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## INSTANTANEOUS MEAN-VARIANCE HEDGING AND SHARPE RATIO PRICING IN A REGIME-SWITCHING FINANCIAL MODEL

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□ *We study hedging and pricing of claims in a non-Markovian regime-switching financial model. Our financial market consists of a bank account and a risky asset whose dynamics are driven by a Brownian motion and a multivariate counting process with stochastic intensities. The counting process is used to model the switching behavior for the states of the economy. We assume that the trajectory of the risky asset is continuous between the transition times for the states of the economy and that the value of the risky asset jumps at the time of the transition. We find the hedging strategy that minimizes the instantaneous mean-variance risk of the hedger's surplus, and we set the price so that the instantaneous Sharpe ratio of the hedger's surplus equals a predefined target. We discuss key properties of our optimal price and optimal hedging strategy.*

**Keywords** Backward stochastic differential equations; Counting process; Instantaneous mean-variance risk; Instantaneous Sharpe ratio; Model ambiguity; No-good-deal pricing.

**Mathematics Subject Classification** 49N90; 60H30.

### 1. INTRODUCTION

Pricing and hedging in incomplete markets is the most important subject in the financial literature. Despite numerous papers, there is still a need to develop new pricing and hedging methods and to derive prices and hedging strategies in realistic financial models. In this article, we focus on instantaneous mean-variance hedging and Sharpe ratio pricing of claims in a regime-switching financial model.

Empirical studies show that regime-switching models can explain empirical behaviors of many economic and financial data, especially the long-term behavior of these data; see Hamilton<sup>[18]</sup>, Hardy<sup>[19]</sup>, and Mamon and Elliott<sup>[24]</sup>. The rationale behind the regime-switching framework is that the financial market may switch between a low-volatility state and a high-volatility

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state, or even between more states representing the conditions of the economy. The switching behavior for the states of the financial market can be attributed to structural changes in economic conditions and changes in business environments. It is clear that there are significant fluctuations in economic variables over a long period of time. Hence, the switching behavior of the states of the financial market (the states of the economy) should be particularly incorporated in models used for valuation of long-term derivatives. We point out that the use of regime-switching models has been recommended by the American Academy of Actuaries and the Canadian Institute of Actuaries for valuation of long-term financial guarantees embedded in insurance contracts.

In this article, we consider a non-Markovian regime-switching financial model. The dynamics of a bank account and a risky asset are driven by a Brownian motion and a multivariate counting process with stochastic intensities. The interest rate, drift, volatility, and intensities fluctuate over time, and, in particular, they depend on the state (regime) of the economy, which is modeled by the multivariate counting process. We assume that the trajectory of the risky asset is continuous between the transition times for the states of the economy and that the value of the risky asset jumps at the transition time. Such a dynamics of the risky asset clearly agrees with the idea of the switching behavior for the financial market. Since we use stochastic transition intensities, we can model an effect in which not only the stock price is affected by the transitions between the states of the economy but also the stock price determines the transition intensities; see Elliott et al.<sup>[17]</sup> for a financial motivation of a so-called feedback effect. The goal is to price and hedge unattainable contingent claims in our general regime-switching financial model.

Pricing and hedging in regime-switching models have gained a lot of interest in the literature; see Donnelly and Heunis<sup>[14]</sup>, Elliott et al.<sup>[17]</sup>, Elliott et al.<sup>[16]</sup>, Elliott et al.<sup>[15]</sup>, Siu et al.<sup>[28]</sup>, Siu<sup>[29]</sup>, and Wu and Li<sup>[30]</sup>, where risk minimization, quadratic hedging, multi-period Markowitz optimization, and the Esscher transform are applied. In this article we use a different pricing and hedging objective, and we investigate instantaneous mean-variance hedging and Sharpe ratio pricing. We should point out that in this article we, in fact, consider three pricing and hedging approaches: instantaneous mean-variance hedging and Sharpe ratio pricing, no-good-deal pricing, and pricing and hedging under model ambiguity, which are equivalent under proper specification.

Bayraktar and Young<sup>[2]</sup>, Young<sup>[31]</sup>, and Bayraktar et al.<sup>[3]</sup> were the first to apply instantaneous variance hedging and Sharpe ratio pricing. They find the hedging strategy that minimizes the instantaneous variance (the quadratic variation) of the surplus (the difference between the hedging portfolio and the price of a claim) and set the price so that the instantaneous Sharpe ratio of the surplus equals a predefined target. Bayraktar and Young<sup>[2]</sup> use this approach to price and hedge claims contingent on a

non-tradeable financial risk, and Bayraktar et al.<sup>[3]</sup>, and Young<sup>[31]</sup> use this approach to price stochastic mortality risk in insurance models. Interestingly, the authors show the equivalence between the local variance minimization under the Sharpe ratio constraint and no-good-deal pricing, which was popularized by Cochrane and Saá-Requejo<sup>[8]</sup> and Björk and Slinko<sup>[6]</sup>. Leitner<sup>[22]</sup> deals with an infinitesimal mean-variance risk measure of the surplus and a robust expectation of the terminal surplus under model ambiguity. He finds the hedging strategies that minimize both risk measures and the prices that make the risk measures vanish. Leitner<sup>[22]</sup> shows that both strategies and prices coincide in a diffusion model with a non-tradeable risk factor. Finally, Delong<sup>[11]</sup> considers a general combined financial and insurance model. He derives the optimal hedging strategy and the optimal price by minimizing the infinitesimal mean-variance risk measure of the surplus and by setting the infinitesimal Sharpe ratio of the surplus at a predefined level. Delong<sup>[11]</sup> also shows that the optimal strategies coincide with the strategies derived under no-good-deal pricing and pricing and hedging under model ambiguity. We point out that none of the above papers covers the case of a regime-switching financial market. We are aware that Donnelly<sup>[13]</sup> finds a no-good-deal price of a contingent claim in a regime-switching financial model. However, she considers a Markovian dynamics of the stock without jumps and without the feedback effect of the stock on the transition intensities. She also does not investigate the optimal hedging strategy that can be derived by using instantaneous mean-variance hedging or hedging under model ambiguity. Consequently, to the best of our knowledge the complete characterization of the optimal price and the optimal hedging strategy under instantaneous mean-variance hedging and Sharpe ratio pricing (no-good-deal pricing and robust pricing and hedging under model ambiguity) in a general non-Markovian regime-switching model is still missing. This article fills this gap. We would like to point out that for the first time we derive the optimal price and the optimal hedging strategy under the instantaneous mean-variance hedging and the instantaneous Sharpe ratio pricing objective in a general non-Markovian regime-switching financial model with the stock the value of which changes in a discontinuous way at the transition times and with the feedback effect under which the stock affects the transition intensities.

We apply backward stochastic differential equations (BSDEs) to solve our optimization problems. For a theory of BSDEs, we refer to Crépey<sup>[10]</sup> and Delong<sup>[12]</sup>. We remark that our mathematical techniques are similar to the one used in Delong<sup>[11]</sup>. However, some non-trivial modifications are introduced since our stock price dynamics is not continuous. We characterize the optimal price and the optimal hedging strategy with a unique solution to a nonlinear, Lipschitz BSDE with jumps. It is known that a measure solution (an arbitrage-free representation of the price) may not exist and a comparison principle (monotonicity of the pricing operator) may fail for a BSDE with jumps; see Barles et al.<sup>[1]</sup>, Royer<sup>[27]</sup>, Delong<sup>[11]</sup>, and Delong<sup>[12]</sup>.

However, we provide a simple, new condition under which the optimal price (the solution to the BSDE with jumps) is arbitrage-free and monotone with respect to the terminal claim and the Sharpe ratio. We also interpret the optimal hedging strategy as a delta-hedging strategy with a correction term reflecting the use of the expected profit requirement in the hedging objective.

This article is structured as follows. In Section 2, we introduce the regime-switching financial model. In Section 3, we describe our pricing and hedging approach, and we provide an additional motivation for the instantaneous mean-variance hedging and Sharpe ratio pricing by giving a link to no-good-deal pricing and robust pricing and hedging under model ambiguity. In Section 4, we solve our optimization problem. Key properties of the optimal hedging strategy and the optimal price are investigated in Section 5. A numerical example is discussed in Section 6.

## 2. THE REGIME-SWITCHING FINANCIAL MODEL

We deal with a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a filtration  $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$  and a finite time horizon  $T < \infty$ . We assume that  $\mathcal{F}$  satisfies the usual hypotheses of completeness ( $\mathcal{F}_0$  contains all sets of  $\mathbb{P}$ -measure zero) and right continuity ( $\mathcal{F}_t = \mathcal{F}_{t+}$ ). On the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , we define an  $\mathcal{F}$ -adapted Brownian motion  $W = (W(t), 0 \leq t \leq T)$  and an  $\mathcal{F}$ -adapted multivariate counting process  $N = (N_1(t), \dots, N_I(t), 0 \leq t \leq T)$ .

We consider an economy that can be in one of  $I$  states (regimes) and switches between those states randomly. For  $i = 1, \dots, I$ , the counting process  $N_i$  counts the number of transitions of the economy into the state  $i$ . We assume that

(A1) the counting process  $N_i$  has intensity  $\lambda_i(t)$  where  $\lambda_i : \Omega \times [0, T] \rightarrow [0, \infty)$  is an  $\mathcal{F}$ -predictable, bounded process.

Consequently, the compensated counting process

$$\tilde{N}_i(t) = N_i(t) - \int_0^t \lambda_i(s) ds, \quad 0 \leq t \leq T, \quad i = 1, \dots, I,$$

is an  $\mathcal{F}$ -martingale; see Chapters XI.1 and XI.4 in He et al.<sup>[20]</sup>. We remark that  $\lambda_i(t)$  is an intensity of the transition of the economy into state  $i$  at time  $t$ . Furthermore, let  $J = (J(t), 0 \leq t \leq T)$  denote an  $\mathcal{F}$ -adapted process which indicates the current state of the economy. If the economy is in a regime  $k \in \{1, \dots, I\}$  at the initial point of time, then the dynamics of the process

$J$  is given the stochastic differential equation

$$dJ(t) = \sum_{i=1}^I (i - J(t-)) dN_i(t), \quad J(0) = k \in \{1, \dots, I\}.$$

A time-homogeneous Markov process  $J$  arises if we choose  $\lambda_i(t) = \lambda_i(J(t-))$ .

The financial market consists of a risk-free bank account and a risky asset. The dynamics of the risk-free bank account  $B = (B(t), 0 \leq t \leq T)$  is given by the differential equation

$$\frac{dB(t)}{B(t)} = r(t) dt, \quad B(0) = 1, \quad (2.1)$$

and the dynamics of the risky asset  $S = (S(t), 0 \leq t \leq T)$  is described by the stochastic differential equation

$$\frac{dS(t)}{S(t-)} = \mu(t) dt + \sigma(t) dW(t) + \sum_{i=1}^I \gamma_i(t) d\tilde{N}_i(t), \quad S(0) = 1. \quad (2.2)$$

We assume that

(A2)  $r, \mu, \sigma, (\gamma_i)_{i=1, \dots, I} : \Omega \times [0, T] \rightarrow \mathbb{R}$  are  $\mathcal{F}$ -predictable, bounded processes such that there exists a unique solution  $S$  to (2.2). Moreover,

$$\begin{aligned} \mu(t) &\geq r(t), \quad 0 \leq t \leq T, \\ |\delta(t)|^2 &= |\sigma(t)|^2 + \sum_{i=1}^I |\gamma_i(t)|^2 \lambda_i(t) \geq \epsilon > 0, \quad 0 \leq t \leq T, \\ \gamma_i(t) &> -1, \quad 0 \leq t \leq T, \quad i = 1, \dots, I. \end{aligned}$$

These conditions are standard in financial modeling. The first condition is clear. The second condition is a non-degeneracy condition for the volatility of the risky asset return. The third condition guarantees that the price process  $S$ , which solves (2.2), is strictly positive; see Theorem 4.61 in Jacod and Shiryaev<sup>[21]</sup>. We point out that we deal with a non-Markovian model. All parameters of the model (2.1)–(2.2) are driven by the Brownian motion and the multivariate counting process. The interest rate, drift, volatility, jump amplitudes, and intensities fluctuate, and they depend on the past and current conditions of the economy and the financial market. Let us also notice that if the economy remains in a state, then the dynamics of the risky asset is continuous. However, if a transition into a different state occurs, then

the value of the risky asset changes in a discontinuous way at the time of the transition. In the sequel, we use the following notation for the instantaneous variance of the risky asset return

$$|\delta(t)|^2 = |\sigma(t)|^2 + \sum_{i=1}^I |\gamma_i(t)|^2 \lambda_i(t),$$

and for the instantaneous Sharpe ratio of the risky asset

$$\theta(t) = \frac{\mu(t) - r(t)}{\delta(t)}.$$

The most important and practically relevant example of the financial model (2.1)–(2.2) arises when the coefficients  $r$ ,  $\mu$ ,  $\sigma$ ,  $\gamma_i$  depend only on the current state of the economy and the intensities  $\lambda_i$  depend on the current state of the economy and the current value of the risky asset. In that case we investigate the dynamics

$$\begin{aligned} \frac{dB(t)}{B(t)} &= r(J(t-)) dt, \\ \frac{dS(t)}{S(t-)} &= \mu(J(t-)) dt + \sigma(J(t-)) dW(t) + \sum_{i=1}^I \gamma_i(J(t-)) d\tilde{N}_i(t), \end{aligned} \quad (2.3)$$

where the counting process  $N_i$  has intensity  $\lambda_i(t) = \lambda_i(J(t-), S(t-))$  and  $J$  indicates the current state of the economy. Such a model is called a Markov-regime-switching since  $(S, J)$  is a Markov process. We remark that  $\lambda_i(t) = \lambda_i(J(t-), S(t-))$  denotes an intensity of the transition into state  $i$  at time  $t$  given the economy is in state  $J(t-)$  and the stock price equals  $S(t-)$ . The dependence  $\lambda_i(t) = \lambda_i(J(t-), S(t-))$  models the so-called feedback effect in the market; see Elliott et al.<sup>[17]</sup> for a motivation. The complete probabilistic description of regime-switching models can be found in Crépey<sup>[9]</sup>.

### 3. INSTANTANEOUS MEAN-VARIANCE HEDGING AND SHARPE RATIO PRICING

Let  $\xi$  be a contingent claim in the regime-switching financial market (2.1)–(2.2) that has to be covered at time  $T$ . We are interested in finding a hedging strategy and a price of the claim  $\xi$ .

Let  $\pi = (\pi(t), 0 \leq t \leq T)$  denote a hedging strategy, i.e., the amount of wealth that is invested into the risky asset. We introduce the set of admissible hedging strategies.

**Definition 3.1.** A strategy  $\pi := (\pi(t), 0 \leq t \leq T)$  is called *admissible*, written  $\pi \in \mathcal{A}$ , if it satisfies the conditions:

1.  $\pi : [0, T] \times \Omega \rightarrow \mathbb{R}$  is an  $\mathcal{F}$ -predictable process,
2.  $\mathbb{E}[\int_0^T |\pi(t)|^2 dt] < \infty$ .

The price of the claim is modeled as a solution to a BSDE. We assume that the price process  $Y := (Y(t), 0 \leq t \leq T)$  of the claim  $\xi$  solves the BSDE

$$Y(t) = \xi + \int_t^T (-Y(s-)r(s) - f(s))ds - \int_t^T Z(s)dW(s) - \int_t^T \sum_{i=1}^I U_i(s)d\tilde{N}_i(s), \quad 0 \leq t \leq T, \quad (3.1)$$

where  $f$  is the generator of the equation which has to be determined. The assumption that the price solves a BSDE is reasonable. First of all, we can view the price as a dynamic risk measure, so it should satisfy a BSDE; see Chapter 13 in Delong<sup>[12]</sup>. Second, if the price is calculated as the conditional expected value of the discounted payoff under an equivalent probability measure, then it satisfies a BSDE; see Chapters 3.3 and 3.4 in Delong<sup>[12]</sup>. Hence, our price dynamics (3.1) with the generator  $f$  can be justified. If we decide on the form of the generator  $f$ , then the price of the claim  $\xi$  can be defined. In order to determine the generator  $f$ , we use instantaneous mean-variance hedging and Sharpe ratio pricing.

First, we define the hedging portfolio. The dynamics of the hedging portfolio  $X^\pi := (X^\pi(t), 0 \leq t \leq T)$  under an admissible hedging strategy  $\pi \in \mathcal{A}$  is given by the stochastic differential equation

$$\begin{aligned} dX^\pi(t) &= \pi(t) \left( \mu(t)dt + \sigma(t)dW(t) + \sum_{i=1}^I \gamma_i(t)d\tilde{N}_i(t) \right) \\ &\quad + (X^\pi(t-) - \pi(t))r(t)dt, \\ X^\pi(0) &= x. \end{aligned}$$

Next, we define the surplus process  $S^\pi(t) = X^\pi(t) - Y(t)$ ,  $0 \leq t \leq T$ , which models the profit or the loss of the hedger resulting from the past investment and the future liability. The surplus process can also be called a hedging error. The dynamics of the surplus  $S^\pi$  is described by the stochastic differential equation

$$dS^\pi(t) = (\pi(t)(\mu(t) - r(t)) + S^\pi(t-)r(t))$$



$$-f(t)dt + (\pi(t)\sigma(t) - Z(t))dW(t) \\ + \sum_{i=1}^I (\pi(t)\gamma_i(t) - U_i(t))d\tilde{N}_i(t).$$

By standard properties of stochastic integrals—see Theorems II.20, 28, 29, and 39 in Protter<sup>[26]</sup>—we can derive the expected infinitesimal return on the surplus

$$\begin{aligned} \mathbb{E}[dS^\pi(t) - S^\pi(t-)r(t)dt|\mathcal{F}_{t-}] \\ = \pi(t)(\mu(t) - r(t))dt - f(t)dt, \quad 0 < t \leq T, \end{aligned} \quad (3.2)$$

and the expected infinitesimal quadratic variation of the surplus

$$\begin{aligned} \mathbb{E}[d[S^\pi, S^\pi](t)|\mathcal{F}_{t-}] &= |\pi(t)\sigma(t) - Z(t)|^2 dt \\ &+ \sum_{i=1}^I |\pi(t)\gamma_i(t) - U_i(t)|^2 \lambda_i(t) dt, \quad 0 < t \leq T. \end{aligned} \quad (3.3)$$

Our goal is to find an admissible hedging strategy  $\pi \in \mathcal{A}$  that minimizes the instantaneous mean-variance risk of the surplus

$$\begin{aligned} \rho(S^\pi) &= L(t)\sqrt{\mathbb{E}[d[S^\pi, S^\pi](t)|\mathcal{F}_{t-}]/dt} \\ &\quad - (\mathbb{E}[dS^\pi(t) - S^\pi(t-)r(t)dt|\mathcal{F}_{t-}]/dt), \end{aligned} \quad (3.4)$$

for all  $t \in (0, T]$ , and set the price of  $\xi$  (find the generator  $f$  of the BSDE (3.1)) in such a way that the instantaneous Sharpe ratio of the surplus equals a predefined target  $L$ , i.e.,

$$\frac{\mathbb{E}[dS^\pi(t) - S^\pi(t-)r(t)dt|\mathcal{F}_{t-}]/dt}{\sqrt{\mathbb{E}[d[S^\pi, S^\pi](t)|\mathcal{F}_{t-}]/dt}} = L(t), \quad (3.5)$$

for all  $t \in (0, T]$ . As the result, the instantaneous mean-variance risk of the surplus (3.4) under the optimal price and hedging strategy is set to zero. The hedging and pricing objectives (3.4)–(3.5) are called the instantaneous mean-variance hedging and Sharpe ratio pricing. We shall assume that

(A3)  $L$  is an  $\mathcal{F}$ -predictable process such that  $L(t) \geq \theta(t) + \epsilon$ ,  $0 \leq t \leq T$ , for some  $\epsilon > 0$ .

Since the Sharpe ratio of the surplus  $L$  is an  $\mathcal{F}$ -predictable process, it can depend on the economy and the financial market. In particular, the hedger may use different Sharpe ratios in different states of the economy. Such an assumption is important from the practical point of view since investors have different profit expectations in a bull market and in a bear market. We also require that the Sharpe ratio of the surplus is strictly greater than the Sharpe ratio of the risky asset. Such an assumption is obvious since the hedger trading  $\xi$  would require a Sharpe ratio  $L$  that is strictly greater than the Sharpe ratio  $\theta$ , which can be earned by simply investing in the stock  $S$ .

Bayraktar and Young<sup>[2]</sup>, Young<sup>[31]</sup>, and Bayraktar et al.<sup>[3]</sup> have advocated the instantaneous mean-variance hedging and Sharpe ratio pricing for hedging and pricing financial and insurance risks. Let us remark that the hedging and pricing objectives (3.4)–(3.5) are easy to communicate, are based on the first two moments of the hedging error, are related to the Markowitz portfolio selection problem, and involve a Sharpe ratio that is well understood by investors. These four features already make the instantaneous mean-variance hedging and Sharpe ratio pricing an appealing method for pricing and hedging risks in incomplete markets. Interestingly, the instantaneous mean-variance hedging and Sharpe ratio pricing can be related to no-good-deal pricing and robust pricing and hedging under model ambiguity.

It turns out that the price derived under the instantaneous Sharpe ratio pricing (3.5) is equivalent to the price derived under no-good-deal pricing; see Bayraktar et al.<sup>[3]</sup>, Bayraktar and Young<sup>[2]</sup>, Delong<sup>[11]</sup>, and Young<sup>[31]</sup>. Hence, the theory of no-good-deal pricing gives us an additional justification for the instantaneous mean-variance hedging and Sharpe ratio pricing. The no-good-deal price of the claim  $\xi$  is defined as a solution to the following optimization problem:

$$Y(t) = \sup_{(\psi, \phi) \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}^{\psi, \phi}} \left[ e^{-\int_t^T r(s) ds} \xi \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T, \quad (3.6)$$

where  $\mathbb{Q}^{\psi, \phi}$  is an equivalent martingale measure. Under no-good-deal pricing, we price a claim with a least favorable pricing measure from a set of equivalent martingale measures. The set of equivalent martingale measures is defined by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{\psi, \phi}}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = M^{\psi, \phi}(t), \quad 0 \leq t \leq T, \quad (\psi, \phi) \in \mathcal{Q},$$

where

$$\frac{dM^{\psi,\phi}(t)}{M^{\psi,\phi}(t-)} = -\psi(t)dW(t) - \sum_{i=1}^I \phi_i(t)d\tilde{N}_i(t), \quad M(0) = 1, \quad (\psi, \phi) \in \mathcal{Q},$$

and

$$\begin{aligned} \mathcal{Q} = \Big\{ \mathcal{F}\text{-predictable processes } (\psi, \phi) = (\psi, \phi_1, \dots, \phi_I) \text{ such that} \\ |\psi(t)|^2 + \sum_{i=1}^I |\phi_i(t)|^2 \lambda_i(t) \leq |L(t)|^2, \\ \psi(t)\sigma(t) + \sum_{i=1}^I \phi_i(t)\gamma_i(t)\lambda_i(t) = \mu(t) - r(t), \\ \phi_i(t) < 1, \quad 0 \leq t \leq T, \quad i = 1, \dots, I \Big\}. \end{aligned}$$

Let us briefly explain the conditions from the set  $\mathcal{Q}$ . The third condition is clear as it guarantees that  $M^{\psi,\phi}$  is strictly positive; see Theorem 4.61 in Jacod and Shiryaev<sup>[21]</sup>. Recalling the Girsanov's theorem—see Theorem 2.5.1 in Delong<sup>[12]</sup>—we can derive the dynamics of the stock

$$\begin{aligned} \frac{dS(t)}{S(t-)} &= \left( \mu(t) - \psi(t)\sigma(t) - \sum_{i=1}^I \phi_i(t)\gamma_i(t)\lambda_i(t) \right) dt \\ &\quad + \sigma(t)dW^{\mathbb{Q}^{\psi,\phi}}(t) + \sum_{i=1}^I \gamma_i(t)d\tilde{N}^{\mathbb{Q}^{\psi,\phi}}(t), \end{aligned}$$

and we can observe that the second condition implies that the discounted stock process is a  $\mathbb{Q}^{\psi,\phi}$ -martingale for any  $(\phi, \psi) \in \mathcal{Q}$  and  $M^{\phi,\psi}$  defines a set of equivalent martingale measures for the market (2.1)–(2.2). Finally, by the Girsanov's theorem and standard arguments for BSDEs—see Chapters 3.3 and 3.4 in Delong<sup>[12]</sup>—we deduce that any arbitrage-free price process  $Y^{\psi,\phi}(t) = \mathbb{E}^{\mathbb{Q}^{\psi,\phi}}[e^{-\int_t^T r(s)ds} \xi | \mathcal{F}_t]$ ,  $0 \leq t \leq T$ ,  $(\psi, \phi) \in \mathcal{Q}$ , has the dynamics

$$\begin{aligned} dY^{\psi,\phi}(t) &= Y^{\psi,\phi}(t-)r(t)dt + Z^{\psi,\phi}(t)\psi(t)dt + \sum_{i=1}^I U_i^{\psi,\phi}(t)\phi_i(t)\lambda_i(t)dt \\ &\quad + Z^{\phi,\psi}(t)dt + \sum_{i=1}^I U_i^{\psi,\phi}(t)d\tilde{N}_i(t), \end{aligned}$$

$$Y^{\psi,\phi}(T) = \xi. \quad (3.7)$$

We can derive the bound for the instantaneous Sharpe ratio of the arbitrage-free price process  $Y^{\psi,\phi}$  of  $\xi$ :

$$\begin{aligned} & \left| \frac{\mathbb{E}[dY^{\psi,\phi}(t) - Y^{\psi,\phi}(t-)r(t)dt | \mathcal{F}_{t-}]/dt}{\sqrt{\mathbb{E}[d[Y^{\psi,\phi}, Y^{\psi,\phi}](t) | \mathcal{F}_{t-}]/dt}} \right| \\ &= \frac{|Z^{\psi,\phi}(t)\psi(t) + \sum_{i=1}^I U_i^{\psi,\phi}(t)\phi_i(t)\lambda_i(t)|}{\sqrt{|Z^{\psi,\phi}(t)|^2 + \sum_{i=1}^I |U_i^{\psi,\phi}(t)|^2 \lambda_i(t)}}, \\ &\leq \sqrt{|\psi(t)|^2 + \sum_{i=1}^I |\phi_i(t)|^2 \lambda_i(t)}, \end{aligned}$$

and we conclude that the first condition in  $\mathcal{Q}$  implies that the instantaneous Sharpe ratio of an arbitrage-free price process of the claim  $\xi$  is bounded by  $L$ . The process  $L$  defines a so-called no-good-deal range in the financial market, and it represents the bound on possible gains in the financial market measured by the instantaneous Sharpe ratio. The existence of such a maximal gain  $L$  is justified by empirical financial data; see Cochrane and Saá-Requejo<sup>[8]</sup> and Björk and Slinko<sup>[6]</sup> for motivation. Hence, under the no-good-deal pricing (3.6) we price the claim  $\xi$  with a least favorable pricing measure under the Sharpe ratio constraint that excludes too high (and unrealistic) gains that could be earned (but only theoretically) by writing the contract with an arbitrary high price. We remark that by the least favorable pricing measure, we mean a measure that leads to the highest expected payoff from the claim.

We solve the no-good-deal pricing problem (3.6) in Section 5.1, and we observe the equivalence between (3.6) and the instantaneous Sharpe ratio pricing (3.5).

The price and the hedging strategy derived under the instantaneous mean-variance hedging and Sharpe ratio pricing (3.4)–(3.5) also coincide with the price and the hedging strategy derived under robust pricing and hedging under model ambiguity; see Leitner<sup>[22]</sup>, Delong<sup>[11]</sup>, and Pelsser<sup>[25]</sup>. The objective of robust pricing and hedging under model ambiguity gives us another justification for using the objective of instantaneous mean-variance hedging and Sharpe ratio pricing. Let us introduce a set that consists of equivalent measures defined by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{\psi,\phi}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = M^{\psi,\phi}(t), \quad 0 \leq t \leq T, \quad (\psi, \phi) \in \mathcal{P},$$

and

$$\frac{dM^{\psi,\phi}(t)}{M^{\psi,\phi}(t-)} = -\psi(t)dW(t) - \sum_{i=1}^I \phi_i(t)d\tilde{N}_i(t), \quad M(0) = 1, \quad (\psi, \phi) \in \mathcal{P},$$

$$\mathcal{P} = \left\{ \mathcal{F}\text{-predictable processes } (\psi, \phi) = (\psi, \phi_1, \dots, \phi_I) \text{ such that} \right.$$

$$|\psi(t)|^2 + \sum_{i=1}^I |\phi_i(t)|^2 \lambda_i(t) \leq |L(t)|^2,$$

$$\phi_i(t) < 1, \quad 0 \leq t \leq T, \quad i = 1, \dots, I \left. \right\}.$$

Clearly,  $\mathcal{Q} \subset \mathcal{P}$ . We now define the price and the hedging strategy as a solution to the following robust optimization problem

$$Y(t) = \inf_{\pi \in \mathcal{A}} \sup_{(\psi, \phi) \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}^{\psi, \phi}} \left[ - \left( e^{-\int_t^T r(s) ds} X^\pi(T) - X(t) - e^{-\int_t^T r(s) ds} F \right) | \mathcal{F}_t \right], \quad 0 \leq t \leq T. \quad (3.8)$$

The set  $\mathcal{P}$  represents different beliefs (different assumptions) about the parameters or the evolution of the risk factors in our model. One way of determining the set  $\mathcal{P}$  for ambiguity modeling is to specify confidence sets around the estimates of the parameters and to take for  $\mathcal{P}$  the class of all measures that are consistent with these confidence sets. Then the process  $L$  can be interpreted as an estimation error. Alternatively, the elements of  $\mathcal{P}$  can be interpreted as prior models that describe probabilities of future scenarios for the risk factors. Then the process  $L$  can define the range of equivalent probabilities for every scenario. Hence, under the objective of pricing and hedging under model ambiguity (3.8), we aim to find a hedging strategy for the claim  $\xi$  that minimizes the expected shortfall in the terminal surplus under a least favorable measure describing future scenarios, and we price the claim  $\xi$  with a value that offsets this worst expected shortfall.

The equivalence between the instantaneous mean-variance hedging and Sharpe ratio pricing (3.4)–(3.5) and the pricing and hedging under model ambiguity (3.8) is not proved in this article. Details can be obtained from the authors upon request. We remark that the proof of the equivalence can be established by modifying the steps of the proofs from Becherer<sup>[5]</sup>, Delong<sup>[11]</sup>, and Chapter 12.1 in Delong<sup>[12]</sup>.

Let us point out that the theories of no-good-deal pricing and pricing and hedging under model ambiguity provide us with additional interpretations of the Sharpe ratio coefficient  $L$  that is used in our mean-variance objective.

#### 4. THE OPTIMAL PRICE AND THE OPTIMAL HEDGING STRATEGY

We characterize the optimal hedging strategy and the optimal price process that solve (3.4)–(3.5) with a solution to a BSDE. In order to use the theory of BSDEs, we assume that the weak property of predictable representation holds; see Proposition 7.5 in Crépey<sup>[9]</sup> and Chapter XIII.2 in He et al.<sup>[20]</sup>, i.e.,

(A4) every  $(\mathbb{P}, \mathcal{F})$  local martingale  $M$  has the representation

$$M(t) = M(0) + \int_0^t \mathcal{Z}(s) dW(s) + \int_0^t \sum_{i=1}^I \mathcal{U}_i(s) d\tilde{N}_i(s) \quad 0 \leq t \leq T,$$

with  $\mathcal{F}$ -predictable processes  $(\mathcal{Z}, \mathcal{U}_1, \dots, \mathcal{U}_I)$  which are integrable in the Itô sense.

This assumption is satisfied if we define the probability space and the driving processes in an appropriate way; see Becherer<sup>[4]</sup> and Crépey<sup>[9]</sup>.

We present the main theorem of this article.

**Theorem 4.1.** *We investigate the instantaneous mean-variance hedging and Sharpe ratio pricing (3.4)–(3.5) of the claim  $\xi$ . Let  $\xi$  be an  $\mathcal{F}$ -measurable claim such that  $\mathbb{E}[|\xi|^2] < \infty$ , and assume that (A1)–(A4) hold. Consider the BSDE*

$$\begin{aligned} Y(t) = \xi &+ \int_t^T \left( -Y(s)r(s) - \frac{Z(s)\sigma(s) + \sum_{i=1}^I U_i(s)\gamma_i(s)\lambda_i(s)}{\delta(s)}\theta(s) \right. \\ &\quad \left. + \sqrt{|L(s)|^2 - |\theta(s)|^2} \right. \\ &\quad \left. \times \sqrt{|Z(s)|^2 + \sum_{i=1}^I |U_i(s)|^2 \lambda_i(s) - \frac{|Z(s)\sigma(s) + \sum_{i=1}^I U_i(s)\gamma_i(s)\lambda_i(s)|^2}{|\delta(s)|^2}} \right) ds \\ &\quad - \int_t^T Z(s) dW(s) - \int_t^T \sum_{i=1}^I U_i(s) d\tilde{N}_i(s), \quad 0 \leq t \leq T, \end{aligned} \quad (4.1)$$

with its unique solution  $(Y, Z, U_1, \dots, U_I)$ . The optimal admissible hedging strategy  $\pi^* \in \mathcal{A}$  for  $\xi$  is of the form

$$\begin{aligned} \pi^*(t) = & \frac{Z(t)\sigma(t) + \sum_{i=1}^I U_i(t)\gamma_i(t)\lambda_i(t)}{|\delta(t)|^2} \\ & + \frac{\theta(t)}{\delta(t)\sqrt{|L(t)|^2 - |\theta(t)|^2}} \end{aligned}$$

$$\times \sqrt{|Z(t)|^2 + \sum_{i=1}^I |U_i(t)|^2 \lambda_i(t) - \frac{|Z(t)\sigma(t) + \sum_{i=1}^I U_i(t)\gamma_i(t)\lambda_i(t)|^2}{|\delta(t)|^2}},$$

$$0 \leq t \leq T, \quad (4.2)$$

and the price process of  $\xi$  is given by  $Y$ .

**Proof.** *Step 1)* First, we find the optimal solution to our optimization problem (3.4). By (3.2) and (3.3), we have to find a minimizer of the function

$$h(\pi) = L \sqrt{|\pi\sigma - z|^2 + \sum_{i=1}^I |\pi\gamma_i - u_i|^2 \lambda_i} - \pi(\mu - r).$$

Since (A3) holds, then  $\lim_{\pi \rightarrow +\infty} h(\pi) = +\infty$  and  $\lim_{\pi \rightarrow -\infty} h(\pi) = +\infty$ . Consequently, there exists an odd number of extreme points of  $h$  and at least one minimizer of  $h$ . We can notice that the function  $h$  is differentiable everywhere, except at  $\pi = \frac{z}{\sigma}$  if  $\frac{z}{\sigma} = \frac{u_i}{\gamma_i}$ . Hence, let us find stationary points of  $h$  by solving the equation

$$0 = h'(\pi) = L \frac{(\pi\sigma - z)\sigma + \sum_{i=1}^I (\pi\gamma_i - u_i)\gamma_i\lambda_i}{\sqrt{|\pi\sigma - z|^2 + \sum_{i=1}^I |\pi\gamma_i - u_i|^2 \lambda_i}} - (\mu - r). \quad (4.3)$$

Given that the stationary point  $\pi$  must satisfy

$$0 \leq (\pi\sigma - z)\sigma + \sum_{i=1}^I (\pi\gamma_i - u_i)\gamma_i\lambda_i = \pi \left( \sigma^2 + \sum_{i=1}^I \gamma_i^2 \lambda_i \right) - z\sigma - \sum_{i=1}^I u_i \gamma_i \lambda_i, \quad (4.4)$$

we end up with the quadratic equation

$$\frac{(\mu - r)^2}{L^2} \left( |\pi\sigma - z|^2 + \sum_{i=1}^I |\pi\gamma_i - u_i|^2 \lambda_i \right) = \left| (\pi\sigma - z)\sigma + \sum_{i=1}^I (\pi\gamma_i - u_i)\gamma_i\lambda_i \right|^2,$$

which after easy, but tedious, calculations reduces to

$$\begin{aligned} & \pi^2 \left( \sigma^2 + \sum_{i=1}^I \gamma_i \lambda_i \right) \left( \left( \frac{\mu - r}{L} \right)^2 - \sigma^2 - \sum_{i=1}^I \gamma_i \lambda_i \right) \\ & + 2\pi \left( z\sigma + \sum_{i=1}^I u_i \gamma_i \lambda_i \right) \left( \sigma^2 + \sum_{i=1}^I \gamma_i \lambda_i - \left( \frac{\mu - r}{L} \right)^2 \right) \end{aligned}$$

$$+\left(\frac{\mu-r}{L}\right)^2\left(z^2+\sum_{i=1}^I u_i^2\lambda_i\right)-\left(z\sigma+\sum_{i=1}^I u_i\gamma_i\lambda_i\right)^2=0. \quad (4.5)$$

We can calculate

$$\begin{aligned} \Delta &= 4\left(\sigma^2+\sum_{i=1}^I \gamma_i^2\lambda_i-\left(\frac{\mu-r}{L}\right)^2\right)\left(\frac{\mu-r}{L}\right)^2\left(\sigma^2+\sum_{i=1}^I \gamma_i^2\lambda_i\right) \\ &\quad \left(z^2+\sum_{i=1}^I u_i^2\lambda_i-\frac{(z\sigma+\sum_{i=1}^I u_i\gamma_i\lambda_i)^2}{\sigma^2+\sum_{i=1}^I \gamma_i^2\lambda_i}\right), \end{aligned}$$

and we obtain that the quadratic equation (4.5) has two roots

$$\begin{aligned} \pi_1^* &= \frac{z\sigma+\sum_{i=1}^I u_i\gamma_i\lambda_i}{\sigma^2+\sum_{i=1}^I \gamma_i^2\lambda_i} \\ &\quad + \frac{\mu-r}{(\sigma^2+\sum_{i=1}^I \gamma_i^2\lambda_i)\sqrt{L^2-\frac{(\mu-r)^2}{\sigma^2+\sum_{i=1}^I \gamma_i^2\lambda_i}}}\sqrt{z^2+\sum_{i=1}^I u_i^2\lambda_i-\frac{(z\sigma+\sum_{i=1}^I u_i\gamma_i\lambda_i)^2}{\sigma^2+\sum_{i=1}^I \gamma_i^2\lambda_i}}, \\ \pi_2^* &= \frac{z\sigma+\sum_{i=1}^I u_i\gamma_i\lambda_i}{\sigma^2+\sum_{i=1}^I \gamma_i^2\lambda_i} \\ &\quad - \frac{\mu-r}{(\sigma^2+\sum_{i=1}^I \gamma_i^2\lambda_i)\sqrt{L^2-\frac{(\mu-r)^2}{\sigma^2+\sum_{i=1}^I \gamma_i^2\lambda_i}}}\sqrt{z^2+\sum_{i=1}^I u_i^2\lambda_i-\frac{(z\sigma+\sum_{i=1}^I u_i\gamma_i\lambda_i)^2}{\sigma^2+\sum_{i=1}^I \gamma_i^2\lambda_i}}. \end{aligned}$$

It is straightforward to check that only  $\pi_1^*$  satisfies (4.4). By the properties of  $h$ , we can now conclude that  $\pi_1^*$  is the unique minimizer of the function  $h$ . From (3.5) we immediately deduce that the optimal generator  $f^*$  of the BSDE is given by the formula

$$f^* = \pi_1^*(\mu-r) - L\sqrt{|\pi_1^*\sigma - z|^2 + \sum_{i=1}^I |\pi_1^*\gamma_i - u_i|^2\lambda_i},$$

and recalling (4.3), we derive

$$f^* = \pi_1^*(\mu-r) - \frac{L^2}{\mu-r}\left((\pi_1^*\sigma - z)\sigma + \sum_{i=1}^I (\pi_1^*\gamma_i - u_i)\gamma_i\lambda_i\right).$$

Substituting  $\pi_1^*$ , we obtain the generator of our BSDE.



*Step 2)* We prove the existence of a unique solution to the BSDE (4.1). We can notice that the strategy

$$\tilde{\pi}^*(t, Z(t), U(t)) = \frac{Z(t)\sigma(t) + \sum_{i=1}^I U_i(t)\gamma_i(t)\lambda_i(t)}{|\delta(t)|^2}, \quad (4.6)$$

is the unique minimizer of the quadratic variation of the surplus (3.3), and we have

$$\begin{aligned} & \sqrt{|\tilde{\pi}^*(t, Z(t), U(t))\sigma(t) - Z(t)|^2 + \sum_{i=1}^I |\tilde{\pi}^*(t, Z(t), U(t))\gamma_i(t) - U_i(t)|^2 \lambda_i(t)} \\ &= \sqrt{|Z(t)|^2 + \sum_{i=1}^I |U_i(t)|^2 \lambda_i(t) - \frac{|Z(t)\sigma(t) + \sum_{i=1}^I U_i(t)\gamma_i(t)\lambda_i(t)|^2}{|\delta(t)|^2}}. \end{aligned} \quad (4.7)$$

We can now show that the generator  $f$  of the BSDE (4.1) is Lipschitz continuous in the sense that

$$\begin{aligned} & |f(t, Y(t), Z(t), U(t)) - f(t, Y'(t), Z'(t), U'(t))|^2 \\ &= |Y(t)r(t) - Y'(t)r(t) \\ &\quad + \tilde{\pi}^*(t, Z(t), U(t))(\mu(t) - r(t)) - \tilde{\pi}^*(t, Z'(t), U'(t))(\mu(t) - r(t)) \\ &\quad - \sqrt{|L(t)|^2 - |\theta(t)|^2} \\ &\quad \times \sqrt{|\tilde{\pi}^*(t, Z(t), U(t))\sigma(t) - Z(t)|^2 + \sum_{i=1}^I |\tilde{\pi}^*(t, Z(t), U(t))\gamma_i(t) - U_i(t)|^2 \lambda_i(t)} \\ &\quad + \sqrt{|L(t)|^2 - |\theta(t)|^2} \\ &\quad \times \sqrt{|\pi^*(t, Z'(t), U'(t))\sigma(t) - Z'(t)|^2 + \sum_{i=1}^I |\pi^*(t, Z'(t), U'(t))\gamma_i(t) - U'_i(t)|^2 \lambda_i(t)}|^2 \\ &\leq K \left( |\tilde{\pi}^*(t, Z(t), U(t)) - \tilde{\pi}^*(t, Z'(t), U'(t))|^2 \right. \\ &\quad \left. + |Y(t) - Y'(t)|^2 + |Z(t) - Z'(t)|^2 + \sum_{i=1}^I |U_i(t) - U'_i(t)|^2 \lambda_i(t) \right) \\ &\leq K \left( |Y(t) - Y'(t)|^2 + |Z(t) - Z'(t)|^2 + \sum_{i=1}^I |U_i(t) - U'_i(t)|^2 \lambda_i(t) \right), \end{aligned}$$

where we use the representation (4.7), the boundedness assumptions (A2), and the inequality

$$|\sqrt{x^2 + a^2} - \sqrt{y^2 + b^2}|^2 \leq |x - y|^2 + |a - b|^2. \quad (4.8)$$

Since the generator  $f$  is Lipschitz continuous and the terminal condition  $\xi$  is square integrable, we can conclude that there exists a unique solution to the BSDE (4.1); see Theorem 3.1.1 in Delong<sup>[12]</sup>.

*Step 3)* We are left with showing the admissability of the optimal strategy. The standard result on the solution to a BSDE—see Theorem 3.1.1 in Delong<sup>[12]</sup>—yields that the process  $Y$  is  $\mathcal{F}$ -adapted,  $(Z, U_1, \dots, U_I)$  are  $\mathcal{F}$ -predictable, and

$$\begin{aligned} \mathbb{E}\left[\sup_{t \in [0, T]} |Y(t)|^2\right] &< \infty, \quad \mathbb{E}\left[\int_0^T |Z(s)|^2 ds\right] < \infty, \\ \mathbb{E}\left[\sum_{i=1}^I \int_0^T |U_i(s)|^2 \lambda_i(s) ds\right] &< \infty. \end{aligned}$$

Hence, it is straightforward to deduce that  $\pi^* \in \mathcal{A}$ .  $\square$

We succeed in characterizing the optimal hedging strategy and the optimal price process with a unique solution to a nonlinear BSDE that has a Lipschitz generator.

## 5. PROPERTIES OF THE PRICE AND THE HEDGING STRATEGY

In this section, we investigate some important properties of the price process and the hedging strategy.

### 5.1. The Arbitrage-Free Representation of the Price and No-Good-Deal Pricing

From the point of view of the arbitrage-free pricing theory, the crucial point is to check whether the price process (4.1) can be represented in the form

$$Y(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s) ds} \xi \mid \mathcal{F}_t\right], \quad 0 \leq t \leq T,$$

where  $\mathbb{Q}$  is an equivalent martingale measure. Looking at the BSDE (4.1), it is rather difficult to guess the form of the equivalent martingale measure and prove the arbitrage-free representation. Since we expect that the no-good-deal price coincides with the price (4.1) derived under the instantaneous mean-variance hedging and Sharpe ratio pricing, we now solve the no-good-deal pricing problem (3.6). As a by-product, we show the equivalence of the two pricing approaches, and we obtain the arbitrage-free representation of the price (4.1).

**Theorem 5.1.1.** *Let  $\xi$  be an  $\mathcal{F}$ -measurable claim such that  $\mathbb{E}[|\xi|^2] < \infty$ , and assume that (A1)–(A4) hold. Consider the BSDE (4.1) with its unique solution  $(Y, Z, U_1, \dots, U_I)$ . Let*

$$\sqrt{|L(t)|^2 - |\theta(t)|^2} + \frac{|\gamma_i(t)|\sqrt{\lambda_i(t)}}{\delta(t)}\theta(t) < \sqrt{\lambda_i(t)}, \quad 0 \leq t \leq T, \quad i = 1, \dots, I, \quad (5.1)$$

*on the set  $\{\lambda_i(t) > 0\}$ . The optimal equivalent martingale measure  $\mathbb{Q}^{\psi^*, \phi^*}$  which solves the optimization problem (3.6) is determined by the processes*

$$\begin{aligned} \psi^*(t) &= \theta(t)\mathbf{1}_{\{\forall_{i=1, \dots, I} \lambda_i(t) = 0\}} \\ &\quad + \frac{Z(t) - \sigma(t)K_1^*(t, Z(t), U(t))}{2K_2^*(t, Z(t), U(t))}\mathbf{1}_{\{\exists_{i=1, \dots, I} \lambda_i(t) > 0\}}, \quad 0 \leq t \leq T, \\ \phi_i^*(t) &= \frac{U_i(t) - \gamma_i(t)K_1^*(t, Z(t), U(t))}{2K_2^*(t, Z(t), U(t))}\mathbf{1}_{\{\lambda_i(t) > 0\}}, \quad 0 \leq t \leq T, \quad i = 1, \dots, I, \end{aligned} \quad (5.2)$$

where

$$\begin{aligned} K_2^*(t, Z(t), U(t)) &= \\ &= -\frac{1}{2} \frac{\sqrt{|Z(t)|^2 + \sum_{i=1}^I |U_i(t)|^2 \lambda_i(t) - \frac{|Z(t)\sigma(t) + \sum_{i=1}^I U_i(t)\gamma_i(t)\lambda_i(t)|^2}{|\delta(t)|^2}}}{\sqrt{|L(t)|^2 - |\theta(t)|^2}}, \\ K_1^*(t, Z(t), U(t)) &= -\frac{\theta(t)}{\delta(t)} 2K_2^*(t, Z(t), U(t)) \\ &\quad + \frac{Z(t)\sigma(t) + \sum_{i=1}^I U_i(t)\gamma_i(t)\lambda_i(t)}{|\delta(t)|^2}. \end{aligned}$$

Moreover, the process  $Y$  coincides with the optimal value function of the optimization problem (3.6), and we have

$$Y(t) = \sup_{(\psi, \phi) \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}^{\psi, \phi}} \left[ e^{-\int_t^T r(s) ds} \xi | \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}^{\psi^*, \phi^*}} \left[ e^{-\int_t^T r(s) ds} \xi | \mathcal{F}_t \right] \quad 0 \leq t \leq T.$$

**Proof.** *Step 1)* First, we solve the optimization problem

$$\begin{aligned} z\psi + \sum_{i=1}^I u_i \phi_i \lambda_i &\rightarrow_{\psi, \phi_1, \dots, \phi_I} \min \\ \psi\sigma + \sum_{i=1}^I \phi_i \gamma_i \lambda_i &= \mu - r, \end{aligned}$$

$$|\psi|^2 + \sum_{i=1}^I |\phi_i|^2 \lambda_i \leq L^2. \quad (5.3)$$

If  $\lambda_i = 0$ ,  $i = 1, \dots, I$ , then we immediately get the optimal solution. Let us now assume w.l.o.g. that  $\lambda_i > 0$ ,  $i = 1, \dots, I$ . We can also assume that  $u_i \lambda_i \neq 0$ ,  $i = 1, \dots, I$ . Indeed, if  $\lambda_i(t) > 0$  and the terminal condition or the generator of the BSDE (4.1) depends on regime  $i$ , which is the case in the regime-switching economy, then  $U_i(t) \neq 0$ . We can notice that  $\psi = \frac{\mu-r}{\sigma^2 + \sum_{i=1}^I \gamma_i^2 \lambda_i} \sigma$ ,  $\phi_i = \frac{\mu-r}{\sigma^2 + \sum_{i=1}^I \gamma_i^2 \lambda_i} \gamma_i$ ,  $i = 1, \dots, I$ , are the non-regular point of the constraints. The value of the objective function at the non-regular point is equal to

$$\frac{z\sigma + \sum_{i=1}^I u_i \gamma_i \lambda_i}{\sigma^2 + \sum_{i=1}^I \gamma_i^2 \lambda_i} (\mu - r). \quad (5.4)$$

Let us now deal with regular points of the constraints. We introduce the Lagrangian

$$\begin{aligned} F(\psi, \phi, K_1, K_2) = & z\psi + \sum_{i=1}^I u_i \phi_i \lambda_i - K_1 \left( \psi\sigma + \sum_{i=1}^I \phi_i \gamma_i \lambda_i - \mu - r \right) \\ & - K_2 \left( |\psi|^2 + \sum_{i=1}^I |\phi_i|^2 \lambda_i - L^2 \right). \end{aligned}$$

The first-order conditions yield the set of equations

$$\begin{aligned} z - K_1 \sigma - 2K_2 \psi &= 0, \\ u_i \lambda_i - K_1 \gamma_i \lambda_i - 2K_2 \phi_i \lambda_i &= 0, \quad i = 1, \dots, I, \\ \psi \sigma + \sum_{i=1}^I \phi_i \gamma_i \lambda_i &= \mu - r, \\ |\psi|^2 + \sum_{i=1}^I |\phi_i|^2 \lambda_i &= L^2, \end{aligned} \quad (5.5)$$

and the second-order condition gives us that the minimum in (5.3) is attained for  $K_2 < 0$ . From (5.5), we easily obtain

$$\psi = \frac{z - K_1 \sigma}{2K_2},$$

$$\begin{aligned}
\phi_i &= \frac{u_i - K_1 \gamma_i}{2K_2}, \quad i = 1, \dots, I, \\
K_1 &= -\frac{\mu - r}{\sigma^2 + \sum_{i=1}^I |\gamma_i|^2 \lambda_i} 2K_2 + \frac{z\sigma + \sum_{i=1}^I u_i \gamma_i \lambda_i}{\sigma^2 + \sum_{i=1}^I |\gamma_i|^2 \lambda_i}, \\
4K_2^2 &= \left| \frac{z - K_1 \sigma}{L} \right|^2 + \sum_{i=1}^I \left| \frac{u_i - K_1 \gamma_i}{L} \right|^2 \lambda_i.
\end{aligned} \tag{5.6}$$

Substituting the formula for  $K_1$  into the last equation in (5.6), we can derive the quadratic equation

$$\begin{aligned}
&4K_2^2 \left( L^2 - \frac{(\mu - r)^2}{\sigma^2 + \sum_{i=1}^I |\gamma_i|^2 \lambda_i} \right) \\
&= \left| z - \sigma \frac{z\sigma + \sum_{i=1}^I u_i \gamma_i \lambda_i}{\sigma^2 + \sum_{i=1}^I |\gamma_i|^2 \lambda_i} \right|^2 + \sum_{i=1}^I \left| u_i - \gamma_i \frac{z\sigma + \sum_{i=1}^I u_i \gamma_i \lambda_i}{\sigma^2 + \sum_{i=1}^I |\gamma_i|^2 \lambda_i} \right|^2 \lambda_i.
\end{aligned}$$

Recalling (4.6)–(4.7), we get the optimal  $K_2^* < 0$ . We calculate the value of the objective function (5.3) at the regular point  $(\psi^*, \phi^*)$ . Using the formulas from (5.6), we derive

$$\begin{aligned}
z\psi^* + \sum_{i=1}^I \phi_i^* u_i \lambda_i &= \frac{z\sigma + \sum_{i=1}^I u_i \gamma_i \lambda_i}{\sigma^2 + \sum_{i=1}^I |\gamma_i|^2 \lambda_i} (\mu - r) \\
&\quad - \sqrt{L^2 - \left( \frac{\mu - r}{\sigma^2 + \sum_{i=1}^I |\gamma_i|^2 \lambda_i} \right)^2} \\
&\quad \times \sqrt{z^2 + \sum_{i=1}^I |u_i|^2 \lambda_i - \frac{\left( z\sigma + \sum_{i=1}^I u_i \gamma_i \lambda_i \right)^2}{\sigma^2 + \sum_{i=1}^I |\gamma_i|^2 \lambda_i}}.
\end{aligned} \tag{5.7}$$

Since (5.7) is less than (5.4), we conclude that  $(\psi^*, \phi^*)$  is the optimal solution to (5.3).

*Step 2)* We now find the optimal solution to our no-good-deal optimization problem (3.6). We choose  $(\psi, \phi) \in \mathcal{Q}$ . Let  $Y^{\psi, \phi}(t) = \mathbb{E}^{\mathbb{Q}^{\psi, \phi}} \left[ e^{-\int_t^T r(s) ds} \xi \mid \mathcal{F}_t \right]$ ,  $0 \leq t \leq T$ . Recalling results on linear BSDEs—see Propositions 3.3.1 and 3.4.1 in Delong<sup>[12]</sup>—we conclude that the process

$Y^{\psi,\phi}$  can be characterized as a unique solution to the BSDE

$$\begin{aligned} dY^{\psi,\phi}(t) &= Y^{\psi,\phi}(t)r(t)dt + Z^{\psi,\phi}(t)\psi(t)dt + \sum_{i=1}^I U_i^{\psi,\phi}(t)\phi_i(t)\lambda_i(t)dt \\ &\quad + Z^{\phi,\psi}(t)dW(t) + \sum_{i=1}^I U_i^{\psi,\phi}(t)d\tilde{N}_i(t), \\ Y^{\psi,\phi}(T) &= \xi. \end{aligned} \quad (5.8)$$

Let us consider the process  $Y$  that solves the BSDE (4.1). The existence of a unique solution  $Y$  to (4.1) is established in Theorem 4.1. By (5.4) we can deal with the dynamics

$$\begin{aligned} dY(t) &= Y(t)r(t)dt + Z(t)\psi^*(t)dt + \sum_{i=1}^I U_i(t)\phi_i^*(t)\lambda_i(t)dt \\ &\quad + Z(t)dW(t) + \sum_{i=1}^I U_i(t)d\tilde{N}_i(t), \\ Y(T) &= \xi, \end{aligned} \quad (5.9)$$

where  $(\psi^*, \phi^*)$  are defined in (5.2). By the Girsanov's theorem—see Theorem 2.5.1 in Delong<sup>[12]</sup>—we derive

$$\begin{aligned} d(Y^{\psi,\phi}(t) - Y(t)) &= (Y^{\psi,\phi}(t) - Y(t))r(t)dt \\ &\quad + (Z(t)\psi(t) + \sum_{i=1}^I U_i(t)\phi_i(t)\lambda_i(t) - Z(t)\psi^*(t) + \sum_{i=1}^I U_i(t)\phi_i^*(t)\lambda_i(t))dt \\ &\quad + (Z^{\psi,\phi}(t) - Z(t))dW^{\mathbb{Q}^{\psi,\phi}} + \sum_{i=1}^I (U_i^{\psi,\phi}(t) - U_i(t))d\tilde{N}_i^{\mathbb{Q}^{\psi,\phi}}(t), \\ Y^{\psi,\phi}(T) - Y(T) &= 0, \end{aligned}$$

where  $W^{\mathbb{Q}^{\psi,\phi}}$  and  $\tilde{N}^{\mathbb{Q}^{\psi,\phi}}$  are the  $\mathbb{Q}^{\psi,\phi}$ -Brownian motion and the  $\mathbb{Q}^{\psi,\phi}$ -compensated counting process. The result established in Step 1 and the comparison principle for BSDEs—see Theorem 3.2.2 in Delong<sup>[12]</sup>—yield that  $Y^{\phi,\psi}(t) \leq Y(t)$ ,  $0 \leq t \leq T$ . By (4.6), (4.7), and (5.6), we get

$$\phi_i^*(t) = \frac{U_i(t) - \gamma_i(t)\left(-\frac{\theta(t)}{\delta(t)}2K_2(t, Z(t), U(t)) + \tilde{\pi}^*(t, Z(t), U(t))\right)}{2K_2(t, Z(t), U(t))}$$

$$= \frac{\gamma_i(t)}{\delta(t)} \theta(t) - \frac{(U_i(t) - \tilde{\pi}^*(t, Z(t), U(t)) \gamma_i(t)) \sqrt{|L(t)|^2 - |\theta(t)|^2}}{\sqrt{|\tilde{\pi}^*(t, Z(t), U(t)) \sigma(t) - Z(t)|^2 + \sum_{i=1}^I |U_i(t) - \tilde{\pi}^*(t, Z(t), U(t)) \gamma_i(t)|^2 \lambda_i}},$$

and our condition (5.1) implies that  $|\phi_i^*(t)| < 1$ ,  $0 \leq t \leq T$ ,  $i = 1, \dots, I$ . Hence,  $(\phi^*, \psi^*) \in \mathcal{Q}$ . Since there exists a unique solution to the BSDE (5.8), we deduce that  $Y^{\phi^*, \psi^*}(t) = Y(t)$ ,  $0 \leq t \leq T$ , and we finally conclude

$$\sup_{(\phi, \psi) \in \mathcal{Q}^{\phi, \psi}} Y^{\phi, \psi}(t) = Y^{\phi^*, \psi^*}(t) = Y(t), \quad 0 \leq t \leq T.$$

□

In Theorem 5.1, we formulate a condition that guarantees that our optimal price process (4.1) is arbitrage-free, and we provide its arbitrage-free representation. We point out that in a general model with jumps, the instantaneous Sharpe ratio pricing can lead to arbitrage prices, and some conditions have to be introduced to exclude arbitrage prices; see Delong<sup>[11]</sup> and Chapter 10.4 in Delong<sup>[12]</sup>. Such a condition is proposed in (5.1). From the mathematical point of view, condition (5.1) guarantees that there exists an equivalent martingale measure that solves the no-good-deal optimization problem or that there exists a measure solution to the BSDE (4.1) that characterizes the price. We are aware that (5.1) is not optimal, yet we believe that it should be sufficient in many financial applications. One can notice that our arbitrage-free pricing condition (5.1) is satisfied if the surplus's Sharpe ratios  $L$  is not too large (compared to  $\theta$ ), the stock's Sharpe ratio  $\theta$  is not too large (only required if  $\gamma \neq 0$ ), and transition intensities  $\lambda$  are not too small. Those assumptions should be fulfilled in many cases. Let us remark that Lo<sup>[23]</sup> estimates monthly Sharpe ratios for different assets in the range of (0.14, 1.26), and Hardy<sup>[19]</sup> estimates the intensity of transition into a “bad” state at 0.5 and the intensity of transition into a “good” state at 6, whereas Hamilton<sup>[18]</sup> estimates those intensities at 0.5 and 1. With those estimates of  $\theta$  and  $\lambda$ , our condition (5.1) is satisfied for many values of  $L$  and  $\gamma$ .

It should not be surprising that our Sharpe ratio pricing objective can lead to arbitrage strategies. One could have expected that high values of the Sharpe ratio  $L$  can lead to arbitrage prices since they intuitively lead to high prices. Consequently, high values of the Sharpe ratio should be excluded from considerations to guarantee arbitrage-free pricing. As already noted, condition (5.1) excludes high values of the Sharpe ratio  $L$ . As an example illustrating this remark, let us consider pricing of an insurance contract that pays 1 if a policyholder survives a given period. If the insurer requires

a Sharpe ratio  $L$ , then this profit expectation has to be reflected in the price—in this case, in the value of the death probability. If the insurer requires a very high profit, then such a high profit can be realized only by using zero death probability in pricing. Yet zero death probability is not an equivalent death probability, and the arbitrage must arise.

## 5.2. Monotonicity of the Price

It is clear that a reasonable pricing operator should be monotone with respect to the terminal claim, in the sense that a more severe claim should be valued at a higher price. Moreover, since the process  $L$  appearing in (3.5) is interpreted as a Sharpe ratio, we should also expect that the higher the Sharpe ratio the hedger requires, the higher the price of the claim should be. Such properties of our optimal price process (4.1) could be established provided that we could apply a comparison principle for BSDEs. However, it is well known that a comparison principle for BSDEs with jumps does not always hold; see Barles et al.<sup>[1]</sup>. Consequently, the price process (4.1) may not satisfy the monotonic properties in all cases. In the next theorem, we prove a comparison principle for the BSDE (4.1) under which our optimal price process fulfills the desirable monotonic properties.

**Theorem 5.2.1.** *Let  $\xi, \xi'$  be  $\mathcal{F}$ -measurable claims such that  $\mathbb{E}[|\xi|^2] < \infty, \mathbb{E}[|\xi'|^2] < \infty$ . Assume that (A1)–(A4) hold and*

$$\sqrt{|L(t)|^2 - |\theta(t)|^2} + \frac{|\gamma_i(t)|\sqrt{\lambda_i(t)}}{\delta(t)}\theta(t) < \sqrt{\lambda_i(t)}, \quad 0 \leq t \leq T, \\ i = 1, \dots, I, \quad (5.10)$$

*on the set  $\{\lambda_i(t) > 0\}$ . Let  $Y$  and  $Y'$  denote the solutions to the BSDEs (4.1) with terminal conditions  $\xi$  and  $\xi'$  and coefficients  $L$  and  $L'$ . If  $\xi \leq \xi'$  and  $L(t) \leq L'(t), 0 \leq t \leq T$ , then  $Y(t) \leq Y'(t), 0 \leq t \leq T$ .*

**Proof.** Let  $f$  denote the generator of the BSDE (4.1). Recalling (4.6)–(4.7), we can notice that

$$f(Y(t), Z(t), U_1(t), \dots, U_I(t)) - f(Y(t), Z(t), U'_1(t), \dots, U'_I(t)) \\ = \sum_{i=1}^I \frac{\Gamma_i(t)}{(U_i(t) - U'_i(t))\lambda_i(t)} \mathbf{1}_{\{(U_i(t) - U'_i(t))\lambda_i(t) \neq 0\}} (U_i(t) - U'_i(t))\lambda_i(t),$$

where

$$\Gamma_i(t) = \frac{\gamma_i(t)\theta(t)}{\delta(t)} (U_i(t) - U'_i(t))\lambda_i(t)$$



$$\begin{aligned}
& -\sqrt{|L(t)|^2 - |\theta(t)|^2} \\
& \times \left( \sqrt{|\tilde{\pi}_i(t)\sigma(t) - Z(t)|^2 + \sum_{j=1}^I |\tilde{\pi}_i(t)\gamma_j(t) - U_{j,i}(t)|^2 \lambda_j(t)} \right. \\
& \left. - \sqrt{|\tilde{\pi}_{i+1}(t)\sigma(t) - Z(t)|^2 + \sum_{j=1}^I |\tilde{\pi}_{i+1}(t)\gamma_j(t) - U_{j,i+1}(t)|^2 \lambda_j(t)} \right), \quad (5.11)
\end{aligned}$$

and we introduce

$$\begin{aligned}
& \tilde{\pi}_i(t) \\
& = \frac{Z(t)\sigma(t) + U'_1(t)\gamma_1(t)\lambda_1(t) + \cdots + U'_{i-1}(t)\gamma_{i-1}(t)\lambda_{i-1}(t) + U_i(t)\gamma_i(t)\lambda_i(t) + \cdots + U_I(t)\gamma_I(t)\lambda_I(t)}{|\delta(t)|^2}, \\
& U_{j,i}(t) = U_i(t)\mathbf{1}\{j \geq i\} + U'_i(t)\mathbf{1}\{j < i\}.
\end{aligned}$$

In order to apply a comparison principle for BSDEs with jumps, we have to control the coefficients  $\frac{\Gamma_i(t)}{(U_i(t) - U'_i(t))\lambda_i(t)}$ . We show that condition (5.10) implies  $|\frac{\Gamma_i(t)}{(U_i(t) - U'_i(t))\lambda_i(t)}| < 1$ ,  $0 \leq t \leq T$ ,  $i = 1, \dots, I$ , which is a sufficient condition for the application of the comparison principle from Theorem 3.2.2 in Delong<sup>[12]</sup>; see also Royer<sup>[27]</sup>. It is straightforward to notice that

$$\begin{aligned}
& |\Gamma_i(t)| \leq \frac{|\gamma_i(t)|}{\delta(t)} \theta(t) |U_i(t) - U'_i(t)| \lambda_i(t) \\
& + \sqrt{|L(t)|^2 - |\theta(t)|^2} \\
& \times \left| \sqrt{|\tilde{\pi}_i(t)\sigma(t) - Z(t)|^2 + \sum_{j=1}^I |\tilde{\pi}_i(t)\gamma_j(t) - U_{j,i}(t)|^2 \lambda_j(t)} \right. \\
& \left. - \sqrt{|\tilde{\pi}_{i+1}(t)\sigma(t) - Z(t)|^2 + \sum_{j=1}^I |\tilde{\pi}_{i+1}(t)\gamma_j(t) - U_{j,i+1}(t)|^2 \lambda_j(t)} \right|. \quad (5.12)
\end{aligned}$$

Using inequality (4.8) and the definitions of  $\tilde{\pi}_i$  and  $U_{i,j}$ , we get

$$\begin{aligned}
& \left| \sqrt{|\tilde{\pi}_i(t)\sigma(t) - Z(t)|^2 + \sum_{j=1}^I |\tilde{\pi}_i(t)\gamma_j(t) - U_{j,i}(t)|^2 \lambda_j(t)} \right. \\
& \left. - \sqrt{|\tilde{\pi}_{i+1}(t)\sigma(t) - Z(t)|^2 + \sum_{j=1}^I |\tilde{\pi}_{i+1}(t)\gamma_j(t) - U_{j,i+1}(t)|^2 \lambda_j(t)} \right|^2
\end{aligned}$$

$$\begin{aligned}
&\leq |\tilde{\pi}_i(t)\sigma(t) - \tilde{\pi}_{i+1}(t)\sigma(t)|^2 \\
&\quad + \sum_{j=1}^I |\tilde{\pi}_i(t)\gamma_j(t) - \tilde{\pi}_{i+1}(t)\gamma_j(t) - U_{j,i}(t) + U_{j,i+1}(t)|^2 \lambda_j(t) \\
&= \left| \frac{(U_i(t) - U'_i(t))\gamma_i(t)\lambda_i(t)}{|\delta(t)|^2} \right|^2 |\sigma(t)|^2 \\
&\quad + \left| \frac{(U_i(t) - U'_i(t))\gamma_i(t)\lambda_i(t)}{|\delta(t)|^2} \gamma_i(t) - U_i(t) + U'_i(t) \right|^2 \lambda_i(t) \\
&\quad + \sum_{j=1, j \neq i}^I \left| \frac{(U_i(t) - U'_i(t))\gamma_i(t)\lambda_i(t)}{|\delta(t)|^2} \gamma_j(t) \right|^2 \lambda_j(t) \\
&= |(U_i(t) - U'_i(t))\lambda_i(t)|^2 \frac{|\sigma(t)|^2 + \sum_{j \neq i} |\gamma_j(t)|^2 \lambda_j(t)}{\delta^2 \lambda_i(t)} \\
&\leq |(U_i(t) - U'_i(t))\lambda_i(t)|^2 \frac{1}{\lambda_i(t)}. \tag{5.13}
\end{aligned}$$

Combining (5.12) with (5.13), we can derive

$$\left| \frac{\Gamma_i(t)}{(U_i(t) - U'_i(t))\lambda_i(t)} \right| \leq \frac{|\gamma_i(t)|}{\delta(t)} \theta(t) + \sqrt{|L(t)|^2 - |\theta(t)|^2} \frac{1}{\sqrt{\lambda_i(t)}}.$$

Hence, (5.10) implies  $|\frac{\Gamma_i(t)}{(U_i(t) - U'_i(t))\lambda_i(t)}| < 1$ ,  $0 \leq t \leq T$ ,  $i = 1, \dots, I$ , and the comparison now follows from Theorem 3.2.2 in Delong<sup>[12]</sup>.  $\square$

Let us remark that the comparison principle is proved under the same assumption (5.10) that guarantees the arbitrage-free representation of the price; see (5.1).

Even though our pricing operator (4.1) may lead to arbitrage opportunities and may fail the monotonicity property in some cases, we still believe that the instantaneous mean-variance hedging and Sharpe ratio pricing is a financially sound pricing and hedging objective in view of the advantages presented in Section 3. It should be pointed out that one of the goals of Theorems 5.1.1 and 5.2.1 is to introduce a condition that guarantees arbitrage-free instantaneous Sharpe ratio pricing, and the inevitability of such a condition in our case agrees with the intuition as discussed at the end of Section 5.1.

### 5.3. The Markov-Regime-Switching Model

In practical applications we deal with Markovian models. In this section, we establish the relation between the solution to the BSDE (4.1) and the

solution to a partial integro-differential equation. Such a relation allows us to interpret the optimal hedging strategy.

**Theorem 5.3.1.** *Consider the Markov-regime-switching financial model (2.3) with the payoff  $\xi = F(S(T), J(T))$  and the Sharpe ratio  $L(t) = L(J(t-))$ . Assume that  $\mathbb{E}[|\xi|^2] < \infty$ , and let (A1)–(A4) hold. If there exists a unique classical solution  $V$  with a uniformly bounded derivative  $V_s(t, s, i)$  to the system of nonlinear PIDEs*

$$\begin{aligned}
 & V_t(t, s, i) + V_s(t, s, i)s\mu(i) + \frac{1}{2}V_{ss}(t, s, i)s^2\sigma^2(i) \\
 & + \sum_{j \neq i} (V(t, s + s\gamma_j(i), j) - V(t, s, i) - V_s(t, s, i)s\gamma_j(i))\lambda_j(i, s) \\
 & = V(t, s, i)r(i) \\
 & + \frac{V_s(t, s, i)s\sigma^2(i) + \sum_{j \neq i} (V(t, s + s\gamma_j(i), j) - V(t, s, i))\gamma_j(i)\lambda_j(i, s)}{\delta(i)}\theta(i) \\
 & - \sqrt{L^2(i) - \theta^2(i)}\sqrt{g(V(t, s, i))}, \quad (t, s) \in [0, T) \times (0, \infty), i = 1, \dots, I, \\
 & V(T, s, i) = F(s, i), \quad s \in (0, \infty), i = 1, \dots, I,
 \end{aligned} \tag{5.14}$$

where

$$\begin{aligned}
 g(V(t, s, i)) = & |V_s(t, s, i)|^2 s^2 \sigma^2(i) \\
 & + \sum_{j \neq i} |V(t, s + s\gamma_j(i), j) - V(t, s, i)|^2 \lambda_j(i, s) \\
 & - \frac{|V_s(t, s, i)s\sigma^2(i) + \sum_{j \neq i} (V(t, s + s\gamma_j(i), j) - V(t, s, i))\gamma_j(i)\lambda_j(i, s)|^2}{\delta^2(i)},
 \end{aligned}$$

then the solution to the BSDE (4.1) can be characterized as

$$\begin{aligned}
 Y(t) &= V(t, S(t), J(t)), \quad 0 \leq t \leq T, \\
 Z(t) &= V_s(t, S(t-), J(t-))S(t-)\sigma(J(t-)), \quad 0 \leq t \leq T, \\
 U_i(t) &= \left( V(t, S(t-) + S(t-)\gamma_i(J(t-)), i) \right. \\
 & \quad \left. - V(t, S(t-), J(t-)) \right) \mathbf{1}\{i \neq J(t-)\} \quad 0 \leq t \leq T, \quad i = 1, \dots, I.
 \end{aligned}$$

**Proof.** From the Markov property of the system we can deduce that  $Y(t) = V(t, S(t), J(t))$ ,  $0 \leq t \leq T$ , for some measurable function  $V$ ; see Corollary 2.3 and Remark 2.4 in Barles et al.<sup>[1]</sup>. Assuming that  $V$  is sufficiently smooth,

we can apply the Itô's formula and we get the dynamics

$$\begin{aligned}
dV(t, S(t), J(t)) = & V_t(t, S(t-), J(t-)) dt \\
& + V_s(t, S(t-), J(t-)) S(t-) \mu(J(t-)) dt \\
& + V_s(t, S(t-), J(t-)) S(t-) \sigma(J(t-)) dW(t) \\
& + \frac{1}{2} V_{ss}(t, S(t-), J(t-)) S^2(t-) \sigma^2(J(t-)) \\
& + \sum_{j \neq J(t-)} \left( V(t, S(t-) + S(t-) \gamma_j(J(t-)), j) \right. \\
& \left. - V(t, S(t-), J(t-)) \right) d\tilde{N}_j(t) \\
& + \sum_{j \neq J(t-)} \left( V(t, S(t-) + S(t-) \gamma_j(J(t-)), j) \right. \\
& \left. - V(t, S(t-), J(t-)) \right. \\
& \quad \left. - V_s(t, S(t-), J(t-)) S(t-) \right. \\
& \quad \left. \times \gamma_j(J(t-)) \right) \lambda_j(J(t-), S(t-)) dt. \tag{5.15}
\end{aligned}$$

The result now follows by comparing the terms in (5.15) and (4.1).  $\square$

Since the process  $Y$  models the price of the claim, the function  $V$  determines the value of the claim given the current value of the underlying risk factors. It is straightforward to notice that the optimal hedging strategy (4.2) consists of two terms. In view of Theorem 5.3.1, the first term is based on the change in the price of the claim that results from a continuous change in the stock value (the interpretation of the control process  $Z$ ) and on the change in the price of the claim that results from a discontinuous change in the stock value induced by a transition of the economy into a new state (the interpretation of the control processes  $U$ ). Hence, the first term of the optimal hedging strategy (5.4) is a delta-hedging strategy. The second term of the optimal hedging strategy (4.2) can be seen as a correction factor for the delta-hedging strategy. Recalling the interpretation of the strategy (4.6), we can deduce that the correction term arises since the hedger optimizes the mean-variance risk measure of the surplus instead of minimizing the variance of the surplus. The correction term reflects the use of the expected profit in the hedging objective. It leads to a higher expected profit of the surplus and also to a higher variance of the surplus.

We could have used Hamilton-Jacobi-Bellman equations to solve our optimization problems in a Markovian framework. However, it would be difficult to establish the existence of a unique classical (viscosity) solution  $V$  to

**TABLE 1** The parameters of the Markov-regime-switching model

| State $i$ | $r(i)$ | $\mu(i)$ | $\sigma(i)$ | $\gamma(i)$ | $\lambda(i)$ |
|-----------|--------|----------|-------------|-------------|--------------|
| 1         | 0.03   | 0.07     | 0.10        | -0.10       | 2            |
| 2         | 0.01   | 0.02     | 0.25        | 0.05        | 5            |

the system of PIDEs (5.14), or the existence of a unique classical (viscosity) solution would be established under very strong assumptions on the parameters. Hence, we believe that our approach based on BSDEs is mathematically more tractable. Moreover, solving the PIDE (5.14) numerically by a finite difference method is generally less efficient than solving the BSDE (4.1) by Monte Carlo methods. This remark points out an important advantage in using BSDEs instead of PIDEs in solving our pricing and hedging problem. Finally, in our general non-Markovian model we can only use the approach based on BSDEs.

## 6. NUMERICAL EXAMPLE

In this last section we present some numerical results that show a possible application of our pricing method. We consider the Markov-regime-switching model (2.3) with 2 states of economy and the parameters that are specified in Table 1. In particular, the stock's Sharpe ratios are equal to  $\theta(1) = 0.24$  and  $\theta(2) = 0.04$ .

We are interested in pricing 1-year call options with various strikes  $Q$ . The initial price of the derivative is determined by the solution  $Y(0)$  to the BSDE (4.1). The BSDE has to be solved numerically. We apply discrete-time approximation and least squares Monte Carlo. In our example, the solution to the BSDE (4.1) can be derived by using the backward recursion

$$Y(1) = (S(T) - Q)^+,$$

$$Z_i(t_k) = \frac{1}{h} \mathbb{E}[Y(t_{k+1})(W(t_{k+1}) - W(t_k)) | S(t_k) = s, J(t_k) = i], \quad i = 1, 2,$$

$$U_1(t_k) = \frac{1}{\lambda_1(2)h} \mathbb{E}[Y(t_{k+1})(\tilde{N}_1(t_{k+1}) - \tilde{N}_1(t_k)) | S(t_k) = s, J(t_k) = 2],$$

$$U_2(t_k) = \frac{1}{\lambda_2(1)h} \mathbb{E}[Y(t_{k+1})(\tilde{N}_2(t_{k+1}) - \tilde{N}_2(t_k)) | S(t_k) = s, J(t_k) = 1],$$

$$Y_1(t_k) = \frac{1}{1 + r(1)h} \left\{ \mathbb{E}[Y(t_{k+1}) | S(t_k) = s, J(t_k) = 1] \right. \\ \left. - \left( \frac{Z_1(t_k)\sigma(1) + U_2(t_k)\gamma_2(1)\lambda_2(1)}{\delta(1)} \theta(1) \right) \right\}$$

**TABLE 2** The prices of the call options with strike  $Q$  and the Sharpe ratios  $L(1) = 0.4, L(2) = 0.2$  in the Markov-regime-switching model

| The strike | $Q = 80$ | $Q = 90$ | $Q = 100$ | $Q = 110$ | $Q = 120$ |
|------------|----------|----------|-----------|-----------|-----------|
| The price  | 24.561   | 16.677   | 10.381    | 5.857     | 2.928     |

$$\begin{aligned}
 & -\sqrt{|L(1)|^2 - |\theta(1)|^2} \\
 & \cdot \sqrt{|Z_1(t_k)|^2 + |U_2(t_k)|^2 \lambda_2(1) - \frac{|Z_1(t_k)\sigma(1) + U_2(t_k)\gamma_2(1)\lambda_2(1)|^2}{|\delta(1)|^2}} \bigg) h \bigg\}, \\
 Y_2(t_k) = & \frac{1}{1 + r(2)h} \bigg\{ \mathbb{E}[Y(t_{k+1})|S(t_k) = s, J(t_k) = 2] \\
 & - \bigg( \frac{Z_2(t_k)\sigma(2) + U_1(t_k)\gamma_1(2)\lambda_1(2)}{\delta(2)} \theta(2) \\
 & -\sqrt{|L(2)|^2 - |\theta(2)|^2} \\
 & \cdot \sqrt{|Z_2(t_k)|^2 + |U_1(t_k)|^2 \lambda_1(2) - \frac{|Z_2(t_k)\sigma(2) + U_1(t_k)\gamma_1(2)\lambda_1(2)|^2}{|\delta(i)|^2}} \bigg) h \bigg\},
 \end{aligned}$$

where  $0 = t_0 < t_1 < \dots < t_{k-1} < t_k = 1$  and  $h$  is a time-discretization step. The processes  $Y, Z, U$  are next approximated with regression functions; see Bouchard and Elie<sup>[7]</sup>.

The prices of the call options with various strikes for the Sharpe ratios  $L(1) = 0.4, L(2) = 0.2$  are given in Table 2. Monotonicity of the price with respect to the strike can be observed. In Table 3 we also present the prices of the call options in two classical Black-Scholes model with the parameters  $(r(i), \sigma(i))$  determined by the state 1 and 2. We might have expected that the price in the regime-switching model should be between the prices in the Black-Scholes models. However, under our pricing method the hedger specifies his expected profit reflected by the Sharpe ratio  $L$  which increases

**TABLE 3** The prices of the call options with strike  $Q$  in the complete Black-Scholes model with parameters  $(r, \sigma)$

| The strike                    | $Q = 80$ | $Q = 90$ | $Q = 100$ | $Q = 110$ | $Q = 120$ |
|-------------------------------|----------|----------|-----------|-----------|-----------|
| The BS price for (0.03, 0.10) | 22.381   | 13.038   | 5.581     | 1.596     | 0.299     |
| The BS price for (0.01, 0.25) | 22.891   | 15.830   | 10.405    | 6.532     | 3.948     |

**TABLE 4** The prices of the call options with the strike  $Q = 100$  and Sharpe ratios  $(L(1), L(2))$  in the Markov-regime-switching model

|                   |                                |                              |                              |                              |                              |
|-------------------|--------------------------------|------------------------------|------------------------------|------------------------------|------------------------------|
| The Sharpe ratios | $L(1) = 0.24$<br>$L(2) = 0.04$ | $L(1) = 0.3$<br>$L(2) = 0.1$ | $L(1) = 0.5$<br>$L(2) = 0.3$ | $L(1) = 0.7$<br>$L(2) = 0.5$ | $L(1) = 1.2$<br>$L(2) = 1.2$ |
| The price         | 9.827                          | 10.082                       | 10.660                       | 11.226                       | 13.168                       |

the price. Hence, our price can be between the Black-Scholes prices or above the higher price. The relation between our prices in the regime-switching model and the prices in the Black-Scholes models can be observed by comparing the results in Table 2 and 3. In Table 4 we find the prices of the call options with the strike  $Q = 100$  for various Sharpe ratios. Monotonicity of the price with respect to the hedger's Sharpe ratio can be observed. We remark that the pair  $(0.24, 0.04)$  is the lowest Sharpe ratio and  $(1.2, 1.2)$  is the highest Sharpe ratio (assuming that  $L(1) \geq L(2)$ ), which can be used under the assumptions of Theorems 4.1, 5.1.1, and 5.2.1. Consequently, in our example for all reasonable values of the hedger's Sharpe ratios, the arbitrage-free condition (5.1) and the monotonicity condition (5.10) are fulfilled.

## 7. CONCLUSION

We have studied hedging and pricing of contingent claims in a non-Markovian regime-switching financial model. We have derived the hedging strategy that minimizes the instantaneous mean-variance risk of the hedger's surplus and the price under which the instantaneous Sharpe ratio of the hedger's surplus equals a predefined target. The optimal hedging strategy and the optimal price process have been characterized with a unique solution to a nonlinear, Lipschitz BSDE with jumps. We have discussed key properties of the price and the hedging strategy.

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