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Insurance: Mathematics and Economics 35 (2004) 369–398

INSURANCE MATHEMATICS & ECONOMICS

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Pricing Rate of Return Guarantees in Regular Premium Unit Linked Insurance

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Received October 2003; received in revised form June 2004; accepted 13 July 2004

Abstract

We derive general pricing formulas for Rate of Return Guarantees in Regular Premium Unit Linked Insurance under stochastic interest rates. Our main contribution focusses on the effect of stochastic interest rates. First, we show the effect of stochastic interest rates can be interpreted as, what is known in the financial community as, a convexity correction. Second we link the LIBOR Market Model to our model of the economy. This allows us to find guarantee prices consistent with observed cap and swaption prices. Numerical results show the effect of this more sophisticated interest rate modelling is considerable. We also consider ways of approximating Asian option values through tight bounds. We show that we can obtain accurate bounds in spite of the high volatility induced by the long maturities of the guarantees.

JEL classification: G12; G13; G22

SIBC classification: IE43; IE50

Keywords: Return guarantee; Average rate option; Convexity correction; LIBOR Market Model

1. Introduction

Unit Linked (UL) insurance is a form of insurance where the policyholder bears the investment risk. The premiums are invested in several investment funds which usually invest a large percentage of their money in stocks. Sometimes

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0167-6687/$ – see front matter © 2004 Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2004.07.003

the policyholder is even allowed to invest directly in stocks. Rate of return guarantees in a UL context can therefore be considered as some kind of stock option. Many insurance companies have given guarantees on UL contracts in the beginning of the nineties, not realizing the risk attached to this product characteristic. With the current bearish stock markets and Fair Value calculations at the center of attention it should be realized that these options are in or at least at the money. The results in this paper can help quantify the risk attached to the guarantees and provide in the need for market values of insurance liabilities.

A single premium UL contract with maturity guarantee can be viewed as a stock along with a put option on that stock (Brennan and Schwartz, 1976). Bacinello and Ortu (1993) and Nielsen and Sandmann (1995, 1996a,b, 2002b) analyze the periodic premium contract with maturity guarantee from a fair premium principle perspective.

Our contribution to the existing literature is threefold. First we take a different approach to the insurance contract as we let the cost and mortality deductions be exogenously given. This is the case in practice and differs from the approach by Nielsen and Sandmann (1995, 1996a, 2002b) as they derive the existence of Fair Premium principles for the guarantee within the contract. The payment of multiple premiums makes the option payoff dependent on the stock price at different time points, which leads to an analogy with Asian options. We show that for a generic structure of the cost and mortality deductions the structure of the option payoff remains Asian like. In our approach we take full account of all insurance aspects of the contract. We will apply option pricing techniques to the context of UL products using Change of Numeraire methods. We derive a general pricing formula for the guarantee. It turns out the guarantee can be expressed as a put option on a stochastically weighted average of the stock price at maturity.

Second, we make the analogy of the guarantee with Asian options explicit by proving equality between prices of both contracts in a constant interest rate environment. This setup allows for stochastic (stationary) volatility however the analogy brakes down when stochastic interest rates are introduced. We discuss the cause of the differences that arise when interest rates become stochastic. This already gives some intuition on how these contracts can best be hedged.

Third and most importantly, we argue for a more general setup of the randomness of the term structure. This is done in the following steps. Choosing a convexity correction approach we specify a quite general lognormal model of the economy. In this model we derive results for the Levy (1992) approximation to the price of the guarantee, next to an upper and lower bound to this price extending work by Rogers and Shi (1995), Nielsen and Sandmann (2002a), Thompson (1998). It turns out the effect of stochastic interest rates can be interpreted as a convexity correction. Then we show how our setup can be linked with the popular LIBOR Market Model (LMM). For long term options, typically encountered in life insurance, the convexity correction effect of stochastic interest has a high impact on the price of the option. Realizing this, we conjecture that it is of interest to use more sophisticated term structure models. We provide in the need to have a stock option pricing model which has its term structure part in accordance with the dominant term structure model in the option pricing literature, the LMM. Our results provide the link between the standard Black–Scholes stock model and the Black pricing model for Caps (and Swaptions). We provide approximate expressions for the forward bond volatility in a LMM. Building on arguments by Brace et al. (2001) we show that forward bond prices are approximately lognormal in the LMM. It thus seems natural to use the LMM not only for interest rate derivative purposes but also in pricing stock options. Numerical pricing results using real data suggest that more general term structure models can produce non negligible price differences when compared with single (and two) factor Hull–White models.

Finally we show, using an empirical example, the impact of our more general setup for the price of the guarantee. Furthermore, we show that we can obtain a tight lower bound using the method by Rogers and Shi (1995), recently generalized by Nielsen and Sandmann (2002a). A tight upper bound is obtained by generalizing the method by Thompson (1998) to the case of rate of return guarantees. This is of independent interest since these methods have not yet been tested for the maturities encountered in life insurance. It turns out the generalized Thompson upper bound can be made extremely tight and hence can be used for pricing.

Convexity correction or convexity adjustment is frequently used in the financial industry to value payments made “at the wrong time point” (e.g. an interest rate known at time T is paid at a later time S) or in a different currency (e.g. a foreign interest rate paid in domestic currency). It is shown in a review article by Pelsser (2003) that convexity correction has its basis in a change of measure associated with a change of numeraire. The advantage of convexity correction is a product based pricing approach. It results in an analytical pricing formula and the determinants of the price are apparent from the respective formulas instead of hidden in the equations/simulations of some general pricing framework.

To arrive at our results we extensively use the Change of Numeraire techniques developed by Geman et al. (1995). They extend the original ideas of Harrison and Kreps (1979) and Harrison and Pliska (1981). Specifically, they show that for any self-financing portfolio of assets with strict positive value (called a numeraire), there exists an equivalent measure under which asset prices normalized by this portfolio are martingales. Hence this portfolio can be used as numeraire. They also show how to derive the Radon-Nikodym derivative associated with any change of numeraire. An introduction to changes of numeraire and convexity correction is given in Appendix A.

The remainder of the paper is organized as follows. First, in Section 2 we describe the financial and insurance aspects of the Regular Premium Unit Linked contract and derive our general pricing formula. In Section 3 we make the analogy of the UL guarantee with an Asian option explicit and discuss some hedging issues. Section 4 derives the Levy approximation and shows the interpretation of the effect of stochastic interest rates as a convexity correction. To obtain prices consistent with the popular LMM we derive expressions for the forward bond volatility in terms of LIBOR rate volatilities in Section 5. In Section 6, we discuss and generalize pricing bounds to the case of rate of return guarantees. Results in a parameterized framework are given in Section 7. Numerical results showing the effect of stochastic interest rates and implications of more general interest rate dynamics are given in Section 8. Section 9 concludes.

2. Contract definition and general pricing formula

The Unit Linked concept refers to the way the policy holders' premiums are invested. The net premiums are invested based on the choice of the policyholder. Common practice is to let the policyholder choose between selected investment funds. Some insurers even give the possibility to invest in individual stocks. In this construction the policyholder bears the investment risk. This means he also has to account for the losses. This type of insurance has high potential profitability because profit is based on equity investments instead of fixed income. However, policy holders are in for a disappointment in times of economic downfall. This is where the Rate of Return Guarantee comes in.

Typical for a Unit Linked contract is that the reserve is not administered in money but in units of several investment funds or stocks. The reserve in money terms is the number of units times the price of each unit. This reserve is termed the *fundvalue*. A gross premium is paid at regular intervals until expiry of the insurance contract. After cost deduction for investment and administration costs and mortality risk premiums, an investment premium results. For each investment premium, units of each funds chosen by the policy holder are purchased at the prevailing price at the payment date. In the presence of a guarantee, the fundvalue at expiry is compared to a guaranteed amount. This amount is likely to be determined by factors like the height of the premium, guaranteed return and cost and mortality deductions, but could as well be exogenous. In this paper we assume that the policyholder invests only in a single investment fund or stock. In this way we can restrict ourselves to a single stock price process. This assumption is not at all restrictive. Our results can be generalized in a straightforward manner to include investments in multiple investment funds or stocks.[1] After this introduction to UL insurance, let us fix some notation.

Let S_t be the price of a unit at time t. This should be thought of as a stock price or stock index. Let the start of the contract be at $t_0 = 0$ and let t_i, $i = 0, \ldots, n-1$ be the time points at which a premium P_i is credited to

[1] From our later results it can be seen that investing in multiple assets means the Guarantee turns into an Asian option on a basket of assets. The techniques used to value basket options are similar to those used in Asian option pricing, hence the problem can be solved analogously. Since the weights of each asset in the basket sum to one, it is interesting to note that considering multiple assets only changes the result in Corollary 4. Theorem 2 remains unchanged. Pricing bounds can be obtained by generalization of the results in Section 6.

the reserve.[2] When writing premium, we mean *investment* premium, so costs and mortality charge are taken into account. Since these cost deductions may depend on the fundvalue at each time t_i (the investment premium P_i may depend on the evolution of S prior to t_i), this means P_i is in general path dependent. We don't make this dependence explicit in our notation, as we will show we can remove it, under weak assumptions, later. Furthermore, let $T = t_n$ be the expiry date of the contract and K be the guaranteed amount at expiry. Then, at time t_i, $i = 0, 1, \dots, n-1$, the policyholder purchases P_i/S_{t_i} units and each unit has value S_T at expiry. The fundvalue at the expiry date is $\mathrm{FV}_n = \sum_{i=0}^{n-1} P_i(S_T/S_{t_i})$. At each payment date, t_i, prior to expiry the fundvalue is given by $\mathrm{FV}_i = \sum_{j=0}^{i-1} P_j(S_{t_i}/S_{t_j})$. Since the policyholder is entitled to a minimum payment of K, conditional upon survival of the insured until time T the payoff of the contract at maturity equals,

$$\max(\mathrm{FV}_n, K) = \max\left(\sum_{i=0}^{n-1} P_i\frac{S_T}{S_{t_i}}, K\right) = \sum_{i=0}^{n-1} P_i\frac{S_T}{S_{t_i}} + \left(K - \sum_{i=0}^{n-1} P_i\frac{S_T}{S_{t_i}}\right)^+ \tag{1}$$

From this formula we can draw our first conclusion; the value of the guarantee is represented by a put option on $\sum_{i=0}^{n-1} P_i(S_T/S_{t_i})$, which can be interpreted as some stochastically weighted average of the unit (i.e. stock) price at expiry. Making use of the put-call parity the insurance contract can also be interpreted as a traditional endowment insurance on the amount K, with an upside potential depending on the stochastically weighted average of the stock price. The quantity $\ln(S_T/S_{t_i})$ represents the logreturn of the investment fund over the period $[t_i, T]$. With a minimum guaranteed rate of return of say, R, we are likely to find insurers calculate the guaranteed amount at time T according to, $K = \sum_{i=0}^{n-1} P_i(R)\,\mathrm{e}^{R(T-t_i)}$. Here off course the P_i depend on the choice of R through the cost deduction scheme, hence we write $P_i(R)$.

Dependent on the insurer, the contract could also have a guarantee implicit if the insured dies before the end of the contract. The convention is adopted that payments to the policyholder are made at the end of the period. If the insured dies in the interval $[t_{i-1}, t_i)$ and the guaranteed amount in that case is K_i, then the payoff of the contract equals,

$$\max(\mathrm{FV}_i, K_i) = \max\left(\sum_{j=0}^{i-1} P_j\frac{S_{t_i}}{S_{t_j}}, K_i\right) = \sum_{j=0}^{i-1} P_j\frac{S_{t_i}}{S_{t_j}} + \left(K_i - \sum_{j=0}^{i-1} P_j\frac{S_{t_i}}{S_{t_j}}\right)^+ \tag{2}$$

which results in a payoff of the guarantee of,

$$\left(K_i - \sum_{j=0}^{i-1} P_j\frac{S_{t_i}}{S_{t_j}}\right)^+ \tag{3}$$

Again, a put option on a particular weighted average of the stock price at termination of the contract. The payoff is similar to that of a guarantee at maturity in a contract with maturity at time t_i. Valuation of this payoff is analogous to that of (1). In case of a specified guaranteed rate of return, the value of K_i is also likely to be determined by an algorithm similar to the one determining K.

At this point we introduce our generic form of the investment premium. It is necessary to make some assumption on the investment premiums since cost deductions could depend on the fundvalue and hence make the investment premium stochastic. This would make the path dependency of the option even more complicated. Our assumption makes the dependence on the fundvalue explicit and maintains the structure of the payoff.

[2] We assume there are no possibilities to surrender. This means, conditional upon survival of the insured until T, there is no insecurity about premium payments.

In practice these products have the following generic form of the investment premium[3] as a function of gross premium, GP_i, fixed costs, FC_i and fundvalue related cost deduction (including mortality charges), c_i, here the GP_i's, FC_i's and c_i's are deterministic,[4]

$$P_i = \mathrm{GP}_i - \mathrm{FC}_i - c_i \mathrm{FV}_i \tag{4}$$

$$P_i = NP_i - c_i \sum_{j=0}^{i-1} P_j \frac{S_{t_i}}{S_{t_j}} \tag{5}$$

and hence the payoff of the guarantee is given by,

$$\left(K - \sum_{i=0}^{n-1} \left(NP_i - c_i \sum_{j=0}^{i-1} P_j \frac{S_{t_i}}{S_{t_j}} \right) \frac{S_T}{S_{t_i}} \right)^+ \tag{6}$$

Since $\mathrm{FV}_0 = 0$ we have that $NP_0 = \mathrm{GP}_0 - \mathrm{FC}_0$ and hence $P_0 = NP_0$. In this form the path dependency of the option seems to get out of control, however rewriting gives back the original structure,

$$\left(K - \sum_{i=0}^{n-1} \tilde{P}_i^{(n)} \frac{S_T}{S_{t_i}} \right)^+ \tag{7}$$

where $\tilde{P}_i^{(n)} = NP_i \cdot \prod_{j=1}^{n-i-1} (1 - c_{n-j})$ which is deterministic. The proof of this relation is given in Appendix B. If $c_i \equiv c$ then, $\tilde{P}_i = NP_i(1-c)^{n-i-1}$. The effect that we see is that because of the way the investment premiums are determined at each time t_i we already know how much of the gross premium minus fixed costs in terms of value at time T is devoted to fundvalue related loadings. The interpretation of the guarantee is still that of a put option on a particular stochastically weighted average of the stock price.

Recently, attempts have been made to include stochastic mortality rates and a market price of mortality risk in the pricing of options embedded in life insurance products, see Milevsky and Promislow (2001), Jiang et al. (2001). We adopt this approach here and give results in terms of risk-neutral mortality probabilities.[5] Furthermore, we apply common practice and assume independence between mortality and the financial markets. This enables us to consecutively take expectations with respect to mortality and financial risk. In case of a linear dependence on mortality this results in treatment of mortality probabilities as known constants. We should off course distinguish between risk-neutral mortality probabilities and those used to determine investment premiums and possibly the guaranteed amount at any date. The latter probabilities are known and are part of the product. For the former, only estimates can be used. They do not play a role in the product, only in the pricing formula.

Let M_x denote the time of mortality of the policy holder, where x is the age of the policyholder at the issue of the contract and $D(t, T)$ be the price at time t of a zero coupon bond with maturity date T. Furthermore let $\boldsymbol{Q}^T$ denote the T-Forward measure. Throughout the paper we will use $E^{\boldsymbol{X}}$ and $E_t^{\boldsymbol{X}}$ for expectation and conditional expectation with respect to some probability measure $\boldsymbol{X}$. Using the aforementioned assumptions on mortality we can write for

[3] The function we pose can always be considered as a first order approximation of the true investment premium (as a function of the fundvalue).

[4] They can be considered to parameterize the contract together with t_i and K_i, $i = 0, 1, \ldots, n$.

[5] More formally, we give results in terms of expected mortality where expectation is taken under the risk neutral measure. Since we also assume mortality is independent of the financial markets, for mortality, this equals expectation under the T-forward measure. If one then adopts the view that mortality risk can be diverisfied by increasing the number of policies hence assume investors are risk-neutral with respect to mortality, risk neutral mortality probabilities equal real world mortality probabilities.

the, time t, price G_t of the guarantee at maturity,[6]

$$G_t = D(t,T)E_t^{\mathbf{Q}^T}\left[\left(K - \sum_{i=0}^{n-1}\tilde{P}_i^{(n)}\frac{S_T}{S_{t_i}}\right)^+ I_{[M_x>T]}\right] \tag{8}$$

$$G_t = D(t,T)E_t^{\mathbf{Q}}[I_{[M_x>T]}]E_t^{\mathbf{Q}^T}\left[\left(K - \sum_{i=0}^{n-1}\tilde{P}_i^{(n)}\frac{S_T}{S_{t_i}}\right)^+\right] \tag{9}$$

$$G_t = {}_{T-t}p_{x+t}D(t,T)E_t^{\mathbf{Q}^T}\left[\left(K - \sum_{i=0}^{n-1}\tilde{P}_i^{(n)}\frac{S_T}{S_{t_i}}\right)^+\right] \tag{10}$$

where $I_{[\cdot]}$ is an indicator function. Furthermore ${}_{T-t}p_{x+t} \equiv E_t^{\mathbf{Q}}[I_{[M_z>T]}]$ is the $T-t$ year risk-neutral survival probability of an $x+t$ year old. Formula (9) illustrates the effect of the independence assumption between mortality and financial markets. In the remainder of the paper we assume only a guarantee at the maturity date of the contract is given. Guarantees given upon termination of the contract at an earlier date, for instance because of death of the insured, can be priced as follows. As before, let an amount K_i be guaranteed when the insured dies in the interval $(t_{i-1}, t_i]$ then, again using the assumptions on mortality, the total price of the guarantee in the contract is,

$$G_t^* = \sum_{i=1}^{n} {}_{t_{i-1}-t}p_{x+t}\ {}_{(t_i-t_{i-1})}q_{(x+t_{i-1})}E_t^{\mathbf{Q}^{t_i}}\left[\left(K_i - \sum_{j=0}^{i-1}\tilde{P}_j^{(i)}\frac{S_{t_i}}{S_{t_j}}\right)^+\right] + {}_{T-t}p_{x+t}E_t^{\mathbf{Q}^T}\left[\left(K - \sum_{j=0}^{n-1}\tilde{P}_j^{(n)}\frac{S_T}{S_{t_j}}\right)^+\right] \tag{11}$$

Where ${}_{(t_i-t_{i-1})}q_{(x+t_{i-1})} = E_t^{\mathbf{Q}}[M_x \in (t_{i-1}, t_i] | M_x > t_{i-1}]$, the risk-neutral probability of mortality in the time interval $(t_{i-1}, t_i]$ given that the insured has survived until time t_{i-1}.[7] The elements in the sum correspond to guarantees upon death and the lone term to the guarantee at maturity. Now the guarantee can be interpreted as a portfolio of put options (with different maturities) on a stochastically weighted average of the stock price.

Formulas (10) and (11) give the general price of the guarantee in a Regular Premium Unit Linked contract. The next section shows the similarity of this price with that of an Asian option. Hence, using the general pricing formula (10) as a starting point, guarantee prices can be obtained by extending pricing methods for Asian options to include UL Guarantees.

3. Relationship with Asian options

The dependency of the guarantee payoff on the stock price at different time points leads to an analogy with Asian options. An average price Asian is an option on the average of the stock price at different time points.

[6] Starting from the well-known risk neutral valuation formula, in which the money market account is the numeraire, we can use the Change of Numeraire Theorem to find the price of the guarantee if the zero bond with maturity T is used as numeraire. These type of numeraire changes were first introduced by El Karoui and Rochet (1989), Jamshidian (1991).

[7] This follows since $E_t^{\mathbf{Q}}[I_{[M_x\in(t_{i-1},t_i]]}] = E_t^{\mathbf{Q}}[I_{M_x\in(t_{i-1},t_i]}|M_x > t_{i-1}]E_t^{\mathbf{Q}}[I_{[M_x>t_{i-1}]}]$. In a general model of mortality these probabilities are time inhomogeneous functions. A simple case for example in which one can already see this effect is a model in which mortality exhibits a decreasing trend. This is a simple model for the longevity effect. The one-year mortality probability of a sixty year old at this moment is not the same as it will be in 30 years. The latter will (most likely) be lower.

The strong relationship between an Asian option and the guarantee can be summarized in the following proposition,

Proposition 1. *Assume markets are arbitrage free and complete. Also assume that the stock price follows a diffusion process. Furthermore return volatility, σ_S, and the short rate, r, are constant. Consider the regular premium UL contract with $NP_i = S_0/n$ and $c_i = 0$, $\forall i$, hence $\tilde{P}_i^{(n)} = S_0/n$, and assume for simplicity, $t_i - t_{i-1} = 1$. Then, ignoring mortality, we have equality between the prices of an average price Asian Put, with strike K, and the Rate of Return Guarantee of the UL contract with the same strike. More formally,*

$$\mathrm{e}^{-rT} E^{Q}\left[\left(K - \frac{1}{n}\sum_{i=1}^{n} S_{t_i}\right)^{+}\right] = \mathrm{e}^{-rT} E^{Q}\left[\left(K - \sum_{i=0}^{n-1} \tilde{P}_i^{(n)} \frac{S_T}{S_{t_i}}\right)^{+}\right] \tag{12}$$

Proof. If we can establish equality in distribution under $\boldsymbol{Q}$ between $1/n \sum_{i=1}^{n} S_{t_i}$ and $\sum_{i=0}^{n-1} \tilde{P}_i (S_T/S_{t_i})$ we are done. We know the stock price is given by,

$$S_t = S_0 \exp([r - \tfrac{1}{2}\sigma_S^2]t + \sigma_S W_t)$$

where W_t is a Brownian motion under $\boldsymbol{Q}$. Consider the vectors of lognormal random variables, $A = [S_{t_1}/n \ \ldots \ S_{t_n}/n]$ and $\mathrm{UL} = [\tilde{P}_i^{(n)}(S_T/S_{t_{n-1}}) \ \ldots \ \tilde{P}_i^{(n)}(S_T/S_{t_0})]$. Equality in distribution of A and UL implies equality in distribution of $A \cdot \mathbf{1}$ and $\mathrm{UL} \cdot \mathbf{1}$, where $\mathbf{1}$ is a $(n \times 1)$ vector of ones. Since these are lognormal r.v.'s we have equality in distribution if we can show the first two moments are equal under the risk neutral measure. Using $\tilde{P}_i^{(n)} = S_0/n$ and $t_i = i$, straightforward calculations give,

$$E(A)_{(i)} = E(UL)_{(i)} = \frac{S_0}{n}\,\mathrm{e}^{ri}$$

where $E(A)_{(i)}$ and $E(\mathrm{UL})_{(i)}$ denote the i$^{\text{th}}$ element of the first moment of A and UL resp. which is a vector. For the second moment we have,

$$E(A'\,A)_{(i,j)} = E([\mathrm{UL}]'\,[\mathrm{UL}])_{(i,j)} = \left(\frac{S_0}{n}\right)^2 \mathrm{e}^{r(i+j)+\sigma_S^2 \min(i,j)}$$

where $E(A'\,A)_{(i,j)}$ and $E([\mathrm{UL}]'\,[\mathrm{UL}])_{(i,j)}$ denote the element in the ith row and jth column of the second (non central) moment of A and UL resp. This completes the proof. □

Proposition 1 illustrates the strong similarities between Asian options and the UL Guarantee. It shows that in this perfect Black–Scholes world there is exactly the same randomness in the ith fixing of the stock price as there is in the $(n - i)$th premium payment. We can generalize this result to allow for stationary stochastic volatility. It can however not be generalized to allow for stochastic interest rates or for any non stationary time dependence in both volatility or interest rates. The randomness in forward stock prices is complementary in the two contracts. The Asian is sensitive for forward stock price movements over the intervals $[0, t_i]$ $i = 1, 2, \ldots, n$, whereas the guarantee is sensitive over the intervals $[t_i, T]$, $i = 0, 1, \ldots, n - 1$. We can say that, within the simplified setting of proposition 1, "time runs in opposite directions" for the two types of options. The UL Guarantee is an Asian which "starts at time T and expires at zero". In the Asian option, the risk of each individual term in the summation, S_{t_i}, runs from time zero to t_i. This is mainly stock price risk, represented through the choice of (sum of the stochastic parts of the) forward stock return as conditioning variable. In our UL contract we can split the risk of each individual term, S_T/S_{t_i}, in interest rate risk, related to some forward bond price, from time zero to t_i and forward stock price risk from time t_i to T.

This latter observation should also provide direction on how to hedge these type of options in a 'quick and dirty' way. At the start of the contract the risk is mainly interest rate related. This risk could be hedged by using caps/floors. However as time progresses the risk becomes more and more stock related. The instruments that come to mind are (forward starting) stock options. Especially forward starting options have the characteristics to be a good hedge. It is shown in the next section that the implied volatility of these options also arises in the volatility of the fundvalue. We have established close similarity between the guarantee and an Asian option. Next, we analyze the effect of stochastic interest rates on the guarantee value using a well-known approximation method (Levy, 1992). More accurate approximation methods are considered in Section 6.

4. Effect of stochastic interest rates

In this section we analyze the effect of stochastic interest rates on the guarantee value. This is done within the context of the Levy (1992) approximation. This consists of approximating the distribution of $\sum_{i=0}^{n-1} \tilde{P}_i^{(n)} S_T/S_{t_i}$ by a lognormal one with the same mean and variance. We choose the Levy approximation because it is simple and allows for a nice financial economic interpretation of the effect of stochastic interest rates. Under the assumption of lognormality of the weighted sum of stock prices, the effect of stochastic interest rates on the guarantee value can be inferred from the effect of stochastic interest rates on the first two moments. For this purpose we derive the first and second moment of $\sum_{i=0}^{n-1} \tilde{P}_i^{(n)} S_T/S_{t_i}$ under the T-Forward measure. This also gives us the variance of $\ln(\sum_{i=0}^{n-1} \tilde{P}_i^{(n)} (S_T/S_{t_i}))$ since for a lognormal variable Y we have that $\mathrm{Var}(\ln Y) = \ln(EY^2/(EY)^2)$. Since the first moment doesn't show the effect of stochastic interest rates explicitly, this implies it is isolated in the second moment. This, off course, holds only under the assumption of lognormality. The Levy approximation is not very accurate for the maturities typical for UL contracts. Therefore, we use our results in this section mainly for expositional purposes. However, we have approximate lognormality and the effect is similar for the more accurate methods of Section 6. Note that we don't restrict ourselves to formulas for the Levy approximation at the time of writing of the contract. Approximate prices for the guarantee obtained through the Levy approximation during the time of the contract are implicit in the expressions for $\mu_{\mathrm{FV}}(t) \equiv E_t^{Q^T}[\sum_{i=0}^{n-1} \tilde{P}_i^{(n)} S_T/S_{t_i}]$ and $\sigma_{\mathrm{FV}}^2 \equiv \mathrm{Var}_t^{Q^T}[\ln \sum_{i=0}^{n-1} \tilde{P}_i^{(n)} (S_T/S_{t_i})]$. Results on the first moment are presented in Theorem 2. Lemma 3 and Corollary 4 give the (partial) expressions for the second moment. If we define F_t^T, the T-forward stock price process, as follows $F_t^T \equiv S_t/D(t,T)$ then without any modelling assumptions on stock or bond price dynamics from the general Change of Numeraire Theorem[8] results:

Theorem 2. *Let $t_j \le t < t_{j+1}$, $j \in \{0, 1, 2, \ldots, n-1\}$. Let the assumptions of the Change of Numeraire Theorem hold. Then the time t conditional first moment of the fundvalue at maturity, $\mu_{\mathrm{FV}}(t)$, is given by,*

$$\mu_{\mathrm{FV}}(t) \equiv E_t^{Q^T}\left[\sum_{i=0}^{n-1} \tilde{P}_i^{(n)} \frac{S_T}{S_{t_i}}\right] = \sum_{i=0}^{j} \tilde{P}_i^{(n)} \frac{F_t^T}{S_{t_i}} + \sum_{i=j+1}^{n-1} \tilde{P}_i^{(n)} \frac{D(t,t_i)}{D(t,T)} \qquad (13)$$

Proof. From the martingale property of numeraire adjusted asset prices and the Tower Law of conditional expectation we have, for $t \le t_i$

$$\begin{aligned} E_t^{Q^T}\left[\frac{S_T}{S_{t_i}}\right] &= E_t^{Q^T}\left[\frac{1}{S_{t_i}} E_{t_i}^{Q^T}\left[\frac{S_T}{D(T,T)}\right]\right] = E_t^{Q^T}\left[\frac{1}{S_{t_i}} \frac{S_{t_i}}{D(t_i,T)}\right] = E_t^{Q^T}\left[\frac{1}{D(t_i,T)}\right] \\ &= E_t^{Q^T}\left[\frac{D(t_i,t_i)}{D(t_i,T)}\right] = \frac{D(t,t_i)}{D(t,T)} \end{aligned} \qquad (14)$$

[8] See Appendix A.

and for $t > t_i$,

$$E_t^{Q^T}\left[\frac{S_T}{S_{t_i}}\right] = \frac{1}{S_{t_i}} E_t^{Q^T}\left[\frac{S_T}{D(T,T)}\right] = \frac{1}{S_{t_i}} F_t^T = \frac{S_t}{S_{t_i} D(t,T)} \tag{15}$$

□

The interpretation of (14) is clear. The expectation, conditional on the information until time t, under the T-Forward measure, of the return on the asset in the period $[t_i, T]$, where $t \leq t_i$, equals the continuously compounded forward rate for that period. In a non stochastic interest rate environment we have $E_t^{Q^T}[S_T/S_{t_i}] = E_t^{Q}[S_T/S_{t_i}] = \exp(\int_{t_i}^T r_s \, \mathrm{d}s)$. The interpretation of (15) is the following: the quantity S_t/S_{t_i} represents the return which is already locked in at time t, the quantity $1/D(t,T) = D(t,t)/D(t,T)$ has the same interpretation as (14).

For the second moment we need some assumptions on price dynamics. Because all observable volatilities in the market are those of forward quantities, like forward stock prices, forward LIBOR and Swap rates, we start modelling at this level. Furthermore, we assume the volatility of forward stock and bond prices to be a deterministic function of time. In a parameterized model, this amounts to a lognormal stock price process and a Gaussian interest rate model. We do not assume the volatilities to be constant to be able to adapt to variation in implied volatilities. Especially since we make a connection between our lognormal model of the economy and the popular LIBOR Market Model in the next section. We assume the T-Forward stock price, F_t^T, and the T-Forward bond price with maturity U, $D^T(t,U) \equiv D(t,U)/D(t,T)$, to follow the dynamics,

$$\mathrm{d}F_t^T = \sigma_F(t) F_t^T \, \mathrm{d}W_t^T \tag{16}$$

$$\mathrm{d}D^T(t,U) = \sigma_U(t) D^T(t,U) \, \mathrm{d}W_t^{\mathrm{UT}} \tag{17}$$

where W_t^T and W_t^{UT} (for all relevant U) are Brownian Motions under the T-Forward measure[9] and σ_F and σ_U are deterministic functions of time. Correlations between those Brownian Motions are given by $\mathrm{d}W_t^T \, \mathrm{d}W_t^{\mathrm{UT}} = \rho_{F,U}(t) \, \mathrm{d}t$ and $\mathrm{d}W_t^{\mathrm{UT}} \, \mathrm{d}W_t^{VT} = \rho_{UV}(t) \, \mathrm{d}t$. In our notation we implicitly assume we are working under the T-Forward measure, using the bond with maturity T as numeraire. When using time points t_i and t_j we will write W^{iT}, $\sigma_i(t)$, $\rho_{F,i}(t)$ and $\rho_{ij}(t)$ for $W^{t_i T}$, $\sigma_{t_i}(t)$, $\rho_{F,t_i}(t)$ and $\rho_{t_i t_j}(t)$ respectively.

Start by writing the trivial result,

$$E_t^{Q^T}\left[\left(\sum_{i=0}^{n-1} \tilde{P}_i^{(n)} \frac{S_T}{S_{t_i}}\right)^2\right] = \sum_{i=0}^{n-1} (\tilde{P}_i^{(n)})^2 E_t^{Q^T}\left[\left(\frac{S_T}{S_{t_i}}\right)^2\right] + 2\sum_{i=0}^{n-2}\sum_{j>i} \tilde{P}_i^{(n)} \tilde{P}_j^{(n)} E_t^{Q^T}\left[\frac{S_T^2}{S_{t_i} S_{t_j}}\right] \tag{18}$$

now we have the following lemma (the proof is given in Appendix B),

Lemma 3. *Let again the conditions of the Change of Numeraire Theorem hold. Then for $t \leq t_i \leq t_j$, we have*

$$E_t^{Q^T}\left[\frac{S_T^2}{S_{t_i} S_{t_j}}\right] = D^T(t,t_i) D^T(t,t_j) \cdot \exp\left(\int_t^{t_i} \rho_{ij}(s)\sigma_i(s)\sigma_j(s) \, \mathrm{d}s\right)$$
$$\times \exp\left(\int_{t_i}^{t_j} \rho_{F,j}(s)\sigma_F(s)\sigma_j(s) \, \mathrm{d}s + \int_{t_j}^{T} \sigma_F^2(u) \, \mathrm{d}u\right) \tag{19}$$

We can split the long expression in (19) in three parts corresponding to the three integrals. The argument of each integral is the instantaneous covariance of $\ln(S_T / S_{t_i})$ and $\ln(S_T / S_{t_j})$ in the relevant time intervals. The first integral, ranging from t to t_i, corresponds to the correlation between the normalized bonds with maturity t_i and t_j. This can

[9] The martingale property of F_t^T and $D^T(t,S)$ follows from the Change of Numeraire Theorem in combination with the no-arbitrage condition.

be explained by the fact that the uncertainty in the quantities S_T/S_{t_i} and S_T/S_{t_j} is driven by the corresponding T-forward bond processes. Second, $\int_{t_i}^{t_j} \rho_{F,j}(s)\sigma_F(s)\sigma_j(s)\,\mathrm{d}s$, represents the covariance between the forward asset price and $D(t,t_j)/D(t,T)$. Since after t_i, S_{t_i} is fixed and the risk in S_T/S_{t_i} is now represented by the T-forward asset price. The risk is now twofold and measured by the quadratic covariation of the forward stock and forward bond price processes. Finally after t_j, we are left with pure equity risk, i.e. both S_{t_i} and S_{t_j} are known, as $\int_{t_j}^{T} \sigma_F^2(u)\,\mathrm{d}u$ represents the implied volatility of a forward start option.

At this point the effect of stochastic interest rates is clear. When interest rates are deterministic, there's no risk in S_T/S_{t_i} in the interval $[0, t_i]$ so the first and second integral in (19) should equal zero.[10] These two integrals, the quadratic covariation of two forward bond prices and the quadratic covariation of the forward stock price and a forward bond price respectively, are precisely the effect of stochastic interest rates on the volatility of the fundvalue and hence (in the lognormal approximation) on the guarantee value.

Lemma 3 gives us the expression for $E_t^{Q^T}[S_T^2/S_{t_i}S_{t_j}]$ assuming $t \le t_i \le t_j$. Modifications of this result for e.g. $t > t_i$ are easy to obtain. This gives us:

Corollary 4. *Let $t_k \le t < t_{k+1}$. Then under the assumption of deterministic volatilities for the T-forward asset price and T-forward bond prices we obtain for the second moment of $\sum_{i=0}^{n-1} \tilde{P}_i^{(n)} S_T/S_{t_i}$,*

$$
\begin{aligned}
\mu_{\mathrm{FV},2}(t) \equiv E_t^{Q^T}\left[\left(\sum_{i=0}^{n-1} \tilde{P}_i^{(n)} \frac{S_T}{S_{t_i}}\right)^2\right] &= \sum_{i=0}^{k} (\tilde{P}_i^{(n)})^2 \left[\frac{F_t^T}{S_{t_i}}\right]^2 \exp\left(\int_t^T \sigma_F^2(u)\,\mathrm{d}u\right) \\
&+ \sum_{i=k+1}^{n-1} (\tilde{P}_i^{(n)})^2 [D^T(t,t_i)]^2 \exp\left(\int_t^{t_i} \sigma_i^2(s)\,\mathrm{d}s + \int_{t_i}^{T} \sigma_F^2(u)\,\mathrm{d}u\right) \\
&+ 2\sum_{i=0}^{k-1}\sum_{j=i+1}^{k} \tilde{P}_i^{(n)} \tilde{P}_j^{(n)} \frac{(F_t^T)^2}{S_{t_i}S_{t_j}} \exp\left(\int_t^T \sigma_F^2(u)\,\mathrm{d}u\right) \\
&+ 2\sum_{i=0}^{k}\sum_{j=k+1}^{n-1} \tilde{P}_i^{(n)} \tilde{P}_j^{(n)} \frac{F_t^T D^T(t,t_j)}{S_{t_i}} \cdot \exp\left(\int_t^{t_j} \rho_{F,j}(s)\sigma_F(s)\sigma_j(s)\,\mathrm{d}s + \int_{t_j}^{T} \sigma_F^2(u)\,\mathrm{d}u\right) \\
&+ 2\sum_{i=k+1}^{n-2}\sum_{j=i+1}^{n-1} \tilde{P}_i^{(n)} \tilde{P}_j^{(n)} D^T(t,t_i) D^T(t,t_j) \\
&\cdot \exp\left(\int_t^{t_i} \rho_{ij}(s)\sigma_i(s)\sigma_j(s)\,\mathrm{d}s + \int_{t_i}^{t_j} \rho_{F,j}(s)\sigma_F(s)\sigma_j(s)\,\mathrm{d}s + \int_{t_j}^{T} \sigma_F^2(u)\,\mathrm{d}u\right)
\end{aligned} \tag{20}
$$

The expression for the second moment consists of five summations. The first summation consists of those terms of the first summation in (18) for which $t \ge t_i$, the second summation consists of the terms with $t < t_i$.The third to fifth terms correspond to parts of the double summations in (18), with $t_i < t_j \le t$, $t_i \le t < t_j$ and $t < t_i < t_j$ respectively.

Now, $\sigma_{\mathrm{FV}}^2(t) = \ln(\mu_{\mathrm{FV},2}(t)/[\mu_{\mathrm{FV}}(t)]^2)$, "the implied guarantee volatility". Theorem 2 and Corollary 4together with the approximate lognormality of the fundvalue result in the following approximate pricing formula for the guarantee (cf. Levy, 1992),

$$G_t \approx D(t,T)[K\Phi(-d_t + \sigma_{\mathrm{FV}}(t)) - \mu_{\mathrm{FV}}(t)\Phi(-d)] \tag{21}$$

[10] This also follows from the fact that, when interest rates are deterministic, forward bond volatilities are zero.

where,

$$d_t = \frac{\ln(\mu_{\mathrm{FV}}(t)/K) + 1/2\sigma^2_{\mathrm{FV}}(t)}{\sigma_{\mathrm{FV}}(t)} \tag{22}$$

and $\Phi(\cdot)$ is defined by the standard normal distribution.

4.1. Convexity correction interpretation of stochastic interest rates

We have isolated the effect of stochastic interest rates (in the Levy approximation it only shows up in the implied guarantee volatility). Next we will show how it is related to change of measure and convexity correction. We give an interpretation to the expectations $E^{\boldsymbol{Q}^T}_{t_i}[F^T_{t_j} D^T(t_j, t_j)]$ and $E^{\boldsymbol{Q}^T}[(D(t_i, t_i)/D(t_i, T))(D(t_i, t_j)/D(t_i, T))]$ (assume again that $i < j$). With regard to the first expectation, we have, defining $\tilde{F}^T_{t_i} = E^{\boldsymbol{Q}^{t_j}}_{t_i}[F^T_{t_j}]$ and using the Change of Numeraire Theorem,

$$E^{\boldsymbol{Q}^T}_{t_i}[F^T_{t_j} D^T(t_j, t_j)] = \frac{D(t_i, t_j)}{D(t_i, T)} E^{\boldsymbol{Q}^{t_j}}_{t_i}[F^T_{t_j}] = \frac{D(t_i, t_j)}{D(t_i, T)} \tilde{F}^T_{t_i} = \frac{D(t_i, t_j)}{D(t_i, T)} F^T_{t_i} \cdot \text{Convexity Correction} \tag{23}$$

Where $\tilde{F}^T_{t_i}$ is the convexity corrected forward asset price and the convexity correction is given by $\tilde{F}^T_{t_i}/F^T_{t_i}$. We see (comparing (23) and (B.9) from the proof in Appendix C) that the expression $\exp(\int_{t_i}^{t_j} \rho_{F,j}(s)\sigma_F(s)\sigma_j(s)\,\mathrm{d}s)$ is the convexity correction arising from taking the expectation of the T-Forward Asset Price “under the wrong measure” (i.e. not under the measure associated with the asset which is used to normalize the stock price). It is comparable with a LIBOR-in-arrears payment. Instead of the value of a forward LIBOR rate paid at an “earlier” time point, we have the value of a forward stock price paid at an “earlier” time point.

To interpret the second expectation, note that, conditional upon the information at time t, $(D(t_i, T)/D(t_i, t_j))(D(t, t_j)/D(t, T))$ is the Radon-Nikodym derivative for a change of measure from the T-Forward measure to the t_j-Forward measure. We can write,

$$\begin{aligned} E^{\boldsymbol{Q}^T}_t[D^T(t_i, t_i) D^T(t_i, t_j)] &= E^{\boldsymbol{Q}^T}_t\left[\frac{D(t_i, t_i)}{D(t_i, T)} \frac{D(t_i, t_j)}{D(t_i, T)}\right] = \frac{D(t, t_j)}{D(t, T)}(1 + E^{\boldsymbol{Q}^{t_j}}_t[L_{t_i T}(t_i)]) \\ &= \frac{D(t, t_j)}{D(t, T)}(1 + \tilde{L}_{t_i T}(t_i)) = \frac{D(t, t_j)}{D(t, T)} \frac{D(t, t_i)}{D(t, T)} \cdot \text{Convexity Correction} \end{aligned} \tag{24}$$

where $L_{t_i T}(t) = (D(t, t_i) - D(t, T))/D(t, T)$, the forward LIBOR rate for the time period (t_i, T) and $\tilde{L}_{t_i T}(t_i)$ is the convexity corrected forward LIBOR rate. Because $i < j$ and hence $L_{t_i T}(t_i)$ is known at t_i, $D(t, t_j) E^{\boldsymbol{Q}^{t_j}}_t[L_{t_i T}(t_i)]$ is the time t value of a payment of $L_{t_i T}(t_i)$, at a later time t_j. It can be seen from (24) and (B.11), from the proof in Appendix B, that the expression $\exp(\int_t^{t_i} \rho_{ij}(s)\sigma_i(s)\sigma_j(s)\,\mathrm{d}s)$ can then be explained as the convexity correction arising from taking the expectation of the LIBOR payment “under the wrong measure”.[11] Note that with deterministic interest rates there would not be “a wrong measure” since then $\boldsymbol{Q}^T = \boldsymbol{Q}^{t_i} = \boldsymbol{Q}^{t_j} = \boldsymbol{Q}$ and hence there would not be any convexity correction. This is in accordance with our previous claim that the effect of stochastic interest rates coincides with the first two integrals in (19).

Since $\sigma_F(t)$ is the instantaneous volatility of the forward asset price at time t and $\sqrt{\int_s^T \sigma^2_F(u)\,\mathrm{d}u/(T - s)}$ can be given the interpretation of the implied volatility of a forward start option with maturity date T starting at $s > t$. This motivates the use of forward starting options in hedging these contracts. We will illustrate the effect of the convexity correction in Section 8. Up till now we have not yet specified how we envision the implementation of the model in (16) and especially (17). This will be the subject of the next section.

[11] This is mentioned only for interpretation, these kind of LIBOR payments are not traded in the market.

5. Calibration of forward bond volatilities

Nowadays the LIBOR Market Model (LMM) is the dominant term structure model to price interest rate derivatives. The ability to fit cap and swaption prices better than e.g. short rate models, is one of its advantages. Therefore, it is of interest to obtain stock option prices (like the UL guarantee under consideration in this paper) which are consistent with the LMM. We link the forward bond volatilities in (17) to LIBOR rate volatilities. To rewrite forward bond volatility we use a method developed and tested by both Hull and White (2000), Brace et al. (2001). They conclude that, although assuming both swap and LIBOR rates lognormal is mutually inconsistent, swap rates are approximately lognormal in the LMM. Based upon their arguments we conclude that forward bond prices are also approximately lognormal in the LMM. This gives us accurate approximations to the forward bond volatilities. We can use these in our lognormal model to obtain guarantee prices consistent with observed cap and swaption prices.

The forward LIBOR rate at time t starting from T with maturity S is defined as,

$$L_{TS}(t) = \frac{1}{\alpha_{TS}}\left(\frac{D(t,T)-D(t,S)}{D(t,S)}\right) \tag{25}$$

Where α_{TS} is the daycount fraction for the period (T, S). In the interest rate market only LIBOR rates with a specific tenor $S-T$ are traded. Let N be the number of LIBOR rates under consideration and let the tenor be $\Delta_L T$, then we define $L_j(t) = L_{T_j^L T_{j+1}^L}(t)$ and $\alpha_j^L = \alpha_{T_j^L T_{j+1}^L}$, where $T_j^L = j\Delta_L T$, $j = 0, 1, \ldots, N+1$ are the so called *reset dates*. Now an m-factor lognormal version of the LIBOR Market Model (LMM) developed independently by Miltersen et al. (1997), Brace et al. (1997) poses the following dynamics for the forward LIBOR rates L_j,

$$\mathrm{d}L_j(t) = \sum_{q=1}^{m} \sigma_{L_j}^q(t) L_j(t)\,\mathrm{d}W_t^{j+1,q} \tag{26}$$

Where the $W^{j+1,q}$'s are Brownian Motions (which can be assumed uncorrelated, since we can rotate factors) under $\boldsymbol{Q}^{T_{j+1}^L}$, the T_{j+1}^L-Forward measure. The $\sigma_{L_j}^q(t)$'s are deterministic functions of time. The popularity of the LMM stems from the fact that the parameters of the model (the $\sigma_{L_j}^q(t)$'s) can be chosen such that the model exactly matches observed cap prices in the market.

A full factor LMM assumes m, the number of Brownian Motions, equals N. Hence we can construct the model such that each LIBOR rate is driven by its own Brownian Motion. Furthermore, assuming a stationary volatility and correlation structure, we have, $\sigma_{L_j}^q(t) = \sigma(T_j^L - t)$ and $dW_t^{j+1}\,\mathrm{d}W_t^{i+1} = \rho(T_j^L - t, T_i^L - t)\,\mathrm{d}t$. Summarizing the model becomes,

$$\mathrm{d}L_j(t) = \sigma(T_j^L - t) L_j(t)\,\mathrm{d}W_t^{j+1}, \quad j = 1, \ldots, N \tag{27}$$

$$\mathrm{d}W_t^{j+1}\,\mathrm{d}W_t^{k+1} = \rho(T_j^L - t, T_k^L - t)\,\mathrm{d}t$$

This is the model we use to obtain numerical results later. We will now discuss how to obtain approximate forward bond volatilities in this model. Results for other parameterizations of the LMM can be obtained in a similar manner.

We assume the dates of the forward bond prices under consideration in the pricing of the guarantee coincide at least partly with the reset dates of LIBOR rates.[12] Define $\tilde{t} := \{\{t_i\}_{i=1}^n\}$ and $\widetilde{T_L} := \{\{T_i^L\}_{i=1}^{N+1}\}$ then our assumption

[12] Implicit in this assumption is that we consider pricing at payment dates only.

boils down to assuming $\tilde{t} \subseteq \widetilde{T_L}$. Next define, $G_{j,i} := \{j : T_j^L \in [t_i, T)\}$. Then the relationship with bond prices is,[13]

$$D^T(t, t_i) = \frac{D(t, t_i)}{D(t, T)} = \prod_{j \in G_{j,i}} [1 + \alpha_j^L L_j(t)] \tag{28}$$

Now we have,

$$\frac{1}{D^T(t, t_i)} \frac{\partial D^T(t, t_i)}{\partial L_j(t)} = \frac{\alpha_j^L}{1 + \alpha_j^L L_j(t)}$$

Applying Itô's Lemma to (28), bearing in mind (26) gives,[14]

$$\mathrm{d}D^T(t, t_i) = \begin{cases} \cdots \, \mathrm{d}t + \displaystyle\sum_{j \in G_{j,i}} \frac{1}{D^T(t, t_i)} \frac{\partial D^T(t, t_i)}{\partial L_j(t)} \sigma(T_j^L - t) L_j(t) D^T(t, t_i) \, \mathrm{d}W_t^{j+1} \\ \cdots \, \mathrm{d}t + \displaystyle\sum_{j \in G_{j,i}} \frac{\alpha_j^L \, \sigma(T_j^L - t) L_j(t)}{1 + \alpha_j^L L_j(t)} D^T(t, t_i) \, \mathrm{d}W_t^{j+1} \end{cases} \tag{29}$$

This leads to a variance rate $\sigma^2_{D_i^T}$ of $D^T(t, t_i)$ of,

$$\begin{aligned} \sigma^2_{D_i^T}(t) = & \sum_{j \in G_{j,i}} \left(\frac{\alpha_j^L \sigma(T_j^L - t) L_j(t)}{1 + \alpha_j^L L_j(t)} \right)^2 \\ & + 2 \sum_{j \in G_{j,i}} \sum_{k > j} \rho(T_j^L - t, T_k^L - t) \left(\frac{\alpha_j^L \sigma(T_j^L - t) L_j(t)}{1 + \alpha_j^L L_j(t)} \right) \left(\frac{\alpha_k^L \, \sigma(T_k^L - t) L_k(t)}{1 + \alpha_k^L L_k(t)} \right) \end{aligned} \tag{30}$$

Hull and White suggest to approximate (30), which is a stochastic quantity, by a constant, effectively replacing the forward LIBOR rate by their time zero values. This leads to an approximate variance rate of,

$$\begin{aligned} \tilde{\sigma}^2_{D_i^T}(t) = & \sum_{j \in G_{j,i}} \left(\frac{\alpha_j^L \sigma(T_j^L - t) L_j(0)}{1 + \alpha_j^L L_j(0)} \right)^2 \\ & + 2 \sum_{j \in G_{j,i}} \sum_{k > j} \rho(T_j^L - t, T_k^L - t) \left(\frac{\alpha_j^L \sigma(T_j^L - t) L_j(0)}{1 + \alpha_j^L L_j(0)} \right) \left(\frac{\alpha_k^L \sigma(T_k^L - t) L_k(0)}{1 + \alpha_k^L L_k(0)} \right) \end{aligned} \tag{31}$$

This is in line with the approach of Brace et al. (2001), Brace and Womersley (2000), who observe that $\alpha_j^L \, L_j(t)/1 + \alpha_j^L L_j(t)$ is a low variance $\boldsymbol{Q}^{T_{j+1}^L}$-martingale and use this to approximate swaption volatilities in the LMM. We use it to approximate forward bond volatilities in the LMM. Expression (31) gives the variance rate at time t. A frequent assumption in applications is that correlation and volatility are constant between reset dates.

[13] We could do the exact same thing in the context of the Swap Market Model (SMM). However the LMM seems to be preferable over the SMM in terms of out of sample pricing performance (see De Jong et al., 2001).

[14] We do not calculate the drift term, or equivalently specify the Brownian Motion, since it is irrelevant for our purposes. We are interested in the quadratic variation terms.

For example we can approximate the variance of $D^T(t, t_i)$ over the interval $[0, T_l]$ by:

$$\int_0^{T_l} \sigma_i^2(s)\,\mathrm{d}s \approx \sum_{k=0}^{l-1} \alpha_k^L \tilde{\sigma}_i^2(T_k^L)$$

which is what we use in Section 8.

The literature on the subject of LMM calibration is rapidly growing. Ideally one would calibrate the model not only to caps but also to swaption prices. This, and for example the inclusion of more factors, will give a better fit to the correlation structure of the forward LIBOR rates. In general, to calibrate a LMM we must specify volatility functions for the LIBOR rates and a correlation matrix for the Brownian Motions such that the option prices are fitted by the model. Important with non-vanilla products, such as ours, is that the calibrated model fits the correlation structure of the relevant rates well. Therefore, in our numerical examples we use calibration results for a multifactor model. Also important in the calibration process are the users goals. Usually, traders prefer exact calibration since they want their models to replicate observed prices exactly, whereas for risk management (or reserving) purposes the user might want to protect against overfitting and use a parsimonious model and non-exact calibration for better out of sample performance. The current state of the art LMM calibration is based on semidefinite programming techniques, see Brace and Womersley (2000), d'Aspremont (2002).

6. Bounds on the price of the guarantee

We have explored interpretation and more accurate modelling of the effect of stochastic interest rates in the context of the Levy approximation. However, this method is not accurate enough to be used for pricing of contracts with high maturity (such as guarantees). In this section, we will provide bounds tight enough to be used for actual pricing. In the literature several techniques exist to bound the price of an arithmetic Asian option. A very accurate lower bound to the price of an Asian option is the method of Rogers and Shi (1995), recently generalized to allow for stochastic interest rates by Nielsen and Sandmann (2002a) and applied to an Equity Linked contract in Nielsen and Sandmann (2002b). Results on pricing bounds are also derived in Simon et al. (2000) and Dhaene et al. (2002a,b). It is important to note the bounds derived in these papers are valid in a general setting, whereas the previously mentioned authors only consider the particular case of lognormal prices. We provide additional results to adapt existing methods to the case of guarantees in regular premium UL contracts in the model (16) and (17). The method by Rogers and Shi only provides a tight lower bound. Therefore, we generalize the upper bound by Thompson (1998) to the case of the regular premium UL Guarantee at the same time allowing interest rates to be stochastic. Besides the upper bound by Thompson we also discuss and compare some other well-known upper bounds. Results on the tightness of these bounds are presented and discussed.

6.1. Lower bound

For ease of notation assume $\tilde{P}_i^{(n)} = 1$. To obtain a price for the guarantee we are interested in the following expectation and lower bound,

$$E_t^{Q^T}\left[\left(K - \sum_{i=0}^{n-1} \frac{S_T}{S_{t_i}}\right)^+\right] \geq E_t^{Q^T}\left[\left(E_t^{Q^T}\left[K - \sum_{i=0}^{n-1} \frac{S_T}{S_{t_i}} | Z\right]\right)^+\right] \tag{32}$$

The method uses a conditioning variable to derive a lower bound to the price which is extremely tight. The approach by Rogers and Shi develops according to the following steps. First choose Z to be a standard normal random variable. Then calculate $E_t^{\mathbf{Q}^T}[K - \sum_{i=0}^{n-1} S_T/S_{t_i}|Z]$ by summing individual expectations.[15] This expectation will be a convex function in Z (since sum of convex functions is itself convex and the individual expectations are convex functions). Second, using the ideas of Jamshidian (1989) split the option on a sum into a portfolio of options by solving, $E_t^{\mathbf{Q}^T}[K - \sum_{i=0}^{n-1} S_T/S_{t_i}|Z] = 0$ for Z. For details, see Nielsen and Sandmann (2002a), Definition 1 and Theorem 1.

In our case, the second step of the approach by Nielsen and Sandmann will be exactly the same. With the results from the first step we will be able to do a Jamshidian decomposition on the conditioning variable Z. This simplifies the pricing problem from one of an option on a sum to that of a sum of options. The first step is however a bit different in this case.

To apply a Jamshidian decomposition we must calculate $E_t^{\mathbf{Q}^T}[\sum_{i=0}^{n-1} S_T/S_{t_i}|Z]$. At this point we introduce the following shorthand notation,

$$\int_t^T \bar{\sigma}_i(s)\,\mathrm{d}\bar{W}_s^i \equiv \int_t^{\max(t,t_i)} \sigma_i(s)\,\mathrm{d}W_s^{iT} + \int_{\max(t,t_i)}^T \sigma_F(s)\,\mathrm{d}W_s^T \tag{33}$$

$$E_t^{\mathbf{Q}^T}\left[\frac{S_T}{S_{t_i}}\right] \equiv \bar{\mu}_i(t) = \frac{D(t,t_i)}{D(t,T)} I_{[0,t_i]}(t) + \frac{S_t/D(t,T)}{S_{t_i}} I_{[t_i,T]}(t) \tag{34}$$

This enables us to write the volatility terms we encounter in the remainder using a single integral. This also implies,

$$\int_t^T \bar{\sigma}_i^2(s)\,\mathrm{d}s \equiv \int_t^{\max(t,t_i)} \sigma_i^2(s)\,\mathrm{d}s + \int_{\max(t,t_i)}^T \sigma_F^2(s)\,\mathrm{d}s \tag{35}$$

Using the developed notation, with the advantage becoming clear straight away, we can immediately write,

$$\frac{S_T}{S_{t_i}} = \bar{\mu}_i(t)\exp\left(-\frac{1}{2}\int_t^T \bar{\sigma}_i^2(s)\,\mathrm{d}s + \int_t^T \bar{\sigma}_i(s)\,\mathrm{d}\bar{W}_s^i\right) \tag{36}$$

This gives, using well-known relations for the conditional expectation of normal random variables,[16]

$$E_t^{\mathbf{Q}^T}\left[\frac{S_T}{S_{t_i}}|Z\right] = \bar{\mu}_i(t)\exp\left(-\frac{1}{2}\int_t^T \bar{\sigma}_i^2(s)\,\mathrm{d}s + \mu_{i|Z}(t)Z + \frac{1}{2}\sigma_{i|Z}^2(t)\right) \tag{37}$$

$$E_t^{\mathbf{Q}^T}\left[\frac{S_T}{S_{t_i}}|Z\right] = \bar{\mu}_i(t)\exp\left(\mu_{i|Z}(t)Z - \frac{1}{2}\mu_{i|Z}(t)^2\right) \tag{38}$$

where,

$$\mu_{i|Z}(t) = E_t^{\mathbf{Q}^T}\left[Z\int_t^T \bar{\sigma}_i(s)\,\mathrm{d}\bar{W}_s^i\right] \tag{39}$$

$$\sigma_{i|Z}^2(t) = \int_t^T \bar{\sigma}_i^2(s)\,\mathrm{d}s - \mu_{i|Z}(t)^2 \tag{40}$$

Notice that the randomness in S_T/S_{t_i} over the interval $[t, t_i]$ is given by $\int_t^{\max(t,t_i)} \sigma_i(s)\,\mathrm{d}W_s^{iT}$. So until the premium is paid, the risk for the insurer is pure interest rate risk.

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[15] Since we are in the lognormal framework in these calculations one can use standard relations for conditional expectations of normally distributed random variables.

[16] $E(X|Z) = E(X) + (\mathrm{Cov}(X,Z)/\mathrm{Var}(Z))[Z - E(Z)]$ and $\mathrm{Var}(X|Z) = \mathrm{Var}(X) - ([\mathrm{Cov}(X,Z)]^2/\mathrm{Var}(Z))$.

6.2. Choice of conditioning variable

The approximate solution to the pricing problem, i.e. the lower bound in (32), depends on the choice of Z. A sensible choice, and indeed the one Rogers and Shi make, is that conditioning variable for which the variance of the conditional payoff is "small". In the Asian option the risk of each individual term in the summation, S_{t_i}, runs from time zero to t_i. This is mainly stock price risk, represented through the choice of (sum of the stochastic parts of the) forward stock return as conditioning variable (see Nielsen and Sandmann, 2002a). In our UL contract we can split the risk of each individual term, S_T/S_{t_i}, in interest rate risk, related to some forward bond price, from time zero to t_i and forward stock price risk from time t_i to T. This is reflected by the choice of conditioning variable, which is (the sum of the stochastic parts of) the forward bond return over the interval $[0, t_i]$ and the forward stock return over $[t_i, T]$. Summarizing we have,

$$Z = \frac{1}{\alpha_t}\sum_{i=0}^{n-1}\left\{\int_t^T \bar{\sigma}_i(s)\,\mathrm{d}\bar{W}_s^i\right\} \tag{41}$$

where the relevant forward bond volatilities are taken zero if $t > t_i$ and α_t is a normalizing constant. Observe that our choice of Z amounts to conditioning on the stochastic parts of the "return" on the forward stock and bond prices. We have,

$$\alpha_t = \sum_{j=0}^{n-1}\mathrm{Cov}_t^{Q^T}\left[\int_t^T \bar{\sigma}_j(s)\,\mathrm{d}\bar{W}_s^j;\int_t^T \bar{\sigma}_j(s)\,\mathrm{d}\bar{W}_s^j\right] + 2\sum_{j=0}^{n-2}\sum_{k=j+1}^{n-1}\mathrm{Cov}_t^{Q^T}\left[\int_t^T \bar{\sigma}_k(s)\,\mathrm{d}\bar{W}_s^k;\int_t^T \bar{\sigma}_j(s)\,\mathrm{d}\bar{W}_s^j\right] \tag{42}$$

and, for $i \le j$

$$\mathrm{Cov}_t^{Q^T}\left[\int_t^T \bar{\sigma}_i(s)\,\mathrm{d}\bar{W}_s^i;\int_t^T \bar{\sigma}_j(s)\,\mathrm{d}\bar{W}_s^j\right] = \int_t^{\max(t,t_i)} \rho_{ij}(s)\sigma_i(s)\sigma_j(s)\,\mathrm{d}s + \int_{\max(t,t_i)}^{\max(t,t_j)} \rho_{F,j}(s)\sigma_F(s)\sigma_j(s)\,\mathrm{d}s + \int_{\max(t,t_j)}^T \sigma_F^2(s)\,\mathrm{d}s \tag{43}$$

which is the (logarithm of the) convexity correction (first two terms) and the implied volatility of a forward starting stock option (third term) present in the second moment of the fund value. Furthermore,

$$\mu_{i|Z}(t) = E_t^{Q^T}\left[Z\int_t^T \bar{\sigma}_i(s)\,\mathrm{d}\bar{W}_s^i\right] \tag{44}$$

$$\begin{aligned} &= \frac{1}{\alpha_t}\left[\sum_{j=0}^{i}\int_t^{\max(t,t_j)} \rho_{ij}(s)\sigma_i(s)\sigma_j(s)\,\mathrm{d}s + \int_{\max(t,t_j)}^{\max(t,t_i)} \rho_{F,i}(s)\sigma_F(s)\sigma_i(s)\,\mathrm{d}s + \int_{\max(t,t_i)}^T \sigma_F^2(s)\,\mathrm{d}s\right] \\ &\quad + \frac{1}{\alpha_t}\left[\sum_{j=i+1}^{n-1}\int_t^{\max(t,t_i)} \rho_{ij}(s)\sigma_i(s)\sigma_j(s)\,\mathrm{d}s + \int_{\max(t,t_i)}^{\max(t,t_j)} \rho_{Fj}(s)\sigma_F(s)\sigma_j(s)\,\mathrm{d}s + \int_{\max(t,t_j)}^T \sigma_F^2(s)\,\mathrm{d}s\right] \end{aligned} \tag{45}$$

which are again the expressions for the convexity correction and the implied volatility of a forward start option in the second moment of the fund value at maturity.

It is important to note, that for reasonable correlation values (mainly for the correlation between forward stock and forward bond processes) the coefficients of Z are positive for all i. This results in *a unique solution* to the equation, $\sum_{i=0}^{n-1} E_t^{Q^T}[S_T/S_{t_i}|Z] - K = 0$. This simplifies the calculations of the lower bound.[17] Let Z^* be the unique solution to $\sum_{i=0}^{n-1} E_t^{Q^T}[S_T/S_{t_i}|Z] - K = 0$. The following lower bound to the price of the guarantee results,

$$G_t \geq D(t,T)\left\{K\Phi(Z^*) - \sum_{i=0}^{n-1}\left[\bar{\mu}_i(t)\Phi(Z^* - \mu_{i|Z}(t))\right]\right\}$$

see Theorem 1 in Nielsen and Sandmann (2002a). We will refer to this lower bound as the Rogers and Shi Lower Bound or RSLB.

6.3. Upper bound

Throughout the literature several candidate upper bounds exist for the price of Asian options and analogous rate of return guarantees in premium paying contracts. Results in Nielsen and Sandmann (2003) suggest no particular bound clearly dominates the other although a refinement of the upper bound by Rogers and Shi has the best overall performance for the maturity and volatility considered by these authors. Not much is known however about the performance of these bounds in the case of the guarantees considered in this paper and especially the maturities associated with these contracts. Besides the refinement of the Rogers and Shi upper bound, based on Nielsen and Sandmann (2002b, 2003), we consider here a slight generalization of the upper bound by Thompson (1998). Numerical results show that this latter bound outperforms most, if not all, other bounds in the literature. Extensive results on the generalization and subsequent optimization of Thompson's upper bound can be found in Lord (2003).

6.3.1. Refinement of Rogers and Shi upper bound

For the refined Rogers and Shi upper bound (denoted by RSUB+) we have the following result which slightly generalizes the result in Nielsen and Sandmann (2002b) to our model (16) and (17). Define,

$$d_t \equiv \frac{\ln(K/n) - (1/n)\ln\left[\prod_{i=0}^{n-1}\tilde{P}_i^{(n)}\bar{\mu}_i(t)\exp\left(-(1/2)\int_t^T \bar{\sigma}_i^2(s)\,\mathrm{d}s\right)\right]}{(\alpha_t/n)} \quad (46)$$

then the correction ε_t on the Rogers and Shi lower bound which gives the refined upper bound is given by,

$$\varepsilon_t = \frac{1}{2}\Phi(d_t)^{1/2}\left[D(t,T)^2\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\tilde{P}_i^{(n)}\tilde{P}_j^{(n)}\bar{\mu}_i(t)\bar{\mu}_j(t)\exp(\mu_{i|Z}(t)\mu_{j|Z}(t))\{\exp(\sigma_{i,j|Z}^2(t)) - 1\}\right.$$
$$\left.\times\ \Phi(d_t - \{\mu_{i|Z}(t) + \mu_{j|Z}(t)\})\right]^{1/2} \quad (47)$$

The original upper bound by Rogers and Shi (denoted by RSUB) corresponds to the situation where $d_t = +\infty$. This will lead to a correction on the lower bound which is independent of the strike.

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[17] Thompson (1998) derives an analytical solution to the equation, $\sum_{i=0}^{n-1} E_t^{Q^T}[S_{t_i}|Z] - K = 0$ in the case of an Asian option based on interchanging exponentiation and summation. This doesn't work in our case since the terms in the exponent are not small enough.

6.3.2. Generalized Thompson upper bound

The upper bound Thompson proposed is based on the observation that,

$$E_t^{Q^T}\left[\left(K-\sum_{i=0}^{n-1}\tilde{P}_i^{(n)}\frac{S_T}{S_{t_i}}\right)^+\right]\leq\sum_{i=0}^{n-1}E_t^{Q^T}\left[\left(f_iK-\tilde{P}_i^{(n)}\frac{S_T}{S_{t_i}}\right)^+\right] \tag{48}$$

under the condition that $\sum_{i=0}^{n-1}f_i=1$ and f is some (stochastic) function of i. Slightly generalizing the ideas in Thompson (1998), see Lord (2003) for a full discussion, we propose to use,

$$f_i=m_i+\beta\left(\int_t^T\bar{\sigma}_i(s)\,\mathrm{d}\bar{W}_s^i-\frac{1}{n}\sum_{j=0}^{n-1}\int_t^T\bar{\sigma}_j(s)\,\mathrm{d}\bar{W}_s^j\right) \tag{49}$$

The first part, m_i, is deterministic. The second part represents the total return minus the individual return on a premium paid at time t_i. The original upper bound in Thompson (1998) has $\beta=1$ (denoted by THSTOCH). When $\beta=0$, the function f becomes deterministic and the upper bound resulting from optimal choice of f (denoted by THDET) coincides with that of proposition 1 in Nielsen and Sandmann (2003) and the comonotonic upper bound by Simon et al. (2000). This upper bound has the interpretation of a portfolio of forward starting options with optimally chosen strikes. A naive approach would be to chose $f_i=\tilde{P}_i/\left(\sum_{j=0}^{n-1}\tilde{P}_j^{(n)}\right)$ (denoted by THNAIVE). The optimal choice of f is given by,

$$f_i=\frac{\tilde{P}_i^{(n)}}{K}\bar{\mu}_i(t)\exp\left(\gamma\sqrt{\int_t^T\bar{\sigma}_i^2(s)\,\mathrm{d}s}-\frac{1}{2}\int_t^T\bar{\sigma}_i^2(s)\,\mathrm{d}s\right) \tag{50}$$

where γ is set to satisfy $\sum_{i=0}^{n-1}f_i=1$.

The upper bound resulting from a general choice of β is the following summation of integrals (for a formal derivation refer to Thompson (1998) or Lord (2003)), where $\phi(\cdot)$ denotes the standard normal density,

$$G_t\leq\sum_{i=0}^{n-1}\int_{-\infty}^{\infty}\left\{a(i,X_i)\Phi\left(\frac{a(i,X_i)}{b(i,X_i)}\right)+b(i,X_i)\phi\left(\frac{a(i,X_i)}{b(i,X_i)}\right)\right\}\frac{1}{\sqrt{\int_t^T\bar{\sigma}_i^2(s)\,\mathrm{d}s}}\phi\left(\frac{X_i}{\sqrt{\int_t^T\bar{\sigma}_i^2(s)\,\mathrm{d}s}}\right)\mathrm{d}X_i \tag{51}$$

where,

$$a(i,X_i)=K\left(m_i-\frac{\beta}{n}\frac{\mu_{i|Z}(t)}{\int_t^T\bar{\sigma}_i^2(s)\,\mathrm{d}s}X_i+\beta X_i\right)-\tilde{P}_i^{(n)}\bar{\mu}_i(t)\exp\left(-\frac{1}{2}\int_t^T\bar{\sigma}_i^2(s)\,\mathrm{d}s+X_i\right)$$

$$b(i,X_i)=\frac{K}{n}\beta\alpha_t\sqrt{1-\frac{\mu_{i|Z}(t)}{\int_t^T\bar{\sigma}_i^2(s)\,\mathrm{d}s}}$$

$$m_i=\frac{1}{K}\left(\tilde{P}_i^{(n)}\bar{\mu}_i(t)\exp\left(-\frac{1}{2}\int_t^T\bar{\sigma}_i^2(s)\,\mathrm{d}s\right)+\gamma\sqrt{\mathrm{Var}_t^{Q^T}[N_i]}\right)$$

$$\mathrm{Var}_t^{Q^T}[N_i]=c_i^2\int_t^T\bar{\sigma}_i^2(s)\,\mathrm{d}s+2c_i\beta\frac{K}{n}\mu_{i|Z}(t)+\left(\beta\frac{K}{n}\right)^2\alpha_t^2$$

$$c_i = \tilde{P}_i^{(n)} \overline{\mu}_i(t) \exp\left(-\frac{1}{2}\int_t^T \overline{\sigma}_i^2(s)\,\mathrm{d}s\right) - \beta K$$

$$\gamma = \frac{K - \sum_{i=0}^{n-1} \tilde{P}_i^{(n)} \overline{\mu}_i(t) \exp\left(-(1/2)\int_t^T \overline{\sigma}_i^2(s)\,\mathrm{d}s\right)}{\sum_{i=0}^{n-1} \sqrt{\mathrm{Var}^{Q^T}[N_i]}}$$

The integrals in (51) have to be solved numerically. For an indication of the computational times of the different upper bounds refer to Lord (2003). The tightness of the upper bound for a general choice of β is largely dependent on the variance of the stochastic term. This variance must not be to large. Since variance increases with maturity we naively set $\beta = 1/T$ and denote the resulting upper bound by (THSTOCH+). Numerical results show that this approach already outperforms the other bounds we considered.

6.4. Accuracy of upper and lower bounds

Numerical results on the tightness of for rate of return guarantees in premium paying UL contracts are given in Table 2. Pricing results for the Levy approximation and several pricing bounds are given for several contract maturities and guarantee levels. The column headed MC price holds the price obtained by Monte Carlo simulation. The Levy approximation for the guarantee prices behaves similar to that for Asian options. The quality deteriorates with increasing volatility and the approximation is best for at the money options. Our results on pricing bounds confirm the results in Nielsen and Sandmann (2003) for Asian options. RSUB+ outperforms all other bounds considered by Nielsen and Sandmann (RSUB, THDET, THNAIVE). We can see that THSTOCH, corresponding to $\beta = 1$, is not tight at all. This is induced by the high volatility in the stochastic term of *f*. The Thompson bound corresponding to $\beta = 0$ performs much better. However we see that when we reduce the volatility in the stochastic term by setting $\beta = 1/T$ we already obtain an extremely tight upper bound, even outperforming RSUB+.[18] Further discussion on the optimal choice of β and other versions of Thompson's upper bound can be found in Lord (2003).

7. Results in a Black–Scholes Hull–White model

In this section we derive the results of Sections 4 and 6 in the context of a combined Black–Scholes Hull–White (BSHW) model. We assume the stock price has a constant volatility and the short rate follows a Hull–White (Hull and White, 1990) process under the risk neutral measure. Furthermore we assume the (instantaneous) correlation between the stock and the short rate equals ρ, i.e. $\mathrm{Corr}^{Q}[d(\ln S_t); dr_t] = \rho dt$. Explicitly, we have the following SDEs for the stockprice and the shortrate,

$$\mathrm{d}S_t = r_t S_t\,\mathrm{d}t + \sqrt{1-\rho^2}\sigma_S S_t\,\mathrm{d}W_{1,t} + \rho\sigma_S S_t\,\mathrm{d}W_{2,t} \tag{52}$$

$$\mathrm{d}r_t = (\theta_t - a r_t)\,\mathrm{d}t + \sigma_r\,\mathrm{d}W_{2,t}$$

where $W_{1,t}$ and $W_{2,t}$ are independent Brownian Motions under the risk neutral measure. Essential to all of our preceding results is the covariance $E_t^{Q^T}[\int_t^T \overline{\sigma}_i(s)\,\mathrm{d}\overline{W}_s^i \int_t^T \overline{\sigma}_j(s)\,\mathrm{d}\overline{W}_s^j]$. From this quantity all results in Sections 4 and 6 are derived.

[18] We even obtain values below the MC price, this is due to simulation error. The upper bound values are well within the confidence band around the MC price.

Our result in this section extends the results of Nielsen and Sandmann (1996b) for Asian options to Guarantees in Regular Premium UL insurance. But more importantly, using the results and ideas of Section 4, their results can quite easily be restated in terms of forward stock and forward bond volatilities. This enables us to interpret their results in terms of convexity adjusted quantities and obtain additional insight in the determinants of the price of Asian options. See also Appendix C.

In a Black–Scholes Hull–White model all volatilities of forward prices are a deterministic function of time. We essentially parameterize our general lognormal model in (16) and (17). The convexity correction can then be interpreted as parameterized convexity correction. We now proceed with the derivation.

It is not difficult to derive that under the stated assumptions of constant stock volatility and a correlation between stock and short rate of ρ, the T-forward stock and T-forward bond price follow the dynamics,

$$\mathrm{d}F_t^T = \sqrt{1-\rho^2}\sigma_S F_t^T\,\mathrm{d}W_{1,t}^T + \rho\sigma_S F_t^T\,\mathrm{d}W_{2,t}^T + \sigma_r B(t,T)F_t^T\,\mathrm{d}W_{2,t}^T \tag{53}$$

$$\mathrm{d}D^T(t,S) = -\sigma_r[B(t,S)-B(t,T)]D^T(t,S)\,\mathrm{d}W_{2,t}^T \tag{54}$$

where $W_{1,t}^T$ and $W_{2,t}^T$ are independent Brownian Motions under the T-Forward measure and $B(t,T)=(1/a)[1-\mathrm{e}^{-a(T-t)}]$. For (53) we can write equivalently (in weak SDE solution terms):

$$\mathrm{d}F_t^T = \sqrt{\sigma_S^2 + 2\rho\sigma_S\sigma_r B(t,T) + \sigma_r^2 B(t,T)^2}F_t^T\,\mathrm{d}Z_t^T \tag{55}$$

Where Z_t^T is a Brownian Motion under the T-Forward Measure. This makes the forward asset price volatility direct. From $\ln E^{Q^T}[F_t^T D^T(t,S)] = \int_0^t \rho_{F,S}(s)\sigma_F(s)\sigma_S(s)\mathrm{d}s + \ln(F_0^T D^T(0,S))$, the instantaneous covariance between F_t^T and $D^T(t,S)$, can be shown to equal,

$$\rho_{F,S}(t)\sigma_F(t)\sigma_S(t) = -\rho\sigma_S\sigma_r[B(t,S)-B(t,T)] + \sigma_r^2 B(t,T)^2 - \sigma_r^2 B(t,T)B(t,S) \tag{56}$$

Instead of inferring the volatilities of interest from market data we parameterize them according to the results above. Then if we parameterize the forward stock price volatility and forward bond price volatility in (16) and (17) according to (53) and (54) respectively, we obtain (also assuming that $i<j$ and $t<t_i$),

$$\begin{aligned}
&E_t^{Q^T}\left[\int_t^T \bar{\sigma}_i(s)\,\mathrm{d}\bar{W}_s^i \int_t^T \bar{\sigma}_j(s)\,\mathrm{d}\bar{W}_s^j\right]\\
&\quad= \left(\sigma_r^2\int_t^{t_i}[B(s,T)-B(s,t_i)][B(s,T)-B(s,t_j)]\,\mathrm{d}s\right)\\
&\qquad+\left(-\rho\sigma_S\sigma_r\int_{t_i}^{t_j} B(s,t_j)-B(s,T)\,\mathrm{d}s + \sigma_r^2\int_{t_i}^{t_j} B(s,T)^2 - B(s,T)B(s,t_j)\,\mathrm{d}s\right)\\
&\qquad+\left(\sigma_S^2(T-t_j) + 2\rho\sigma_r\sigma_S\int_{t_j}^T B(s,T)\,\mathrm{d}s + \sigma_r^2\int_{t_j}^T B(s,T)^2\,\mathrm{d}s\right)
\end{aligned} \tag{57}$$

We can split this long expression in three parts corresponding to the three integrals in (19), namely first $\sigma_r^2\int_t^{t_i}[B(s,T)-B(s,t_i)][B(s,T)-B(s,t_j)]\,\mathrm{d}s$ corresponds to $\int_t^{t_i}\rho_{ij}(s)\sigma_i(s)\sigma_j(s)\,\mathrm{d}s$, the correlation between the bonds with maturity t_i and t_j normalized by the bond with maturity T, since in a one factor model the correlation between bonds (and hence forward bond prices) equals one. This expression is direct from (54), the forward bond volatility. Second, $-\rho\sigma_S\sigma_r\int_{t_i}^{t_j}[B(s,t_j)-B(s,T)]\,\mathrm{d}s + \sigma_r^2\int_{t_i}^{t_j}B(s,T)^2 - B(s,T)B(s,t_j)\,\mathrm{d}s$ corresponds to $\int_{t_i}^{t_j}\rho_{F,j}(s)\sigma_F(s)\sigma_j(s)\,\mathrm{d}s$, the covariance between the forward asset price and $D(t,t_j)/D(t,T)$. Finally, $\sigma_S^2(T-t_j) + 2\rho\sigma_S\sigma_r\int_{t_j}^T B(s,T)\,\mathrm{d}s + \sigma_r^2\int_{t_j}^T B(s,T)^2\,\mathrm{d}s$ corresponds to $\int_{t_j}^T\sigma_F^2(u)\,\mathrm{d}u$, the implied volatility of a forward start option. The same approach can be followed for any other Gaussian interest rate model in combination with a Geometric Brownian Motion for the stock.

8. Numerical results

This section provides numerical results for both the pricing approach discussed in Sections 4 and 6 as well as the one based on the Black–Scholes Hull–White (HW) model. To obtain numerical results on the price of a guarantee according to the approach in Section 5 which have empirical relevance, we use the estimation results of De Jong et al. (2002) on a full-factor LMM. A motivation for the use of this model can be found in their paper. We take the S&P100 as the investment funds. We take the CBOE S&P100 implied volatility index as our forward stock volatility estimate. We estimated, using sample estimates, the correlation between forward stock and bond prices using weekly S&P100 and interest rate data for the same period over which the LMM parameters were estimated, namely January 1995 to June 1999. We assumed that, $\rho_{F,i}(t) \equiv \rho(T-t, T-t_i)$, only the remaining time to maturity of the relevant zeros is taken into account. A time homogeneous correlation matrix results. To let the HW parameters, like the LMM parameters, be representative for the whole sample we calibrated the HW parameters to a set of implied 'at-the-money-forward' swaption volatilities generated by the calibrated LMM using the term structure of the latest observation in our sample. These swaption volatilities were calculated using the method described by Hull and White (2000). We minimized the sum of squared errors of the swaption volatility implied by HW model prices with respect to the LMM implied swaption volatility. The following results for the parameters were obtained, $a = 0.0349$ and $\sigma_r = 0.0116$. To estimate correlation between short rate and stock price we used time series data on 3 month interest rate and the S&P100. A correlation coefficient of -0.0200 resulted. For pricing we used an implied forward stock volatility equal to the CBOE implied volatility index of the S&P100 (equal to 21.01%) and the term structure corresponding to the latest observation in our sample. We can consider the results as prices at end of June 1999.

The following parameterization of the insurance contract is taken, $\mathrm{GP}_i \equiv \mathrm{GP} = 100$, $\mathrm{FC}_i = 5 + 25[i \leq 3]$, $c_i \equiv c = 0.02$ and yearly premium payments. Only a guaranteed amount at maturity is considered. The prices we calculated do not take into account survival rates of the insured. These can be very different over countries, age and gender.[19] Furthermore in the light of our results in Section 2, no clear-cut way is known to us to determine risk neutral survival probabilities.

We proceed as follows, first we visualize the effect of stochastic interest rates using the volatility of the fundvalue. Since the mean of the fundvalue is independent of modelparameters there's a one-to-one correspondence between the price and volatility in the lognormal approximation. Second, keeping in mind the importance of stochastic interest rates, we look at differences between prices obtained using the LMM for the calculation of forward bond volatility versus prices obtained using the BSHW model of Section 7. Our results show this effect is non-negligible.

Fig. 1 shows the convexity correction effect on the volatility of the fundvalue at maturity in the BSHW model. The effect of stochastic interest rates on the volatility (i.e. the convexity correction) increases with maturity. We see that the volatility first decreases because of an averaging effect. The volatility is an average of the volatility of each premium payment. This volatility is highest for the first payment and lowest for the last payment (see the result in Lemma 3, stock volatility is greater than bond volatility). Then the volatility increases again due to a time effect. For maturities typical in the context of pension funds this completely cancels out the averaging effect, partly because of the effect of stochastic interest rates. This time effect is induced by the increasing number of cross correlations (at a rate equal to T^2) between premium payments at longer maturities. The sudden fall in volatility after three years

[19] A typical 10 year survival rate of a 35 year old male is 98%, for a 50 year old male this is 93%. These numbers are based on a Dutch Mortality table over the years 1990–1995. So, for reasonable maturities and age below 50, mortality will not influence the prices shown here dramatically. However for products like these in the context of pension funds the maturity of the contracts is a lot higher and the influence of mortality increases. We can say though that in general the maturity of a contract like the one discussed here doesn't go beyond age 65, which is usually when years are starting to count in terms of survival probabilities.

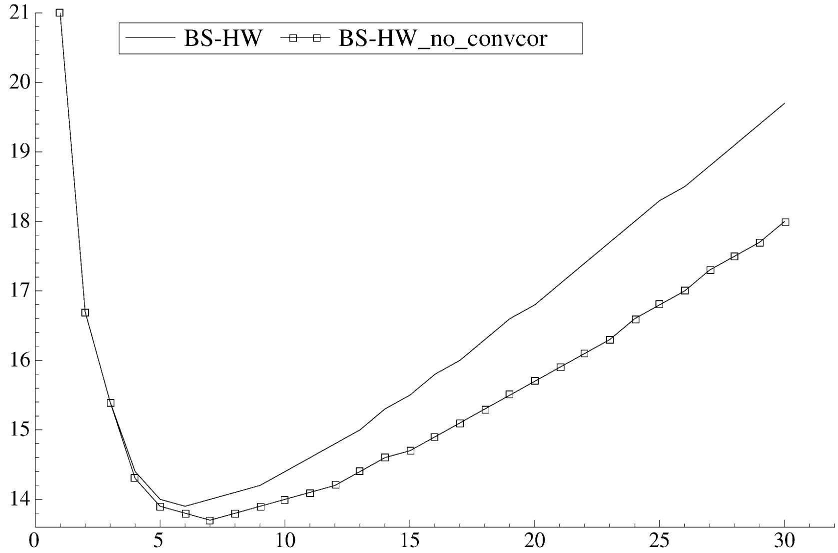

Fig. 1. Volatility for Rate of Return Guarantees calculated using the BS-HW model with and without the convexity correction effect. The normal line shows the volatility whereas the boxed line shows the volatility without the convexity correction. As can be seen from the graph, the convexity correction effect increases with maturity.

can be explained by the structure of the fixed cost deductions. For the first three years fixed costs are 30% of gross premium. After that fixed costs are only 5% of gross premium. This is similar to reality where insurers let policy holders pay for their acquisition costs. As the main part of the volatility is caused by forward stock volatility, the premium payments in the first years are contributing most. Because of the acquisition costs the *relative* effect of the earlier premium payments decreases and hence volatility decreases.

For a contract with a maturity of 10 years, in the BSHW model the convexity correction is 40 volatility basis points on a volatility of 14.4%. For a contract with maturity 30 (very common in the context of pension funds) this is even higher at 171 vol. bp. (1.71%) on a volatility of 19.7%. When we use the LMM, the convexity correction is 29 vol. bp for a contract with a 10 year maturity. These numbers are not surprising since convexity correction is a second order effect. It remains important however since insurers are working with large portfolios and the effect increases with maturity. To illustrate this let us consider an insurance portfolio of 250,000 policies with an average premium of $1000 a year and a maturity guarantee of 3%. Let the maturities (of 6 to 10 years) be equally distributed over the policies. The convexity correction amounts to a difference in reserve/Fair Value of 3.2% or $2,600,000.–.

A comparison of implied volatilities for LMM and Hull–White results is given in Fig. 2. On the horizontal axis is the maturity of the contract. Although forward bond volatility is higher in the LMM, the effect of stochastic interest rates is higher in the HW model. Due to faster decorrelation[20] between forward LIBOR rates (see correlation parameterization of LMM model), and hence forward bond prices, both forward bond *co*variance and the *co*variance between forward bond and forward stock price are lower in the LMM based model for long maturities. This implies considerable overpricing for long maturities by the HW model. This effect becomes more pronounced with increasing maturity of the contract. However due to higher volatility of short term forward bonds we have underpricing of the BSHW model for short maturities. Table 1 shows the percentage difference of the convexity correction of the BSHW model compared with LMM based results for different contract maturities. It shows that the BSHW model

[20] This decorrelation is fast in comparison with the Hull–White setting in which all rates are perfectly correlated.

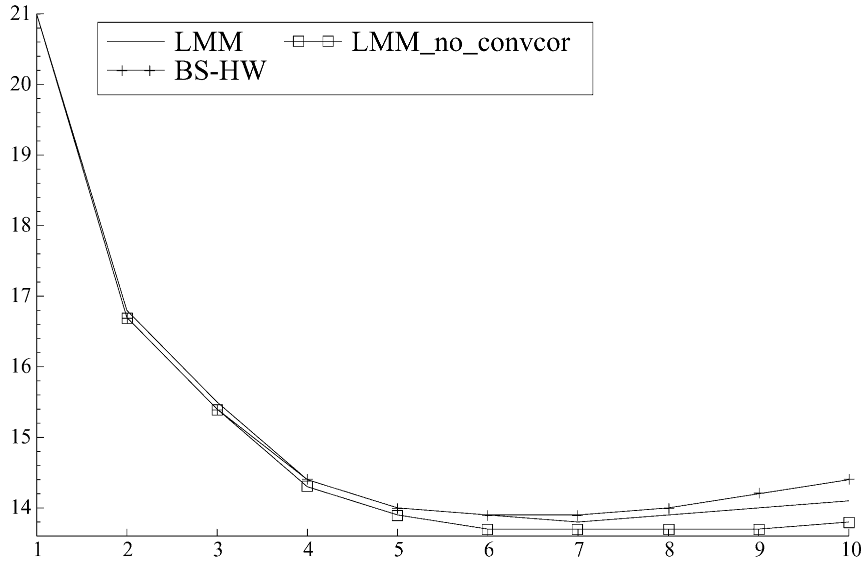

Fig. 2. Volatility for guarantees, percentage points, in regular premium contract for different maturities. The straight line represents the implied volatility with the convexity correction based on the LMM. The boxed line represents the volatility without the convexity correction. For completeness, the line with plusses is the volatility based on the BS-HW model.

Table 1
Percentage differences between convexity corrections based on the BSHW model and the LMM

Maturity	Percentage difference (%)	*Maturity*	Percentage difference (%)
3	−56	7	11
4	−39	8	22
5	−20	9	32
6	−3	10	38

For short maturities the BSHW model underestimates the effect of stochastic interest rates, for long maturities the effect is overestimated.

underprices contracts with short maturity, illustrated by the low convexity correction compared with LMM and overprices long maturity contracts, illustrated by the high convexity correction compared with LMM.[21]

We stress that effects of using the LMM are similar for the pricing bounds both qualitatively and quantitatively (see Table 3). However the advantage of the Levy approximation is that we can explain and interpret these differences. We can directly see the effect of bond volatility and correlation in (19). Using the methods of Section 6 the explanation is hidden behind an involved procedure.

In Table 2 we present results for guarantee prices for the BSHW model in money terms and as a percentage of discounted premiums for various contract lengths and guarantee levels, we let $K = \sum_{i=0}^{n-1} \tilde{P}_i^{(n)} \mathrm{e}^{R(T-t_i)}$. As we would expect, prices increase with a higher guaranteed rate of return. The price of the guarantee increases with maturity for guaranteed rate of 3% and 6%. For a guaranteed rate of 0% (essentially a "no-loss" guarantee) the price decreases with maturity.

Results on the absolute and relative prices of the rate of return guarantee in the LMM framework discussed in Section 6 are presented in Table 3. We see that prices are lower than for the BSHW framework. This is due to reduced convexity correction effect on the volatility. For contracts with a maturity of 10 years the percentage differences are,

[21] We have experimented with a 2 factor Hull–White model but this leads to the same conclusions. The resulting model underprices short maturity guarantees and overprices long maturity guarantees compared to the LMM.

Table 2
Monte Carlo results, Levy approximation and price bounds for Rate of Return Guarantees in Regular Premium UL Insurance, BSHW model (1,000,000 paths, $\sigma_s = 21.01\%$, $a = 0.0349$, $\sigma_r = 0.0116$, $\rho = -0.02$)

Guarantee level (%)	MC price	MC s.e.	% Net Prem. (%)	Levy	Levy error (%)	RSLB	RSLB error (%)	THSTOCH	THSTOCH+	THSTOCH+ error (%)	THDET	THNAIVE	RSUB	RSUB+	RSUB+ error (%)
Contract maturity 5 years															
0	13.10	0.03	3.89	13.60	3.86	12.94	−1.22	56.12	13.12	0.16	16.78	17.18	15.87	13.78	5.20
3	21.33	0.03	6.34	21.77	2.06	21.14	−0.91	73.15	21.30	−0.13	25.72	28.15	24.07	22.21	4.14
6	32.87	0.04	9.77	33.17	0.90	32.65	−0.68	93.12	32.81	−0.20	37.79	39.74	35.59	33.96	3.31
Contract maturity 10 years															
0	19.42	0.04	3.17	21.79	12.18	19.00	−2.20	279.67	19.68	1.34	26.74	27.35	31.34	20.23	4.13
3	40.31	0.07	6.58	42.80	6.17	39.71	−1.50	379.11	40.34	0.06	50.19	51.30	52.05	41.53	3.03
6	76.27	0.10	12.45	78.30	2.66	75.48	−1.03	502.16	76.21	−0.08	88.11	90.69	87.83	78.08	2.38
Contract maturity 20 years															
0	20.00	0.05	2.15	27.93	39.65	19.47	−2.65	855.07	21.49	7.42	28.75	29.82	91.31	20.89	4.44
3	60.19	0.10	6.46	71.13	18.16	59.03	−1.93	1363.86	60.74	0.90	74.85	80.27	130.87	62.30	3.50
6	156.50	0.18	16.80	168.03	7.37	154.36	−1.36	2130.18	157.44	0.60	176.77	194.61	226.20	162.38	3.76
Contract maturity 30 years															
0	15.98	0.04	1.48	27.73	73.57	15.55	−2.69	1169.22	18.71	17.13	23.24	24.06	230.89	16.99	6.34
3	64.09	0.10	5.93	83.73	30.65	62.75	−2.09	2248.70	65.37	2.01	78.61	87.53	278.09	67.64	5.54
6	216.09	0.23	19.99	240.93	11.49	212.62	−1.61	4290.09	219.25	1.46	238.80	270.34	427.96	229.81	6.35

Table 3
Levy approximation, Rogers and Shi lower and extended upper bound for Rate of Return Guarantees in Regular Premium UL insurance LMM consistent pricing

Guarantee level (%)	Levy (LMM)	vs. BSHW (%)	RSLB (LMM) (%)	vs. BSHW	RSUB+ (LMM)	vs. BSHW (%)
Contract maturity 5 years						
0	13.58	−0.19	12.95	0.11	13.78	−0.02
3	21.74	−0.14	21.15	0.05	22.20	−0.04
6	33.14	−0.10	32.66	0.02	33.94	−0.05
Contract maturity 10 years						
0	20.61	−5.43	18.15	−4.45	19.33	−4.44
3	41.14	−3.88	38.42	−3.25	40.21	−3.18
6	76.22	−2.65	73.78	−2.26	76.39	−2.16

Table 4
Relative prices (as a percentage of discounted premiums) of rate of return guarantees for Single Premium contracts (rows 1 and 2) and regular premium contracts with a guaranteed amount on every invested premium (rows 3 and 4)

Maturity (year)	Guaranteed rate (%)		
	0 (%)	3 (%)	6 (%)
5	4.6	8.4	14.1
10	2.8	7.4	16.8
5	4.6	7.0	10.4
10	5.2	9.9	18.1

−5.43%, −3.88% and −2.65%, for guarantee levels of 0%, 3% and 6% respectively. In line with results in Table 1 the overpricing is highest for the lowest guarantee levels (which are of greater importance in practice).

To find out how these prices compare with those of other rate of return guarantees, we compared with single premium contracts and regular premium contracts with a guaranteed amount on every invested premium. In the latter case, the guarantee is a sum of forward starting put options. The prices of these forward starting put options can be calculated using the techniques of Section 4. We calculated prices using the convexity correction approach based on the LMM. Since all prices are relative to the amount invested, we only present relative prices. The results are shown in Table 4. We see that for both single premium contracts and regular premium contracts with a guaranteed amount on every invested premium, prices are higher than for the contract analyzed in this paper. This is because the possibility of averaging out losses over the lifetime of the contract is eliminated for the former two contracts. For single premium contracts the price is higher because the total amount is subject to forward stock volatility for the whole maturity. For regular premium contracts the invested premium is only subject to stock volatility for the remaining maturity after premium payment.

We conclude that the convexity correction on the volatility derived in Section 4 is important in the context of pricing insurance liabilities. This effect is especially important for contracts with long maturities. Furthermore, it seems that a one factor model interest rate model tends to overestimate prices. We recommend guarantee pricing based on the LMM approach suggested in Section 5 or some other multi-factor model.

9. Conclusion

In recent years Unit Linked insurance has become a more prominent part of life insurance business. Hence it is of interest to be able to price guarantees in these products. Our results can be used to price and hedge guarantees without making restrictive assumptions about the stochastic processes of the underlying instruments. We have derived, using Change of Numeraire techniques, a general pricing formula for Rate of Return Guarantees in a Regular Premium

Unit Linked Insurance contract. We show the guarantee is equivalent to a put option on some stochastically weighted average of the stock price at maturity. Furthermore, we derive some results on and discuss the analogy of the guarantee with Asian options. The main contribution of our paper focusses on the effect of stochastic interest rates. In the context of the Levy approximation we derive general expressions for this effect and show it has the interpretation of a convexity correction. We show how we can obtain guarantee prices in accordance with the popular LIBOR Market Model. This enables one to find prices of the guarantee which are consistent with both observed stock option prices and observed cap and swaption prices. We extend earlier results on pricing bounds of Asian options to UL Guarantees and stochastic interest rates. Numerical results show non-negligible prices of guarantees. This also illustrates the importance of the convexity correction arising from stochastic interest rates. We also find a one factor interest rate model overestimates prices in comparison with LMM consistent pricing. Overpricing increases with the maturity of the contract and is highest for guarantee levels most relevant to the industry.

Acknowledgements

The authors would like to thank Peter Boswijk, Frank de Jong, Roger Lord, an anonymous referee, participants at the Symposium on Insurance and Finance 2003 in Bergen, participants at the Insurance: Mathematics and Economics Conference 2003 in Lyon for comments and suggestions.

Appendix A. Change of Numeraire and convexity correction

In a complete and arbitrage-free market the unique value of any financial claim equals the expectation of the payoff normalized by the money market account under some equivalent measure, Harrison and Kreps (1979), Harrison and Pliska (1981). Since under this intended probability measure the expected return on all assets equals the risk free rate, the probability measure is termed the Risk Neutral measure, denoted here by $\boldsymbol{Q}$, and expectation with respect to this measure is called Risk Neutral expectation. In this context, the normalizing asset (in these papers the money market account) is called the *Numeraire*. Geman et al. (1995) show how not only the money market account, but every strictly positive self-financing portfolio of traded assets, can be used as a numeraire. Their Change of Numeraire theorem shows how an expectation under a probability measure $\boldsymbol{Q}^N$ associated with numeraire N is related to an expectation under an equivalent probability measure $\boldsymbol{Q}^M$ associated with numeraire M. As a by-product all normalized assets are martingales under the probability measure associated with the numeraire. To be more specific their theorem states that in a complete and arbitrage-free market, for any numeraires N and M with associated measures $\boldsymbol{Q}^N$ and $\boldsymbol{Q}^M$ respectively, the following holds for the price of an asset H at time t,

$$H(t) = N(t)E_t^N\left[\frac{H(T)}{N(T)}\right] = M(t)E_t^M\left[\frac{H(T)}{M(T)}\right] \tag{A.1}$$

Where E_t^N, E_t^M denotes expectation conditional on the information available at time t under $\boldsymbol{Q}^N$ and $\boldsymbol{Q}^M$ respectively. The Radon-Nikodym derivative associated with a Change of Measure from $\boldsymbol{Q}^N$ to $\boldsymbol{Q}^M$ is given by,

$$\frac{d\boldsymbol{Q}^M}{d\boldsymbol{Q}^N} = \frac{M(T)/M(t)}{N(T)/N(t)} \tag{A.2}$$

Hence if the price of an asset with payoff $H(T)$, known at time T, can be calculated by taking a Risk-Neutral expectation, it can be equivalently and sometimes more conveniently calculated by changing numeraires.

Many (particular interest rate) derivatives can be characterized as *exotic* European options. This means that the price of the option is determined by the joint distribution of a few relevant interest rates at one point in time. A possible approach in the case of interest rates is to specify a full (multi-factor) model, estimate the parameters and calculate the price, possibly analytic or otherwise by numerical techniques like Monte Carlo simulation. The danger of this approach is that it can lead to very unrealistic correlation structures between the relevant rates (see Rebonato, 1998 Chapter 3 and 1999). Contrary to this approach convexity correction focuses the modelling as closely as possible on the problem at hand. Of the joint distribution of the relevant rates, the marginal distributions are almost always taken lognormal, hence the joint distribution is characterized by the marginals and some correlation parameter(s). In this paper we use the idea of convexity correction to interpret our results on the pricing of Rate of Return Guarantees. The application of this idea to stock options is a novelty in the literature. This leads (under weak assumptions) to a very interpretable expression of the implied volatility to use in the Levy approximation for which expressions are derived in section 5. An introductory example applying the ideas of convexity correction to the pricing of a displaced LIBOR payment, the so called LIBOR in arrears payment, can be found in Pelsser (2003). An example of how convexity correction arises in the pricing of an Arithmetic Asian option is given in Appendix C. This already hints at the approach we take to determine the value of the guarantee. For more on convexity correction see Pelsser (2003) and the references therein.

Appendix B. Proofs

First we show that:

$$\sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}} = \sum_{i=0}^{n-1} \tilde{P}_i^{(n)} \frac{S_T}{S_{t_i}} \tag{B.1}$$

where, $P_i = \mathrm{GP}_i - \mathrm{FC}_i - c_i \sum_{j=0}^{i-1} P_j S_{t_i}/S_{t_j}$, $i > 1$, $P_0 = \mathrm{GP}_0 - \mathrm{FC}_0 = NP_0$ and $\tilde{P}_i^{(n)} = NP_i \cdot \prod_{j=1}^{n-i-1}(1 - c_{n-j}), i = 0, \dots, n-1$. This implies in particular that $\tilde{P}_{n-1}^{(n)} = NP_{n-1}$ and $\tilde{P}_i^{(n)} = \tilde{P}_i^{(n-1)}(1 - c_{n-1}), i = 0, \dots, n-2$.

To prove this we first show it holds for $n = 1$,

$$\sum_{i=0}^{1-1} P_i \frac{S_{t_1}}{S_{t_i}} = NP_0 \frac{S_{t_1}}{S_{t_i}} = \sum_{i=0}^{1-1} \tilde{P}_i^{(1)} \frac{S_{t_1}}{S_{t_i}} \tag{B.2}$$

Now we show, given the relationship (B.1.) holds for $n-1$, it holds for n. We have, using $P_i = \mathrm{GP}_i - \mathrm{FC}_i - c_i \sum_{j=0}^{i-1} P_j S_{t_i}/S_{t_j}$,

$$\sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}} = \sum_{i=0}^{n-1} P_i \frac{S_{t_n}}{S_{t_i}} = NP_{n-1} \frac{S_{t_n}}{S_{t_{n-1}}} + \frac{S_{t_n}}{S_{t_{n-1}}}(1 - c_{n-1}) \sum_{i=0}^{n-2} P_i \frac{S_{t_{n-1}}}{S_{t_i}} \tag{B.3}$$

since, by assumption, (B.1.) holds for $n-1$, $\sum_{i=0}^{n-2} P_i S_{t_{n-1}}/S_{t_i} = \sum_{i=0}^{n-2} \tilde{P}_i^{(n-1)} S_{t_{n-1}}/S_{t_i}$, hence,

$$\sum_{i=0}^{n-1} P_i \frac{S_{t_n}}{S_{t_i}} = \tilde{P}_{n-1}^{(n)} \frac{S_{t_n}}{S_{t_{n-1}}} + \sum_{i=0}^{n-2} \tilde{P}_i^{(n-1)}(1 - c_{n-1}) \frac{S_{t_n}}{S_{t_i}} \tag{B.4}$$

$$\sum_{i=0}^{n-1} P_i \frac{S_{t_n}}{S_{t_i}} = \tilde{P}_{n-1}^{(n)} \frac{S_{t_n}}{S_{t_{n-1}}} + \sum_{i=0}^{n-2} \tilde{P}_i^{(n)} \frac{S_{t_n}}{S_{t_i}} \tag{B.5}$$

$$\sum_{i=0}^{n-1} P_i \frac{S_{t_n}}{S_{t_i}} = \sum_{i=0}^{n-1} \tilde{P}_i^{(n)} \frac{S_T}{S_{t_i}} \tag{B.6}$$

This proves (B.1.) by induction.

Second, we give the proof of Lemma 3.

Proof. For $t \leq t_i \leq t_j$, using the Tower Law of conditional expectation as in Theorem 2, we have

$$E_t^{Q^T}\left[\frac{S_T^2}{S_{t_i}S_{t_j}}\right] = E_t^{Q^T}\left[\frac{1}{S_{t_i}}E_{t_i}^{Q^T}\left[\frac{1}{S_{t_j}}E_{t_j}^{Q^T}[S_T^2]\right]\right] \tag{B.7}$$

now from (16) we obtain for the last part of this expression,

$$E_{t_j}^{Q^T}[S_T^2] = E_{t_j}^{Q^T}[(F_T^T)^2] = (F_{t_j}^T)^2 \exp\left(\int_{t_j}^T \sigma_F^2(u)\,\mathrm{d}u\right)$$

Continuing we obtain,

$$\begin{aligned} E_t^{Q^T}\left[\frac{S_T^2}{S_{t_i}S_{t_j}}\right] &= \exp\left(\int_{t_j}^T \sigma_F^2(u)\,\mathrm{d}u\right) E_t^{Q^T}\left[\frac{1}{S_{t_i}}E_{t_i}^{Q^T}\left[\frac{1}{S_{t_j}}(F_{t_j}^T)^2\right]\right] \\ &= \exp\left(\int_{t_j}^T \sigma_F^2(u)\,\mathrm{d}u\right) E_t^{Q^T}\left[\frac{1}{S_{t_i}}E_{t_i}^{Q^T}\left[F_{t_j}^T D^T(t_j, t_j)\right]\right] \end{aligned} \tag{B.8}$$

Now both F_t^T and $D^T(t, t_j)$ are martingales under the T-forward measure, using the solutions to (16) and (17) gives,

$$E_{t_i}^{Q^T}[F_{t_j}^T D^T(t_j, t_j)] = F_{t_i}^T D^T(t_i, t_j) \exp\left(\int_{t_i}^{t_j} \rho_{F,j}(s)\sigma_F(s)\sigma_j(s)\,\mathrm{d}s\right) \tag{B.9}$$

Plugging this expression in (B.8.) gives,

$$\begin{aligned} E_t^{Q^T}\left[\frac{S_T^2}{S_{t_i}S_{t_j}}\right] &= \exp\left(\int_{t_j}^T \sigma_F^2(u)\,\mathrm{d}u\right)\exp\left(\int_{t_i}^{t_j} \rho_{F,j}(s)\sigma_F(s)\sigma_j(s)\,\mathrm{d}s\right) E_t^{Q^T}\left[\frac{1}{S_{t_i}}F_{t_i}^T D^T(t_i, t_j)\right] \\ &= \exp\left(\int_{t_j}^T \sigma_F^2(u)\,\mathrm{d}u\right)\exp\left(\int_{t_i}^{t_j} \rho_{F,j}(s)\sigma_F(s)\sigma_j(s)\,\mathrm{d}s\right) E_t^{Q^T}\left[D^T(t_i, t_i)D^T(t_i, t_j)\right] \end{aligned} \tag{B.10}$$

Again using (17), we obtain,

$$E_t^{Q^T}[D^T(t_i, t_i)D^T(t_i, t_j)] = D^T(t, t_i)D^T(t, t_j) \exp\left(\int_t^{t_i} \rho_{ij}(s)\sigma_i(s)\sigma_j(s)\,\mathrm{d}s\right) \tag{B.11}$$

This gives the desired result. □

Appendix C. Arithmetic stock price average

In this appendix we are concerned with the expectation of the arithmetic average of the stock price assuming stochastic interest rates under the T-Forward measure. We encounter this in the calculation of the price of an Asian option. The time zero price of an arithmetic Asian option maturing at time T, strike K, with the average taken over the time points $t_i, i = 1, \dots, n, t_n = T$ is given by,

$$V^{\text{Asian}}(t) = D(t,T) E_t^{Q^T}\left[\left(\sum_{i=1}^{n} S_{t_i} - K\right)^+\right] \tag{C.1}$$

If one would use the Levy approximation to calculate this price one would first be interested in $E_t^{Q^T}\sum_{i=1}^{n} S_{t_i}$. This problem is not more difficult than the calculation of $E_t^{Q^T} S_{t_i}$ for $i < n$. This is very similar to the problem of pricing LIBOR in arrears, but in this case we are dealing with a 'displaced' stock price (a stock price at a time point which doesn't correspond to the forward measure under which the expectation is taken). Observe that both $F_t^{t_i} \equiv S_t/D(t,t_i)$ and $D^{t_i}(t,T) \equiv D(t,T)/D(t,t_i)$ are martingales under the t_i-Forward measure. The convexity correction approach to this valuation problem is to assume both t_i-Forward stock and t_i-Forward bond prices with maturity T have volatilities, σ_F and σ_T respectively, which are deterministic functions of time. This implies both forward stock and forward bond prices are lognormal. Also assume t_i-forward stock and t_i-forward bond (maturity T) prices are correlated with correlation $\rho_{F,T}$. Use the Change of Numeraire theorem, the martingale property and the assumption of lognormality, in the given order, to obtain,

$$\begin{aligned} E^{Q^T}[S_{t_i}] &= \frac{D(0,t_i)}{D(0,T)} E^{Q^{t_i}}\left[S_{t_i}\frac{D(t_i,T)}{D(t_i,t_i)}\right] = \frac{D(0,t_i)}{D(0,T)}\frac{S_0}{D(0,t_i)}\frac{D(0,T)}{D(0,t_i)}\exp\left(\int_0^{t_i}\rho_{F,T}(s)\sigma_F(s)\sigma_T(s)\,\mathrm{d}s\right) \\ &= F_0^{t_i}\exp\left(\int_0^{t_i}\rho_{F,T}(s)\sigma_F(s)\sigma_T(s)\,\mathrm{d}s\right) \end{aligned} \tag{C.2}$$

or assuming volatilities and correlation constant,

$$E^{Q^T}[S_{t_i}] = F_0^{t_i}\exp(\rho_{F,T}\sigma_F\sigma_T t_i) \tag{C.3}$$

So the expected stock price is the forward stock price times some convexity correction. The advantages of using convexity correction techniques are clear. The determinants of the price can be seen from the formulas in an eyesight and the price can be written in terms of readily observable implied volatilities. The volatilities can be taken from implied stock option volatility and cap or swaption volatility (as we have shown in Section 7). The correlation can be obtained from timeseries data.

Now if we consider a Black–Scholes Hull–White model and observing that in a model in which the short rate follows a Hull and White model the volatility of $D^{t_i}(t,T)$ equals $-\sigma_r[B(t,T) - B(t,t_i)]$ (see Section 6) it follows that the expectation of the 'displaced' stock price equals,

$$\begin{aligned} E^{Q^T}[S_{t_i}] = \frac{S_0}{D(0,t_i)}\exp\Bigg\{&-\sigma_r^2\int_0^{t_i} B(s,T)B(s,t_i)\,\mathrm{d}s + \sigma_r^2\int_0^{t_i} B^2(s,t_i)\,\mathrm{d}s \\ &+ \rho\sigma_S\sigma_r\int_0^{t_i} B(s,t_i) - B(s,T)\,\mathrm{d}s\Bigg\} \end{aligned} \tag{C.4}$$

Comparing with (C.2) and (C.3) we can interpret these integrals in (C.4) as the quadratic covariance of $\ln(F_t^{t_i})$ and $\ln(D^{t_i}(t,T))$. This is completely in line with the results in Nielsen and Sandmann (1996a,b).

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