Accounting for Stochastic Interest Rates, Stochastic Volatility and a General Dependency Structure in the Valuation of Forward Starting Options

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A quantitative analysis on the pricing of forward starting options under stochastic volatility and stochastic interest rates is performed. The main finding is that forward starting options not only depend on future smiles, but also directly on the evolution of the interest rates as well as the dependency structures among the underlying asset, the interest rates, and the stochastic volatility: compared to vanilla options, dynamic structures such as forward starting options are much more sensitive to model specifications such as volatility, interest rate, and correlation movements. We conclude that it is of crucial importance to take all these

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factors explicitly into account for a proper valuation and risk management of these securities. The performed analysis is facilitated by deriving closed-form formulas for the valuation of forward starting options, hereby taking the stochastic volatility, stochastic interest rates as well the dependency structure between all these processes explicitly into account. The valuation framework is derived using a probabilistic approach, enabling a fast and efficient evaluation of the option price by Fourier inverting the forward starting characteristic functions. Jrl Fut Mark 31:103–125, 2011

INTRODUCTION

Due to the increasing popularity for exotic structures like cliquets and ratchet options, the pricing of forward starting options (which can be seen as natural building blocks for these contracts) has recently attracted a lot of attention from both academics and practitioners. Forward starting options belong to the class of path-dependent European-style contracts in the sense that they not only depend on the terminal value of the underlying asset, but also on the asset price at an intermediate point (often dubbed as "strike determination date"). Typically, a forward starting contract gives the holder a call (or put) option with a strike that is set equal to a fixed proportion of the underlying asset price at this intermediate date. A special form of this option is that on the (future) return of the underlying, which can be seen as a call option on the ratio of the stock price at maturity and the intermediate date. This is often being used by insurance companies to hedge unit-linked guarantees embedded in life insurance products. Additionally, structured products involving forward starting options (like cliquet and ratchet structures) are often tailored for investors seeking upside potential, while keeping protection against downside movements.

Though forward starting options seem quite simple exotic derivatives, their valuation can be demanding, depending on the underlying model. Our pricing takes into account two important factors in the pricing of forward starting options: stochastic volatility and stochastic interest rates, while also taking into account the correlation between those processes explicitly. It is hardly necessary to motivate the inclusion of stochastic volatility in a derivative pricing model. Stochastic interest rates are crucial for the pricing of forward starting options because securities with forward starting features often have a long-dated maturity and are therefore much more interest rate sensitive, e.g. see Guo and Hung (2008) or Kijima and Muromachi (2001). The addition of interest rates as a stochastic factor has been the subject of empirical investigations most notably by Bakshi, Cao and Chen (2000). These authors show that the hedging performance of delta hedging strategies of long-maturity options improves when taking stochastic interest rates into account.

The pricing of forward starting options was first considered by Rubinstein (1991) who provides a closed-form solution for the pricing of forward starting options based on the assumptions of the Black and Scholes (1973) model. Lucic (2003), Hong (2004) and Kruse and Nögel (2005) relax the constant volatility assumption and consider the pricing of forward starting options under Heston (1993) stochastic volatility. The pricing of forward starting options under stochastic volatility with independent stochastic interest rates was considered by Guo and Hung (2008), Ahlip and Rutkowski (2009) and Nunes and Alcaria (2009). The framework employed in this study distinguishes itself from these models by a closed-form pricing formula and an explicit, rather than implicit, incorporation of the correlation between the underlying and the term structure of interest rates. The flexibility of the stochastic volatility model with (correlated) stochastic rates and the pricing of vanilla call options in such a framework has been covered in Ahlip (2008) and van Haastrecht, Lord, Pelsser and Schrager (2009).

The main goal of this work is performing a quantitative analysis on the pricing of forward starting options under stochastic volatility and stochastic interest rates. In particular we want to investigate the impact of stochastic volatility, stochastic interest rates as well as a realistic dependency structure between all the underlying processes on the valuation of these securities. The analysis is made possible by developing a closed-form solution for the price of a forward starting option in a model in which the instantaneous stochastic volatility is given by the Schöbel and Zhu (1999) model and the interest rates follow Hull and White (1993) dynamics. We explicitly incorporate the correlation between the underlying stock and the term structure of interest rates, which is an important empirical characteristic that needs to be taken into account for the pricing and hedging of long-term options, e.g. see Bakshi et al. (2000) or Piterbarg (2005). The setup of the study is as follows: we discuss the modelling framework in second section. The pricing of forward starting options and corresponding option pricing formulas is considered in subsequent section. In penultimate section we discuss the implementation of these formulas and analyze the valuation and risk management of forward starting option under stochastic volatility, stochastic interest rates, and a general correlation structure. Finally, we conclude in last section.

THE MODELLING FRAMEWORK

Under the risk-neutral measure Q, which uses a money market account as numeraire, we assume that $S(t)$ has the following Schöbel–Zhu–Hull–White dynamics

$$dS(t) = r(t)S(t)\,dt + \nu(t)S(t)\,dW_S^Q(t),\ S(0) = S_0, \tag{1}$$

$$\nu(t) = \kappa(\psi - \nu(t))\,dt + \tau\,dW_\nu^Q(t),\ \nu(0) = \nu_0, \tag{2}$$

see van Haastrecht et al. (2009). Here $\nu(t)$, driver for the stochastic volatility $|\nu(t)$ of the stock, follows an Ornstein–Uhlenbeck process. The parameters of the volatility process are the positive constants κ (mean reversion), $\nu(0)$ (short-term mean), ψ (long-term mean) and τ (volatility of the volatility). The interest rates are given by a one-factor Hull–White (1993) model, which can be expressed as

$$r(t) = \beta(t) + x(t),\ r(0) = r_0, \tag{3}$$

$$dx(t) = -ax(t)\,dt + \sigma dW_r^Q(t), x(0) = 0. \tag{4}$$

Here a (mean reversion) and σ (volatility) are the positive parameters of the model, and where $\beta(t)$ can be used to exactly fit the current term structure of interest rates, e.g. see Pelsser (2000) or Brigo and Mercurio (2006) for further details. The model allows for a general correlation structure between all driving model factors, i.e. the correlation matrix between of the Brownian motions $W_S(t)$, $W_r(t)$, $W_\nu(t)$ is given by

$$\begin{pmatrix} 1 & \rho_{Sr} & \rho_{S\nu} \\ \rho_{Sr} & 1 & \rho_{r\nu} \\ \rho_{S\nu} & \rho_{r\nu} & 1 \end{pmatrix}. \tag{5}$$

Even though the dynamics incorporate stochastic interest rates, stochastic volatility, and a general correlation, one can still obtain closed-form formulas for European option prices, which is a big advantage in the calibration, see van Haastrecht et al. (2009).

At first sight, one curious property of the model is that the volatility process $\nu(t)$ affects the sign of the instantaneous correlation between $\nu(t)$ and $\ln x(t)$. Indeed, one can show that

$$\mathrm{Corr}(d\ln x(t), d\nu(t)) = \frac{\rho_{x\nu}\nu(t)\tau}{\sqrt{\nu^2(t)\tau^2}} = \rho_{x\nu}\,\mathrm{sgn}(\nu(t))\,dt, \tag{6}$$

This effect is visualized in Figure 1, where we have plotted a sample path of $x(t)$, $\nu(t)$, and $|\nu(t)|$.

Indeed, when $\nu(t)$ is negative and decreasing, the asset price is increasing, contrary to what one would expect from the parameter configuration. The key lies therein that $\nu(t)$ should not be interpreted as the volatility of the underlying asset: it is merely a latent variable which drives the true volatility of the asset,

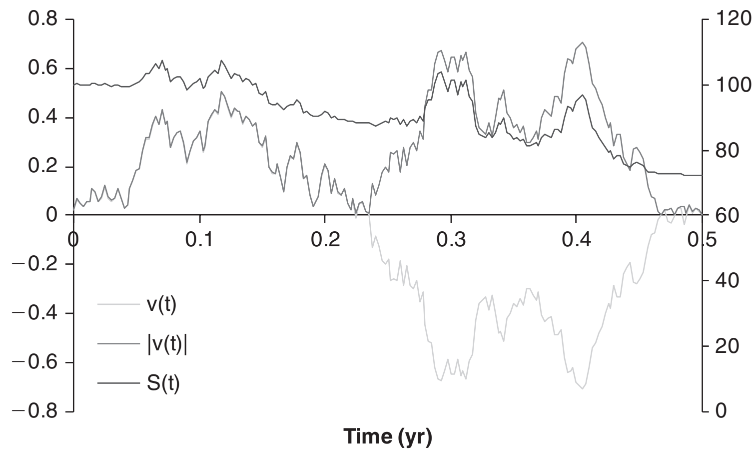

FIGURE 1
Sample path of $S(t)$, $\nu(t)$ and $|\nu(t)|$. SZ parameters are $\kappa = \tau = 1$, $\nu(0) = \psi = 25\%$, $x(0) = 100$.

the true volatility being defined as the square root of the variance. Indeed if one applies the Ito–Tanaka theorem to derive the dynamics of $|\nu(t)|$, e.g. see Revuz and Yor (1999), this leads to an instantaneous correlation of

$$\mathrm{Corr}(d\ln x(t),\, d|\nu(t)|) = \frac{\rho_{x\nu}|\nu(t)|\tau}{\sqrt{\nu^2(t)\tau^2}} = \rho_{x\nu}\,dt, \tag{7}$$

as we would like it to be.

FORWARD STARTING OPTIONS

Forward starting options are contracts which not only on depend on the terminal value of the underlying asset, but also on the asset price at an intermediate time between the current time and its expiry time. Kruse and Nögel (2005) consider two types of forward starting options under the Heston (1993) model: European forward starting call options on the underlying asset and on the underlying return. The first structure is prevalent in Employee stock option schemes, whereas the second category forms a building block for cliquet, ratchet, and Unit-Linked insurance options. In both contracts a premium is paid on the purchase date; however, the option's life will only start on an intermediate date (in-between the purchase and expiry date, dubbed as the strike determination time). Thus, the terminal payoff of these options depends on the underlying asset price at both the maturity and the start date of the underlying option. The next definitions formalize these options.

Definition 1: The terminal payoff of a European forward starting call option on the underlying asset price S, with a percentage strike of K, strike determination time T_{i-1} and maturity T_i is given by

$$[S(T_i) - KS(T_{i-1})]^+. \tag{8}$$

Definition 2: The terminal payoff of a European forward starting call option on the return of the underlying asset price S, with an absolute strike of K, determination time T_{i-1} and maturity T_i is given by

$$\left[\frac{S(T_i)}{S(T_{i-1})} - K\right]^+. \tag{9}$$

Pricing formulas for these options are provided in the following two sections.

Pricing Formula for a Forward Starting Option on the Underlying Asset

We can express the price of the forward starting call option price $C_F(T_{i-1}, T_i)$ on the underlying asset, i.e. with terminal payoff (8), in the following expectation under the risk-neutral measure $\mathcal{Q}$

$$C_F(T_{i-1}, T_i) = \mathbb{E}^{\mathcal{Q}}\left[e^{-\int_t^{T_i} r(u)\,du}(S(T_i) - KS(T_{i-1}))^+ | \mathcal{F}_t\right]. \tag{10}$$

Instead of evaluating the expected discounted payoff under the risk-neutral bank account measure, we can also change the underlying probability measure to evaluate this expectation under the stock price probability measure $\mathcal{Q}^S$ (e.g. see Geman, Karoui, & Rochet, 1996), i.e. with the stock price S as numeraire. Hence, conditional on time t, we can evaluate the price of the forward starting option (10) as

$$\begin{aligned}
C_F(T_{i-1}, T_i) &= S(t)\,\mathbb{E}^{\mathcal{Q}^S}\left[\frac{1}{S(T_i)}(S(T_i) - KS(T_{i-1}))^+ \middle| \mathcal{F}_t\right] \\
&= S(t)\mathbb{E}^{\mathcal{Q}^S}\left[\left(1 - K\frac{S(T_{i-1})}{S(T_i)}\right)^+ \middle| \mathcal{F}_t\right] \\
&= S(t)K\mathbb{E}^{\mathcal{Q}^S}\left[\left(\frac{1}{K} - \frac{S(T_{i-1})}{S(T_i)}\right)^+ \middle| \mathcal{F}_t\right],
\end{aligned} \tag{11}$$

where the last line can be interpreted as a put option with strike $1/K$ on the ratio $S(T_{i-1})/S(T_i)$. For the pricing of the forward starting option on the underlying asset, it suffices to know the characteristic function $\phi_F(T_{i-1}, T_i, u)$ of ln $S(T_{i-1})/S(T_i)$ under the stock price probability measure $\mathcal{Q}^S$. The characteristic function solution is given by the following closed-form expression

$$\phi_F(T_{i-1}, T_i, u) = \exp\left[a_0 + a_1\mu_x + \frac{1}{2}a_1^2\sigma_x^2(1 - \rho_{x\nu}^2(t, T_{i-1}))\right]$$
$$\times \frac{\exp\left[a_2\mu_\nu + a_3\mu_\nu^2 + \dfrac{(a_1\sigma_x\rho_{x\nu}^2(t, T_{i-1}) + a_2\sigma_\nu + 2a_3\mu_\nu\sigma_\nu)^2}{2(1 - 2a_3\sigma_\nu^2)}\right]}{\sqrt{1 - 2a_3\sigma_\nu^2}}, \quad (12)$$

see Appendix A.

Following Heston (1993), Carr and Madan (1999), Levis (2001) and Lord and Kahl (2008), we can express the option (11) with log strike $k := \ln 1/K$, in terms of the (T-forward) characteristic function as

$$C_F(T_{i-1}, T_i, k) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}(e^{(\alpha - iv)k}\psi_F(T_{i-1}, T_i, v))\,dv, \quad (13)$$

$$\psi_F(T_{i-1}, T_i, v) := \frac{\phi_F(T_{i-1}, T_i, v + (\alpha - 1)i)}{(iv - \alpha)(iv - \alpha + 1)}, \quad (14)$$

where $\alpha > 1$ has been introduced for Fourier Transform regularization.

Note that in principle it is also possible, following the lines of Rubinstein (1991), Guo and Hung (2008) and Ahlip and Ruthowski (2009), to express the forward starting option price as the expected value of a future call option price, which can be evaluated using similar techniques as the evaluation of formula (11), but results in a pricing formula containing two integrals. However, working out the equivalent expectation (11) results in a pricing formula which only contains one integral. Not only does this make the corresponding implementation more efficient, but even more importantly it has been shown in Andersen and Andreasen (2002) and Lord and Kahl (2008) that the double integral formulation suffers from numerical instabilities whereas the single integral can be implemented in a numerically very stable way.

Pricing Formula for a Forward Starting Option on the Underlying Return

For the price $C_R(T_{i-1}, T_i)$ of the forward starting call option on the return of the underlying asset, i.e. with terminal payoff (8), the following expectation under the T_i-forward measure holds

$$C_R(T_{i-1}, T_i) = P(t, T_i)\mathbb{E}^{Q^{T_i}}\left[\left(\frac{S(T_i)}{S(T_{i-1})} - K\right)^+ \middle| \mathcal{F}_t\right], \quad (15)$$

that is, where the corresponding numeraire is now the (pure) discount bond $P(t, T_i)$ maturing at time T_i. For the pricing of the forward starting return option, it suffices to know the characteristic function $\phi_R(T_{i-1}, T_i, u)$ of

$\ln S(T_i)/S(T_{i-1})$ under the stock price probability measure $\mathcal{Q}^S$. A closed-form solution for this characteristic function is given by

$$\phi_R(T_{i-1}, T_i, u) = \exp\left[b_0 + b_1\mu_x + \frac{1}{2}b_1^2\sigma_x^2(1 - \rho_{x\nu}^2(t, T_{i-1}))\right]$$
$$\times \frac{\exp\left[b_2\mu_\nu + b_3\mu_\nu^2 + \dfrac{(b_1\sigma_x\rho_{x\nu}^2(t, T_{i-1}) + b_2\sigma_\nu + 2b_3\mu_\nu\sigma_\nu)^2}{2(1 - 2b_3\sigma_\nu^2)}\right]}{\sqrt{1 - 2b_3\sigma_\nu^2}}, \tag{16}$$

see Appendix B.

Consequently, the forward starting option (11) with log strike $k := \ln K$, can be expressed in terms of the (T-forward) characteristic function as

$$C_R(T_{i-1}, T_i, k) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}(e^{-(\alpha+iv)k}\psi(T_{i-1}, T_i, v))\, dv, \tag{17}$$

$$\psi_R(T_{i-1}, T_i, v) := \frac{\phi_R(T_{i-1}, T_i, v - (\alpha + 1)i)}{(\alpha + iv)(\alpha + 1 + iv)}, \tag{18}$$

see Carr and Madan (1999), Lewis (2001), and Lord and Kahl (2008) where $\alpha \in \mathbb{R}^+$ has been introduced for Fourier Transform regularization, see Carr and Madan (1999), Lewis (2001), and Lord and Kahl (2008).

NUMERICAL RESULTS

To investigate the impact of stochastic volatility and stochastic interest rates on the prices of forward starting options, we will consider the following numerical test cases. As the prices of forward starting options can be calculated in closed-form, a Monte Carlo benchmark against the pricing formulas (13)–(17) forms a standard test case for their implementation. We then explicitly investigate the impact and parameter sensitivities of stochastic interest rates and stochastic volatility on the prices of forward starting options. Finally, we tackle the issue of model risk and compare our framework with the Black and Scholes (1973) and Heston (1993) model, respectively, considered in Rubinstein (1991) and Guo and Hung (2008) for the valuation of forward starters.

Implementation of the Option Pricing Formulas

In this section we consider the practical implementation of the pricing formulas (13) and (17); both the implementation of the inverse Fourier transform, as well as the calculation of the characteristic function underlying this transform, deserve some attention. For the calculation of the inverse Fourier transform we refer the reader to Lord and Kahl (2008), Kilin (2006), and van Haastrecht et al. (2009), where this topic is covered in great detail. Instead we focus on the application-specific calculation of the characteristic functions (12) and (16). The calculation of the characteristic functions (12) and (16) is trivial up to the calculation of the constants $A(u, t, T)$ of (A14) and $L(u, t, T)$ of (B6), which involves the calculation of a numerical integral. Hence we focus on the calculation of $A(u, t, T)$, but a completely analogous reasoning holds for the calculation of $L(u, t, T)$.

It is possible to write a closed-form expression for the remaining integral in (A14). As the ordinary differential equation for $D(u, t, T)$ is exactly the same as in the Heston (1993) or Schöbel and Zhu (1999) model, it will involve a complex logarithm and should therefore be evaluated as outlined in Lord and Kahl (2007) in order to avoid any discontinuities. The main problem however lies in the integrals over $C(u, t, T)$ and $C^2(u, t, T)$, which will involve the Gaussian hypergeometric ${}_2F_1(a, b, c; z)$. The most efficient way to evaluate this hypergeometric function (according to Press & Flannery, 1992) is to integrate the defining differential equation. Since all of the terms involved in $D(u, t, T)$ are also required in $C(u, t, T)$, numerical integration of the second part of (A14) seems to be the most efficient method for evaluating $A(u, t, T)$. Note that we hereby conveniently avoid any issues regarding complex discontinuities altogether. It remains to have a closer look at the implementation of the numerical integral of $A(u, t, T)$ and $L(u, t, T)$.

We compute the prices for short- and long-term forward starting option for a range of strikes and where we use fixed-point Gaussian–Legendre quadrature to compute the numerical integral in (A14) and (B6). Hereby, we vary the number of quadrature points to determine how many points are needed in the test cases to obtain a certain accuracy. The numerical results together with the corresponding Monte Carlo estimates (using 10^6 sample paths) can be found in Table and below.

From the tables we see that the characteristic functions (12) and (16) underlying the option price formulas can be calculated very accurately, using only a small number of quadrature points; the prices of short-term options (Table I) and long-term options (Table II) can be calculated within a basis point of accuracy by using respectively just two and eight quadrature points for the calculation of the integral in $A(u, t, T)$ and $L(u, t, T)$. Note hereby that the corresponding Monte Carlo confidence interval is also larger in test case of Table II, due to the longer-dated maturity. Combining the efficient calculation of characteristic functions (12) and (16) with the efficient Fourier inversion techniques, we can all in all conclude the pricing of forward starting options can be done fast, highly accurate and in closed-form using the latter methods.

TABLE I

Closed-Form Solution Prices (CF(N)) Using N Quadrature Points for $A(u, T_1, T_2)$ in (32) and Monte Carlo Prices (MC) of the Forward Starting Call Option (11) for $t = 0$, $T_1 = 5$, $T_2 = 15$ and $P(t, T_1) = P(t, T_2) = 1.0$ and Model Parameters $\kappa = 1.00$, $\nu(0) = \psi = 0.20$, $a = 0.02$, $\sigma = 0.01$, $\tau = 0.50$, $\rho_{Sv} = -0.70$, $\rho_{Sr} = 0.30$ and $\rho_{rv} = 0.15$

strike level (%)	*CF(4)*	*CF(8)*	*CF(16)*	*CF(1024)*	*MC (±95% Interval)*
50	65.31	65.26	65.26	65.26	65.30 (±0.31)
75	53.94	53.85	53.85	53.85	53.89 (±0.29)
100	44.97	44.85	44.85	44.85	44.90 (±0.27)
125	37.80	37.65	37.65	37.65	37.71 (±0.25)
150	32.00	31.82	31.82	31.82	31.89 (±0.24)

TABLE II

Closed-Form Solution Prices (CF(N)) Using N Quadrature Points for $L(u, T_1, T_2)$ in (50) and Monte Carlo Prices (MC) of the Forward Starting Return Call Option (15) for $t = 0$, $T_1 = 1$, $T_2 = 2$ and $P(t, T_1) = P(t, T_2) = 1.0$ and Model Parameters $\kappa = 0.30$, $\nu(0) = \psi = 0.15$, $\tau = 0.20$, $a = 0.05$, $\sigma = 0.01$, $\rho_S = -0.40$, $\rho_{Srv} = 0.20$ and $\rho_{rv} = 0.10$

strike level (%)	*CF(1)*	*CF(2)*	*CF(4)*	*CF(1024)*	*MC (±95% Interval)*
50	50.23	50.24	50.24	50.24	50.27 (±0.05)
75	26.77	26.79	26.79	26.79	26.80 (±0.04)
100	8.56	8.39	8.39	8.39	8.39 (±0.03)
125	2.07	2.04	2.04	2.04	2.05 (±0.02)
150	0.69	0.69	0.69	0.69	0.69 (±0.01)

Impact of Stochastic Interest Rates and Stochastic Volatility

In this section we will cover the impact of stochastic volatility and (correlated) stochastic interest rates on the prices of forward starting options. That is, we investigate qualitative aspects of our extended framework in comparison to deterministic (or independent) interest rates and volatility assumptions. Rubinstein (1991) considered the pricing of a vanilla forward starting option in the Black and Scholes (1973) framework; as both interest rates and volatilities are deterministic in this model, the prices of forward starting options are (up to deterministic discounting effects) equal for all forward starting dates. The constant volatility assumption has been relaxed by Lucic (2003), Hong (2004), and Kruse and Nögel (2005), who consider the pricing of forward starting options under Heston (1993) stochastic volatility. The impact of stochastic volatility can be seen from the top graphs of Figure 2.

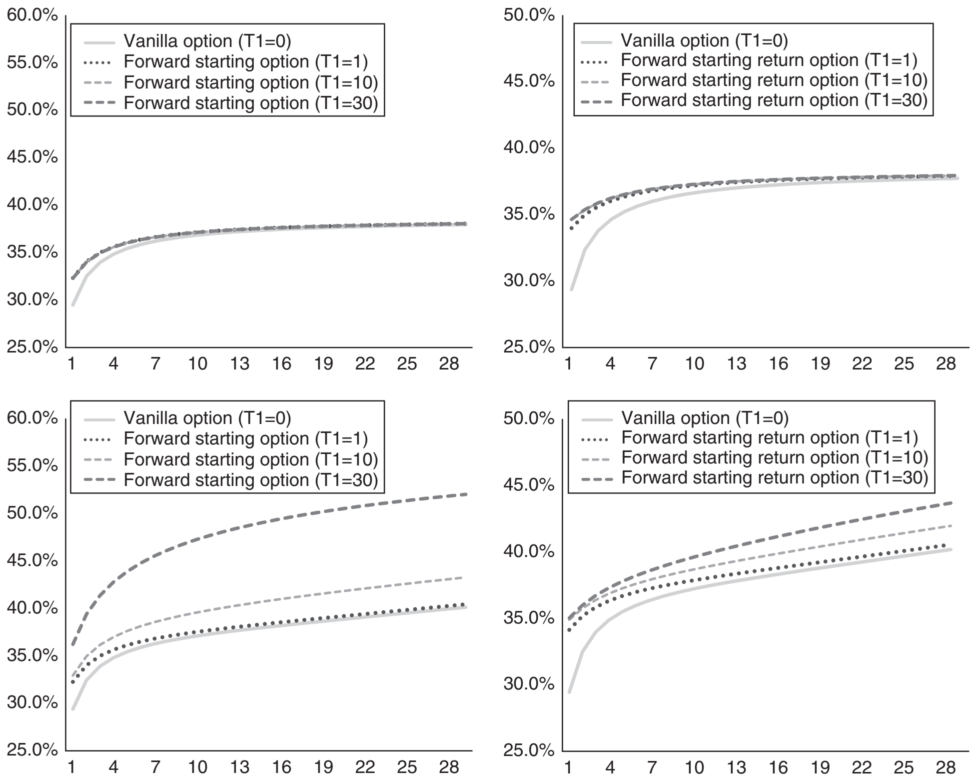

FIGURE 2

The figures plot, for different option maturities, the impact of stochastic interest rates on the forward implied volatility structure of an underlying call (left pictures) and return call option (right pictures). Parameters are $\kappa = 1.0$, $\nu(t) = \psi = 0.20$, $= \tau = 0.5$, $\rho_{S_\nu} = -0.70$, $\rho_{Sr} = \rho_{r\nu} = 0$ and $P(t, s) = \exp(-0.04(s - t))$ for all $s > t$. The top figures plot the volatility structure for deterministic interest rates, while the bottom figures plot the volatility structure for stochastic interest rates case with parameters $a = 0.02$ and $\sigma = 0.01$.

Compared to constant volatility, the addition of stochastic volatility increases the future uncertainty about the underlying option price which is hence reflected in higher implied volatilities for longer forward starting dates. Intuitively this effect is rather appealing as this coincides with market prices for forward starting structures where the writer of such an option wants to be compensated for the extra (future) volatility risk he is exposed to. Furthermore it is interesting to note from the figures that these effects are more apparent where the underlying option has a short maturity, which effect may be explained by the mean reverting property of stochastic volatility that is less severe for a short-term option hence increasing the future volatility risk. Finally, note from the top two graphs of Figure 2 that with deterministic rates the long-term uncertainty approaches a limit (or a stationary state) as the forward starting date or the underlying option maturity increase. For example the implied volatilities for forward starting options with a forward date of 10 and 30 years are exactly equal, which is counterintuitive as the term structure of implied volatilities remains increasing for long-term options and in general does not flatten out nor approaches a limit, for instance see the implied volatility quotes in long-maturity equity markets (readily available from MarkIT or Bloomberg) or the over-the-counter FX quotes in Piterbarg (2005) or Andreasen (2006).

More likely, the discrepancy with the way the market and the latter models look at long-term implied volatility structures suggests that these models lack an extra factor in their pricing frameworks; this conjecture is supported by Guo and Hung (2008) and Kijima and Muromachi (2001), who claim that stochastic interest rates are crucial for the pricing of forward starting options as these securities are often much more interest rate sensitive due to their long-term nature. In fact if we look at the bottom graphs of Figure 2, where we add stochastic interest rates to the framework with stochastic volatility, we see that the implied volatilities increase for longer forward starting and maturity dates. These model effects also correspond with a general feature of the interest rate market: the market's view on the uncertainty of long-maturity bonds is often much higher than that of shorter maturity bonds, reflecting the increasing impact of stochastic interest rates for long-dated structures. In this sense stochastic interest rates do seem to incorporate the larger uncertainty the writers of the forward starting options are exposed to.

The addition of stochastic interest rates as an independent factor for the pricing of forward starting options has been investigated in Guo and Hung (2008) and Nunes and Alcaria (2009). Though one step in the right direction, the independency assumption is certainly not supported by empirical analysis (e.g. see Baur, 2009) nor do the exotic option markets (such as hybrid equity-interest rate options) price these derivatives in this way, e.g. see Andreasen (2007) or Antonov, Arneguy, and Audet (2008); from Figure C1 of Appendix C and Figure D1 of Appendix D, we see that correlated stochastic interest rates can have a big impact on the prices of forward starting options. From Figure C1 we can see that for a positive rate-asset correlation coefficient the prices of forward starting options increase and vice versa for a negative correlation coefficient. In particular note from Figure C1 that, though the correlation coefficient between the interest rate and the stock also affects the implied volatility structure of the current time vanilla options, the effects on the prices of forward starting options are much more pronounced. Forward starting options are thus not only more interest rate and volatility sensitive, but are also much more exposed to correlation risks. This is not surprising as a joint movement in both the interest rate and the asset price not only affects future discounting, but more importantly also the (joint) asset price distribution. All in all, we can conclude that because forward starting options are very sensitive to future interest rate movements, volatility smiles as well as their dependency structure with the underlying asset, it is very important to take all these stochastic quantities into account for proper pricing and risk management of these derivatives.

CONCLUSION

We have performed a quantitative analysis on the valuation of forward starting options, where we explicitly accounted for stochastic volatility, stochastic interest rates as well as a general dependency structure between all underlying processes. The analysis was made possible by the development of closed-form formulas involving the pricing of the two main forward starting structures, currently present in the literature and the financial markets. Using a probabilistic approach, we derived closed-form expressions for the characteristic functions of the assets underlying the forward starting options. We then demonstrated how forward starting options can be priced efficiently and in closed-form by Fourier inverting these forward starting characteristic functions. An additional advantage of this technique is that our modelling framework can include jumps as a trivial extension, as we already work in the Fourier option pricing domain.

Our results are of great practical importance as the derivative markets for long-dated dynamic securities such as forward starting options have grown very rapidly over the last decade; compared to vanilla options, these structures directly depend on future volatility smiles, the term structure of interest rates as well as their dependency structure with the underlying asset. Moreover, as these contracts often incorporate long-dated maturities, we found that it is of crucial importance to take stochastic interest rates, volatility, and a general correlation structure into account for a proper valuation and hedging of these securities: not doing so leads to serious mispricings, not to mention the potential for hedge errors. Compared to other models, the analysis performed in our framework stands out by modelling both the stochastic volatility and interest rates, as well as taking a general correlation structure between all underlying drivers explicitly into account.

Besides investigating the behaviorof these dynamic derivatives, our formulas can also be used to directly price or hedge financial contracts. For instance unit-linked guarantees embedded in life insurance products, being sold in large amounts by insurance companies, can be priced in closed-form relying on our formulas. The same applies for cliquet options, which are heavily traded in over-the-counter markets, and CEO/employee stock option plans. Furthermore, there is a big intercourse between forward starting options considered here and over-the-counter exotic structures such as ratchet options and pension contracts, as these form the natural building blocks and hedge instruments for such contracts. Finally, as all the above-mentioned products explicitly depend on future volatility smiles, the term structure of interest rates as well as their dependency structure with the underlying asset, we judge that a proper valuation framework should account for all these characteristics.

APPENDIX A: DERIVATION OF THE CHARACTERISTIC FUNCTION UNDER THE STOCK PRICE MEASURE

This appendix derives the characteristic function of Equation (12). To this end, we define the forward asset price F as

$$F(t, T) = \frac{S(t)}{P(t, T)}, \tag{A1}$$

where $P(t, T)$ denotes the price of a (pure) discount bond at time t maturing at time T. Using the tower law of conditional expectations, i.e. conditioning on the time T_{i-1}, we can therefore express the characteristic function ϕ_F of ln $S(T_{i-1})/S(T_i)$ under the stock price measure as

$$\begin{aligned}\phi_F(T_{i-1}, T_i, u) &:= \mathbb{E}^{Q^S}\left[\exp\left(iu \ln \frac{S(T_{i-1})}{S(T_i)}\right) \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^{Q^S}[e^{iu \ln P(T_{i-1},T_i) + iu \ln F(T_{i-1},T_i)} \mathbb{E}^{Q^S}\{e^{i(-u)\ln F(T_i,T_i)} | \mathcal{F}_{T_{i-1}}\} | \mathcal{F}_t]. \end{aligned} \tag{A2}$$

This characteristic function can be evaluated in two steps.

First, in the Gaussian rate model (e.g. see Brigo & Mercurio, 2006) we have for $P(t, T)$

$$P(t, T) = A_{HW}(t, T) e^{-B_{HW}(t,T)x(t)}, \tag{A3}$$

with A_{HW}, B_{HW} defined in (A11) and (A12).

Secondly, note that the inner expectation in (A2) is just the characteristic function of ln $= F(T_i, T_i)$ evaluated at the point $-u$. To determine its solution, we follow Heston (1993) and reduce the problem of finding the characteristic function of the forward log-asset price dynamics to solving a partial differential equation. The Feynman–Kac theorem implies that the characteristic function

$$f(t, y, \nu) = \mathbb{E}^{Q^S}[\exp(iuy(T)) | \mathcal{F}_t], \tag{A4}$$

is given by the solution of the partial differential equation,

$$0 = f_t + \frac{1}{2}\nu_F^2(t)(f_{yy} + f_y) + \kappa(\xi(t) - \nu(t))f_\nu$$

$$+ (\rho_{S\nu}\tau\nu(t) + \rho_{r\nu}\tau\sigma B_{HW}(t, T))f_{y\nu} + \frac{1}{2}\tau^2 f_{\nu\nu}, \tag{A5}$$

$$f(T, y, \nu) = \exp(iuy(T)), \tag{A6}$$

where the subscripts denote partial derivatives. To solve for this characteristic function explicitly, we guess the functional form

$$f(t, y, \nu) = \exp\left[A(u, t, T) + B(u, t, T)y(t) + C(u, t, T)\nu(t) + \frac{1}{2}D(u, t, T)\nu^2(t)\right], \tag{A7}$$

and substitute this in (A5). This reduces it to a system of ordinary differential equations for A, B, C, and D, which can be solved for in closed form, see (A14)–(A17).

By substituting the characteristic function solution (A7) and (A3) in (A2), the characteristic function $\phi_F(T_{i-1}, T_i, u)$ can be expressed completely in terms of the Gaussian factors $x(T_{i-1})$ and $\nu^2(T_{i-1})$, i.e.

$$\phi_F(T_{i-1}, T_i, u) =: \mathbb{E}^{Q^S}[\exp\{a_0 + a_1 x(T_{i-1}) + a_2\nu(T_{i-1}) + a_3\nu^2(T_{i-1})\}|\mathcal{F}_t]. \tag{A8}$$

Because the above expression is a Gaussian quadratic form of the variables $x(T_{i-1})$ and $\nu(T_{i-1})$, one can evaluate this expectation completely in terms of the means μ_x, μ_ν, variances σ_x^2, σ_ν^2 and correlation $\rho_{x\nu}(t, T_{i-1})$ of these two state variables, e.g. see Feuerverger and Wong (2000) or Glasserman (2003). A straightforward evaluation (e.g. by completing the square or by integration the exponential affine function against the bivariate normal distribution) of this Gaussian quadratic expectation results in the characteristic function $\phi_F(T_{i-1}, T_i, u)$ of (12).

Here, the constants $a_0, \ldots, a_3$ are given by

$$a_0 := iu \ln A_{HW}(T_{i-1}, T_i) + A(-u, T_{i-1}, T_i),\ a_1 := -iu B_{HW}(T_{i-1}, T_i), \tag{A9}$$

$$a_2 := C(-u, T_{i-1}, T_i)\ a_3 := \frac{1}{2}D(-u, T_{i-1}, T_i), \tag{A10}$$

where the bond price formula of the Hull and White (1993) model can be obtained with

$$A_{\mathrm{HW}}(T_{i-1}, T_i) = \frac{P^M(t, T_i)}{P^M(t, T_{i-1})}\exp\left[\frac{1}{2}(V(T_{i-1}, T_i) - V(t, T_i) + V(t, T_{i-1}))\right] \tag{A11}$$

$$B_{\mathrm{HW}}(T_{i-1}, T_i) = \frac{1 - e^{-a(T_i - T_{i-1})}}{a} \tag{A12}$$

$$V(t, T) = \frac{\sigma^2}{a^2}\left[(T - t) + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right], \tag{A13}$$

and the solutions to the ordinary differential equations of the characteristic function solution (A7) by

$$\begin{aligned} A(u, t, T) &= \frac{1}{2}u(i - u)V(t, T) \\ &+ \int_t^T \left[(\tilde{\kappa}\tilde{\psi} + \rho_{r\nu} iu\tau\sigma B_{HW}(s, T))C(s) + \frac{1}{2}\tau^2(C^2(s) + D(s))\right]ds, \end{aligned} \tag{A14}$$

$$B(u, t, T) = iu, \tag{A15}$$

$$C(u, t, T) = u(i-u)\frac{((\gamma_3 - \gamma_4 e^{-2\gamma(T-t)}) - (\gamma_5 e^{-a(T-t)} - \gamma_6 e^{-(2\gamma+a)(T-t)}) - \gamma_7 e^{-\gamma(T-t)})}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}} \tag{A16}$$

$$D(u, t, T) = u(i - u)\frac{1 - e^{-2\gamma(T-t)}}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}}, \tag{A17}$$

with

$$\gamma = \sqrt{(\tilde{\kappa} - \rho_{S\nu}\tau iu)^2 - \tau^2 u(i - u)},\ \gamma_1 = \gamma + (\tilde{\kappa} - \rho_{S\nu}\tau iu), \tag{A18}$$

$$\gamma_2 = \gamma - (\tilde{\kappa} - \rho_{S\nu}\tau iu),\ \gamma_3 = \frac{\rho_{Sr}\sigma\gamma_1 + \tilde{\kappa}a\tilde{\psi} + \rho_{r\nu}\sigma\tau iu}{a\gamma},$$

$$\gamma_4 = \frac{\rho_{Sr}\sigma\gamma_2 - \tilde{\kappa}a\tilde{\psi} - \rho_{r\nu}\sigma\tau iu}{a\gamma},\ \gamma_5 = \frac{\rho_{Sr}\sigma\gamma_1 + \rho_{r\nu}\sigma\tau iu}{a(\gamma - a)},$$

$$\gamma_6 = \frac{\rho_{Sr}\sigma\gamma_2 - \rho_{r\nu}\sigma\tau iu}{a(\gamma + a)},\ \gamma_7 = (\gamma_3 - \gamma_4) - (\gamma_5 - \gamma_6),$$

and where $\tilde{\kappa}$, $\tilde{\psi}$ are defined as $\tilde{\kappa}: = \kappa - \rho_{S\nu}\tau$, $\tilde{\psi} := \kappa\psi/\tilde{\kappa}$.

Finally, we provide the moments of $\nu(T_{i-1})$ and $x(T_{i-1})$ under the stock price measure. Using Fubini's theorem, Ito's isometry and some tedious, but straightforward algebra, one can show that (conditional on time t) the pair $\nu(T_{i-1})$, $x(T_{i-1})$ follows a bivariate normal distribution with means μ_ν, μ_x, variances σ_ν^2, σ_x^2 and correlation $\rho_{x\nu}(t, T_{i-1})$, respectively, given by

$$\mu_\nu = \tilde{\psi} + (\nu(t) - \tilde{\psi})e^{\tilde{\kappa}(T_{i-1}-t)} \tag{A19}$$

$$\sigma_\nu^2 = \frac{\tau^2}{2\tilde{\kappa}}(1 - e^{-2\tilde{\kappa}\,(T_{t-1}-t)}, \tag{A20}$$

$$\mu_x = \rho_{Sr}\sigma = \left(\frac{\tilde{\psi}}{a}[1 - e^{-a(T_{i-1}-t)}] + \frac{\nu(0) - \tilde{\psi}}{(a-\tilde{\kappa})}[e^{-\tilde{\kappa}\,(T_{i-1}-t)} - e^{-a(T_{i-1}-t)}]\right), \tag{A21}$$

$$\sigma_x^2 = \sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2 \tag{A22}$$

$$\rho_{x\nu}(t, T_{i-1}) = \frac{\rho_{r\nu}\sigma\tau}{\sigma_x\sigma_\nu(a + \tilde{\kappa})}[1 - e^{-(a+\tilde{\kappa})(T_{i-1}-t)}], \tag{A23}$$

where

$$\sigma_1 = \sigma\sqrt{\frac{1 - e^{-2a(T_{i-1}-t)}}{2a}}, \tag{A24}$$

$$\sigma_2 = \frac{\rho_{Sr}\sigma_\tau}{a - \tilde{\kappa}}\sqrt{\frac{1}{2\tilde{\kappa}} + \frac{1}{2a} + \frac{1}{\tilde{\kappa}+a} - \frac{e^{-2\tilde{\kappa}\,(T_{i-1}-t)}}{2\tilde{\kappa}} - \frac{e^{-2a(T_{i-1}-t)}}{2a} + \frac{2e^{(\tilde{\kappa}+a)(T_{i-1}-t)}}{(\tilde{\kappa}+a)}}, \tag{A25}$$

$$\rho_{12} = \rho_{r\nu}\frac{\sigma^2\rho_{Sr}\tau}{\sigma_1\sigma_2(a - \tilde{\kappa})}\left[\frac{1 - e^{-(a+\tilde{\kappa})(T_{i-1}-t)}}{(a + \tilde{\kappa})} - \frac{1-e^{-2a(T_{i-1}-t)}}{2a}\right]. \tag{A26}$$

APPENDIX B: DERIVATION OF THE CHARACTERISTIC FUNCTION UNDER THE T_i-FORWARD MEASURE

This appendix derives the characteristic function of Equation (12). The characteristic function ϕ_R of $\ln S(T_i)/S(T_{i-1})$ under the T_i-forward probability measure can be obtained using similar arguments as in Appendix A. That is, using the tower law of conditional expectations, i.e. conditioning on the time T_{i-1} filtration $\mathcal{F}_{T_{i-1}}$, we have that

$$\begin{aligned}\phi_R(T_{i-1}, T_i, u) &:= \mathbb{E}^{Q^{T_i}}\left[\exp\left(iu\ln\frac{S(T_i)}{S(T_{i-1})}\right) \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^{Q^{T_i}}[e^{-iu\ln P(T_{i-1},\,T_i)-iu\ln F(T_{i-1},\,T_i)}\mathbb{E}^{Q^{T_i}}\{e^{iu\ln F(T_i,T_i)}|\mathcal{F}_{T_{i-1}}\}|\mathcal{F}_t].\end{aligned} \tag{B1}$$

The inner expectation is the characteristic function of $\ln = F(T_i, T_i)$, now under the T_i-forward measure, evaluated at the point u. Its solution $f(t, y, v)$ can obtained using analogous techniques as in Appendix B, and is given by

$$f(t, y, \nu) = \exp\left[L(u,t,T) + M(u,t,T)y(t) + N(u,t,T)\nu(t) + \frac{1}{2}O(u,t,T)\nu^2(t)\right]. \tag{B2}$$

Substituting this solution and the Gaussian bond formula (A3) in (B1) we obtain that

$$\phi_R(T_{i-1}, T_i, u) = \mathbb{E}^{\mathcal{Q}^{T_i}}[\exp\{b_0 + b_1 x(T_{i-1}) + b_2\nu(T_{i-1}) + b_3\nu^2(T_{i-1})\}|\mathcal{F}_t], \tag{B3}$$

the only difference with the Gaussian quadratic form (A8) being the dynamics of the processes $x(T_{i-1})$ and $\nu(T_{i-1})$, which now instead need to be evaluated under the T_i-forward measure. The solution for the characteristic function $\phi_R(T_{i-1}, T_i, u)$ is therefore given by the same evaluation of the Gaussian quadratic expectation, resulting in the characteristic function (16).

Here, the constants $b_0, \ldots, b_3$ are here given by

$$b_0 := -iu \ln A_{\mathrm{HW}}(T_{i-1}, T_i) + L(u, T_{i-1}, T_i),\ b_1 := iuB_{\mathrm{HW}}(T_{i-1}, T_i), \tag{B4}$$

$$b_2 := N(u, T_{i-1}, T_i)\ b_3 := \frac{1}{2}O(u, T_{i-1}, T_i). \tag{B5}$$

where A_{HW}, B_{HW} are defined in (A11) and (A12). The characteristic function solution (B2) can be obtained by

$$\begin{aligned} L(u,t,T) &= -\frac{1}{2}u(i+u)V(t,T) \\ &+ \int_t^T\left[(\kappa\psi + \rho_{r\nu}(iu-1)\tau\sigma B_{\mathrm{HW}}(s,T))N(s) + \frac{1}{2}\tau^2(N^2(s) + O(s))\right]ds, \end{aligned} \tag{B6}$$

$$M(u,t,T) = iu, \tag{B7}$$

$$N(u,t,T) = -u(i+u)\frac{((\delta_3 - \delta_4 e^{-2\delta(T-t)}) - (\delta_5 e^{-a(T-t)} - \delta_6 e^{-(2\delta+a)(T-t)}) - \delta_7 e^{-\delta(T-t)})}{\delta_1 + \delta_2 e^{-2\delta(T-t)}}, \tag{B8}$$

$$O(u,t,T) = -u(i+u)\frac{1 - e^{-2\delta(T-t)}}{\delta_1 + \delta_2 e^{-2\delta(T-t)}}, \tag{B9}$$

with $V(t, T)$ as in (A13) and where

$$\delta = \sqrt{(\kappa - \rho_{S\nu}\tau iu)^2 + \tau^2 u(i+u)},\ \delta_1 = \delta + (\kappa - \rho_{S\nu}\tau iu), \tag{B10}$$

$$\delta_2 = \delta - (\kappa - \rho_{S\nu}\tau iu),\ \delta_3 = \frac{\rho_{Sr}\sigma\delta_1 + \kappa a\psi + \rho_{r\nu}\sigma\tau(iu - 1)}{a\delta},$$

$$\delta_4 = \frac{\rho_{Sr}\sigma\delta_2 - \kappa a\psi - \rho_{r\nu}\sigma\tau(iu - 1)}{a\delta},\ \delta_5 = \frac{\rho_{Sr}\sigma\delta_1 + \rho_{r\nu}\sigma\tau(iu - 1)}{a(\delta - a)},$$

$$\delta_6 = \frac{\rho_{Sr}\sigma\delta_2 - \rho_{r\nu}\sigma\tau(iu - 1)}{a(\delta + a)},\ \delta_7 = (\delta_3 - \delta_4) - (\delta_5 - \delta_6).$$

Finally, we provide the moments of the pair $\nu(T_{i-1})$, $x(T_{i-1})$ under the T_i-forward measure, which (conditional on time t) follows a bivariate normal distribution with means μ_ν, μ_x, variances σ_ν^2, σ_x^2 and correlation $\rho_{x\nu}(t, T_{i-1})$, respectively, given by

$$\mu_\nu = \nu(t)e^{-\kappa(T_{i-1}-t)} + \left(\psi - \frac{\rho_{r\nu}\sigma\tau}{a\kappa}\right)(1 - e^{-\kappa(T_{i-1}-t)}) - \frac{\rho_{r\nu}\sigma\tau}{a(\kappa + a)}(e^{-a(T_i-t)-\kappa(T_{i-1}-t)} - e^{-a(T_i-T_{i-1})}), \tag{B11}$$

$$\sigma_\nu^2 = \frac{\tau^2}{2\kappa}(1 - e^{-2\kappa(T_{i-1}-t)}), \tag{B12}$$

$$\mu_x = x(t)e^{-a(T_{i-1}-t)} - M^{T_i}(t, T_{i-1}), \tag{B13}$$

$$\sigma_x^2 = \frac{\sigma^2}{2a}(1 - e^{-2a(T_{i-1}-t)}), \tag{B14}$$

$$\rho_{x\nu}(t, T_{i-1}) = \frac{\rho_{r\nu}\sigma\tau}{\sigma_x\sigma_\nu(a + \kappa)}[1 - e^{-(a+\kappa)(T_{i-1}-t)}]. \tag{B15}$$

APPENDIX C: IMPACT OF THE RATE-ASSET CORRELATION COEFFICIENT ON THE FORWARD STARTING OPTIONS

Impact of the rate-asset correlation ρ_{Sr} on the (forward) implied volatility structure for different underlying call option maturities is given in Figure C1.

APPENDIX D: IMPACT OF THE RATE-VOLATILITY CORRELATION COEFFICIENT ON THE FORWARD STARTING OPTIONS

Impact of the rate-volatility correlation $\rho_{r\nu}$ on the (forward) implied volatility structure for different underlying call option maturities is given in Figure D1.

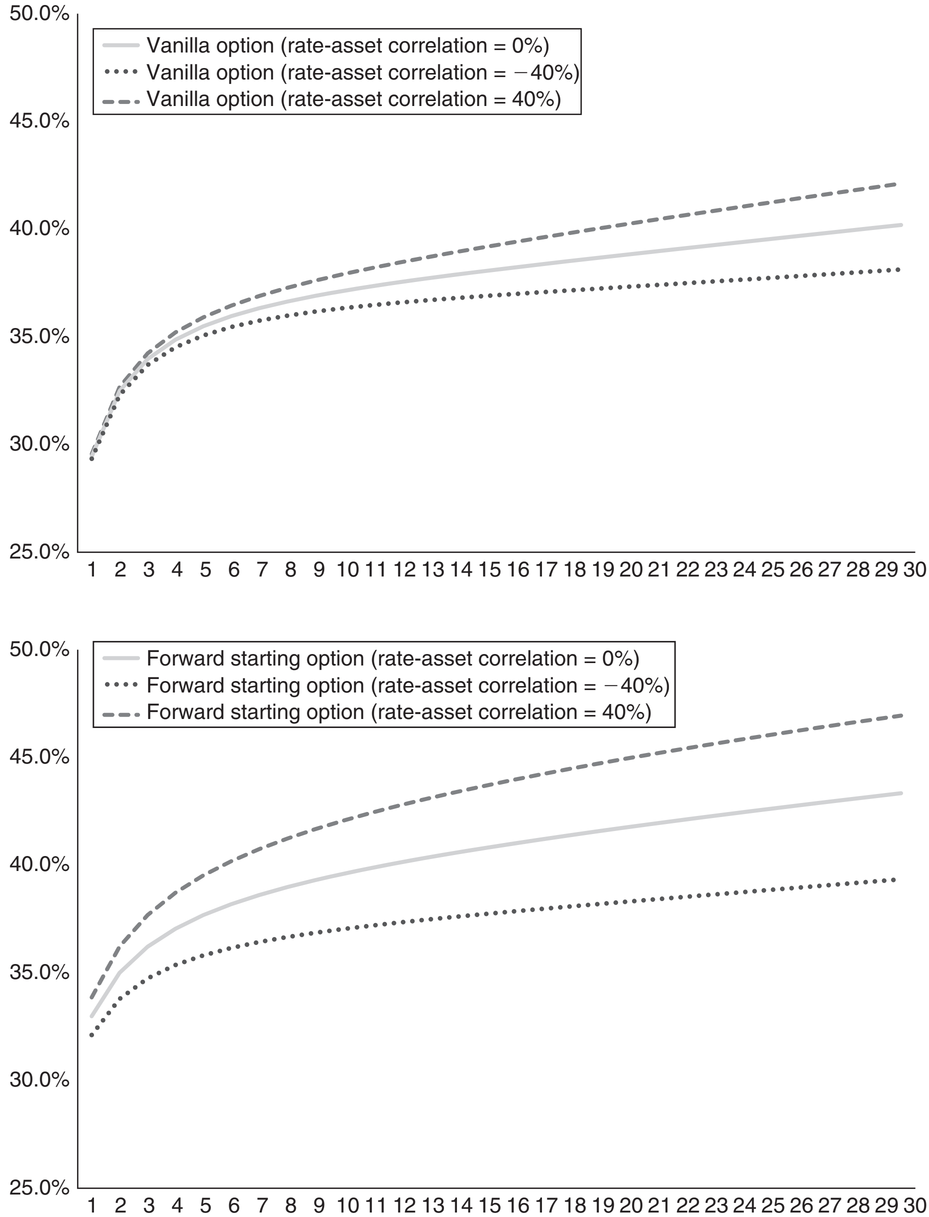

FIGURE C1

Impact of the rate-asset correlation ρ_{Sr} on the (forward) implied volatility structure for different underlying call option maturities. Parameters are $\kappa = 1.0$, $= \nu(t) = \psi = 0.20$, $\tau = 0.5$, $a = 0.02$, $\sigma = 0.01$, $= \rho_{S\nu} = -0.70$, $\rho_{r\nu} = 0$ and $P(t, s) = \exp(-0.04 = (s > t))$ for all $s > t$. The top figure shows the impact of this correlation on the volatilities of the current time (vanilla) options, whereas the bottom figure plots these volatility structures for forward starting call options with strike determination date $T_1 = 10$ year.

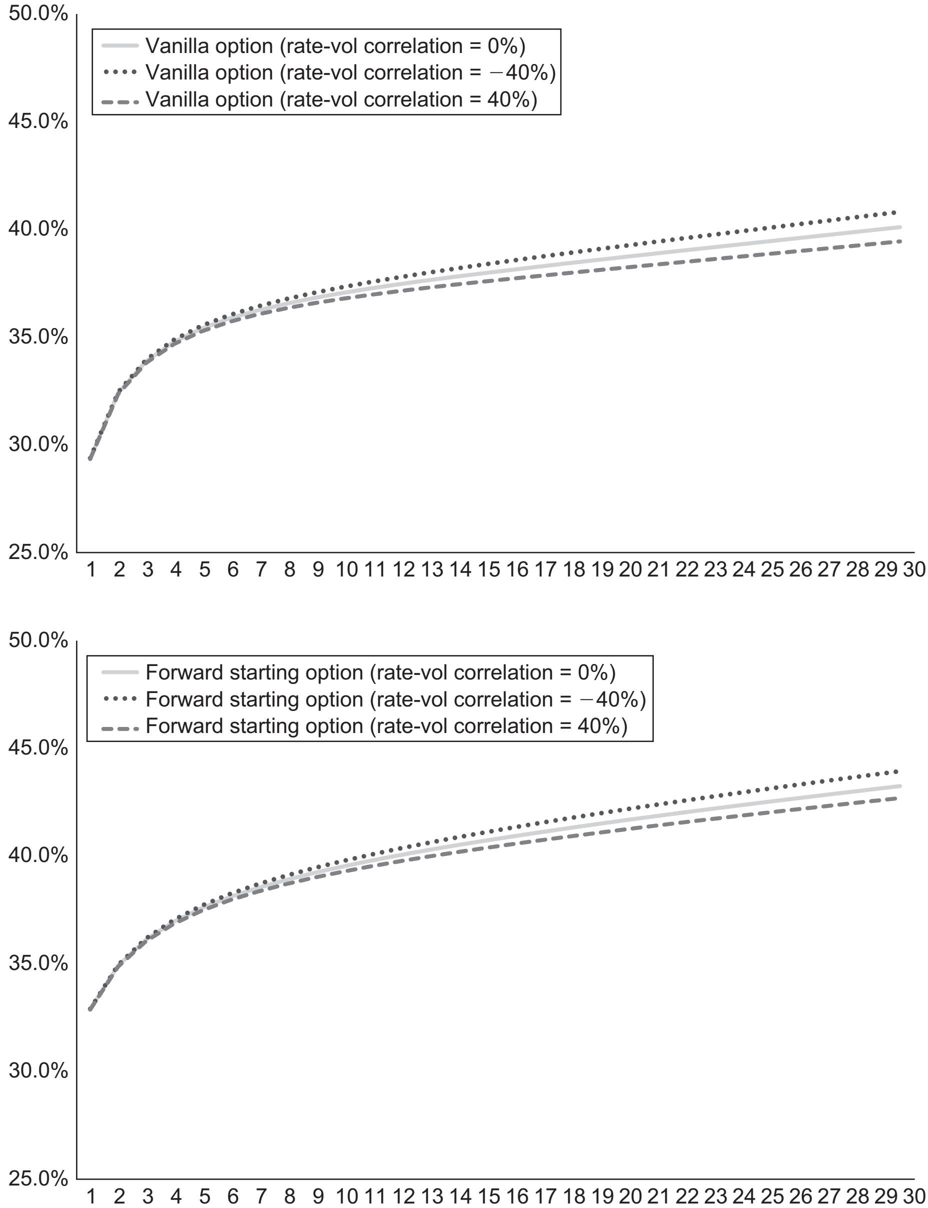

FIGURE D1

Impact of the rate-volatility correlation $\rho_{r\nu}$ on the (forward) implied volatility structure for different underlying call option maturities. Parameters are $\kappa = 1.0, = \nu(t) = \psi = 0.20, \tau = 0.5, a = 0.02, \sigma = 0.01, = \rho_{S\nu} = -0.70, \rho_{r\nu} = 0$ and $P(t, s) = \exp(-0.04 = (s - t))$ for all $s > t$. The top figure graphs the impact of this = correlation on the volatilities of the current time (vanilla) options, whereas the bottom figure plots these volatility structures for forward starting call options with strike determination date $T_1 = 10$ year.

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