

# Time-consistent and market-consistent evaluations

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## TIME-CONSISTENT AND MARKET-CONSISTENT EVALUATIONS

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We consider evaluation methods for payoffs with an inherent financial risk as encountered for instance for portfolios held by pension funds and insurance companies. Pricing such payoffs in a way consistent to market prices typically involves combining actuarial techniques with methods from mathematical finance. We propose to extend standard actuarial principles by a new market-consistent evaluation procedure which we call “two-step market evaluation.” This procedure preserves the structure of standard evaluation techniques and has many other appealing properties. We give a complete axiomatic characterization for two-step market evaluations. We show further that in a dynamic setting with continuous stock prices every evaluation which is time-consistent and market-consistent is a two-step market evaluation. We also give characterization results and examples in terms of  $g$ -expectations in a Brownian-Poisson setting.

KEY WORDS: actuarial valuation principles, financial risk, market-consistency, time-consistency.

### 1. INTRODUCTION

We investigate evaluation methods for payoffs with an inherent financial risk and propose a new market-consistent procedure to evaluate these payoffs. Our procedure yields the extension of many standard actuarial principles into both time-consistent and market-consistent directions.

Many insurance companies sell products which depend on financial as well as non-financial risk. Typical examples are equity-linked insurance contracts or catastrophe insurance futures and bonds. Pricing such payoffs in a way consistent to market prices usually involves combining actuarial techniques with methods from mathematical finance. The minimal conditions which any market-consistent evaluation should satisfy is that a purely financial replicable payoff should be equal to the amount necessary to replicate it.

Standard actuarial premium principles are typically based on a pooling argument which justifies applying the law of large numbers to price using the expectation with

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respect to the physical measure and then to take an additional risk load. With these principles one usually considers a static premium calculation problem: what is the price today of an insurance contract with payoff at time  $T$ ? See for example the textbooks by Bühlmann (1970), Gerber (1979), or Kaas et al. (2008). Also, the concept of convex risk measures and the closely related one of monetary utility functions have been studied in such a static setting. See for example Föllmer and Schied (2002), Frittelli and Rosazza Gianin (2002), Jouini, Schachermayer, and Touzi (2008), and Filipovic and Kupper (2008).

In financial pricing one usually considers a “dynamic” pricing problem: how does the price evolve over time until the final payoff date  $T$ ? This dynamic perspective is driven by the focus on hedging and replication. This literature started with the seminal paper of Black and Scholes (1973) and has been immensely generalized to broad classes of securities and stochastic processes; see Delbaen and Schachermayer (1994).

In recent years, researchers have begun to investigate risk measures in a dynamic setting, the central question being the construction of time-consistent (“dynamic”) risk measures. See Riedel (2004), Roorda, Schumacher, and Engwerda (2005), Cheridito, Delbaen, and Kupper (2006), Rosazza Gianin (2006), Artzner et al. (2007). In a dynamical context *time-consistency* is a natural approach to glue together static evaluations. It means that the same value is assigned to a financial position regardless of whether it is calculated over two time periods at once or in two-steps backwards in time. In a recent paper Jobert and Rogers (2008) show how time-consistent valuations can be constructed via backward induction of static one-period risk-measures (or “valuations”). See also Hardy and Wirth (2005) for an example with the risk measure given by Average Value at Risk.

An important branch of literature considers risk measures/valuations in a so-called market-consistent setting. This started with the pricing of contracts in an incomplete-market setting, where one seeks to extend the arbitrage-free pricing operators (which are only defined in a complete-market setting) to the larger space of (partially) unhedgeable contracts. One approach to evaluate the payoff in this situation is by utility indifference pricing: the investor pays the amount such that he is no worse off in expected utility terms than he would have been without the claim. The paper by Hodges and Neuberger (1989) is often cited for the root-idea of this stream of literature. For other contributions in this direction, see for instance Henderson (2002), Young and Zariphopoulou (2002), Hobson (2004), Musiela and Zariphopoulou (2004a), Monoyios (2006), and the recent book by Carmona (2009).

Several papers deal with the extension of the arbitrage-free pricing operators using (local) risk-minimization techniques and the related notion of minimal martingale measures; see Föllmer and Schweizer (1989), Schweizer (1995), Delbaen and Schachermayer (1996). A rich duality theory has been developed that establishes deep connections between utility maximization and minimization over martingale measures; see Cvitanic and Karatzas (1992), Kramkov and Schachermayer (1999). A very elegant summary is given by Rogers (2001). Another stream of the literature, where the class of martingale measures considered is restricted, is given by the works on good-deal bound pricing, see Cochrane and Saá-Requejo (2000), Černý and Hodges (2002), and Björk and Slinko (2006).

Using utility-indifference (and duality) methods, the market-consistency of pricing operators is automatically induced. However, an explicit formal definition of market-consistent pricing operators has only begun to emerge recently; see Kupper, Cheridito, and Filipovic (2008), Malamud, Trubowitz, and Wüthrich (2008), Barrieu and El Karoui (2005, 2009), and Knispel, Stahl, and Weber (2011).

In this paper we investigate well-known actuarial premium principles such as the variance principle and the standard-deviation principle, and study their extension into both time-consistent and market-consistent directions. To do this, we introduce the concept of *two-step market evaluations* and study their properties. Two-step market evaluations convert any evaluation principle into a market-consistent one by applying the actuarial principle to the residual risk which remains after having conditioned on the future development of the stock price. This operator splitting preserves the structure and the computationally tractability of the original actuarial evaluation. Furthermore, we get some appealing properties like numeraire invariance. We are able to give a complete axiomatic characterization for two-step market evaluations and show that these axioms are satisfied in a setting where the stock process is continuous and the insurance process is revealed at fixed time instances (or more generally has predictable jumps). This provides a strong argument for the use of two-step market evaluations. We also consider some time-consistent extensions of our market-consistent evaluations to continuous time in a Brownian-Poisson setting. For this we need some results from the theory of backward stochastic differential equations (BSDEs), also called *g*-expectations. For background material on BSDEs we refer to El Karoui, Peng, and Quenez (1997).

The paper is organized as follows: In Section 2 we define conditional evaluations, give some background material, and recall some of the most standard actuarial principles. In Section 3 the notion of market-consistency is defined and two-step market evaluations are introduced and motivated. We give a complete axiomatic characterization for two-step market evaluations. In Section 4 it is shown that in a dynamic setting with continuous stock prices every evaluation which is time-consistent and market-consistent can be viewed as a two-step market evaluation. In Section 5 we extend our evaluations to a continuous-time setup with processes with jumps. Section 6 gives a summary and conclusions. Section 7 contains the proofs of our results.

## 2. CONDITIONAL EVALUATIONS

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Equalities and inequalities between random variables are understood in the  $\mathbb{P}$ -almost sure sense unless explicitly stated otherwise. The space of bounded random variables will be denoted by  $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$  ( $L^\infty(\mathcal{F})$  for short). The space of bounded, nonnegative random variables will be denoted by  $L_+^\infty(\mathcal{F})$ . The space of random variables which are integrable with respect to  $\mathbb{P}$  will be denoted by  $L^1(\Omega, \mathcal{F}, \mathbb{P})$  ( $L^1(\mathcal{F})$  for short). *Financial and insurance positions* are represented by random variables  $H \in L^\infty(\mathcal{F})$  where  $H(\omega)$  is the discounted net loss of the position at maturity under the scenario  $\omega$ . Now given a  $\sigma$ -algebra  $\mathcal{G} \subset \mathcal{F}$ , with information available to the agent, we can define a *conditional evaluation*:

**DEFINITION 2.1.** A mapping  $\Pi_{\mathcal{G}} : L^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})$  is called a  $\mathcal{G}$ -conditional evaluation if the following axioms hold:

- Normalization:  $\Pi_{\mathcal{G}}(0) = 0$ .
- $\mathcal{G}$ -Cash invariance:  $\Pi_{\mathcal{G}}(H + m) = \Pi_{\mathcal{G}}(H) + m$  for  $H \in L^\infty(\mathcal{F})$  and  $m \in L^\infty(\mathcal{G})$ .
- $\mathcal{G}$ -Convexity: For  $H_1, H_2 \in L^\infty(\mathcal{F})$   $\Pi_{\mathcal{G}}(\lambda H_1 + (1 - \lambda)H_2) \leq \lambda \Pi_{\mathcal{G}}(H_1) + (1 - \lambda)\Pi_{\mathcal{G}}(H_2)$  for all  $\lambda \in L^\infty(\mathcal{G})$  with  $0 \leq \lambda \leq 1$ .
- $\mathcal{G}$ -Local property:  $\Pi_{\mathcal{G}}(I_A H_1 + I_{A^c} H_2) = I_A \Pi_{\mathcal{G}}(H_1) + I_{A^c} \Pi_{\mathcal{G}}(H_2)$  for all  $H \in L^\infty(\mathcal{F})$  and  $A \in \mathcal{G}$ .

- **Fatou property:** For any bounded sequence  $(H_n)$  which converges a.s. to  $H$

$$\Pi_{\mathcal{G}}(H) \leq \liminf_n \Pi_{\mathcal{G}}(H_n).$$

Normalization guarantees that the null position does not require any capital reserves. If  $\Pi$  is not normal then the agent can consider the operator  $\Pi(H) - \Pi(0)$  without changing his preferences. Convexity, which under cash invariance is equivalent to quasiconvexity, says that diversification should not be penalized. Cash invariance gives the interpretation of  $\Pi(H)$  as a capital reserve. The local property is motivated in the following way. Since the agent has the information given by  $\mathcal{G}$  he knows if the event  $A$  has happened or not and should adjust his evaluation accordingly. If  $\Pi$  satisfies

- **Monotonicity:** For  $H_1, H_2 \in L^\infty(\mathcal{F})$  with  $H_1 \leq H_2$   $\Pi_{\mathcal{G}}(H_1) \leq \Pi_{\mathcal{G}}(H_2)$

then we will call  $\Pi$  a monotone conditional evaluation. Monotonicity postulates that if in a.s. any scenario  $H_2$  causes a greater higher loss than  $H_1$  then the premium charged for  $H_2$  (or the capital reserve held) should be greater than for  $H_1$ . Note that if  $\Pi_{\mathcal{G}}$  is monotone then  $\rho_{\mathcal{G}}(H) := \Pi_{\mathcal{G}}(-H)$  defines a conditional convex risk measure and  $U_{\mathcal{G}}(H) := -\rho_{\mathcal{G}}(H)$  defines a conditional monetary utility function. For the definition of a convex risk measure, see Föllmer and Schied (2002) or Frittelli and Rosazza Gianin (2002). In particular, all results in this paper also hold (with obvious change of signs) for conditional convex risk measures and conditional monetary utility functions.

**REMARK 2.2.** It has been pointed out to us by Patrick Cheridito that the local property is also implied by  $\mathcal{G}$ -convexity. Indeed, if  $\Pi_{\mathcal{G}}$  is  $\mathcal{G}$ -convex, then we have for  $A \in \mathcal{G}$  that clearly  $\Pi_{\mathcal{G}}(1_A H_1 + 1_{A^c} H_2) \leq 1_A \Pi_{\mathcal{G}}(H_1) + 1_{A^c} \Pi_{\mathcal{G}}(H_2)$ . In particular,  $1_A \Pi_{\mathcal{G}}(1_A H_1 + 1_{A^c} H_2) \leq 1_A \Pi_{\mathcal{G}}(H_1)$ . The other direction follows by setting  $\tilde{H} = 1_A H_1 + 1_{A^c} H_2$ . Then as before

$$1_A \Pi_{\mathcal{G}}(H_1) = 1_A \Pi_{\mathcal{G}}(1_A \tilde{H} + 1_{A^c} H_1) \leq 1_A \Pi_{\mathcal{G}}(\tilde{H}).$$

Switching the role of  $H_1$  and  $H_2$  yields then the desired conclusion.

Other possible axioms which we will consider in a dynamic setting are as follows:

- **$\mathcal{G}$ -positive homogeneity:** For  $H \in L^\infty(\mathcal{F})$   $\Pi_{\mathcal{G}}(\lambda H) = \lambda \Pi_{\mathcal{G}}(H)$  for all  $\lambda \in L_+^\infty(\mathcal{G})$ .
- **Continuity:** For any bounded sequence  $(H_n)$  which converges a.s. to  $H$

$$\Pi_{\mathcal{G}}(H) = \lim_n \Pi_{\mathcal{G}}(H_n).$$

- **$p$ -norm boundedness:** There exists  $p \in (1, \infty)$ ,  $\lambda \in L_+^\infty(\mathcal{G})$ , and a measure  $\bar{\mathbb{P}}$  having the same zero sets as  $\mathbb{P}$  such that for  $H \in L^\infty(\mathcal{F})$

$$\Pi_{\mathcal{G}}(H) \leq \lambda \int (|H| + |H|^p) d\bar{\mathbb{P}}_{\mathcal{G}}.$$

We will also refer to the continuity axiom as “continuity with respect to a.s. bounded convergence,” if there is any ambiguity. If  $\Pi$  is a  $\mathcal{G}$ -conditional evaluation which is additionally assumed to be positively homogeneous then we call  $\Pi$  a  $\mathcal{G}$ -conditional *coherent* evaluation. For a further discussion of these axioms see also Artzner et al. (1999). Note that many similar axioms for premium principles can be found in the literature, see for instance Deprez and Gerber (1985) or Kaas et al. (2008). Conditional evaluations in a dynamic setting have been considered for instance in Frittelli and Gianin

(2004), Roorda et al. (2005), Ruszczynski and Shapiro (2006), Delbaen (2006), Artzner et al. (2007), Klöppel and Schweizer (2007), Jobert and Rogers (2008), Barrieu and El Karoui (2009), and Cheridito and Kupper (2011). Classical examples of (conditional) evaluations are, see for instance Kaas et al. (2008):

EXAMPLE 2.3.

- Conditional mean-variance principle:

$$\Pi_{\mathcal{G}}^v(H) = \mathbb{E}_{\mathcal{G}}[H] + \frac{1}{2}\alpha \text{Var}_{\mathcal{G}}[H], \quad \alpha \geq 0.$$

- Conditional standard-deviation principle:

$$\Pi_{\mathcal{G}}^{st}(H) = \mathbb{E}_{\mathcal{G}}[H] + \beta\sqrt{\text{Var}_{\mathcal{G}}[H]}, \quad \beta \geq 0.$$

- Conditional semideviation principle:

$$\Pi_{\mathcal{G}}^s(X) = \mathbb{E}_{\mathcal{G}}[H] + \lambda|\mathbb{E}_{\mathcal{G}}[(H - \mathbb{E}_{\mathcal{G}}[H])_+]^q|^{1/q}, \quad \lambda \geq 0, \quad q \in [1, \infty),$$

where  $x_+$  is 0 if  $x < 0$  and  $x$  else.

- Conditional average value at risk principle:

$$\Pi_{\mathcal{G}}^{AV@R}(H) = \mathbb{E}_{\mathcal{G}}[H] + \delta AV@R_{\mathcal{G}}^{\alpha}(H - \mathbb{E}_{\mathcal{G}}[H]), \quad \delta \geq 0$$

where  $AV@R_{\mathcal{G}}^{\alpha}(H) = \frac{1}{\alpha} \int_0^{\alpha} V@R_{\mathcal{G}}^{\lambda}(H) d\lambda$ ,  $\alpha \in (0, 1]$  and  $V@R_{\mathcal{G}}^{\lambda}(H)$  corresponds to computing the Value at Risk of  $H$  at the confidence level  $\lambda$  with the available information  $\mathcal{G}$ .

- Conditional exponential principle:

$$\Pi_{\mathcal{G}}^v(H) = \gamma \log(\mathbb{E}_{\mathcal{G}}[\exp\{H/\gamma\}]), \quad \gamma > 0.$$

Apart from the exponential principle the evaluations above are generally not monotone. However, they are continuous,  $\mathcal{G}$ -convex,  $\mathcal{G}$ -cash invariant and satisfy the local property. In particular, they are  $\mathcal{G}$ -conditional evaluations. The standard-deviation principle, the average value at risk principle and the semideviation principle additionally satisfy  $\mathcal{G}$ -positive homogeneity while the mean-variance principle is  $p$ -norm bounded (with  $\bar{\mathbb{P}} = \mathbb{P}$ ). The average value at risk principle and the semideviation principle are monotone if  $\lambda$  or  $\delta$  are in  $[0, 1]$ .

We will need some duality results. For a  $\sigma$ -algebra  $\mathcal{G} \subset \mathcal{F}$ , denote  $\mathcal{Q}_{\mathcal{G}} = \{\xi \in L^1(\mathcal{F}) | \mathbb{E}_{\mathcal{G}}[\xi] = 1\}$  and  $\mathcal{Q}_{\mathcal{G}}^+ := \{\xi \in L_+^1(\mathcal{F}) | \mathbb{E}_{\mathcal{G}}[\xi] = 1\}$ . In other words, given the information  $\mathcal{G}$ ,  $\mathcal{Q}_{\mathcal{G}}$  is the set of all signed measures and  $\mathcal{Q}_{\mathcal{G}}^+$  is the set of all probability measures. Therefore, conditional on our starting information  $\mathcal{G}$ , we may identify every  $\xi \in \mathcal{Q}_{\mathcal{G}}$  with a signed measure, and every  $\xi \in \mathcal{Q}_{\mathcal{G}}^+$  with a probability measure. For instance, for  $\xi \in \mathcal{Q}_{\mathcal{G}}^+$  we can define the corresponding conditional probability measure  $\mathbb{Q}_{\mathcal{G}}^{\xi}(A) := \mathbb{E}_{\mathcal{G}}[\xi I_A]$ , and its conditional density as  $\frac{d\mathbb{Q}_{\mathcal{G}}^{\xi}}{d\mathbb{P}_{\mathcal{G}}} := \xi$ .

Recall that by standard duality results we have that  $\Pi_{\mathcal{G}}$  is coherent if and only if

$$(2.1) \quad \Pi_{\mathcal{G}}(H) = \text{ess sup}_{\xi \in M} \mathbb{E}_{\mathcal{G}}[\xi H],$$

for a unique closed, convex set  $M \subset \mathcal{Q}_{\mathcal{G}}$ . For the precise definition of the essential supremum, see the Appendix.  $\xi$  is often interpreted as a weighting function for the

different scenarios  $\omega$ , or as a test or stress measure. By taking the supremum, a worst-case approach is being taken. For instance in the good-deal bound literature mentioned in the introduction the supremum is taken over all pricing kernels with a density admitting a variance smaller than a certain constant.

Generally it holds for all conditional evaluations that for  $\Pi_{\mathcal{G}}^*$  defined by

$$\Pi_{\mathcal{G}}^*(\xi) = \text{ess sup}_{H \in L^\infty(\mathcal{F})} \{\mathbb{E}_{\mathcal{G}}[\xi H] - \Pi_{\mathcal{G}}(H)\}$$

we have that

$$(2.2) \quad \Pi_{\mathcal{G}}(H) = \text{ess sup}_{\xi \in \mathcal{Q}_{\mathcal{G}}} \{\mathbb{E}_{\mathcal{G}}[\xi H] - \Pi_{\mathcal{G}}^*(\xi)\},$$

see for instance Delbaen (2006) or Cheridito and Kupper (2011).  $\Pi_{\mathcal{G}}^*$  is also called the penalty function corresponding to  $\Pi_{\mathcal{G}}$ , and  $-\Pi_{\mathcal{G}}^*(\xi)$  may be seen as the plausibility of the density  $\xi$ . The dual representations (2.1) and (2.2) are often interpreted as robust expectations with respect to different priors, see Hansen and Sargent (2001), or Hansen and Sargent (2007).

Of course our results also hold without conditioning on  $\mathcal{G}$ . In this case  $\mathcal{G}$  is chosen to be the trivial  $\sigma$ -algebra. However, for our dynamic analysis in the later sections it will simplify matters if we do our analysis conditional on some information available to the agent.

### 3. MARKET-CONSISTENT PRICING

#### 3.1. Market-Consistency and Two-Step Market Evaluations

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be the underlying probability space. Let  $\mathcal{G} \subset \mathcal{F}$  be a  $\sigma$ -algebra whose information is initially available to the agent. Let  $S = (S^1, \dots, S^m)$  be the  $n$ -dimensional price process of  $n$  traded stocks and denote by  $\tilde{\mathcal{F}}^S \subset \mathcal{F}$  the  $\sigma$ -algebra generated by  $S$ . Furthermore, we denote by  $\mathcal{F}^S$  the  $\sigma$ -algebra given by the stock process and our starting information  $\mathcal{G}$ , i.e.,  $\mathcal{F}^S := \tilde{\mathcal{F}}^S \vee \mathcal{G} := \sigma(\tilde{\mathcal{F}}^S, \mathcal{G})$ . The financial market given by the  $n$ -dimensional stock process  $S$  should be arbitrage free and complete, i.e., all derivatives which conditional on  $\mathcal{G}$  only depend on  $S$  are perfectly hedgeable and there exists a unique probability measure  $\mathbb{Q}_{\mathcal{G}} \in \mathcal{Q}_{\mathcal{G}}^+$  such that  $S$  is a martingale under  $\mathbb{Q}_{\mathcal{G}}$  (componentwise). Furthermore,  $\mathbb{Q}_{\mathcal{G}}$  is assumed to be a.s. equivalent to  $\mathbb{P}_{\mathcal{G}}$ , in the sense that its conditional density is positive.

Since  $\mathcal{F}^S$  is in general a strict subset of  $\mathcal{F}$ , the market given by all  $\mathcal{F}$ -measurable payoffs is incomplete. For instance, we could have an untraded insurance process which is correlated with the traded assets  $S$  but not perfectly replicable.

In the financial market the martingale measure  $\mathbb{Q}_{\mathcal{G}}$  defines the (linear) no-arbitrage pricing operator  $\Pi_{\mathcal{G}}^f : L^\infty(\mathcal{F}^S) \rightarrow L^\infty(\mathcal{G})$  given by

$$\Pi_{\mathcal{G}}^f(H^S) := \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[H^S] := \int_{\Omega} H^S(\omega) \mathbb{Q}_{\mathcal{G}}(d\omega) = \mathbb{E}_{\mathcal{G}}[\xi^{\mathbb{Q}_{\mathcal{G}}} H],$$

where  $\xi^{\mathbb{Q}_{\mathcal{G}}}$  is the density in  $\mathcal{Q}_{\mathcal{G}}^+$  with which  $\mathbb{Q}_{\mathcal{G}}$  may be identified. Note that only the martingale measure on  $\mathcal{F}^S$  is unique while on the filtration  $\mathcal{F}$  there can be infinitely many martingale measures.



The next definition extends the notion of cash invariance to all assets traded in the financial market. For identical or similar notions of market-consistency see also Cont (2006), Kupper et al. (2008), Malamud et al. (2008), or Artzner and Eisele (2010).

**DEFINITION 3.1.** An evaluation is called market-consistent if for any financial payoff  $H^S \in L^\infty(\mathcal{F}^S)$  and  $H \in L^\infty(\mathcal{F})$

$$\Pi_G(H^S + H) = \mathbb{E}^{\mathbb{Q}_G}[H^S] + \Pi_G(H).$$

Market-consistency postulates that liquidly traded assets and payoffs replicable by them should not carry any risk as they can be converted to cash at any time. It follows immediately from the definition that a market-consistent evaluation cannot be “improved” by hedging.

**REMARK 3.2.** Our definition of market-consistency requires that we have liquidly traded assets constituting a complete financial market. If the financial market is not complete then there are two possibilities which may still validate our approach:

- (1) One could use certain financial derivatives as additional hedging instruments to make the financial market complete. For results in this direction, see Jacod and Protter (2010) and the references therein. Note that in our setting  $S$  has to be finite-dimensional. Therefore, only finitely many additional hedging instruments are allowed. However, many stochastic volatility models like the Heston model can be completed in this way.
- (2) One could remove certain financial assets as possible hedging instruments. In some cases the remaining assets might constitute a complete financial market.

The next proposition shows that market-consistency is already implied by the assumption that *purely* hedgeable derivatives should be valued according to the amount of capital necessary to replicate them. Furthermore, it shows that, in case that  $\Pi_G$  is monotone, market-consistency is equivalent with the no-arbitrage principle in the entire market.

**PROPOSITION 3.3.** For a conditional evaluation  $\Pi_G : L^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})$  the following statements are equivalent:

- (i)  $\Pi_G(H^S) = \mathbb{E}^{\mathbb{Q}_G}[H^S]$  for any financial payoff  $H^S \in L^\infty(\mathcal{F}^S)$ .
- (ii) There exists a penalty function<sup>1</sup>  $c_G : \mathcal{Q}_{\mathcal{F}^S} = \{Z \in L^1(\mathcal{F}) | \mathbb{E}_{\mathcal{F}^S}[Z] = 1\} \rightarrow \mathbb{R} \cup \{\infty\}$  such that we have for every  $H \in L^\infty(\mathcal{F})$

$$\Pi_G(H) = \text{ess sup}_{Z \in \mathcal{Q}_{\mathcal{F}^S}} \{\mathbb{E}^{\mathbb{Q}_G}[ZH] - c_G(Z)\}.$$

- (iii)  $\Pi_G$  is market-consistent.

Furthermore, in the case that  $\Pi_G$  is additionally assumed to be monotone, market-consistency is equivalent with any payoff being evaluated between its super- and subreplication price.

In particular, in the case of monotonicity, we could have also defined market-consistency by stating that the evaluation of every payoff should respect the no-arbitrage principle.

<sup>1</sup>A function  $c$  is called a penalty function if it is convex, lower semicontinuous, and  $\text{ess inf } c = 0$ .



Examples of market-consistent evaluations arise if an agent *starts* with a (usually non-market-consistent) evaluation  $\Pi_{\mathcal{G}}$  and then tries to reduce his risk by hedging. Specifically denote by  $\mathcal{A}$  all admissible hedging strategies  $\pi$  (defined in an appropriate way), by  $\mathcal{M}_{\mathcal{G}} \subset \mathcal{Q}_{\mathcal{G}}^+ := \mathcal{Q}_{\mathcal{G}} \cap L_+^1(\mathcal{F})$  the set of all local martingale measures, by  $\tilde{S}$  the discounted stock process, and by  $\Pi_{\mathcal{G}}^*$  the penalty function corresponding to  $\Pi_{\mathcal{G}}$ . It can be shown, see for instance Barrieu and El Karoui (2009), or Toussaint and Sircar (2011), that under appropriate assumptions

$$(3.1) \quad \bar{\Pi}_{\mathcal{G}}(H) := \operatorname{ess\,inf}_{\pi \in \mathcal{A}} \bar{\Pi}_{\mathcal{G}}(H + (\pi \cdot \tilde{S})_T) = \operatorname{ess\,sup}_{\mathbb{P}_{\mathcal{G}} \in \mathcal{M}_{\mathcal{G}}} \left\{ \bar{\mathbb{E}}_{\mathcal{G}}[H] - \Pi_{\mathcal{G}}^* \left( \frac{d\bar{\mathbb{P}}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} \right) \right\}.$$

Note that by definition  $\bar{\Pi}_{\mathcal{G}}$  is market-consistent. Therefore, one way of obtaining market-consistent evaluations is to intersect the test measures in the dual representations (2.1) and (2.2) above with local martingale measures. This is a class of examples which arises naturally when starting with an evaluation  $\Pi_{\mathcal{G}}$ . However, the new evaluation  $\bar{\Pi}_{\mathcal{G}}$  may be hard to compute, and the structure and interpretation of the original evaluation  $\Pi_{\mathcal{G}}$  is lost. For instance, if payoffs are evaluated using the mean-variance principle then  $\bar{\Pi}_{\mathcal{G}}(H) = \operatorname{ess\,sup}_{\mathbb{P}_{\mathcal{G}} \in \mathcal{M}_{\mathcal{G}}} \{ \bar{\mathbb{E}}_{\mathcal{G}}[H] - \frac{1}{2\alpha} C_{\mathcal{G}}(\bar{\mathbb{P}}_{\mathcal{G}} | \mathbb{P}_{\mathcal{G}}) \}$ , where  $C_{\mathcal{G}}$  is the relative Gini index defined by  $C_{\mathcal{G}}(\bar{\mathbb{P}}_{\mathcal{G}} | \mathbb{P}_{\mathcal{G}}) = \mathbb{E}_{\mathcal{G}}[(\frac{d\bar{\mathbb{P}}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}})^2 - 1]$ . For an overview about mean-variance hedging, see Schweizer (2010) and the reference therein. Now note that two important reasons for the popularity of the mean-variance principle are: (a) it has a straightforward interpretation; (b) it is easy to compute. However, while the new evaluation,  $\bar{\Pi}_{\mathcal{G}}(H)$  is market-consistent (since it uses risk adjusted probabilities), it is neither easy to compute nor does it directly relate to the variance of the payoff  $H$  from which we started.

Consequently, in this paper we propose a new class of market-consistent evaluations which we will call two-step market evaluations. Extending standard actuarial principles with two-step market evaluations will have the advantage that the extensions can be computed easily and that the interpretation of our starting principles can be preserved. Furthermore, we will show that two-step market evaluations in general have many other appealing properties. A strong argument for the use of two-step market evaluations will be provided in the section 4. There we show that any insurance company which wants to apply a market-consistent and time-consistent evaluation, has to use a two-step market evaluation, in a setting where the stock process is continuous and the insurance process is only revealed at fixed time instances (or more generally at predictable stopping times).

We will start with evaluations like the one from our Examples 2.3. Then we will give the corresponding market-consistent evaluations which do not arise from hedging but from operator splitting. Namely, in a first step we compute the value of the position  $H$  by replacing the measure  $\mathbb{P}|_{\mathcal{G}}$  with the measure  $\mathbb{P}|_{\mathcal{F}^S}$ , i.e., we compute mean-variance principle, the standard-deviation principle, etc., of  $H$ , conditional on  $\mathcal{G}$  and the values of the stocks  $S$ . Then for every different future value of the stock price we get a different evaluation. However, since payoff depending only on the stock prices are perfectly hedgeable one could argue that these remaining evaluations do not contain any risk. Therefore, the total value of the position  $H$  should be equal to the initial capital needed to hedge the different evaluations, obtained in the first step, which depend on  $S$ . This corresponds to taking in a second step the expectation with respect to the risk adjusted probability measure  $\mathbb{Q}_{\mathcal{G}}$  coming from the financial market. This procedure is computationally tractable and preserves the evaluation principles considered in the beginning.

These principles are applied to the risk which remains after having conditioned on  $S$ . We then get the following market-consistent examples:

EXAMPLE 3.4.

- Two-step mean-variance principle:

$$\Pi_G^v(H) = \mathbb{E}^{\mathbb{Q}_G} \left[ \mathbb{E}_{\mathcal{F}^S}[H] + \frac{1}{2} \alpha \text{Var}_{\mathcal{F}^S}[H] \right], \quad \alpha \geq 0.$$

- Two-step standard-deviation principle:

$$\Pi_G^s(H) = \mathbb{E}^{\mathbb{Q}_G} [\mathbb{E}_{\mathcal{F}^S}[H] + \beta \sqrt{\text{Var}_{\mathcal{F}^S}[H]}], \quad \beta \geq 0.$$

- Two-step semideviation principle:

$$\Pi_G^s(X) = \mathbb{E}^{\mathbb{Q}_G} \left[ \mathbb{E}_{\mathcal{F}^S}[H] + \lambda \left| \mathbb{E}_{\mathcal{F}^S}[(H - \mathbb{E}_{\mathcal{F}^S}[H])_+^q] \right|^{1/q} \right], \quad \lambda \geq 0, \quad q \in [1, \infty).$$

- Two-step average value at risk principle:

$$\Pi_G^{AV@R}(H) = \mathbb{E}^{\mathbb{Q}_G} [\mathbb{E}_{\mathcal{F}^S}[H] + \delta AV@R_{\mathcal{F}^S}^\alpha(H - \mathbb{E}_{\mathcal{F}^S}[H])], \quad \delta \geq 0.$$

- Two-step exponential principle:

$$\Pi_G(H) = \mathbb{E}^{\mathbb{Q}_G} [\gamma \log(\mathbb{E}_{\mathcal{F}^S}[\exp\{H/\gamma\}])], \quad \gamma > 0.$$

A standard deviation principle, which is different than the one above but is also obtained by first conditioning on the future stock price, is considered in Møller (2002). The last example is known in the literature as the indifference price of  $H$  under an exponential utility function. It arises in an incomplete market when an agent maximizes his exponential utility through dynamic trading, see for instance Musiela and Zariphopoulou (2004b). The indifference price for a claim  $H$  is then defined as the amount of cash the agent is willing to pay for the right to receive  $H$  such that he is no worse off in expected utility terms than he would have been without the claim. For references on indifference pricing see the introduction. The examples above motivate the following definition:

DEFINITION 3.5. We call a  $\Pi_G : L^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})$  a two-step market evaluation if there exists an  $\mathcal{F}^S$ -conditional valuation  $\Pi_{\mathcal{F}^S} : L^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F}^S)$  such that

$$(3.2) \quad \Pi_G(H) = \mathbb{E}^{\mathbb{Q}_G} [\Pi_{\mathcal{F}^S}(H)].$$

Note that in case that there is no financial market, i.e.,  $S = 0$ , our two-step evaluations reduce to the standard actuarial principles. On the other hand if  $\mathcal{F}^S = \mathcal{F}$ , i.e., if the financial market gives the entire filtration, of course  $\Pi_G(H) = \mathbb{E}^{\mathbb{Q}_G}[H]$ .

EXAMPLE 3.6. Another example for a two-step evaluation defined above arises when combining hedging with an average value at risk principle. Specifically set

$$\Pi_G(H) := \text{ess inf}_{\pi \in \mathcal{A}} \Pi_G^{AV@R}(H + (\pi \cdot \tilde{S})_T).$$

Define

$$M := \left\{ \bar{\mathbb{Q}}_G \in \mathcal{M}_G \left| 1 - \delta \leq \frac{d\bar{\mathbb{Q}}_G}{d\mathbb{P}_G} \leq 1 + \delta \frac{1 - \alpha}{\alpha} \right. \right\},$$

where  $\delta > 0$  and the risk level  $\alpha \in (0, 1]$  have been introduced in Examples 2.3. (Note that if  $\delta = 0$  then  $\Pi_G^{AV@R}(H) = \mathbb{E}_G[H]$ .)

From (3.1) and the dual representation of the average-value at risk principle (see for instance Föllmer and Schied 2004) it follows that

$$\Pi_G(H) = \text{ess sup}_{\tilde{\mathbb{Q}}_G \in \mathcal{M}} \mathbb{E}^{\tilde{\mathbb{Q}}_G}[H].$$

Let

$$M' := \left\{ Z \in \mathcal{Q}_{\mathcal{F}^S}^+ \mid (1 - \delta) \frac{d\mathbb{P}_G}{d\mathbb{Q}_G} \leq Z \leq \frac{d\mathbb{P}_G}{d\mathbb{Q}_G} \left( 1 + \delta \frac{1 - \alpha}{\alpha} \right) \right\}.$$

Note that  $\frac{d\mathbb{Q}_G}{d\mathbb{P}_G} M' = M$ . Furthermore,  $M'$  satisfies the concatenation property on  $\mathcal{F}^S$ , also called rectangular property or  $m$ -stability.<sup>2</sup> It may be seen that as a result

$$\Pi_{\mathcal{F}^S}(H) := \text{ess sup}_{Z \in M'} \mathbb{E}_{\mathcal{F}^S}[ZH]$$

is an  $\mathcal{F}^S$ -conditional evaluation and it holds that

$$\Pi_G(H) = \mathbb{E}^{\mathbb{Q}_G} [\Pi_{\mathcal{F}^S}(H)],$$

compare also with Theorem 3.10 below. In particular, combining hedging with the average-value at risk principle gives an example of the two-step procedure we explained above.

**EXAMPLE 3.7.** Our last example is given by the super-replication price of a contingent claim. The super-replication price is given by

$$\Pi_G(H) := \text{ess sup}_{\tilde{\mathbb{Q}}_G \in \mathcal{M}_G} \mathbb{E}^{\tilde{\mathbb{Q}}_G}[H],$$

It is straightforward to check that  $\mathcal{M}_G = \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} \mathcal{Q}_{\mathcal{F}^S}^+$ . Clearly,

$$\Pi_{\mathcal{F}^S}(H) := \text{ess sup}_{Z \in \mathcal{Q}_{\mathcal{F}^S}^+} \mathbb{E}_{\mathcal{F}^S}[ZH]$$

is an  $\mathcal{F}^S$ -conditional evaluation. It computes the essential supremum conditional on  $\mathcal{F}^S$ . Since  $\mathcal{Q}_{\mathcal{F}^S}^+$  is  $m$ -stable it can be seen that

$$\Pi_G(H) = \mathbb{E}^{\mathbb{Q}_G} [\Pi_{\mathcal{F}^S}(H)].$$

In particular, the super-replication price is a two-step evaluation.

**REMARK 3.8.** Note that equity linked insurance payoffs are typically of the form  $H = Y_T^{(n)} f(S_T)$  where  $f(S_T)$  is a financial derivative and  $Y_T^{(n)}$  are the number of policy holder who survived up to time  $T$  (out of an initial cohort of  $n$ ), see for instance Møller (2002). In the special case that the financial and the mortality risk are independent a two-step market evaluation would yield

$$\Pi_G(H) = \mathbb{E}^{\mathbb{Q}_G} [\Pi_{\mathcal{F}^S}(f(S_T) Y_T^{(n)})] = \mathbb{E}^{\mathbb{Q}_G}[f(S_T)] \Pi_G(Y_T^{(n)}).$$

<sup>2</sup>The rectangular property or  $m$ -stability on a  $\sigma$ -algebra  $\tilde{\mathcal{G}}$  postulates that for every  $A \in \mathcal{F}^S$  and  $Z_1, Z_2 \in M'$  we have that  $I_A Z_1 + I_{A^c} Z_2 \in M'$ , see for instance Chen and Epstein (2002) or Delbaen (2006).

Note that the structure obtained in this special case is similar to the market-consistent valuation method suggested in chapter 2.6 in Wüthrich, Bühlmann, and Furrer (2008).

Two-step market valuations provide a rich class of market-consistent evaluations with a clear underlying intuition. They appear as indifference-price with an exponential utility, in hedging with an average value at risk principle, and in the super-replication price of an contingent claim. Two-step market valuations are also useful in optimization problems since the maximum may be computed by a two-step procedure using Bellman's principle: first one can compute a value function conditioned on the stock process, and then in a second step one can compute the optimum by maximizing the value function obtained in Step 1 under the pricing measure.

Note that contrary to other evaluations two-step market evaluations can be directly converted into an equivalent evaluation which takes the stock process as the numeraire. That is, assume payoffs  $\tilde{H}$  are expressed in units of the  $i$ -th stock,  $S^i$  for  $i \in \{1, \dots, n\}$ , i.e.,  $\tilde{H} = H/S_T^i$ . An agent who wants to use the evaluation  $\Pi_G$  but wants to express everything in units of  $S^i$  obviously should use the evaluation  $\tilde{\Pi}_G$  which satisfies  $S_0^i \tilde{\Pi}_G(\tilde{H}) = \Pi_G(H)$ . The evaluation  $\tilde{\Pi}_G$  often might not be easy to characterize directly, for instance with a dual representation and a penalty function. However, a two-step market evaluation remains a two-step market evaluation under the change of numeraire. Moreover, one can just define the new penalty function of  $\tilde{\Pi}_{\mathcal{F}^S}$  as the penalty function of  $\Pi_{\mathcal{F}^S}$  in units of  $S_T^i$ . That is we set

$$\tilde{\Pi}_{\mathcal{F}^S}^*(\xi) := \frac{\Pi_{\mathcal{F}^S}^*(\xi)}{S_T^i} \text{ and } \tilde{\Pi}_{\mathcal{F}^S}(\tilde{H}) := \text{ess sup}_{\xi \in L^1(\mathcal{F})} \{ \mathbb{E}_{\mathcal{F}^S}[\tilde{H}] - \tilde{\Pi}_{\mathcal{F}^S}^*(\xi) \}.$$

Denote by  $\tilde{\mathbb{Q}}_G$  the unique equivalent martingale measure on  $\mathcal{F}^S$  with numeraire  $S^i$  that is

$$\frac{d\tilde{\mathbb{Q}}_G}{d\mathbb{P}_G} := \frac{S_T^i}{S_0^i} \frac{d\mathbb{Q}_G}{d\mathbb{P}_G}.$$

Then we obtain

$$\tilde{\Pi}_G(\tilde{H}) = \frac{\Pi_G(H)}{S_0^i} = \mathbb{E}^{\mathbb{Q}_G} \left[ \frac{S_T^i}{S_0^i} \text{ess sup}_{\xi \in L^1(\mathcal{F})} \{ \mathbb{E}_{\mathcal{F}^S}[\tilde{H}] - \tilde{\Pi}_{\mathcal{F}^S}^*(\xi) \} \right] = \mathbb{E}^{\tilde{\mathbb{Q}}_G} [\tilde{\Pi}_{\mathcal{F}^S}(\tilde{H})].$$

We will summarize the last paragraph in the following proposition:

**PROPOSITION 3.9.** *If  $\Pi_G$  is a two-step market evaluation then  $\tilde{\Pi}_G$  is a two-step market evaluation as well. Furthermore, the penalty function of  $\tilde{\Pi}_{\mathcal{F}^S}$  is given by the penalty function of  $\Pi_{\mathcal{F}^S}$  converted into units of stock  $i$ .*

Note that for evaluations which are not two step an easy conversion of the penalty function as given in the last proposition usually only works if the numeraire is deterministic.

It is not hard to see that a two-step market valuation is *always* market-consistent, see the theorems below. On the other hand our Example 3.12 below shows that a market-consistent valuation is not necessarily a two-step market valuation.

### 3.2. An Axiomatic Characterization of Two-Step Market Evaluations

Notice that the Examples 3.4 all satisfy the *market local property*, i.e.:

- For every  $A \in \mathcal{F}^S$  and  $H \in L_+^\infty(\mathcal{F})$

$$(3.3) \quad \Pi_{\mathcal{G}}(H) = \Pi_{\mathcal{G}}(I_A H) + \Pi_{\mathcal{G}}(I_{A^c} H).$$

The binary options  $I_A H$  and  $I_{A^c} H$  can be seen as insurances against the events  $A$  or  $A^c$ , respectively. For example if  $A$  happens then the owner of the option  $I_A H$  gets a nonnegative amount  $H$  which possibly depends on the insurance process. If the evaluation  $\Pi_{\mathcal{G}}$  is assumed to be sublinear, the value of the insurance contract  $I_A H$  plus the value of the insurance contract  $I_{A^c} H$  should be larger than the value of  $H = I_A H + I_{A^c} H$ . The economic reason is that if the valuation of  $H$  is decomposed into the sum of the evaluations of the binary insurance contracts then the additional uncertainty given by the event  $A$  for each binary contract, should lead to an increase of the total insurance premium. The local property however postulates that the uncertainty added to the payoff  $H$  by an event from the financial market should not carry any extra premium. We will see in Section 4 that, in a setting with a correlated stock and insurance process, the local property is satisfied for a time-consistent and market-consistent evaluation if the stock process is continuous and the value of the insurance process for is revealed at fixed time instances (or more generally at predictable stopping times).

Now the question in which we are interested in the remainder of this section, is the following: Given a conditional evaluation  $\Pi_{\mathcal{G}}$  does then its market-consistency and the market local property imply that every position  $H$  has to be priced with a two-step market valuation? Our results below will actually show that this statement holds.

The following theorem shows that market-consistency and the market local property in the coherent case are equivalent to two-step market evaluations. Furthermore, it gives an explicit formula for  $\Pi_{\mathcal{F}^S}$ .

**THEOREM 3.10.** *The following statements are equivalent:*

- $\Pi_{\mathcal{G}}$  is a coherent market-consistent  $\mathcal{G}$ -conditional evaluation which satisfies the market local property.
- There exists an  $\mathcal{F}^S$ -conditional coherent evaluation  $\Pi_{\mathcal{F}^S} : L^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F}^S)$  such that  $\Pi_{\mathcal{G}}(H) = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^S}(H)]$ . Furthermore,  $\Pi_{\mathcal{F}^S}(H) = \text{ess sup}_{Z \in M} \mathbb{E}_{\mathcal{F}^S}[ZH]$ , with

$$(3.4) \quad M' := \left( \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} \right)^{-1} M \subset \{Z \in L^1(\mathcal{F}) | \mathbb{E}_{\mathcal{F}^S}[Z] = 1\} = \mathcal{Q}_{\mathcal{F}^S},$$

and  $M$  given by (2.1).

The assumptions of the theorem are satisfied for all our examples above except for the mean-variance and the exponential market-consistent principles. The reason is that these do not satisfy positive homogeneity. However, in the case that the filtration  $\mathcal{F}^S$  is generated by countably many sets, and  $\Pi_{\mathcal{G}}$  is continuous, and monotone or  $p$ -norm bounded, we can prove that an evaluation has to be a two-step market evaluation *without the assumption of positive homogeneity*.

**THEOREM 3.11.** *Suppose that  $\mathcal{F}^S$  is generated by countably many sets. Then the following statements are equivalent:*

- $\Pi_{\mathcal{G}}$  is a monotone, continuous market-consistent  $\mathcal{G}$ -conditional evaluation which satisfies the market local property.

- There exists a monotone, continuous  $\mathcal{F}^S$ -conditional evaluation  $\Pi_{\mathcal{F}^S} : L^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F}^S)$  such that

$$\Pi_{\mathcal{G}}(H) = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}} [\Pi_{\mathcal{F}^S}(H)].$$

Furthermore, if  $\Pi_{\mathcal{G}}$  is additionally assumed to be  $p$ -norm bounded then the statement also holds without the monotonicity assumption in (i) and (ii).

The mean-variance principle is not monotone but it is  $p$ -norm bounded. In particular, Theorems 3.10–3.11 include all our examples. Theorem 3.11 further applies to the exponential principle, and the average value at risk and the semi-deviation principle if  $\lambda$  and  $\delta$  only take values between 0 and 1.

The proof of the last theorem heavily relies on the assumed continuity of our evaluations which was not needed in Theorem 3.10. The evaluation  $\Pi_{\mathcal{F}^S}$  will be obtained only as an a.s. limit using the martingale convergence theorem without an explicit formula.

Note that the market-consistency of an evaluation  $\Pi$  does not necessarily imply that we get a representation as in Theorems 3.10 and 3.11, since there are market-consistent evaluations not satisfying the market local property. This can be seen from the following example:

**EXAMPLE 3.12.** Assume that  $\mathcal{G}$  is trivial and let  $Z_1$  and  $Z_2$  be densities independent of  $S$  with  $Z_1 \neq Z_2$ . Then  $Z_i$  are also independent of  $\frac{d\mathbb{Q}}{d\mathbb{P}}$ . Now suppose that the agent is not sure if he should trust the density  $\frac{d\mathbb{Q}}{d\mathbb{P}} Z_1$  or  $\frac{d\mathbb{Q}}{d\mathbb{P}} Z_2$ . Therefore, he decides to take a worst-case approach over all convex combinations of  $\frac{d\mathbb{Q}}{d\mathbb{P}} Z_1$  or  $\frac{d\mathbb{Q}}{d\mathbb{P}} Z_2$ . That is

$$(3.5) \quad \Pi(H) := \max_{\mathbb{P} \in M} \bar{\mathbb{E}}[H] = \max_{i=1,2} \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} Z_i H \right],$$

with  $M = \{\lambda \frac{d\mathbb{Q}}{d\mathbb{P}} Z_1 + (1-\lambda) \frac{d\mathbb{Q}}{d\mathbb{P}} Z_2 | \lambda \in [0, 1]\}$ . It is straightforward to check using the independence of  $Z_i$  and  $S$  that  $\Pi$  is a market-consistent, coherent evaluation. Let  $H \geq 0$  be  $\mathcal{F}$ - but not  $\mathcal{F}^S$ -measurable. Assume without loss of generality that the maximum in (3.5) is attained in  $i = 1$ . Now choose a set  $A \in \mathcal{F}^S$  such that

$$(3.6) \quad \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} Z_2 I_A H \right] > \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} Z_1 I_A H \right].$$

Since the maximum in (3.5) is attained in  $i = 1$  we must have that

$$\mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} Z_2 I_{A^c} H \right] < \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} Z_1 I_{A^c} H \right].$$

But then we get

$$(3.7) \quad \begin{aligned} \Pi(HI_A) + \Pi(HI_{A^c}) &= \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} Z_2 I_A H \right] + \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} Z_1 I_{A^c} H \right] \\ &> \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} Z_1 I_A H \right] + \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} Z_1 I_{A^c} H \right] = \Pi(H). \end{aligned}$$

One possible choice for  $A$  and  $H$  would be to define  $H = I_A I_{\{Z_2 > Z_1\}} + I_{A^c} I_{\{Z_1 > Z_2\}}$ . Then (3.6) always holds if  $A$  is chosen to be a nonzero set with  $A \subset \{Z_2 > Z_1\}$ . Furthermore, one can ensure that  $\Pi(H) = \mathbb{E}[Z_1 H]$  by choosing  $A$  such that  $\mathbb{Q}(A^c)$  is sufficiently close to one. In particular, (3.7) holds and therefore  $\Pi$  defined in (3.5) does not satisfy the

market local property. By the direction (ii)  $\Rightarrow$  (i) of Theorem 3.10 this implies that  $\Pi$  is not a two-step market evaluation.

#### 4. DYNAMIC EVALUATIONS IN CONTINUOUS TIME: TIME-CONSISTENCY AND MARKET-CONSISTENCY

In this section we will give sufficient conditions in which the market local property holds in a *dynamic* setting. Specifically we will obtain that in certain dynamic settings time-consistency and market-consistency imply that all evaluations have to be two-step market evaluations. Time-consistency in a dynamic setting often has strong implications. For instance, for general preferences the indifference price of an agent with time-consistent dynamic preferences are recursive if and only if the preferences are cash-invariant, see theorem 3.4, Cheridito and Kupper (2009).

Subsequently, we fix a finite time horizon  $T > 0$ . Throughout the rest of the paper we assume that additional to the stocks  $(S_t)_{0 \leq t \leq T} = ((S_t^1, \dots, S_t^n))_{0 \leq t \leq T}$ , we have an untraded insurance process  $(Y_t)_{0 \leq t \leq T}$ . For the sake of simplicity let us assume that the insurance process is one-dimensional. (The generalization is straightforward.) Let  $\tilde{\mathcal{F}}^S$  be filtration generated by  $S$ , and let  $\tilde{\mathcal{F}}^Y$  be the filtration generated by  $Y$ . We again assume that the financial market is complete while the entire market is incomplete. Denote by  $\mathbb{Q}$  the unique martingale measure on  $\tilde{\mathcal{F}}_T^S$  with density  $\frac{d\mathbb{Q}}{d\mathbb{P}}$ .

Define the total information which is available as  $\mathcal{F} := \tilde{\mathcal{F}}^S \vee \tilde{\mathcal{F}}^Y := \sigma(\tilde{\mathcal{F}}^S \cup \tilde{\mathcal{F}}^Y)$ . Setting  $\mathbb{Q}(A) := \mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}} I_A]$  for  $A \in \mathcal{F}_T$ , we can extend  $\mathbb{Q}$  canonically to the whole filtration. We call a collection of mappings  $(\Pi_\sigma)_{0 \leq \sigma \leq T}$ , a continuous-time *dynamic evaluation* if it has the following properties:

- For all stopping times  $\sigma$ ,  $\Pi_\sigma$  is an  $\mathcal{F}_\sigma$ -conditional evaluation.
- *Time-Consistency*: For every  $H \in L^\infty(\mathcal{F}_T)$ :

$$\Pi_\sigma(H) = \Pi_\sigma(\Pi_\tau(H)) \text{ for all stopping times } \sigma \leq \tau.$$

In a dynamical context time-consistency is a natural assumption to glue together static risk measures. It means that the same risk is assigned to a financial position regardless of whether it is calculated over two time periods at once or in two-steps backwards in time. For a general analysis of weaker notions of time-consistency see, e.g., Roorda et al. (2005).

**REMARK 4.1.** Alternative names for our definition of time-consistency would have been “recursiveness” or “tower property.” Note that in the literature often the following notion of time-consistency is used: if an asset  $H_1$  is preferred to an asset  $H_2$  under all possible scenarios at some time  $\tau$  then  $H_1$  should also have been preferred at every time  $\sigma$  before  $\tau$ . Let us call the latter definition of time-consistency property (TC). Now if  $\Pi$  is monotone it is well known that our notion of time-consistency is in fact equivalent to property (TC). Furthermore, since (TC) implies the monotonicity of  $\Pi$ , our notion of time-consistency includes (TC) as a special case, but it can also be applied to nonmonotone evaluations.

Time-consistent evaluations have been discussed in continuous time by Peng (2004), Frittelli and Gianin (2004), Rosazza Gianin (2006), Delbaen (2006), Klöppel and Schweizer (2007), Bion-Nadal (2008), Bion-Nadal (2009), and Barrieu and El



Karoui (2009). Duffie and Epstein (1992), Chen and Epstein (2002), and Maccheroni, Marinacci, and Rustichini (2006) deal with dynamic preferences using similar notions of time-consistency.

Given a dynamic evaluation  $(\Pi_\sigma)$  we define  $\Pi_{\sigma,\tau}$  to be equal to  $\Pi_\sigma$  restricted to  $L^\infty(\mathcal{F}_\tau)$ , i.e.,  $\Pi_{\sigma,\tau} = \Pi_\sigma|_{L^\infty(\mathcal{F}_\tau)}$ . Next we will assume that the insurance process  $Y$  a.s. *jumps only at finitely many predictable time instances*, say  $0 \leq \tau_1 \leq \tau_2 \leq \dots$  and there is no additional randomness added between the jumps, i.e.,  $\bar{\mathcal{F}}_{\tau_{i+1}-}^Y = \bar{\mathcal{F}}_{\tau_i}^Y$  for all  $i$ . Recall that a stopping time  $\tau$  is predictable if there exists a sequence of stopping times  $\tau^n < \tau$  such that  $\tau^n \uparrow \tau$ . Hitting times of continuous processes are predictable while jump times of Lévy processes (or more general, strong Markov Feller processes) are not. On the set where  $Y$  does not jump at all we set  $\tau_1 = T$ .<sup>3</sup> One example could be given by a setting in which the insurance process  $Y$  is only updated at finitely many *fixed* time instances,  $t_1 < \dots < t_k$ . Another possibility could be that damages occur at *unpredictable* stopping times  $\bar{\tau}_i$ , but the amount of money the insurance will have to pay is not clear right away. Instead the insurance needs some additional time, say  $\varepsilon > 0$ , to agree to a certain amount and to pay it out at  $\bar{\tau}_i + \varepsilon$ , respectively.

We will use the following definition in a dynamic setting:

**DEFINITION 4.2.** We will say that a conditional continuous-time evaluation  $(\Pi_\sigma)_{\sigma \in [0, T]}$  is market-consistent if for every stopping time  $\sigma$  and every financial payoffs  $H^S \in L^\infty(\bar{\mathcal{F}}_T^S \vee \mathcal{F}_\sigma)$  and  $H \in L^\infty(\mathcal{F}_T)$

$$\Pi_\sigma(H^S + H) = \mathbb{E}_{\mathcal{F}_\sigma}^{\mathbb{Q}}[H^S] + \Pi_\sigma(H).$$

Note that this definition of market-consistency coincides with the definition in the static case with  $\mathcal{G} = \mathcal{F}_\sigma$  and  $\bar{\mathcal{F}}^S = \bar{\mathcal{F}}_T^S$ .

Now for every stopping time  $\sigma$  we define  $\tau_\sigma$  to be the time of the next jump after  $\sigma$ , i.e.,  $\tau_\sigma := \inf\{t > \sigma \mid \Delta Y_t > 0\} \wedge T$ . Denote  $\mathcal{F}_{\tau_\sigma}^S := \bar{\mathcal{F}}_{\tau_\sigma}^S \vee \mathcal{F}_\sigma$ . That is  $\mathcal{F}_{\tau_\sigma}^S$  is the  $\sigma$ -algebra which includes all the information (of both stock and insurance process) up to time  $\sigma$  and additionally the information of the stock process up to the time of the next jump of  $Y$ .

**THEOREM 4.3.** Suppose that  $S$  is continuous and the insurance process is as described above. Let  $(\Pi_\sigma)_{0 \leq \sigma \leq T}$  be a time-consistent and market-consistent evaluation such that for every  $\sigma$ ,  $\Pi_\sigma(\cdot)$  is continuous. Then for every stopping time  $\sigma$ , we have that  $\Pi_{\sigma,\tau_\sigma}$  satisfies the market-local property. In particular, if  $(\Pi_\sigma)$  is additionally assumed to be either monotone,  $p$ -norm bounded, or positively homogeneous then for every stopping time  $\sigma$  there exists an  $\mathcal{F}_{\tau_\sigma}^S$ -conditional evaluation  $\Pi_{\mathcal{F}_{\tau_\sigma}^S} : L^\infty(\mathcal{F}_{\tau_\sigma}) \rightarrow L^\infty(\mathcal{F}_{\tau_\sigma}^S)$  such that

$$\Pi_{\sigma,\tau_\sigma}(H) = \mathbb{E}_{\mathcal{F}_\sigma}^{\mathbb{Q}}[\Pi_{\mathcal{F}_{\tau_\sigma}^S}(H)].$$

This theorem shows in particular that in a setting where the agent just observes the insurance process at finitely many time instances, *every* market-consistent and time-consistent evaluation has to admit a representation of the form  $\mathbb{E}_{\mathcal{F}_\sigma}^{\mathbb{Q}}[\Pi_{\mathcal{F}_{\tau_\sigma}^S}(H)]$  at every stopping time  $\sigma$ . In other words, an agent who wants to use a time-consistent and market-consistent evaluation *has to* apply a two-step market evaluation.

Theorem 4.3 also yields the following corollary:

<sup>3</sup>Similarly we can set some  $\tau_i(\omega)$  equal to  $T$  if not all paths,  $(Y_t(\omega))_t$ , have the same number of jumps.

**COROLLARY 4.4.** *Suppose that  $S$  is continuous, and that the insurance process  $Y$  a.s. jumps only at finitely many fixed time instances, say  $0 \leq t_1 \leq t_2 \leq \dots \leq t_k$ , and  $\bar{\mathcal{F}}_{t_{k+1}-}^Y = \bar{\mathcal{F}}_{t_k}^Y$ . If  $(\Pi_\sigma)_{0 \leq \sigma \leq T}$  is a time-consistent and market-consistent evaluation which is either monotone,  $p$ -norm bounded or positively homogeneous then for every  $s \in [0, T]$  there exists an  $\mathcal{F}_{t_i}^S$ -conditional evaluation  $\Pi_{\mathcal{F}_{t_i}^S} : L^\infty(\mathcal{F}_{t_i}) \rightarrow L^\infty(\mathcal{F}_{t_i}^S)$  such that*

$$\Pi_{s, t_i}(H) = \mathbb{E}_{\mathcal{F}_s}^{\mathbb{Q}}[\Pi_{\mathcal{F}_{t_i}^S}(H)],$$

where  $t_i$  is the next time instance after  $s$  at which the insurance process jumps.

We have restricted our analysis to payoffs rather than payment streams. However, if  $H$  is a discrete payment streams which pays cash amounts  $(H_{\eta_1}, \dots, H_{\eta_k})$  at stopping times  $\eta_1 < \eta_2 < \dots < \eta_k$ , we could consider evaluations  $\Pi_\sigma$  mapping payment streams, starting paying amounts from time  $\sigma$  on, to  $L^\infty(\mathcal{F}_\sigma)$ . In this case time-consistency could be defined as

$$\Pi_\sigma(H) = \Pi_\sigma(HI_{[0, \tau)}) + \Pi_\tau(HI_{[\tau, T]}) \quad \text{for all } \sigma \leq \tau,$$

see also Cheridito et al. (2006), or Jobert and Rogers (2008). By a proof analogue to the one for Theorem 4.3 one can then show that for an evaluation satisfying similar properties as above time-consistency and market-consistency entail

$$\Pi_{\sigma, \tau_\sigma}(HI_{[\sigma, \tau_\sigma]}) = \mathbb{E}_{\mathcal{F}_\sigma}^{\mathbb{Q}}[\Pi_{\mathcal{F}_{\tau_\sigma}^S}(HI_{[\sigma, \tau_\sigma]})] = \mathbb{E}_{\mathcal{F}_\sigma}^{\mathbb{Q}}[\Pi_{\mathcal{F}_{\tau_\sigma}^S}(H_{\tau_\sigma})] + \mathbb{E}_{\mathcal{F}_\sigma}^{\mathbb{Q}}\left[\sum_{\sigma \leq \eta < \tau_\sigma} H_\eta\right].$$

## 5. MARKET-CONSISTENT EVALUATIONS IN CONTINUOUS TIME

### 5.1. Results on Market-Consistent BSDEs

In a continuous time Brownian-Poisson setting we can provide examples of time-consistent and market-consistent evaluations by  $g$ -expectations. It is well known that  $g$ -expectations induce time-consistent evaluations, see for instance Peng (2004), Frittelli and Gianin (2004), Rosazza Gianin (2006), Bion-Nadal (2008), Barrieu and El Karoui (2009), or El Karoui and Ravanelli (2009). In this section we will give a complete characterization of  $g$ -expectations which are market-consistent.

Suppose the filtration  $\mathcal{F}$  is generated by the following independent processes: an  $n$ -dimensional standard Brownian motion  $W^f$ , a  $d$ -dimensional standard Brownian motion  $W$ , and a Poisson random counting measure  $N(ds, dx)$  defined on  $[0, T] \times \mathbb{R} \setminus \{0\}$ . We denote the corresponding compensator by

$$\hat{N}(ds, dx) = \nu(dx) ds.$$

We assume that the measure  $\nu(dx)$  is nonnegative and satisfies

$$\int_{\mathbb{R} \setminus \{0\}} (|x|^2 \wedge 1) \nu(dx) < \infty.$$

Denote  $\tilde{N}(ds, dx) := N(ds, dx) - \hat{N}(ds, dx)$ .

Suppose that we have a bond  $B$  with  $B_0 = 1$  and  $dB_t = rB_t dt$ . We assume that the stock process  $S = (S^1, \dots, S^n)$  is given by

$$dS_t^i = S_t^i(\mu^i(t, S_t) dt + \tilde{\sigma}^i(t, S_t) dW_t^f), \quad S_0^i = s_0^i,$$

with  $s_0^i > 0$  and  $\mu^i : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\tilde{\sigma}^i : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  for  $i = 1, \dots, n$ . Note that we used vector notation for the stochastic integral. Define the vector  $\mu = (\mu^i)_{i=1, \dots, n}$  taking values in  $\mathbb{R}^n$ . The rows of the matrix  $\tilde{\sigma} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  should be given by  $(\tilde{\sigma}^i)_{i=1, \dots, n}$ . We assume that  $\mu$  and  $\tilde{\sigma}$  are uniformly bounded. Furthermore,  $\tilde{\sigma}$  should be invertible, and uniformly elliptic, i.e., there exists  $K_1, K_2 > 0$  such that  $K_1 I_n \leq \tilde{\sigma} \tilde{\sigma}^\top \leq K_2 I_n$ . Furthermore, we need standard measurability and Lipschitz continuity assumptions on  $\mu$  and  $\tilde{\sigma}$ . Then it is well known that a solution  $S$  for the SDE above exists and that the corresponding financial market, consisting of  $(S, B)$ , is complete. Generally payoffs can depend on  $(W^f, W, \tilde{N})$  and may not be replicable.

Let  $\tilde{\mathcal{F}}^S$  be the filtration generated by  $S$ . Denote the market price of uncertainty by  $\theta_t = \tilde{\sigma}^{-1}(t, S_t)(\mu(t, S_t) - re)$ , where  $e$  is an  $n$ -dimensional vector consisting of ones. As in the sections before we will denote by  $H$  hedgeable and unhedgeable discounted payoffs.

We will consider evaluations of  $H$  given by the solutions of BSDEs. Denote by  $\mathcal{P}$  the predictable  $\sigma$ -algebra on the entire filtration. Let

$$\mathcal{H}_m^2 := \left\{ Z = (Z^1, \dots, Z^n) \in \mathcal{P} \mid \mathbb{E} \left[ \int_0^T |Z_s|^2 ds \right] < \infty \right\},$$

where we denote by  $|\cdot|$  the Euclidean norm.

Let  $\mathcal{S}^2$  be the space of all one-dimensional optional processes whose path-maximum is square integrable with respect to  $\mathbb{P}$ . Let  $L^2(\nu(dx))$  be the space all  $\mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable functions mapping from  $\mathbb{R} \setminus \{0\}$  to  $\mathbb{R}$ , which are square integrable with respect to  $\nu$ , where, as usual, two functions are identified if they are equal  $\nu$  a.s. Define  $L^2(\nu(dx) \times d\mathbb{P} \times ds)$  as all  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable functions which are square-integrable with respect to  $\nu(dx) \times d\mathbb{P} \times ds$ . Now suppose that we have a suitably measurable function  $g : [0, T] \times \Omega \times \mathbb{R}^2 \times L^2(\nu(dx)) \rightarrow \mathbb{R}$ .

A solution of the BSDE with driver  $g(t, \omega, z^f, z, \tilde{z})$  and terminal condition  $H \in L^\infty(\mathcal{F}_T)$  is a quadruple of processes  $(Y(H), Z^f, Z, \tilde{Z}) \in \mathcal{S}^2 \times \mathcal{H}_n^2 \times \mathcal{H}_d^2 \times L^2(\nu(dx) \times d\mathbb{P} \times ds)$  such that

$$dY_t(H) = -g(t, Z_t^f, Z_t, \tilde{Z}_t) dt + Z_t^f dW_t^f + Z_t dW_t + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_t(x) \tilde{N}(dt, dx) \quad \text{and}$$

$$Y_T(H) = H.$$

Often BSDEs are also written in the following equivalent form:

$$\begin{aligned} Y_t(H) = & H + \int_t^T g(s, Z_s^f, Z_s, \tilde{Z}_s) ds \\ & - \int_t^T Z_s^f dW_s^f - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_s(x) \tilde{N}(ds, dx). \end{aligned}$$

Since the terminal condition is given at maturity time  $T$ , BSDEs have to be computed backwards in time. As in many applications a terminal reward is specified and solutions of BSDEs satisfy a dynamic programming principle, BSDEs are often applied to solve problems in stochastic optimal control and mathematical finance, see apart from the

papers mentioned above for instance El Karoui et al. (1997), Lazrak and Quenez (2003), Hamadène and Jeanblanc (2007), and Porchet, Touzi, and Warin (2009); or for the discrete time case Madan, Pistorius, and Schoutens (2013).

Subsequently, we will always assume that the BSDE which we consider has a unique solution. This is for instance the case if  $g(t, 0, 0, 0)$  is in  $L^2(d\mathbb{P} \times dt)$ , and  $g$  is uniformly Lipschitz continuous; that is there exists  $K > 0$  such that

$$|g(t, z_1^f, z_1, \tilde{z}_1) - g(t, z_0^f, z_0, \tilde{z}_0)| \leq K \left( |z_1^f - z_0^f| + |z_1 - z_0| + \sqrt{\int_{\mathbb{R} \setminus \{0\}} |\tilde{z}_1(x) - \tilde{z}_0(x)|^2 \nu(dx)} \right),$$

see for instance Royer (2006) and the reference therein.

EXAMPLE 5.1. Let  $H$  be a bounded payoff and define  $Y_t(H) = \mathbb{E}_{\mathcal{F}_t}[H]$ . Then by the martingale representation theorem (see, e.g., Jacod and Shiryaev 1987, section 3, theorem 4.34) there exist predictable square-integrable processes  $Z^f$ ,  $Z$ , and  $\tilde{Z}$  such that  $Y$  satisfies

$$dY_t(H) = Z_t^f dW_t^f + Z_t dW_t + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_t(x) \tilde{N}(dt, dx) \quad \text{and} \quad Y_T(H) = H.$$

This is the simplest BSDE with  $g = 0$ .

Hence, a conditional expectation may be seen as a BSDE with  $g = 0$ . This is why BSDEs are being referred to as  $g$ -expectations. The name should express that a BSDE may be viewed as generalized (usually nonlinear) conditional expectation with an additional drift.

EXAMPLE 5.2. Let  $H$  be a bounded payoff and define  $Y_t(H) = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Q}}[H]$ . Then by the martingale representation theorem and by the Girsanov theorem  $Y(H)$  satisfies

$$dY_t(H) = -\theta_t Z_t^f dt + Z_t^f dW_t^f + Z_t dW_t + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_t(x) \tilde{N}(dt, dx) \quad \text{and} \quad Y_T(H) = H.$$

This is a linear BSDE with  $g(t, z^f, z, \tilde{z}) = \theta_t z^f$ .

Subsequently, we will write

$$\mathcal{E}_t^g(H) = Y_t(H).$$

In a Markovian setting  $g$ -expectations correspond to semilinear parabolic PDEs (or PIDEs in the case of jumps), see for instance El Karoui et al. (1997) in a Brownian setting (see Barles, Buckdahn, and Pardoux 1997 in the case of jumps).

It may be seen that if  $g$  is convex and  $g(t, 0, 0, 0) = 0$  then the evaluation defined by

$$\Pi_\sigma(H) := \mathcal{E}_\sigma^g(H),$$

is normal, monotone, cash invariant, convex, time-consistent, and satisfies the local property. Hence,  $g$ -expectations give us an abundance of time-consistent, continuous-time evaluations. There are also certain sufficient conditions under which, in a Brownian filtration, a time-consistent evaluation is induced as the solution of a  $g$ -expectation, see Coquet et al. (2002).

The following theorem gives a complete characterization of market-consistent evaluations given by  $g$ -expectations:

**THEOREM 5.3.** *A  $g$ -expectation is market-consistent if and only if  $g(t, z^f, z, \tilde{z}) - \theta_t z^f$  does not depend on  $z^f$   $d\mathbb{P} \times dt$  a.s.*

## 5.2. Examples of Market-Consistent BSDEs

To get an interpretation of  $g$  we will consider some examples. The next proposition gives a *dynamic* market-consistent extension of the exponential principle. It follows directly from Theorem 5.3 above and theorem 2 in Morlais (2010). (Morlais (2010) also allows for a hedging set  $\mathcal{C}$  and an initial capital amount  $x$ . In our case this becomes  $\mathcal{C} = \{0\}$  and  $x = 0$ .)

**PROPOSITION 5.4.** *Define the evaluation  $\Pi$  as the solution of the BSDE with driver function*

$$g(t, z^f, z, \tilde{z}) = \theta_t z^f + \frac{1}{2\gamma} |z|^2 + \gamma \int_{\mathbb{R} \setminus \{0\}} \left[ \exp \left\{ \frac{\tilde{z}(x)}{\gamma} \right\} - \frac{\tilde{z}(x)}{\gamma} - 1 \right] \nu(dx).$$

*Then (i)  $\Pi$  is market-consistent, and (ii) for pure insurance risk (i.e., terminal conditions independent of  $S$ )  $\Pi$  corresponds to the exponential principle from Example 2.3.*

Other examples of the driver function  $g$  can be obtained by looking at one-period evaluations in discrete time defined recursively. Namely, suppose that we have an equispaced time grid  $I = \{0, h, 2h, \dots, T\}$  where we assumed without loss of generality that  $T$  is a multiple of  $h$ . The filtration  $(\mathcal{F}_{ih})_{i=0,1,\dots,T/h}$  is generated by  $(W_{ih}^f, W_{ih}, \tilde{N}((0, ih], dx))_{i=1,\dots,T/h}$ . Define  $S_0^{h,j} = s_0^j, S_{(i+1)h}^{h,j} = S_{ih}^{h,j} (1 + \mu^j(ih, S_{ih}^h)h + \tilde{\sigma}^j(ih, S_{ih}^h)\Delta W_{(i+1)h}^f)$  for  $i = 1, \dots, T/h$  and  $j = 1, \dots, n$ , and  $S^h = (S^{h,1}, \dots, S^{h,n})$ . Denote further  $\mathcal{F}_{(i+1)h}^{S^h} = \tilde{\mathcal{F}}_{(i+1)h}^{S^h} \vee \mathcal{F}_{ih}$ , where  $\tilde{\mathcal{F}}^{S^h}$  is the filtration generated by  $S^h$ . In other words  $\mathcal{F}^{S^h}$  is the information of the (discrete-time) stock process together with the previous values of the insurance process. Let  $\mathbb{Q}^h$  be the measure (with  $\tilde{\mathcal{F}}^{S^h}$ -measurable density) such that  $\Delta W_{(i+1)h}^{f,*} := \Delta W_{(i+1)h}^f - \theta_{ih}h$  is a martingale.

Now we can use evaluations from our Examples 3.4 over one period and glue them together recursively. Using our two-step procedures for the one-periodic evaluations could be natural, in particular, if the stock process can be observed before the insurance process. Suppose for instance for a moment that we are at time  $ih$  and the stock process,  $S^h$ , can be observed at time  $(i + \frac{1}{2})h$ , whereas the insurance process is revealed after the stock process at time  $(i + 1)h$ . Since data from the financial market can be observed almost continuously, while data from insurance companies are typically observed less frequently, this may not be an unreasonable assumption, see also Section 4. Of course our insurance process will possibly be effected by the financial market through its correlation to  $W^f$ . However, it will not be completely predictable due to its dependence on the jumps and  $W$ . In this situation if an evaluation  $(\Pi_\sigma)_{\sigma \in [0, T]}$  is time-consistent, then market-consistency would imply that for the aggregated evaluation  $(\Pi_{ih})_{i=0,1,\dots,T/h}$  we have

$$\Pi_{ih}(H_{(i+1)h}) = \Pi_{ih}(\Pi_{(i+\frac{1}{2})h}(H_{(i+1)h})) = \mathbb{E}_{\mathcal{F}_{ih}}^{\mathbb{Q}^h}[\Pi_{(i+\frac{1}{2})h}(H_{(i+1)h})].$$

The last equation holds by market-consistency since  $\Pi_{(i+\frac{1}{2})h}(H_{(i+1)h})$ , is  $\mathcal{F}_{(i+\frac{1}{2})h}^{S^h} = \bar{\mathcal{F}}_{(i+\frac{1}{2})h}^{S^h} \vee \mathcal{F}_{ih}$ -measurable. Therefore, the setting outlined above would indeed give rise to applying a two-step market evaluation at time-instances  $ih$ , to evaluate payoffs up to time  $(i+1)h$ . We will calculate one example explicitly by considering at time instances  $ih$  the mean-variance two-step market evaluation

$$\Pi_{ih,(i+1)h}^v(H_{(i+1)h}) = \mathbb{E}_{\mathcal{F}_{ih}}^{\mathbb{Q}^h} \left[ \mathbb{E}_{\mathcal{F}_{(i+1)h}^{S^h}} [H_{(i+1)h}] + \frac{1}{2} \alpha \text{Var}_{\mathcal{F}_{(i+1)h}^{S^h}} [H_{(i+1)h}] \right], \quad \alpha \geq 0.$$

To obtain a multi-period evaluation  $(\Pi_{ih}^h)_{i=0,1,\dots,T/h}$  on the whole filtration with  $\Pi_{ih}^h : L^\infty(\mathcal{F}_T) \rightarrow L^\infty(\mathcal{F}_{ih})$  we will define  $\Pi^h$  recursively by setting

$$(5.1) \quad \Pi_T^h(H) = H \quad \text{and} \quad \Pi_{ih}^h(C) = \Pi_{ih}^v(\Pi_{(i+1)h}^h(H)) \quad \text{for } i = 0, 1, \dots, T/h - 1.$$

The following proposition is proved in the Appendix:

**PROPOSITION 5.5.** *Suppose that the evaluation  $\Pi^h$  is constructed by (5.1), i.e., using locally the mean-variance two-step market evaluation. Then for every terminal payoff  $H \in L^\infty(\mathcal{F}_T)$  there exists predictable  $(Z^{h,f}, Z^h, \tilde{Z}^h)$  and a martingale  $(L_{ih}^h)_i$  orthogonal (under  $\mathbb{Q}^h$ ) to  $(W_{ih}^f, W_{ih}, \tilde{N}((0, ih], dx))_i$  such that for all  $i$  we have*

$$(5.2) \quad \begin{aligned} \Pi_{ih}^h(H) = & H + \sum_{j=i}^{T/h-1} \left( \left[ \theta_{jh} Z_{jh}^{h,f} + \frac{\alpha}{2} \left( |Z_{jh}^h|^2 + \int_{\mathbb{R} \setminus \{0\}} |\tilde{Z}_{jh}^h(x)|^2 v(dx) \right) \right] h \right. \\ & + \frac{\alpha}{2} \mathbb{E}_{\mathcal{F}_{jh}}^{\mathbb{Q}^h} \left[ \left( \Delta L_{(j+1)h}^h - \mathbb{E}_{\mathcal{F}_{(j+1)h}^{S^h}} [\Delta L_{(j+1)h}^h] \right)^2 \right] \Big) - \sum_{j=i}^{T/h-1} Z_{jh}^{h,f} \Delta W_{(j+1)h}^f \\ & - \sum_{j=i}^{T/h-1} Z_{jh}^h \Delta W_{(j+1)h} - \sum_{j=i}^{T/h-1} \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_{jh}^h(x) \tilde{N}((jh, (j+1)h], dx) - (L_T^h - L_{ih}^h). \end{aligned}$$

In particular,  $\Pi^h$  satisfies a discrete-time BSDE.

From Proposition 5.5 we may infer that

$$\begin{aligned} \mathbb{E}_{\mathcal{F}_{jh}} [\Delta \Pi_{(j+1)h}^h(H)] = & - \left[ \theta_{jh} Z_{jh}^{h,f} + \frac{\alpha}{2} \left( |Z_{jh}^h|^2 + \int_{\mathbb{R} \setminus \{0\}} |\tilde{Z}_{jh}^h(x)|^2 v(dx) \right) \right] h \\ & - \frac{\alpha}{2} \mathbb{E}_{\mathcal{F}_{jh}}^{\mathbb{Q}^h} \left[ \left( \Delta L_{(j+1)h}^h - \mathbb{E}_{\mathcal{F}_{(j+1)h}^{S^h}} [\Delta L_{(j+1)h}^h] \right)^2 \right]. \end{aligned}$$

Note that the orthogonal martingale terms  $\Delta L_{(j+1)h}^h$  arise because the discretized Brownian motions do not have the representation property. However, the continuous time Brownian motions and the Poisson random measure do have the representation property. Therefore, if we ignore the  $L^h$  then an analogous infinitesimal way of charging the risk in continuous time would be an evaluation which satisfies

$$\mathbb{E}_{\mathcal{F}_t} [d\Pi_t(H)] = - \left[ \theta_t Z_t^f + \frac{\alpha}{2} \left( |Z_t|^2 + \int_{\mathbb{R} \setminus \{0\}} |\tilde{Z}_t(x)|^2 v(dx) \right) \right] dt.$$

This corresponds to an evaluation given by the solution of a BSDE with driver function

$$g(t, z, \tilde{z}) = \theta_t z^f + \frac{\alpha}{2} \left( |z|^2 + \int_{\mathbb{R} \setminus \{0\}} |\tilde{z}(x)|^2 \nu(dx) \right).$$

REMARK 5.6. The analogy stated above only corresponds to the local way of charging risk. A global correspondence of charging risk would involve proving a convergence result for the whole path. In the case that the driver function is Lipschitz continuous it can actually be shown in a purely Brownian setting that after an appropriate scaling the whole path of the discrete time evaluations converges to the corresponding solution of the BSDE; see Stadje (2010). However, in Proposition 5.5. the driver function is quadratic in  $z$ . In this case already in the purely Brownian setting an extension by convergence may not always be possible. Cheridito and Stadje (2010) give an example which shows that in a setting where the discrete time filtration is generated by a Bernoulli random walk it may happen that the discrete time  $H^n$  are uniformly bounded and converge to  $H \in L^\infty(\mathcal{F})$  but the discrete time evaluations explode.

## 6. SUMMARY AND CONCLUSIONS

In this paper we have studied the extension of standard actuarial principles in time-consistent and market-consistent directions by introducing a new market-consistent evaluation procedure which we call “two-step market evaluation.” On the one hand, two-step market evaluations sometimes arise when an agent starts with an evaluation that is not market-consistent, such as the average value at risk or the exponential premium principle, and then engages in hedging. On the other hand, market-consistent evaluations can also be defined directly by applying a standard evaluation technique, conditional on the stock process. In this case the structure of many standard evaluation techniques can be preserved.

We have shown that two-step market evaluations are invariant if the stock is taken as a numeraire. In Theorems 3.8 and 3.9 a complete axiomatic characterization for two-step market evaluations is provided. Moreover, we have proved that in a dynamic setting with a continuous stock prices process and an insurance process being revealed at predictable times every evaluation which is time-consistent and market-consistent is a two-step market evaluation up to the next predictable time, which gives a strong argument for their use. We have also characterized market-consistency in terms of  $g$ -expectations and studied the extension of the mean-variance and the exponential principle to continuous time, in a setting with jumps. Our analysis shows that two-step evaluations can provide a useful, computationally tractable tool for market-consistent valuations.

## APPENDIX A: TECHNICAL MATERIAL AND PROOFS

### A.1. Background to the Essential Supremum

The first part of the appendix is basically a summary of the definitions and results given in A.5 Föllmer and Schied (2004). Consider a family of random variables  $M$  on a given probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Now if  $M$  is countable then  $Z^*(\omega) = \sup_{Z \in M} Z(\omega)$  is also measurable. However, measurability is not guaranteed if  $M$  is uncountable. Even if the pointwise supremum is measurable it might not be the right concept when we focus on a.s. properties. For instance if  $\mathbb{P}$  is the Lebesgue measure on  $\Omega = [0, 1]$  and



$M = \{I_{\{x\}} | 0 \leq x \leq 1\}$  then  $\sup_{Z \in M} Z = 1$  while  $Z = 0$  a.s. for all  $Z \in M$ . This suggest the following notion of an *essential supremum* defined in terms of almost sure inequalities. This result can be found as theorem A.32 in Föllmer and Schied (2004).

**THEOREM A.1.** *Let  $M$  be any set of random variables on  $(\Omega, \bar{\mathcal{F}}, \mathbb{P})$ .*

(a) *There exists a random variable  $Z^*$  such that*

$$Z^* \geq Z \text{ } \mathbb{P}\text{-a.s. for all } Z \in M. \quad (A)$$

*Moreover,  $Z^*$  is a.s. unique in the following sense: Any other random variable  $\hat{Z}$  with property (A) satisfies  $\hat{Z} \geq Z^*$   $\mathbb{P}$ -a.s.*

(b) *Suppose that  $M$  is directed upwards, i.e., for  $Z_1, Z_2 \in M$  there exists  $Z_3 \in M$  with  $Z_3 \geq \max(Z_1, Z_2)$ . Then there exists an increasing sequence  $Z_1 \leq Z_2 \leq \dots$  in  $M$  such that  $Z^* = \lim_{\eta} Z_{\eta}$   $\mathbb{P}$ -a.s.*

**DEFINITION A.2.** The random variable  $Z^*$  in the theorem above is called the *essential supremum* of  $M$  and we write:

$$Z^* = \text{ess sup}_{Z \in M} Z.$$

We define the essential infimum similarly.

If the probability space is finite the essential supremum corresponds to the pointwise supremum taken over all atoms.

**LEMMA A.3.** *If  $M$  satisfies the concatenation property, i.e., for every  $A \in \bar{\mathcal{F}}$  and  $Z_1, Z_2 \in M$  we have that  $Z_1 I_A + Z_2 I_{A^c} \in M$ , then  $M$  is directed upwards.*

*Proof.* Define

$$Z^* = Z_1 I_{\{Z_1 \geq Z_2\}} + Z_2 I_{\{Z_1 < Z_2\}}.$$

By the concatenation property  $Z^* \in M$ , and by definition  $Z^* \geq \max(Z_1, Z_2)$ . □

## A.2. Proofs of the results in Section 3.1

*Proof of Proposition 3.3.* (i)  $\Rightarrow$  (ii) By (2.2) we have that

$$\Pi_{\mathcal{G}}(H) = \text{ess sup}_{\xi \in L^1(\mathcal{F})} \{\mathbb{E}_{\mathcal{G}}[\xi H] - \Pi_{\mathcal{G}}^*(\xi)\}.$$

Furthermore,

$$\begin{aligned} (A.1) \quad \Pi_{\mathcal{G}}^*(\xi) &= \text{ess sup}_{H \in L^\infty(\mathcal{F})} \{\mathbb{E}_{\mathcal{G}}[\xi H] - \Pi_{\mathcal{G}}(H)\} \\ &\geq \text{ess sup}_{H \in L^\infty(\mathcal{F}^S)} \{\mathbb{E}_{\mathcal{G}}[\xi H] - \Pi_{\mathcal{G}}(H)\} \\ &= \text{ess sup}_{H \in L^\infty(\mathcal{F}^S)} \{\mathbb{E}_{\mathcal{G}}[\mathbb{E}_{\mathcal{F}^S}[\xi] H] - \Pi_{\mathcal{G}}(H)\}. \end{aligned}$$

The last term in (A.1) is the dual of  $\Pi_{\mathcal{G}}$  restricted to  $\mathcal{F}^S$  evaluated at  $\mathbb{E}_{\mathcal{F}^S}[\xi]$ . Now by assumption  $\Pi_{\mathcal{G}}(H^S) = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[H^S]$  is linear. Thus, its dual penalty function,  $(\Pi_{\mathcal{G}}|_{\mathcal{F}^S})^* : L^1(\mathcal{F}^S) \rightarrow \mathbb{R} \cup \{\infty\}$ , must be equal to the indicator function which is zero if the input argument is  $\frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}}$ , and infinity else. But then, by the inequality above,  $\Pi_{\mathcal{G}}^*(\xi)$  must be

equal to infinity as well if  $\mathbb{E}_{\mathcal{F}^S}[\xi] \neq \frac{d\mathbb{Q}_G}{d\mathbb{P}_G}$ . Thus, it is sufficient to consider  $\xi$  of the form  $\xi = \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} Z$  for a  $Z := \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} \in \mathcal{Q}_{\mathcal{F}^S}$ . Defining for  $Z \in \mathcal{Q}_{\mathcal{F}^S}$ ,  $c_G(Z) = \Pi_G^*\left(\frac{d\mathbb{Q}_G}{d\mathbb{P}_G} Z\right)$ , we have indeed that

$$\begin{aligned}\Pi_G(H) &= \text{ess sup}_{\xi \in L^1(\mathcal{F})} \{ \mathbb{E}_G[\xi H] - \Pi_G^*(\xi) \} \\ &= \text{ess sup}_{Z \in \mathcal{Q}_{\mathcal{F}^S}} \left\{ \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} Z H \right] - \Pi_G^* \left( \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} Z \right) \right\} \\ &= \text{ess sup}_{Z \in \mathcal{Q}_{\mathcal{F}^S}} \{ \mathbb{E}^{\mathbb{Q}_G}[ZH] - c_G(Z) \}.\end{aligned}$$

(ii)  $\Rightarrow$  (iii): It is by (ii)

$$\begin{aligned}\Pi_G(H^S + H) &= \text{ess sup}_{Z \in \mathcal{Q}_{\mathcal{F}^S}} \left\{ \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} Z (H^S + H) \right] - c_G(Z) \right\} \\ &= \text{ess sup}_{Z \in \mathcal{Q}_{\mathcal{F}^S}} \left\{ \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} Z H^S \right] + \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} Z H \right] - c_G(Z) \right\} \\ &= \text{ess sup}_{Z \in \mathcal{Q}_{\mathcal{F}^S}} \left\{ \mathbb{E}_G \left[ \mathbb{E}_{\mathcal{F}^S} \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} Z H^S \right] \right] + \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} Z H \right] - c_G(Z) \right\} \\ &= \text{ess sup}_{Z \in \mathcal{Q}_{\mathcal{F}^S}} \left\{ \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} H^S \mathbb{E}_{\mathcal{F}^S}[Z] \right] + \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} Z H \right] - c_G(Z) \right\} \\ &= \text{ess sup}_{Z \in \mathcal{Q}_{\mathcal{F}^S}} \left\{ \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} H^S \right] + \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} Z H \right] - c_G(Z) \right\} \\ &= \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} H^S \right] + \text{ess sup}_{Z \in \mathcal{Q}_{\mathcal{F}^S}} \left\{ \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} Z H \right] - c_G(Z) \right\} \\ &= \mathbb{E}^{\mathbb{Q}_G}[H^S] + \Pi_G(H),\end{aligned}$$

where we have used in the fourth equation that  $d\mathbb{Q}_G/d\mathbb{P}_G$  and  $H^S$ , by assumption, are  $\mathcal{F}^S$ -measurable. In the fifth equation we have used that  $\mathbb{E}_{\mathcal{F}^S}[Z] = 1$ . This proves (ii) $\Rightarrow$ (iii). The direction (iii) $\Rightarrow$ (i) is clear using that by normalization  $\Pi_G(0) = 0$ .

Finally let us show that if  $\Pi_G$  is additionally assumed to be monotone, then market-consistency is equivalent to any payoff being evaluated between its super- and sub-replication price. Clearly, if any payoff is evaluated between its super- and sub-replication price then for any  $H^S \in L^\infty(\mathcal{F}^S)$  we must have that  $\Pi_G(H^S) = \mathbb{E}^{\mathbb{Q}_G}[H^S]$ , as the financial market is assumed to be complete. Hence, by the direction (i) $\Rightarrow$ (iii) shown above, indeed  $\Pi_G$  is market-consistent. On the other hand if  $\Pi_G$  is market-consistent and monotone then by standard duality results the penalty function in (ii) must have a domain in  $\mathcal{Q}_{\mathcal{F}^S}^+$ . Also note that the set defined by  $M := \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} \mathcal{Q}_{\mathcal{F}^S}^+ = \{ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} Z | Z \in \mathcal{Q}_{\mathcal{F}^S}^+ \}$  is equal to the set of all local martingale measures  $\mathcal{M}_G$ . (Actually for our proof we only need that  $M \subset \mathcal{M}_G$ .) This yields that

$$\begin{aligned}\Pi_G(H) &= \text{ess sup}_{Z \in \mathcal{Q}_{\mathcal{F}^S}^+} \{ \mathbb{E}^{\mathbb{Q}_G}[ZH] - c_G(Z) \} \leq \text{ess sup}_{Z \in \mathcal{Q}_{\mathcal{F}^S}^+} \mathbb{E}^{\mathbb{Q}_G}[ZH] \\ &= \text{ess sup}_{\tilde{\mathbb{P}}_G \in \mathcal{M}_G} \tilde{\mathbb{E}}_G[H],\end{aligned}$$

where we have used in the first equality that (ii) holds as  $\Pi_G$  is market-consistent. In the inequality we applied that  $c_G \geq 0$ . In the last equality we used that  $M = \mathcal{M}_G$ . Hence, indeed  $\Pi_G(H)$  is smaller than the super-replication price of  $H$ . To show that  $\Pi_G(H)$  is greater than the sub-replication price, note that as  $c$  is a penalty function we must

have  $\text{ess inf}_{Z \in \mathcal{Q}_{\mathcal{F}^S}^+} c_{\mathcal{G}}(Z) = 0$ . Now clearly,  $\mathcal{Q}_{\mathcal{F}^S}^+$  satisfies the concatenation property from Lemma A.3. Thus, there exists a sequence  $Z^n \in \mathcal{Q}_{\mathcal{F}^S}^+$  such that a.s.  $\lim_n c_{\mathcal{G}}(Z^n) = 0$ . This entails that

$$\begin{aligned} \Pi_{\mathcal{G}}(H) &= \text{ess sup}_{Z \in \mathcal{Q}_{\mathcal{F}^S}^+} \{\mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[ZH] - c_{\mathcal{G}}(Z)\} \\ &\geq \lim_n \{\mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[Z^n H] - c_{\mathcal{G}}(Z^n)\} \\ &\geq \liminf_n \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[Z^n H] + \liminf_n -c_{\mathcal{G}}(Z^n) = \liminf_n \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[Z^n H] \\ &\geq \text{ess inf}_{\tilde{\mathbb{P}}_{\mathcal{G}} \in \mathcal{M}_{\mathcal{G}}} \tilde{\mathbb{E}}_{\mathcal{G}}[H]. \end{aligned}$$

Hence,  $\Pi_{\mathcal{G}}(H)$  is greater than the sub-replication price. The proposition is proved.  $\square$

Theorems 3.10 and 3.11 may be seen as versions of the Radon-Nikodym theorem with a nonlinear part  $\Pi_{\mathcal{F}^S}$  and without assumptions like monotonicity or continuity. We will need the following lemma. Its proof is straightforward doing an induction over  $r$ .

**LEMMA A.4.** *Suppose that  $\Pi_{\mathcal{G}}$  satisfies market-consistency and the local property. Then for disjoint sets  $C_1, \dots, C_r \in \mathcal{F}^S$  and payoffs  $H_1, \dots, H_r \in L^\infty(\mathcal{F})$ , we have*

$$\Pi_{\mathcal{G}}(H_1 I_{C_1} + \dots + H_r I_{C_r}) = \Pi_{\mathcal{G}}(H_1 I_{C_1}) + \dots + \Pi_{\mathcal{G}}(H_r I_{C_r}).$$

The next lemma will also be useful.

**LEMMA A.5.** *A  $\tilde{\mathcal{G}}$ -conditional evaluation satisfies  $\Pi_{\tilde{\mathcal{G}}}(I_A H) = I_A \Pi_{\tilde{\mathcal{G}}}(H)$  for every  $A \in \tilde{\mathcal{G}}$ .*

*Proof.* It is

$$\Pi_{\tilde{\mathcal{G}}}(I_A H) = \Pi_{\tilde{\mathcal{G}}}(I_A H + I_{A^c} 0) = I_A \Pi_{\tilde{\mathcal{G}}}(H) + I_{A^c} \Pi_{\tilde{\mathcal{G}}}(0) = I_A \Pi_{\tilde{\mathcal{G}}}(H),$$

by normalization and the local property of  $\Pi_{\tilde{\mathcal{G}}}$ .  $\square$

For the proof of Theorem 3.10 we will also need the following lemma:

**LEMMA A.6.** *In the setting of Theorem 3.10, the set  $M'$  defined by (3.4) has the concatenation property in the sense that  $Z_1, Z_2 \in M'$  implies that for any  $A \in \mathcal{F}^S$  we have that  $Z_1 I_A + Z_2 I_{A^c} \in M'$ . In particular,  $I_A M' + I_{A^c} M' := \{Z_1 I_A + Z_2 I_{A^c} \mid Z_1 \in M', Z_2 \in M'\} = M'$ .*

*Proof.* For  $Z_1, Z_2 \in M'$  we have

$$\begin{aligned} &(\text{A.2}) \quad \Pi_{\mathcal{G}}^* \left( \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} (Z_1 I_A + Z_2 I_{A^c}) \right) \\ &= \text{ess sup}_{H \in L^\infty(\mathcal{F})} \left\{ \mathbb{E}_{\mathcal{G}} \left[ \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} (Z_1 I_A + Z_2 I_{A^c}) H \right] - \Pi_{\mathcal{G}}(H) \right\} \\ &= \text{ess sup}_{H \in L^\infty(\mathcal{F})} \left\{ \mathbb{E}_{\mathcal{G}} \left[ \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} Z_1 (I_A H) \right] + \mathbb{E}_{\mathcal{G}} \left[ \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} Z_2 (I_{A^c} H) \right] - \Pi_{\mathcal{G}}(H I_A + H I_{A^c}) \right\} \end{aligned}$$

$$\begin{aligned}
&= \operatorname{ess\,sup}_{HI_A, HI_{A^c} \in L^\infty(\mathcal{F})} \left\{ \mathbb{E}_{\mathcal{G}} \left[ \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} Z_1(I_A H) \right] - \Pi_{\mathcal{G}}(HI_A) \right. \\
&\quad \left. + \mathbb{E}_{\mathcal{G}} \left[ \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} Z_2(I_{A^c} H) \right] - \Pi_{\mathcal{G}}(HI_{A^c}) \right\} \\
&= \operatorname{ess\,sup}_{HI_A \in L^\infty(\mathcal{F})} \left\{ \mathbb{E}_{\mathcal{G}} \left[ \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} Z_1(HI_A) \right] - \Pi_{\mathcal{G}}(HI_A) \right\} \\
&\quad + \operatorname{ess\,sup}_{HI_{A^c} \in L^\infty(\mathcal{F})} \left\{ \mathbb{E}_{\mathcal{G}} \left[ \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} Z_2(HI_{A^c}) \right] - \Pi_{\mathcal{G}}(HI_{A^c}) \right\},
\end{aligned}$$

where the first equation holds by the definition of  $\Pi^*$ . The third equation holds because of Lemma A.4 for  $r = 2$ . (A.2) yields that

$$\begin{aligned}
\Pi_{\mathcal{G}}^* \left( \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} (Z_1 I_A + Z_2 I_{A^c}) \right) &\leq \operatorname{ess\,sup}_{H \in L^\infty(\mathcal{F})} \left\{ \mathbb{E}_{\mathcal{G}} \left[ \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} Z_1 H \right] - \Pi_{\mathcal{G}}(H) \right\} \\
&\quad + \operatorname{ess\,sup}_{H \in L^\infty(\mathcal{F})} \left\{ \mathbb{E}_{\mathcal{G}} \left[ \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} Z_2 H \right] - \Pi_{\mathcal{G}}(H) \right\} \\
&= \Pi_{\mathcal{G}}^* \left( \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} Z_1 \right) + \Pi_{\mathcal{G}}^* \left( \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} Z_2 \right) = 0 + 0 = 0,
\end{aligned}$$

where in the last line we have used that  $\Pi_{\mathcal{G}}^* \left( \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} Z_i \right) = 0$ . The reason for this is that  $\Pi^*$  is zero on  $M$  and infinity else. Therefore, the fact that  $Z_i \in M'$  implies by (3.4) that  $\frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} Z_i \in \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} M' = M$  for  $i = 1, 2$ .

Since  $\Pi_{\mathcal{G}}^*$  is only takes the values zero and infinity we must have that  $\Pi_{\mathcal{G}}^* \left( \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} (Z_1 I_A + Z_2 I_{A^c}) \right) = 0$ . Thus, we can conclude that indeed  $\frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} (Z_1 I_A + Z_2 I_{A^c}) \in M$ . Therefore,

$$Z_1 I_A + Z_2 I_{A^c} \in \left( \frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}_{\mathcal{G}}} \right)^{-1} M = M'. \quad \square$$

*Proof of Theorem 3.10.* (ii) $\Rightarrow$ (i): It is

$$\Pi_{\mathcal{G}}(H^S + H) = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^S}(H^S + H)] = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[H^S] + \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^S}(H)] = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[H^S] + \Pi_{\mathcal{G}}(H),$$

where we have used  $\mathcal{F}^S$ -conditional cash invariance in the second equation. This shows market-consistency. Moreover,

$$\begin{aligned}
\Pi_{\mathcal{G}}(I_A H_1 + I_{A^c} H_2) &= \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^S}(I_A H_1 + I_{A^c} H_2)] \\
&= \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[I_A \Pi_{\mathcal{F}^S}(H_1) + I_{A^c} \Pi_{\mathcal{F}^S}(H_2)] \\
&= \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^S}(I_A H_1) + \Pi_{\mathcal{F}^S}(I_{A^c} H_2)] = \Pi_{\mathcal{G}}(I_A H_1) + \Pi_{\mathcal{G}}(I_{A^c} H_2),
\end{aligned}$$

where we used the  $\mathcal{F}^S$ -local property in the second and Lemma A.5 in the third equation.

(i) $\Rightarrow$ (ii): By Proposition 3.3 (iii) $\Rightarrow$ (ii) and positive homogeneity, market-consistency implies that

$$\begin{aligned}
 (A.3) \quad \Pi_G(H) &= \text{ess sup}_{Z \in M} \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} ZH \right] \\
 &= \text{ess sup}_{Z \in M} \mathbb{E}_G \left[ \mathbb{E}_{\mathcal{F}^S} \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} ZH \right] \right] = \text{ess sup}_{Z \in M} \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} \mathbb{E}_{\mathcal{F}^S}[ZH] \right],
 \end{aligned}$$

where we used in the third equation that  $\frac{d\mathbb{Q}_G}{d\mathbb{P}_G}$  is  $\mathcal{F}^S$ -measurable. Define

$$\Pi_{\mathcal{F}^S}(H) := \text{ess sup}_{Z \in M} \mathbb{E}_{\mathcal{F}^S}[ZH].$$

Clearly,  $\Pi_{\mathcal{F}^S}$  is normal,  $\mathcal{F}^S$ -convex,  $\mathcal{F}^S$ -cash invariant and  $\mathcal{F}^S$ -positively homogeneous. The  $\mathcal{F}^S$ -local property is satisfied because for  $A \in \mathcal{F}^S$  we have

$$\begin{aligned}
 \Pi_{\mathcal{F}^S}(I_A H_1 + I_{A^c} H_2) &= \text{ess sup}_{Z \in M} \mathbb{E}_{\mathcal{F}^S}[Z I_A H_1 + Z I_{A^c} H_2] \\
 &= \text{ess sup}_{Z \in M} \mathbb{E}_{\mathcal{F}^S}[Z I_A H_1] + \mathbb{E}_{\mathcal{F}^S}[Z I_{A^c} H_2] \\
 &= \text{ess sup}_{Z_1 \in M', Z_2 \in M} \mathbb{E}_{\mathcal{F}^S}[Z_1 I_A H_1] + \mathbb{E}_{\mathcal{F}^S}[Z_2 I_{A^c} H_2] \\
 &= \text{ess sup}_{Z_1 \in M'} \mathbb{E}_{\mathcal{F}^S}[Z_1 I_A H_1] + \text{ess sup}_{Z_2 \in M} \mathbb{E}_{\mathcal{F}^S}[Z_2 I_{A^c} H_2] \\
 &= I_A \text{ess sup}_{Z_1 \in M'} \mathbb{E}_{\mathcal{F}^S}[Z_1 H_1] + I_{A^c} \text{ess sup}_{Z_2 \in M} \mathbb{E}_{\mathcal{F}^S}[Z_2 H_2] \\
 &= I_A \Pi_{\mathcal{F}^S}(H_1) + I_{A^c} \Pi_{\mathcal{F}^S}(H_2),
 \end{aligned}$$

where we used in the third equation that Lemma A.6 implies that  $M' = \{Z_1 I_A + Z_2 I_{A^c} \mid Z_1 \in M', Z_2 \in M\}$ . Hence, indeed  $\Pi_{\mathcal{F}^S}$  is an  $\mathcal{F}^S$ -conditional evaluation. Finally, let us prove that

$$\Pi_G(H) = \mathbb{E}^{\mathbb{Q}_G}[\Pi_{\mathcal{F}^S}(H)].$$

Notice that if we could show that

$$(A.4) \quad \text{ess sup}_{Z \in M} \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} \mathbb{E}_{\mathcal{F}^S}[ZH] \right] = \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} \text{ess sup}_{Z \in M} \mathbb{E}_{\mathcal{F}^S}[ZH] \right]$$

then we are done, since the left-hand side of (A.4) is equal to  $\Pi_G(H)$  by (A.3), while the right-hand side is equal to  $\mathbb{E}^{\mathbb{Q}_G}[\Pi_{\mathcal{F}^S}(H)]$  by the definition of  $\Pi_{\mathcal{F}^S}$ . So let us show (A.4). Clearly,

$$\text{ess sup}_{Z \in M} \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} \mathbb{E}_{\mathcal{F}^S}[ZH] \right] \leq \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} \text{ess sup}_{Z \in M} \mathbb{E}_{\mathcal{F}^S}[ZH] \right].$$

Let us prove “ $\geq$ .” It is well known, see also Theorem A.1 and Lemma A.3, that the concatenation property implies that there exists a sequence  $Z_n \in M'$  with  $\mathbb{E}_{\mathcal{F}^S}[Z_1 H] \leq \mathbb{E}_{\mathcal{F}^S}[Z_2 H] \leq \dots$  such that  $\lim_n \mathbb{E}_{\mathcal{F}^S}[Z_n H] = \text{ess sup}_{Z \in M} \mathbb{E}_{\mathcal{F}^S}[ZH]$ . Therefore, by the monotone convergence theorem

$$\begin{aligned}
 \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} \text{ess sup}_{Z \in M} \mathbb{E}_{\mathcal{F}^S}[ZH] \right] &= \lim_n \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} \mathbb{E}_{\mathcal{F}^S}[Z_n H] \right] \\
 &\leq \text{ess sup}_{Z \in M} \mathbb{E}_G \left[ \frac{d\mathbb{Q}_G}{d\mathbb{P}_G} \mathbb{E}_{\mathcal{F}^S}[ZH] \right].
 \end{aligned}$$

This shows (A.4). This proves Theorem 3.10.  $\square$

For the proof of Theorem 3.11 we will need the following Corollary of Lemma A.4:

**COROLLARY A.7.** *Suppose that  $\Pi_{\mathcal{G}}$  is continuous, market-consistent, and satisfies the local property. Then for disjoint sets  $C_1, C_2, \dots \in \mathcal{F}^S$  and a payoff  $H \in L^\infty(\mathcal{F})$  we have*

$$\Pi_{\mathcal{G}}\left(H \sum_{i=1}^{\infty} I_{C_i}\right) = \sum_{i=1}^{\infty} \Pi_{\mathcal{G}}(H I_{C_i}).$$

*In particular, if  $\Pi_{\mathcal{G}}$  is additionally assumed to be monotone (or  $p$ -norm bounded) then for every  $H \in L^\infty(\mathcal{F})$  the mapping  $C \rightarrow \Pi_{\mathcal{G}}(H I_C)$  is a real-valued, (signed) measure on  $\mathcal{F}^S$ .*

*Proof of Theorem 3.11.* (ii) $\Rightarrow$ (i): Continuity, convexity, and monotonicity in the case that  $\Pi_{\mathcal{F}^S}$  is monotone, are straightforward. The other properties are seen analogously as in the proof of Theorem 3.10.

(i) $\Rightarrow$ (ii): By assumption  $\mathcal{F}^S = \mathcal{G} \vee \sigma(A_1, A_2, \dots)$  and we will assume without loss of generality that  $A_i \neq A_j$  if  $i \neq j$ . For  $n \in \mathbb{N}$  we define the filtration  $\mathcal{F}^n$  as the smallest  $\sigma$ -algebra containing  $\mathcal{G}$  and the events  $A_1, \dots, A_n$ . Now let us define the partitions corresponding to  $A_1, \dots, A_n$  recursively in a standard way. For  $n = 1$ ,  $\mathcal{F}^1$  is generated by  $\mathcal{G}$ , and the partition given by  $B_1^1 := A_1$ ,  $B_2^1 := A_1^c$ . Moreover,  $\mathcal{F}^{n+1}$  is generated by  $\mathcal{G}$  and the partition

$$B_i^{n+1} := B_i^n \cap A_{n+1} \quad \text{and} \quad B_{2^n+i}^{n+1} := B_i^n \cap A_{n+1}^c \quad i = 1, \dots, 2^n.$$

Of course,  $\mathcal{F}^1 \subset \mathcal{F}^2 \subset \dots \subset \mathcal{F}^S$ .

Set  $q_k^n = \mathbb{Q}_{\mathcal{G}}(B_k^n)$ . Note that  $q_k^n$  are  $\mathcal{G}$ -measurable random variables summing up to one for a.s. all fixed  $\omega$ . Define

$$(A.5) \quad \Pi_{\mathcal{F}^n}(H) := \sum_{k=1}^{2^n} \frac{I_{B_k^n}}{q_k^n} \Pi_{\mathcal{G}}(H I_{B_k^n}),$$

where we set  $0/0 = 0$ . Note that if  $q_k^n = \mathbb{Q}_{\mathcal{G}}[B_k^n] = 0$  on a nonzero set  $C \in \mathcal{G}$ , then  $\mathbb{P}_{\mathcal{G}}[B_k^n] = 0$  on  $C$  as well (as  $\mathbb{Q}_{\mathcal{G}}$  is equivalent to  $\mathbb{P}_{\mathcal{G}}$ ) and therefore,  $\Pi_{\mathcal{G}}(H I_{B_k^n}) = 0$  on  $C$ , too. In particular,  $\Pi_{\mathcal{F}^n}$  is well defined.

Next, every set  $A \in \mathcal{F}^n$  can be written as  $A = (B_{k_1}^n \cap G_1) \cup \dots \cup (B_{k_r}^n \cap G_r)$ , for a  $r \in \{1, \dots, 2^n\}$ ,  $1 \leq k_1 < k_2 < \dots < k_r \leq 2^n$  and  $G_i \in \mathcal{G}$ . As  $\Pi_{\mathcal{G}}(0) = 0$ , it is straightforward to check using Definition (A.5) that

$$(A.6) \quad \Pi_{\mathcal{F}^n}(I_A H) = I_A \Pi_{\mathcal{F}^n}(H).$$

Furthermore, by Lemma A.4 for every  $H \in L^\infty(\mathcal{F})$

$$\mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^n}(H)] = \sum_{k=1}^{2^n} \frac{\mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[I_{B_k^n}]}{q_k^n} \Pi_{\mathcal{G}}(H I_{B_k^n}) = \Pi_{\mathcal{G}}(H).$$

We will need the following lemma which is a version of Proposition A.12 in Föllmer and Schied (2004). □

LEMMA A.8. Suppose that  $\hat{\mathbb{P}}$  and  $\bar{\mathbb{P}}$  share the same zero sets, and that  $\mathcal{F}_0 \subset \mathcal{F}$ . Then for any bounded  $\mathcal{F}$ -measurable  $H$

$$\hat{\mathbb{E}}_{\mathcal{F}_0}[H] = \frac{1}{\bar{\mathbb{E}}_{\mathcal{F}_0}[d\hat{\mathbb{P}}/d\bar{\mathbb{P}}]} \bar{\mathbb{E}}_{\mathcal{F}_0} \left[ \frac{d\hat{\mathbb{P}}}{d\bar{\mathbb{P}}} H \right] \quad \bar{\mathbb{P}} \quad \text{and} \quad \hat{\mathbb{P}} \quad \text{a.s.}$$

Next we will show the following lemma:

LEMMA A.9. Under our assumptions, for every  $H \in L^\infty(\mathcal{F})$  the process  $M_n = \Pi_{\mathcal{F}^n}(H)$  is a  $\mathbb{Q}_{\mathcal{G}}$ -uniformly integrable<sup>4</sup> martingale.

*Proof.* It may be seen from standard arguments that  $M_n = \Pi_{\mathcal{F}^n}(H)$  is a  $\mathbb{Q}_{\mathcal{G}}$ -martingale. Let us see that  $M_n$  is uniformly integrable under the measure  $\mathbb{Q}_{\mathcal{G}}$ . First of all note that in the case that  $\Pi_{\mathcal{G}}$  is monotone we have that

$$\Pi_{\mathcal{F}^n}(H) \leq \sum_{k=1}^{2^n} \frac{I_{B_k^n}}{q_k^n} \Pi_{\mathcal{G}}(|H|_\infty I_{B_k^n}) = \sum_{k=1}^{2^n} \frac{I_{B_k^n}}{q_k^n} \|H\|_\infty q_k^n = \|H\|_\infty,$$

where we used market-consistency in the first equation. Similarly, it is seen that  $\Pi_{\mathcal{F}^n}(H) \geq -\|H\|_\infty$ . Hence, if  $\Pi_{\mathcal{F}^n}$  is monotone then for fixed  $H$ ,  $\Pi_{\mathcal{F}^n}(H)$  is bounded uniformly. In particular,  $M_n$  is uniformly integrable.

In the case that  $\Pi_{\mathcal{G}}$  is  $p$ -norm bounded, notice that

$$\begin{aligned} |\Pi_{\mathcal{F}^n}(H)| &\leq \lambda \left| \sum_{k=1}^{2^n} \frac{I_{B_k^n}}{q_k^n} \int (|H| + |H|^p) I_{B_k^n} d\bar{\mathbb{P}}_{\mathcal{G}} \right| \\ &\leq \lambda (\|H\|_\infty + \|H\|_\infty^p) \left| \sum_{k=1}^{2^n} I_{B_k^n} \frac{\bar{\mathbb{P}}_{\mathcal{G}}(B_k^n)}{\mathbb{Q}_{\mathcal{G}}(B_k^n)} \right| = \lambda (\|H\|_\infty + \|H\|_\infty^p) \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}} \left[ \frac{d\bar{\mathbb{P}}_{\mathcal{G}}}{d\mathbb{Q}_{\mathcal{G}}} \middle| \mathcal{F}^n \right]. \end{aligned}$$

Since the last term is uniformly integrable,  $\Pi_{\mathcal{F}^n}(H)$  is uniformly integrable as well.  $\square$

Hence, if  $\Pi_{\mathcal{G}}$  is monotone or  $p$ -norm bounded we may conclude by the martingale convergence theorem that  $M_n = \Pi_{\mathcal{F}^n}(H)$  converges a.s. and in  $L^1(\mathbb{Q}_{\mathcal{G}})$  to a random variable  $M_\infty$ . Set

$$\Pi_{\mathcal{F}^S}(H) := M_\infty = \lim_n \Pi_{\mathcal{F}^n}(H).$$

Now  $\Pi_{\mathcal{G}}(H) = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^S}(H)]$  will follow from the following lemma:

LEMMA A.10. For all  $A \in \mathcal{F}^S$  and  $H \in L^\infty(\mathcal{F})$ ,  $\Pi_{\mathcal{F}^S}(H)$  satisfies the characteristic equation:

$$(A.7) \quad \Pi_{\mathcal{G}}(I_A H) = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[I_A \Pi_{\mathcal{F}^S}(H)].$$

In particular,  $\Pi_{\mathcal{G}}(H) = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^S}(H)]$ . Furthermore, for every  $H \in L^\infty(\mathcal{F})$ ,  $\Pi_{\mathcal{F}^S}(H)$  is the unique  $\mathcal{F}^S$ -measurable, and a.s.  $\mathbb{Q}_{\mathcal{G}}$ -integrable random variable satisfying (A.7).

<sup>4</sup>Meaning that  $\lim_N \sup_n \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[I_{\{|\Pi_{\mathcal{F}^n}(H)| > N\}} \Pi_{\mathcal{F}^n}(H)] = 0$ .



*Proof.* Clearly for every  $n$   $\Pi_{\mathcal{F}^n}$  is  $\mathcal{F}^n$ - and hence also  $\mathcal{F}^S$ -measurable. This entails that its limit,  $\Pi_{\mathcal{F}^S}$ , is  $\mathcal{F}^S$ -measurable. Furthermore, for every  $A \in \mathcal{F}^{n_0} \subset \mathcal{F}^{n_0+1} \subset \dots$  by (A.6)

$$(A.8) \quad \Pi_{\mathcal{G}}(I_A H) = \lim_n \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^n}(I_A H)] = \lim_n \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[I_A \Pi_{\mathcal{F}^n}(H)] = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[I_A \Pi_{\mathcal{F}^S}(H)],$$

where we have used in the third equality that  $\Pi_{\mathcal{F}^n}(H)$  converges to  $\Pi_{\mathcal{F}^S}(H)$  in  $L^1(\mathbb{Q}_{\mathcal{G}})$ . Now by Corollary A.7 the left-hand side and the right-hand side are signed measures (measuring sets  $A \in \mathcal{F}^S$ ). As by (A.8) they both agree on  $\bigcup_n \mathcal{F}^n$ , which is closed under intersection and generates  $\mathcal{F}^S$ , they must agree on the entire filtration  $\mathcal{F}^S$ . Uniqueness follows from equation (A.7) using standard arguments.

Now all what is left to prove is the following lemma:

LEMMA A.11.  $\Pi_{\mathcal{F}^S}$  is a continuous,  $\mathcal{F}^S$ -conditional evaluation which is monotone if  $\Pi_{\mathcal{G}}$  is monotone.

*Proof.* By construction,  $\Pi_{\mathcal{F}^S}$  is normalized as  $\Pi_{\mathcal{F}^n}$  are. If  $\Pi_{\mathcal{G}}$  is monotone then the  $\Pi_{\mathcal{F}^n}$  are monotone as well which implies that  $\Pi_{\mathcal{F}^S}$  is monotone.

Next, let us check that  $\Pi_{\mathcal{F}^S}$  satisfies the  $\mathcal{F}^S$ -local property. It is necessary and sufficient that for every  $A \in \mathcal{F}^S$  and  $H \in L^\infty(\mathcal{F})$

$$\Pi_{\mathcal{F}^S}(HI_A) = I_A \Pi_{\mathcal{F}^S}(H).$$

We will prove the equality by showing that the right-hand side satisfies the characteristic equation (A.7) for the left-hand side. So let  $A' \in \mathcal{F}^S$ . It is

$$\Pi_{\mathcal{G}}(I_{A'}(I_A H)) = \Pi_{\mathcal{G}}(I_{A' \cap A} H) = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[I_{A' \cap A} \Pi_{\mathcal{F}^S}(H)] = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[I_{A'}(I_A \Pi_{\mathcal{F}^S}(H))],$$

where we have used (A.7) in the second equation. This shows that  $I_A \Pi_{\mathcal{F}^S}(H)$  satisfies the characteristic equation of  $I_A H$  and hence by the uniqueness stated in Lemma A.10 indeed  $\Pi_{\mathcal{F}^S}(HI_A) = I_A \Pi_{\mathcal{F}^S}(H)$ .

To see  $\mathcal{F}^S$ -cash invariance of  $\Pi_{\mathcal{F}^S}$  assume that for a  $m \in L^\infty(\mathcal{F}^S)$  and  $H \in L^\infty(\mathcal{F})$  we have that the set  $A = \{\Pi_{\mathcal{F}^S}(H + m) \stackrel{(<)}{>} \Pi_{\mathcal{F}^S}(H) + m\}$  has positive measure under  $\mathbb{P}_{\mathcal{G}}$ . Then  $A$  has also positive measure under  $\mathbb{Q}_{\mathcal{G}}$  and

$$\begin{aligned} \Pi_{\mathcal{G}}(HI_A + mI_A) &= \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^S}((H + m)I_A)] = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^S}(H + m)I_A] \\ &\stackrel{(<)}{>} \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[(\Pi_{\mathcal{F}^S}(H) + m)I_A] \\ &= \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^S}(HI_A)] + \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[mI_A] = \Pi_{\mathcal{G}}(HI_A) + \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[mI_A], \end{aligned}$$

where we have used in the second equation the  $\mathcal{F}^S$ -local property of  $\Pi_{\mathcal{F}^S}$ , which we have proved above. This is a contradiction to the market-consistency of  $\Pi_{\mathcal{G}}$ . Thus, indeed  $\Pi_{\mathcal{F}^S}$  is  $\mathcal{F}^S$ -cash invariant.

Next, for  $H \in L^\infty(\mathcal{F})$ , and  $\lambda \in L^\infty(\mathcal{G})$  with  $0 \leq \lambda \leq 1$

$$\begin{aligned}
 (A.9) \quad \Pi_{\mathcal{F}^n}(\lambda H_1 + (1 - \lambda)H_2) &= \sum_{k=1}^{2^n} \frac{I_{B_k^n}}{q_k^n} \Pi_{\mathcal{G}}(\lambda I_{B_k^n} H_1 + (1 - \lambda)I_{B_k^n} H_2) \\
 &\leq \sum_{k=1}^{2^n} \frac{I_{B_k^n}}{q_k^n} (\lambda \Pi_{\mathcal{G}}(I_{B_k^n} H_1) + (1 - \lambda) \Pi_{\mathcal{G}}(I_{B_k^n} H_2)) \\
 &= \lambda \Pi_{\mathcal{F}^n}(H_1) + (1 - \lambda) \Pi_{\mathcal{F}^n}(H_2),
 \end{aligned}$$

where we have used the convexity of  $\Pi_{\mathcal{G}}$  in the inequality. In particular,  $\Pi_{\mathcal{F}^S}$  is  $\mathcal{G}$ -convex as limit of  $\mathcal{G}$ -convex functionals. Next, let  $\lambda^{\mathcal{F}^n} \in L^\infty(\mathcal{F}^n)$  with  $0 \leq \lambda^{\mathcal{F}^n} \leq 1$ . Then there exists disjoint sets  $A_1^n, \dots, A_r^n \in \sigma(A_1, \dots, A_n)$  and constants  $\lambda_1^n, \dots, \lambda_r^n \in L^\infty(\mathcal{G})$  taking values in  $[0, 1]$  with  $\lambda^{\mathcal{F}^n} = \sum_{i=1}^r \lambda_i^n I_{A_i^n}$ . By adding an additional set with an additional constant equal to zero if necessary, we may assume without loss of generality that  $\Omega = \bigcup_{i=1}^r A_i^n$ . From Lemma A.4 it follows that for sets  $C_1, \dots, C_r \in \mathcal{F}^S$

$$\begin{aligned}
 (A.10) \quad \Pi_{\mathcal{F}^n} \left( \sum_{i=1}^r I_{C_i} H_i \right) &= \sum_{k=1}^{2^n} \frac{I_{B_k^n}}{q_k^n} \left( \Pi_{\mathcal{G}} \left( \sum_{i=1}^r I_{B_k^n} I_{C_i} H_i \right) \right) \\
 &= \sum_{i=1}^r \sum_{k=1}^{2^n} \frac{I_{B_k^n}}{q_k^n} (\Pi_{\mathcal{G}}(I_{B_k^n} I_{C_i} H_i)) = \sum_{i=1}^r \Pi_{\mathcal{F}^n}(I_{C_i} H_i).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \Pi_{\mathcal{F}^n}(\lambda^{\mathcal{F}^n} H_1 + (1 - \lambda^{\mathcal{F}^n})H_2) &= \Pi_{\mathcal{F}^n} \left( \sum_{i=1}^r I_{A_i^n} (\lambda_i^n H_1 + (1 - \lambda_i^n)H_2) \right) \\
 &= \sum_{i=1}^r \Pi_{\mathcal{F}^n}(\lambda_i^n I_{A_i^n} H_1 + (1 - \lambda_i^n)I_{A_i^n} H_2) \\
 &\leq \sum_{i=1}^r \lambda_i^n \Pi_{\mathcal{F}^n}(I_{A_i^n} H_1) + (1 - \lambda_i^n) \Pi_{\mathcal{F}^n}(I_{A_i^n} H_2) \\
 &= \sum_{i=1}^r \lambda_i^n I_{A_i^n} \Pi_{\mathcal{F}^n}(H_1) + \sum_{i=1}^r (1 - \lambda_i^n) I_{A_i^n} \Pi_{\mathcal{F}^n}(H_2) \\
 &= \lambda^{\mathcal{F}^n} \Pi_{\mathcal{F}^n}(H_1) + (1 - \lambda^{\mathcal{F}^n}) \Pi_{\mathcal{F}^n}(H_2),
 \end{aligned}$$

where we have used (A.10) in the second equation, (A.9) in the inequality, and the  $\mathcal{F}^n$ -local property for  $\Pi_{\mathcal{F}^n}$ , proved in (A.6), in the third equation. Hence, indeed  $\Pi_{\mathcal{F}^n}$  is  $\mathcal{F}^n$ -convex.

Next, if  $\lambda^{\mathcal{F}^m} \in L^\infty(\mathcal{F}^m)$  with  $0 \leq \lambda^{\mathcal{F}^m} \leq 1$ , and  $m \in \mathbb{N}$  then clearly  $\lambda^{\mathcal{F}^m} \in \mathcal{F}^n$  for every  $n \geq m$ . This entails

$$\begin{aligned}
 (A.11) \quad \Pi_{\mathcal{F}^S}(\lambda^{\mathcal{F}^m} H_1 + (1 - \lambda^{\mathcal{F}^m})H_2) &= \lim_n \Pi_{\mathcal{F}^n}(\lambda^{\mathcal{F}^m} (H_1 + (1 - \lambda^{\mathcal{F}^m})H_2)) \\
 &\leq \lim_n \lambda^{\mathcal{F}^m} \Pi_{\mathcal{F}^n}(H_1) + (1 - \lambda^{\mathcal{F}^m}) \Pi_{\mathcal{F}^n}(H_2) \\
 &= \lambda^{\mathcal{F}^m} \Pi_{\mathcal{F}^S}(H_1) + (1 - \lambda^{\mathcal{F}^m}) \Pi_{\mathcal{F}^S}(H_2).
 \end{aligned}$$

Finally, to see that  $\Pi_{\mathcal{F}^S}$  is  $\mathcal{F}^S$ -convex let  $\lambda^{\mathcal{F}^S} \in L^\infty(\mathcal{F}^S)$  with  $0 \leq \lambda^{\mathcal{F}^S} \leq 1$ . Then also  $0 \leq \mathbb{E}_{\mathcal{F}^n}[\lambda^{\mathcal{F}^S}] \leq 1$ . Furthermore, by the martingale convergence theorem  $\mathbb{E}_{\mathcal{F}^n}[\lambda^{\mathcal{F}^S}]$  converges to  $\mathbb{E}_{\mathcal{F}^S}[\lambda^{\mathcal{F}^S}] = \lambda^{\mathcal{F}^S}$  a.s. We will prove  $\mathcal{F}^S$ -convexity by contradiction. Let  $A := \{\Pi_{\mathcal{F}^S}(\lambda^{\mathcal{F}^S} H_1 + (1 - \lambda^{\mathcal{F}^S}) H_2) > \lambda^{\mathcal{F}^S} \Pi_{\mathcal{F}^S}(H_1) + (1 - \lambda^{\mathcal{F}^S}) \Pi_{\mathcal{F}^S}(H_2)\}$  and suppose that  $A$  is a nonzero set. Then clearly

(A.12)

$$\begin{aligned} \Pi_{\mathcal{G}}(I_A(\lambda^{\mathcal{F}^S} H_1 + (1 - \lambda^{\mathcal{F}^S}) H_2)) &= \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^S}(I_A(\lambda^{\mathcal{F}^S} H_1 + (1 - \lambda^{\mathcal{F}^S}) H_2))] \\ &= \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[I_A(\Pi_{\mathcal{F}^S}(\lambda^{\mathcal{F}^S} H_1 + (1 - \lambda^{\mathcal{F}^S}) H_2))] \\ &> \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[I_A(\lambda^{\mathcal{F}^S} \Pi_{\mathcal{F}^S}(H_1) + (1 - \lambda^{\mathcal{F}^S}) \Pi_{\mathcal{F}^S}(H_2))], \end{aligned}$$

where we used the local property for  $\Pi_{\mathcal{F}^S}$ . On the other hand,

$$\begin{aligned} \Pi_{\mathcal{G}}(I_A(\lambda^{\mathcal{F}^S} H_1 + (1 - \lambda^{\mathcal{F}^S}) H_2)) &= \lim_n \Pi_{\mathcal{G}}(\mathbb{E}_{\mathcal{F}^n}[\lambda^{\mathcal{F}^S}] I_A H_1 + (1 - \mathbb{E}_{\mathcal{F}^n}[\lambda^{\mathcal{F}^S}]) I_A H_2) \\ &= \lim_n \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\Pi_{\mathcal{F}^S}(\mathbb{E}_{\mathcal{F}^n}[\lambda^{\mathcal{F}^S}] I_A H_1 + (1 - \mathbb{E}_{\mathcal{F}^n}[\lambda^{\mathcal{F}^S}]) I_A H_2)] \\ &\leq \lim_n \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\mathbb{E}_{\mathcal{F}^n}[\lambda^{\mathcal{F}^S}] \Pi_{\mathcal{F}^S}(I_A H_1) + (1 - \mathbb{E}_{\mathcal{F}^n}[\lambda^{\mathcal{F}^S}]) \Pi_{\mathcal{F}^S}(I_A H_2)] \\ &= \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}[\lambda^{\mathcal{F}^S} I_A \Pi_{\mathcal{F}^S}(H_1) + (1 - \lambda^{\mathcal{F}^S}) I_A \Pi_{\mathcal{F}^S}(H_2)], \end{aligned}$$

where we have used the continuity of  $\Pi_{\mathcal{G}}$  in the first equality and (A.11) in the inequality. This is a contradiction to (A.12). Therefore, the set  $A$  must be a zero set and the  $\mathcal{F}^S$ -convexity is proved.

Now let us show that  $\Pi_{\mathcal{F}^S}$  is continuous. In the case that  $\Pi_{\mathcal{G}}$  is monotone,  $\Pi_{\mathcal{F}^S}$  is monotone as well and the continuity of  $\Pi_{\mathcal{F}^S}$  follows from the characteristic equation using standard arguments. On the other hand if  $\Pi_{\mathcal{G}}$  is  $p$ -norm bounded we get for  $H \in L^\infty(\mathcal{F})$

$$\begin{aligned} |\Pi_{\mathcal{F}^n}(H)| &\leq \lambda \left| \sum_{k=1}^{2^n} \frac{I_{B_k^n}}{q_k^n} \int (|H| + |H|^p) I_{B_k^n} d\bar{\mathbb{P}}_{\mathcal{G}} \right| \\ &= \lambda \left| \sum_{k=1}^{2^n} \frac{I_{B_k^n}}{q_k^n} \int (|H| + |H|^p) I_{B_k^n} \frac{d\bar{\mathbb{P}}_{\mathcal{G}}}{d\mathbb{Q}_{\mathcal{G}}} d\mathbb{Q}_{\mathcal{G}} \right| = \lambda \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}} \left[ (|H| + |H|^p) \frac{d\bar{\mathbb{P}}_{\mathcal{G}}}{d\mathbb{Q}_{\mathcal{G}}} \middle| \mathcal{F}^n \right]. \end{aligned}$$

Therefore,

(A.13)

$$\begin{aligned} |\Pi_{\mathcal{F}^S}(H)| &= \lim_n |\Pi_{\mathcal{F}^n}(H)| \\ &\leq \lambda \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}} \left[ (|H| + |H|^p) \frac{d\bar{\mathbb{P}}_{\mathcal{G}}}{d\mathbb{Q}_{\mathcal{G}}} \middle| \mathcal{F}^S \right] \end{aligned}$$

$$\begin{aligned}
&= \lambda \mathbb{E}^{\mathbb{Q}_G} \left[ \frac{d\bar{\mathbb{P}}_G}{d\mathbb{Q}_G} \middle| \mathcal{F}^S \right] \frac{\mathbb{E}^{\mathbb{Q}_G} \left[ (|H| + |H|^p) \frac{d\bar{\mathbb{P}}_G}{d\mathbb{Q}_G} \middle| \mathcal{F}^S \right]}{\mathbb{E}^{\mathbb{Q}_G} \left[ \frac{d\bar{\mathbb{P}}_G}{d\mathbb{Q}_G} \middle| \mathcal{F}^S \right]} \\
&= \lambda \mathbb{E}^{\mathbb{Q}_G} \left[ \frac{d\bar{\mathbb{P}}_G}{d\mathbb{Q}_G} \middle| \mathcal{F}^S \right] \int (|H| + |H|^p) d\bar{\mathbb{P}}_{\mathcal{F}^S}.
\end{aligned}$$

The third equation holds by Lemma A.8. Now by (A.13) we can extend  $\Pi_{\mathcal{F}^S}$  to  $L^p(\Omega, \mathcal{F}, \bar{\mathbb{P}})$  by setting

$$\tilde{\Pi}_{\mathcal{F}^S}(H) = \limsup_{N \rightarrow \infty} \limsup_{m \rightarrow \infty} \Pi_{\mathcal{F}^S}(-m \vee H \wedge N).$$

Note that we have that  $\tilde{\Pi}_{\mathcal{F}^S}$  is  $\mathcal{F}^S$ -convex (as the lim sup of  $\mathcal{F}^S$ -convex functionals), and agrees with  $\Pi_{\mathcal{F}^S}$  on  $L^\infty(\mathcal{F})$ . By (A.13), for every  $H$  we have that  $\tilde{\Pi}_{\mathcal{F}^S}(H)$  is real-valued and uniformly bounded in any  $L^p(\Omega, \mathcal{F}, \bar{\mathbb{P}}_{\mathcal{F}^S})$  environment around  $H$ .

It follows then from standard arguments for convex functionals, see for instance theorem 2.2.9 in Zălinescu (2002), that convergence of  $H_n$  to  $H$  in  $L^p(\Omega, \mathcal{F}, \bar{\mathbb{P}}_{\mathcal{F}^S})$  implies that  $\tilde{\Pi}_{\mathcal{F}^S}(H_n)$  converges to  $\tilde{\Pi}_{\mathcal{F}^S}(H)$ . Since  $\tilde{\Pi}_{\mathcal{F}^S}$  and  $\Pi_{\mathcal{F}^S}$  agree on  $L^\infty(\mathcal{F})$  we may conclude that indeed  $\Pi_{\mathcal{F}^S}$  is continuous with respect to bounded a.s. convergence. The lemma is proved.  $\square$

Lemma A.10 and Lemma A.11 imply the theorem.

### A.3. Proofs of the Results in Section 4

For the proof of Theorem 4.3 we will need the following lemma:

**LEMMA A.12.** *In the setting of Theorem 4.3, let  $\tau$  be a stopping time such that  $\sigma \leq \tau < \tau_\sigma$ . Let  $H$  be a bounded,  $\mathcal{F}_\tau$ -measurable payoff. Then  $\Pi_\sigma(H) = \mathbb{E}_{\mathcal{F}_\sigma}^{\mathbb{Q}}[H]$ .*

*Proof.* As  $\bar{\mathcal{F}}_{\tau_{i+1}-}^Y = \bar{\mathcal{F}}_{\tau_i}^Y$  for all  $i$ ,  $H$  is  $\bar{\mathcal{F}}_\tau^S \vee \mathcal{F}_\sigma$ -measurable. Consequently, the lemma follows directly from the definition of market-consistency.  $\square$

*Proof of Theorem 4.3.* As the jump times of  $Y$  for the  $i$ -th jump,  $(\tau_i)$ , are predictable, respectively,  $\tau_\sigma$  has to be predictable as well. This is seen as follows: First of all note that by the predictability of  $\tau_i$  there exists sequences of stopping times  $(\tau_i^n)$  with  $\tau_i^n < \tau_i$  and  $\tau_i^n \uparrow \tau_i$  as  $n \rightarrow \infty$ . Define  $\sigma^n := \sum_{i=1}^\infty I_{\{\tau_i = \tau_\sigma\}}(\tau_i^n \vee \sigma)$ . Let  $\tau_0 : 0$ . Then  $\{\tau_i = \tau_\sigma\} = \{\tau_{i-1} \leq \sigma < \tau_i\} \in \mathcal{F}_\sigma$  for  $i = 1, 2, \dots$  (Since  $\tau_\sigma$  is the first jump after time  $\sigma$ , we know at time  $\sigma$ , if we have observed  $i - 1$  jumps so far, so that the next jump will be the  $i$ -th one.) Therefore,

$$\{\sigma^n \leq t\} = \bigcup_{i=1}^\infty (\{\tau_i^n \leq t\} \cap \{\sigma \leq t\} \cap \{\tau_i = \tau_\sigma\}) \in \mathcal{F}_t.$$

Thus,  $\sigma^n$  is indeed a stopping time. Furthermore, clearly  $\sigma^n < \tau_\sigma$  and

$$\sigma^n \uparrow \sum_{i=1}^{\infty} I_{\{\tau_i = \tau_\sigma\}}(\tau_i \vee \sigma) = \sum_{i=1}^{\infty} I_{\{\tau_i = \tau_\sigma\}}(\tau_\sigma \vee \sigma) = \sum_{i=1}^{\infty} I_{\{\tau_i = \tau_\sigma\}} \tau_\sigma = \tau_\sigma,$$

where we have used that  $\sigma < \tau_\sigma$ . Hence,  $\tau_\sigma$  is indeed predictable.

Next, let  $H \in L^\infty(\mathcal{F}_{\tau_\sigma})$  and  $A \in \mathcal{F}_{\sigma^m}$  for an  $m \in \mathbb{N}$ . By time-consistency and the local property of  $(\Pi_\sigma)$  we get for all  $n \geq m$

$$\Pi_\sigma(H) = \Pi_\sigma(\Pi_{\sigma^n}(HI_A + HI_{A^c})) = \Pi_\sigma(I_A \Pi_{\sigma^n}(H) + I_{A^c} \Pi_{\sigma^n}(H)).$$

Next observe that for  $n \geq m$ ,  $I_A \Pi_{\sigma^n}(H) + I_{A^c} \Pi_{\sigma^n}(H)$  is  $\mathcal{F}_{\sigma^n}$ -measurable. By Lemma A.12 this implies that

$$\begin{aligned} \Pi_\sigma(I_A \Pi_{\sigma^n}(H) + I_{A^c} \Pi_{\sigma^n}(H)) &= \mathbb{E}_{\mathcal{F}_\sigma}^\mathbb{Q} [I_A \Pi_{\sigma^n}(H) + I_{A^c} \Pi_{\sigma^n}(H)] \\ &= \mathbb{E}_{\mathcal{F}_\sigma}^\mathbb{Q} [I_A \Pi_{\sigma^n}(H)] + \mathbb{E}_{\mathcal{F}_\sigma}^\mathbb{Q} [I_{A^c} \Pi_{\sigma^n}(H)] \\ &= \mathbb{E}_{\mathcal{F}_\sigma}^\mathbb{Q} [\Pi_{\sigma^n}(I_A H)] + \mathbb{E}_{\mathcal{F}_\sigma}^\mathbb{Q} [\Pi_{\sigma^n}(I_{A^c} H)] \\ &= \Pi_\sigma(\Pi_{\sigma^n}(I_A H)) + \Pi_\sigma(\Pi_{\sigma^n}(I_{A^c} H)) \\ &= \Pi_\sigma(I_A H) + \Pi_\sigma(I_{A^c} H), \end{aligned}$$

where we used Lemma A.5 in the third and time-consistency in the last equation. Hence, for every  $A \in \mathcal{F}_{\sigma^m}$  we have that

$$(A.14) \quad \Pi_\sigma(H) = \Pi_\sigma(I_A H) + \Pi_\sigma(I_{A^c} H).$$

Since by assumption  $S$  is continuous and  $\tilde{\mathcal{F}}_{\tau_{i+1}-}^Y = \tilde{\mathcal{F}}_{\tau_i}^Y$  for all  $i$ ,  $\bigcup_m \mathcal{F}_{\sigma^m}$  is a generating system for  $\mathcal{F}_{\tau_\sigma}^S$ . By the continuity of  $\Pi_\sigma$  this entails that (A.14) holds for all  $A \in \mathcal{F}_{\tau_\sigma}^S$ . Thus, indeed  $\Pi_\sigma$  restricted to  $\mathcal{F}_{\tau_\sigma}^S$  satisfies the market-local property. This proves the first part of the theorem. The second part follows from Theorems 3.10 and 3.11.

#### A.4. Proofs of the Results in Section 5.1

*Proof of Theorem 5.3.* First assume that  $g(t, z^f, z, \tilde{z}) - \theta_t z^f$  does not depend on  $z^f$ . We will prove that for every  $\tau$ ,  $\mathcal{E}_\tau^g$  is market-consistent, by using Proposition 3.3 (i)  $\Rightarrow$  (iii). Without loss of generality assume that  $\tau = 0$ . Let  $H^S \in L^\infty(\tilde{\mathcal{F}}_T^S)$ . Denote by  $W^{*,f}$  the Brownian motion under  $\mathbb{Q}$ , i.e.,  $W_s^{*,f} = W_s^f - \int_0^s \theta_u du$ , where the integral is defined componentwise. Since the financial market is complete there exists a predictable  $n$ -dimensional process  $Z^f \in L^2(\tilde{\mathcal{F}}^S, d\mathbb{Q} \times du)$  such that  $H^S = \int_0^T Z_s^f dW_s^{*,f}$ . Define  $Y_s = \mathbb{E}_{\tilde{\mathcal{F}}_s^S}^\mathbb{Q}[H^S] = \int_0^s Z_u^f dW_u^{*,f}$ . Clearly, for every stopping time  $\sigma$  we have that

$$\mathbb{E}_{\tilde{\mathcal{F}}_\tau^S} \left[ \int_\sigma^T |Z_u^f|^2 dW_u^{*,f} \right] \leq 2 \|Y\|_{S^\infty}^2 \leq 2 \|H\|_\infty^2.$$

Predictable processes satisfying such a boundedness property are also called bounded mean oscillations (BMOs), see Kazamaki (1994). As  $\theta$  is bounded it follows from theorem 3.24 in Barrieu and El Karoui (2009) that  $Z^f$  is a BMO under  $\mathbb{P}$ . In particular,

$Z^f \in L^2(\bar{\mathcal{F}}^S, d\mathbb{P} \times du)$ . Now we get

$$\begin{aligned} dY_s &= -g(s, 0, 0, 0)ds + Z_s^f dW_s^{*,f} \\ &= -(g(s, Z_s^f, 0, 0) - \theta_s Z_s^f)ds + Z_s^f dW_s^{*,f} = -g(s, Z_s^f, 0, 0)ds + Z_s^f dW_s^f, \end{aligned}$$

where we used that  $g(s, 0, 0, 0) = 0$  in the first equation. In the second equation we applied that  $g(s, 0, 0, 0) = g(s, z^f, 0, 0) - \theta_s z^f$ , as by assumption  $g(s, z^f, 0, 0) - \theta_s z^f$  does not depend on  $z^f$ . In the last equation we used the definition of  $W^{*,f}$ . This entails that  $Y_s = \mathbb{E}_{\bar{\mathcal{F}}_s}^{\mathbb{Q}}[H]$  solves the BSDE with terminal condition  $H$  and driver  $g$ . Therefore, indeed for every  $H^S$  we have that  $\mathbb{E}^{\mathbb{Q}}[H^S] = \mathcal{E}_0^g(H^S)$ , and it follows from Proposition 3.3 (i) $\Rightarrow$ (iii) (with  $\mathcal{G} = \{\Omega, \emptyset\}$ ) that  $\mathcal{E}_0^g$  is market-consistent. For general  $\tau$  the argument is similar.

Now let us prove the other direction. For arbitrary  $\bar{z}^f \in \mathbb{R}^n$  define

$$\bar{g}(t, z^f, z, \tilde{z}) := g(t, z^f + \bar{z}^f, z, \tilde{z}) - \theta_t \bar{z}^f.$$

We need to show that  $\bar{g} = g$ . Let  $(\mathcal{E}_t^g(H + \bar{z}^f W_T^{*,f}), Z^f, Z, \tilde{Z})$  be the solution of the BSDE with terminal condition  $H + \bar{z}^f W_T^{*,f}$  and driver function  $g$ . Note that the process  $Y_t^* := \mathcal{E}_t^g(H + \bar{z}^f W_T^{*,f}) - \bar{z}^f W_t^{*,f}$  is equal to  $H$  at time  $T$ . On the other hand we have

$$\begin{aligned} dY_t^* &= -(g(t, Z_t^f, Z_t, \tilde{Z}_t) - \theta_t \bar{z}^f)dt + (Z_t^f - \bar{z}^f)dW_t^f + Z_t dW_t + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_t(x) \tilde{N}(dt, dx) \\ &= -\bar{g}(t, Z_{new,t}^f, Z_t, \tilde{Z}_t)dt + Z_{new,t}^f dW_t^f + Z_t dW_t + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_t(x) \tilde{N}(dt, dx), \end{aligned}$$

where  $Z_{new}^f := Z^f - \bar{z}^f$ . Therefore,  $Y^*$  solves the BSDE with terminal condition  $H$  and driver  $\bar{g}$ . Hence, for every  $t$  we have  $\mathcal{E}_t^g(H + \bar{z}^f W_T^{*,f}) - \bar{z}^f W_t^{*,f} = Y_t^* = \mathcal{E}_t^{\bar{g}}(H)$ . Denote by  $\tilde{S}$  the vector of the discounted stock prices. By market-consistency we can conclude that for every  $H \in L^\infty(\mathcal{F}_T)$

$$\begin{aligned} \text{(A.15)} \quad \mathcal{E}_t^g(H) &= \mathcal{E}_t^g\left(H + \int_0^T \bar{z}^f \tilde{\sigma}^{-1}(t, e^{rt} \tilde{S}_t) d\tilde{S}_t\right) - \int_0^t \bar{z}^f \tilde{\sigma}^{-1}(t, e^{rt} \tilde{S}_t) d\tilde{S}_t \\ &= \mathcal{E}_t^g(H + \bar{z}^f W_T^{*,f}) - \bar{z}^f W_t^{*,f} = \mathcal{E}_t^{\bar{g}}(H), \end{aligned}$$

for all  $t$ . Next choose  $z^f \in \mathbb{R}^n$ ,  $z \in \mathbb{R}^d$ , and  $\tilde{z} \in L^2(\nu(dx))$ . Set

$$H := -\int_0^T g(s, z^f, z, \tilde{z})ds + z^f W_T^f + z W_T + \int_0^T \int_{\mathbb{R} \setminus \{0\}} \tilde{z}(x) \tilde{N}(ds, dx).$$

Notice that  $-\int_0^t g(s, z^f, z, \tilde{z})ds + \int_0^t z^f dW_s^f + \int_0^t z dW_s + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \tilde{z}(x) \tilde{N}(ds, dx)$  by definition is the solution of the BSDE with terminal condition  $H$  and driver  $g$ . In

particular,  $\mathcal{E}_0^g(H) = 0$ . By (A.15) this yields  $\mathcal{E}_0^{\tilde{g}}(H) = \mathcal{E}_0^g(H) = 0$  and

$$\begin{aligned}
 & - \int_0^t g(s, z^f, z, \tilde{z}) ds + \int_0^t z^f dW_s^f + \int_0^t z dW_s + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \tilde{z}(x) \tilde{N}(ds, dx) \\
 & = \mathcal{E}_t^g(H) = \mathcal{E}_t^{\tilde{g}}(H) \\
 & = \mathcal{E}_0^{\tilde{g}}(H) - \int_0^t \tilde{g}(s, Z_s^f, Z_s, \tilde{Z}_s) ds + \int_0^t Z_s^f dW_s^f + \int_0^t Z_s dW_s \\
 & \quad + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_s(x) \tilde{N}(ds, dx) \\
 & = - \int_0^t \tilde{g}(s, Z_s^f, Z_s, \tilde{Z}_s) ds + \int_0^t Z_s^f dW_s^f + \int_0^t Z_s dW_s + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_s(x) \tilde{N}(ds, dx),
 \end{aligned}$$

where  $(Z^f, Z, \tilde{Z})$  belong to the solution of the  $\tilde{g}$ -expectation with terminal condition  $H$ . By the uniqueness of the decomposition of semimartingales this entails that

$$\begin{aligned}
 (A.16) \quad & \int_0^t z^f dW_s^f + \int_0^t z dW_s + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \tilde{z}(x) \tilde{N}(ds, dx) \\
 & = \int_0^t Z_s^f dW_s^f + \int_0^t Z_s dW_s + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_s(x) \tilde{N}(ds, dx),
 \end{aligned}$$

and

$$(A.17) \quad \int_0^t g(s, z^f, z, \tilde{z}) ds = \int_0^t \tilde{g}(s, Z_s^f, Z_s, \tilde{Z}_s) ds.$$

Taking for instance the quadratic covariation with respect to the components of  $W^f$ ,  $W$ , and with respect to  $\tilde{N}$  in (A.16), respectively, we may conclude that  $Z_t^f = z^f$ ,  $Z_t = z$ ,  $d\mathbb{P} \times dt$  a.s., and  $\tilde{Z}_t = \tilde{z}$ ,  $v(dx) \times d\mathbb{P} \times dt$ . But then (A.17) yields that for a.s. all  $\omega$

$$\int_0^t g(s, z^f, z, \tilde{z}) ds = \int_0^t \tilde{g}(s, z^f, z, \tilde{z}) ds, \quad \text{for all } t \in [0, T]$$

and therefore  $g(t, z^f, z, \tilde{z}) = \tilde{g}(t, z^f, z, \tilde{z})$  for a.s. all  $\omega$  for Lebesgue a.s. all  $t$ .  $\square$

## A.5. Proofs of the Results in Section 5.2

*Proof of Proposition 5.5.* First of all note that since  $\frac{d\mathbb{Q}^h}{d\mathbb{P}}$  is  $\sigma(W_t^{f,*} | 0 \leq t \leq T)$ -measurable, we have that  $W$  and  $\tilde{N}$  have the same joint distribution under  $\mathbb{Q}^h$  as under  $\mathbb{P}$  (since they are independent of  $W^{f,*}$ ).

By well-known projection results, there exists adapted  $Z_{ih}^{h,f} : \Omega \rightarrow \mathbb{R}^n$ ,  $Z_{ih}^h : \Omega \rightarrow \mathbb{R}^d$ , measurable with respect to  $\mathcal{F}_{ih}$ ,  $\tilde{Z}_{ih}^h : \Omega \times \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ , measurable with respect to  $\mathcal{F}_{ih} \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ , and a real-valued  $\mathbb{Q}^h$ -martingale  $(L_{ih}^h)_i$  which is orthogonal (under  $\mathbb{Q}^h$ ) to  $W_{ih}^{f,*}$ ,  $W_{ih}$ , and  $\tilde{N}([0, ih], dx)$ , such that

$$\begin{aligned}
 \Pi_{(i+1)h}(H) &= \mathbb{E}_{\mathcal{F}_{ih}}^{\mathbb{Q}^h}[\Pi_{(i+1)h}(H)] + Z_{ih}^{h,f} \Delta W_{(i+1)h}^{f,*} + Z_{ih}^h \Delta W_{(i+1)h} \\
 &\quad + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_{ih}^h(x) \tilde{N}([ih, (i+1)h], dx) + \Delta L_{(i+1)h}^h.
 \end{aligned}$$

For the sake of simplicity we will omit the superscript  $h$  for the  $Z^{h,f}$ ,  $Z^h$ , and  $\tilde{Z}^h$  in the sequel.

It follows that

$$\begin{aligned}
 & \text{(A.18)} \\
 \Delta \Pi_{(i+1)h}(H) &= \Pi_{(i+1)h}(H) - \Pi_{ih}(H) \\
 &= \Pi_{(i+1)h}(H) - \Pi_{ih,(i+1)h}^v(\Pi_{(i+1)h}(H)) \\
 &= Z_{ih}^f \Delta W_{(i+1)h}^{f,*} + Z_{ih} \Delta W_{(i+1)h} \\
 &\quad + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_{ih}(x) \tilde{N}((ih, (i+1)h], dx) + \Delta L_{(i+1)h} \\
 &\quad - \Pi_{ih}^v \left( Z_{ih}^f \Delta W_{(i+1)h}^{f,*} + Z_{ih} \Delta W_{(i+1)h} \right. \\
 &\quad \left. + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_{ih}(x) \tilde{N}((ih, (i+1)h], dx) + \Delta L_{(i+1)h} \right) \\
 &= Z_{ih}^f \Delta W_{(i+1)h}^{f,*} + Z_{ih} \Delta W_{(i+1)h} + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_{ih}(x) \tilde{N}((ih, (i+1)h], dx) + \Delta L_{(i+1)h} \\
 &\quad - \Pi_{ih,(i+1)h}^v \left( Z_{ih} \Delta W_{(i+1)h} + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_{ih}(x) \tilde{N}((ih, (i+1)h], dx) + \Delta L_{(i+1)h} \right) \\
 &\stackrel{!}{=} Z_{ih}^f \Delta W_{(i+1)h}^f + Z_{ih} \Delta W_{(i+1)h} + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_{ih}(x) \tilde{N}((ih, (i+1)h], dx) + \Delta L_{(i+1)h} \\
 &\quad - \left[ \theta_{ih} Z_{ih}^f + \frac{\alpha}{2} |Z_{ih}|^2 + \frac{\alpha}{2} \int_{\mathbb{R} \setminus \{0\}} |\tilde{Z}_{ih}(x)|^2 v(dx) \right] h \\
 &\quad - \frac{\alpha}{2} \mathbb{E}_{\mathcal{F}_{ih}}^{\mathbb{Q}^h} \left[ (\Delta L_{(i+1)h} - \mathbb{E}_{\mathcal{F}_{(i+1)h}^{S^h}} [\Delta L_{(i+1)h}])^2 \right],
 \end{aligned}$$

where we have used (5.1) in the third equation. Furthermore, we applied cash invariance in the third and market-consistency in the fourth equation. To see that the last equation holds denote by  $\text{Cov}_{\mathbb{P}_{\mathcal{F}_{(i+1)h}^{S^h}}}^{\mathbb{P}}(X_1, X_2)$  the covariance of  $X_1$  and  $X_2$  with respect to  $\mathbb{P}_{\mathcal{F}_{(i+1)h}^{S^h}}$ .

Since all random variables are  $\mathcal{F}_{(i+1)h}$ -measurable we may assume that  $\frac{d\mathbb{Q}^h}{d\mathbb{P}}$  is  $\mathcal{F}_{(i+1)h}^{S^h}$ -measurable.

It is

$$\begin{aligned}
 & \text{(A.19)} \\
 \Pi_{ih,(i+1)h}^v & \left( Z_{ih} \Delta W_{(i+1)h} + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_{ih}(x) \tilde{N}((ih, (i+1)h], dx) + \Delta L_{(i+1)h} \right) \\
 &= \frac{\alpha}{2} \mathbb{E}_{\mathcal{F}_{ih}}^{\mathbb{Q}^h} \left[ \text{Var}_{\mathcal{F}_{(i+1)h}^{S^h}} \left( Z_{ih} \Delta W_{(i+1)h} + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_{ih}(x) \tilde{N}((ih, (i+1)h], dx) + \Delta L_{(i+1)h} \right) \right] \\
 &= \frac{\alpha}{2} \left( h |Z_{ih}|^2 + h \int_{\mathbb{R} \setminus \{0\}} |\tilde{Z}_{ih}(x)|^2 v(dx) + \mathbb{E}_{\mathcal{F}_{ih}}^{\mathbb{Q}^h} \left[ (\Delta L_{(i+1)h} - \mathbb{E}_{\mathcal{F}_{(i+1)h}^{S^h}} [\Delta L_{(i+1)h}])^2 \right] \right. \\
 &\quad \left. + 2 \sum_{j=1}^d Z_{ih}^j \mathbb{E}_{\mathcal{F}_{ih}}^{\mathbb{Q}^h} \left[ \text{Cov}_{\mathbb{P}_{\mathcal{F}_{(i+1)h}^{S^h}}} (\Delta W_{(i+1)h}^j, \Delta L_{(i+1)h}) \right] \right)
 \end{aligned}$$



$$\begin{aligned}
& + 2\mathbb{E}_{\mathcal{F}_{ih}}^{\mathbb{Q}^h} \left[ \text{Cov}_{\mathbb{P}_{\mathcal{F}_{(i+1)h}}^{S^h}} \left( \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_{ih}(x) \tilde{N}((ih, (i+1)h], dx), \Delta L_{(i+1)h} \right) \right] \\
& + 2 \sum_{j=1}^d Z_{ih}^j \mathbb{E}_{\mathcal{F}_{ih}}^{\mathbb{Q}^h} \left[ \text{Cov}_{\mathbb{P}_{\mathcal{F}_{(i+1)h}}^{S^h}} \left( \Delta W_{(i+1)h}^j, \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_{ih}(x) \tilde{N}((ih, (i+1)h], dx) \right) \right],
\end{aligned}$$

where we have used that  $W$  and  $\tilde{N}$  are independent of  $S^h$  and that  $\frac{d\mathbb{Q}^h}{d\mathbb{P}}$  is  $\mathcal{F}_{(i+1)h}^{S^h}$ -measurable.

Hence, to prove (A.18) the only thing what remains left is to show that the covariance terms in (A.19) are zero. By Lemma A.8 (with  $\hat{\mathbb{P}} = \mathbb{Q}^h$  and  $\bar{\mathbb{P}} = \mathbb{P}$ ) we get for  $j = 1, \dots, d$

$$\mathbb{E}_{\mathcal{F}_{(i+1)h}^{S^h}}^{\mathbb{Q}^h} [\Delta W_{(i+1)h}^j] = \mathbb{E}_{\mathcal{F}_{(i+1)h}^{S^h}} [\Delta W_{(i+1)h}^j] = \mathbb{E}_{\mathcal{F}_{ih}} [\Delta W_{(i+1)h}^j] = 0,$$

where we used the independence of  $S^h$  and  $W$  in the second equation. Hence, since  $L_{(i+1)h}$  and  $\Delta W_{(i+1)h}$  are orthogonal under  $\mathbb{Q}_{\mathcal{F}_{ih}}^h$ , we may conclude that

$$\begin{aligned}
\mathbb{E}_{\mathcal{F}_{ih}}^{\mathbb{Q}^h} [\text{Cov}_{\mathbb{P}_{\mathcal{F}_{(i+1)h}}^{S^h}} (\Delta W_{(i+1)h}^j, \Delta L_{(i+1)h})] &= \mathbb{E}_{\mathcal{F}_{ih}}^{\mathbb{Q}^h} [\Delta W_{(i+1)h}^j \Delta L_{(i+1)h}] \\
&= \text{Cov}_{\mathbb{Q}_{\mathcal{F}_{ih}}^h} (\Delta W_{(i+1)h}^j, \Delta L_{(i+1)h}) = 0.
\end{aligned}$$

Similarly, it may be seen that the second and third covariance terms in (A.18) are zero. Thus, (A.18) is proved. From (A.18) we may finally conclude

$$\begin{aligned}
\Pi_T(H) - \Pi_{ih}(H) &= \sum_{j=i}^{T/h-1} \Delta \Pi_{(j+1)h}(H) \\
&= \sum_{j=i}^{T/h-1} \left( Z_{jh}^f \Delta W_{(j+1)h}^f + Z_{jh} \Delta W_{(j+1)h} \right. \\
&\quad \left. + \int_{\mathbb{R} \setminus \{0\}} \tilde{Z}_{jh}(x) \tilde{N}((jh, (j+1)h], dx) + \Delta L_{(j+1)h} \right) \\
&\quad - \sum_{j=i}^{T/h-1} \left[ \theta_{jh} Z_{jh}^f + \frac{\alpha}{2} |Z_{jh}|^2 + \frac{\alpha}{2} \int_{\mathbb{R} \setminus \{0\}} |\tilde{Z}_{jh}(x)|^2 \nu(dx) \right] h \\
&\quad + \frac{\alpha}{2} \mathbb{E}_{\mathcal{F}_{jh}}^{\mathbb{Q}^h} \left[ (\Delta L_{(j+1)h} - \mathbb{E}_{\mathcal{F}_{(j+1)h}^{S^h}} [\Delta L_{(j+1)h}])^2 \right].
\end{aligned}$$

Since by construction  $\Pi_T(H) = H$  the proposition is proved.  $\square$

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