$. We can simplify (15) by switching to logarithmic coordinates and rotating the Brownian motions $W_x^T(t)$ and $W_r^T(t)$ to $W_F^T(t)$. Defining $y(t) := \log\big(F^T(t)\big)$ and an application of Ito's lemma yields

$$\mathrm{d}y(t) = -\frac{1}{2}\nu_F^2(t)\mathrm{d}t + \nu_F(t)\mathrm{d}W_F^T(t), \tag{17}$$

$$\mathrm{d}\nu(t) = \kappa\big(\xi(t) - \nu(t)\big)\mathrm{d}t + \tau\mathrm{d}W_\nu^T(t) \tag{18}$$

with

$$\nu_F^2(t) := \nu^2(t) + 2\rho_{xr}\nu(t)\sigma B_r(t,T) + \sigma^2 B_r^2(t,T) \tag{19}$$

$$\xi(t) := \Big(\psi - \frac{\rho_{r\nu}\sigma\tau}{\kappa}B_r(t,T)\Big). \tag{20}$$

Notice that we now have reduced the system (1) of the three variables $x(t)$, $r(t)$ and $\nu(t)$ under the risk-neutral measure, to the system (17) of two variables $y(t)$ and $\nu(t)$ under the T-forward measure. What remains is to find the characteristic function of the reduced system of variables.

Determining the characteristic function of the forward log-asset price

We will now determine the characteristic function of the reduced system (17), which we will do by means of a partial differential approach. That is, we apply the Feynman–Kac theorem and reduce the problem of finding the characteristic of the forward log-asset price dynamics to solving a partial differential equation; that is, the Feynman–Kac theorem implies that the characteristic function

$$f(t,y,\nu) = \mathbb{E}^{\mathcal{Q}^T}\Big[\exp\big(\mathrm{i}uy(T)\big)|\mathcal{F}_t\Big], \tag{21}$$

is given by the solution of the following partial differential equation

$$0 = f_t - \frac{1}{2}\nu_F^2(t)f_y + \kappa\big(\xi(t)-\nu(t)\big)f_\nu + \frac{1}{2}\nu_F^2(t)f_{yy} + \big(\rho_{x\nu}\tau\nu(t) + \rho_{r\nu}\tau\sigma B_r(t,T)\big)f_{y\nu} + \frac{1}{2}\tau^2 f_{\nu\nu}, \tag{22}$$

$$f(T,y,\nu) = \exp\Big(\mathrm{i}uy(T)\Big), \tag{23}$$

where the subscripts denote partial derivatives and we took into account that the covariance term $\mathrm{d}y(t)\mathrm{d}\nu(t)$ is equal to

$$\mathrm{d}y(t)\mathrm{d}\nu(t) = \big(\nu(t)\mathrm{d}W_x^T(t) + \sigma B_r(t,T)\mathrm{d}W_r^T(t)\big)\big(\tau\mathrm{d}W_\nu^T(t)\big) = \big(\rho_{x\nu}\tau\nu(t) + \rho_{r\nu}\tau\sigma B_r(t,T)\big)\mathrm{d}t, \tag{24}$$

and to ease the notation we dropped the explicit (t,y,ν)-dependence for f.

Due to the affine structure of the model, we can solve the defining partial differential equation (22) subject to the boundary condition (23), which leads to the following proposition.

Proposition 2.1. *The characteristic function of T-forward log-asset price of the SZHW model is given by the closed-form solution in* Box I.

Proof. The model we are considering is not an affine model in $y(t)$ and $\nu(t)$, but it is if we enlarge the state space to include $\nu^2(t)$:

$$f(t,y,v) = \exp\Big[A(u,t,T) + B(u,t,T)y(t) + C(u,t,T)v(t) + \frac{1}{2}D(u,t,T)v^2(t)\Big], \tag{25}$$

where:

$$A(u,t,T) = -\frac{1}{2}u(\mathrm{i}+u)V(t,T) + \int_t^T \Big[\Big(\kappa\psi + \rho_{rv}(\mathrm{i}u-1)\tau\sigma B_r(s,T)\Big)C(s) + \frac{1}{2}\tau^2\Big(C^2(s)+D(s)\Big)\Big]\mathrm{d}s \tag{26}$$

$$B(u,t,T) = \mathrm{i}u, \tag{27}$$

$$C(u,t,T) = -u(\mathrm{i}+u)\frac{\Big(\big(\gamma_3 - \gamma_4 \mathrm{e}^{-2\gamma(T-t)}\big) - \big(\gamma_5 \mathrm{e}^{-a(T-t)} - \gamma_6 \mathrm{e}^{-(2\gamma+a)(T-t)}\big) - \gamma_7 \mathrm{e}^{-\gamma(T-t)}\Big)}{\gamma_1 + \gamma_2 \mathrm{e}^{-2\gamma(T-t)}}, \tag{28}$$

$$D(u,t,T) = -u(\mathrm{i}+u)\frac{1-\mathrm{e}^{-2\gamma(T-t)}}{\gamma_1 + \gamma_2 \mathrm{e}^{-2\gamma(T-t)}}, \tag{29}$$

with:

$$\gamma = \sqrt{(\kappa - \rho_{xv}\tau\mathrm{i}u)^2 + \tau^2 u(\mathrm{i}+u)}, \qquad \gamma_1 = \gamma + (\kappa - \rho_{xv}\tau\mathrm{i}u), \tag{30}$$

$$\gamma_2 = \gamma - (\kappa - \rho_{xv}\tau\mathrm{i}u), \qquad \gamma_3 = \frac{\rho_{xr}\sigma\gamma_1 + \kappa a\psi + \rho_{rv}\sigma\tau(\mathrm{i}u-1)}{a\gamma},$$

$$\gamma_4 = \frac{\rho_{xr}\sigma\gamma_2 - \kappa a\psi - \rho_{rv}\sigma\tau(\mathrm{i}u-1)}{a\gamma}, \qquad \gamma_5 = \frac{\rho_{xr}\sigma\gamma_1 + \rho_{rv}\sigma\tau(\mathrm{i}u-1)}{a(\gamma-a)},$$

$$\gamma_6 = \frac{\rho_{xr}\sigma\gamma_2 - \rho_{rv}\sigma\tau(\mathrm{i}u-1)}{a(\gamma+a)}, \qquad \gamma_7 = (\gamma_3 - \gamma_4) - (\gamma_5 - \gamma_6),$$

and:

$$V(t,T) = \frac{\sigma^2}{a^2}\left((T-t) + \frac{2}{a}\mathrm{e}^{-a(T-t)} - \frac{1}{2a}\mathrm{e}^{-2a(T-t)} - \frac{3}{2a}\right). \tag{31}$$

Box I.

$$\mathrm{d}y(t) = -\frac{1}{2}v_F^2(t)\mathrm{d}t + v_F(t)\mathrm{d}W_F^T(t) \tag{32}$$

$$\mathrm{d}v(t) = \kappa\big(\xi(t) - v(t)\big)\mathrm{d}t + \tau\mathrm{d}W_v^T(t) \tag{33}$$

$$\begin{aligned}\mathrm{d}v^2(t) &= 2v(t)\mathrm{d}v(t) + \tau^2\mathrm{d}t \\ &= 2\kappa\Big(\frac{\tau^2}{2\kappa} + \xi(t)v(t) - v^2(t)\Big)\mathrm{d}t + 2\tau v(t)\mathrm{d}W_v^T(t).\end{aligned} \tag{34}$$

We can find the characteristic function of the T-forward log price by solving the partial differential equation (22) for the joint distribution $f(t,y,v)$ with the corresponding boundary condition (23); substituting the partial derivatives of the functional form (25) into (22) provides us four ordinary differential equations containing the functions $A(t)$, $B(t)$, $C(t)$ and $D(t)$. Solving this system yields the above solution, see Appendix A. □

We note that the strip of regularity of the SZHW characteristic function is the same as that of the Schöbel and Zhu (1999) model, for which we refer the reader to Lord and Kahl (2007).

3. Impact of stochastic interest rates and correlation

To gain some insights into the impact of the correlated stochastic rates and corresponding parameter sensitivities we will look at the at-the-money implied volatility structure which we compute for different parameter settings. Besides comparing different parameter settings of the SZHW model, we also make a comparison with the classical Schöbel and Zhu (1999) model to determine the impact of stochastic rates in general. The behaviour of the 'non-interest rate' parameters are similar to other stochastic volatility models like Heston (1993) and Schöbel and Zhu (1999), that is the volatility of the volatility lifts the wings of the volatility smile, the correlation between the stock process and the volatility process can incorporate a skew, and the short and long-term vol determine the level of the implied volatility structure. The impact of stochastic rates and the corresponding correlation can be found in the Fig. 1.

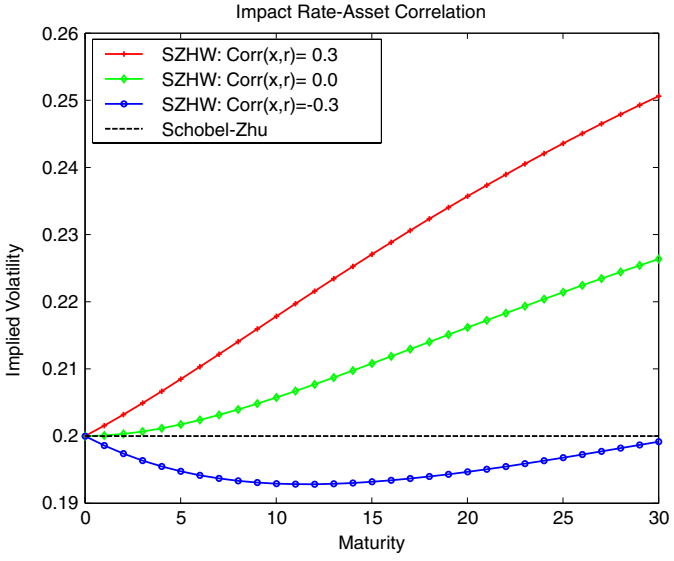

Fig. 1. Impact of ρ_{xr} on at-the-money implied volatilities. The graph corresponds to the (degenerate) Black–Scholes–Hull–White case with parameter values $r(t) = 0.05$, $a = 0.05$, $\sigma = 0.01$, $v(0) = \psi = 0.20$, $\rho_{xv} = 0.0$ and constant volatility process.

From Fig. 1, one can see that the stochastic interest rates add extra flexibility to the modeling framework; by changing the

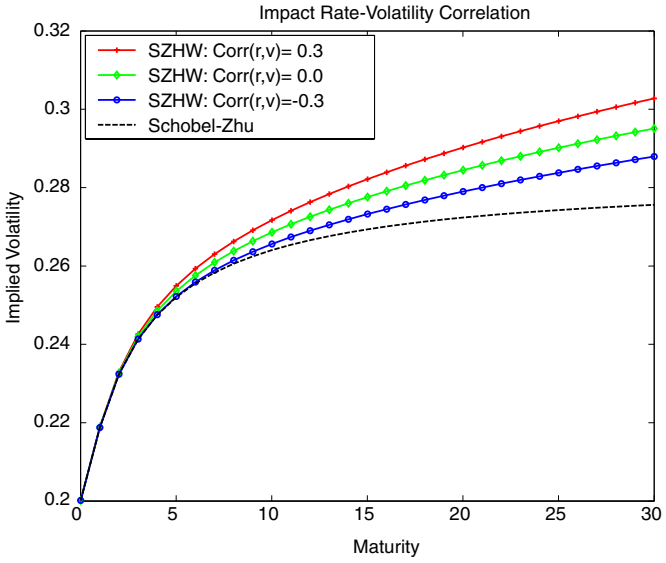

Fig. 2. Impact of ρ_{xv} on at-the-money implied volatilities. The graph corresponds to the parameter values $r(t) = 0.05$, $a = 0.05$, $\sigma = 0.01$, $v(0) = \psi = 0.20$, $\rho_{xr} = 0.0$ and with volatility process with a mean reversion coefficient of $\kappa = 0.5$ and volatility of volatility $\tau = 0.2$.

rate-asset correlation one can create an upward (or an initially downward) sloping term structure of volatility, even in case the volatility process is constant. If we compare the case with zero correlation between the equity and interest rate drivers with the ordinary process with deterministic rates, we see that the stochastic rates make the term structure upward sloping. Note that this is in correspondence with empirical data, which shows higher at-the-money volatilities the longer the maturities go. The effect becomes more apparent for maturities larger than five years; while for one year the effect of uncorrelated stochastic rates is below a basis point, the effect on a five year option is already more than ten basis points which increases to a couple of hundred basis points for a thirty year option. These model effects also correspond with a general feature of the interest rate market: the market's view on the uncertainty of long-maturity bonds is often much higher than that of shorter bond, hence reflecting the increasing impact of stochastic interest rates for long-maturity equity options. Finally, we note that for higher positive values of linear correlation coefficient between equity and the interest rate component, the impact of stochastic rates becomes more apparent.

From Fig. 2, one can see that the effect of the correlation coefficient between the drivers of the rate and volatility process is similar, however the impact on the implied volatility structure is less severe and different in sign: a positive correlation coefficient causes a dampening effect, whereas a negative correlation increases the overall volatility, which effect can also be seen from the volatility dynamics (16). Note hereby that the increasing term structure for $\rho_{rv} =$, in the Fig. 2 is mainly caused by the Schöbel–Zhu stochastic volatility process in comparison to the deterministic volatility process used in Fig. 1. In comparison to the Schöbel and Zhu (1999) model, we can see that the stochastic interest rates increase the slope of the term structure. More importantly, the implied volatilities do not die out, but remain upward sloping, which behaviour often corresponds with implied volatility quotes in long-maturity equity (e.g. see MarkIT) or FX (e.g. see Andreasen (2006)) options. However for strong positive correlation values this might be the other way around. In contrast to the first picture, we see somewhat smaller effects: for example the increasing effect of stochastic rates is even larger than that of the dampening effect of a positive correlation of 30% between the rate and volatility drivers. Again we see that the effects of stochastic rates become more apparent for longer maturities.

In general, we can see from Figs. 1 and 2 that stochastic rates have a significant impact on the backbone of the implied volatility structure and add extra flexibility to the modeling framework. The effects become more apparent for larger maturities and for larger absolute values of the correlation coefficients. Hereby the effect of correlation coefficient between equity and interest rates seems to be the most determinant factor. One can then use these degrees of freedom in several ways: either one jointly calibrates these parameters to implied volatility surfaces (or some other options), or one can first calibrate these and then use the other parameters to calibrate the remainder of the model. In our opinion this choice has to depend on the exotic product: if the correlations are of larger impact on an exotic product (e.g. on a hybrid equity-interest rate product) than on short-dated vanilla calls, it might then be preferable to use a historical estimate for the correlation coefficient at the cost of a slightly worse calibration result. One way or the other, the SZHW stands out by the additional freedom it offers by explicitly modelling the correlation coefficient between the underlying, the stochastic volatility and the stochastic interest rates.

3.1. Relationship with the Heston model

It was already noted by Heston (1993) in this famous 1993-paper, that an Ornstein–Uhlenbeck process for the volatility is closely related to a square-root process for the variance process. If the volatility follows an Ornstein–Uhlenbeck process as in (1):

$$\mathrm{d}v(t) = \kappa\big(\psi - v(t)\big)\mathrm{d}t + \tau \mathrm{d}W_v(t),$$

then Ito's lemma shows that the variance process $v^2(t)$ follows the process

$$\mathrm{d}v^2(t) = 2\kappa\left(\frac{\tau^2}{2\kappa} + \psi v(t) - v^2(t)\right)\mathrm{d}t + 2\tau v(t)\mathrm{d}W_v(t). \tag{35}$$

Since the variance process of the Heston model has the following dynamics

$$\mathrm{d}v_H^2(t) = \kappa_H\Big(\psi_H - v_H^2(t)\Big)\mathrm{d}t + \tau_H v_H(t)\mathrm{d}W_v(t), \tag{36}$$

one can easily establish a relationship between the Heston and the Schöbel–Zhu model; in the case the long-term mean of the volatility process of (1) $\psi = 0$, Schöbel–Zhu model equals the Heston model in which $\kappa_H = 2\kappa$, $\tau_H = 2\tau$ and $\psi_H = \frac{\tau^2}{2\kappa}$. The overlap of the models is restricted to this very special case.

4. Calculating the inverse Fourier transform

In Lord and Kahl (2007) the practical calculation of the inverse Fourier transform (7) is discussed in great detail

$$C_T(k) = P(t,T)\frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big(\mathrm{e}^{-(\alpha+\mathrm{i}v)k}\psi_T(v)\Big)\mathrm{d}v + R\Big(F^T(t), K, \alpha(k)\Big). \tag{37}$$

They recommend that

- Any truncation error is avoided by appropriately transforming the range of integration to a finite interval.
- An adaptive integration algorithm is used, hereby allowing the discretization error to be of a prescribed maximum size.
- The damping parameter α is chosen such that the integrand is minimized in $v = 0$, which typically leads to much more accurate prices for options which have long maturities and/or are away from the at-the-money level.

By changing variables from v to $g(v)$, which maps $[0, \infty) \mapsto [0, 1]$, the pricing equation (37) becomes

$$C_T(k) = P(t,T)\frac{1}{\pi}\int_0^1 \mathrm{Re}\Big(\mathrm{e}^{-\left(\alpha+\mathrm{i}g(v)\right)k}\psi_T\big(g(v)\big)\cdot g'(v)\Big)\mathrm{d}v + R\Big(F^T(t),K,\alpha(k)\Big). \tag{38}$$

However one carefully has to choose the transformation function g such that the integrand remains finite over the range of integration, as it is in (37). To find such a transformation, we analyse the limiting behaviour of the characteristic function. In particular, suppose that the characteristic function of the SZHW model for large values of u behaves as

$$\phi_T(u) \propto \exp\Big(\phi_r(u)+\mathrm{i}\phi_i(u)\Big), \tag{39}$$

with both $\phi_r(u)$ and $\phi_i(u)$ functions on the real line. The integrand in (37) will then have the following asymptotics

$$\mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}(u-\mathrm{i}\alpha)k}\frac{\phi_T\big(u-(\alpha+1)\mathrm{i}\big)}{(\alpha+\mathrm{i}u)(\alpha+\mathrm{i}+\mathrm{i}u)}\right) \propto \frac{\mathrm{e}^{-\alpha k+\psi_r\left(u-(\alpha+1)\mathrm{i}\right)}}{u^2}\cdot\cos\Big(ku-\psi_i\big(u-(\alpha+1)\mathrm{i}\big)\Big). \tag{40}$$

In the remainder we will determine ψ_r, which will tell us which transformation function is suitable to use. Lord and Kahl (2007) already supplies a number of intermediary results for the Schöbel and Zhu (1999) model, but as the notation we use here is slightly different, we will briefly restate these results. For large values of u, only γ, γ_1 and γ_2 in (30) are $O(u)$, whereas γ_3 to γ_6 tend to a constant, and γ_7 is actually $O\big(\frac{1}{u}\big)$. The limits we require here are

$$\lim_{u\to\infty}\frac{\gamma(u)}{u} = \tau\sqrt{1-\rho_{xv}} =: \gamma(\infty), \tag{41}$$

$$\lim_{u\to\infty}\frac{\gamma_1(u)}{u} = \gamma(\infty)-\mathrm{i}\rho_{xv}\tau =: \gamma_1(\infty), \tag{42}$$

$$\lim_{u\to\infty}\gamma_3(u) = \sigma\frac{\rho_{xr}\gamma(\infty)+\mathrm{i}\tau\left(\rho_{rv}-\rho_{xr}\rho_{xv}\right)}{a\gamma(\infty)} =: \gamma_3(\infty), \tag{43}$$

$$\lim_{u\to\infty}\gamma_5(u) = \sigma\frac{\rho_{xr}\gamma(\infty)+\mathrm{i}\tau\left(\rho_{rv}-\rho_{xr}\rho_{xv}\right)}{a\gamma(\infty)} =: \gamma_5(\infty). \tag{44}$$

We find that the limiting behaviour for $C(u,t,T)$ in (28) follows from

$$\begin{aligned}\lim_{u\to\infty}\frac{C(u,t,T)}{u} &= -\frac{\gamma_3(\infty)-\gamma_5(\infty)\mathrm{e}^{-a(T-t)}}{\gamma_1(\infty)}\\ &= -\frac{i\rho_{rv}+\rho_{xr}\left(\sqrt{1-\rho_{xv}^2}-\mathrm{i}\rho_{xv}\right)}{\tau\left(1-\rho_{xv}^2-\mathrm{i}\rho_{xv}\sqrt{1-\rho_{xv}^2}\right)}\frac{\sigma}{\tau}B_r(t,T)\\ &\equiv C(\infty)\frac{\sigma}{\tau}B_r(t,T).\end{aligned} \tag{45}$$

From the above result, the limiting behaviour of $D(u,t,T)$ in (29) for large values of u follows as

$$\lim_{u\to\infty}\frac{D(u,t,T)}{u} = -\frac{1}{\gamma_1(\infty)}. \tag{46}$$

Finally, we need to analyse $A(t) = A(u,t,T)$ in (26). Its defining ODE (103) can be found in Appendix A, i.e.

$$\begin{aligned}\frac{\partial A(u,t,T)}{\partial t} &= -\Big[\kappa\xi(t)+iu\rho_{rv}\tau\sigma B_r(t,T)\Big]C(u,t,T)\\ &\quad+\frac{1}{2}u\big(\mathrm{i}+u\big)\sigma^2B_r^2(t,T)\\ &\quad-\frac{1}{2}\tau^2\big(C^2(u,t,T)+D(u,t,T)\big).\end{aligned} \tag{47}$$

The first derivative of $A(u,t,T)$ behaves as $O(u^2)$ for large values of u, as can be seen from

$$\lim_{u\to\infty}\frac{1}{u^2}\frac{\partial A(u,t,T)}{\partial t} = \frac{1}{2}\Big(1-C^2(\infty)-2i\rho_{rv}C(\infty)\Big)\sigma^2B_r^2(t,T). \tag{48}$$

Finally, together with the boundary condition $A(u,T,T)=0$, we have

$$\begin{aligned}\lim_{u\to\infty}\frac{A(u,t,T)}{u^2} &= -\int_t^T\lim_{u\to\infty}\frac{1}{u^2}\frac{\partial A(u,s,T)}{\partial s}\mathrm{d}s\\ &= -\frac{1}{2}V(t,T)\cdot\Big(1-C^2(\infty)-2i\rho_{rv}C(\infty)\Big)\\ &\equiv -A(\infty),\end{aligned} \tag{49}$$

where $V(t,T)$ denotes the integrated bond variance, i.e. as defined in (31). One can show that $\mathrm{Re}\Big(A(\infty)\Big)\geq 0$ as $V(t,T)\geq 0$ and:

$$\mathrm{Re}\Big(C^2(\infty)+2i\rho_{rv}C(\infty)\Big) = \frac{\rho_{xr}^2-2\rho_{rv}\rho_{xr}\rho_{xv}+\rho_{rv}^2(4\rho_{xv}^2-3)}{1-\rho_{xv}^2}\leq 1. \tag{50}$$

This follows by maximising the right-hand side with respect to the constraint that the three correlations constitute a positive semi-definite correlation matrix. For example, the maximum is achieved when $\rho_{xr}=-\frac{1}{2}\sqrt{3}$, $\rho_{xv}=-\frac{1}{2}$ and $\rho_{rv}=0$.

The above analysis determines ϕ_r as

$$\phi_r\Big(u-(\alpha+1)\mathrm{i}\Big) = -\mathrm{Re}\Big(A(\infty)\Big)\cdot u^2. \tag{51}$$

One can conclude that the tail behaviour of the characteristic function of the SZHW model is quite different from that of the Schöbel and Zhu (1999) model; whereas the decay in the Schöbel–Zhu model is only exponential, the decay here resembles that of a Gaussian characteristic function, caused by the addition of a Gaussian short rate process. Clearly, if σ (the volatility of the short rate) is zero, $A(\infty)=0$ and the decay of the characteristic function becomes exponential once again. As the tail behaviour of the characteristic function is of the same from as that of the Black and Scholes (1973) characteristic function, an appropriate transformation function is, as in Lord and Kahl (2007),

$$g(u) = -\frac{\ln u}{\sqrt{A(\infty)}}, \tag{52}$$

which can be used in the pricing equation (38).

5. Forward starting options

Due to the popularity of forward starting options such as cliquets, the pricing of forward starting options has recently attracted the attention of both practitioners and academics (e.g. see Lucić (2003), Hong (2004), Kruse and Nögel (2005) and Brigo and Mercurio (2006)). In this section we will show how one can price forward starting options within the SZHW framework; following Hong (2004), we consider the (forward) log return of the asset price x:

$$z(T_{i-1},T_i) := \log\Big(\frac{x(T_i)}{x(T_{i-1})}\Big). \tag{53}$$

Since

$$\log x(t) = y(t)+\log P(t,T_i), \tag{54}$$

we can express (53) also in terms of the T_i-forward log-asset price $y(t)$, i.e.

$$z(T_{i-1},T_i) = y(T_i)-y(T_{i-1})-\log P(T_{i-1},T_i). \tag{55}$$

We are then interested in the following forward starting call option with strike $K = \exp(k)$ on the return $\frac{x(T_i)}{x(T_{i-1})}$,

$$C_{T_{i-1},T_i}(k) = \mathbb{E}^{\mathcal{Q}}\left[\exp\left[-\int_t^{T_i} r(u)\mathrm{d}u\right]\left(\frac{x(T_i)}{x(T_{i-1})} - K\right)^+ \middle| \mathcal{F}_t\right] = P(t,T_i)\mathbb{E}^{\mathcal{Q}^T}\left[\left(F^{T_i}_{T_{i-1},T_i}(T_i) - K\right)^+ | \mathcal{F}_t\right], \tag{56}$$

where

$$F^{T_i}_{T_{i-1},T_i}(T_i) := \exp\left[z(T_{i-1},T_i)\right]$$

denotes the forward return between T_{i-1} and T_i under the T_i-forward measure. Note that the above expression is nothing more than some call option under the T-forward measure. Therefore, as noted by Hong (2004), the pricing of forward starting options can be reduced to finding the characteristic function of the log forward return under the T-forward measure; by replacing the log-asset price by the forward log-return one can directly apply the pricing equation (7) or (38), i.e. by replacing the corresponding characteristic function by $\psi_{T_{i-1},T_i}(v)$: the characteristic function (under the T_i-forward measure) of the forward log-return between T_{i-1} and T_i. What remains to be done for the pricing of forward starting options is the derivation of this forward characteristic function, which we will deal with in the following subsection.

5.1. Forward characteristic function

We will now derive the forward characteristic function of the forward log return $z^{T_i}_{T_{i-1},T_i} = y(T_i) - y(T_{i-1}) - \log P(T_{i-1})$ in the SZHW model. In the derivation we will use the now following corollary.

Corollary 5.1. *Let Z be a standard normal distributed random variable, furthermore let p and q be two positive constants. Then the moment-generating function, provided that $uq < 1$, of $Y := pZ + \frac{q}{2}Z^2$ is given by*

$$\phi_Y(u,p,q) := \mathbb{E}\exp(uY) = \frac{\exp\left(\frac{p^2u^2}{2-2uq}\right)}{\sqrt{1-uq}}. \tag{57}$$

Proof. Either by completing the square and using properties of the non-central chi-squared distribution or by direct integration of an exponential affine form against the normal distribution, e.g. see Johnson et al. (1994) or Glasserman (2003). □

Before we can apply the above corollary we first need to rewrite the characteristic function of the log-return $y(T_i) - y(T_{i-1})$ in the form of the above corollary. To simplify the notation we write $B := iu$, $A(T_{i-1}) := A(u, T_{i-1}, T_i)$, $C(T_{i-1}) := C(u, T_{i-1}, T_i)$ and $D(T_{i-1}) := D(u, T_{i-1}, T_i)$. By using the tower law for conditional expectations and the (conditional) characteristic function of the SZHW model one can then obtain

$$\begin{aligned}
&\phi_{T_{i-1},T_i}(u) \\
&= \mathbb{E}^{\mathcal{Q}^T}\left\{\exp\left(\mathrm{i}u\left[y(T_i) - y(T_{i-1}) - \log P(T_{i-1},T_i)\right]\right)\middle|\mathcal{F}_t\right\} \qquad (58)\\
&= \mathbb{E}^{\mathcal{Q}^T}\left\{\mathbb{E}^{\mathcal{Q}^T}\left[\exp\left(\mathrm{i}u\left[y(T_i) - y(T_{i-1}) - \log P(T_{i-1},T_i)\right]\right)\middle|\mathcal{F}_{T_{i-1}}\right]\middle|\mathcal{F}_t\right\}\\
&= \exp\left[A(T_{i-1}) - \mathrm{i}uA_r(T_{i-1},T_i)\right]\\
&\quad\times\mathbb{E}^{\mathcal{Q}^T}\left\{\exp\left[\mathrm{i}uB_r(T_{i-1},T_i)r(T_{i-1}) + C(T_{i-1})\nu(T_{i-1})\right.\right.\\
&\quad\left.\left.+\frac{1}{2}D(T_{i-1})\nu^2(T_{i-1})\right]\middle|\mathcal{F}_t\right\}.
\end{aligned}$$

Since the pair $\left(r(T_{i-1}), \nu(T_{i-1})\right)$, conditional on its standard time-t filtration, follows a joint Gaussian distribution with means μ_r, μ_ν (see (65), (67)) and variances σ_r^2, σ_ν^2 (see (66), (68)), we can write the sum of dependent normal variates $r(T_{i-1})$, $\nu(T_{i-1})$ in terms of two independent standard normal distributions Z_1 and Z_2 (e.g. by a Cholesky decomposition):

$$\begin{aligned}
&\mathrm{i}ubr(T_{i-1}) + c\nu(T_{i-1}) + \frac{1}{2}\mathrm{d}\nu^2(T_{i-1})\\
&\overset{d}{=} iub\left(\mu_r + \sigma_r\left[\rho_{r\nu}(t,T_{i-1})Z_1 + \sqrt{1-\rho^2_{r\nu}(t,T_{i-1})}Z_2\right]\right)\\
&\quad + c\left(\mu_\nu + \sigma_\nu Z_1\right) + \frac{1}{2}d\left(\mu_\nu + \sigma_\nu Z_1\right)^2\\
&= \mathrm{i}ub\mu_r + c\mu_\nu + \frac{1}{2}\mathrm{d}\mu_\nu^2 + \mathrm{i}ub\sigma_r\sqrt{1-\rho^2_{r\nu}(t,T_{i-1})}Z_2\\
&\quad + \left[c\sigma_\nu + d\mu_\nu\sigma_\nu + \mathrm{i}ub\rho_{r\nu}(t,T_{i-1})\sigma_r\right]Z_1 + \frac{1}{2}\mathrm{d}\sigma_\nu^2Z_1^2, \qquad (59)
\end{aligned}$$

where the correlation $\rho_{r\nu}(t,T_{i-1})$ between $r(T_{i-1})$ and $\nu(T_{i-1})$ over the interval $[t,T_{i-1}]$ is given by

$$\rho_{r\nu}(t,T_{i-1}) = \frac{\rho_{r\nu}\sigma\tau}{\sigma_r\sigma_\nu(a+\kappa)}\left[1 - \mathrm{e}^{-(a+\kappa)(T_{i-1}-t)}\right]. \tag{60}$$

Hence using the independence of Z_1 and Z_2 and Eqs. (59)–(58), one can find the following expression for the forward characteristic function

$$\begin{aligned}
&\phi_{T_{i-1},T_i}(u) = \exp\left[A(T_{i-1}) + \mathrm{i}u\left(B_r(T_{i-1},T_i)\mu_r - A_r(T_{i-1},T_i)\right)\right.\\
&\quad\left.+ C(T_{i-1})\mu_\nu + \frac{1}{2}D(T_{i-1})\mu_\nu^2\right]\\
&\quad\times\mathbb{E}^{\mathcal{Q}^T}\left\{\exp\left[\mathrm{i}uB_r(T_{i-1},T_i)\sigma_r\sqrt{1-\rho^2_{r\nu}(t,T_{i-1})}Z_2\right]\middle|\mathcal{F}_t\right\}\\
&\quad\times\mathbb{E}^{\mathcal{Q}^T}\left\{\exp\left(\left[C(T_{i-1})\sigma_\nu + D(T_{i-1})\mu_\nu\sigma_\nu\right.\right.\right.\\
&\quad\left.\left.\left.+\mathrm{i}uB_r(T_{i-1},T_i)\rho_{r\nu}(t,T_{i-1})\sigma_r\right]Z_1 + \frac{1}{2}D(T_{i-1})\sigma_\nu^2Z_1^2\right)\middle|\mathcal{F}_t\right\}. \qquad (61)
\end{aligned}$$

Hence we come to the following proposition

Proposition 5.2. *Conditional on the current time t, the characteristic function of the forward log return $z(T_{i-1},T_i)$ under the T_i-forward measure is given by the following closed-form solution:*

$$\begin{aligned}
\phi_{T_{i-1},T_i}(u) &= \exp\left[A(T_{i-1}) + \mathrm{i}u\left[B_r(T_{i-1},T_i)\mu_r - A_r(T_{i-1},T_i)\right]\right.\\
&\quad\left.+ C(T_{i-1})\mu_\nu + \frac{1}{2}D(T_{i-1})\mu_\nu^2\right]\\
&\quad\times\phi_{Z_2}\left(\mathrm{i}uB_r(T_{i-1},T_i)\sigma_r\sqrt{1-\rho^2_{r\nu}(t,T_{i-1})}\right)\\
&\quad\times\phi_Y\left(1, P(T_{i-1}), Q(T_{i-1})\right) \qquad (62)
\end{aligned}$$

with

$$P(T_{i-1}) = C(T_{i-1})\sigma_\nu + D(T_{i-1})\mu_\nu\sigma_\nu + \mathrm{i}u\rho_{r\nu}(t,T_{i-1})B_r(T_{i-1},T_i)\sigma_r,$$
$$Q(T_{i-1}) = D(T_{i-1})\sigma_\nu^2,$$
$$\phi_{Z_2}(y) = \exp\left(\frac{y^2}{2}\right),$$

and where $\phi_Y\Big(1, P(T_{i-1}), Q(T_{i-1})\Big)$*, provided that* $Q(T_{i-1}) < 1$*, is given by Corollary 5.1.*

Proof. The result follows directly by evaluating the expectations from expression (61) for the moment-generating function of the standard Gaussian distribution Z_2 evaluated in the point $iuB_r(T_{i-1}, T_i)\sigma_r\sqrt{1-\rho_{rv}^2(t, T_{i-1})}$, while the second expectation is the moment generating function of the random variable $Y = P(T_{i-1})Z_1 + \frac{Q(T_{i-1})}{2}Z_1^2$ evaluated in the unit point, for which (provided that $Q(T_{i-1}) < 1$) an analytical expression is given by Corollary 5.1. □

What yet remains, is to determine (conditional on the time-t) the T_i-forward mean and variance of the Ornstein–Uhlenbeck processes $r(T_{i-1})$ and $v(T_{i-1})$. Before we do this, we briefly address the strip of regularity and decay of the characteristic function.

The strip of regularity of (62) is once again determined by $C(T_{i-1})$, see Andersen and Piterbarg (2007) for a detailed analysis in case of the Heston model, and Lord and Kahl (2007) for the SZ model. The difference with the SZ and SZHW models is the additional condition that $Q(T_{i-1}) < 1$, which is imposed by Corollary 5.1.

The decay of the characteristic function is slightly different than our analysis for the SZHW model. We will briefly mention how to derive the exact behaviour, though we do not provide all details for reasons of brevity. For large values of u, the characteristic function will behave like $\frac{\exp\left(-C_1u^2\right)}{\sqrt{1+C_2u^2}}$, where C_1 and C_2 are constants. Both $A(T_i - 1)$, ϕ_{Z_2} and ϕ_Y contribute to the exponential term, whereas only the latter contributes to the square root term.

5.2. Moments of the Hull–White short interest rate

To determine the moments of the Hull–White short interest rate under the T_i-forward measure, for a certain time $T_{i-1} \leq T_i$ and conditional on the filtration at time t, one can consider the following transformation of variables (see e.g. Pelsser (2000) or Brigo and Mercurio (2006))

$$r(T_{i-1}) = \alpha(T_{i-1}) + \beta(T_{i-1}), \tag{63}$$

with β a driftless Ornstein–Uhlenbeck process and where

$$\alpha(T_{i-1}) = \mathrm{e}^{-aT_{i-1}}\left(r(t) + \int_t^{T_{i-1}} \mathrm{e}^{au}\theta(s)\mathrm{d}u\right),$$

which, in case one wants to fit the initial term structure of interest rates evolves into

$$\alpha(T_{i-1}) = f(t, T_{i-1}) + \frac{\sigma^2}{2a^2}\left(1 - \mathrm{e}^{-aT_{i-1}}\right)^2.$$

A solution for $\beta(T_{i-1})|\beta(t)$ under the T_i-forward measure is given by

$$\begin{aligned}\beta(T_{i-1}) &= \beta(t)\mathrm{e}^{-a(T_{i-1}-t)} - M^{T_i}(t, T_{i-1}) \\ &\quad + \sigma\int_t^{T_{i-1}} \mathrm{e}^{-a(T_{i-1}-u)}\mathrm{d}W_r^{T_i}(u),\end{aligned}$$

where

$$\begin{aligned}M^{T_i}(t, T_{i-1}) &= \frac{\sigma^2}{a^2}\left(1 - \mathrm{e}^{-a(T_{i-1}-t)}\right) \\ &\quad - \frac{\sigma^2}{2a^2}\left(\mathrm{e}^{-a(T_i-T_{i-1})} - \mathrm{e}^{-a(T_i+T_{i-1}-2t)}\right).\end{aligned} \tag{64}$$

Hence, from Ito's isometry, we immediately have that $r(T_{i-1})$, under the T_i-forward measure (conditional on time t), is normally distributed with mean μ_r and variance σ_r^2 given by

$$\mu_r = \beta(t)\mathrm{e}^{-a(T_{i-1}-t)} - M^{T_i}(t, T_{i-1}) + \alpha(T_{i-1}), \tag{65}$$

$$\sigma_r^2 = \frac{\sigma^2}{2a}\left(1 - \mathrm{e}^{-2a(T_{i-1}-t)}\right), \tag{66}$$

which can hence be used in Proposition 5.2.

5.3. Moments of the Schöbel–Zhu volatility process

To determine the first two moments of the Schöbel–Zhu volatility process, under the T_i-forward measure, for a certain time $T_{i-1} \leq T_i$ and conditional on the filtration at time t, one can integrate the dynamics of (16) to obtain

$$\begin{aligned}v(T_{i-1}) &= v(t)\mathrm{e}^{-\kappa(T_{i-1}-t)} + \int_t^{T_{i-1}} \kappa\xi(u)\mathrm{e}^{-\kappa(T_{i-1}-u)}\mathrm{d}u \\ &\quad + \int_t^{T_{i-1}} \tau\mathrm{e}^{-\kappa(T_{i-1}-u)}\mathrm{d}W_v^T(u),\end{aligned}$$

where $\xi(u) := \psi - \frac{\rho_{rv}\sigma\tau}{a\kappa}\left(1 - \mathrm{e}^{a(T_i-u)}\right)$. Hence under the T_i-forward measure, we have the following for the mean and standard deviation of v:

$$\begin{aligned}\mu_v &= v(t)\mathrm{e}^{-\kappa(T_{i-1}-t)} + \left(\psi - \frac{\rho_{rv}\sigma\tau}{a\kappa}\right)\left(1 - \mathrm{e}^{-\kappa(T_{i-1}-t)}\right) \\ &\quad - \frac{\rho_{rv}\sigma\tau}{a(\kappa+a)}\left(\mathrm{e}^{-a(T_i-t)-\kappa(T_{i-1}-t)} - \mathrm{e}^{-a(T_i-T_{i-1})}\right),\end{aligned} \tag{67}$$

$$\sigma_v^2 = \frac{\tau^2}{2\kappa}\left(1 - \mathrm{e}^{-2\kappa(T_{i-1}-t)}\right), \tag{68}$$

which can hence be used in Proposition 5.2.

6. Schöbel–Zhu–Hull–White Foreign Exchange model

In this section we present the Schöbel–Zhu–Hull–White Foreign Exchange (SZHW-FX) model. That is, we introduce a domestic and a foreign exchange currency, which are modelled by Hull–White processes. We model the exchange rate process by geometric motion where we let the volatility follow an Ornstein–Uhlenbeck process. Moreover we allow all factors to be correlated with each other.

Notation is as follows: we let $x(t)$ denote the Foreign Exchange (FX) rate, with volatility v, between the domestic currency r_1 and the foreign currency r_2. The risk-neutral FX dynamics of the Schöbel–Zhu–Hull–White (SZHW) then read:

$$\begin{aligned}\mathrm{d}x(t) &= x(t)\big(r_1(t) - r_2(t)\big)\mathrm{d}t \\ &\quad + x(t)v(t)\mathrm{d}W_x(t), \quad x(0) = x_0,\end{aligned} \tag{69}$$

$$\mathrm{d}r_1(t) = \big(\theta_1(t) - a_1r_1(t)\big)\mathrm{d}t + \sigma_1\mathrm{d}W_{r_1}(t), \quad r_1(0) = r_0^1, \tag{70}$$

$$\begin{aligned}\mathrm{d}r_2(t) &= \big(\theta_2(t) - a_2r_2(t) - \rho_{xr_2}v(t)\sigma_2\big)\mathrm{d}t \\ &\quad + \sigma_2\mathrm{d}W_{r_2}(t), \quad r_2(0) = r_0^2,\end{aligned} \tag{71}$$

$$\mathrm{d}v(t) = \kappa\big(\psi - v(t)\big)\mathrm{d}t + \tau\mathrm{d}W_v(t), \quad v(0) = v_0, \tag{72}$$

where $a_i, \sigma_i, \kappa, \psi, \tau$ are positive parameters. Hence the domestic and the (shifted) foreign interest rate markets are modelled by Hull–White models and the exchange rate is modelled by a Schöbel–Zhu stochastic volatility model. $W(t) = \big(W_x(t), W_{r_1}, W_{r_2}(t), W_v(t)\big)$ denotes a Brownian motion under the risk-neutral measure $\mathcal{Q}$ with a positive covariance matrix:

$$\mathrm{Var}\big(\widetilde{W}(t)\big) = \begin{pmatrix} 1 & \rho_{xr_1} & \rho_{xr_2} & \rho_{xv} \\ \rho_{xr_1} & 1 & \rho_{r_1r_2} & \rho_{r_1v} \\ \rho_{xr_2} & \rho_{r_1r_2} & 1 & \rho_{r_2v} \\ \rho_{xv} & \rho_{r_1v} & \rho_{r_2v} & 1 \end{pmatrix} t. \tag{73}$$

We will now show that the above model dynamics yield a closed-form expression for the price of an European FX-option with strike K and maturity T. Hence we consider:

$$\mathbb{E}^{\mathcal{Q}}\left[\frac{\big(w(x(T) - K)\big)^+}{N_1(T)}\middle|\mathcal{F}_t\right], \tag{74}$$

where $w = \pm 1$ for a call/put option and with

$$N_1(T) = \exp\left[\int_t^T r(u)\mathrm{d}u\right] \tag{75}$$

denotes the bank-account in the domestic economy. We can also represent the expectation (76) in the domestic T-forward measure $\mathcal{Q}^T$ associated with a domestic zero-coupon bond option $P_1(t,T)$ which matures at time T, hence we obtain

$$\mathbb{E}^{\mathcal{Q}}\left[\frac{\big(w(x(T)-K)\big)^+}{N_1(T)}\middle|\mathcal{F}_t\right] = P_1(t,T)\mathbb{E}^{\mathcal{Q}^T}\Big[\big(w(FFX^T(T)-K)\big)^+|\mathcal{F}_t\Big], \tag{76}$$

where

$$FFX_T(t) = \frac{x(t)P_2(t,T)}{P_1(t,T)} \tag{77}$$

denotes the forward FX-rate under the domestic T-forward measure.

The Hull–White model yields analytical expressions for the above prices of the zero-coupon discount bonds, i.e.

$$P_i(t,T) = \exp\big[A_i(t,T) - B_i(t,T)r_i(t)\big] \quad \text{with: } B_i(t,T) := \frac{1-\mathrm{e}^{-a_i(T-t)}}{a_i}, \tag{78}$$

where $A_i(t,T)$ is affine function. Hence we can express the forward FX-rate as

$$FFX_T(t) = \frac{x(t)\exp\big[A_2(t,T) - B_2(t,T)r_2(t)\big]}{\exp\big[A_1(t,T) - B_1(t,T)r_1(t)\big]}. \tag{79}$$

Note that under their own risk-neutral measures (where we the money market bank account of their own currency is used as numeraire) the discount bond prices follows the processes

$$\frac{\mathrm{d}P_i(t,T)}{P_i(t,T)} = r_i(t)\mathrm{d}t - \sigma_i B_i(t,T)\mathrm{d}W_{r_i}(t), \tag{80}$$

hence, by an application of Ito's lemma, we find the following dynamics for the T-forward stock price process

$$\begin{aligned}\frac{\mathrm{d}FFX_T(t)}{FFX_T(t)} &= \big(\sigma_1^2 B_1^2(t,T) + \rho_{xr_1}\nu(t)\sigma_1 B_1(t,T) \\ &\quad - \rho_{r_1r_2}\sigma_2 B_2(t,T)\sigma_1 B_1(t,T)\big)\mathrm{d}t \\ &\quad + \nu(t)\mathrm{d}W_x(t) + \sigma_1 B_1(t,T)\mathrm{d}W_{r_1}(t) \\ &\quad - \sigma_2 B_2(t,T)\mathrm{d}W_{r_2}(t).\end{aligned} \tag{81}$$

By definition the forward FX-rate is a martingale process under the domestic T-forward measure. This is achieved by defining the following transformations of the Brownian motion(s):

$$\begin{aligned}\mathrm{d}W_{r_1}(t) &\mapsto \mathrm{d}W^T_{r_1}(t) - \sigma_1 B_1(t,T)\mathrm{d}t,\\ \mathrm{d}W_{r_2}(t) &\mapsto \mathrm{d}W^T_{r_2}(t) - \rho_{r_1r_2}\sigma_1 B_1(t,T)\mathrm{d}t,\\ \mathrm{d}W_x(t) &\mapsto \mathrm{d}W^T_x(t) - \rho_{xr_1}\sigma_1 B_1(t,T)\mathrm{d}t,\\ \mathrm{d}W_\nu(t) &\mapsto \mathrm{d}W^T_\nu(t) - \rho_{r_1\nu}\sigma_1 B_1(t,T)\mathrm{d}t.\end{aligned}$$

Hence under the domestic T-forward measure the forward FX-rate and the associated volatility process are given by

$$\begin{aligned}\frac{\mathrm{d}FFX_T(t)}{FFX_T(t)} &= \nu(t)\mathrm{d}W^T_x(t) + \sigma_1 B_1(t,T)\mathrm{d}W^T_{r_1} \\ &\quad - \sigma_2 B_2(t,T)\mathrm{d}W^T_{r_2}(t)\end{aligned} \tag{82}$$

$$\mathrm{d}\nu(t) = \kappa\Big(\psi - \frac{\rho_{r_1\nu}\sigma_1\tau}{\kappa}B_1(t,T) - \nu(t)\Big)\mathrm{d}t + \tau\mathrm{d}W^T_\nu(t). \tag{83}$$

We can simplify (82) by switching to logarithmic coordinates and rotating the Brownian motions $W^T_x(t)$, $W^T_{r_1}(t)$ and $W^T_{r_2}(t)$ to $W^T_F(t)$. Defining $y(t) := \log\big(FFX_T(t)\big)$ and an application of Ito's lemma yields

$$\mathrm{d}y(t) = -\frac{1}{2}\nu_F^2(t)\mathrm{d}t + \nu_F(t)\mathrm{d}W^T_F(t) \tag{84}$$

$$\mathrm{d}\nu(t) = \kappa\Big(\xi(t) - \nu(t)\Big)\mathrm{d}t + \tau\mathrm{d}W^T_\nu(t), \tag{85}$$

with:

$$\begin{aligned}\nu_F^2(t) &:= \nu^2(t) + \sigma_1^2 B_1^2(t,T) + \sigma_2^2 B_2^2(t,T) \\ &\quad + 2\rho_{xr_1}\nu(t)\sigma_1 B_1(t,T) - 2\rho_{xr_2}\nu(t)\sigma_2 B_2(t,T) \\ &\quad - 2\rho_{r_1r_2}\sigma_1 B_1(t,T)\sigma_2 B_2(t,T)\end{aligned} \tag{86}$$

$$\xi(t) := \psi - \frac{\rho_{r_1\nu}\sigma_1\tau}{\kappa}B_1(t,T). \tag{87}$$

Notice that we have now reduced the system (69) of the variables $x(t)$, $r_1(t)$, $r_2(t)$, $\nu(t)$ under the domestic risk-neutral measure, to the system (84) of variables $y(t)$ and $\nu(t)$ under the domestic T-forward measure. What now remains is to determine the characteristic function of this reduced system.

Determining the characteristic function of the forward log-FX rate

We will now determine the characteristic function of the forward FX rate. Since this calculation goes in a similar spirit as the calculation of the ordinary characteristic function of the Schöbel–Zhu–Hull–White model of Section 2, we restrict ourselves to the most important steps. Again we apply the Feynman–Kac theorem and reduce the search for the characteristic function of the forward-FX rate dynamics to solving a partial differential equation. That is, we try to determine the Kolmogorov backward partial differential equation of the joint probability function $f = f(t,y,\nu)$. To this end we need to take into account the following covariance term

$$\begin{aligned}\mathrm{d}y(t)\mathrm{d}\nu(t) &= \Big(\nu(t)\mathrm{d}W^T_x(t) + \sigma_1 B_1(t,T)\mathrm{d}W^T_{r_1}(t) \\ &\quad - \sigma_2 B_2(t,T)\mathrm{d}W^T_{r_1}(t)\Big)\Big(\tau\mathrm{d}W^T_\nu(t)\Big) \\ &= \Big(\rho_{x\nu}\tau\nu(t) + \rho_{r_1\nu}\tau\sigma_1 B_1(t,T) - \rho_{r_2\nu}\tau\sigma_2 B_2(t,T)\Big)\mathrm{d}t.\end{aligned} \tag{88}$$

Hence using (84) and (88), the Feynman–Kac theorem then implies that the solution of the following PDE

$$\begin{aligned}0 &= f_t - \frac{1}{2}\nu_F^2(t)f_y + \kappa\big(\xi(t)-\nu(t)\big)f_\nu + \frac{1}{2}\nu_F^2(t)f_{yy} \\ &\quad + \Big(\rho_{x\nu}\tau\nu(t) + \rho_{r_1\nu}\tau\sigma_1 B_1(t,T) - \rho_{r_2\nu}\tau\sigma_2 B_2(t,T)\Big)f_{y\nu} \\ &\quad + \frac{1}{2}\tau^2 f_{\nu\nu},\end{aligned} \tag{89}$$

subject to the terminal boundary condition $f(T,y,\nu) = \exp(iuy(T))$, equals the characteristic function of the forward FX-rate dynamics. Solving the above system hence leads to the following proposition.

Proposition 6.1. *The characteristic function of domestic T-forward log SZHW-FX-rate is given by the closed-form solution in* Box II.

Proof. See Appendix B. □

The strip of regularity and the decay of the characteristic function can be determined analogous to the SZHW model. The function $C(u,t,T)$ once again determines the strip of regularity, whereas $A(u,t,T)$ ensures the characteristic function decays like $\exp\big(-C(u,t,T)u^2\big)$, where the exact constant follows from a similar analysis to that in Section 4.

$$f(t,y,v)=\exp\Big[A(t)+B(t)y(t)+C(t)v(t)+\frac{1}{2}D(t)v^2(t)\Big], \tag{90}$$

where:

$$A(u,t,T)=\frac{1}{2}\big(B^2-B\big)V_{FX}(t,T)+\int_t^T\Big[\Big(\kappa\psi+\rho_{r_1v}(iu-1)\tau\sigma_1B_1(s,T)-\rho_{r_2v}iu\tau\sigma_2B_2(s,T)\Big)C(s)+\frac{1}{2}\tau^2\Big(C^2(s)+D(s)\Big)\Big]ds, \tag{91}$$

$$B=iu, \tag{92}$$

$$\begin{aligned}C(u,t,T)\;=\;&-u\big(i+u\big)\times\frac{\Big(\big(\gamma_3-\gamma_4e^{-2\gamma(T-t)}\big)-\big(\gamma_5e^{-a_1(T-t)}-\gamma_6e^{-(2\gamma+a_1)(T-t)}\big)-\gamma_7e^{-\gamma(T-t)}\Big)}{\gamma_1+\gamma_2e^{-2\gamma(T-t)}}\\&+u\big(i+u\big)\times\frac{\Big(\big(\gamma_8-\gamma_9e^{-2\gamma(T-t)}\big)-\big(\gamma_{10}e^{-a_2(T-t)}-\gamma_{11}e^{-(2\gamma+a_2)(T-t)}\big)-\gamma_{12}e^{-\gamma(T-t)}\Big)}{\gamma_1+\gamma_2e^{-2\gamma(T-t)}},\end{aligned} \tag{93}$$

$$D(u,t,T)=-u\big(i+u\big)\frac{1-e^{-2\gamma(T-t)}}{\gamma_1+\gamma_2e^{-2\gamma(T-t)}} \tag{94}$$

with:

$$\gamma=\sqrt{(\kappa-\rho_{xv}\tau B)^2-\tau^2(B^2-B)}, \tag{95}$$

$$\gamma_1=\gamma+(\kappa-\rho_{xv}\tau B),\qquad \gamma_2=\gamma-(\kappa-\rho_{xv}\tau B),$$

$$\gamma_3=\frac{\rho_{xr_1}\sigma_1\gamma_1+\kappa a_1\psi+\rho_{r_1v}\sigma_1\tau(iu-1)}{a_1\gamma},\qquad \gamma_4=\frac{\rho_{xr_1}\sigma_1\gamma_2-\kappa a_1\psi-\rho_{r_1v}\sigma_1\tau(iu-1)}{a_1\gamma},$$

$$\gamma_5=\frac{\rho_{xr_1}\sigma_1\gamma_1+\rho_{r_1v}\sigma_1\tau(iu-1)}{a_1(\gamma-a_1)},\qquad \gamma_6=\frac{\rho_{xr_1}\sigma_1\gamma_2-\rho_{r_1v}\sigma_1\tau(iu-1)}{a_1(\gamma+a_1)},$$

$$\gamma_8=\frac{\rho_{xr_2}\sigma_2\gamma_1+\rho_{r_2v}\sigma_2\tau B}{a_2\gamma},\qquad \gamma_9=\frac{\rho_{xr_2}\sigma_2\gamma_2-\rho_{r_2v}\sigma_2\tau B}{a_2\gamma},$$

$$\gamma_{10}=\frac{\rho_{xr_2}\sigma_2\gamma_1+\rho_{r_2v}\sigma_2\tau B}{a_2(\gamma-a_2)},\qquad \gamma_{11}=\frac{\rho_{xr_2}\sigma_2\gamma_2-\rho_{r_2v}\sigma_2\tau B}{a_2(\gamma+a_2)},$$

$$\gamma_7=(\gamma_3-\gamma_4)-(\gamma_5-\gamma_6),\qquad \gamma_{12}=(\gamma_8-\gamma_9)-(\gamma_{10}-\gamma_{11})$$

and:

$$\begin{aligned}V_{FX}(t,T)\;:=\;&\frac{\sigma_1^2}{a_1^2}\Big((T-t)+\frac{2}{a_1}e^{-a_1(T-t)}-\frac{1}{2a_1}e^{-2a_1(T-t)}-\frac{3}{2a_1}\Big)+\frac{\sigma_2^2}{a_2^2}\Big((T-t)+\frac{2}{a_2}e^{-a_2(T-t)}-\frac{1}{2a_2}e^{-2a_2(T-t)}-\frac{3}{2a_2}\Big)\\&-2\rho_{r_1r_2}\frac{\sigma_1\sigma_2}{a_1a_2}\Big((T-t)+\frac{e^{-a_1(T-t)}-1}{a_1}+\frac{e^{-a_2(T-t)}-1}{a_2}-\frac{e^{-(a_1+a_2)(T-t)}-1}{a_1+a_2}\Big).\end{aligned} \tag{96}$$

Box II.

7. Conclusion

We have introduced the SZHW model which allows for the pricing of insurance contracts under both stochastic volatility and stochastic interest rates in conjunction with an explicit incorporation of the correlation between the underlying asset and the term structure of interest rates. As insurance contracts typically involve long maturities, they are are much more sensitive to changes in the interest rates and the volatility. Therefore, having the flexibility to correlate the underlying asset price with both the stochastic volatility and the stochastic interest rates yields a more realistic model, which is of practical importance for the pricing and hedging of long-term options.

Our model incorporates the closed-form pricing of European options by Fourier transforming the conditional characteristic function of the asset price in closed-form. We extensively considered the numerical implementation of the pricing formulas which enables a fast and accurate valuation of European options, which is a big advantage for the calibration (and sensitivity analysis) of the model to market prices. We have also derived a closed-form pricing formula for forward starting options, which allows for a calibration of the model to forward smiles.

The SZHW model will be especially useful in the pricing and risk management of insurance contracts and other long-maturity exotic derivatives. Examples include pension products, variable and guaranteed annuities, rate of return guarantees, unit-linked contracts and exotic options like PRDC FX options which have a long-term nature. For these products it is especially important to consider the risk of the underlying in conjunction with the interest rate risk of the contract. Given empirical data on option prices our model can be used to examine the pricing and especially hedging performance of stochastic volatility models while correcting for interest rate risk. An empirical study on the relative performance of the SZHW model versus other stochastic volatility models, as well as the relative benefit of the modelling of stochastic interest rates (covered earlier by Bakshi et al. (1997)), is beyond the scope of this paper, and is left for future research.

Appendix A. Deriving the log asset price characteristic function

In this appendix we will show that the partial differential equation (22)

$$f_t+\kappa\big(\xi(t)-v(t)\big)f_v+\frac{1}{2}v_F^2(t)\big(f_{yy}-f_y\big)$$

$$+\big(\rho_{xv}\tau v(t)+\rho_{rv}\tau\sigma B_r(t,T)\big)f_{yv}+\frac{1}{2}\tau^2 f_{vv}=0, \tag{97}$$

subject to the terminal boundary condition

$$f(T,y,v)=\psi(y,v):=\exp\Big(\mathrm{i}uy(T)\Big),$$

has a solution given by (25)–(29).

To lighten the notation, we from here on omit the explicit dependence on u and T in the A, B, C, D terms and hence write $A(t)$ instead of $A(u,t,T)$ for these terms. Using the ansatz

$$f(t,y,v)=\exp\Big[A(t)+B(t)y(t)+C(t)v(t)+\frac{1}{2}D(t)v^2(t)\Big], \tag{98}$$

we find the following partial derivatives for $f=f(t,y,v)$:

$$f_t=f\cdot\left(A'(t)+B'(t)y(t)+C'(t)v(t)+\frac{1}{2}D'(t)v^2(t)\right),$$
$$f_y=fB(t),$$
$$f_v=f\cdot\big(C(t)+D(t)v(t)\big),\qquad f_{yy}=fB^2(t),$$
$$f_{yv}=fB(t)\big(C(t)+D(t)v(t)\big)$$
$$\begin{aligned}f_{vv}&=f\cdot\big(C(t)+D(t)v^2(t)\big)\\&=f\cdot\big(C^2(t)+D(t)+2C(t)D(t)v(t)+D^2(t)v^2(t)\big).\end{aligned}$$

Substituting these partial derivatives into the partial differential equation (97) then gives

$$\begin{aligned}&\left(A'(t)+B'(t)y(t)+C'(t)v(t)+\frac{1}{2}D'(t)v^2(t)\right)\\&+\kappa\big(\xi(t)-v(t)\big)\big(C(t)+D(t)v(t)\big)\\&+\frac{1}{2}\Big(v^2(t)+2\rho_{xr}v(t)\sigma B_r(t,T)+\sigma^2B_r^2(t,T)\Big)\big(B^2(t)-B(t)\big)\\&+\big(\rho_{xv}\tau v(t)+\rho_{rv}\tau\sigma B_r(t,T)\big)B(t)\big(C(t)+D(t)v(t)\big)\\&+\frac{1}{2}\tau^2\big(C^2(t)+D(t)+2C(t)D(t)v(t)+D^2(t)v^2(t)\big)=0.\end{aligned} \tag{99}$$

Collecting terms for $y(t)$,$v(t)$, and $\frac{1}{2}v^2(t)$ then yields the following four ordinary differential equations for the functions $A(t),\ldots,D(t)$:

$$0=B'(t)\Rightarrow B(t):=B, \tag{100}$$

$$0=D'(t)-2\big(\kappa-\rho_{xv}\tau B\big)D(t)+\tau^2D^2(t)+(B^2-B), \tag{101}$$

$$\begin{aligned}0&=C'(t)+\big(\rho_{xv}\tau B-\kappa+\tau^2D\big)C(t)+\rho_{xr}\sigma B_r(t,T)\big(B^2-B\big)\\&\quad+\big(\kappa\xi(t)+\rho_{rv}\tau\sigma B_r(t,T)B\big)D(t),\end{aligned} \tag{102}$$

$$\begin{aligned}0&=A'(t)+\big(\kappa\xi(t)+\rho_{rv}\tau\sigma B_r(t,T)B\big)C(t)\\&\quad+\frac{1}{2}\sigma^2B_r^2(t,T)\big(B^2-B\big)+\frac{1}{2}\tau^2\big(C^2(t)+D(t)\big).\end{aligned} \tag{103}$$

As already noted in Eq. (100), it immediately that follows $B(t)=B$ equals a constant since its derivative is zero. Subject to the boundary condition (97) we then find

$$B=\mathrm{i}u. \tag{104}$$

The second Eq. (101) yields a Riccati equation with constant coefficients with boundary condition $D(T)=0$:

$$\begin{aligned}D'(t)&=-(B^2-B)+2(\kappa-\rho_{xv}\tau B)D(t)-\tau^2D^2(t)\\&=:q_0+q_1D(t)+q_2D^2(t).\end{aligned}$$

Making the substitution $D(t)=\frac{-v'(t)}{q_2v(t)}$ transforms the Riccati equation into the following second order linear differential equation with constant coefficients:

$$v''(t)-q_1v'(t)+q_0q_2v(t)=0, \tag{105}$$

which solution is given by

$$v(t)=\gamma_1\exp\big[\lambda_+(T-t)\big]+\gamma_2\exp\big[\lambda_-(T-t)\big],$$
$$\lambda_\pm=-\frac{q_1}{2}\pm\sqrt{q_1^2-4q_0q_2}.$$

Hence defining $\gamma=\sqrt{q_1^2-4q_0q_2}$ we find:

$$\begin{aligned}D(t)&=\frac{-v'(t)}{q_2v(t)}=-\frac{1}{\tau^2}\frac{\gamma_1\gamma_2\mathrm{e}^{\gamma(T-t)}-\gamma_1\gamma_2\mathrm{e}^{-\gamma(T-t)}}{\gamma_1\mathrm{e}^{\gamma(T-t)}+\gamma_2\mathrm{e}^{-\gamma(T-t)}}\\&=(B^2-B)\frac{\mathrm{e}^{\gamma(T-t)}-\mathrm{e}^{-\gamma(T-t)}}{\gamma_1\mathrm{e}^{\gamma(T-t)}+\gamma_2\mathrm{e}^{-\gamma(T-t)}}\\&=-u\big(\mathrm{i}+u\big)\frac{1-\mathrm{e}^{-2\gamma(T-t)}}{\gamma_1+\gamma_2\mathrm{e}^{-2\gamma(T-t)}}\end{aligned} \tag{106}$$

$$\text{with: }\gamma=\sqrt{(\kappa-\rho_{xv}\tau B)^2-\tau^2(B^2-B)}, \tag{107}$$

$$\gamma_1=\gamma+\frac{1}{2}q_1=\gamma+(\kappa-\rho_{xv}\tau B), \tag{108}$$

$$\gamma_2=\gamma-\frac{1}{2}q_1=\gamma-(\kappa-\rho_{xv}\tau B). \tag{109}$$

Here the constants in Eq. (106) are determined from the identity $(\gamma+\frac{1}{2}q_1)(\gamma-\frac{1}{2}q_1)=-(B^2-B)\tau^2$ and the boundary condition $D(T)=0$.

The third Eq. (102) looks pretty daunting, but is merely a first order linear ordinary differential equation of the form $C'(t)+g(t)C(t)+h(t)=0$. Subject to the boundary condition $C(T)=0$ and using (20), we can hence represent a solution for $C(t)$ as:

$$C(t)=\int_t^T h(s)\exp\left[\int_t^s g(w)\mathrm{d}w\right]\mathrm{d}s, \tag{110}$$

$$\text{with: }g(w)=-(\kappa-\rho_{xv}\tau B)+\tau^2D(w), \tag{111}$$

$$\begin{aligned}h(s)&=\rho_{xr}\sigma B_r(s,T)\big(B^2-B\big)+\big(\kappa\xi(s)+\rho_{rv}\tau\sigma B_r(s,T)B\big)D(s)\\&=\rho_{xr}\sigma B_r(s,T)\big(B^2-B\big)\\&\quad+\big(\kappa\psi+\rho_{rv}(B-1)\tau\sigma B_r(s,T)\big)D(s).\end{aligned} \tag{112}$$

We first consider the integral over g: dividing Eq. (101) by $D(t)$, rearranging terms and integrating we find the surprisingly simple solution:

$$\begin{aligned}\int g(w)\mathrm{d}w&=\int-(\kappa-\rho_{xv}\tau B)+\tau^2D(w)\mathrm{d}w\\&=\int(\kappa-\rho_{xv}\tau B)-\frac{(B^2-B)}{D(w)}-\frac{D'(w)}{D(w)}\mathrm{d}w\\&=\log\Big(\gamma_1\mathrm{e}^{\gamma(T-w)}+\gamma_2\mathrm{e}^{-\gamma(T-w)}\Big)+C,\end{aligned} \tag{113}$$

where C denotes the integration constant. Hence taking the exponent and filling in the required integration boundaries yields

$$\exp\left[\int_t^s g(w)\mathrm{d}w\right]=\frac{\gamma_1\mathrm{e}^{\gamma(T-s)}+\gamma_2\mathrm{e}^{-\gamma(T-s)}}{\gamma_1\mathrm{e}^{\gamma(T-t)}+\gamma_2\mathrm{e}^{-\gamma(T-t)}}, \tag{114}$$

and after a straightforward calculation we get $C(t)$ as in Box III.

Finally, by solving Eq. (103), we find the following expression for $A(t)$:

$$\begin{aligned}A(t)&=\int_t^T\frac{1}{2}\big(B^2-B\big)\sigma^2B_r^2(s,T)\mathrm{d}s\\&\quad+\int_t^T\left[\Big(\kappa\xi(t)+\rho_{rv}\tau\sigma B_r(s,T)B\Big)C(s)\right.\\&\quad\left.+\frac{1}{2}\tau^2\Big(C^2(s)+D(s)\Big)\right]\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
C(t) &= \frac{1}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}} \int_t^T h(s)\Big(\gamma_1 e^{\gamma(T-s)} + \gamma_2 e^{-\gamma(T-s)}\Big)ds \\
&= \big(B^2 - B\big)\frac{\Big(\big(\gamma_3 e^{\gamma(T-t)} - \gamma_4 e^{-\gamma(T-t)}\big) - \big(\gamma_5 e^{(\gamma-a)(T-t)} - \gamma_6 e^{-(\gamma+a)(T-t)}\big) - \gamma_7\Big)}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}} \\
&= -u\big(i+u\big)\frac{\Big(\big(\gamma_3 - \gamma_4 e^{-2\gamma(T-t)}\big) - \big(\gamma_5 e^{-a(T-t)} - \gamma_6 e^{-(2\gamma+a)(T-t)}\big) - \gamma_7 e^{-\gamma(T-t)}\Big)}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}},
\end{aligned} \tag{115}
$$

with $\gamma, \gamma_1, \ldots, \gamma_7$ as defined in (30).

Box III.

$$
\begin{aligned}
&= -\frac{1}{2}u\big(i+u\big)V(t,T) \\
&\quad + \int_t^T \Big[\Big(\kappa\psi + \rho_{rv}(iu-1)\tau\sigma B_r(s,T)\Big)C(s) \\
&\quad + \frac{1}{2}\tau^2\Big(C^2(s) + D(s)\Big)\Big]ds
\end{aligned} \tag{116}
$$

where $V(t,T)$ can be found by simple integration and is given by

$$
V(t,T) = \frac{\sigma^2}{a^2}\left((T-t) + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right). \tag{117}
$$

It is possible to write a closed-form expression for the remaining integral in (116). As the ordinary differential equation for $D(s)$ is exactly the same as in the Heston (1993) or Schöbel and Zhu (1999) model, it will involve a complex logarithm and should therefore be evaluated as outlined in Lord and Kahl (2008) in order to avoid any discontinuities. The main problem however lies in the integrals over $C(s)$ and $C^2(s)$, which will involve the Gaussian hypergeometric ${}_2F_1(a,b,c;z)$. The most efficient way to evaluate this hypergeometric function (according to *Numerical Recipes*, Press and Flannery (1992)) is to integrate the defining differential equation. Since all of the terms involved in $D(u)$ are also required in $C(u)$, numerical integration of the second part of (116) seems to be the most efficient method for evaluating $A(t)$. Hereby we conveniently avoid any issues regarding complex discontinuities altogether.

Appendix B. Deriving the log FX-rate characteristic function

In this appendix we will prove that the partial differential equation (89), i.e.

$$
\begin{aligned}
0 &= f_t + \kappa\big(\xi(t) - v(t)\big)f_v + \frac{1}{2}v_F^2(t)\big(f_{yy} - f_y\big) \\
&\quad + \Big(\rho_{xv}\tau v(t) + \rho_{r_1 v}\tau\sigma_1 B_1(t,T) - \rho_{r_2 v}\tau\sigma_2 B_2(t,T)\Big)f_{yv} \\
&\quad + \frac{1}{2}\tau^2 f_{vv},
\end{aligned} \tag{118}
$$

subject to the terminal boundary condition $f(T,y,\sigma) = \exp\big(iuy(T)\big)$ has a solution given by (90)–(95); we follow the same approach as in Appendix A, that is we use the ansatz (90), find the corresponding partial derivatives and substitute these in the PDE (118).

Expanding $v_F^2(t)$ according to (86) and collecting the terms for $y(t)$, $v(t)$ and $\frac{1}{2}v^2(t)$ yields the following system of ordinary differential equations for the functions $A(t), \ldots, D(t)$:

$$
0 = B'(t) \Rightarrow B(t) := B, \tag{119}
$$

$$
0 = D'(t) - 2\big(\kappa - \rho_{xv}\tau B\big)D(t) + \tau^2 D^2(t) + \big(B^2 - B\big), \tag{120}
$$

$$
\begin{aligned}
0 &= C'(t) + \big(\rho_{xv}\tau B - \kappa + \tau^2 D\big)C(t) \\
&\quad + \Big(\rho_{xr_1}\sigma_1 B_1(t,T) - \rho_{xr_2}\sigma_2 B_2(t,T)\Big)\big(B^2 - B\big) \\
&\quad + \Big(\kappa\xi(t) + \big(\rho_{r_1 v}\tau\sigma_1 B_1(t,T) - \rho_{r_2 v}\tau\sigma_2 B_2(t,T)\big)B\Big)D(t),
\end{aligned} \tag{121}
$$

$$
\begin{aligned}
0 &= A'(t) + \big(\kappa\xi(t) + \rho_{r_1 v}\tau\sigma_1 B_1(t,T)B - \rho_{r_2 v}\tau\sigma_2 B_2(t,T)B\big)C(t) \\
&\quad + \Big(\frac{1}{2}\sigma_1^2 B_1^2(t,T) + \frac{1}{2}\sigma_2^2 B_2^2(t,T) - \rho_{r_1 r_2}\sigma_1 B_1(t,T)\sigma_2 B_2(t,T)\Big) \\
&\quad \times \big(B^2 - B\big) + \frac{1}{2}\tau^2\big(C^2(t) + D(t)\big).
\end{aligned} \tag{122}
$$

Hence we end up with an analogue system of ordinary differential equations as in Appendix A: the first two differential equations (119) and (120) for B and $D(t)$ are equivalent to (100) and (101) whose solutions are given in the Eqs. (104) and (106)–(109). The third Eq. (121) for $C(t)$ looks pretty daunting, but is again merely a first order linear differential equation of the form $C'(t)+g(t)C(t)+h(t) = 0$, with associated boundary condition $C(T) = 0$. Hence expanding $\xi(t)$ according to (87), we can represent a solution for $C(t)$ as:

$$
C(t) = \int_t^T h(s)\exp\left[\int_t^s g(w)dw\right]ds, \tag{123}
$$

$$
\text{with: } g(w) = -(\kappa - \rho_{xv}\tau B) + \tau^2 D(w), \tag{124}
$$

$$
\begin{aligned}
h(s) &= \Big(\rho_{xr_1}\sigma_1 B_1(s,T) - \rho_{xr_2}\sigma_2 B_2(s,T)\Big)\big(B^2 - B\big) \\
&\quad + \Big(\kappa\xi(s) + \big(\rho_{r_1 v}\tau\sigma_1 B_1(s,T) - \rho_{r_2 v}\tau\sigma_2 B_2(s,T)\big)B\Big)D(s) \\
&= \rho_{xr_1}\sigma_1 B_1(s,T)\big(B^2 - B\big) + \Big(\kappa\psi + \rho_{r_1 v}(B-1)\tau\sigma_1 B_1(s,T)\Big)D(s) \\
&\quad - \rho_{xr_2}\sigma_2 B_2(s,T)\big(B^2 - B\big) - \big(\rho_{r_2 v}B\tau\sigma_2 B_2(s,T)\big)D(s).
\end{aligned} \tag{125}
$$

Now notice that the integral over g is equivalent to (113), hence its solution is given by Eq. (114), i.e.

$$
\exp\left[\int_t^s g(w)dw\right] = \frac{\gamma_1 e^{\gamma(T-s)} + \gamma_2 e^{-\gamma(T-s)}}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}}, \tag{126}
$$

with γ, γ_1 and γ_2 defined in (95). Substituting this expression into (123) we find (after a long but straightforward calculation) $C(t)$ as in Box IV.

Finally, by solving Eq. (122), we find the following expression for $A(t)$:

$$
\begin{aligned}
A(t) &= \int_t^T \frac{1}{2}\big(B^2 - B\big)\Big(\sigma_1^2 B_1^2(s,T) + \sigma_2^2 B_2^2(s,T) \\
&\quad - 2\rho_{r_1 r_2}\sigma_1 B_1(s,T)\sigma_2 B_2(s,T)\Big)ds \\
&\quad + \int_t^T \Big[\kappa\Big(\xi(s) + \rho_{r_1 v}B\tau\sigma_1 B_1(t,T) - \rho_{r_2 v}B\tau\sigma_2 B_2(t,T)\Big)C(s) \\
&\quad + \frac{1}{2}\tau^2\Big(C^2(s) + D(s)\Big)\Big]ds
\end{aligned}
$$

$$
\begin{aligned}
C(t) &= \left(B^2 - B\right)\frac{\left(\left(\gamma_3 e^{\gamma(T-t)} - \gamma_4 e^{-\gamma(T-t)}\right) - \left(\gamma_5 e^{(\gamma-a_1)(T-t)} - \gamma_6 e^{-(\gamma+a_1)(T-t)}\right) - \gamma_7\right)}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}} \\
&\quad - \left(B^2 - B\right)\frac{\left(\left(\gamma_8 e^{\gamma(T-t)} - \gamma_9 e^{-\gamma(T-t)}\right) - \left(\gamma_{10} e^{(\gamma-a_2)(T-t)} - \gamma_{11} e^{-(\gamma+a_2)(T-t)}\right) - \gamma_{12}\right)}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}} \\
&= -u(i+u)\frac{\left(\left(\gamma_3 - \gamma_4 e^{-2\gamma(T-t)}\right) - \left(\gamma_5 e^{-a_1(T-t)} - \gamma_6 e^{-(2\gamma+a_1)(T-t)}\right) - \gamma_7 e^{-\gamma(T-t)}\right)}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}} \\
&\quad + u(i+u)\frac{\left(\left(\gamma_8 - \gamma_9 e^{-2\gamma(T-t)}\right) - \left(\gamma_{10} e^{-a_2(T-t)} - \gamma_{11} e^{-(2\gamma+a_2)(T-t)}\right) - \gamma_{12} e^{-\gamma(T-t)}\right)}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}},
\end{aligned} \tag{127}
$$

with $\gamma, \gamma_1, \ldots, \gamma_{12}$ as defined in (95).

Box IV.

$$
\begin{aligned}
&= \frac{1}{2}\left(B^2 - B\right)V_{FX}(t,T) + \int_t^T \Big[\left(\kappa\psi + \rho_{r_1 v}(iu-1)\tau\sigma_1 B_1(s,T)\right. \\
&\quad \left. - \rho_{r_2 v} iu\tau\sigma_2 B_2(s,T)\right)C(s) + \frac{1}{2}\tau^2\left(C^2(s) + D(s)\right)\Big]ds,
\end{aligned} \tag{128}
$$

where $V_{FX}(t,T)$ can found by simple integration and is given by:

$$
\begin{aligned}
&V_{FX}(t,T) \\
&:= \frac{\sigma_1^2}{a_1^2}\left((T-t) + \frac{2}{a_1}e^{-a_1(T-t)} - \frac{1}{2a_1}e^{-2a_1(T-t)} - \frac{3}{2a_1}\right) \\
&\quad + \frac{\sigma_2^2}{a_2^2}\left((T-t) + \frac{2}{a_2}e^{-a_2(T-t)} - \frac{1}{2a_2}e^{-2a_2(T-t)} - \frac{3}{2a_2}\right) \\
&\quad - 2\rho_{r_1 r_2}\frac{\sigma_1\sigma_2}{a_1 a_2}\left((T-t) + \frac{e^{-a_1(T-t)} - 1}{a_1}\right. \\
&\quad \left. + \frac{e^{-a_2(T-t)} - 1}{a_2} - \frac{e^{-(a_1+a_2)(T-t)} - 1}{a_1 + a_2}\right).
\end{aligned} \tag{129}
$$

Analogue to (116), integrating over the $C(s)$ and $C^2(s)$ terms in (128) seems to be the most efficient method to evaluate $A(t)$.

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