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Insurance: Mathematics and Economics 47 (2010) 266–277

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Valuation of guaranteed annuity options using a stochastic volatility model for equity prices☆

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ARTICLE INFO

Article history:
Received April 2010
Received in revised form
June 2010
Accepted 29 June 2010

Keywords:
Guaranteed annuity option (GAO)
Guaranteed Minimum Income Benefit (GMIB)
Stochastic volatility
Stochastic interest rates
Variable annuities

ABSTRACT

Guaranteed annuity options are options providing the right to convert a policyholder's accumulated funds to a life annuity at a fixed rate when the policy matures. These options were a common feature in UK retirement savings contracts issued in the 1970's and 1980's when interest rates were high, but caused problems for insurers as the interest rates began to fall in the 1990's. Currently, these options are frequently sold in the US and Japan as part of variable annuity products. The last decade the literature on pricing and risk management of these options evolved. Until now, for pricing these options generally a geometric Brownian motion for equity prices is assumed. However, given the long maturities of the insurance contracts a stochastic volatility model for equity prices would be more suitable. In this paper explicit expressions are derived for prices of guaranteed annuity options assuming stochastic volatility for equity prices and either a 1-factor or 2-factor Gaussian interest rate model. The results indicate that the impact of ignoring stochastic volatility can be significant.

1. Introduction

Life insurers often include embedded options in the terms of their products. One of the most familiar embedded options is the Guaranteed Annuity Option (GAO). A GAO provides the right to convert a policyholder's accumulated funds to a life annuity at a fixed rate when the policy matures. These options were a common feature in retirement savings contracts issued in the 1970's and 1980's in the United Kingdom (UK). According to Bolton et al. (1997) the most popular guaranteed conversion rate was about 11%. Due to the high interest rates at that time, the GAOs were far out of the money. However, as the interest rate levels decreased in the 1990's and the (expected) mortality rates improved, the value of the GAOs increased rapidly and amongst others led to the downfall of Equitable Life in 2000. Currently, similar options are frequently sold under the name Guaranteed Minimum Income Benefit (GMIB) in the US and Japan as part of variable annuity products. The markets for variable annuities in the US and Japan have grown explosively over the past years, and growth in Europe is also expected, see Wyman (2007).

The last decade the literature on pricing and risk management of these options evolved. Approaches for risk management and hedging of GAOs were described in Dunbar (1999), Yang (2001), Wilkie et al. (2003) and Pelsser (2003). The pricing of GAOs and GMIBs has been described by several authors, for example van Bezooyen et al. (1998), Boyle and Hardy (2001), Ballotta and Haberman (2003), Boyle and Hardy (2003), Biffis and Millossovich (2006), Chu and Kwok (2007), Bauer et al. (2008) and Marshall et al. (2009). In most of these papers, the focus is on unit linked deferred annuity contracts purchased originally by a single premium. Generally a standard geometric Brownian motion is assumed for equity prices. However, Ballotta and Haberman (2003) and Chu and Kwok (2007) noted that, given the long maturities of the insurance contracts, a stochastic volatility model for equity prices would be more suitable.

In this paper explicit expressions are derived for prices of GAOs, assuming stochastic volatility for equity prices and (of course) stochastic interest rates. The model used for this is the Schöbel–Zhu Hull–White (SZHW) model, introduced in van Haastrecht et al. (2009). The model combines the stochastic

☆ The authors would like to thank Mark Davis, Peter Forsyth, Peter Spreij, participants of the 9th Winter school on Mathematical Finance and two anonymous referees for their valuable comments and suggestions.

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doi:10.1016/j.insmatheco.2010.06.007

volatility model of Schöbel and Zhu (1999) with the 1-factor Gaussian interest rate model of Hull and White (1993), taking the correlation structure between those processes explicitly into account. Furthermore, this is extended to the case of a 2-factor Gaussian interest rate model.

The remainder of the paper is organized as follows. First, in Section 2 the characteristics of the GAO are given. Section 3 describes the SZHW model to be used for the pricing of the GAO. In Section 5 explicit pricing formulas are derived for the GAOs given an underlying SZHW model. These results are extended to a 2-factor Hull–White model in Section 6. In Section 7 two numerical examples are worked out: the first shows the impact of stochastic volatility on the pricing of the GAO, whilst the second example deals with a comparison of the efficiency of our explicit formula for the 2-factor model with existing methods in the literature. Conclusions are given in Section 8.

2. Guaranteed annuity contract

A GAO gives the holder the right to receive at the retirement date T either a cash payment equal to the investment in the equity fund $S(T)$ or a life annuity of this investment against the guaranteed rate g. A rational policy holder would choose the greater of the two assets. In other words, if at inception, the policy holder is aged x and the normal retirement date is at time T, then the annuity value at maturity is $S(T)+H(T)$ with GAO payoff $H(T)$ equal to

$$H(T) := \left(gS(T)\sum_{i=0}^{n} c_i P(T,t_i) - S(T)\right)^+, \tag{1}$$

provided that the policy holder is still alive at that time. Here g is the guaranteed rate, $P(T,t_i)$ the zero-coupon bond at time T maturing at t_i and c_i the insurance amounts for time i multiplied by the probability of survival from time T until time t_i for the policyholder. Without loss of generality, we will use unit insured amounts in the remainder of this paper. Furthermore, we assume that the survival probabilities are independent of the equity prices and interest rates. Note that

$$H(T) = gS(T)\left(\sum_{i=0}^{n} c_i P(T,t_i) - K\right)^+, \tag{2}$$

where $K := 1/g$ and $(x)^+ := \max(x,0)$. This last equality shows that one can interpret the GAO as a quanto call option with strike K on the zero-coupon bond portfolio $\sum_{i=0}^{n} c_i P(T,t_i)$ which is payed out using the exchange rate/currency $S(T)$, e.g. see Boyle and Hardy (2003). Under the risk-neutral measure $\mathcal{Q}$, which uses the money market account $B(T)$,

$$B(T) := \exp\left(\int_0^T r(u)\mathrm{d}u\right) \tag{3}$$

as numeraire, the price of this option can be expressed as

$$C(T) = {}_{(R-X)}P_X\, \mathbb{E}^{\mathcal{Q}}\left[\exp\left(-\int_0^T r(u)\mathrm{d}u\right) gS(T) \times \left(\sum_{i=0}^{n} c_i P(T,t_i) - K\right)^+\right], \tag{4}$$

where ${}_{(R-X)}P_X$ denotes the probability that the policy holder aged X survives $R-X$ years, i.e. until the retirement age R at time T. To derive an explicit expression for the GAO of (4), it is more convenient to measure payments in terms of units of stock instead of money market values. Mathematically, we can establish this by using the equity price $S(T)$ as numeraire and changing from the risk-neutral measure to the equity-price measure $\mathcal{Q}^S$, see Geman et al. (1996). Under the equity-price measure $\mathcal{Q}^S$, the GAO price is then given by

$$C(T) = {}_{(R-X)}P_X\, gS(0)\mathbb{E}^{\mathcal{Q}^S}\left[\left(\sum_{i=0}^{n} c_i P(T,t_i) - K\right)^+\right]. \tag{5}$$

To evaluate this expectation we need to take into account the dynamics of the zero-coupon bonds prices $P(T,t_i)$ under the equity price measure.

Apart from the guaranteed rate, the drivers of the GAO price are the interest rates, the equity prices, the correlation between those, and the survival probabilities. The combined model for interest rates and equity prices is explained in Section 3. This model needs an assumption for the correlation, which could be derived from historical data. Note that if it is assumed that equity prices and interest rates are independent, it does not matter which model is assumed for equity prices.[1] Both from historical data as well as from market quotes, one however rarely finds that the equity prices and interest rates behave in an independent fashion. As this dependency structure is one of the main driver for the GAO price and its sensitivities, a non-trivial structure therefore has to be taken into account for a proper pricing and risk management of these derivatives, e.g. see Boyle and Hardy (2003), Ballotta and Haberman (2003) or Baur (2009).

3. The Schöbel–Zhu–Hull–White model

The model used in this paper is the Schöbel–Zhu Hull–White (SZHW) model, introduced in van Haastrecht et al. (2009). The model combines the stochastic volatility model of Schöbel and Zhu (1999) with the 1-factor Gaussian interest rate model of Hull and White (1993), taking explicitly into account the correlation between these processes. In the SZHW model, the process for equity price $S(t)$ under the risk-neutral measure $\mathcal{Q}$ is:

$$\frac{\mathrm{d}S(t)}{S(t)} = r(t)\mathrm{d}t + \nu(t)\mathrm{d}W_S^{\mathcal{Q}}(t), \quad S(0) = S_0, \tag{6}$$

$$\nu(t) = \kappa\left(\psi - \nu(t)\right)\mathrm{d}t + \tau \mathrm{d}W_\nu^{\mathcal{Q}}(t), \quad \nu(0) = \nu_0. \tag{7}$$

Here $\nu(t)$, which follows an Ornstein–Uhlenbeck process, is the (instantaneous) stochastic volatility of the equity $S(t)$. The parameters of the volatility process are the positive constants κ (mean reversion), $\nu(0)$ (short-term mean), ψ (long-term mean) and τ (volatility of the volatility). We assume the interest rates are given by a one-factor Hull and White (1993) process, whose dynamics under $\mathcal{Q}$ can be parameterized by

$$r(t) = \alpha(t) + x(t), \quad r(0) = r_0, \tag{8}$$

$$\mathrm{d}x(t) = -ax(t)\mathrm{d}t + \sigma \mathrm{d}W_x^{\mathcal{Q}}(t), \quad x(0) = 0. \tag{9}$$

Here a (mean reversion) and σ (volatility) are the positive parameters of the model, and where $\alpha(t)$ can be used to fit the current term structure of interest rates exactly, e.g. see Pelsser (2000) or Brigo and Mercurio (2006). Under the above dynamics for the equity, volatility and interest rates there exist closed-form calibration formulas for the prices of European equity options, e.g. see van Haastrecht et al. (2009). Moreover the model allows for a general correlation structure, i.e.

$$\mathrm{d}W_\nu^{\mathcal{Q}}(t)\mathrm{d}W_S^{\mathcal{Q}}(t) = \rho_{\nu S}\mathrm{d}t, \quad \mathrm{d}W_x^{\mathcal{Q}}(t)\mathrm{d}W_S^{\mathcal{Q}}(t) = \rho_{xS}\mathrm{d}t, \quad \mathrm{d}W_x^{\mathcal{Q}}(t)\mathrm{d}W_\nu^{\mathcal{Q}}(t) = \rho_{x\nu}\mathrm{d}t, \tag{10}$$

[1] Explicit pricing formulas, for this case, under one and two-factor Gaussian interest rates are provided in Appendix C.

where ρ_{vS}, ρ_{xS} and ρ_{xv} are the instantaneous correlation parameters between the Brownian motions of the equity price, the stochastic volatility and the interest rate. Having the flexibility to correlate the equity price with both stochastic volatility and stochastic interest rates yields a realistic model, which is of practical importance for the pricing and hedging of options with long-term exposures such as guaranteed annuities, e.g. see Boyle and Hardy (2003).

It is hardly necessary to motivate the inclusion of stochastic volatility in a long-term derivative pricing model. First, compared to constant volatility models, stochastic volatility models are significantly better able to fit the market's option data, e.g. see Andreasen (2006) or Andersen and Brotherton-Ratcliffe (2001). Second, as stochastic interest rates and stochastic volatility are empirical phenomena, the addition of these factors yields a more realistic model, which becomes important for the pricing and especially the hedging of long-term derivatives. The addition of stochastic volatility and stochastic interest rates as stochastic factors is important when considering long-maturity equity derivatives and has been the subject of empirical investigations most notably by Bakshi et al. (2000). These authors show that the hedging performance of delta hedging strategies of long-maturity options improves when stochastic volatility and stochastic interest rates are taken into account.

Stochastic volatility models have been described by several others, for example Stein and Stein (1991), Heston (1993), Schöbel and Zhu (1999), Duffie et al. (2000, 2003), van der Ploeg (2006) and van Haastrecht et al. (2009). Also regime-switching models are suggested in the literature for the pricing of equity-linked insurance policies, e.g. see Hardy (2001) and Brigo and Mercurio (2006). In the limit of an infinite number of regimes these models again converge to a continuous-time stochastic volatility model, however in discrete time they can benefit from a greater analytical tractability. A proper model assessment, greatly depends on the properties of the embedded options in the insurance contract.

To investigate the impact of using a stochastic volatility model on the pricing of GAOs, note that the GAO directly depends on the stochastic interest rates, the underlying equity fund and the correlation between the rates and the equity. For the pricing of GAOs we therefore choose to use the SZHW model over other stochastic volatility models, as this model distinguishes itself models by an explicit incorporation of the correlation between the underlying equity fund and the term structure of interest rates, whilst maintaining a high degree of analytical tractability.

In Section 7 the impact of stochastic volatility on the pricing of GAOs is investigated. That is, we compare the pricing of GAOs in the SZHW stochastic volatility model with the Black–Scholes Hull–White (BSHW) constant volatility model. The BSHW process for equity prices $S(t)$ under the risk neutral measure $\mathcal{Q}$ is:

$$\frac{\mathrm{d}S(t)}{S(t)} = r(t)\mathrm{d}t + \sigma_S \mathrm{d}W_S^{\mathcal{Q}}(t), \quad S(0) = S_0, \tag{11}$$

where the interest rate process $r(t)$ follows Hull and White (1993) dynamics as in (8) and with the instantaneous correlation between Brownian motions of the interest rate and the equity price equal to

$$\mathrm{d}W_S^{\mathcal{Q}}(t)\mathrm{d}W_x^{\mathcal{Q}}(t) = \rho_{xS}\mathrm{d}t. \tag{12}$$

In the following section both the SZHW and BSHW model are calibrated to market data.

4. Calibration of the SZHW and BSHW model

To come up with a fair analysis of the impact of stochastic volatility on the pricing of GAOs, we first calibrate the BSHW and SZHW model to the same market's option data per end July 2007. This is done by first calibrating the interest rate parameters, then estimating the effective correlation between the interest rates and equity, and finally we specify the equity components of the BSHW/SZHW model. We detail the calibration approach in the following.

4.1. Interest rates

First we calibrate the Hull and White (1993) interest rate models to EU and US swaption markets. The option prices and corresponding swap curves are obtained from Bloomberg. Here a total of 151 swaption prices, which are contributed by various issuers and maintained by Bloomberg, can be found for different tenors and maturities ranging from 1 to 30 years. For the calibration of the interest rate model we used close (mid) swaption prices 31st of July 2007. We calibrate the Hull and White (1993) models to these prices by minimizing the sum of the squared differences between the model's and the market's swaption implied volatilities. For the US market, the mean average price error is 1.88% and for the EU market 1.34% which is very good given the large set of option prices that is fitted using only 2 interest rate parameters.

4.2. Terminal correlation

After calibrating the interest rate component, we need to calibrate the equity and correlation parameters. For the equity component of the GAO we assume a large stock index, for which the Eurostoxx50 index (EU) and the S&P500 (US) are used. The Eurostoxx50 consists of 50 large European companies is traded on the Dow–Jones exchange, whilst the S&P500 is maintained by Standard&Poors and consists of NASDAQ and NYSE denoted shares. The effective 10 years correlation between the log equity returns and the interest rates is determined by time series analysis of the 10-year swap rate and the log returns of the EuroStoxx50 (EU) and S&P500 (US) index over the period from February 2002 to July 2007. For the EU and the US this resulted in correlation coefficients of 34.65% and 14.64% between the interest rates and the log equity returns.

It is well known that it is hard to calibrate the correlation coefficient. Furthermore large bid-ask spreads have to be paid to hedge this risk, which shows that the markets for correlation risks are unfortunately not very liquid. As a result, additional capital needs to be reserved in order to protect against this unhedgeable risk.

4.3. Equity

Using the interest rate parameters and the effective correlation parameter determined in the previous steps, the equity specific parameters are calibrated to option prices on the EuroStoxx50 and S&P500 index. These option prices are obtained from the implied volatility service of MarkIT, a financial data provider, which provides (mid) implied volatility quotes by averaging quotes from a large number of issuers. For large indices a total of 94 liquid quotes are available for 10 maturities ranging from 1 month up to 15 years, and 10 strikes ranging from 60% to 200%.

To aid a fair comparison between the models, the SZHW model is calibrated in such a way that the effective correlation between interest rates and equity prices is equal to that of the BSHW process. Finally, as the considered GAO in Section 7 only depends on terminal asset price distribution after 10 years, we have calibrated the equity model to market option prices maturing in 10 years time. This estimation is performed by minimizing the sum of absolute differences between market's and model's implied volatilities. The calibration results to the Eurostoxx50 and S&P500 can be found in Table 1.

The tables show that SZHW is significantly better in capturing the market's implied volatility structure and provides an ex-

Table 1
Comparison of the calibration results for the SZHW and BSHW model for 10-year call options with different strikes. Calibrations are performed on market data for options of major indices at the end of July 2007: for EU index the EuroStoxx50 is used, whereas for US index this is the S&P500.

Strike	Market (%)	SZHW (%)	BSHW (%)
Implied volatility, 10-year call options, EU			
80	27.8	27.9	26.4
90	27.1	27.1	26.4
95	26.7	26.7	26.4
100	26.4	26.4	26.4
105	26.0	26.0	26.4
110	25.7	25.7	26.4
120	25.1	25.1	26.4
Implied volatility, 10-year call options, US			
80	27.5	27.5	25.8
90	26.6	26.6	25.8
95	26.2	26.2	25.8
100	25.8	25.8	25.8
105	25.4	25.4	25.8
110	25.0	25.0	25.8
120	24.3	24.4	25.8

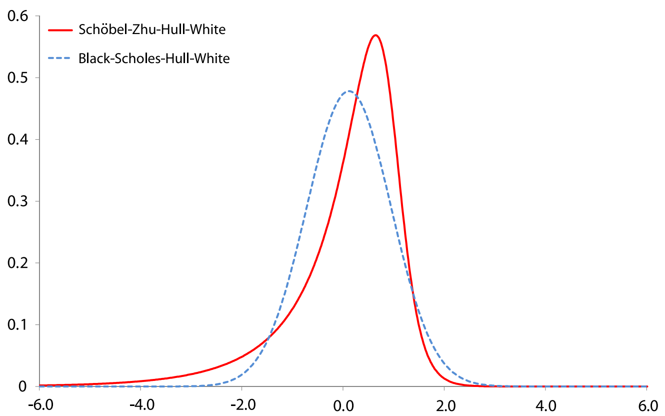

Fig. 1. Risk-neutral density of the log-asset price for the SZHW and BSHW model, calibrated to EU market option data.

tremely good fit. The fit of the BSHW model is relatively poor. Furthermore, a direct consequence of the log-normal distribution of the BSHW model, it that the asset returns have thin tails, which does not correspond to historical data nor to the market's view on long-term asset returns. The SZHW model provides a more realistic picture on the market's view on long-term asset returns as it can incorporate heavy-tailed returns. The latter can be made especially clear by looking at the risk-neutral densities of the log-asset price of the SZHW and BSHW model. These are plotted in Fig. 1 for the BSHW and SZHW model, calibrated to EU option prices.

Clearly, the SZHW model incorporates the skewness and heavy-tails seen in option markets (e.g. see Bakshi et al. (1997)) a lot more realistically than the BSHW model. The effects of these log-asset price distributions on the pricing of GAOs, combined with correlated interest rates, are extensively analyzed in Section 7.

5. Pricing the guaranteed annuity option under stochastic volatility and stochastic interest rates

For the pricing of the GAO in the SZHW model, i.e. the evaluation of (5), we need to consider the pricing of zero-coupon bonds in the Gaussian rate model. In the Hull and White (1993) model, one has the following expression for the time-T price of a zero-coupon bond $P(T, t_i)$ maturing at time t_i:

$$P(T, t_i) = A(T, t_i)e^{-B(T,t_i)x(T)}, \tag{13}$$

where

$$A(T, t_i) = \frac{P^M(0, t_i)}{P^M(0, T)} \times \exp\left[\frac{1}{2}\left(V(T, t_i) - V(0, t_i) + V(0, T)\right)\right], \tag{14}$$

$$B(T, t_i) = \frac{1 - e^{-a(t_i-T)}}{a}, \tag{15}$$

$$V(T, t_i) = \frac{\sigma^2}{a^2}\left((t_i - T) + \frac{2}{a}e^{-a(t_i-T)} - \frac{1}{2a}e^{-2a(t_i-T)} - \frac{3}{2a}\right), \tag{16}$$

and with $P^M(0, s)$ denoting the market's time zero discount factor maturing at time s. Using (13), we have for the GAO price (5) under the equity price measure $\mathbb{Q}^S$:

$$C(T) = {}_{(R-X)}P_X\, gS(0)\mathbb{E}^{\mathbb{Q}^S} \times \left[\left(\sum_{i=0}^{n} c_i A(T, t_i)e^{-B(T,t_i)x(T)} - K\right)^+\right]. \tag{17}$$

To further evaluate this expression, we first have to consider the dynamics of $x(T)$ under the equity price measure $\mathbb{Q}^S$ in the SZHW model.

5.1. *Taking the equity price as numeraire*

To change the money market account numeraire into the equity price numeraire, we need to calculate the corresponding Radon–Nikodým derivative (e.g. see Geman et al. (1996)), which is given by

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}} = \frac{S(T)B(0)}{S(0)B(T)} = \exp\left[-\frac{1}{2}\int_0^T \nu^2(u)du + \int_0^T \nu(u)dW_S^{\mathbb{Q}}(u)\right]. \tag{18}$$

The multi-dimensional version of Girsanov's theorem (e.g. see Oksendal (2005)) hence implies that

$$dW_S^{\mathbb{Q}^S}(t) \mapsto dW_S^{\mathbb{Q}}(t) - \nu(t)dt, \tag{19}$$

$$dW_x^{\mathbb{Q}^S}(t) \mapsto dW_x^{\mathbb{Q}}(t) - \rho_{xS}\nu(t)dt, \tag{20}$$

$$dW_\nu^{\mathbb{Q}^S}(t) \mapsto dW_\nu^{\mathbb{Q}}(t) - \rho_{\nu S}\nu(t)dt, \tag{21}$$

are $\mathbb{Q}^S$ Brownian motions. Hence under $\mathbb{Q}^S$ one has the following model dynamics for the volatility and interest rate process

$$dx(t) = -ax(t)dt + \rho_{xS}\sigma\nu(t)dt + \sigma dW_x^{\mathbb{Q}^S}(t), \quad x(0) = 0, \tag{22}$$

$$\begin{aligned} d\nu(t) &= \kappa(\psi - \nu(t))\,dt + \rho_{\nu S}\tau\nu(t)dt + \tau dW_\nu^{\mathbb{Q}^S}(t) \\ &= \widetilde{\kappa}(\widetilde{\psi} - \nu(t))dt + \tau dW_\nu^{\mathbb{Q}^S}(t), \quad \nu(0) = \nu_0, \end{aligned} \tag{23}$$

where $\widetilde{\kappa} := \kappa - \rho_{\nu S}\tau$, $\widetilde{\psi} := \frac{\kappa\psi}{\widetilde{\kappa}}$. After some calculations, conditional on the current time filtration $\mathcal{F}_0$, one can show that:

$$\nu(T) = \widetilde{\psi} + (\nu(0) - \widetilde{\psi})e^{-\widetilde{\kappa}T} + \tau\int_0^T e^{-\widetilde{\kappa}(T-u)}dW_\nu^{\mathbb{Q}^S}(u), \tag{24}$$

$$\begin{aligned} x(T) &= \rho_{xS}\sigma\left(\frac{\widetilde{\psi}}{a}\left[1 - e^{-aT}\right] + \frac{\nu(0) - \widetilde{\psi}}{a - \widetilde{\kappa}}\left[e^{-\widetilde{\kappa}T} - e^{-aT}\right]\right) \\ &\quad + \frac{\rho_{xS}\sigma\tau}{(a - \widetilde{\kappa})}\int_0^T \left[e^{-\widetilde{\kappa}(T-u)} - e^{-a(T-u)}\right]dW_\nu^{\mathbb{Q}^S}(u) \\ &\quad + \sigma\int_0^T e^{-a(T-u)}dW_x^{\mathbb{Q}^S}(u). \end{aligned} \tag{25}$$

Using Ito's isometry and Fubini's theorem, we have that $x(T)$ (conditional on $\mathcal{F}_0$) is normally distributed with mean μ_x and variance σ_x^2 given by

$$\mu_x = \rho_{xS}\sigma\left(\frac{\widetilde{\psi}}{a}\left[1-\mathrm{e}^{-aT}\right]+\frac{\nu(0)-\widetilde{\psi}}{(a-\widetilde{\kappa})}\left[\mathrm{e}^{-\widetilde{\kappa}T}-\mathrm{e}^{-aT}\right]\right), \tag{26}$$

$$\sigma_x^2 = \sigma_1^2+\sigma_2^2+2\rho_{12}\sigma_1\sigma_2 \tag{27}$$

where

$$\sigma_1 = \sigma\sqrt{\frac{1-\mathrm{e}^{-2aT}}{2a}}, \tag{28}$$

$$\sigma_2 = \frac{\rho_{xS}\sigma\tau}{a-\widetilde{\kappa}} \times\sqrt{\frac{1}{2\widetilde{\kappa}}+\frac{1}{2a}-\frac{2}{(\widetilde{\kappa}+a)}-\frac{\mathrm{e}^{-2\widetilde{\kappa}T}}{2\widetilde{\kappa}}-\frac{\mathrm{e}^{-2aT}}{2a}+\frac{2\mathrm{e}^{-(\widetilde{\kappa}+a)T}}{(\widetilde{\kappa}+a)}}, \tag{29}$$

$$\rho_{12} = \rho_{x\nu}\frac{\sigma^2\rho_{xS}\tau}{\sigma_1\sigma_2(a-\widetilde{\kappa})}\left[\frac{1-\mathrm{e}^{-(a+\widetilde{\kappa})T}}{(a+\widetilde{\kappa})}-\frac{1-\mathrm{e}^{-2aT}}{2a}\right]. \tag{30}$$

5.2. Explicit formula for the GAO price

Using the results from the previous paragraph, we can now further evaluate the expression (17) for the GAO price in the SZHW model: as the zero-coupon bond price is a monotone function of one state variable, $x(T)$, one can use the Jamshidian (1989) result and write the call option (17) on the sum of zero-coupon bonds as a sum of zero-coupon bond call options: let x^* solve

$$\sum_{i=0}^{n} c_i A(T,t_i)\mathrm{e}^{-B(T,t_i)x^*} = K, \tag{31}$$

and let

$$K_i := A(T,t_i)\mathrm{e}^{-B(T,t_i)x^*}. \tag{32}$$

Using Jamshidian (1989), we have that the price of GAO is equal to the price of a sum of zero-coupon bond options, i.e.

$$C(T) = {}_{(R-X)}P_X\, gS(0)\mathbb{E}^{\mathcal{Q}^S}\times\left[\sum_{i=0}^{n} c_i\left(A(T,t_i)\mathrm{e}^{-B(T,t_i)x(T)}-K_i\right)^+\right]. \tag{33}$$

As $x(T)$ is normally distributed, we have that $P(T,t_i) = A(T,t_i)\,\mathrm{e}^{-B(T,t_i)x(T)}$ is log-normally distributed. Provided that we know the mean M_i and variance V_i of $\ln P(T,t_i)$ under $\mathcal{Q}^S$, one can directly express the above expectation in terms of the Black and Scholes (1973) formula, i.e.

$$C(T) = {}_{(R-X)}P_X\, gS(0)\sum_{i=0}^{n} c_i\left[F_i N\left(d_1^i\right)-K_i N\left(d_2^i\right)\right], \tag{34}$$

$$F_i = \mathrm{e}^{M_i+\frac{1}{2}V_i}, \tag{35}$$

$$d_1^i = \frac{\ln\left(F_i/K_i\right)+\frac{1}{2}V_i}{\sqrt{V_i}}, \tag{36}$$

$$d_2^i = d_1^i-\sqrt{V_i}. \tag{37}$$

To determine M_i and V_i, recall from (26) and (27) that $x(T)$ is normally distributed with mean μ_x and variance σ_x. Hence with $P(T,t_i) = A(T,t_i)\mathrm{e}^{-B(T,t_i)x(T)}$, one can directly obtain that the mean M_i and variance V_i of $\ln P(T,t_i)$ are given by

$$M_i = \ln A(T,t_i)-B(T,t_i)\mu_x, \tag{38}$$

$$V_i = B^2(T,t_i)\sigma_x^2. \tag{39}$$

Hence under the SZHW dynamics (6)–(9), we have derived the explicit formula (34) for the price of a GAO under stochastic volatility and correlated stochastic interest rates. With this result, we are able to investigate the impact of stochastic volatility on the pricing of GAOs, which will be the subject of Section 7.1.

6. Extension to two-factor interest rates

A one-factor assumption for the short interest rate unfortunately that all future interest rates are driven by one factor. As reported in Brigo and Mercurio (2006), principal components analysis show that the full interest rate curve is (depending on the currency) typically driven by two or more factors. When calibrating calibrating to European swaption prices, it is demonstrated in that a two-factor Gaussian model gives significantly better fits and produces more realistic future interest rate curves. Furthermore, as noted in Chu and Kwok (2007), the one-factor assumption typically leads to a full correlation of all future interest rates. In particular these authors recommend to use a two-factor interest rate model for the pricing of long-term derivatives and GAO contracts in particular. In this section, we therefore generalize the setting of the previous section from one to two-factor Gaussian interest rates. That is under the risk-neutral measure $\mathcal{Q}$, we assume the following dynamics for the short interest rate process:

$$r(t) = \varphi(t)+x(t)+y(t), \quad r(0) = r_0, \tag{40}$$

$$\mathrm{d}x(t) = -ax(t)\mathrm{d}t+\sigma\mathrm{d}W_x^{\mathcal{Q}}(t), \quad x(0) = 0, \tag{41}$$

$$\mathrm{d}y(t) = -by(t)\mathrm{d}t+\eta\mathrm{d}W_y^{\mathcal{Q}}(t), \quad y(0) = 0, \tag{42}$$

$$\mathrm{d}W_x^{\mathcal{Q}}(t)\mathrm{d}W_y^{\mathcal{Q}}(t) = \rho_{xy}\mathrm{d}t. \tag{43}$$

Here a, b (mean reversion) and σ, η (volatility) are the positive parameters of the model and $|\rho_{xy}| \leq 1$. The deterministic function $\varphi(t)$ can be used to exactly fit the current term structure of interest rates, e.g. see Brigo and Mercurio (2006). Much of the analytical structure of the one-factor Gaussian is preserved in this two-factor setting. For example time T zero-coupon bond prices maturity at time t_i are given by

$$P(T,t_i) = A(T,t_i)\mathrm{e}^{-B(a,T,t_i)x(T)-B(b,T,t_i)y(T)}, \tag{44}$$

where

$$A(T,t_i) = \frac{P^M(0,t_i)}{P^M(0,T)}\times\exp\left[\frac{1}{2}\left(V(T,t_i)-V(0,t_i)+V(0,T)\right)\right], \tag{45}$$

$$B(z,T,t_i) = \frac{1-\mathrm{e}^{-z(t_i-T)}}{z}, \tag{46}$$

$$\begin{aligned} V(T,t_i) = {} & \frac{\sigma^2}{a^2}\left[(t_i-T)+\frac{2}{a}\mathrm{e}^{-a(t_i-T)}-\frac{1}{2a}\mathrm{e}^{-2a(t_i-T)}-\frac{3}{2a}\right] \\ & +\frac{\eta^2}{b^2}\left[(t_i-T)+\frac{2}{b}\mathrm{e}^{-b(t_i-T)}-\frac{1}{2b}\mathrm{e}^{-2b(t_i-T)}-\frac{3}{2b}\right] \\ & +2\rho_{xy}\frac{\sigma\eta}{ab}\left[(t_i-T)+\frac{\mathrm{e}^{-a(t_i-T)}-1}{a}\right. \\ & \left.+\frac{\mathrm{e}^{-b(t_i-T)}-1}{b}-\frac{\mathrm{e}^{-(a+b)(t_i-T)}-1}{a+b}\right]. \end{aligned} \tag{47}$$

Substituting the zero-coupon bond expression (44) into the pricing Eq. (5) and evaluating this expectation, results in the following explicit expression for the GAO price:

$$C(T) = {}_{(R-X)}P_X\, gS(0)\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x\sqrt{2\pi}}\times\left[F_i(x)N\left(h_2(x)\right)-KN\left(h_1(x)\right)\right]\mathrm{d}x, \tag{48}$$

where N denotes the cumulative standard normal distribution function and with

$$h_1(x) := \frac{y^* - \mu_y}{\sigma_y\sqrt{1-\rho_{xy}^2}} - \frac{\rho_{xy}(x-\mu_x)}{\sigma_x\sqrt{1-\rho^2}}, \tag{49}$$

$$h_2(x) := h_1(x) + B(b,T,t_i)\sigma_y\sqrt{1-\rho_{xy}^2}, \tag{50}$$

$$\lambda_i(x) := c_i A(T,t_i)\mathrm{e}^{-B(a,T,t_i)x}, \tag{51}$$

$$\kappa_i(x) := -B(b,T,t_i)\left[\mu_y - \frac{1}{2}\sigma_y^2(1-\rho_{xy}^2)B(b,T,t_i) + \rho_{xy}\sigma_y\frac{(x-\mu_x)}{\sigma_x}\right], \tag{52}$$

$$F_i(x) := \sum_{i=0}^{n}\lambda_i(x)\mathrm{e}^{\kappa_i(x)}, \tag{53}$$

and with y^* the unique solution of

$$\sum_{i=0}^{n}\lambda_i(x)\mathrm{e}^{-B(b,T,t_i)y^*} = K. \tag{54}$$

The proof of (48) is given in Appendix A.

In the pricing formula (48) it remains to determine the first two moments of $x(T)$ and $y(T)$ and the (terminal) correlation between $x(T)$ and $y(T)$, under the equity price measure $\mathbb{Q}^S$. These are given in Appendix B. Note that in the pricing formula (48), one is integrating a Gaussian probability density function against a bounded function. Because the Gaussian density functions decays very rapidly,[2] one can therefore truncate the integration domain in an implementation of (48) to a suitable number of standard deviations σ_x around the mean μ_x.

7. Numerical examples

In this section two numerical examples are given. In Section 7.1 the values of the GAO using the stochastic volatility model described in 3 are compared with values that result when a geometric Brownian motion is assumed for equity prices. Section 7.2 deals with sensitivity analyses of different risk drivers. In Section 7.3 our approach for two-factor interest rate models is compared with the methods described in Chu and Kwok (2007).

7.1. Comparison results SZHW model and Black–Scholes Hull–White model

In this section the impact of stochastic volatility of equity prices is shown for an example policy. The results for the SZHW model given in (6)–(9) are compared with a model that combines a Black–Scholes process for equity prices with a one-factor Hull White model for interest rates, the so-called Black–Scholes-Hull–White (BSHW) model given in (11)–(12). The SZHW and BSHW models are both calibrated to market information (implied volatilities and interest rates) per end July 2007, see Section 4.

In the example, the policyholder is 55 years old, the retirement age is 65, giving the maturity T of the GAO option of 10 years. Furthermore, $S(0)$ is assumed to be 100. The survival rates are based on the PNMA00 table of the Continuous Mortality Investigation (CMI) for male pensioners.[3]

In Table 2 the prices for the GAO are given for both models. The prices are given for different guaranteed rates g. As mentioned in Section 1, the most popular conversion rate in the 1970s and 1980s was around 11%. Currently, the GMIB contracts in the US and Japan will be based on at-the-money guaranteed conversion rates. The results for the SZHW model are obtained using the explicit expressions given in (34)–(39). The pricing formula for the BSHW is a special case of this, and is also derived in Ballotta and Haberman (2003). The results are determined for EU data and US data with an equity-interest rate correlation of respectively 0.3465 and 0.1464 (see Section 4). The table presents the total value of the GAO as well as the time value. The time value is determined as the difference between the total value and the (forward) intrinsic value. The latter is determined by setting all volatilities to zero. While the total value gives the impact on the total prices, the time value gives more insight in the relative impact of the models (since those only have impact the time value). Also, the time value of the GAO is often reported separately, for example within Embedded Value reporting of insurers.

The table shows that the use of a stochastic volatility model such as the SZHW model has a significant impact on the total value of the GAO. The value increases with 4%–50% for a EU data and 4%–17% for a US data, depending on the level of the guarantee.

These price differences are not caused by a volatility effect as both models are calibrated to the same market data in Section 3. Fig. 1 of Section 3, however showed that the distribution of equity prices under a SZHW process has a heavy left tail, but also relatively more mass on the right of the distribution compared to the BSHW

Table 2
Comparison of GAO total values and time values of the SZHW and BSHW model for different guaranteed rates g. In the examples: at-the-money guaranteed rate g is 8.21% (EU) and 8.44% (US), effective correlation between the stock price and the interest rates is 34.65% (EU) and 14.64% (US).

Srike g (%)	SZHW	BSHW	Rel. Diff (%)
Total value, EU			
8.23	3.82	3.07	+24.5
7	0.59	0.39	+50.7
8	2.89	2.26	+28.0
9	8.40	7.25	+15.8
10	17.02	15.53	+9.6
11	27.37	25.69	+6.5
12	38.30	36.47	+5.0
13	49.35	47.37	+4.2
Time value, EU			
8.23	3.82	3.07	+24.5
7	0.59	0.39	+50.7
8	2.89	2.26	+28.0
9	2.43	1.29	+88.9
10	−0.11	−1.60	−93.0
11	−0.93	−2.60	−64.4
12	−1.17	−2.99	−61.0
13	−1.28	−3.26	−60.7
Total value, US			
8.44	5.43	4.84	+12.0
7	1.04	0.88	+18.0
8	3.54	3.11	+13.6
9	8.53	7.74	+10.3
10	16.06	14.90	+7.8
11	25.42	23.96	+6.1
12	35.73	34.06	+4.9
13	46.43	44.58	+4.1
Time value, US			
8.44	5.43	4.84	+12.0
7	1.04	0.88	+18.0
8	3.54	3.11	+13.6
9	7.27	6.47	+12.3
10	4.15	2.99	+39.1
11	2.86	1.40	+104.2
12	2.53	0.86	+195.5
13	2.58	0.74	+250.1

[2] For instance, 99.9999% of the probability mass of a Gaussian density function lies within five standard deviations around its mean.

[3] This table is available at: http://www.actuaries.org.uk/knowledge/cmi/cmi_tables/00_series_tables.

Table 3
Comparison of the simulated distribution of discounted payoffs for the SZHW and BSHW model: reported are the probabilities that the discounted payoff lies in a specific interval.

Payoff	SZHW (%)	BSHW (%)	Diff (%)
0	58.3	58.5	−0.2
(0, 1]	7.5	5.2	2.2
(1, 10]	22.0	26.3	−4.3
(10, 20]	7.2	6.8	0.4
(20, 30]	2.7	1.9	0.8
(30, 40]	1.2	0.7	0.4
(40, 50]	0.5	0.3	0.2
(50, 60]	0.3	0.1	0.1
(60, 70]	0.2	0.1	0.1
(70, 80]	0.1	0.1	0.0
(80, 90]	0.1	0.0	0.0
(90, 100]	0.0	0.0	0.0
(100, 110]	0.0	0.0	0.0
>110	0.1	0.0	0.1

process. Given a positive correlation between equity prices and interest rates, and the fact that the GAO pays off when interest rates are low, this means that for the SZHW model there will be some very low payoffs for equity prices in the left tail, but relatively higher payoffs for the remaining scenarios. This is illustrated in Table 3. For the EU data and $g = 8.23\%$, 50,000 Monte Carlo simulations are generated for both models and the discounted payoffs are segmented in specific intervals.

The table shows that indeed:

- SZHW has relatively much payoffs in the interval (0, 1) due to the heavy left tail.
- For the remaining intervals, SZHW has more mass to the right, illustrated by the less frequent payoffs in the interval (1, 10) and more frequent payoffs in the intervals greater than 10.

Since the models only have an impact on the time value, the relative changes in time value for in-the-money GAOs are higher, which is also illustrated in Table 2. One might wonder why the time values for the EU data are negative for high levels of g. The reason for this is that due to the positive correlation between interest rates and equity prices, higher equity volatility means that there is a higher chance of lower payoffs, leading to a lower total option value compared to the intrinsic value. For the US data no negative time values are reported. Reason for this is that due to the lower correlation between interest rates and equity prices, the effect described above is less significant than the positive impact of interest rates on the time value.

7.2. Impact of different risk drivers

As noted in Section 2, we assume that the survival probabilities are independent of the equity prices and interest rates. It is interesting though to see the impact of significant changes in those survival probabilities on the GAO price and to compare it with the impact of changes in equity prices and interest rates. To get a feeling about this, we apply shocks for these risk drivers as defined in the technical specifications of the Quantitative Impact Study 5 (QIS5) of CEIOPS.[4] QIS5 is the basis for the Solvency 2, a new risk-based framework for regulatory required solvency capital. The shocks are aimed to represent the 99.5% percentile on a 1 year horizon.

Table 4 shows the impact of 2 shifts in survival probabilities. The shifts are based on a 25% reduction of mortality rates (longevity risk) and a 15% increase in mortality rates (mortality risk). Table 5 shows the impact of 2 shifts in the yield curve. The shifts are given in Appendix D. Table 6 shows the impact of shocks of +39% and −/−39% on equity prices.

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Table 4
Impact of survival probabilities on GAO total value.

Strike g (%)	SZHW	Longevity	Mortality
Total value, EU			
8.23	3.82	7.28	2.53
7	0.59	1.61	0.31
8	2.89	5.82	1.85
9	8.40	13.63	6.17
10	17.02	24.01	13.72
11	27.37	35.49	23.38
12	38.30	47.25	33.86
13	49.35	59.05	44.52
Total value, US			
8.44	5.43	9.08	3.88
7	1.04	2.88	0.85
8	3.54	7.61	3.07
9	8.53	15.22	7.71
10	16.06	25.05	14.90
11	25.42	36.13	24.01
12	35.73	47.76	34.14
13	46.43	59.56	44.68

Table 5
Impact of changes in yield curve on GAO total value.

Strike g (%)	SZHW	Rates down	Rates up
Total value, EU			
8.23	3.82	9.91	1.11
7	0.59	2.49	0.10
8	2.89	8.10	0.77
9	8.40	17.38	3.24
10	17.02	28.80	8.65
11	27.37	40.94	16.71
12	38.30	53.23	26.19
13	49.35	65.54	36.13
Total value, US			
8.44	5.43	10.48	1.54
7	1.04	3.42	0.25
8	3.54	8.83	1.15
9	8.53	17.20	3.58
10	16.06	27.69	8.21
11	25.42	39.27	15.05
12	35.73	51.26	23.50
13	46.43	63.37	32.79

Table 6
Impact of shocks on equity prices on GAO total value.

Strike g (%)	SZHW	Equity up	Equity down
Total value, EU			
8.23	3.82	5.31	2.33
7	0.59	0.81	0.36
8	2.89	4.01	1.76
9	8.40	11.67	5.12
10	17.02	23.66	10.38
11	27.37	38.04	16.70
12	38.30	53.24	23.36
13	49.35	68.60	30.10
Total value, US			
8.44	5.43	7.54	3.31
7	1.04	1.44	0.63
8	3.54	4.92	2.16
9	8.53	11.86	5.20
10	16.06	22.33	9.80
11	25.42	35.33	15.51
12	35.73	49.67	21.80
13	46.43	64.53	28.32

Although the impact differs for different strike levels, the tables show that the impact of the different risk drivers is reasonably similar for this particular example. Table 4 shows that indeed the

GAO value increases significantly when a shift down is applied to the mortality rates. A shift up in mortality rate has the opposite effect on the value of the GAO. Similar effects can be seen in Table 5 for the yield curve shifts. Note that the impact of the yield curve shifts is (coincidently) approximately equal to a 1% shift in the strike level. Of course, higher (lower) equity prices will lead to higher (lower) payments, as shown in Table 6. But for low strike prices, the impact of changes in equity prices is less than the impact of interest rates and longevity

7.3. Comparison results of the two-factor model with Chu and Kwok (2007)

A special case of our modeling framework is considered in Chu and Kwok (2007), namely an equity model with constant volatility. Chu and Kwok (2007) argue that for a two-factor Gaussian interest rate model no analytical pricing formulas exist. Therefore they propose three approximation methods for the valuation of GAOs:

1. *Method of minimum variance duration*: This method approximates the annuity with a single zero-coupon bond and minimizes the approximation error by choosing the maturity of the zero-coupon bond to be equal to the stochastic duration.
2. *Edgeworth expansion*: This method makes use of the Edgeworth approximation of the probability distribution of the value of the annuity (see Collin-Dufresne and Goldstein (2002)).
3. *Affine approximation*: This method approximates the conditional distributions of the risk factors in affine diffusions.

In the paper the runtimes and approximation errors are compared with benchmark results using Monte Carlo simulations and the method of minimum variance duration comes out most favorably. The other approximation methods do have very long runtime, the Edgeworth expansion method requires even more time then a Monte Carlo simulation.

However, as we have shown in Section 6, it is possible to derive an explicit expression where only a single numerical integration is needed for the case of a two-factor Gaussian interest rate model. It takes hardly any runtime (a couple of hundreds of seconds) to do this numerical integration, whilst it provides exact results. The parameter setting used is the same as in Chu and Kwok (2007) and is given in Appendix E. Table 7 shows a comparison of the results for the different methods and a Monte Carlo simulation with 1000,000 sample paths, which are compared to the exact GAO prices obtained by the quasi closed-form expression in (48).

The table shows that the approximation methods considered by Chu and Kwok (2007) break down for higher interest rates, where the guarantee is out-the-money. Note hereby that the first moment of the underlying distribution is main driving factor for the option price, while for the price of out-of-the-money options the higher moments play a more important role, e.g. see Brigo and Mercurio (2006). Taking into account that the mean of the underlying annuity is determined exactly in the approximations, this implies that these methods have severe difficulties in estimating the higher moments of the underlying distribution, resulting in poor an approximation quality of the out-of-money GAOs, see Table 7.

The explicit (quasi closed-form) exact formula (48) does give highly accurate prices for GAOs across for all strike levels. Both the Monte Carlo method as the explicit formula are unbiased. Differences between the Monte Carlo method and the exact formula are sampling errors as we can see that the 95% confidence interval of the Monte Carlo method for all cases is overlapping with the price of the explicit exact formula. Typically such Monte Carlo noise increases for out-of-the-money options (e.g. see Glasserman (2003)) as can also be seen from Table 7 for the considered GAOs. The careful reader may notice that in the above example the sign of the difference between the Monte Carlo price and the exact formula is always negative, which is due to the fact that the same set of Monte Carlo paths is used for all strikes.

Where the Affine approximation method and the Edgeworth expansion method take require a very long runtime (according to Chu and Kwok (2007), the runtime of the Edgeworth expansion is even larger than of the Monte Carlo method with 100,000 sample paths), the runtime the explicit quasi-closed-form method is comparable to the method of minimum variance duration and takes only a few hundreds of a second. The closed-form exact approach proposed in Section 6 is preferable compared to the approaches described in Chu and Kwok (2007), as it gives exact GAO prices over all strike levels whilst being computationally very efficient.

8. Conclusion

In this paper explicit formulas for the pricing of GAOs using a stochastic volatility model for equity prices are given. The considered framework further allows for 1-factor and 2-factor Gaussian interest rates, whilst taking the correlation between the equity, the stochastic volatility and the stochastic interest rates explicitly into account. The basis for the explicit formulas for GAOs lies in the fact that under the equity price measure, the GAO can be written in terms of an option on a sum of coupon bearing bonds: after some calculations the Jamshidian (1989) result can be used that expresses the latter option on a sum into a sum of options which can be priced in closed-form. For 1-factor interest rates the price of a GAO can be expressed as sum of Black and Scholes (1973) options, whereas an explicit expression using a single integral can be established for the case of a two-factor Gaussian interest rate model.

A special case of our modeling framework, that is an equity model with constant volatility, is considered in Chu and Kwok (2007). These authors argue that for a two-factor Gaussian interest rate model no analytical pricing formulas exist and propose several approximation methods for the valuation of GAOs. In this paper we did derive an exact quasi-closed-form pricing formula in terms of a single numerical integral, which called for a comparison between these valuation methods. The numerical results show that the use of the quasi closed-form exact approach is preferable compared to the approaches described in Chu and Kwok (2007), as it gives exact GAO prices over all strike levels whilst being computational very efficient to compute.

Because GAOs generally involve long-dated maturities and the annuity payoff is directly linked to the performance of an equity fund, it is important for a proper pricing and risk management of such products to consider realistic returns for the equity fund combined with a non-trivial dependency structure with the underlying interest rates. Using US and the EU market option data, we investigated the effects of a stochastic volatility model for the pricing of GAOs. Time-series analysis between the considered equity funds (S&P500 for US and EuroStoxx50 for EU) and the long term interest rates revealed a substantial positive correlation. We then calibrated the stochastic and the constant volatility model to the market's options and this correlation, making sure that the implied correlation between the terminal asset price and the interest rates is equal in both frameworks for a fair comparison. For both markets, the results indicate that the impact of using a stochastic volatility model is significant. From the sensitivity analysis it followed that the volatility risk might also be dominated by changes in mortality or interest rates, depending on the moneyness of the contract. In the considered empirical test cases we found that, the prices for the GAOs using a stochastic volatility model for equity prices are considerably higher in comparison to the constant volatility model, especially for GAOs with out-of-the-money strikes.

Table 7
Comparison between the explicit (quasi closed-form) exact formula in (48), the method of minimum variance duration, the Edgeworth expansion, the affine approximation and a Monte Carlo simulation. Numbers in parentheses are the relative differences compared to the exact formula for the GAO price. Values lying within the 95% confidence interval of the Monte Carlo estimates are starred (*) and made bold.

r_0 (%)	Strike level (%)	Exact	Min. var. duration	Edgeworth expansion	Affine approx	Monte Carlo ($\pm$95% interval)
0.5	127	**11.800***	**11.810*** (0.1%)	**11.816*** (0.1%)	**11.7913*** (−0.1%)	11.792
1.0	123	**9.756***	**9.771*** (0.2%)	**9.750*** (−0.1%)	**9.7412*** (−0.1%)	9.749
1.5	118	**7.874***	**7.896*** (0.3%)	**7.848*** (−0.3%)	**7.8529*** (−0.3%)	7.868
2.0	114	**6.169***	6.195 (0.4%)	6.1293 (−0.6%)	**6.142*** (−0.4%)	6.163
2.5	110	**4.661***	4.686 (0.5%)	4.6199 (−0.9%)	4.631 (−0.6%)	4.656
3.0	106	**3.373***	3.391 (0.5%)	3.3408 (−1.0%)	3.346 (−0.8%)	3.368
3.5	103	**2.322***	**2.327*** (0.2%)	2.300 (−0.9%)	**2.304*** (−0.7%)	2.317
4.0	99	**1.510***	**1.501*** (−0.6%)	1.490 (−1.3%)	**1.506*** (−0.3%)	1.507
4.5	96	**0.921***	0.901 (−2.2%)	0.8942 (−2.9%)	0.931 (1.0%)	0.920
5.0	93	**0.525***	0.498 (−5.1%)	0.4922 (−6.2%)	0.544 (3.6%)	0.524
5.5	90	**0.278***	0.252 (−9.4%)	–	–	0.278
6.0	88	**0.136***	0.115 (−15.4%)	–	–	0.135
6.5	85	**0.061***	0.047 (−23.3%)	–	–	0.061
7.0	83	**0.025***	0.017 (−32.8%)	–	–	0.025

Appendix A. Pricing of a coupon bearing option under two-factor interest rates

Let the pair $(x(T), y(T))$ follow a bivariate normal distribution, i.e. with means μ_x, μ_y, variances σ_x^2, σ_y^2 and correlation ρ. The probability density function $f(x, y)$ of $(x(T), y(T))$ is hence given by

$$f(x,y) = \frac{\exp\left\{-\frac{1}{2(1-\rho_{xy}^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho_{xy}\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}}. \quad (55)$$

Furthermore, let the time T price of the zero-coupon bond $P(T, t_i)$ maturing at time t_i be given by

$$P(T, t_i) = A(T, t_i)\mathrm{e}^{-B(a,T,t_i)x(T)-B(b,T,t_i)y(T)}. \quad (56)$$

We then come to the following proposition.

Proposition A.1. *The expected value of the coupon-bearing option maturing at time T, paying coupons c_i at times $i = 0, \ldots, n$ and with strike K is given by a one-dimensional integral, i.e.*

$$\begin{aligned}
&\mathbb{E}\left\{\left(\sum_{i=0}^n c_i P(T,t_i) - K\right)^+\right\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{i=0}^n c_i A(T,t_i)\mathrm{e}^{-B(a,T,t_i)x(T)-B(b,T,t_i)y(T)} - K\right)^+ \\
&\quad\times f(x,y)\mathrm{d}y\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x\sqrt{2\pi}}\left[F_i(x)N(h_2(x)) - KN(h_1(x))\right]\mathrm{d}x \\
&=: G\left(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho_{xy}\right), \quad (57)
\end{aligned}$$

where N denotes the cumulative standard normal distribution function, with

$$h_1(x) := \frac{y^* - \mu_y}{\sigma_y\sqrt{1-\rho_{xy}^2}} - \frac{\rho_{xy}(x-\mu_x)}{\sigma_x\sqrt{1-\rho_{xy}^2}},$$

$$h_2(x) := h_1(x) + B(b,T,t_i)\sigma_y\sqrt{1-\rho_{xy}^2},$$

$$\lambda_i(x) := c_i A(T,t_i)\mathrm{e}^{-B(a,T,t_i)x},$$

$$\begin{aligned}
\kappa_i(x) &:= -B(b,T,t_i) \\
&\quad\times\left[\mu_y - \frac{1}{2}\sigma_y^2(1-\rho_{xy}^2)B(b,T,t_i) + \rho_{xy}\sigma_y\frac{(x-\mu_x)}{\sigma_x}\right],
\end{aligned}$$

$$F_i(x) := \sum_{i=0}^n \lambda_i(x)\mathrm{e}^{\kappa_i(x)}$$

and with y^* the unique solution of

$$\sum_{i=0}^n \lambda_i(x)\mathrm{e}^{-B(b,T,t_i)y^*} = K.$$

Proof. The result is analogous to the derivation of the swaption price under the G2++ model, we therefore refer to equation (4.31) in Brigo and Mercurio (2006) on pp. 158–159 and the corresponding proof on pp. 173–175. □

Appendix B. Moments and terminal correlation of the two-factor Gaussian interest rates

To determine the moments of $x(T)$ and $y(T)$ under the equity price measure, we need to consider the dynamics of (40), there stated under the risk-neutral measure $\mathcal{Q}$, under the equity price measure $\mathcal{Q}^S$. To change the underlying numeraire (e.g. see Geman et al. (1996)), we calculate the corresponding Radon–Nikodým derivative which is given by

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{Q}^S}{\mathrm{d}\mathcal{Q}} &= \frac{S(T)B(0)}{S(0)B(T)} \\
&= \exp\left[-\frac{1}{2}\int_0^T \nu^2(u)\mathrm{d}u + \int_0^T \nu(u)\mathrm{d}W_S^{\mathcal{Q}}(u)\right]. \quad (58)
\end{aligned}$$

The multi-dimensional version of Girsanov's theorem (e.g. see Oksendal (2005)) hence implies that

$$\mathrm{d}W_S^{\mathcal{Q}^S}(t) \mapsto \mathrm{d}W_S^{\mathcal{Q}}(t) - \nu(t)\mathrm{d}t, \quad (59)$$

$$\mathrm{d}W_x^{\mathcal{Q}^S}(t) \mapsto \mathrm{d}W_x^{\mathcal{Q}}(t) - \rho_{xS}\nu(t)\mathrm{d}t, \quad (60)$$

$$\mathrm{d}W_y^{\mathcal{Q}^S}(t) \mapsto \mathrm{d}W_y^{\mathcal{Q}}(t) - \rho_{yS}\nu(t)\mathrm{d}t, \quad (61)$$

$$\mathrm{d}W_\nu^{\mathcal{Q}^S}(t) \mapsto \mathrm{d}W_\nu^{\mathcal{Q}}(t) - \rho_{\nu S}\nu(t)\mathrm{d}t, \quad (62)$$

are $\mathcal{Q}^S$ Brownian motions. Hence under $\mathcal{Q}^S$ one has the following model dynamics for the volatility and interest rate process

$$\mathrm{d}x(t) = -ax(t)\mathrm{d}t + \rho_{xS}\sigma\nu(t)\mathrm{d}t + \sigma\mathrm{d}W_x^{\mathcal{Q}^S}(t), \quad x(0) = 0, \quad (63)$$

$$\mathrm{d}y(t) = -ay(t)\mathrm{d}t + \rho_{yS}\eta\nu(t)\mathrm{d}t + \eta\mathrm{d}W_y^{\mathcal{Q}^S}(t), \quad y(0) = 0, \quad (64)$$

$$\mathrm{d}\nu(t) = \widetilde{\kappa}\left(\widetilde{\psi} - \nu(t)\right)\mathrm{d}t + \tau\mathrm{d}W_\nu^{\mathcal{Q}^S}(t), \quad \nu(0) = \nu_0, \quad (65)$$

where $\widetilde{\kappa} := \kappa - \rho_{\nu S}\tau$, $\widetilde{\psi} := \frac{\kappa\psi}{\widetilde{\kappa}}$. Integrating the latter dynamics (conditional on the current time filtration $\mathcal{F}_0$) yields the following explicit solutions:

$$\nu(T) = \widetilde{\psi} + (\nu(0) - \widetilde{\psi})e^{-\widetilde{\kappa}T} + \tau\int_0^T e^{-\widetilde{\kappa}(T-u)} dW_\nu^{\mathbb{Q}^S}(u), \tag{66}$$

$$\begin{aligned} x(T) &= \rho_{xS}\sigma\left(\frac{\widetilde{\psi}}{a}\left[1 - e^{-aT}\right] + \frac{\nu(0)-\widetilde{\psi}}{a-\widetilde{\kappa}}\left[e^{-\widetilde{\kappa}T} - e^{-aT}\right]\right) \\ &\quad + \frac{\rho_{xS}\sigma\tau}{(a-\widetilde{\kappa})}\int_0^T \left[e^{-\widetilde{\kappa}(T-u)} - e^{-a(T-u)}\right] dW_\nu^{\mathbb{Q}^S}(u) \\ &\quad + \sigma\int_0^T e^{-a(T-u)} dW_x^{\mathbb{Q}^S}(u), \end{aligned} \tag{67}$$

$$\begin{aligned} y(T) &= \rho_{yS}\eta\left(\frac{\widetilde{\psi}}{b}\left[1 - e^{-bT}\right] + \frac{\nu(0)-\widetilde{\psi}}{b-\widetilde{\kappa}}\left[e^{-\widetilde{\kappa}T} - e^{-bT}\right]\right) \\ &\quad + \frac{\rho_{yS}\eta\tau}{(b-\widetilde{\kappa})}\int_0^T \left[e^{-\widetilde{\kappa}(T-u)} - e^{-b(T-u)}\right] dW_\nu^{\mathbb{Q}^S}(u) \\ &\quad + \eta\int_0^T e^{-b(T-u)} dW_y^{\mathbb{Q}^S}(u). \end{aligned} \tag{68}$$

Using Ito's isometry, one has that the $x(T)$, $y(T)$ (conditional on $\mathcal{F}_0$) is normally distributed with means μ_x, μ_y and variance σ_x^2, σ_y^2 and correlation $\rho_{xy}(T)$ given by

$$\mu_x := \rho_{xS}\sigma\left(\frac{\widetilde{\psi}}{a}\left[1 - e^{-aT}\right] + \frac{\nu(0)-\widetilde{\psi}}{(a-\widetilde{\kappa})}\left[e^{-\widetilde{\kappa}T} - e^{-aT}\right]\right), \tag{69}$$

$$\mu_y := \rho_{yS}\sigma\left(\frac{\widetilde{\psi}}{b}\left[1 - e^{-bT}\right] + \frac{\nu(0)-\widetilde{\psi}}{(b-\widetilde{\kappa})}\left[e^{-\widetilde{\kappa}T} - e^{-bT}\right]\right), \tag{70}$$

$$\begin{aligned} \sigma_x^2 &:= \sigma_1^2(\sigma, a) + \sigma_2^2(\sigma, a, \rho_{xS}) + 2\rho_{12}(\sigma, a, \rho_{x\nu}, \rho_{xS}) \\ &\quad \times \sigma_1(\sigma, a)\sigma_2(\sigma, a, \rho_{xS}), \end{aligned} \tag{71}$$

$$\begin{aligned} \sigma_y^2 &:= \sigma_1^2(\eta, b) + \sigma_2^2(\eta, b, \rho_{yS}) + 2\rho_{12}(\eta, b, \rho_{y\nu}, \rho_{yS}) \\ &\quad \times \sigma_1(\eta, b)\sigma_2(\eta, b, \rho_{yS}), \end{aligned} \tag{72}$$

$$\rho_{xy} := \frac{\mathrm{Cov}\,(x(T), y(T))}{\sigma_x\sigma_y}, \tag{73}$$

where

$$\sigma_1(\lambda, z) := \lambda\sqrt{\frac{1 - e^{-2zT}}{2z}},$$

$$\sigma_2(\lambda, z, \rho) := \frac{\rho\lambda\tau}{z-\widetilde{\kappa}}\sqrt{\frac{1}{2\widetilde{\kappa}} + \frac{1}{2z} - \frac{2}{(\widetilde{\kappa}+z)} - \frac{e^{-2\widetilde{\kappa}T}}{2\widetilde{\kappa}} - \frac{e^{-2zT}}{2z} + \frac{2e^{-(\widetilde{\kappa}+z)T}}{(\widetilde{\kappa}+z)}},$$

$$\begin{aligned} \rho_{12}(\lambda, z, \rho_1, \rho_2) &:= \rho_1\frac{\lambda^2\rho_2\tau}{\sigma_1(\lambda, z)\sigma_2(\lambda, z, \rho_2)(z-\widetilde{\kappa})} \\ &\quad \times \left[\frac{1 - e^{-(z+\widetilde{\kappa})T}}{(z+\widetilde{\kappa})} - \frac{1 - e^{-2zT}}{2z}\right], \end{aligned}$$

$$\begin{aligned} &\mathrm{Cov}\,(x(T), y(T)) \\ &:= \rho_{xy}\sigma\eta\left[\frac{1 - e^{-(a+b)T}}{(a+b)}\right] \\ &\quad + \rho_{x\nu}\sigma\frac{\rho_{yS}\eta\tau}{(b-\widetilde{\kappa})}\left[\frac{1 - e^{-(a+\widetilde{\kappa})T}}{(a+\widetilde{\kappa})} - \frac{1 - e^{-(a+b)T}}{(a+b)}\right] \\ &\quad + \rho_{y\nu}\eta\frac{\rho_{xS}\sigma\tau}{(a-\widetilde{\kappa})}\left[\frac{1 - e^{-(b+\widetilde{\kappa})T}}{(b+\widetilde{\kappa})} - \frac{1 - e^{-(a+b)T}}{(a+b)}\right] \\ &\quad + \frac{\rho_{xS}\sigma\tau}{(a-\widetilde{\kappa})}\frac{\rho_{yS}\eta\tau}{(b-\widetilde{\kappa})}\left[\frac{1 - e^{-2\widetilde{\kappa}T}}{2\widetilde{\kappa}} + \frac{1 - e^{-(a+b)T}}{(a+b)}\right. \\ &\quad \left. - \frac{1 - e^{-(a+\widetilde{\kappa})T}}{(a+\widetilde{\kappa})} - \frac{1 - e^{-(b+\widetilde{\kappa})T}}{(b+\widetilde{\kappa})}\right]. \end{aligned}$$

Appendix C. Special case: pricing formulas with an independent equity price process or pure interest rate guaranteed annuities

If one does not link the GAO to the performance of an equity fund (e.g. seen in the Netherlands) the expression (4) for the GAO price can be simplified significantly. One then has that the GAO price is given by

$$\begin{aligned} C(T) &= {}_{(R-X)}P_X\,\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r(u)du\right)\right. \\ &\quad \left. \times\; g\left(\sum_{i=0}^n c_iP(T, t_i) - K\right)^+\right] \end{aligned} \tag{74}$$

$$= {}_{(R-X)}P_X\,gP(0, T)\mathbb{E}^{\mathbb{Q}^T}\left[\left(\sum_{i=0}^n c_iP(T, t_i) - K\right)^+\right], \tag{75}$$

where the above expectation is taken with respect to the T-forward measure $\mathbb{Q}^T$, which uses the zero-coupon bond price maturing at time T as numeraire. Moreover, also in case one assumes the equity price is independent from the annuity, e.g. according to Boyle and Hardy (2003) and Pelsser (2003), one ends up with the same expectation as (74); to obtain this price, one only has to multiply formula (74) with the expected future equity price $\mathbb{E}^{\mathbb{Q}^T}[S(T)]$. In the following sections we will derive explicit expressions for the GAO price under both one-factor and two-factor Gaussian interest rates.

C.1. Hull–White model

Under $\mathbb{Q}^T$, one has the following expression for the stochastic process $x(T)$, driving the short interest rate (e.g. see Brigo and Mercurio (2006) and Pelsser (2000)):

$$x(T) = \mu_x^T + \sigma\int_0^T e^{-a(T-u)} dW_x^{\mathbb{Q}^T}(u), \tag{76}$$

hence from Ito's isometry, we have that $x(T)$ is normally distributed with mean μ_x and variance σ_x^2 given by

$$\mu_x^T := -\frac{\sigma^2}{a^2}\left[1 - e^{-aT}\right] + \frac{\sigma^2}{2a^2}\left[1 - e^{-2aT}\right], \tag{77}$$

$$\sigma_x^T := \sigma\sqrt{\frac{1 - e^{-2aT}}{2a}}. \tag{78}$$

Just as in Section 5, we have that $x(T)$ is normally distributed, i.e. with the same variance σ_x^2, but with a different mean μ_x^T. Hence completely analogous to Section 5, one can use the Jamshidian (1989) result and write the call option on the sum of zero-coupon bonds as a sum of zero-coupon bond call options: let x^* solve

$$\sum_{i=0}^n c_iA(T, t_i)e^{-B(T,t_i)x^*} = K, \tag{79}$$

and let

$$K_i := A(T, t_i)e^{-B(T,t_i)x^*}. \tag{80}$$

Using Jamshidian (1989), we can have that the price of a GAO is given by the price of a sum of zero-coupon bond options, i.e.

$$\begin{aligned} C(T) &= {}_{(R-X)}P_X\,gP(0, T) \\ &\quad \times \mathbb{E}^{\mathbb{Q}^T}\left[\sum_{i=0}^n c_i\left(A(T, t_i)e^{-B(T,t_i)x(T)} - K_i\right)^+\right]. \end{aligned} \tag{81}$$

As the bond price again follows a log-normal distribution in the Gaussian model, one can express GAO price in terms of the Black and Scholes (1973) formula, i.e.

$$C(T) = {}_{(R-X)}P_X\, gP(0,T)\sum_{i=0}^{n} c_i \left[F_i N\left(d_1^i\right) - K_i N\left(d_2^i\right)\right], \tag{82}$$

$$F_i = \mathrm{e}^{M_i+\frac{1}{2}V_i}, \tag{83}$$

$$d_1^i = \frac{\ln(F_i/K_i) + \frac{1}{2}V_i}{\sqrt{V_i}}, \tag{84}$$

$$d_2^i = d_1^i - \sqrt{V_i}, \tag{85}$$

where

$$M_i = \ln A(T,t_i) - B(T,t_i)\mu_x^T, \tag{86}$$

$$V_i = B^2(T,t_i)\left(\sigma_x^T\right)^2, \tag{87}$$

and note that the above expression only deviates from (34) in the different means and variances for the $x(T)$ process.

C.2. Gaussian two-factor model

Under $\mathbb{Q}^T$, one has the following expression for the stochastic processes $x(T)$, $y(T)$ that drive the short interest rate (e.g. see Brigo and Mercurio (2006)):

$$\begin{aligned} x(T) &= \mu_x^T + \sigma\int_0^T \mathrm{e}^{-a(T-u)}\mathrm{d}W_x^{\mathbb{Q}^T}(u), \\ y(T) &= \mu_y^T + \sigma\int_0^T \mathrm{e}^{-b(T-u)}\mathrm{d}W_y^{\mathbb{Q}^T}(u), \end{aligned} \tag{88}$$

hence $x(T), y(T)$ is normally distributed with means μ_x^T, μ_y^T, variances σ_x^2, σ_y^2 and correlation $\rho_{xy}(T)$ given by

$$\begin{aligned} \mu_x^T := & -\left(\frac{\sigma^2}{a^2} + \rho_{xy}\frac{\sigma\eta}{ab}\right)\left[1-\mathrm{e}^{-aT}\right] \\ & + \frac{\sigma^2}{2a^2}\left[1-\mathrm{e}^{-2aT}\right] + \frac{\rho_{xy}\sigma\eta}{b(a+b)}\left[1-\mathrm{e}^{-(a+b)T}\right], \end{aligned} \tag{89}$$

$$\begin{aligned} \mu_y^T := & -\left(\frac{\eta^2}{b^2} + \rho_{xy}\frac{\sigma\eta}{ab}\right)\left[1-\mathrm{e}^{-bT}\right] \\ & + \frac{\eta^2}{2b^2}\left[1-\mathrm{e}^{-2bT}\right] + \frac{\rho_{xy}\sigma\eta}{a(a+b)}\left[1-\mathrm{e}^{-(a+b)T}\right], \end{aligned} \tag{90}$$

$$\sigma_x := \sigma\sqrt{\frac{1-\mathrm{e}^{-2aT}}{2a}}, \tag{91}$$

$$\sigma_y := \eta\sqrt{\frac{1-\mathrm{e}^{-2bT}}{2b}}, \tag{92}$$

$$\rho_{xy}(T) := \rho_{xy}\frac{\sigma\eta}{\sigma_x\sigma_y}\left[\frac{1-\mathrm{e}^{-(a+b)T}}{(a+b)}\right]. \tag{93}$$

Hence analogously to Section 6, one has that the GAO price is given by

$$C(T) = {}_{(R-X)}P_X\, gP(0,T)G\left(\mu_x^T, \mu_y^T, \sigma_x, \sigma_y, \rho_{xy}(T)\right), \tag{94}$$

where G is given by an explicit expression, i.e. defined by Eq. (57) of Appendix A.

Appendix D. Yield curve shocks

In Section 7.2 the 2 shocks given in Table 8 are applied to the yield curves. These shocks are aimed to represent the 99.5% percentile on a 1 year horizon in the Quantitative Impact Study 5 of CEIOPS.

Table 8
Percentage yield changes for the up and down sensitivity shock in QIS5.

Maturity	Up (%)	Down (%)
1	75	−75
2	65	−65
3	56	−56
4	50	−50
5	46	−46
6	42	−42
7	39	−39
8	36	−36
9	33	−33
10	31	−31
11	30	−30
12	29	−29
13	28	−28
14	28	−28
15	27	−27
16	28	−28
17	28	−28
18	28	−28
19	29	−29
20	29	−29
21	29	−29
22	30	−30
23	30	−30
24	30	−30
25	30	−30
26	30	−30
27	30	−30
28	30	−30
29	30	−30
30	30	−30

Appendix E. Model setup of the Chu and Kwok (2007) case

In this appendix we describe the numerical input of the example being used in Chu and Kwok (2007). We also report the relative differences between the GAO price obtained by their methods and the explicit (quasi-closed-form) exact expression in (48) for that example; note that as the Black–Scholes G2++ model, used in Chu and Kwok (2007), is special case of the Schöbel–Zhu G2++ considered in Section 6, we can one on one translate their parameters into our modeling framework. As the notation is slightly different, we explicitly provide this translation into our modeling framework.

The GAO is specified using the guaranteed rate $g = 9$, the current age of the policy holder $x = 50$ and his retirement age is $r = 65$, with corresponding probability of survival ${}_{(R-X)}P_X = 0.9091$ and time to expiry for the GAO equal to $T = 15$ years. The equity price is modeled by the Black and Scholes (1973) model with parameters:

$$q = 5\%, \quad S(0) = 100\exp(-q\cdot T) = 47.24, \quad \sigma_S = 10\%, \tag{95}$$

where q denotes the continuous dividend rate and σ_S the constant volatility of the equity price. The current (continuous) yield curve is given by (97) and for the G2++ interest rate model (e.g. see Brigo and Mercurio (2006)) the following parameters are used:

$$a = 0.77, \quad b = 0.08, \quad \sigma = 2\%, \quad \eta = 1\%, \quad \rho_{xy} = -0.7 \tag{96}$$

where the correlations between equity and interest rate drivers given by $\rho_{xS} = 0.5$ and $\rho_{yS} = 0.0071$. Finally, the i-year survival probabilities c_i from policy holder's retirement age 65 are provided in Table 9.

In Section 7.3 we compared the prices of the explicit solution (48) and estimates obtained using 1000,000 Monte Carlo simulations with the Minimum Variance, the Edgeworth and Affine Approximation method which are used in Chu and Kwok (2007). These results can be found in Table 7, where a comparison is

Table 9
i-year survival probabilities c_i from policy holder's retirement age 65. A maximum age of 100 is assumed, that is for all $j > 35$: $c_j = 0$.

c_0	1.0000	c_9	0.8304	c_{18}	0.4889	c_{27}	0.0998
c_1	0.9871	c_{10}	0.8018	c_{19}	0.4414	c_{28}	0.0725
c_2	0.9730	c_{11}	0.7708	c_{20}	0.3934	c_{29}	0.0503
c_3	0.9578	c_{12}	0.7374	c_{21}	0.3454	c_{30}	0.0330
c_4	0.9411	c_{13}	0.7015	c_{22}	0.2981	c_{31}	0.0203
c_5	0.9229	c_{14}	0.6632	c_{23}	0.2523	c_{32}	0.0115
c_6	0.9029	c_{15}	0.6226	c_{24}	0.2088	c_{33}	0.0059
c_7	0.8808	c_{16}	0.5798	c_{25}	0.1684	c_{34}	0.0027
c_8	0.8567	c_{17}	0.5351	c_{26}	0.1319	c_{35}	0.0011

given for different levels r_0 of the yield curve provided by the (continuous) yields

$$Y(T) = r_0 + 0.04(1 - e^{-0.2T}). \tag{97}$$

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