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Abstract

In the context of claims problems, we formulate an invariance axiom of a rule with respect to its "partial implementation": having applied the rule to a problem and distributed their awards to some of the claimants, we consider the problem obtained by setting the claims of these claimants equal to zero and decreasing the endowment by the sum of their awards; we require that in this problem the rule assigns to each remaining claimant the same amount as it did initially. We formulate several variants of this requirement of "partial-implementation invariance" and a "converse" of it. We investigate how it relates to known axioms and ask whether it is preserved under certain "operators" that have been defined on the space of rules. Our main result is a fixed-population characterization of a family of rules introduced and characterized by Young (1987) in a variable-population framework, known as the "parametric rules".

Keywords: claims problems; partial-implementation invariance; consistency; Young's rules

JEL classification: D63, D71, D74

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1 Introduction

In the context of the search for rules to adjudicate the conflicting claims that a group of agents have on a resource, we introduce an invariance axiom that expresses the robustness of a rule to its "partial implementation". Specifically, once an awards vector has been calculated for some claims problem by applying a chosen rule, let us imagine that some of the claimants involved pick up their awards, and let us reassess the situation at this point. These claimants having been taken care of, we set their claims equal to 0. Their accounts are closed, so to speak.[1] We now face a problem in which the claims of the other claimants have their original values – there is no reason to revise them – but the endowment has been decreased by the sum of the awards to the claimants who have received their awards. We require that in this problem, the rule assign to each remaining claimant what it had initially assigned to them. Thus, the partial implementation of the rule does not create the need for adjustments in their awards. We refer to the requirement as "partial-implementation invariance".

Our goal is to understand its implications. We formulate several variants of it, and define a converse of it. We investigate how these axioms relate to known axioms. We investigate whether the axiom is preserved by certain "operators" that have been defined on the space of rules.

Partial-implementation invariance is conceptually and mathematically related to the well-known axiom of "consistency". Consistency can be seen as an expression of the same robustness objective, but in the scenario underlying it, after some claimants have received their awards, they leave, this resulting in a problem with fewer claimants. The axiom we are considering here is a fixed-population axiom. Consistency has played an important role in the theory concerning the adjudication of conflicting claims. A central result in a paper that has played a key role in the development of this theory is the first characterization of a rule that rationalizes the recommendation made in the Talmud for some numerical examples presented there (Aumann and Maschler, 1985) – this rule is now known as the Talmud rule. It is also the main requirement in a characterization due to Young (1987) of a family of rules that he defined. We derive parallel characterizations of the Talmud rule and of Young's family in which our invariance axiom also plays a key role. However, there is an important difference between the models in Aumann and Maschler (1985) and Young (1987) and our setting. As we noted, consistency is a variable-population axiom. The option of varying

[1]We could also say that claimant i's claim is "neutralized" by the operation underlying the definition.

population in proofs is a powerful tool that is not available to us.

This paper is organized as follows. Section 2 introduces the model and the axioms that we will invoke. Section 3 introduces partial-implementation invariance and its converse. Section 4 establishes characterizations of the Talmud rule and of Young's rules by invoking partial-implementation invariance. Section 5 concludes. Appendix A provides the complete proof of our characterization of Young's rules.

2 The model

A group of agents have claims on a resource that is insufficient to honor all of these claims. How much should each agent receive? We search for systematic ways of coming up with a division. The formal model is as follows. There is a finite population N of agents ($|N| \geq 2$), called **claimants**, having claims on a resource, called the **endowment**. The endowment is not sufficient to fully honor all claims. A **claims problem** with claimant set N (O'Neill, 1982) is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. Let $\mathcal{C}^N$ be the class of all problems with claimant set N and $\mathcal{C}^N_2$ the subclass of problems in which at most two claimants have a non-zero claim. A **rule** associates with each problem a vector $x \in \mathbb{R}^N$ satisfying the very natural and self-explanatory **non-negativity** and **claims boundedness** requirements $0 \leqq x \leqq c$, as well as the **balance** requirement $\sum_{i \in N} x_i = E$. Let $\boldsymbol{X(c, E)}$ denote the set of vectors satisfying these requirements. Our generic notation for a rule is S. We may also denote by S a correspondence. In such a case, we explicitly state that we are dealing with a correspondence.

We also work with a generalization of the model in which the population of claimants may vary. Denoting by $\mathbb{N}$ the set of natural numbers, let $\mathbb{N}^* \subseteq \mathbb{N}$ be a non-empty set of "potential" claimants. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}^*$. Here, a rule is a mapping defined on $\mathcal{C} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ and taking its values in $\bigcup_{N \in \mathcal{N}} \mathbb{R}^N$. The same requirements of *non-negativity*, *claims boundedness*, and *balance* are imposed on rules. Given $N' \subseteq N$, we denote by $0_{N'}$ the vector in $\mathbb{R}^{N'}$ whose coordinates are all equal to 0. Throughout the paper, unless explicitly stated, rules and correspondences are defined on $\mathcal{C}^N$.

Here are a number of axioms of rules. We start with the fairness requirement that two claimants with equal claims be awarded equal amounts:

Equal treatment of equals: For each $(c, E) \in \mathcal{C}^N$ and each pair $i, j \in N$ such that $c_i = c_j$,

$$S_i(c, E) = S_j(c, E).$$

A strengthening of this axiom is invariance with respect to the renaming of claimants:

Anonymity: For each $(c, E) \in \mathcal{C}^N$, each bijection $\pi : N \to N$, and each $i \in N$,

$$S_i(c, E) = S_{\pi(i)} \left((c_{\pi^{-1}(j)})_{j \in N}, E \right).$$

Another strengthening of *equal treatment of equals* is that of two claimants, (i) the larger one should be assigned at least as much as the smaller one and (ii) the loss incurred by the larger one should be at least as large as the loss incurred by the smaller one (Aumann and Maschler, 1985):

Order preservation: For each $(c, E) \in \mathcal{C}^N$ and each pair $i, j \in N$ such that $c_i \leq c_j$,

$$S_i(c, E) \leq S_j(c, E) \text{ and } c_i - S_i(c, E) \leq c_j - S_j(c, E).$$

A group version of this axiom can also be defined: of two groups of claimants, (i) the one with the larger aggregate claim should be assigned in total at least as much as the one with the smaller aggregate claim, and (ii) the loss incurred in total by the first group should be at least as large as the loss incurred in total by the second group (Thomson, 1988):

Group order preservation: For each $(c, E) \in \mathcal{C}^N$ and each pair $N', N'' \subseteq N$ such that $\sum_{i \in N'} c_i \leq \sum_{i \in N''} c_i$,

$$\sum_{i \in N'} S_i(c, E) \leq \sum_{i \in N''} S_i(c, E) \text{ and } \sum_{i \in N'} (c_i - S_i(c, E)) \leq \sum_{i \in N''} (c_i - S_i(c, E)).$$

Next, no claimant should ever be awarded less when the endowment increases.

Endowment monotonicity: For each pair $(c, E), (c, E') \in \mathcal{C}^N$ such that $E < E'$,

$$S(c, E) \leq S(c, E').$$

We obtain a strict version of the axiom, **strict endowment monotonicity**, by requiring that under the same hypotheses, the inequality appearing in the conclusion be strict for each claimant whose claim is positive.

The following requirement is that small changes in problems not lead to large changes in the chosen awards vector:

Continuity: For each $(c, E) \in \mathcal{C}^N$, $S(c, E)$ is jointly continuous in c and E.

We now define various invariance axioms. The first requirement is that truncating claims at the endowment not affect the chosen awards vector (Dagan and Volij, 1993):[2]

Claims truncation invariance: For each $(c, E) \in \mathcal{C}^N$,

$$S(c, E) = S(t(c, E), E),$$

where $t(c, E) \equiv (\min\{c_i, E\})_{i \in N}$.

The second requirement is that any problem be equivalently solved in either one of the following two ways, (i) directly, or (ii) in two steps, as follows: first, each claimant receives their "minimal right", namely whatever is left over after all other claimants have been fully compensated if possible and nothing otherwise; second, the amount awarded to them in the problem in which each claimant's claim has been revised down by the claimant's minimal right and the endowment has been revised down by the sum of everyone's minimal rights (Curiel et al., 1987):[3]

Minimal rights first: For each $(c, E) \in \mathcal{C}^N$,

$$S(c, E) = m(c, E) + S(c - m(c, E), E - \sum_{i \in N} m_i(c, E)),$$

where $m(c, E) \equiv (\max\{E - \sum_{j \in N \setminus \{i\}} c_j, 0\})_{i \in N}$.

[2]The idea of truncation is mentioned by Aumann and Maschler (1985) and developed by Curiel et al. (1987). Dagan and Volij (1993) refer to this axiom as "independence of irrelevant claims". The expression *claims truncation invariance* appears in Thomson (2003).

[3]Curiel et al. (1987) refer to this axiom as the "minimal rights property". This axiom is also known as "v-separability" (Dagan, 1996) and "composition from minimal rights" (Herrero and Villar, 2001). Thomson (2003) uses the expression *minimal rights first*.

The next two invariance axioms are conceptually the closest to the new axioms introduced in the next section. One is the requirement that the choice made for a problem by a rule should be "confirmed" by the rule in each of the "reduced" problems that result when some claimants have received their awards and left (Aumann and Maschler, 1985; Young, 1987). Because this is a variable-population axiom, a formal statement requires that the model be generalized. Accordingly, rules are now defined on $\mathcal{C}$:

Consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subseteq N$,

$$S_{N'}(c, E) = S\left(c_{N'}, E - \sum_{i \in N \setminus N'} S_i(c, E)\right).$$

We also define weaker variants of *consistency*. First is the version of *consistency* for which all but two claimants leave the scene, **bilateral consistency**. Also, a very weak form of *consistency* is that if some claimants' claims are 0 and these claimants leave, in the resulting reduced problem, each of the remaining claimants should be assigned what they were assigned initially, **null claims consistency** (Axiom A4 in Appendix C of O'Neill, 1982).[4] The following two requirements are of intermediate strength between *consistency* and *null claims consistency*. First, if some claimants are assigned 0, and whether or not their claims are 0, and they leave, in the resulting reduced problem each remaining claimant should be assigned the same amount as initially, **null-compensation consistency** (Thomson, 2019). Second, if some claimants are fully compensated, and they leave with their awards, in the resulting reduced problem, each remaining claimant should be assigned the same amount as initially, **full-compensation consistency** (Thomson, 2019).[5]

We also have a "converse" of *consistency*, which says the following: suppose that an awards vector for a problem is such that, for each two-claimant subset of the claimants it involves, it chooses the restriction of the vector to that population for the associated reduced problem this population faces; then the requirement is that it be chosen for the initial problem. Again, rules are defined on $\mathcal{C}$:

[4] This axiom is also known as "dummy" (Chun, 1988), "zero out" (Dagan and Volij, 1993), "independence of null demands" (Moulin, 2000), "null consistency" (Ju, 2003), and "limited consistency" (Thomson, 2003). Thomson (2019) uses the expression *null claims consistency*.

[5] *Null-compensation consistency* and *full-compensation consistency* are also known as "zero-award-out-consistency" and "full-award-out-consistency", respectively (Ju and Moreno-Ternero, 2017).

Converse consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $x \in X(c, E)$, if for each $N' \subseteq N$ with $|N'| = 2$, $x_{N'} = S(c_{N'}, E - \sum_{i \in N \setminus N'} x_i)$, then $x = S(c, E)$.

Weak converse consistency differs from *converse consistency* only in that the hypothesis is written for each proper subset of N, as opposed to each subset of size two. An easy argument by induction on population size shows that *converse consistency* and *weak converse consistency* are in fact equivalent.

3 Partial-implementation invariance

A decision having been made on how to solve a claims problem, imagine that it is implemented in two steps, some claimants collecting their awards first. These claimants having been accommodated, we set their claims equal to 0. Accordingly, we adjust the endowment down by the sum of their awards. This yields a well-defined revised problem. We apply the rule to it. By definition of a rule, any claimant whose claim has been set equal to 0 is assigned 0. Our robustness requirement on the rule is that it assign to each of the other claimants the same amount as it did initially.

Partial-implementation invariance: For each $(c, E) \in \mathcal{C}^N$ and each $N' \subseteq N$,

$$S_{N'}(c, E) = S_{N'}\left((c_{N'}, 0_{N \setminus N'}), E - \sum_{i \in N \setminus N'} S_i(c, E)\right).$$

Partial-implementation invariance is vacuously satisfied if $|N| \leq 2$.

The term "implementation" in the expression we chose to designate the axiom should be understood as in common language. It should not be given the technical meaning it has in the theory of mechanism design à la Hurwicz-Maskin.

A weaker version of the axiom can be defined for situations in which all but two claimants have been assigned their awards. Let us refer to it as **bilateral partial-implementation invariance**.

We could also meaningfully limit its applications to situations in which the claimants whose accounts are settled are assigned nothing. Again, we would replace the claim of such a claimant by 0, but this time we would keep the endowment constant. We would then reapply the rule, and require that each remaining claimant be assigned what they were assigned initially. Let us use the expression **null-compensation partial-implementation invariance** for the requirement.

Alternatively, we could only consider the departure of claimants who have been fully compensated. The expression **full-compensation partial-implementation invariance** seems appropriate here.[6]

It is easy to see that *partial-implementation invariance* is satisfied by the following rules, in the definitions of which $(c, E) \in \mathcal{C}^N$ is an arbitrary problem and $\lambda \in \mathbb{R}_+$ is chosen so as to achieve balance: the **proportional rule**, $\boldsymbol{P}$, makes awards proportional to claims: $P(c, E) \equiv \lambda c$; the **constrained equal awards rule**, $\boldsymbol{CEA}$, makes awards as equal as possible subject to no one receiving more than their claim: $CEA(c, E) \equiv (\min\{c_i, \lambda\})_{i \in N}$; the **constrained equal losses rule**, $\boldsymbol{CEL}$, makes the losses claimants incur as equal as possible subject to no one receiving a negative amount: $CEL(c, E) \equiv (\max\{c_i - \lambda, 0\})_{i \in N}$. A hybrid of the constrained equal awards and the constrained equal losses rules is the **Talmud rule**, $\boldsymbol{T}$: if the endowment is no larger than the half-sum of the claims, this rule selects the awards vector that the constrained equal awards rule would select for the problem in which all claims are divided by two; otherwise, it selects the sum of the vector of half-claims and the awards vector that the constrained equal losses rule would select for the problem in which all claims are divided by two and the endowment is adjusted down by the half-sum of the claims: for each $(c, E) \in \mathcal{C}^N$, if $E \leq \sum_{i \in N} \frac{c_i}{2}$, then $T(c, E) \equiv CEA(\frac{c}{2}, E)$, and if $E > \sum_{i \in N} \frac{c_i}{2}$, then $T(c, E) \equiv \frac{c}{2} + CEL(\frac{c}{2}, E - \sum_{i \in N} \frac{c_i}{2})$. Our final example consists of a family, the **sequential priority rules**: to each order on the claimant set is associated such a rule; the claimant who is first is assigned their claim if possible and the endowment otherwise; the claimant who is second is assigned their claim if possible and the remainder otherwise, and so on: let $\mathcal{O}^N$ be the set of priority orders over N. Given $\prec \in \mathcal{O}^N$, the **sequential priority rule relative to** $\boldsymbol{\prec}$, $\boldsymbol{SP^{\prec}}$, is defined by setting for each $(c, E) \in \mathcal{C}^N$, $SP^{\prec}(c, E) \equiv (\min\{c_i, \max\{E - \sum_{j \in N: j \prec i} c_j, 0\}\})_{i \in N}$.

However, some rules violate *partial-implementation invariance*. An example is the **random arrival rule**, $\boldsymbol{RA}$, which selects the average of the awards vectors selected by the sequential priority rules: for each $(c, E) \in \mathcal{C}^N$, $RA(c, E) \equiv \frac{1}{|N|!}(\sum_{\prec \in \mathcal{O}^N} \min\{c_i, \max\{E - \sum_{j \in N: j \prec i} c_j, 0\}\})_{i \in N}$.

[6] These variants should be seen as counterparts of variants of the axiom of consistency.

Example 1. *The random arrival rule violates partial-implementation invariance.* We consider the so-called Three-Wives problem of the Talmud (O'Neill, 1982) – their claims are $c \equiv (100, 200, 300)$ – when the endowment is $E \equiv 200$. We calculate that $RA(c, E) = (33\frac{1}{3}, 83\frac{1}{3}, 83\frac{1}{3})$. Let us assign her award to wife 2, reduce her claim to 0 and adjust the endowment down by her award. In the revised problem, the claims vector is $(100, 0, 300)$ and the endowment $116\frac{2}{3}$. There are three orders of arrival in which wife 1 is assigned a positive amount, namely 1-2-3, 1-3-2, and 2-1-3. In each case, she is in fact fully compensated. That is obvious for the first two orders because the endowment exceeds her claim; it is also true for the third one because wife 2's claim is now 0 and she has been assigned 0; thus, when wife 1 comes in, the endowment has its initial value. In each of the other orderings, wife 1 comes after wife 3 and because wife 3's claim exceeds the endowment, there is nothing left for wife 1. Altogether, wife 1 receives $\frac{300}{6} \neq 33\frac{1}{3}$.

The **minimal overlap rule**, $\boldsymbol{MO}$, is another rule that violates *partial-implementation invariance.* For each problem in which the endowment is no larger than the largest claim – focusing on that case will suffice for our purposes – the rule selects the awards vector that is defined as follows: the smallest claimant receives $\frac{1}{|N|}$ of their claim; the second smallest claimant receives that amount plus $\frac{1}{|N|-1}$ of the difference between their claim and the smallest claim; the third smallest claimant receives that amount plus $\frac{1}{|N|-2}$ of the difference between their claim and the second smallest claim, and so on. Thus, the awards vector is obtained by nesting claims and dividing equally among all claimants claiming a particular part of the endowment that part of it, each claimant's award being calculated as a sum of the partial compensations the claimant receives for the various sections into which the endowment is partitioned.

Example 2. *The minimal overlap rule violates partial-implementation invariance.* We consider the same claims vector as in Example 1 but set $E \equiv 300$. Because the largest claim is equal to the endowment, the awards vector is obtained by nesting claims: $MO(c, E) = (33\frac{1}{3}, 33\frac{1}{3} + 50, 33\frac{1}{3} + 50 + 100)$. Let us assign her award to wife 2, reduce her claim to 0 and adjust the endowment down by her award. Of course, in the revised problem, the largest claim still exceeds the endowment, and nesting claims is what is needed to obtain the awards vector. The revised endowment being larger than wife 1's claim, she gets $\frac{100}{2} \neq 33\frac{1}{3}$.

A strengthening of *partial-implementation invariance* is **claims-and-endowment separability** (Chun, 1999), which says that if the data of a problem change but the claims of a subgroup N' of claimants remain the same (the claims of the members of $N \setminus N'$ and the endowment may change) and in total, the members of N' still receive the same amount as initially, then each of them should receive the same amount as initially.[7] We obtain *partial-implementation invariance* by imposing two restrictions on the changes in the parameters of a problem. First, the claims of the members of $N \setminus N'$ are reduced to 0. Second, the endowment changes by the sum of their initial awards. It follows from these restrictions that the sum of the awards to the members of N' remains the same, an equality that does not have to be written out as a separate hypothesis then.

Although *partial-implementation invariance* can also be described as a "consistency" requirement, it is obviously not the same as what we called *consistency*. First, as already noted, it is a fixed-population axiom whereas *consistency* is a variable-population axiom. A rule defined on $\mathcal{C}$ may be *partial-implementation invariant* without being *consistent*. Indeed, *partial-implementation invariance* imposes no cross-population restrictions whereas *consistency* is meant to discipline rules as population varies. Consider for example the rule that coincides with the proportional rule for all problems with at most three claimants and with the constrained equal awards rule for problems involving at least four claimants. The rule satisfies the former but not the latter.

Because *consistency* is meant to relate the recommendations a rule makes across populations, any possible logical relation between the two axioms has to be mediated by at least one variable-population axiom. The following lemma confirms this observation. It involves *null claims consistency*.

Lemma 1. Consider a rule defined on $\mathcal{C}$. For such a rule, *partial-implementation invariance* and *null claims consistency* together imply *consistency*. Also, *consistency* implies *partial-implementation invariance*.

[7] The expression *claims-and-endowment separability* is due to Thomson (2019). Chun (1999) refers to this axiom as "separability". In the terminology of Thomson (2023), *partial-implementation invariance*, *claims-and-endowment separability*, and *consistency* are "post-application" axioms. By contrast, the order-preservation axioms, *endowment monotonicity*, and *null claims consistency* are "pre-application" axioms. The rule appears in the hypotheses of an axiom of the first type, whereas the hypotheses of an axiom of the second type make no reference to the rule.

Proof. For each assertion, let S be a rule satisfying the hypotheses. For the first assertion, let $N \in \mathcal{N}$, $(c, E) \in \mathcal{C}^N$, and $N' \subseteq N$. By *partial-implementation invariance*,

$$S_{N'}(c, E) = S_{N'}\left((c_{N'}, 0_{N\setminus N'}), E - \sum_{i \in N\setminus N'} S_i(c, E)\right),$$

and by *null claims consistency*,

$$S_{N'}\left((c_{N'}, 0_{N\setminus N'}), E - \sum_{i \in N\setminus N'} S_i(c, E)\right) = S\left(c_{N'}, E - \sum_{i \in N\setminus N'} S_i(c, E)\right).$$

For the second assertion, let $N \in \mathcal{N}$, $(c, E) \in \mathcal{C}^N$, and $N' \subseteq N$. By *consistency*,

$$S_{N'}(c, E) = S\left(c_{N'}, E - \sum_{j \in N\setminus N'} S_j(c, E)\right).$$

Also, by *consistency*, for each $i \in N'$,

$$S_i\left(c_{N'}, E - \sum_{j \in N\setminus N'} S_j(c, E) - \sum_{j \in N\setminus N'} 0\right) = S_i\left((c_{N'}, 0_{N\setminus N'}), E - \sum_{j \in N\setminus N'} S_j(c, E)\right).$$

□

Because on $\mathcal{C}$, the proportional, constrained equal awards, and constrained equal losses rules are *consistent*, we deduce from this lemma that for each $N \in \mathcal{N}$, they are *partial-implementation invariant* on $\mathcal{C}^N$, as we noted earlier.

One can define a converse of *partial-implementation invariance* in the obvious way: consider a rule; given a problem $(c, E) \in \mathcal{C}^N$, let x be an awards vector for it; suppose that the restriction of x to each two-claimant subgroup of N is chosen by the rule for the problem in which the claims of the members of the complementary subgroup are set equal to 0 and the endowment is revised down by the sum of their coordinates of x; then x should be chosen for (c, E).

Converse partial-implementation invariance: For each $(c, E) \in \mathcal{C}^N$ and each $x \in X(c, E)$, if for each $N' \subseteq N$ with $|N'| = 2$, $x_{N'} = S_{N'}((c_{N'}, 0_{N \setminus N'}), E - \sum_{i \in N \setminus N'} x_i)$, then $x = S(c, E)$.

A weak form of *converse partial-implementation invariance* can also be stated – let us call it **weak converse partial-implementation invariance** – that differs only in that the hypotheses are written for all proper subgroups of N as opposed to all subgroups of two claimants. These two versions are parallel to *converse consistency* and *weak converse consistency*. Just like these two axioms, *weak converse partial-implementation invariance* and *converse partial-implementation invariance* are in fact equivalent.

The proportional, constrained equal awards, constrained equal losses, and Talmud rules, as well as the sequential priority rules all satisfy *converse partial-implementation invariance*. On the other hand, the random arrival and minimal overlap rules again violate the axiom:

Example 3. *The random arrival rule violates converse partial-implementation invariance.* Let $N \equiv \{1, 2, 3\}$ and $(c, E) \in \mathcal{C}^N$ be equal to $((2, 4, 5), 5)$. Let $x \equiv (1, 2, 2)$. Then for each $N' \subset N$ with $|N'| = 2$, $x_{N'} = RA_{N'}((c_{N'}, 0_{N \setminus N'}), E - \sum_{i \in N \setminus N'} x_i)$. However, $RA(c, E) = (\frac{5}{6}, \frac{11}{6}, \frac{14}{6}) \neq x$.

Example 4. *The minimal overlap rule violates converse partial-implementation invariance.* We consider the same problem as in Example 3. Since on $\mathcal{C}_2^N$, the minimal overlap rule coincides with the random arrival rule, the awards vector x also satisfies the hypotheses of *converse partial-implementation invariance* for the minimal overlap rule. However, $MO(c, E) = (\frac{2}{3}, \frac{2}{3} + \frac{2}{2}, \frac{2}{3} + \frac{2}{2} + 1) = (\frac{2}{3}, \frac{5}{3}, \frac{8}{3}) \neq x$.

Here is a logical relation between *converse partial-implementation invariance* and *converse consistency*:

Lemma 2. Consider a rule defined on $\mathcal{C}$. For such a rule, *converse consistency* and *null claims consistency* together imply *converse partial-implementation invariance.*

Proof. Let S be a rule satisfying the hypothesis. Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{C}^N$, and $x \in X(c, E)$ be such that for each $N' \subseteq N$ with $|N'| = 2$, $x_{N'} = S_{N'}((c_{N'}, 0_{N \setminus N'}), E - \sum_{i \in N \setminus N'} x_i)$. By *null claims consistency*, for each $N' \subseteq N$ with $|N'| = 2$, $x_{N'} = S(c_{N'}, E - \sum_{i \in N \setminus N'} x_i)$. By *converse consistency*, $x = S(c, E)$. □

An **operator** on the space of rules is a mapping that associates with each rule another rule (Thomson and Yeh, 2008). Central are the following: the **duality operator** associates with each rule S the rule S^d defined by setting for each $(c, E) \in \mathcal{C}^N$, $S^d(c, E) \equiv c - S(c, \sum_{i \in N} c_i - E)$.

The **claims truncation operator** associates with each rule S the rule S^t defined by setting for each $(c, E) \in \mathcal{C}^N$, $S^t(c, E) \equiv S(t(c, E), E)$, where $t(c, E) \equiv (\min\{c_i, E\})_{i \in N}$.

The **attribution of minimal rights operator** associates with each rule S the rule S^m defined by setting for each $(c, E) \in \mathcal{C}^N$, $S^m(c, E) \equiv m(c, E) + S(c - m(c, E), E - \sum_{i \in N} m_i(c, E))$, where $m(c, E) \equiv (\max\{E - \sum_{j \in N \setminus \{i\}} c_j, 0\})_{i \in N}$.

The **convexification operator** associates with each list of rules $(S^k)_{k \in K}$, where K is a non-empty and finite set, and each list $(\lambda^k)_{k \in K}$ with $\lambda^k \in \mathbb{R}_+$ for each $k \in K$ and $\sum_{k \in K} \lambda^k = 1$ the rule $\omega((S^k)_{k \in K}, (\lambda^k)_{k \in K})$ defined by setting for each $(c, E) \in \mathcal{C}^N$, $\omega((S^k)_{k \in K}, (\lambda^k)_{k \in K})(c, E) \equiv \sum_{k \in K} \lambda^k S^k(c, E)$.

The **composition of the claims truncation and attribution of minimal rights operators** associates with each rule S the rule S^{mot} defined by setting for each $(c, E) \in \mathcal{C}^N$, $S^{mot}(c, E) \equiv m(c, E) + S(t(c - m(c, E), E - \sum_{i \in N} m_i(c, E)), E - \sum_{i \in N} m_i(c, E))$.

The following lemma answers the question of whether *partial-implementation invariance* is preserved by these operators.

Lemma 3. *Partial-implementation invariance* is preserved by the duality operator. It is preserved by none of the claims truncation, attribution of minimal rights, and convexification operators, nor by the composition of the claims truncation and attribution of minimal rights operators.

Proof. • Duality operator.
Let $(c, E) \in \mathcal{C}^N$, $N' \subseteq N$, and $i \in N$. Showing that

$$S_i(c, E) = S_i\left((c_{N'}, 0_{N \setminus N'}), E - \sum_{j \in N \setminus N'} S_j(c, E)\right),$$

is equivalent to showing that

$$c_i - S_i^d(c, \sum_{j \in N} c_j - E) = c_i - S_i^d\left((c_{N'}, 0_{N \setminus N'}), \sum_{j \in N'} c_j - (E - \sum_{j \in N \setminus N'} (c_j - S_j^d(c, \sum_{k \in N} c_k - E))\right),$$

equivalently that

$$S_i^d(c, \sum_{j\in N} c_j - E) = S_i^d\left((c_{N'}, 0_{N\setminus N'}), \sum_{j\in N} c_j - E - \sum_{j\in N\setminus N'} S_j^d(c, \sum_{k\in N} c_k - E)\right),$$

which, after setting $F \equiv \sum_{j\in N} c_j - E$, is equivalent to

$$S_i^d(c, F) = S_i^d\left((c_{N'}, 0_{N\setminus N'}), F - \sum_{j\in N\setminus N'} S_j^d(c, F)\right),$$

which is the statement that S^d satisfies *partial-implementation invariance*.

• Claims truncation operator.
The proportional rule satisfies *partial-implementation invariance* but when subjected to the claims truncation operator, it does not. Indeed, let $N \equiv \{1,2,3\}$ and $(c,E) \in \mathcal{C}^N$ be equal to $((2,4,6),6)$. Then, $P^t(c,E) = P(c,E) = (1,2,3)$. We have that

$$P_1^t((c_{\{1,2\}}, 0), E - P_3^t(c,E)) = P_1^t((2,4,0),3) = P_1((2,3,0),3) = \frac{6}{5} \neq 1 = P_1^t((2,4,6),6).$$

• Attribution of minimal rights operator.
The constrained equal awards rule satisfies *partial-implementation invariance* but when subjected to the attribution of minimal rights first operator, it does not. Indeed, let $N \equiv \{1,2,3\}$ and $(c,E) \in \mathcal{C}^N$ be equal to $((1,2,3),4)$. Then $CEA^m(c,E) = (1,1,2)$. However,

$$CEA_1^m\left((c_{\{1,3\}}, 0), E - CEA_2^m(c,E)\right) = CEA_1^m((1,0,3),3) = \frac{1}{2} \neq 1 = CEA_1^m(c,E).$$

• Convexification operator.
The constrained equal awards and the constrained equal losses rules satisfy *partial-implementation invariance* but their simple average, Av, does not.[8] Indeed, let $N \equiv \{1, 2, 3\}$ and $(c, E) \in \mathcal{C}^N$ be equal to $((100, 200, 300), 240)$. Then

$$Av(c, E) = \frac{(80, 80, 80) + (0, 70, 170)}{2} = (40, 75, 125).$$

However,

$$Av_1\left((c_{\{1,2\}}, 0), E - Av_3(c, E)\right) = \frac{57.5 + 7.5}{2} = 32.5 \neq 40 = Av_1(c, E).$$

• Composition of the claims truncation and attribution of minimal rights operators.
We used the constrained equal awards rule to show that *partial-implementation invariance* is not preserved under the attribution of minimal rights first operator. Now, note that the rule is *claims truncation invariant*, so $CEA^{mot} = CEA^m$. □

The next lemma addresses the issue of preservation of *converse partial-implementation invariance* by the four operators encountered earlier.

Lemma 4. *Converse partial-implementation invariance* is preserved by the duality operator. It is preserved by none of the claims truncation, attribution of minimal rights, and convexification operators, nor by the composition of the claims truncation and attribution of minimal rights operators.

Proof. • Duality operator.
Let $(c, E) \in \mathcal{C}^N$ and $x \in X(c, E)$. Suppose that for each $N' \subseteq N$ with $|N'| = 2$,

$$x_{N'} = S^d_{N'}((c_{N'}, 0_{N \setminus N'}), E - \sum_{i \in N \setminus N'} x_i).$$

[8] Here is the formal definition of the average of the constained equal awards and the constrained equal losses rules: for each $(c, E) \in \mathcal{C}^N$,

$$Av(c, E) \equiv \frac{CEA(c, E) + CEL(c, E)}{2}.$$

We prove that $x = S^d(c, E)$. By definition of S^d, our hypothesis can be written as follows: For each $N' \subset N$ with $|N'| = 2$,

$$x_{N'} = c_{N'} - S\left((c_{N'}, 0_{N\setminus N'}), \sum_{i\in N'} c_i - \sum_{i\in N'} x_i\right),$$

equivalently for each $N' \subset N$ with $|N'| = 2$,

$$(c - x)_{N'} = S\left((c_{N'}, 0_{N\setminus N'}), \sum_{i\in N'} (c_i - x_i)\right).$$

The vector $c - x$ satisfies the hypotheses of *converse partial-implementation invariance* for S and the well-defined problem $(c, \sum_{i\in N}(c_i - x_i))$. Since S is *converse partial-implementation invariant*,

$$c - x = S\left(c, \sum_{i\in N}(c_i - x_i)\right).$$

Thus,

$$x = c - S\left(c, \sum_{i\in N} c_i - \sum_{i\in N} x_i\right) = c - S\left(c, \sum_{i\in N} c_i - E\right) = S^d(c, E).$$

• Claims truncation operator.

The proportional rule satisfies *converse partial-implementation invariance* but when subjected to the claims truncation operator, it does not. Indeed, let $N \equiv \{1, 2, 3\}$ and $(c, E) \in \mathcal{C}^N$ be equal to $((2, 4, 6), 3)$. Let $x \equiv (1, 1, 1)$. For each $N' \subset N$ with $|N'| = 2$,

$$x_{N'} = P^t_{N'}\left((c_{N'}, 0_{N\setminus N'}), E - \sum_{k\in N\setminus N'} x_k\right).$$

However,

$$x = (1, 1, 1) \neq \left(\frac{6}{8}, \frac{9}{8}, \frac{9}{8}\right) = P^t(c, E).$$

• Attribution of minimal rights first operator.

The constrained equal awards rule satisfies *converse partial-implementation invariance* but when subjected to the attribution of minimal rights first operator, it does not. Indeed, let $N \equiv \{1, 2, 3\}$ and $(c, E) \in \mathcal{C}^N$ be equal to $((10, 10, 20), 25)$. Let $x \equiv (5, 5, 15)$. Then

$$\begin{aligned}
(x_1, x_2) &= (5, 5) = m_{\{1,2\}}((c_{\{1,2\}}, 0), 10) + CEA_{\{1,2\}}\left((10, 10, 0), 10\right), \\
(x_1, x_3) &= (5, 15) = m_{\{1,3\}}((c_{\{1,3\}}, 0), 20) + CEA_{\{1,3\}}\left((10, 0, 10), 10\right), \\
(x_2, x_3) &= (5, 15) = m_{\{2,3\}}((c_{\{2,3\}}, 0), 20) + CEA_{\{2,3\}}\left((0, 10, 10), 10\right).
\end{aligned}$$

However,

$$x = (5, 5, 15) \neq \left(\frac{20}{3}, \frac{20}{3}, \frac{35}{3}\right) = m(c, E) + CEA\left((10, 10, 15), 20\right).$$

• Convexification operator.

The constrained equal awards and the constrained equal losses rules both satisfy *converse partial-implementation invariance* but their simple average does not. Indeed, let $N \equiv \{1, 2, 3\}$ and $(c, E) \in \mathcal{C}^N$ be equal to $((100, 100, 300), 325)$. Let $x \equiv (65, 65, 195)$. Then

$$\begin{aligned}
(x_1, x_2) &= (65, 65) = \frac{(65, 65) + (65, 65)}{2} = Av_{\{1,2\}}((c_{\{1,2\}}, 0), 130), \\
(x_1, x_3) &= (65, 195) = \frac{(100, 160) + (30, 230)}{2} = Av_{\{1,3\}}((c_{\{1,3\}}, 0), 260), \\
(x_2, x_3) &= (65, 195) = \frac{(100, 160) + (30, 230)}{2} = Av_{\{2,3\}}((c_{\{2,3\}}, 0), 260).
\end{aligned}$$

However,

$$x = (65, 65, 195) \neq \left(\frac{425}{6}, \frac{425}{6}, \frac{1100}{6}\right) = \frac{(100, 100, 125) + (\frac{125}{3}, \frac{125}{3}, \frac{725}{3})}{2} = Av(c, E).$$

• Composition of the claims truncation and attribution of minimal rights operators.

We used the constrained equal awards rule to show that *converse partial-implementation invariance* is not preserved under the attribution of minimal rights first operator. Now, note that the rule is *claims truncation invariant*, so $CEA^{mot} = CEA^m$. □

The definition of *partial-implementation invariance* and its converse can be directly applied to correspondences.

Partial-implementation invariance for correspondences: For each $(c, E) \in \mathcal{C}^N$, each $N' \subseteq N$, and each $x \in S(c, E)$,

$$x_{N'} \in S_{N'}\left((c_{N'}, 0_{N \setminus N'}), E - \sum_{i \in N \setminus N'} x_i\right).$$

Converse partial-implementation invariance for correspondences: For each $(c, E) \in \mathcal{C}^N$ and each $x \in X(c, E)$, if for each $N' \subseteq N$ with $|N'| = 2$, $x_{N'} \in S_{N'}((c_{N'}, 0_{N \setminus N'}), E - \sum_{i \in N \setminus N'} x_i)$, then $x \in S(c, E)$.

Consider now the correspondence that associates with each problem its set of awards vectors satisfying the order preservation requirements and the correspondence that associates with each problem its set of awards vectors satisfying the group order preservation requirements. Both of these correspondences are *partial-implementation invariant*.

According to the next lemma, *partial-implementation invariance* and its *converse* are related in a similar manner to the way in which *consistency* and its *converse* are related.[9] Note that the lemma is written for solution mappings that may or may not be *single-valued*.

Lemma 5. Let S be a solution satisfying *partial-implementation invariance for correspondences* and $\overline{S}$ a solution satisfying *converse partial-implementation invariance for correspondences*. If on $\mathcal{C}_2^N$, $S \subseteq \overline{S}$, this inclusion holds on the entire domain $\mathcal{C}^N$.

Proof. Let S and $\overline{S}$ be two solutions satisfying the hypotheses of the lemma. Let $(c, E) \in \mathcal{C}^N$ and $x \in S(c, E)$. Since S satisfies *partial-implementation invariance for correspondences*, for each $N' \subseteq N$ with $|N'| = 2$,

$$x_{N'} \in S_{N'}\left((c_{N'}, 0_{N \setminus N'}), E - \sum_{i \in N \setminus N'} x_i\right).$$

[9] The lemma relating these last axioms is called the Elevator Lemma in Thomson (2019), in reference to the fact that *consistency* allows us to make a statement about problems on the basis on what is known about larger problems and *converse consistency* allows statement pertaining to the enlargements of problems instead.

Since on $\mathcal{C}_2^N$, $S \subseteq \overline{S}$, we deduce that for each $N' \subseteq N$ with $|N'| = 2$,

$$x_{N'} \in \overline{S}_{N'}\left((c_{N'}, 0_{N \setminus N'}), E - \sum_{i \in N \setminus N'} x_i\right).$$

These are the hypotheses of *converse partial-implementation invariance for correspondences* for $\overline{S}$. Since $\overline{S}$ satisfies *conversely partial-implementation invariance for correspondences*, we conclude that $x \in \overline{S}(c, E)$. □

4 Characterizations

In this section, we turn to characterizations. Given a subdomain $\mathcal{C}' \subseteq \mathcal{C}^N$, a rule S on $\mathcal{C}^N$ is an **extension** of a rule S' on $\mathcal{C}'$ if for each $(c, E) \in \mathcal{C}'$, $S(c, E) = S'(c, E)$. The first characterization is a counterpart of one of the first characterizations in the theory, a characterization of the Talmud rule. That result had been obtained by combining a characterization of the two-claimant version of this rule, known as the "contested garment rule" (Aumann and Maschler, 1985) and "concede-and-divide" (Thomson, 2003), and extending this two-claimant rule to the n-claimant case by invoking *consistency* (Aumann and Maschler, 1985). Note that on $\mathcal{C}^N$, the Talmud rule satisfies *equal treatment of equals*, *claims truncation invariance*, and *minimal rights first*.

Lemma 6. On $\mathcal{C}_2^N$, the Talmud rule is the only rule satisfying *equal treatment of equals*, *claims truncation invariance*, and *minimal rights first*.

Proof. We follow the argument in Dagan (1996). Let S be a rule on $\mathcal{C}_2^N$ satisfying the three axioms of the lemma. Let $((c_0, c_0', 0, \ldots, 0), E) \in \mathcal{C}_2^N$. Without loss of generality, suppose that $c_0 \leq c_0'$.
Case 1: $E \leq c_0$. Then

$$t(c, E) = (E, E, 0, \ldots, 0).$$

By *claims truncation invariance*, $S(c, E) = S((E, E, 0, \ldots, 0), E)$. By *equal treatment of equals*,

$$S((E, E, 0, \ldots, 0), E) = \left(\frac{E}{2}, \frac{E}{2}, 0, \ldots, 0\right) = T(c, E).$$

Case 2: $c_0 < E \leq c'_0$. Then
$$m(c,E) = (0, E - c_0, 0, \ldots, 0).$$
Consider $((c_0, c'_0 - (E - c_0), 0, \ldots, 0), E - (E - c_0)) = ((c_0, c'_0 - (E - c_0), 0, \ldots, 0), c_0) \in \mathcal{C}_2^N$. The fact that $c'_0 - (E - c_0) \geq c_0$ together with *claims truncation invariance* implies that

$$S((c_0, c'_0 - (E - c_0), 0, \ldots, 0), c_0) = S((c_0, c_0, 0, \ldots, 0), c_0).$$

By *equal treatment of equals*,

$$S((c_0, c_0, 0, \ldots, 0), c_0) = \left(\frac{c_0}{2}, \frac{c_0}{2}, 0, \ldots, 0\right).$$

By *minimal rights first*,

$$S(c,E) = (0, E - c_0, 0, \ldots, 0) + \left(\frac{c_0}{2}, \frac{c_0}{2}, 0, \ldots, 0\right) = \left(\frac{c_0}{2}, E - \frac{c_0}{2}, 0, \ldots, 0\right) = T(c,E).$$

Case 3: $c'_0 < E$. Then
$$m(c,E) = (E - c'_0, E - c_0, 0, \ldots, 0).$$

Consider

$$(c', E') \equiv ((c_0 - (E - c'_0), c'_0 - (E - c_0), 0 \ldots, 0), E - (E - c'_0) - (E - c_0)) \in \mathcal{C}_2^N,$$

equivalently
$$(c', E') = (((c_0 + c'_0) - E, (c_0 + c'_0) - E, 0, \ldots, 0), (c_0 + c'_0) - E).$$

By *equal treatment of equals*,

$$S(c', E') = \left(\frac{(c_0 + c'_0) - E}{2}, \frac{(c_0 + c'_0) - E}{2}, 0, \ldots, 0\right).$$

By *minimal rights first*,

$$S(c,E) = (E - c_0', E - c_0, 0, \ldots, 0) + \left(\frac{(c_0 + c_0') - E}{2}, \frac{(c_0 + c_0') - E}{2}, 0, \ldots, 0\right)$$
$$= \left(\frac{E + c_0 - c_0'}{2}, \frac{E + c_0' - c_0}{2}, 0, \ldots, 0\right) = T(c,E).$$

Thus, on $\mathcal{C}_2^N$, $S = T$. □

Lemma 7. Consider a rule S defined on $\mathcal{C}_2^N$ that is *endowment monotonic* and admits an extension satisfying *partial-implementation invariance*. Then this extension is unique.

Proof. Let S be a rule on $\mathcal{C}_2^N$. The proof is by contraposition. Let S' and S'' on $\mathcal{C}^N$ be two different extensions of S satisfying *partial-implementation invariance*. Let $(c,E) \in \mathcal{C}^N$ be such that $S'(c,E) \neq S''(c,E)$. Let $x \equiv S'(c,E)$ and $y \equiv S''(c,E)$. Since $\sum_{i \in N} x_i = \sum_{i \in N} y_i$ and $x \neq y$, there are $i, j \in N$ such that $x_i > y_i$ and $x_j < y_j$. By *partial-implementation invariance*,

$$S'((c_i, c_j, 0, \ldots, 0), x_i + x_j) = (x_i, x_j, 0, \ldots, 0)$$
$$\text{and } S''((c_i, c_j, 0, \ldots, 0), y_i + y_j) = (y_i, y_j, 0, \ldots, 0).$$

Thus

$$S((c_i, c_j, 0, \ldots, 0), x_i + x_j) = (x_i, x_j, 0, \ldots, 0)$$
$$\text{and } S((c_i, c_j, 0, \ldots, 0), y_i + y_j) = (y_i, y_j, 0, \ldots, 0).$$

Thus S violates *endowment monotonicity*. □

Using either (i) Lemmas 5 and 6 and the fact that the Talmud rule satisfies *converse partial-implementation invariance*, or (ii) Lemmas 6 and 7 and the fact that the Talmud rule satisfies *endowment monotonicity*, we establish the following characterization:

Theorem 1. The Talmud rule is the only rule satisfying *equal treatment of equals*, *claims truncation invariance*, *minimal rights first*, and *partial-implementation invariance*.

Young's rules (Young, 1987) were introduced in the study of the variable-population version of the model.[10] They are defined as follows. Let $\overline{\mathbb{R}}$ be the extended reals. Given $\underline{\lambda}, \overline{\lambda} \in \overline{\mathbb{R}}$ with $\underline{\lambda} < \overline{\lambda}$, let $f(\cdot,\cdot)\colon \mathbb{R}_+ \times [\underline{\lambda}, \overline{\lambda}] \to \mathbb{R}_+$ be continuous, nowhere decreasing in the second argument, and such that for each $c_0 \in \mathbb{R}_+$, $f(c_0, \underline{\lambda}) = 0$ and $f(c_0, \overline{\lambda}) = c_0$. Let $\mathcal{F}$ be the family of all such functions.

Young rule associated with $\boldsymbol{f \in \mathcal{F}}$, $\boldsymbol{S^f}$: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, $S^f(c, E)$ is the awards vector $x \in X(c, E)$ for which there is $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ such that for each $i \in N$, $x_i = f(c_i, \lambda)$.

Young's definition can of course be applied to a fixed-population model. In such a model, any Young rule satisfies *partial-implementation invariance*. In generalizing the fixed-population version of Young's rules to the variable-population model, the function f can be allowed to depend on the population of claimants. Any rule so defined will obviously satisfy *partial-implementation invariance*, but in general will fail *consistency*.

One of the most powerful results in the theory concerning the adjudication of conflicting claims is the characterization on $\mathcal{C}$ of Young's family on the basis of *equal treatment of equals*, *continuity*, and *consistency* (Young, 1987). The following is not an exact counterpart of this theorem because it involves the additional axiom of *endowment monotonicity*. It follows Young's argument in its broad outlines but there are significant differences. The fact that it pertains to a fixed population of claimants is the main reason for the complications that arise and the reason why the characterization involves *endowment monotonicity*.

As is the case for Young's characterization, as a stepping stone towards our main result, a characterization in which *strict endowment monotonicity* is imposed can be given (Proposition 1). For that purpose, we first establish the following lemma, which derives *anonymity* from the milder requirement of *equal treatment of equals*.

Lemma 8. For $|N| > 2$, if a rule satisfies *continuity*, *equal treatment of equals*, and *partial-implementation invariance*, then it satisfies *anonymity* on $\mathcal{C}_2^N$.

Proof. Let S be a rule satisfying the hypotheses of the lemma. Let $((c_0, c_0', 0, \ldots, 0), E) \in \mathcal{C}_2^N$. Let

$$(x_0, x_0', 0, \ldots, 0) \equiv S((c_0, c_0', 0, \ldots, 0), E).$$

[10] These rules are also called "parametric".

Then $E = x_0 + x'_0$. By *continuity*, there is $E' \in \mathbb{R}_+$ such that

$$S((c_0, c'_0, c'_0, 0, \ldots, 0), E') = (y_0, y'_0, y''_0, 0, \ldots, 0)$$

and $y_0 + y'_0 = x_0 + x'_0$. By *partial-implementation invariance*, $y_0 = x_0$ and $y'_0 = x'_0$. By *equal treatment of equals*, $y''_0 = y'_0 = x'_0$. Thus

$$S((c_0, c'_0, c'_0, 0, \ldots, 0), E') = (x_0, x'_0, x'_0, 0, \ldots, 0).$$

By *partial-implementation invariance*,

$$S((c_0, 0, c'_0, 0, \ldots, 0), E) = (x_0, 0, x'_0, 0, \ldots, 0).$$

Hence, S satisfies *anonymity* on $\mathcal{C}_2^N$. □

Proposition 1. For $|N| > 2$, the Young rules satisfying *strict endowment monotonicity* are the only rules satisfying *equal treatment of equals*, *continuity*, *strict endowment monotonicity*, and *partial-implementation invariance*.

Proof. Clearly, any *strictly endowment monotonic* Young rule satisfies all the axioms of the proposition. Conversely, let S be a rule satisfying these axioms. Let $f\colon \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ be defined by setting for each $c_0 \in \mathbb{R}_+$ and each $\lambda \in [0,1]$,

$$f(c_0, \lambda) = x_0 \quad \text{if and only if} \quad S((c_0, 1, 0, \ldots, 0), x_0 + \lambda) = (x_0, \lambda, 0, \ldots, 0).$$

By Lemma 8, f is well-defined. Moreover, $f \in \mathcal{F}$.

We claim that f is a Young representation of S. Suppose otherwise. Then there is $(c, E) \in \mathcal{C}^N$ such that for each $\lambda \in [0,1]$,

$$S(c, E) \neq (f(c_i, \lambda))_{i \in N}.$$

Let $i \in N$ with $c_i > 0$. By *continuity*, there is $\lambda \in [0,1]$ such that $S_i(c, E) = f(c_i, \lambda)$. Let $j \in N$ be such that $S_j(c, E) \neq f(c_j, \lambda)$. Let $x \equiv S(c, E)$. By *partial-implementation invariance*,

$$S((c_i, c_j, 0, \ldots, 0), x_i + x_j) = (x_i, x_j, 0, \ldots, 0).$$

By *continuity*, there is $E' \in \mathbb{R}_+$ such that

$$S_i((c_i, c_j, 1, 0, \ldots, 0), E') + S_j((c_i, c_j, 1, 0, \ldots, 0), E') = x_i + x_j.$$

By *partial-implementation invariance*,

$$\begin{aligned} S_i((c_i, c_j, 1, 0, \ldots, 0), E') &= S_i((c_i, c_j, 0, \ldots, 0), x_i + x_j) \\ \text{and } S_j((c_i, c_j, 1, 0, \ldots, 0), E') &= S_j((c_i, c_j, 0, \ldots, 0), x_i + x_j). \end{aligned}$$

Thus

$$S((c_i, c_j, 1, 0, \ldots, 0), E') = (x_i, x_j, E' - x_i - x_j, 0, \ldots, 0).$$

Let $\lambda' \equiv E' - x_i - x_j$. By Lemma 8 and *partial-implementation invariance*,

$$\begin{aligned} S((c_i, 1, 0, \ldots, 0), x_i + \lambda') &= (x_i, \lambda', 0, \ldots, 0) \\ \text{and } S((c_j, 1, 0, \ldots, 0), x_j + \lambda') &= (x_j, \lambda', 0, \ldots, 0). \end{aligned}$$

This means that $x_i = f(c_i, \lambda')$ and $x_j = f(c_j, \lambda')$. By *strict endowment monotonicity*, $\lambda' = \lambda$. Then $S_j(c, E) = f(c_j, \lambda)$. This is a contradiction, so f is a Young representation of S. Hence, S is a Young rule satisfying *strict endowment monotonicity*. □

Here is our main theorem.

Theorem 2. For $|N| > 2$, Young's rules are the only rules satisfying *equal treatment of equals*, *endowment monotonicity*, *continuity*, and *partial-implementation invariance*.

Proof. Clearly, Young's rules satisfy the four axioms of the theorem. Conversely, let S be a rule satisfying these axioms.

For each $\gamma \in \mathbb{R}_+$, each $c_0 \in \mathbb{R}_+$, and each $x_0 \in [0, c_0]$, let

$$g(\gamma; c_0, x_0) \equiv \max \left\{ \chi \in [0, \gamma] | S((c_0, \gamma, 0, \ldots, 0), x_0 + \chi) = (x_0, \chi, 0, \ldots, 0) \right\}.$$

By Lemma 8 and *continuity*, g is well-defined.

Step 1 *For each* $c_0 \in \mathbb{R}_+$ *and each* $x_0 \in [0, c_0]$, $g(\gamma; c_0, x_0)$ *is continuous in* γ.

The proof of this step as well as those of the other steps are in the appendix. Now, for each $c_0 \in \mathbb{R}_+$ and each $x_0 \in [0, c_0]$, let

$$h(c_0, x_0) \equiv \int_{\mathbb{R}_+} g(\gamma; c_0, x_0) e^{-\gamma} d\gamma.$$

By Step 1, h is well-defined. Moreover, for each $c_0 \in \mathbb{R}_+$ and each $x_0 \in [0, c_0]$, $h(c_0, x_0) \in [0, 1]$.

Step 2 *For each* $c_0 \in \mathbb{R}_+$, $h(c_0, x_0)$ *is increasing in* x_0.

For each $c_0 \in \mathbb{R}_+$ and each $x_0 \in [0, c_0]$, let

$$h^-(c_0, 0) \equiv 0, \quad h^-(c_0, x_0) \equiv \lim_{\chi \uparrow x_0} h(c_0, \chi),$$
$$h^+(c_0, c_0) \equiv 1, \quad h^+(c_0, x_0) \equiv \lim_{\chi \downarrow x_0} h(c_0, \chi).$$

For each $c_0 \in \mathbb{R}_+$ and each $\lambda \in [0, 1]$, let $f : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+$ be defined in such a way that

$$f(c_0, \lambda) = x_0 \quad \text{if and only if} \quad h^-(c_0, x_0) \leq \lambda \leq h^+(c_0, x_0).$$

By Step 2, $f \in \mathcal{F}$. Let $(c, E) \in \mathcal{C}^N$. Let $x \equiv S(c, E)$.

Step 3 *For each* $\gamma \in \mathbb{R}_+$, *there is* $\chi \in [0, \gamma]$ *such that for each* $i \in N$,

$$S((\gamma, c_i, 0, \ldots, 0), \chi + x_i) = (\chi, x_i, 0, \ldots, 0).$$

Step 4 *There is* $\lambda \in [0, 1]$ *such that for each* $i \in N$, $h^-(c_i, x_i) \leq \lambda \leq h^+(c_i, x_i)$.

This means that f is a Young representation of S. Hence, S is a Young rule. □

The domain of Young's theorem includes two-claimant problems and the theorem implies that the restriction to the two-claimant case of any rule satisfying his axioms on his domain has to be a two-claimant Young rule. We have noted that *partial-implementation invariance* is silent in the two-claimant case. The remaining axioms of Theorem 2, *equal treatment of equals*, *endowment monotonicity*, and *continuity*, are satisfied by a large family of rules of which the two-claimant Young rules constitute a small subfamily.

The uniqueness parts of both theorems can be strengthened by weakening *partial-implementation invariance* to *bilateral partial-implementation invariance*.

5 Concluding comments

Partial-implementation invariance has an "average" version, similar in spirit to the average version of *consistency* proposed by Dagan and Volij (1997).[11] *Average partial-implementation invariance* can provide the basis for the definition of an operator on the space of rules: it associates with each rule its **average partial-implementation invariant variant**. We will leave the study of this operator to future work.

Claims problems are not the only class of problems to which *partial-implementation invariance* can be applied. We chose to focus on one model but we are hopeful that the concept will be useful in other contexts.

[11] It says that for each problem and each claimant, the award to that claimant should be equal to the average of the awards to this claimant in the derived two-claimant problems in which the claims of all claimants other than that claimant and one other claimant have been set equal to 0 and the endowment has been reduced by the sum of the awards to the other claimants.

A Proof of Theorem 2

Step 1. For each $c_0 \in \mathbb{R}_+$ and each $x_0 \in [0, c_0]$, $g(\gamma; c_0, x_0)$ is continuous in γ.

Proof. Let $c_0 \in \mathbb{R}_+$ and $x_0 \in [0, c_0]$. For each $\gamma \in \mathbb{R}_+$, let $\widehat{g}(\gamma) \equiv g(\gamma; c_0, x_0)$. We show that $\widehat{g}$ is continuous. Let $i \in N$, $c_i \equiv c_0$, and $x_i \equiv x_0$. Then, for each $\gamma \in \mathbb{R}_+$,

$$\widehat{g}(\gamma) = \max \{\chi \in [0, \gamma] | S((c_i, \gamma, 0, \ldots, 0), x_i + \chi) = (x_i, \chi, 0, \ldots, 0)\}.$$

Let $\gamma \in \mathbb{R}_+$. Let $j \in N \setminus \{i\}$ and $c_j \equiv \gamma$. Let $\{c^n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_+$ converging to c_j. Let $k \in N \setminus \{i, j\}$. For each $n \in \mathbb{N}$, let $c_k^n \equiv c^n$. Let $n \in \mathbb{N}$. Let

$$E^n \equiv \max \{E \in \mathbb{R}_+ | S_i((c_i, c_j, c_k^n, 0, \ldots, 0), E) = x_i\}.$$

By *continuity*, E^n is well-defined. Let $x_j \equiv S_j((c_i, c_j, c_k^n, 0, \ldots, 0), E^n)$. By *partial-implementation invariance*,

$$\begin{aligned} S((c_i, c_j, 0, \ldots, 0), x_i + x_j) &= (x_i, x_j, 0, \ldots, 0) \\ \text{and } S((c_i, c_k^n, 0, \ldots, 0), E^n - x_j) &= (x_i, E^n - x_i - x_j, 0, \ldots, 0). \end{aligned}$$

Thus $x_j \leq \widehat{g}(c_j)$ and $E^n - x_i - x_j \leq \widehat{g}(c_k^n)$. Suppose that $x_j < \widehat{g}(c_j)$. By *continuity* and *endowment monotonicity*, there is $E > E^n$ such that

$$S_i((c_i, c_j, c_k^n, 0, \ldots, 0), E) + S_j((c_i, c_j, c_k^n, 0, \ldots, 0), E) = x_i + \widehat{g}(c_j).$$

Let $x' \equiv S((c_i, c_j, c_k^n, 0, \ldots, 0), E)$. By *partial-implementation invariance*,

$$S((c_i, c_j, 0, \ldots, 0), x_i + \widehat{g}(c_j)) = (x'_i, x'_j, 0, \ldots, 0).$$

By definition of $\widehat{g}$, $x'_i = x_i$. This contradicts the definition of E^n, so $x_j = \widehat{g}(c_j)$. Similarly, $E^n - x_i - x_j = \widehat{g}(c_k^n)$. Thus $E^n = x_i + \widehat{g}(c_j) + \widehat{g}(c_k^n)$ and

$$S\left((c_i, c_j, c_k^n, 0, \ldots, 0), x_i + \widehat{g}(c_j) + \widehat{g}(c_k^n)\right) = \left(x_i, \widehat{g}(c_j), \widehat{g}(c_k^n), 0, \ldots, 0\right).$$

By *continuity*,

$$S\left(\left(c_i, c_j, \lim_{n\to\infty} c_k^n, 0, \dots, 0\right), x_i + \widehat{g}(c_j) + \lim_{n\to\infty} \widehat{g}(c_k^n)\right) = \left(x_i, \widehat{g}(c_j), \lim_{n\to\infty} \widehat{g}(c_k^n), 0, \dots, 0\right).$$

Thus

$$S\left((c_i, c_j, c_j, 0, \dots, 0), x_i + \widehat{g}(c_j) + \lim_{n\to\infty} \widehat{g}(c_k^n)\right) = \left(x_i, \widehat{g}(c_j), \lim_{n\to\infty} \widehat{g}(c_k^n), 0, \dots, 0\right).$$

By *equal treatment of equals*, $\lim_{n\to\infty} \widehat{g}(c_k^n) = \widehat{g}(c_j)$. Hence, $\widehat{g}$ is continuous. □

Step 2. For each $c_0 \in \mathbb{R}_+$, $h(c_0, x_0)$ is increasing in x_0.

Proof. Let $c_0 \in \mathbb{R}_+$. By *endowment monotonicity*, $g(\gamma; c_0, x_0)$ is nowhere decreasing in x_0. Let $x_0, x_0' \in [0, c_0]$ be such that $x_0 < x_0'$. Then

$$h(c_0, x_0) = \int_{\mathbb{R}_+} g(\gamma; c_0, x_0) e^{-\gamma} d\gamma \leq \int_{\mathbb{R}_+} g(\gamma; c_0, x_0') e^{-\gamma} d\gamma = h(c_0, x_0').$$

Suppose that $h(c_0, x_0) = h(c_0, x_0')$. Then each nondegenerate interval in $\mathbb{R}_+$ contains γ such that $g(\gamma; c_0, x_0) = g(\gamma; c_0, x_0')$. Let $\{\gamma^n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+$ converging to c_0 such that for each $n \in \mathbb{N}$, $g(\gamma^n; c_0, x_0) = g(\gamma^n; c_0, x_0')$. For each $n \in \mathbb{N}$, let $\chi^n \equiv g(\gamma^n; c_0, x_0)$. By definition of g, for each $n \in \mathbb{N}$,

$$S\left((c_0, \gamma^n, 0, \dots, 0), x_0 + \chi^n\right) = (x_0, \chi^n, 0, \dots, 0)$$
$$\text{and } S\left((c_0, \gamma^n, 0, \dots, 0), x_0' + \chi^n\right) = (x_0', \chi^n, 0, \dots, 0).$$

By *continuity*,

$$S\left((c_0, \lim_{n\to\infty} \gamma^n, 0, \dots, 0), x_0 + \lim_{n\to\infty} \chi^n\right) = \left(x_0, \lim_{n\to\infty} \chi^n, 0, \dots, 0\right)$$
$$\text{and } S\left((c_0, \lim_{n\to\infty} \gamma^n, 0, \dots, 0), x_0' + \lim_{n\to\infty} \chi^n\right) = \left(x_0', \lim_{n\to\infty} \chi^n, 0, \dots, 0\right).$$

Thus

$$S\left((c_0, c_0, 0, \ldots, 0), x_0 + \lim_{n\to\infty} \chi^n\right) = \left(x_0, \lim_{n\to\infty} \chi^n, 0, \ldots, 0\right)$$
$$\text{and } S\left((c_0, c_0, 0, \ldots, 0), x_0' + \lim_{n\to\infty} \chi^n\right) = \left(x_0', \lim_{n\to\infty} \chi^n, 0, \ldots, 0\right).$$

By *equal treatment of equals*, $x_0 = \lim_{n\to\infty} \chi^n = x_0'$. This is a contradiction, so $h(c_0, x_0) < h(c_0, x_0')$. Hence, $h(c_0, x_0)$ is increasing in x_0. □

Step 3. For each $\gamma \in \mathbb{R}_+$, there is $\chi \in [0, \gamma]$ such that for each $i \in N$,

$$S((\gamma, c_i, 0, \ldots, 0), \chi + x_i) = (\chi, x_i, 0, \ldots, 0).$$

Proof. Let $\gamma \in \mathbb{R}_+$. For each $i \in N$, let

$$I_i \equiv \{\chi \in [0, \gamma] \mid S((\gamma, c_i, 0, \ldots, 0), \chi + x_i) = (\chi, x_i, 0, \ldots, 0)\}.$$

By *endowment monotonicity*, I_i is an interval. We show that $\bigcap_{i\in N} I_i \neq \emptyset$.
For each $i \in N$, let $c_i' \equiv \gamma$. For each pair $i, j \in N$, let

$$I_{ij} \equiv \bigcup_{k\in N\setminus\{i,j\}} \{x_k' \in [0, c_k'] \mid S((c_i, c_j, c_k', 0, \ldots, 0), x_i + x_j + x_k') = (x_i, x_j, x_k', 0, \ldots, 0)\}.$$

Let $i, j, k \in N$. By *continuity*, there is $x_k' \in [0, c_k']$ such that

$$S_i((c_i, c_j, c_k', 0, \ldots, 0), x_i + x_j + x_k') + S_j((c_i, c_j, c_k', 0, \ldots, 0), x_i + x_j + x_k') = x_i + x_j.$$

By *partial-implementation invariance*,

$$S((c_i, c_j, c_k', 0, \ldots, 0), x_i + x_j + x_k') = (x_i, x_j, x_k', 0, \ldots, 0).$$

Thus $x_k' \in I_{ij}$. By Lemma 8 and *partial-implementation invariance*, $x_k' \in I_i \cap I_j$. In other words, for each pair $i, j \in N$, $I_i \cap I_j \neq \emptyset$. Hence, $\bigcap_{i\in N} I_i \neq \emptyset$. □

Step 4. There is $\lambda \in [0,1]$ such that for each $i \in N$, $h^-(c_i, x_i) \leq \lambda \leq h^+(c_i, x_i)$.

Proof. By Step 3, for each $\gamma \in \mathbb{R}_+$, there is $\chi \in [0, \gamma]$ such that for each $i \in N$,

$$S((\gamma, c_i, 0, \dots, 0), \chi + x_i) = (\chi, x_i, 0, \dots, 0).$$

For each $\gamma \in \mathbb{R}_+$, let $m(\gamma) \equiv \min_{i \in N} g(\gamma; c_i, x_i)$. Then, for each $\gamma \in \mathbb{R}_+$ and each $i \in N$,

$$S((\gamma, c_i, 0, \dots, 0), m(\gamma) + x_i) = (m(\gamma), x_i, 0, \dots, 0).$$

By Step 1, m is continuous. Let $\lambda \equiv \int_{\mathbb{R}_+} m(\gamma) e^{-\gamma} d\gamma$.
Let $i \in N$. Suppose that $x_i = 0$. Then, for each $\gamma \in \mathbb{R}_+$, $0 \leq m(\gamma) \leq g(\gamma; c_i, x_i)$. Thus

$$h^-(c_i, x_i) = 0 \leq \lambda \leq \int_{\mathbb{R}_+} g(\gamma; c_i, x_i) e^{-\gamma} d\gamma \leq h^+(c_i, x_i).$$

Now, suppose that $x_i > 0$. By definition of g, for each $x'_i \in [0, x_i)$,

$$S((\gamma, c_i, 0, \dots, 0), g(\gamma; c_i, x'_i) + x'_i) = (g(\gamma; c_i, x'_i), x'_i, 0, \dots, 0).$$

By *endowment monotonicity*, for each $x'_i \in [0, x_i)$, $g(\gamma; c_i, x'_i) \leq m(\gamma) \leq g(\gamma; c_i, x_i)$. This implies that for each $x'_i \in [0, x_i)$,

$$\int_{\mathbb{R}_+} g(\gamma; c_i, x'_i) e^{-\gamma} d\gamma \leq \int_{\mathbb{R}_+} m(\gamma) e^{-\gamma} d\gamma \leq \int_{\mathbb{R}_+} g(\gamma; c_i, x_i) e^{-\gamma} d\gamma.$$

Hence, $h^-(c_i, x_i) \leq \lambda \leq h^+(c_i, x_i)$. □

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