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Fair allocation and reallocation: an axiomatic study

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Fair Allocation and Reallocation: An Axiomatic Study

Proefschrift

Ter verkrijging van de graad van doctor aan de Universiteit Maastricht,
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"Alles, was Gegenstand des wissenschaftlichen Denkens überhaupt sein kann, verfällt, sobald es zur Bildung einer Theorie reif ist, der axiomatischen Methode und damit mittelbar der Mathematik."[1]

David Hilbert (1918)

[1]Everything, which may possibly be subject to scientific thought, as soon as it is ready for the development of a theory, comes within the scope of the axiomatic method and thus becomes treatable by means of mathematics.
This translation into English is borrowd from the Introduction to the book "Mathematical Economics, Twenty Papers of Gerard Debreu", Econometric Society Monographs No. 4, ed. Frank Hahn, Cambridge University Press, 1983.

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Notation

Many of the notation we use throughout the thesis are defined in the text were they appear for the first time. Here, we give a short list of some notational conventions we do not introduce explicitly again.

Set theoretical notations

Let A, B be sets.

$A \subseteq B$	A is a subset of B.
$A \subsetneq B$	A is a strict subset of B, *i.e.*, $A \subseteq B$ and $A \neq B$.
$\lvert A \rvert$	cardinality of A.

Natural and real numbers

$\mathbb{N}$	set of natural numbers.
$\mathbb{Q}$	set of rational numbers.
$\mathbb{R}$	set of real numbers.
$\mathbb{R}_+$	$:= \{x \in \mathbb{R} \mid x \geq 0\}$, set of nonnegative real numbers.
$\mathbb{R}^N$, $\mathbb{R}_+^N$	Cartesian product of $\lvert N \rvert$ copies of $\mathbb{R}$, or $\mathbb{R}_+$ respectively, indexed by the elements of N.

Let $a, b \in \mathbb{R}$, $a \leq b$, and $N \subsetneq \mathbb{N}$, N finite.

(a, b) $:= \{x \in \mathbb{R} \,|a < x < b\}$,
open interval between a and b.

$[a, b]$ $:= \{x \in \mathbb{R} \,|a \leq x \leq b\}$,
closed interval between a and b.

$(a, b]$ $:= \{x \in \mathbb{R} \,|a < x \leq b\}$,
$[a, b)$ $:= \{x \in \mathbb{R} \,|a \leq x < b\}$,
half-open intervals between a and b.

Miscellaneous

Let $x, y \in \mathbb{R}^N$.

$x \leq y$ for all $i \in N$, $x_i \leq y_i$.

$\sum_N x_i$ $:= \sum_{i \in N} x_i$.

∞ infinity.

$\blacksquare$ end of a proof.

$\diamond$ end of an example.

$\square$ end of a remark.

Chapter 1

Introduction

1.1 Strategy-proofness and single-peakedness

In many economic and political decision situations the preferences of the agents involved are private information. Depending on the decision mechanism to be used, agents may have an incentive to strategically misrepresent their preferences in order to influence the final outcome to their own advantage, probably at the expense of others. Furthermore, misrepresentation of preferences may yield an outcome that is Pareto-inferior with respect to the true preferences.

Misrepresentation of preferences can be avoided by imposing strategy-proofness on the mechanism. A mechanism (for instance, a voting procedure) is strategy-proof if no agent can ever gain by misrepresenting his preferences, irrespective of the preferences announced by the other agents. An additional advantage of strategy-proofness is that it simplifies the application of the mechanism, because every agent only needs to know his own preference relation in order to determine his optimal strategy.

A first fundamental result concerning strategy-proofness was stated by Gibbard (1973) and Satterthwaite (1975) for the following abstract social choice model. The set of alternatives is finite and all strict preferences over the set of alternatives are admissible. Then, a mechanism with at least three alternatives in its range is strategy-proof if and only if it is dictatorial. However, non-dictatorial strategy-proof mechanisms exist on certain "restricted domains" such as the well-known domain of "single-peaked" preferences: preferences are single-peaked if alternatives can be ordered in such a way that every agent has a single best alternative, his peak, and his welfare decreases monotonically in either direction away from his peak. Single-peaked preferences were introduced by Black (1948) in an election context. For quite some time, contributions to the analysis of strategy-proof mechanisms for single-peaked preferences were concerned mainly with public goods. See, for instance, Dummett and Farquharson (1961), Blin and Satterthwaite (1976), Moulin (1980), Border and Jordan (1983), and Barberà, Gul, and Stacchetti (1993).

In this thesis, we consider several private good models of allocating, or reallocating, an infinitely divisible commodity among a group of agents. Again, strategy-proofness will be a central property in our analysis. However, we are also interested in properties

that express some notion of fairness or solidarity, for instance, no-envy or population-monotonicity. No-envy requires that no agent prefers the share of another agent to his own share. Population-monotonicity requires that after the arrival of new agents, either all agents initially present (weakly) lose together or they all (weakly) gain together. In the sequel, we refer to the analysis of allocation or reallocation mechanisms that satisfy properties of fairness and other "desirable" properties as "fair allocation", or "fair reallocation" respectively.

1.2 Fair allocation and single-peakedness[1]

The literature on strategy-proofness for private good situations also starts off with an impossibility result. For two-agent two-good exchange economies, Hurwicz (1972) showed that even for a restricted domain that contains all translated Cobb-Douglas preferences, essentially no allocation mechanism satisfies strategy-proofness and Pareto optimality. Similar to the public good case, this incompatibility does not occur on some other restricted domains. For instance, there are many situations where preferences are monotonic and the class of mechanisms that satisfy desirable properties from our normative point of view is rich. Examples are bankruptcy (O'Neill, 1982; Aumann and Maschler, 1985), taxation (Young, 1987), and cost and surplus sharing (Moulin, 1987 and 1997).

Another well-known restricted domain for many allocation problems is the domain of single-peaked preferences. A typical example is rationing in a two-good exchange economy when prices are in disequilibrium (see Benassy, 1982, and Barberà and Jackson, 1995). If preferences over the two-dimensional space of bundles are strictly convex, then the restrictions of these preferences to the budget lines are single-peaked. Rationing one of the goods then implies an allocation of the other good, in view of fixed prices and budget constraints. In this context of strategy-proof exchange, Benassy (1982) discussed what is now known as the "uniform rule". It divides the amount that has to be allocated as equally as possible, taking the peaks of the agents as upper bounds in case of excess demand and as lower bounds in case of excess supply. This rule also applies to the problem of allocating a social endowment of some perfectly divisible commodity among a group of agents with single-peaked preferences. Consider the following example.

Example 1.1 ***Allocation of teaching hours:*** A fixed number of teaching hours has to be distributed among the members of a university department. Each member has a preferred number of teaching hours, above which and below which his welfare is decreasing. However, the optimal shares, or peaks, might add up to more (excess demand) or to less (excess supply) than the required total teaching load. Given that we have either excess demand or excess supply, the question we ask is: What is a "fair allocation" of the teaching load?

To be more specific, assume that in the department four staff members, Professors A, B, C, and D, have to share 20 hours of teaching per week. Professor A would

[1]See also the introduction of Chapter 2.

like to teach for 3 hours, Professor B's optimal share is equal to 5 hours, Professor C does not want to teach at all, in contrary to his colleague Professor D who wishes to teach 7 hours. We assume that the Professors' welfares decrease with the distance from their optimal shares. Formally, we denote this allocation problem by the group of Professors $N = \{A, B, C, D\}$ and their peak vector $p(R) = (3, 5, 0, 7)$. Note that this is an allocation problem with excess supply.

First, we consider equal division: the allocation of the teaching load equals $(5, 5, 5, 5)$. However, it is easy to see that this allocation is not Pareto optimal since Professors A and D can reach their peaks by transferring two hours from Professor A to Professor D.

Dividing the 20 hours of teaching in proportion to the peaks yields the following allocation: $(4, 6\frac{2}{3}, 0, 9\frac{1}{3})$. Now, one can argue whether it is fair that everyone but Professor C has to work more than their optimal amount. Furthermore, it is likely that Professor D prefers to teach $6\frac{2}{3}$ hours, the amount that is assigned to his colleague B, than the $9\frac{1}{3}$ hours that were assigned to him. In that case, he envies Professor B, a bad thing for the atmosphere of the department. Besides the fact that the proportional allocation does not satisfy no-envy, it is easy to show that it is not strategy-proof. If, for instance, Professor D declares that not 7, but $5\frac{1}{4}$ is his optimal amount, then he profits from lying and receives as a proportional share his optimal amount of teaching.

Next, we consider the division of the 20 hours of teaching according to the uniform rule we mentioned above, *i.e.*, we divide the teaching hours as equally as possible, taking the peaks of the Professors as lower bounds. We briefly describe how this allocation can be constructed. First, we endow all Professors with their optimal teaching shares $(3, 5, 0, 7)$. Now, we still have to allocate 5 hours of teaching. We start to increase the teaching load of the Professor with the smallest amount to teach until either his teaching load is equal to that of the Professor with the second lowest teaching load or we have assigned all teaching hours. This yields $(3, 5, 3, 7)$. Now, there remain 2 hours of teaching. Again, we increase the teaching load of the Professors with the smallest teaching load until they have a teaching load equal to that of the Professor with the new second lowest teaching load or until we have assigned all teaching hours. This yields the allocation $(4, 5, 4, 7)$ and we are done. Note that this allocation is Pareto optimal and there is no envy between the Professors. ◇

The following example demonstrates the computation of the uniform allocation for an excess demand allocation problem.

Example 1.2 ***Beer allocation:*** At a party there are 5 liters of beer left and the remaining guests $N = \{$a,b,c,d,e$\}$ still wish to consume $p(R) = (2, \frac{1}{2}, 1, 2, 3)$ liters. If free disposal is not allowed, then preferences over the consumption of beer are likely to be single-peaked. Since the sum of the peaks is greater that the amount of beer that is left, we have an allocation problem with excess demand. Now the uniform allocation of this problem can be determined as follows. We start dividing the beer equally until either one of the guests reaches his peak or we have divided all the beer. This yields the allocation $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Now, we still have to allocate $2\frac{1}{2}$ liters of beer. We start

to increase the amount of beer of all guests who still want more equally until, again, either one of these guests reaches his peak or we have divided all the beer. This yields the allocation $(1, \frac{1}{2}, 1, 1, 1)$. Now, we still have to allocate $\frac{1}{2}$ liters of beer. We repeat this procedure once more and finally get the allocation $(1\frac{1}{6}, \frac{1}{2}, 1, 1\frac{1}{6}, 1\frac{1}{6})$. Note that this allocation again is Pareto optimal and there is no envy between the guests. ◇

It turns out that the uniform rule we applied in Examples 1.1 and 1.2 for the allocation of teaching hours and of beer not only satisfies many desirable properties for these specific examples, but for all possible allocation problems. Sprumont (1991) proved that the uniform rule is the only rule that satisfies Pareto optimality, strategy-proofness, and either no-envy or anonymity. The last condition means that an agent's share does not depend on his name. Ching (1994) weakens anonymity to a condition called equal treatment of equals: agents with the same preferences are treated equally. The uniform rule satisfies many other desirable conditions, such as consistency, solidarity, and monotonicity properties. See Thomson (1994a,b, 1995a, 1997a), Sönmez (1994), and Dagan (1995). Further characterizations where obtained by De Frutos and Massó (1994) using a condition of Lorenz maximality, and Otten, Peters, and Volij (1995), applying conditions from bargaining theory. As a conclusion of this research the uniform rule is now accepted as the most important rule for the problem of fair allocation when preferences are single-peaked.

In Chapter 2, we give a review of the most important results concerning this rule. Furthermore, we present the answers to two open questions raised by William Thomson in connection with two characterizations of the uniform rule (Thomson 1995a, 1997a): by means of a single example, we give a positive answer to the question whether replication-invariance in these characterizations is independent from the other characterizing properties.

Proceeding from Chapter 2, we consider two variations of the "classical" allocation problem with single-peaked preferences. In the first variation, which we discuss in the next section, we consider the problem of fair allocation for a class of preferences that can be described as "reversed single-peaked" preferences, or "single-dipped" preferences: preferences are single-dipped if alternatives can be ordered in such a way that every agent has a single worst alternative, his dip, and his welfare increases monotonically in either direction away from the dip. In the second variation of the allocation problem with single-peaked preferences, we assume that instead of a social endowment that has to be allocated, agents have individual endowments that have to be reallocated.

1.3 Fair allocation and single-dippedness[2]

In Chapter 3, we analyze the problem of fair allocation under the assumption that preferences are single-dipped instead of single-peaked. An example of single-dipped preferences are preference relations that can be represented by a strictly convex utility function. Consider, for instance, the following work allocation problem.

[2]See also the introduction of Chapter 3.

Example 1.3 A university department wants to hire at least one of the former Ph.D. students of the department to do a certain amount of teaching. The former Ph.D. students that are available are unemployed and still receive reduced pay from the university.[3] Payment is in proportion to the amount of teaching that is done. However, all extra money that is earned is subtracted from the reduced pay the university provides. Under this assumption, the Ph.D. students (they enjoy their leisure time very much) have single-dipped preferences over the teaching load. ◇

Yet another example is a two-goods exchange economy with fixed prices and strictly quasi-convex utility functions, which induce single-dipped preferences on the budget lines.

Although the domain of single-dipped preferences is less well known than the domain of single-peaked preferences, there are interesting economic situations with underlying single-dipped preferences. For single-dipped preferences in a public good context, we refer to Vickrey (1960), Inada (1964), and Peremans and Storcken (1997). To the best of our knowledge, single-dippedness has only been considered for public good situations.

Assuming for the problem of fair allocation that preferences are single-dipped instead of single-peaked changes the results significantly. It turns out that one of our most important properties, namely Pareto optimality, is not compatible with properties of fairness such as anonymity and no-envy. Furthermore, we show that we must allocate the whole social endowment to a single agent in order to satisfy Pareto optimality, strategy-proofness, and either consistency or one of the solidarity conditions population-monotonicity and "replacement-domination". In addition, we give a full characterization of all allocation rules satisfying the previous combinations of properties. Remarkably, these characterizations are also valid for economies where one indivisible good has to be allocated among a group of agents.

In terms of Example 1.3 we can conclude that only one of the former Ph.D. students is hired to do the teaching job. The question, how this Ph.D. student has to be chosen is answered in Chapter 3.

1.4 Fair reallocation and single-peakedness[4]

Another natural variation of the problem of fair allocation with single-peaked preferences is an allocation, or reallocation, of individual endowments. By way of example, consider again the allocation of teaching hours among the members of a university department as described in Example 1.1. Suppose that the allocation was made for the past term, but that before the next term starts, a reallocation is desirable because preferences have changed over time.

Example 1.4 ***Reallocation of teaching hours:*** Consider the same university department as in Example 1.1, but now assume that there is an existing distribution of

[3] In The Netherlands, unemployed Ph.D. students receive reduced pay, or "wachtgeld" in Dutch, for a certain period after their Ph.D. contract expired.

[4] See also the introduction of Chapter 4.

teaching hours. Again there are four staff members, $N = \{A, B, C, D\}$, who share 20 hours of teaching per week. The individual endowments of teaching from the last term are described by the vector $\omega = (4, 6, 2, 8)$. The preferred teaching load per week the professors announce for the next term is $p(R) = (6, 4, 6, 0)$. Obviously, the existing distribution of teaching hours is not Pareto optimal with respect to the newly announced optima. Hence, by trade all professors could benefit. To be continued. ◇

In economic problems such as the exchange problem mentioned in Section 1.2, individual endowments arise naturally. In Example 1.4, clearly the individual endowments play a crucial role since they discriminate between agents. Another interpretation of this extension can be found in Barberà, Jackson, and Neme (1997), who proceed from the original allocation model with single-peaked preferences and discard anonymity by introducing natural claims or priorities of the agents, for instance in investment situations.

Proceeding from the results we discussed for the original model of fair allocation, we are concerned with the axiomatic analysis of fair reallocation and particularly with an extension of the uniform rule to the reallocation model, the so-called uniform reallocation rule. We revisit Example 1.4.

Example 1.5 ***Reallocation of teaching hours:*** Consider the reallocation problem described in Example 1.4 where Professors A, B, C, and D with peak vector $p(R) = (6, 4, 6, 0)$ have to reallocate the individual teaching endowments $\omega = (4, 8, 2, 6)$. Hence, in this situation where the sum of the peaks is smaller than the total sum of the individual teaching endowments, Professors A and C who want to teach more in the next term can help to improve the situation by taking some of the teaching load of Professors B and D. We are interested in a reallocation of the form $(6, \cdot, 6, \cdot)$. The question is, how to subtract the 6 hours of teaching A and C then receive from B and D. Note that the Professors B and D wish to supply 4 and 6 hours of teaching.

Subtracting in proportion to these supplies yields $(6, 5\frac{3}{5}, 6, 2\frac{2}{5})$ where Professor B supplies $2\frac{2}{5}$ and Professor D supplies $3\frac{3}{5}$ hours of teaching. However, at this reallocation, Professor B would prefer to supply $3\frac{3}{5}$ hours, the amount that his colleague D supplies, instead of the $2\frac{2}{5}$ hours that he is currently supplying. In this situation, he envies Professor D, again, a bad thing for the atmosphere of the department.

Subtracting the 6 hours equally from the individual endowments of the Professors B and D with the peaks as lower bounds, yields the following uniform reallocation $(6, 5, 6, 3)$. ◇

The following example demonstrates the computation of the uniform reallocation for an excess demand reallocation problem.

Example 1.6 ***Beer reallocation:*** At a party there are five guests $N = \{$a,b,c,d,e$\}$ who are each endowed with 1 liter of beer. The amounts of beer they wish to consume are $p(R) = (0, 0, 1\frac{1}{2}, 2, 3)$ liters. Since the sum of the peaks is greater that the amount of beer that is left, we have a reallocation problem with excess demand. Now the uniform reallocation of this problem can be determined as follows. Guests a and b receive no

beer and we start dividing their two liters of beer equally among the remaining guests. We do so until either one of the guests reaches his peak or we have divided all the beer. This yields the reallocation $(0, 0, 1\frac{1}{2}, 1\frac{1}{2}, 1\frac{1}{2})$. Now, we still have to allocate $\frac{1}{2}$ liter of beer. We start to increase the amount of beer of all guests who still want more equally until, again, either one of these guests reaches his peak or we have divided all the beer. This yields the allocation $(0, 0, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{3}{4})$ and we are done. ◇

Similar to the uniform allocation rule, the uniform reallocation rule satisfies many desirable properties such as Pareto optimality, strategy-proofness, and no-envy in terms of net changes of the endowments.

In Chapters 4 and 5 we answer the question to what extent the results as achieved for the allocation model can be generalized to the more general reallocation model. In Chapter 4 we introduce and explore the reallocation model and properties for reallocation rules. Furthermore we derive several characterizations of the uniform reallocation rule and discuss the parallels and the differences to related allocation results. Finally, in Chapter 5, we analyze some variations of the model parameters. We show, for instance, that most of the results remain true if we allow agents to have debts.

1.5 Organization of the thesis

Part I of this thesis is concerned with the model of fair allocation we introduced in Sections 1.2 and 1.3. Chapter 2 is devoted to the problem of fair allocation when preferences are single-peaked. Chapter 3 is devoted to the problem of fair allocation when preferences are single-dipped.

Part II is concerned with the model of fair reallocation we introduced in Section 1.4. The main analysis takes place in Chapter 4. In Chapter 5, we consider some model variations and their impact on the results of Chapter 4.

Chapters 2, 3, and 4 are self-contained. Since Chapter 5 is closely related to Chapter 4, it is only "partly self-contained".

Since the aim of this thesis is an axiomatic analysis of allocation and reallocation models, throughout the remainder of the thesis, axioms, or properties, are typed in *slanted*.

Part I

Fair allocation

Chapter 2

Fair allocation and single-peakedness[1]

We consider the allocation, or division, of some perfectly divisible commodity among a group of agents with single-peaked preferences. This means that each agent has a most preferred share, his peak, below which and above which his welfare is decreasing. A typical example is rationing in a two-good exchange economy when prices are in disequilibrium (*e.g.*, Benassy 1982): if the preferences of the agents over the two-dimensional space of bundles are strictly convex, then the restrictions of these preferences to the budget lines are single-peaked. In this context Benassy (1982) considered the so-called uniform rationing scheme. For the more general class of allocation problems with single-peaked preferences, this solution is known as the uniform rule.

Sprumont (1991) initiated the axiomatic analysis of this class of problems and proved that the uniform rule is the only allocation rule that satisfies *Pareto optimality*, *anonymity*, and *strategy-proofness*. In a second characterization he showed that *anonymity* can be replaced by *no-envy*. Since then, a wide literature has been concerned with the search for rules with appealing properties. An important conclusion of this research is that the uniform rule is now accepted as the most important rule for allocation problems with single-peaked preferences. It is an extension of this rule to allocation problems when agents have individual endowments, the so-called uniform reallocation rule, that we study extensively in the second part of the thesis. Before we do so, however, we give a short review of the most important results concerning the uniform rule.

This chapter is organized as follows. In Section 2.1 we introduce the model and two allocation rules that capture different notions of fairness: the proportional and the uniform rule. Since the uniform rule turns out to be the central allocation rule when preferences are single-peaked, in Section 2.2 we introduce several of its properties. Furthermore, we give an overview of the most important characterizations of this rule.[2]

[1]This chapter is mainly based on the following papers: Ching (1992, 1994), Dagan (1996), Klaus (1997a), Sönmez (1994), Sprumont (1991), and Thomson (1994a,b, 1995a, 1997a,b).

[2]We refer to these characterizations in Part II where we extend the model to situations where agents have individual endowments and axiomatically study so-called reallocation rules. It turns out that in

Finally, in Section 2.3 we give the answers to two open questions raised by William Thomson in connection with two characterizations of the uniform rule (Thomson 1995a, 1997a): by means of a single example, we give a positive answer to the question whether "*replication-invariance*" in these characterizations is independent from the other characterizing properties.

2.1 The model

There is an infinite population of potential agents, indexed by the natural numbers $\mathbb{N}$. Each agent $i \in \mathbb{N}$ is equipped with a continuous and single-peaked preference relation R_i defined over the non-negative real numbers $\mathbb{R}_+$. Single-peakedness of R_i means that there exists a point $p(R_i) \in \mathbb{R}_+$, called the *peak of agent* i, with the following property: for all $x, y \in \mathbb{R}_+$ with $x < y \leq p(R_i)$ or $x > y \geq p(R_i)$, we have $y \, P_i \, x$.[3] Each preference relation R_i can be described in terms of the *preference function* $r_i : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ that assigns to each point $x \in \mathbb{R}_+$ the point on the other side of $p(R_i)$ that is indifferent to x, if such a point exists, and otherwise 0 or ∞, depending on which side of the peak x is. So, for all $x \in \mathbb{R}_+$ the following holds.

$$\text{If } x \leq p(R_i), \text{ then } r_i(x) \geq p(R_i) \text{ and } \begin{cases} r_i(x) \, I_i \, x & \text{if such a point exists, or} \\ r_i(x) = \infty & \text{otherwise.} \end{cases}$$

$$\text{If } x \geq p(R_i), \text{ then } r_i(x) \leq p(R_i) \text{ and } \begin{cases} r_i(x) \, I_i \, x & \text{if such a point exists, or} \\ r_i(x) = 0 & \text{otherwise.} \end{cases}$$

By $\mathcal{R}$ we denote the class of all continuous, single-peaked preference relations over $\mathbb{R}_+$ and by $\mathcal{R}_b \subsetneq \mathcal{R}$ the subclass of preference relations such that for all $R_i \in \mathcal{R}_b$, the corresponding preference function r_i is bounded, *i.e.*, $r_i(0) < \infty$. We call the elements of $\mathcal{R}_b$ *bounded*. By $\mathcal{N}$ we denote the class of non-empty and finite subsets of $\mathbb{N}$. For $N \in \mathcal{N}$, $\mathcal{R}^N$ denotes the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$; $\mathcal{R}_b^N$ has a similar meaning. For profile $R \in \mathcal{R}^N$, $p(R) = (p(R_i))_{i \in N}$ denotes the corresponding *vector of peaks*.

Now, an economy is formalized as follows. Let $\Omega \in \mathbb{R}_+$ be the amount of an infinitely divisible commodity, or the *(social) endowment*, that has to be distributed among a group of agents $N \in \mathcal{N}$ with profile $R \in \mathcal{R}^N$.[4] We call a pair $e = (R, \Omega) \in \mathcal{R}^N \times \mathbb{R}_+$ an *economy*. Let $\mathcal{E}^N = \mathcal{R}^N \times \mathbb{R}_+$ be the class of all economies. Similarly, let $\mathcal{E}_b^N = \mathcal{R}_b^N \times \mathbb{R}_+$. A *feasible allocation for* $e = (R, \Omega) \in \mathcal{E}^N$ is a vector $x \in \mathbb{R}_+^N$ such that $\sum_N x_i = \Omega$. An *allocation rule* φ, or a *rule* for short, is a function that assigns to every $N \in \mathcal{N}$ and

this model of fair reallocation, many results of the allocation model we review in this chapter can be extended.

[3] As usual, $x \, R_i \, y$ is interpreted as "x is weakly preferred to y", and $x \, P_i \, y$ as "x is strictly preferred to y". Furthermore, $x \, I_i \, y$ means that agent i is indifferent between x and y.

[4] Note that free disposal of the commodity is not allowed.

every $e \in \mathcal{E}^N$ a feasible allocation, denoted $\varphi(e)$.[5] Given $i \in N$, we call $\varphi_i(e)$ the *share* of agent i.

A standard requirement for rules is *Pareto optimality:* an allocation assigned by the rule cannot be changed in such a way that no agent is worse off and some agent is better off.

Pareto optimality: For all $N \in \mathcal{N}$ and all $e = (R, \Omega) \in \mathcal{E}^N$, there is no feasible allocation $x \in \mathbb{R}^N_+$, such that for all $i \in N$, $x_i \, R_i \, \varphi_i(e)$, and for some $j \in N$, $x_j \, P_j \, \varphi_j(e)$.

By the single-peakedness of the preferences it is easy to show that a rule is *Pareto optimal* if and only if it is *same-sided*, that is: for all $N \in \mathcal{N}$ and all $e = (R, \Omega) \in \mathcal{E}^N$, either [for all $i \in N$, $\varphi_i(e) \leq p(R_i)$] or [for all $i \in N$, $\varphi_i(e) \geq p(R_i)$]. Sprumont (1991) uses same-sidedness as definition of *Pareto optimality*.

The following two rules embody two different well-known notions of fairness and equity in the literature on fair division, namely the principles of proportionality and of equality.

Young (1988), who investigated the problem of fair division in taxation situations, quotes Aristotle, *Ethics*, Book V, as follows. "What is just ... is what is proportional and what is unjust is what violates the proportion." In our model, the rule that incorporates Aristotle's notion of justice is the proportional rule.

Proportional rule *Pr*: For all $N \in \mathcal{N}$ and all $e = (R, \Omega) \in \mathcal{E}^N$,

$$Pr(e) := \begin{cases} (\frac{\Omega}{|N|}, \ldots, \frac{\Omega}{|N|}) & \text{if } p(R) = (0, \ldots, 0) \in \mathbb{R}^N, \\ \lambda p(R) & \text{otherwise,} \end{cases}$$

where $\lambda \in \mathbb{R}_+$ solves $\sum_N Pr_i(e) = \Omega$.

The next well-known principle of equity we focus on is equal division. However, taking the single-peakedness of the preferences into account, it turns out that the "equal division allocation" might not be *Pareto optimal*. In our opinion, the following rule incorporates the idea of equal division under the additional assumption of *Pareto optimality:* the amount to divide is distributed as equally as possible with all agents' peaks as either lower or upper bounds, depending on whether there is more or less to distribute than the sum of the peaks.

Uniform Rule *U*: For all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $j \in N$,

$$U_j(e) := \begin{cases} \min\{p(R_j), \lambda\} & \text{if } \sum_N p(R_i) \geq \Omega, \\ \max\{p(R_j), \lambda\} & \text{if } \sum_N p(R_i) \leq \Omega, \end{cases}$$

[5]Note that we impose the extra condition of *single-valuedness* on an allocation rule. Most of the results we present here originally were established for *multi-valued* allocation rules; see for instance Ching (1992), Sönmez (1994), and Thomson (1994a,b, 1995a, 1997a). Alternatively, for most results, we could allow for so-called single-plateaued preferences (see for instance Ching 1992) or restrict preferences to the closed interval $[0, \Omega]$.

where $\lambda \in \mathbb{R}_+$ solves $\sum_N U_i(e) = \Omega$.

So, in case of excess demand, *i.e.*, $\sum_N p(R_i) \geq \Omega$, each agent either receives his optimal share or a share that is greater than or equal to the share of each other agent. Similarly, in case of excess supply, *i.e.*, $\sum_N p(R_i) \leq \Omega$, each agent either receives his optimal share or a share that is smaller than or equal to the share of each other agent.[6] Examples demonstrating the computation of the uniform rule U are given in Chapter 1, Examples 1.1 and 1.2.

Another interpretation of this rule is the following "Walrasian interpretation". All agents are asked to maximize their preferences subject to a common upper or lower "budget bound" that is chosen such that feasibility is obtained for this list of maximal shares.[7]

As the following remarks show, the principles underlying the two rules we introduced are also well-known in other models.

Remark 2.1 ***Bankruptcy and taxation:*** Consider the situation where a man dies and leaves behind an estate E which is worth less than the sum of his debts. Every claimant i has a claim of size d_i on E and the problem is how to divide the total estate E among the claimants. Let $(d, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$, where $E \leq \sum_N d_i$, denote such a bankruptcy problem. Aumann and Maschler (1985) study, among other "Talmudic solutions", the so-called constraint equal award solution CEA. For every bankruptcy problem (d, E) it is defined by $CEA(d, E) = (\min\{d_1, \lambda\}, \ldots, \min\{d_n, \lambda\})$, where λ is determined by feasibility. In words, this means that all claimants obtain the same award λ, except that those who claim less than λ get their claim. With a slight abuse of notation (replacing the preference profile by a vector of claims) we can state that $CEA(d, E) = U(d, E)$. Dagan and Volij (1993) associate each bankruptcy problem with a bargaining problem and prove that the Nash bargaining solution induces the constrained equal award solution. Since we can associate in a similar way a bargaining problem to each economy with excess demand, for this subclass of economies, the Nash bargaining solution implies the uniform rule (see also Remark 2.2).

Methods for solving bankruptcy problems are formally equivalent to methods of taxation or loss distribution. In a taxation problem, the total tax that has to be paid corresponds to the estate that has to be divided and the taxable income of the individuals correspond to the individual claims. In this setting the proportional rule is applied under the name "flat tax" and the uniform allocation is applied under the name "head tax" (see, for instance, Young 1987, 1988).[8]

Another class of problems that are closely related are cost sharing problems (see for instance Moulin 1987a, 1997). In a cost sharing problem, the total costs of a

[6]Schummer and Thomson (1997) show that the uniform rule assigns the unique Pareto optimal allocation at which the difference between the smallest and the largest share received by any agent is minimal. Furthermore they show that the uniform rule assigns the unique Pareto optimal allocation at which the variance of the agents' shares is minimal.

[7]I wish to thank William Thomson for bringing this "Walrasian interpretation" of the uniform rule to my attention.

[8]For a survey on bankruptcy and taxation, see Thomson (1995c).

public facility correspond to the estate that has to be divided and the benefits of the individuals correspond to the individual claims. □

Remark 2.2 ***Bargaining and game theory:*** Otten, Peters, and Volij (1996) associate with each economy an auxiliary bargaining problem in such a way that its set of Pareto optimal allocations coincides with the set of feasible and Pareto optimal allocations for the economy. They show that the uniform allocation of the economy coincides with both the Nash and lexicographic egalitarian solution outcomes of the associated bargaining problem. Furthermore, they provide two characterizations of the uniform rule, resembling characterizations of the Nash and lexicographic egalitarian solutions.

De Frutos and Massó (1995) show that the uniform rule is the unique rule that selects a Pareto optimal allocation that "Lorenz dominates" all Pareto optimal allocations. Next, they construct for each economy an auxiliary cooperative game with transferable utility and show that the core of this game equals the set of Pareto optimal allocations of the economy. Furthermore, the game is convex. Since for convex games the lexicographic egalitarian solution as introduced by Dutta and Ray (1989) selects the point in the core that is Lorenz maximal, it follows that the uniform allocation coincides with the Dutta-Ray solution outcome. □

For more examples of rules we refer the interested reader to a survey of fair allocation when preferences are single-peaked by Thomson (1997b), Chapter 11: "Single-peaked preferences". As a conclusion of this survey Thomson states:

> *"Our [...] conclusion is that for this model one solution comes out repeatedly as being best-behaved from a variety of viewpoints. The solution is the uniform rule."*

In the remainder of this chapter we focus on the uniform rule. In the next section we review most of its desirable properties and characterizations.

2.2 The uniform rule: properties and characterizations

One of the most important features of the uniform rule is the fact that it satisfies *strategy-proofness*, one of the properties first analyzed for (allocation) rules by Sprumont (1991). *Strategy-proofness* states that no agent ever benefits from misrepresenting his preferences.[9] Before we give the formal definition, we introduce some notation.

Let $N, M \in \mathcal{N}$ be such that $N \subseteq M$. Let $M \backslash N = \{i \in M \mid i \notin N\}$. For $R \in \mathcal{R}^M$ the *restriction* $(R_i)_{i \in N} \in \mathcal{R}^N$ *of* R *to* N is denoted by R_N. Let $R, \bar{R} \in \mathcal{R}^N$, $i \in N$, and $R_{N \backslash \{i\}} = \bar{R}_{N \backslash \{i\}}$. Then, we call $\bar{R}$ an *i-deviation from R.*

[9] In game theoretical terms a rule is *strategy-proof* if in the direct revelation game form it is a weakly dominant strategy for each agent to announce his true preference relation. Thomson (1990) discusses manipulability and implementation of allocation rules when preferences are single-peaked.

Strategy-proofness: For all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, all $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R, then $\varphi_j(e)\, R_j\, \varphi_j(\bar{e})$.

As already observed by Benassy (1982) in the context of rationing, the uniform rule satisfies *strategy-proofness*. On the other hand, it is easy to show that the proportional rule is not *strategy-proof*.

The next two properties Sprumont (1991) considers are well-known properties. *Anonymity* requires that the agents' shares do not depend on their names. *No-envy* states that no agent strictly prefers the share of another agent to his own share.

Let $N \in \mathcal{N}$. A *permutation* π *on* N is a bijective function $\pi : N \to N$. By Π^N we denote the *set of all permutations on* N. Given $\pi \in \Pi^N$ and $e = (R, \Omega) \in \mathcal{R}^N$, let $R_\pi = (R_{\pi(i)})_{i \in N}$ and $e_\pi = (R_\pi, \Omega)$.

Anonymity: For all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, all $\pi \in \Pi^N$, and all $i \in N$, $\varphi_i(e_\pi) = \varphi_{\pi(i)}(e)$.

No-envy: For all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $i, j \in N$, $\varphi_i(e)\, R_i\, \varphi_j(e)$.

No-envy can be traced back to Foley (1967) who considers it in the context of resource allocation problems.

The following two characterizations are due to Sprumont (1991).

Theorem 2.1 *The uniform rule U is the only rule that satisfies Pareto optimality, strategy-proofness, and anonymity.*

Theorem 2.2 *The uniform rule U is the only rule that satisfies Pareto optimality, strategy-proofness, and no-envy.*

Ching (1992) gives a simple and elegant proof of this last result.

Thomson (1994a) shows that the uniform rule is the only rule that satisfies *Pareto optimality*, *no-envy*, and *peak-onliness*, *i.e.*, the rule only depends on the peaks instead of the complete profile.

Peak-onliness: For all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^N$, if $p(R) = p(\bar{R})$, then $\varphi(e) = \varphi(\bar{e})$.

Theorem 2.3 *The uniform rule U is the only rule that satisfies Pareto optimality, no-envy, and peak-onliness.*

The next property we introduce is *equal treatment of equals*, a weakening of *no-envy*. It requires that if two agents have the same preference relation, then each of them is indifferent between his share and the other agent's share.

Equal treatment of equals: For all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $i, j \in N$, if $R_i = R_j$, then $\varphi_i(e)\, I_i\, \varphi_j(e)$.

If φ is in addition *Pareto optimal*, then *equal treatment of equals* implies the stronger equality condition $\varphi_i(e) = \varphi_j(e)$. Note that *Pareto optimality* and *anonymity* together imply *equal treatment of equals*.

The following strengthening of Theorems 2.1 and 2.2 is due to Ching (1994).

Theorem 2.4 *The uniform U rule is the only rule that satisfies Pareto optimality, strategy-proofness, and equal treatment of equals.*

A summary of the characterizations of the uniform rule by *Pareto optimality*, *strategy-proofness*, and either *peak-onliness* or one of the equity properties *anonymity*, *no-envy*, or *equal treatment of equals*, can be found in Table 2.1.

	Th 2.1	Th 2.2	Th 2.3	Th 2.4
Pareto optimality	×	×	×	×
strategy-proofness	×	×		×
anonymity	×			
no-envy		×	×	
peak-onliness			×	
equal treatment of equals				×

Table 2.1: Characterizations of the uniform rule by *Pareto optimality*, *strategy-proofness*, *peak onliness*, and equity properties.

Next, we discuss so-called "solidarity properties" that describe the effect of certain changes in a single parameter of the economy while the other parameters are kept fixed, *e.g.*, the endowment, the population, or the preferences. If after such a change either all agents that are not responsible for the change (weakly) lose together or all (weakly) gain together, then, depending on which parameter has been changed, the rule satisfies *resource-monotonicity*, *population-monotonicity*[10], or *welfare-domination under preference-replacement*[11]. Thomson (1994a, 1995a, 1997a) shows that these properties are generally incompatible with the properties we introduced above. However, he shows that these incompatibilities only occur when the change in the parameter is such that it turns an economy in which there is "too much" to divide into an economy in which there is "too little" to divide, or *vice versa*. We call a change where this does not

[10]Thomson (1983) introduced *population-monotonicity* in the context of bargaining. For a survey on *population-monotonicity*, see Thomson (1995b).

[11]Moulin (1987b) introduced this "replacement principle" in the context of binary choice with quasi-linear preferences. He calls it "agreement". Thomson (1993, 1996a, 1997a) studies the replacement principle in different settings, *e.g.*, in public and private good economies with single-peaked preferences.

occur *one-sided:* let $N, \bar{N} \in \mathcal{N}$, $e = (R, \Omega) \in \mathcal{E}^N$, and $\bar{e} = (\bar{R}, \bar{\Omega}) \in \mathcal{E}^{\bar{N}}$. If either $[\sum_N p(R_i) \geq \Omega$ and $\sum_{\bar{N}} p(\bar{R}_i) \geq \bar{\Omega}]$ or $[\sum_N p(R_i) \leq \Omega$ and $\sum_{\bar{N}} p(\bar{R}_i) \leq \bar{\Omega}]$, then $\bar{e}$ is a *one-sided change* of e.

Thomson shows that a wide class of rules satisfies the "one-sided" versions of the solidarity properties, *i.e.*, solidarity is only required for one-sided changes. However, the uniform rule turns out to be essentially the only rule that satisfies any of the one-sided solidarity properties, together with *Pareto optimality* and *no-envy*.

First, we consider a one-sided change in the endowment, keeping the set of agents and their preferences fixed (Thomson 1994a). *One-sided resource-monotonicity* states that after such a change, either all agents (weakly) lose or all (weakly) gain.

One-sided resource-monotonicity: For all $N \in \mathcal{N}$, all $e = (R, \Omega)$, and all $\bar{e} = (R, \bar{\Omega}) \in \mathcal{E}^N$, if $\bar{e}$ is a one-sided change of e, then either [for all $i \in N$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in N$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

Next, we consider the arrival of new agents so that this change in the set of agents is one-sided, keeping the preferences of the remaining agents and the endowment fixed (Thomson 1995a). *One-sided population-monotonicity* states that either all agents initially present (weakly) lose or all (weakly) gain.

One-sided population-monotonicity: For all $N, \bar{N} \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^{\bar{N}}$, if $N \subseteq \bar{N}$, $R = \bar{R}_N$, and $\bar{e}$ is a one-sided change of e, then either [for all $i \in N$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in N$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

Finally, we consider a one-sided change of one agent's preference relation, keeping the preferences of the remaining agents, the set of agents, and the endowment fixed (Thomson 1997a). *One-sided welfare-domination under preference-replacement* states that either all remaining agents (weakly) lose or all (weakly) gain.

One-sided welfare-domination under preference-replacement, or one-sided replacement-domination: For all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, all $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R and $\bar{e}$ is a one-sided change of e, then either [for all $i \in N \backslash \{j\}$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in N \backslash \{j\}$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

In the sequel we use the shorter phrase of *one-sided replacement-domination* for *one-sided welfare-domination under preference-replacement*.

On the domain of economies with bounded preferences, $\bigcup_{N \in \mathcal{N}} \mathcal{E}_b^N$, Thomson (1994a) establishes the following characterization of the uniform rule.

Theorem 2.5 *On the domain of economies with bounded preferences the uniform rule U is the only rule that satisfies Pareto optimality, no-envy, and one-sided resource-monotonicity.*

Similar characterizations of the uniform rule by *Pareto optimality*, *no-envy*, and one of the remaining solidarity conditions can be derived if the following well-known condition of *replication-invariance* is added (Thomson 1994b, 1995a, 1997a). *Replication-invariance* states that if an economy is replicated, *i.e.*, the endowment and the preference profile are replicated, then also the allocation assigned by the rule must be replicated. In order to formally describe a replicated economy, we introduce the following notation.

Let $k \in \mathbb{N}$ and $\bar{N}, N_1, \ldots, N_k \in \mathcal{N}$ such that for all $l \in \{1, \ldots k\}$, $N_l \neq \emptyset$. Then, the collection $\{N_1, \ldots, N_k\}$ constitutes a *partition* of $\bar{N}$ if $\bar{N} = \bigcup_{l \in \{1,\ldots k\}} N_l$ and $N_l \cap N_m = \emptyset$ for all $l, m \in \{1, \ldots k\}$, $l \neq m$.

Let $N, \bar{N} \in \mathcal{N}$, $e = (R, \Omega) \in \mathcal{E}^N$, and $\bar{e} = (\bar{R}, \bar{\Omega}) \in \mathcal{E}^{\bar{N}}$ be such that

(i) there exist $k \in \mathbb{N}$ and $N_1, \ldots, N_k \in \mathcal{N}$ such that $N = N_1$ and $\{N_1, \ldots, N_k\}$ constitutes a partition of $\bar{N}$,

(ii) there exist k bijective functions, indexed by $l \in \{1, \ldots k\}$, $b_l \colon N \to N_l$, such that for all $i \in N$, $R_i = \bar{R}_{b_l(i)}$,[12]

(iii) $\bar{\Omega} = k\Omega$.

Then, $\bar{e}$ is a *k-replica of e*. Let $\bar{e} \in \mathcal{E}^{\bar{N}}$ be a k-replica of $e \in \mathcal{E}^N$ and let $x \in \mathbb{R}^N_+$. If $\bar{x} \in \mathbb{R}^{\bar{N}}_+$ is such that for all $i \in N$, $x_i = \bar{x}_{b_l(i)}$ for the functions b_l, $l \in \{1, \ldots k\}$, as described in **(ii)**, then $\bar{x}$ a *k-replica of x*.

Replication-invariance: For all $N, \bar{N} \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $\bar{e} = (\bar{R}, \bar{\Omega}) \in \mathcal{E}^{\bar{N}}$, if $\bar{e}$ is a k-replica of e, then $\varphi(\bar{e})$ is a k-replica of $\varphi(e)$.

Note that if a rule is *replication-invariant*, then is satisfies *anonymity*.

On the domain of economies with bounded preferences, $\bigcup_{N \in \mathcal{N}} \mathcal{E}^N_b$, Thomson (1995a) establishes the following characterization of the uniform rule.

Theorem 2.6 *On the domain of economies with bounded preferences, the uniform rule U is the only rule that satisfies Pareto optimality, no-envy, one-sided population-monotonicity, and replication-invariance.*

Question 1: Following the proof of this Theorem, Thomson asks the question whether *replication-invariance* is independent from the other characterizing properties (see Thomson 1995a, page 242).[13]

[12] Let $l \in \{1, \ldots k\}$ and $i \in N$. Then, agent $b_l(i)$ is the "replica", or the "clone", of agent $i \in N$ in N_l.

[13] Thomson also asks the question whether the uniform rule is the only rule that satisfies *Pareto optimality*, *individual rationality from equal division*, *one-sided population-monotonicity*, and *replication-invariance*.

The answer is that there is a large class of rules satisfying the properties mentioned in the question. Two examples are the "proportional reallocation rule" Pr^r operated from equal division (see Section 4.2) and the "maximally satiating reallocation rule" $\varphi^{\max}$ operated from equal division (see Section 4.6).

	Th 2.5	Th 2.6	Th 2.7
Pareto optimality	×	×	×
no-envy	×	×	×
o-s resource-monotonicity	×		
o-s population-monotonicity		×	
o-s replacement-domination			×
replication-invariance		×	×

Table 2.2: Characterizations of the uniform rule by *Pareto optimality*, *no-envy*, and solidarity properties.

We answer this question in the next section.

The next characterization of the uniform rule is due to Thomson (1997a).

Theorem 2.7 *The uniform rule U is the only rule that satisfies Pareto optimality, no-envy, one-sided replacement-domination, and replication-invariance.*

Question 2: Following the proof of this Theorem, Thomson asks the question whether *replication-invariance* is independent from the other characterizing properties (see Thomson 1997a, page 161).

We answer this question in the next section.

A summary of the characterizations of the uniform rule by *Pareto optimality*, *no-envy*, and one of the solidarity properties *one-sided resource-monotonicity*, *one-sided population-monotonicity*, or *one-sided replacement-domination*, can be found in Table 2.2.

The next property we introduce is *consistency*. Essentially it describes the stability of a recommended allocation in situations where agents leave with their shares: let $N \in \mathcal{N}$ be a group of agents, $e = (R, \Omega) \in \mathcal{E}^N$ an economy for N, and $x \in \mathbb{R}^N_+$ the allocation assigned by the rule. Let $M \subseteq N$ and suppose all agents in $N \backslash M$ leave with the shares assigned to them at x. By doing so they create the *reduced economy* $(R_M, \sum_M x_i) \in \mathcal{E}^M$. Now, by *consistency*, applying the rule to the reduced economy yields the same shares for the remaining agents in M as in the original economy.[14]

Let $M, N \in \mathcal{N}$ such that $M \subseteq N$. For $x \in \mathbb{R}^N$, the *restriction* $(x_i)_{i \in M} \in \mathbb{R}^M$ *of* x *to* M is denoted by x_M.

Consistency: For all $M, N \in \mathcal{N}$ with $M \subseteq N$ and all $e = (R, \Omega) \in \mathcal{E}^N$, $\varphi(e)_M = \varphi(R_M, \sum_M \varphi_i(e))$.

[14]For a survey on *consistency*, see Thomson (1996b).

A weakening of *consistency* is *bilateral consistency* which is obtained from the former condition by limiting its application to subgroups of remaining agents of size two.

Bilateral consistency: For all $M, N \in \mathcal{N}$ with $M \subseteq N$ and $|M| = 2$ and all $e = (R, \Omega) \in \mathcal{E}^N$, $\varphi(e)_M = \varphi(R_M, \sum_M \varphi_i(e))$.

The next characterization of the uniform rule contains the following notion of "*endowment continuity*".

Ω-continuity: For all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all sequences $\{\Omega^k\}_{k \in \mathbb{N}}$ in $\mathbb{R}_+$, if $\lim_{k \to \infty} \Omega^k = \Omega$, then $\lim_{k \to \infty} \varphi(R, \Omega^k) = \varphi(R, \Omega)$.

The next characterization of the uniform rule is due to Thomson (1994b).

Theorem 2.8 *The uniform rule U is the only rule that satisfies Pareto optimality, no-envy, bilateral consistency, and Ω-continuity.*

Next, Thomson (1994b) shows that *no-envy* can be replaced by the following "minimal right" condition: each agent (weakly) prefers his share to the "equal division share" of the economy.

Individual rationality from equal division: For all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $i \in N$, $\varphi_i(e)\, R_i\, \frac{\Omega}{|N|}$.

Theorem 2.9 *The uniform rule U is the only rule that satisfies Pareto optimality, individual rationality from equal division, bilateral consistency, and Ω-continuity.*

Furthermore, Thomson (1994b) shows that in Theorem 2.9, *Ω-continuity* can be replaced by *replication-invariance*.

Theorem 2.10 *The uniform rule U is the only rule that satisfies Pareto optimality, individual rationality from equal division, consistency, and replication-invariance.*

Dagan (1996) presents the following results that can be seen as variants of Theorems 2.8 and 2.9. In contrast with Thomson (1994b), Dagan only considers single-valued rules. Under this "extra assumption" he shows that even if the set of potential agents is bounded from above, but contains at least four agents, the uniqueness of the uniform rule as established in Theorems 2.8, 2.9, and 2.10 is achieved without assuming *Ω-continuity*, or *replication-invariance* respectively.

Theorem 2.11 *Let the set of potential agents contain at least four agents. Then, the uniform rule U is the only rule that satisfies Pareto optimality, no-envy, and bilateral consistency.*

	Th 2.8	Th 2.9	Th 2.10	Th 2.11	Th 2.12	Th 2.13
Pareto optimality	×	×	×	×	×	×
consistency			×			
bilateral consistency	×	×		×	×	
converse consistency						×
individual rationality from equal division		×	×		×	×
no-envy	×			×		
anonymity						×
Ω-continuity	×					
replication-invariance		×	×			

Table 2.3: Characterizations of the uniform rule by *Pareto optimality*, consistency properties, *etc.*

Theorem 2.12 *Let the set of potential agents contain at least four agents. Then, the uniform rule U is the only rule that satisfies Pareto optimality, individual rationality from equal division, and bilateral consistency.*

The last property we introduce is *converse consistency*, a property that determines the desirability of an allocation for an economy on the basis of the desirability of its restrictions to all two-agent reduced economies. So, given a feasible allocation x for an economy, if for every associated two-agent reduced economy the restriction of x to these two agents equals the allocation chosen by the rule for this two-agent reduced economy, then x must be the allocation assigned by the rule.[15]

Converse consistency: For all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all feasible allocations $x \in \mathbb{R}^N_+$, if for all $M \subseteq N$ with $|M| = 2$, $\varphi(R_M, \sum_M x_i) = x_M$, then $\varphi(e) = x$.

The next characterization of the uniform rule is due to Thomson (1994b).

Theorem 2.13 *The uniform rule U is the only rule that satisfies Pareto optimality, individual rationality from equal division, converse consistency, and anonymity.*

A summary of the characterizations of the uniform rule by *Pareto optimality*, *individual rationality from equal division*, *no-envy*, and one of the consistency properties *consistency*, *bilateral consistency*, or *converse consistency*, can be found in Table 2.3.

The following characterizations of the uniform rule, due to Sönmez (1994), do not rely on *Pareto optimality*, a condition that until now was always included.

Theorem 2.14 *The uniform rule U is the only rule that satisfies individual rationality from equal division, one-sided resource-monotonicity, and consistency.*

[15]For a survey on *converse consistency*, see Thomson (1996b).

	Th 2.14	Th 2.15	Th 2.16
individual rationality from equal division	×	×	×
o-s resource-monotonicity	×		×
o-s population-monotonicity		×	
consistency	×	×	
converse consistency			×
replication-invariance		×	

Table 2.4: Characterizations of the uniform rule without *Pareto optimality*.

Theorem 2.15 *The uniform rule U is the only rule that satisfies individual rationality from equal division, one-sided population-monotonicity, consistency, and replication-invariance.*

Theorem 2.16 *The uniform rule U is the only rule that satisfies individual rationality from equal division, one-sided resource-monotonicity, and converse consistency.*

A summary of these last characterizations of the uniform rule without *Pareto optimality* can be found in Table 2.4.

Finally, even though the list of characterizations of the uniform rule we reviewed in this section is impressive (see Tables 2.1–2.4), we cannot claim completeness. For further properties and characterizations of the uniform rule that are related to bargaining theory and cooperative game theory we refer the interested reader to the references given in Remark 2.2.

2.3 The independence of replication-invariance

In this section we answer the open questions whether in the characterizations of the uniform rule stated in Theorems 2.6 and 2.7 *replication-invariance* is independent from the remaining properties (see Section 2.2, Questions 1 and 2). We do so by constructing a rule that satisfies *Pareto optimality*, *no-envy*, *one-sided population-monotonicity*, *one-sided replacement-domination*, but *not replication-invariance*.

Before we define the "absorbing agent" rule φ^a, we introduce some notation.[16] Let $N \in \mathcal{N}$. Then, by $\mathcal{A}^N \subsetneq \mathcal{E}^N$ we denote the following subclass of economies: for $(R, \Omega) \in \mathcal{A}^N$ we have that (i) $\sum_N p(R_i) < \Omega$ and (ii) there exists an agent $j \in N$ such that $p(R_j) > \frac{\Omega}{2}$.

Hence, for $N \in \mathcal{N}$ and $e = (R, \Omega) \in \mathcal{A}^N$ there is (i) too much to distribute and (ii) there is exactly one agent, namely agent j, with a relatively large peak (see Figure 2.1). For such an economy, we construct the allocation $\varphi^a(e)$ as follows.

[16]The a of φ^a stands for "answer", or alternatively for "absorbing agent".

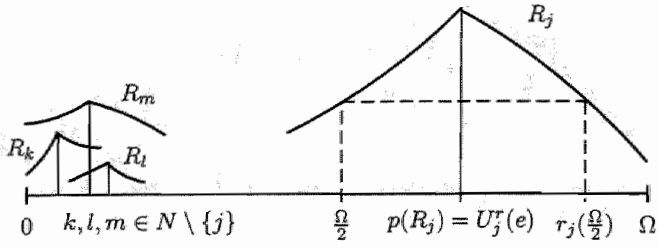

Figure 2.1: An economy $e \in \mathcal{A}^N$.

We start from the uniform allocation $U(e)$ and let agent j "absorb" some part of the excess supply the remaining agents, $i \in N\backslash\{j\}$, experience at their uniform shares, denoted by the list $(U_i(e))_{i \in N\backslash\{j\}}$.

We denote the *excess supply of the agents in* $N\backslash\{j\}$ *at the allocation* $U(e)$ by $s_j(e) = \sum_{N\backslash\{j\}}(U_i(e) - p(R_i))$.

Note that in order to preserve *same-sidedness*, we let agent j absorb at most $s_j(e)$. However, when we let agent j absorb part of, or even the whole excess supply $s_j(e)$, we do not want him to envy the other agents. Therefore, we introduce the sufficiently small upper bound $r_j(\frac{\Omega}{2})$ for agent j's share. Recall that either $r_j(\frac{\Omega}{2})\, I_j\, \frac{\Omega}{2}$ or $r_j(\frac{\Omega}{2}) = \infty$. Note that the shares of all other agents are smaller than $\frac{\Omega}{2}$ and therefore, as long as agent j's share is in between his peak $p(R_j)$ and $r_j(\frac{\Omega}{2})$, he does not envy the other agents.

Summarizing, if $N \in \mathcal{N}$ and $e = (R, \Omega) \in \mathcal{A}^N$, then we obtain $\varphi^a(e)$ from $U(e)$ by letting agent j absorb the amount $a(e) := \min\{s_j(e), r_j(\frac{\Omega}{2}) - p(R_j)\}$ and subtracting this amount uniformly from the uniform shares of the other agents (see Figures 2.2 and 2.3). If $N \in \mathcal{N}$ and $e = (R, \Omega) \notin \mathcal{A}^N$, then we apply the uniform rule.

The rule φ^a: For all $N \in \mathcal{N}$ and all $e = (R, \Omega) \in \mathcal{A}^N$, let $j \in N$ be such that $p(R_j) > \frac{\Omega}{2}$. Then, for all $i \in N$,

$$\varphi_i^a(e) = \begin{cases} p(R_j) + a(e) & \text{if } i = j, \\ U_i(R_{N\backslash\{j\}}, \Omega - \varphi_j^a(e)) & \text{if } i \in N\backslash\{j\}. \end{cases}$$

For all $N \in \mathcal{N}$ and all $e = (R, \Omega) \notin \mathcal{A}^N$,

$$\varphi^a(e) = U(e).$$

The rule φ^a is by construction *same-sided* and therefore *Pareto optimal*. Since we either assign the uniform allocation to an economy or change the uniform allocation in such a way that envy between the agents cannot arise, φ^a satisfies *no-envy*. Furthermore, it is easy to check that the rule φ^a satisfies other well-known properties for division rules, such as *anonymity*, *continuity with respect to the preferences* (Sprumont 1991), *Ω-continuity*, and *individual rationality from equal division*. Given this list of properties

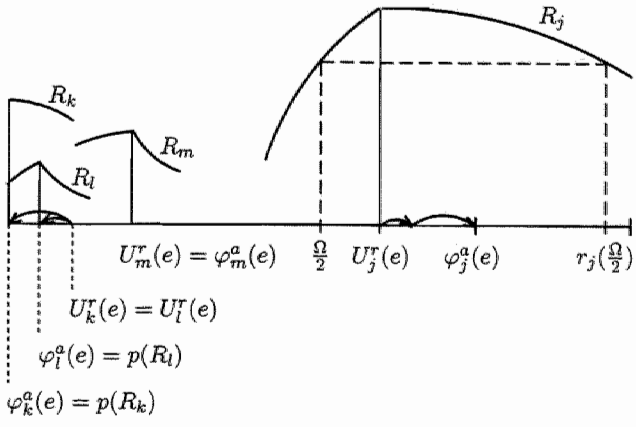

Figure 2.2: The rule φ^a for an economy $e \in \mathcal{A}^N$ with $a(e) = s_j(e)$.

and comparing it to the list of properties given in the characterizations of the uniform rule we recalled in Section 2.2 (*e.g.*, Theorems 2.1, 2.3, 2.5, 2.9, and 2.13) it follows immediately that φ^a satisfies none of the properties *strategy-proofness*, *peak-onliness*, *one-sided resource-monotonicity*, *bilateral consistency*, and *converse consistency*.

The following example demonstrates that the rule φ^a does not satisfy *replication-invariance*.

Example 2.1 Let $N = \{1, 2\}$ and $R \in \mathcal{R}^N$ be such that $p(R) = (1, 4)$, and $r_2(3) = 5$. Let $\Omega = 6$ and consider the economy $e = (R, \Omega)$. Thus, $p(R_1) + p(R_2) < \Omega$ and $p(R_2) > \frac{\Omega}{2}$. Hence, $(R, \Omega) \in \mathcal{A}^N$ and $U(e) = (2, 4)$, $\varphi^a(e) = (1, 5)$.

Let $N_1 = N$ and $N_2 = \{3, 4\}$ be such that $R_3 = R_1$ and $R_4 = R_2$. Let $\bar{N} = N_1 \cup N_2$, $\bar{R} = (R_1, R_2, R_3, R_4)$, and $\bar{\Omega} = 12$. Hence, $(\bar{R}, \bar{\Omega}) \in \mathcal{E}^{\bar{N}}$ is a 2-replica of (R, Ω). Since there exists no $i \in \bar{N}$ such that $p(R_i) > \frac{\bar{\Omega}}{2} = 6$, it follows that $\bar{e} = (\bar{R}, \bar{\Omega}) \notin \mathcal{A}^{\bar{N}}$ and $U(\bar{e}) = (2, 4, 2, 4) = \varphi^a(\bar{e})$. Thus, $\varphi^a(\bar{e})$ is not a 2-replica of $\varphi^a(\bar{e})$. Hence, φ^a is not *replication-invariant*. ◇

Remark 2.3 Note that for all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{A}^N$, all $k \in \mathbb{N}$, and all k-replica $\bar{e} = (\bar{R}, \bar{\Omega}) \in \mathcal{E}^{\bar{N}}$ of e it follows that $\bar{e} \notin \mathcal{A}^{\bar{N}}$ and that $\varphi^a(\bar{e})$ is not a k-replica of $\varphi^a(e)$.□

Lemma 2.1 *The rule φ^a satisfies one-sided population-monotonicity.*

Proof. Let $N, \bar{N} \in \mathcal{N}$, $e = (R, \Omega) \in \mathcal{E}^N$, and $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^{\bar{N}}$ be such that $N \subseteq \bar{N}$ and $R = \bar{R}_N$. Now, it is sufficient to show that, if $\bar{e}$ is a one-sided change of e, then either (i) [for all $i \in N$, $\varphi_i^a(e)\, R_i\, \varphi_i^a(\bar{e})$] or (ii) [for all $i \in N$, $\varphi_i^a(\bar{e})\, R_i\, \varphi_i^a(e)$]. Let $\bar{e}$ be a one-sided change of e.

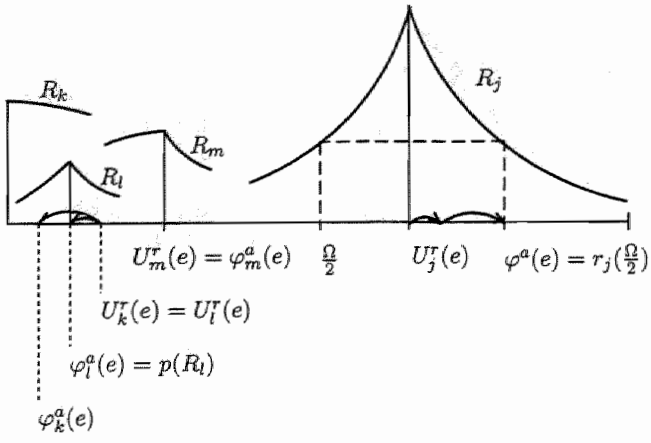

Figure 2.3: The rule φ^a for an economy $e \in \mathcal{A}^N$ with $a(e) = r_j(\frac{\Omega}{2}) - p(R_j)$.

Case 1: $e \notin \mathcal{A}^N$, $\bar{e} \notin \mathcal{A}^{\bar{N}}$. Then, $\varphi^a(e) = U(e)$ and $\varphi^a(\bar{e}) = U(\bar{e})$. Since, the uniform rule is *one-sided population-monotonic*, either (i) or (ii) holds.

Case 2: $e \in \mathcal{A}^N$, $\bar{e} \notin \mathcal{A}^{\bar{N}}$. Hence, $\sum_N p(R_i) < \Omega$ and there exists $j \in N$ such that $p(R_j) > \frac{\Omega}{2}$. Then, $\sum_{\bar{N}} p(\bar{R}_i) \leq \Omega$. Since, $\bar{e} \notin \mathcal{A}^{\bar{N}}$, we have $\sum_{\bar{N}} p(\bar{R}_i) = \Omega$. So, by *Pareto optimality*, for all $i \in N$, $\varphi_i^a(\bar{e}) = p(R_i)$. This implies (ii).

Case 3: $e \notin \mathcal{A}^N$, $\bar{e} \in \mathcal{A}^{\bar{N}}$. Hence, $\sum_N p(R_i) \leq \sum_{\bar{N}} p(\bar{R}_i) < \Omega$ and there exists $j \in \bar{N} \backslash N$ such that $p(\bar{R}_j) > \frac{\Omega}{2}$. Let $R' = \bar{R}_{\bar{N}\backslash\{j\}} \in \mathcal{R}^{\bar{N}\backslash\{j\}}$. Then, for all $i \in N$,

$$\varphi_i^a(e) = U_i(e) \text{ and } \varphi_i^a(\bar{e}) = U_i(R', \Omega - \varphi_j^a(\bar{e})). \tag{2.1}$$

Since the uniform rule is *one-sided resource-monotonic* and (R', Ω), $(R', \Omega - \varphi_j^a(\bar{e})) \in \mathcal{E}^{\bar{N}\backslash\{j\}}$ are such that $\sum_{\bar{N}\backslash\{j\}} p(R'_i) \leq \Omega - \varphi_j^a(\bar{e}) < \Omega$, it follows that for all $i \in N \subseteq \bar{N}\backslash\{j\}$,

$$U_i(R', \Omega - \varphi_j^a(\bar{e}))\, R_i\, U_i(R', \Omega). \tag{2.2}$$

Since the uniform rule is *one-sided population-monotonic* and $e = (R, \Omega) \in \mathcal{E}^N$, $(R', \Omega) \in \mathcal{E}^{\bar{N}\backslash\{j\}}$ are such that $N \subseteq \bar{N}\backslash\{j\}$, $R = R'_N$, and $\sum_{\bar{N}\backslash\{j\}} p(R'_i) \leq \Omega$, it follows that for all $i \in N$,

$$U_i(R', \Omega)\, R_i\, U_i(e). \tag{2.3}$$

Thus, (2.1), (2.2), and (2.3) together imply that for all $i \in N$, $\varphi_i^a(\bar{e})\, R_i\, \varphi_i^a(e)$. Hence, (ii) holds.

Case 4: $e \in \mathcal{A}^N$, $\bar{e} \in \mathcal{A}^{\bar{N}}$. Hence, $\sum_N p(R_i) \leq \sum_{\bar{N}} p(\bar{R}_i) < \Omega$ and there exists $j \in N \subseteq \bar{N}$ such that $p(R_j) > \frac{\Omega}{2}$. Let $R' = R_{N\backslash\{j\}} \in \mathcal{R}^{N\backslash\{j\}}$ and $\bar{R}' = \bar{R}_{\bar{N}\backslash\{j\}} \in \mathcal{R}^{\bar{N}\backslash\{j\}}$. Then,

$$\varphi_i^a(e) \;=\; \begin{cases} p(R_j) + a(e) & \text{if } i = j, \\ U_i(R', \Omega - \varphi_j^a(e)) & \text{if } i \in N\backslash\{j\}, \end{cases} \tag{2.4}$$

$$\varphi_i^a(\bar{e}) \;=\; \begin{cases} p(R_j) + a(\bar{e}) & \text{if } i = j, \\ U_i(\bar{R}', \Omega - \varphi_j^a(\bar{e})) & \text{if } i \in \bar{N}\backslash\{j\}. \end{cases}$$

Note that $U_j(e) = p(R_j) = U_j(\bar{e})$. Therefore, $\Omega - U_j(e) = \sum_{N\backslash\{j\}} U_i(e) = \sum_{\bar{N}\backslash\{j\}} U_i(\bar{e})$ and $s_j(e) \geq s_j(\bar{e})$. Hence, $a(e) \geq a(\bar{e})$ and $p(R_j) \leq \varphi_j^a(\bar{e}) \leq \varphi_j^a(e)$. Thus,

$$\varphi_j^a(\bar{e})\, R_j\, \varphi_j^a(e). \tag{2.5}$$

Now, we have either

(a) $a(\bar{e}) = s_j(\bar{e})$ or

(b) $a(\bar{e}) = r_j(\frac{\Omega}{2}) - p(R_j)$.

Suppose (a) holds. Then, for all $i \in N\backslash\{j\}$, $\varphi_i^a(\bar{e}) = p(R_i)$. Thus, for all $i \in N\backslash\{j\}$,

$$\varphi_i^a(\bar{e})\, R_i\, \varphi_i^a(e). \tag{2.6}$$

Hence, (2.5) and (2.6) imply (ii).

Suppose (b) holds. Then, $a(\bar{e}) = a(e)$ and $\Omega - \varphi_j^a(\bar{e}) = \Omega - \varphi_j^a(e)$. Since the uniform rule is *one-sided population-monotonic* and $(R', \Omega - \varphi_j^a(e)) \in \mathcal{E}^{N\backslash\{j\}}$, $(\bar{R}', \Omega - \varphi_j^a(\bar{e})) \in \mathcal{E}^{\bar{N}\backslash\{j\}}$ are such that $N\backslash\{j\} \subseteq \bar{N}\backslash\{j\}$, $R' = \bar{R}'_{N\backslash\{j\}}$, and $\sum_{\bar{N}\backslash\{j\}} p(\bar{R}_i) \leq \Omega - \varphi_j^a(e)$, it follows that for all $i \in N\backslash\{j\}$,

$$U_i(\bar{R}', \Omega - \varphi_j^a(\bar{e}))\, R_i\, U_i(R', \Omega - \varphi_j^a(e)). \tag{2.7}$$

Thus, by (2.4), for all $i \in N\backslash\{j\}$,

$$\varphi_i^a(\bar{e})\, R_i\, \varphi_i^a(e). \tag{2.8}$$

Hence, (2.5) and (2.8) imply (ii). ■

Lemma 2.2 *The rule φ^a satisfies one-sided replacement-domination.*

Proof. Let $N \in \mathcal{N}$, $e = (R, \Omega)$, $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^N$, and $k \in N$ be such that $\bar{R}$ is a k-deviation from R. It is sufficient to show that if $\bar{e}$ is a one-sided change of e, then either (i) [for all $i \in N\backslash\{k\}$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or (ii) [for all $i \in N\backslash\{k\}$, $\varphi_i^a(\bar{e})\, R_i\, \varphi_i^a(e)$]. Let $\bar{e}$ be a one-sided change of e.

Case 1: $e \notin \mathcal{A}^N$, $\bar{e} \notin \mathcal{A}^N$. Then, $\varphi^a(e) = U(e)$ and $\varphi^a(\bar{e}) = U(\bar{e})$. Since, the uniform rule satisfies *one-sided replacement-domination*, either (i) or (ii) holds.

Case 2: $e \notin \mathcal{A}^N$, $\bar{e} \in \mathcal{A}^N$. Hence, $\bar{R}$ and R are such that $p(R_k) \leq \frac{\Omega}{2}$, $p(\bar{R}_k) > \frac{\Omega}{2}$, and $\sum_N p(R_i) \leq \sum_N p(\bar{R}_i) < \Omega$. Thus, $U_k(e) \leq \frac{\Omega}{2} < p(\bar{R}_k) \leq \varphi_k^a(\bar{R}, \Omega)$. Let $R' = R_{N\backslash\{k\}} \in \mathcal{R}^{N\backslash\{k\}}$. Then, for all $i \in N\backslash\{k\}$,

$$\varphi_i^a(e) = U_i(e) = U_i(R', \Omega - U_k(e)) \text{ and } \varphi_i^a(\bar{e}) = U_i(R', \Omega - \varphi_k^a(\bar{e})). \tag{2.9}$$

Since the uniform rule is *one-sided resource-monotonic* and $(R', \Omega - U_k(e))$, $(R', \Omega - \varphi_k^a(\bar{e})) \in \mathcal{E}^{N\backslash\{k\}}$ are such that $\sum_{N\backslash\{k\}} p(R_i) \leq \Omega - \varphi_k^a(\bar{e}) \leq \Omega - U_k(e)$, it follows that for all $i \in N\backslash\{k\}$,

$$U_i(R', \Omega - \varphi_k^a(\bar{e}))\, R_i\, U_i(R', \Omega - U_k(e)). \tag{2.10}$$

Hence, (2.9) and (2.10) imply (ii).

Case 3: $e \in \mathcal{A}^N$, $\bar{e} \notin \mathcal{A}^N$. Similar to Case 2 by interchanging the roles of $\bar{R}$ and R.

Case 4: $e \in \mathcal{A}^N$, $\bar{e} \in \mathcal{A}^N$. Without loss of generality we assume that $\bar{R}$ is a k-deviation from R such that $p(\bar{R}_k) \geq p(R_k)$. Hence, $\sum_N p(R_i) \leq \sum_N p(\bar{R}_i) < \Omega$ and there exists $j \in N$ such that $p(R_j) > \frac{\Omega}{2}$ and $p(\bar{R}_j) > \frac{\Omega}{2}$. Let $R' = R_{N\setminus\{j\}} \in \mathcal{R}^{N\setminus\{j\}}$ and $\bar{R}' = \bar{R}_{N\setminus\{j\}} \in \mathcal{R}^{N\setminus\{j\}}$. Then,

$$\begin{aligned} \varphi_i^a(e) &= \begin{cases} p(R_j) + a(e) & \text{if } i = j, \\ U_i(R', \Omega - \varphi_j^a(e)) & \text{if } i \in N\setminus\{j\}, \end{cases} \\ \varphi_i^a(\bar{e}) &= \begin{cases} p(\bar{R}_j) + a(\bar{e}) & \text{if } i = j, \\ U_i(\bar{R}', \Omega - \varphi_j^a(\bar{e})) & \text{if } i \in N\setminus\{j\}. \end{cases} \end{aligned} \tag{2.11}$$

Case 4.1: $k = j$. Suppose $\varphi_j^a(e) \leq \varphi_j^a(\bar{e})$. Since the uniform rule is *one-sided resource-monotonic* and $(R', \Omega - \varphi_j^a(e))$, $(\bar{R}', \Omega - \varphi_j^a(\bar{e})) \in \mathcal{E}^{N\setminus\{j\}}$ are such that $\sum_{N\setminus\{j\}} p(R_i) \leq \Omega - \varphi_j^a(\bar{e}) \leq \Omega - \varphi_j^a(e)$, it follows that for all $i \in N\setminus\{j\}$,

$$U_i(\bar{R}', \Omega - \varphi_j^a(\bar{e}))\, R_i\, U_i(R', \Omega - U_j(e)). \tag{2.12}$$

Hence, (2.11) and (2.12) imply (ii).

Suppose $\varphi_j^a(e) \geq \varphi_j^a(\bar{e})$. Then, by similar arguments as for the opposite inequality, *one-sided resource-monotonicity* implies (i).

Case 4.2: $k \neq j$. Since, $p(\bar{R}_k) \geq p(R_k)$ it follows that $s_j(e) \geq s_j(\bar{e})$. Hence, $a(e) \geq a(\bar{e})$ and $\varphi_j^a(\bar{e}) \leq \varphi_j^a(e)$. Thus,

$$\varphi_j^a(\bar{e})\, R_j\, \varphi_j^a(e). \tag{2.13}$$

Now, we have either

(a) $a(\bar{e}) = s_j(\bar{e})$ or

(b) $a(\bar{e}) = r_j(\frac{\Omega}{2}) - p(R_j)$.

Suppose (a) holds. Hence, for all $i \in N\setminus\{j\}$, $\varphi_i^a(\bar{e}) = p(R_i)$. Thus, for all $i \in N\setminus\{j,k\}$,

$$\varphi_i^a(\bar{e})\, R_i\, \varphi_i^a(e). \tag{2.14}$$

Hence, (2.13) and (2.14) imply (ii).

Suppose (b) holds. Hence, $a(\bar{e}) = a(e)$ and $\Omega - \varphi_j^a(\bar{e}) = \Omega - \varphi_j^a(e)$. Since the uniform rule satisfies *one-sided replacement-domination* and $(R', \Omega - \varphi_j^a(e))$, $(\bar{R}', \Omega - \varphi_j^a(\bar{e})) \in \mathcal{E}^{N\setminus\{j\}}$, $k \in N$, are such that $p(\bar{R}_k) \geq p(R_k)$ and $R'_{N'\setminus\{k\}} = \bar{R}'_{N'\setminus\{k\}}$, it follows that for all $i \in N'\setminus\{k\} = N\setminus\{j,k\}$,

$$U_i(\bar{R}', \Omega - \varphi_j^a(\bar{e}))\, R_i\, U_i(R', \Omega - \varphi_j^a(e)). \tag{2.15}$$

Thus, by (2.11) and (2.15), for all $i \in N\setminus\{j,k\}$,

$$\varphi_i^a(\bar{e})\, R_i\, \varphi_i^a(e). \tag{2.16}$$

Hence, (2.13) and (2.16) imply (ii). ■

Now, Lemmas 2.1 and 2.2, together with Example 2.1 establish the answers to Questions 1 and 2.

Chapter 3

Fair allocation and single-dippedness[1]

We consider the allocation, or division, of some perfectly divisible commodity among a group of agents with single-dipped preferences. This means that each agent has a worst share, his dip, below which and above which his welfare is increasing. For example, consider two types of work which have negative cross-effects like, perhaps, teaching and management in a university: combinations of the two types of work may be less preferred than one-type tasks. Yet another example is a two-goods exchange economy with fixed prices and strictly quasi-convex utility functions, which induce single-dipped preferences on the budget lines.

In a public good context Vickrey (1960) refers to the class of single-dipped preferences as "single-troughed" preferences and mentions that this restricted domain has been much less studied than the single-peaked domain. Inada (1964) studies single-dipped preferences over triples of alternatives.[2] Peremans and Storcken (1997) consider the problem of locating a public good with strongly negative externalities. These externalities induce single-dipped preferences on the set of admissible locations. To the best of our knowledge, single-dippedness has only been considered for public good situations.

In Chapter 2 we reviewed the literature on fair allocation when preferences are single-peaked. The uniform rule turned out to satisfy many appealing properties such as *Pareto optimality*, *strategy-proofness*, *no-envy*, *anonymity*, *one-sided population-monotonicity*, *one-sided replacement-domination*, and *consistency*.

As we will show in this chapter, assuming in a private good context that preferences are single-dipped instead of single-peaked changes the results significantly. It turns out that the whole unit must be assigned to a single agent if the allocation rule satisfies *Pareto optimality*, *strategy-proofness*, and one of the properties *replacement-domination*, *consistency*, and *population-monotonicity*. Furthermore, if we impose *Pareto optimality*, other conditions such as *anonymity* or *no-envy* are generally incompatible.

This chapter is organized as follows. In Section 3.1 we introduce the model and give

[1]This chapter is based on Klaus, Peters, and Storcken (1997a) and Klaus (1997a).

[2]In Inada (1964) single-dipped preferences are called single-caved.

a description of all *Pareto optimal* allocations for a given economy. In Section 3.2 we prove that for a fixed set of agents an allocation rule that satisfies *Pareto optimality*, *strategy-proofness*, and *replacement-domination* always assigns the whole amount of the commodity to a single agent. Using this result, we derive a characterization of all allocation rules satisfying these properties. In Section 3.3, this characterization is extended to a variable population setting. In addition, replacing *replacement-domination* by either *consistency* or *population-monotonicity* yield similar characterizations. Remarkably, with only minor adjustments in notation, the characterizations we obtain here are also valid for economies where one indivisible commodity has to be allocated among a group of agents. Since here we are concerned with the allocation of an infinitely divisible commodity, we postpone these results to Appendix A. In Section 3.4 we establish the independence of the properties in the characterizations. Furthermore, we analyze the compatibility of *Pareto optimality* with other properties, mainly *anonymity*, *no-envy*, *equal-treatment*, and *converse consistency*.

3.1 The model

The allocation model for agents with single-dipped preferences is essentially the same as the allocation model for agents with single-peaked preferences we introduced in Chapter 2. In order to keep this chapter self-contained, we reintroduce the whole model and restate the central properties we explore. However, for more details concerning the motivation and the background of the properties we analyze in this section, we refer to Chapter 2.

We start our analysis of this new class of economies with the following fixed population set-up. One unit of some infinitely divisible commodity has to be divided among a set $N = \{1, \ldots, n\}$ of agents.[3] Each agent $i \in N$ is equipped with a single-dipped preference relation R_i defined over the closed interval $[0, 1]$.[4] Single-dippedness of R_i means that there exists a point $d(R_i) \in [0, 1]$, called the *dip of agent* i, with the following property: for all $x, y \in [0, 1]$ with $x < y \leq d(R_i)$ or $x > y \geq d(R_i)$ we have $x \, P_i \, y$.[5]

By $\mathcal{D}$ we denote the class of all single-dipped preference relations over $[0, 1]$. Let $R = (R_1, \ldots, R_n) \in \mathcal{D}^N$ be a *(preference) profile*. Since, for the time being, the set of agents and the amount to divide are fixed we simply designate an *economy* by a profile of preference relations. Thus, the class of economies equals $\mathcal{D}^N$.

Let $\chi^1, \chi^2, \ldots, \chi^n$ denote the unit vectors in $\mathbb{R}^N$, where $\chi^i_j = 1$ if $j = i$ and $\chi^i_j = 0$ if $j \neq i$. Let $X = \{\chi^1, \ldots, \chi^n\}$ and denote by $\mathcal{X}$ its convex hull. Note that $\mathcal{X}$ equals the $(n-1)$-dimensional unit simplex. Then, a *feasible allocation* for $R \in \mathcal{D}^N$ is a vector $x \in \mathcal{X}$. An *allocation rule* Φ, or a *rule* for short, is a function that assigns to every

[3]Note that free disposal of the commodity is not allowed.

[4]Without loss of generality, we may assume that the agents' preference relations are continuous as well.

[5]As usual, $x \, R_i \, y$ is interpreted as "x is weakly preferred to y", and $x \, P_i \, y$ as "x is strictly preferred to y". Furthermore, $x \, I_i \, y$ means that agent i is indifferent between x and y.

$R \in \mathcal{D}^N$ a feasible allocation, denoted $\Phi(R)$. Given $i \in N$, we call $\Phi_i(R)$ the *share* of agent i.

As in Chapter 2, we are interested in rules that assign *Pareto optimal* allocations: an allocation $y \in \mathcal{X}$ is *Pareto optimal* if there is no feasible allocation $x \in \mathcal{X}$ such that for all $i \in N$, $x_i \, R_i \, y_i$, and for some $j \in N$, $x_j \, P_j \, y_j$.

Pareto optimality: For all $R \in \mathcal{D}^N$, there is no feasible allocation $x \in \mathcal{X}$ such that for all $i \in N$, $x_i \, R_i \, \Phi_i(R)$, and for some $j \in N$, $x_j \, P_j \, \Phi_j(R)$.

Just as in the model with single-peaked preferences (see Section 2.1, page 19), there is a simple description of *Pareto optimality* for our model. For this description we introduce the following "partition" of the agents. For an economy $R \in \mathcal{D}^N$ denote by $N_+(R) = \{i \in N \mid 1 \, P_i \, 0\}$ the set of agents who strictly prefer 1 to 0, by $N_0(R) = \{i \in N \mid 0 \, I_i \, 1\}$ the set of agents who are indifferent between 0 and 1, and by $N_-(R) = \{i \in N \mid 0 \, P_i \, 1\}$ the set of agents who strictly prefer 0 to 1.

Hence, for all $R \in \mathcal{D}^N$, $N = N_+(R) \cup N_0(R) \cup N_-(R)$ and the sets $N_+(R)$, $N_0(R)$, and $N_-(R)$ are pairwise disjoint.[6]

Lemma 3.1 *A rule Φ is Pareto optimal if and only if for all $R \in \mathcal{D}^N$ the following holds.*

Case 1 : *If $N_+(R) \neq \emptyset$, then*
for all $i \notin N_+(R)$, $\Phi_i(R) = 0$ and
for all $i \in N_+(R)$, either $\Phi_i(R) = 0$ or $\Phi_i(R) \, P_i \, 0$.[7]

Case 2 : *If $N_+(R) = \emptyset$ and $N_0(R) \neq \emptyset$, then*
for some $j \in N_0(R)$, $\Phi(R) = \chi^j$.

Case 3 : *If $N_-(R) = N$, then*
for all $i \in N$, either $\Phi_i(R) = 1$ or $\Phi_i(R) \, P_i \, 1$.[8]

Proof. Let Φ be a rule and let $R \in \mathcal{D}^N$. It is easy to show that all allocations described in Lemma 3.1 are *Pareto optimal*. It remains to show that they are the only *Pareto optimal* allocations. Let Φ be *Pareto optimal* and let $y = \Phi(R)$.

Case 1: $N_+(R) \neq \emptyset$. Suppose, by contradiction, that $\bar{N} = \{i \in N \mid \text{either } [i \notin N_+(R) \text{ and } y_i \neq 0] \text{ or } [i \in N_+(R) \text{ and } y_i \neq 0, \, 0 \, R_i \, y_i]\} \neq \emptyset$.

Suppose that there exists an agent $j \in N_+(R) \backslash \bar{N}$ such that $y_j \neq 0$. Then, define $x \in \mathcal{X}$ as follows. For $i \in \bar{N}$, $x_i = 0$. For $i \notin \bar{N}$ and $i \neq j$, $x_i = y_i$. Finally, $x_j = y_j + \sum_{\bar{N}} y_i$. The allocation x is a Pareto improvement over y: for all $i \in N$, $x_i \, R_i \, y_i$ and $x_j \, P_j \, y_j$.

Suppose that there exists no agent $j \in N_+(R) \backslash \bar{N}$ such that $y_j \neq 0$. Then, let $k \in N_+(R) \subseteq \bar{N}$ and $x = \chi^k$. The allocation x is a Pareto improvement over y: for all $i \in N$, $x_i \, R_i \, y_i$ and $x_k \, P_k \, y_k$.

[6] However, since some of the sets $N_+(R)$, $N_0(R)$, and $N_-(R)$ are possibly empty, strictly speaking, $\{N_+(R), N_0(R), N_-(R)\}$ may not constitute a partition.

[7] See Figure 3.1 (a).

[8] See Figure 3.1 (b).

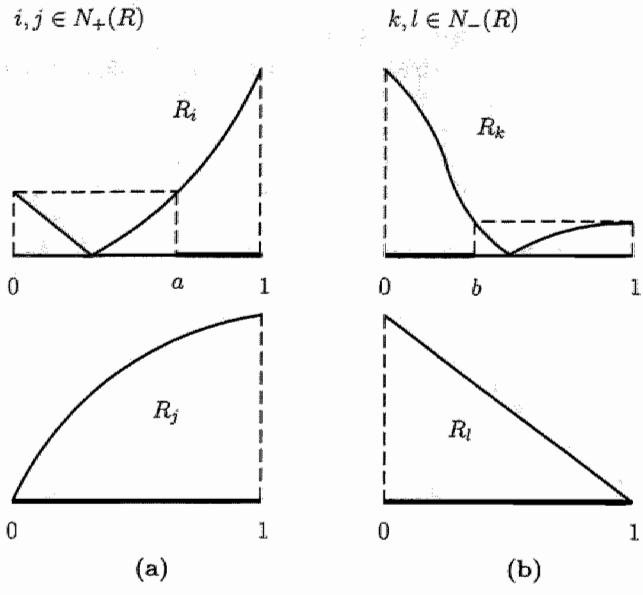

Figure 3.1: Some Pareto optimal shares as described in Lemma 3.1.
(a) Lemma 3.1, Case 1: $N_+(R) \neq \emptyset$. Possible Pareto optimal shares for agents $i, j \in N_+(R)$, x_i, x_j, are : $x_i \in \{0\} \cup (a, 1]$ and $x_j \in [0, 1]$.
(b) Lemma 3.1, Case 3: $N_-(R) = N$. Possible Pareto optimal shares for agents $k, l \in N_-(R)$, y_k, y_l, are : $y_k \in [0, b) \cup \{1\}$ and $y_l \in [0, 1]$.

Case 2: $N_+(R) = \emptyset$ and $N_0(R) \neq \emptyset$. It is easy to see that for all $j \in N_0(R)$, $k \in N_-(R)$, χ^j Pareto dominates χ^k. Now, suppose by contradiction that $\bar{N} = \{i \in N \mid y_i \notin \{0, 1\}\} \neq \emptyset$. Let $j \in \bar{N}$. Then, χ^j is a Pareto improvement over y: for all $i \in N$, $\chi^j_i \, R_i \, y_i$ and $\chi^j_j \, P_j \, y_j$.

Case 3: $N_-(R) = N$. Suppose by contradiction that $\bar{N} = \{i \in N \mid y_i \neq 1$ and $1 \, R_i \, y_i\} \neq \emptyset$. Let $j \in \bar{N}$ and $k \in N$ be such that $y_k \neq 0$. Then, χ^j is a Pareto improvement over y: for all $i \in N$, $\chi^j_i \, R_i \, y_i$ and $\chi^j_k \, P_k \, y_k$. ■

3.2 Fixed population

In this section, in addition to *Pareto optimality*, we are interested in *strategy-proofness* and *replacement-domination*. *Strategy-proofness* states that no agent ever benefits from misrepresenting his preferences (see also Section 2.2, page 22). Before we give the formal definition, we introduce some notation.

Let N, M be such that $M \subseteq N$. Let $N \backslash M = \{i \in N \mid i \notin M\}$. For $R \in \mathcal{D}^N$ the

restriction $(R_i)_{i\in M} \in \mathcal{D}^M$ *of* R *to* M is denoted by R_M. Let $R, \bar{R} \in \mathcal{D}^N$, $i \in N$, and $R_{N\setminus\{i\}} = \bar{R}_{N\setminus\{i\}}$. Then, we call $\bar{R}$ an *i-deviation from* R. By $(R_{-i}, \bar{R}_i)$ we denote the profile obtained from R by replacing R_i by $\bar{R}_i$.

Strategy-proofness: For all $R, \bar{R} \in \mathcal{D}^N$ and all $j \in N$, if $\bar{R}$ is a j-deviation from R, then $\Phi_j(R)\, R_j\, \Phi_j(\bar{R})$.

If under *replacement-domination* an agent's preference relation is "replaced" by some other admissible preference relation, then this unilateral change affects all remaining agents in the same direction, *i.e.*, these agents all (weakly) gain or all (weakly) lose as a result of the replacement (see also Section 2.2, page 24).

Replacement-domination: For all $R, \bar{R} \in \mathcal{D}^N$ and all $j \in N$, if $\bar{R}$ is a j-deviation from R, then either [for all $i \in N\setminus\{j\}$, $\Phi_i(R)\, R_i\, \Phi_i(\bar{R})$] or [for all $i \in N\setminus\{j\}$, $\Phi_i(\bar{R})\, R_i\, \Phi_i(R)$].

Note that *replacement-domination* is trivially satisfied if $|N| = 2$.

The following example shows that the class of *Pareto optimal* and *strategy-proof* rules that does not satisfy *replacement-domination* also contains rules that rely on information that we do not consider as crucial for the problem at hand.

Example 3.1 The following rule $\tilde{\Phi}$ satisfies *Pareto optimality* and *strategy-proofness*. Without loss of generality, we define $\tilde{\Phi}(R)$ for $N = \{1, 2, 3\}$.
Case 1 : $N_+(R) \neq \emptyset$. If $N_+(R) = \{i\}$, then $\tilde{\Phi}(R) = \chi^i$.
If $|N_+(R)| > 1$, then

$$\tilde{\Phi}(R) = \begin{cases} \chi^1 & \text{if } 1\, P_1\, 0, \\ \chi^2 & \text{if } 0\, R_1\, 1 \text{ and } d(R_1) \in \mathbb{Q}, \\ \chi^3 & \text{if } 0\, R_1\, 1 \text{ and } d(R_1) \notin \mathbb{Q}. \end{cases}$$

Case 2 : $N_+(e) = \emptyset$ and $N_0(e) \neq \emptyset$. Then $\tilde{\Phi}(R) = \chi^i$, $i \in N_0(e)$.
Case 3 : $N = N_-(e)$. Then

$$\tilde{\Phi}(R) = \begin{cases} \chi^2 & \text{if } d(R_1) \in \mathbb{Q}, \\ \chi^3 & \text{if } d(R_1) \notin \mathbb{Q}. \end{cases}$$

◇

For an example of a rule that satisfies *Pareto optimality*, *strategy-proofness*, but not *replacement-domination* and that does not always assign the whole unit of the commodity to a single agent, we refer to Section 3.4, Example 3.6.

Next, we show that a rule Φ that satisfies *Pareto optimality*, *strategy-proofness*, and *replacement-domination* assigns the whole unit to a single agent.

Let $\mathcal{B}_\Phi = \{R \in \mathcal{D}^N \mid \Phi(R) \notin X\}$ be the set of economies that yield a *broken allocation* under Φ, *i.e.*, a feasible allocation where no agent obtains the whole unit.

Lemma 3.2 *Let Φ be a rule that satisfies Pareto optimality, strategy-proofness, and replacement-domination. Then, $\mathcal{B}_\Phi = \emptyset$.*

Proof. Let Φ satisfy the properties listed in the lemma and suppose, by contradiction, that $\mathcal{B}_\Phi \neq \emptyset$.

Because $\mathcal{B}_\Phi \neq \emptyset$ and N is finite, there exists a profile $R \in \mathcal{B}_\Phi$ with a maximal number of agents who strictly prefer 0 to 1. So, let R be broken under Φ and for all $R' \in \mathcal{B}_\Phi$, $|N_-(R)| \geq |N_-(R')|$. Let $B_\Phi(R)$ denote the set of agents with broken shares at $\Phi(R)$, *i.e.*, $B_\Phi(R) = \{i \in N \mid \Phi_i(R) \notin \{0,1\}\}$. Hence, $B_\Phi(R) \neq \emptyset$ and $|B_\Phi(R)| \geq 2$.

Because the sets $N_+(R)$, $N_0(R)$, and $N_-(R)$ constitute a partition of N, one of the following cases occurs.

Case 1: $N_+(R) \neq \emptyset$. By *Lemma 3.1*, it follows that $B_\Phi(R) \subseteq N_+(R)$.

Let $i \in B_\Phi(R)$ and $(R_{-i}, \bar{R}_i)$ be an i-deviation from R such that $i \in N_-(R_{-i}, \bar{R}_i)$. Because the number of agents in $N_-(R)$ is maximal for $R \in \mathcal{B}_\Phi$, it follows that $(R_{-i}, \bar{R}_i) \notin \mathcal{B}_\Phi$. So, by *Pareto optimality* and $|B_\Phi(R)| \geq 2$, there is $j \in N_+(R)$ such that $\Phi(R_{-i}, \bar{R}_i) = \chi^j$. By *strategy-proofness*, $j \neq i$. *Replacement-domination* and $B_\Phi(R) \subseteq N_+(R)$ now imply $B_\Phi(R) = \{i, j\}$.

Next, consider an i-deviation R^1 from R, a j-deviation R^2 from R, and an i-deviation R^3 from R^2, such that $1\, P_i^1\, 0\, P_i^1\, \Phi_i(R)$, $1\, P_j^2\, 0\, P_j^2\, \Phi_j(R)$, and $R_i^3 = R_i^1$.

Strategy-proofness for the i-deviation R^1 from R implies $\Phi_i(R) \geq \Phi_i(R^1)$.

By $N_+(R) = N_+(R^1)$ and *Lemma 3.1* it follows that either $\Phi_i(R^1) = 0$ or $\Phi_i(R^1)\, P_i^1\, 0$. The latter can only be true if $\Phi_i(R^1) > \Phi_i(R)$, which contradicts *strategy-proofness*. Hence, $\Phi_i(R^1) = 0$. Note, that R^1 is an i-deviation from $(R_{-i}, \bar{R}_i)$. Thus, by $\Phi(R_{-i}, \bar{R}_i) = \chi^j$ and *replacement-domination*, it follows that $\Phi(R^1) = \chi^j$. By *strategy-proofness* for the j-deviation R^3 from R^1, we have $\Phi(R^3) = \chi^j$. Similarly, by interchanging the roles of i and j, it follows that $\Phi(R^3) = \chi^i$, which is a contradiction.

Case 2: $N_+(R) = \emptyset$ and $N_0(R) \neq \emptyset$. By *Lemma 3.1*, $\Phi(R) = \chi^j$ for some $j \in N_0(R)$. This contradicts $B_\Phi(R) \neq \emptyset$.

Case 3: $N_-(R) = N$. Let $\{i, j\} \in B_\Phi(R)$. Consider an i-deviation R^1 from R, a j-deviation R^2 from R, and an i-deviation R^3 from R^2, such that $0\, P_i^1\, 1 P_i^1\, \Phi_i(R)$, $0\, P_j^2\, 1\, P_j^2\, \Phi_j(R)$, and $R_i^3 = R_i^1$.

Strategy-proofness for the i-deviation R^1 implies $\Phi_i(R) \leq \Phi_i(R^1)$.

By $N_-(R^1) = N$ and *Lemma 3.1*, it follows that either $\Phi_i(R^1) = 1$ or $\Phi_i(R^1)\, P_i^1\, 1$. The latter can only be true if $\Phi_i(R) > \Phi_i(R^1)$, which contradicts *strategy-proofness*. Hence, $\Phi(R^1) = \chi^i$. By *strategy-proofness* and *replacement-domination*, for the j-deviation R^3 from R^1, we have $\Phi(R^3) = \chi^i$. Similarly, by interchanging the roles of i and j, it follows that $\Phi(R^3) = \chi^j$, which is a contradiction. ■

Lemma 3.2 states that for a rule that satisfies *Pareto optimality*, *strategy-proofness*, and *replacement-domination*, the whole unit is assigned to a single agent. The selection of this agent can be described as follows.

Fix a linear order of the set of agents. We ask the agent who is first in this order whether he prefers 1 to 0. If he prefers the whole unit, he receives it and we are done.

If not, we ask the second agent the same question; *etc.* Hence, the first agent according to the fixed order who prefers the whole unit to nothing receives it.

If no agent strictly prefers 1 to 0 and at least one agent is indifferent between 0 and 1, then the unit can be assigned "arbitrarily" to any of the indifferent agents.

If all agents strictly prefer 0 to 1, there exists a preselected agent who receives it. Loosely speaking, there is a "poor guy" who always receives the whole unit when nobody else wants it.

For a formal description, we introduce the following notation.

A *permutation* π *on* N is a bijective function $\pi: N \to N$. By Π^N we denote the *set of all permutations on* N. Since for $\pi \in \Pi^N$ and $i \in N$, $\pi(i)$ can also be interpreted as the position of i in a linear order of N, we alternatively refer to π as to the *linear order* on N. By G^N we denote the set of *choice functions* $g : \mathcal{D}^N \to N$ such that $g(R) \in N_0(R)$ if $N_0(R) \neq \emptyset$.

Selection $f_{\pi,g,k}$: Let $\pi \in \Pi^N$, $g \in G^N$, and $k \in N$. Then the *selection* $f_{\pi,g,k}: \mathcal{D}^N \to N$ (based on π, g, and k) is defined as follows. For all $R \in \mathcal{D}^N$:

Case 1: If $N_+(R) \neq \emptyset$, then $f_{\pi,g,k}(R) = \arg\min\{\pi(i) \mid i \in N_+(R)\}$.

Case 2: If $N_+(R) = \emptyset$ and $N_0(R) \neq \emptyset$, then $f_{\pi,g,k}(R) = g(R)$.

Case 3: If $N = N_-(R)$, then $f_{\pi,g,k}(R) = k$.

Now, we can characterize all rules that satisfy *Pareto optimality*, *strategy-proofness*, and *replacement-domination*.

Theorem 3.1 *A rule Φ satisfies Pareto optimality, strategy-proofness, and replacement-domination if and only if there exist $\pi \in \Pi^N$, $g \in G^N$, and $k \in N$ such that for all $R \in \mathcal{D}^N$,*

$$\Phi(R) = \chi^{f_{\pi,g,k}(R)}.$$

Proof. Let $\pi \in \Pi^N$, $g \in G^N$, and $k \in N$ be such that for all $R \in \mathcal{D}^N$, $\Phi(R) = \chi^{f_{\pi,g,k}(R)}$. We prove that Φ satisfies *Pareto optimality*, *strategy-proofness*, and *replacement-domination*.

By *Lemma 3.1* it follows that $\Phi(R) = \chi^{f_{\pi,g,k}(R)}$ is a *Pareto optimal* allocation.

Next, consider $R \in \mathcal{D}^N$ and an i-deviation $\bar{R}$ from R. In order to prove *strategy-proofness* we consider the following three cases.

Case 1: $i \in N_+(R)$. Hence, $N_+(R) \neq \emptyset$ and $\Phi(R) = \chi^{\arg\min\{\pi(j) \mid j \in N_+(R)\}}$. Let $l = \arg\min\{\pi(j) \mid j \in N_+(R)\}$. If $i \neq l$, then $\arg\min\{\pi(j) \mid j \in N_+(R)\} = \arg\min\{\pi(j) \mid j \in N_+(\bar{R})\}$. Hence, $\Phi(R) = \Phi(\bar{R})$. If $i = l$, then either $\Phi_i(R) = \Phi_i(\bar{R}) = 1$ or $\Phi_i(\bar{R}) = 0$. Hence, $\Phi_i(R)\, R_i\, \Phi_i(\bar{R})$.

Case 2: $i \in N_0(R)$. Because, by the definition of Φ, the whole unit is assigned to one of the agents in N and agent i is indifferent between receiving the whole unit and receiving nothing, $\Phi_i(R)\, R_i\, \Phi_i(\bar{R})$.

Case 3: $i \in N_-(R)$. If $\Phi_i(R) = 0$, agent i cannot improve his outcome by unilateral deviation. If $\Phi_i(R) = 1$, then by *Pareto optimality*, $N = N_-(R)$. So, i equals the

preselected agent of the selection $f_{\pi,g,k}$, *i.e.*, $i = k$. Thus, if $N_-(\bar{R}) = N$, $\Phi_i(R) = 1$. For $N_-(\bar{R}) \neq N$, $\Phi_i(\bar{R}) = 1$ follows by *Pareto optimality*. Hence, $\Phi_i(R)\, R_i\, \Phi_i(\bar{R})$.

Now, by Cases 1, 2, and 3 it follows that for all i-deviations $\bar{R}$ from R, $\Phi_i(R)\, R_i\, \Phi_i(\bar{R})$. This proves *strategy-proofness*.

Let $R \in \mathcal{D}^N$ and consider an i-deviation $\bar{R}$ from R. In order to prove *replacement-domination* we assume by contradiction that agents $l, m \neq i$ exist such that $\Phi_l(R)\, P_l\, \Phi_l(\bar{R})$ and $\Phi_m(\bar{R})\, P_m\, \Phi_m(\bar{R})$.

Because Φ assigns the whole amount to one of the agents, this can only occur if either $[l, m \in N_+(R)$ such that $\Phi_l(R) = \Phi_m(\bar{R}) = 1$ and $\Phi_l(\bar{R}) = \Phi_m(R) = 0]$ or $[l, m \in N_-(R)$ such that $\Phi_l(R) = \Phi_m(\bar{R}) = 0$ and $\Phi_l(\bar{R}) = \Phi_m(R) = 1]$. However, by the definition of Φ by means of the selection $f_{\pi,g,k}$, none of these changes can occur after a unilateral change $\bar{R}$ of an agent $i \neq l, m$. Hence, Φ satisfies *replacement-domination*.

The proof of the remaining part of the theorem is constructive. Suppose Φ is a rule satisfying the properties listed in the theorem. Then, we construct a selection $f_{\pi,g,k} \colon \mathcal{D}^N \to N$ based on **(a)** a permutation $\pi \in \Pi^N$, **(b)** a choice function $g \colon \mathcal{D}^N \to N$, and **(c)** an individual $k \in N$, such that for all $R \in \mathcal{D}^N$, $\Phi(R) = \chi^{f_{\pi,g,k}(R)}$.

By Lemma 3.2 it follows that $\mathcal{B}_\Phi = \emptyset$. Hence, for all $R \in \mathcal{D}^N$, $\Phi(R) = \chi^i$ for some $i \in N$.

(a) Definition of $\pi \in \Pi^N$.

(a.1) Let $R \in \mathcal{D}^N$ be such that $N_+(R) = N$. Then, $\Phi(R) = \chi^{i_1}$ for some $i_1 \in N$. Set $\pi(i_1) = 1$. By *strategy-proofness*, for all i_1-deviations R' from R with $i_1 \in N_+(R')$,

$$\Phi(R') = \chi^{i_1}. \tag{3.1}$$

Let $j \neq i_1$ and consider a j-deviation R'' from R. By *strategy-proofness* and *replacement-domination*,

$$\Phi(R'') = \chi^{i_1}. \tag{3.2}$$

Now, applying (3.1) and (3.2) repeatedly, it follows that for all profiles $\bar{R} \in \mathcal{D}^N$ with $i_1 \in N_+(\bar{R}) \subseteq N$,

$$\Phi(\bar{R}) = \chi^{i_1}.$$

(a.2) Next, let $R \in \mathcal{D}^N$ be such that $N_+(R) = N^1 := N \backslash \{i_1\}$. Then, $\Phi(R) = \chi^{i_2}$ for some $i_2 \in N^1$. Set $\pi(i_2) = 2$. Similarly as in **(a.1)** it can be shown that for all profiles $\bar{R} \in \mathcal{D}^N$ with $i_2 \in N_+(\bar{R}) \subseteq N^1$,

$$\Phi(\bar{R}) = \chi^{i_2}.$$

(a.3) Let $R \in \mathcal{D}^N$ be such that $N_+(R) = N^2 := N^1 \backslash \{i_2\}$. Then $\Phi(R) = \chi^{i_3}$ for some $i_3 \in N^2$. Set $\pi(i_3) = 3$. *etc.*

It is now clear how the linear order $\pi \in \Pi^N$ is constructed step by step by leaving out the agent receiving the whole unit. By the definition of π, it follows that for all $R \in \mathcal{D}^N$ such that $N_+(R) \neq \emptyset$,

$$\Phi(R) = \chi^{\arg\min\{\pi(i) | i \in N_+(R)\}}. \tag{3.3}$$

(b) Definition of $g : \mathcal{D}^N \to N$.

Define a choice function $g: \mathcal{D}^N \to N$ as follows. For all $R \in \mathcal{D}^N$,

$$g(R) := \begin{cases} i & \text{if } \Phi(R) = \chi^i \text{ and } N_+(R) = \emptyset, \\ \min\{i \mid i \in N_0(R)\} & \text{if } N_+(R) \neq \emptyset \text{ and } N_0(R) \neq \emptyset, \\ \min\{i \mid i \in N_+(R)\} & \text{if } N_+(R) \neq \emptyset \text{ and } N_0(R) = \emptyset. \end{cases}$$

Hence, by the definition of g, it follows that for all $R \in \mathcal{D}^N$ such that $N_+(R) = \emptyset$ and $N_0(R) \neq \emptyset$,

$$\Phi(R) = \chi^{g(R)}. \tag{3.4}$$

Note, that the choice function g is not uniquely determined: (3.4) holds for all choice functions $\bar{g}: \mathcal{D}^N \to N$ such that $\bar{g}(R) = i$ if $\Phi(R) = \chi^i$ and $N_+(R) = \emptyset$.

(c) Definition of $k \in N$.

Let $R \in \mathcal{D}^N$ be such that $N_-(R) = N$. Then, $\Phi(R) = \chi^i$ for some $i \in N$. By *strategy-proofness*, it follows that for all i-deviations R' from R,

$$\Phi(R') = \chi^i. \tag{3.5}$$

Let $j \neq i$ and consider a j-deviation R'' from R with $N_-(R'') = N$. Then, by *strategy-proofness* and *replacement-domination*,

$$\Phi(R'') = \chi^i. \tag{3.6}$$

Now, applying (3.5) and (3.6) repeatedly, it follows that for all $\bar{R} \in \mathcal{D}^N$ with $N_-(\bar{R}) = N$,

$$\Phi(\bar{R}) = \chi^i.$$

Hence, by setting $k = i$, for all $R \in \mathcal{D}^N$ with $N_-(R) = N$,

$$\Phi(R) = \chi^k. \tag{3.7}$$

Now, equations (3.3), (3.4), and (3.7) imply that for all $R \in \mathcal{D}^N$, $\Phi(R) = \chi^{f_{\pi,g,k}(R)}$. ■

Remark 3.1 Restricting our model to situations where agents are not indifferent between 0 and 1 and at least one agent prefers 1 to 0, the class of rules that satisfy *Pareto optimality*, *strategy-proofness*, and *replacement-domination* reduces to the serial dictatorships described in Papai (1996) for the problem of allocating a single indivisible object to one agent in a group. □

3.3 Variable population

Next, we extend the model and allow for changes in the set of agents as well as in the amount that has to be distributed.

There is a finite population of potential agents, indexed by $\mathbb{P}$.[9] Each agent $i \in \mathbb{P}$ is equipped with a single-dipped preference relation over the non-negative real numbers $\mathbb{R}_+$.[10] By $\mathcal{D}$ we now denote the class of all single-dipped preference relations over $\mathbb{R}_+$. By $\mathcal{P}$ we denote the class of non-empty subsets of $\mathbb{P}$. For $N \in \mathcal{P}$, $\mathcal{D}^N$ denotes the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{D}$.

Now, an economy is formalized as follows. Let $\Omega \in \mathbb{R}_+$ be the amount of an infinitely divisible commodity, or the *(social) endowment*, that has to be distributed among a group of agents $N \in \mathcal{P}$ with profile $R \in \mathcal{D}^N$. We call a pair $e = (R, \Omega) \in \mathcal{D}^N \times \mathbb{R}_+$ an *economy*. Let $\mathcal{E}^N = \mathcal{D}^N \times \mathbb{R}_+$ be the class of all economies. A *feasible allocation for* $e = (R, \Omega) \in \mathcal{E}^N$ is a vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = \Omega$. An *allocation rule* φ, or a *rule* for short, is a function that assigns to every $N \in \mathcal{P}$ and every $e \in \mathcal{E}^N$ a feasible allocation, denoted $\varphi(e)$. As before, $\varphi_i(e)$ denotes the *share* of agent i.

The definitions of *Pareto optimality*, *strategy-proofness*, and *replacement-domination* can straightforwardly be adopted to this generalized model, and Theorem 3.1 can easily be extended. However, some extra notation is needed.

Let $N \in \mathcal{P}$. For $e = (R, \Omega) \in \mathcal{E}^N$, let $N_+(e) = \{i \in N \mid \Omega P_i 0\}$; $N_0(e)$ and $N_-(e)$ are understood similarly. A selection $f^N_{\pi,g,k} \colon \mathcal{E}^N \to N$ can similarly be defined as before, by replacing R by e and $\mathcal{D}^N$ by $\mathcal{E}^N$. As before, for a given $N \in \mathcal{P}$ and $i \in N$, χ^i denotes the i-th unit vector of $\mathbb{R}^N$. Since we use the unit vectors in the description of allocations for a given economy, it is unambiguously determined to which "index set" of agents the unit vectors belong.

Corollary 3.1 *A rule φ satisfies Pareto optimality, strategy-proofness, and replacement-domination if and only if for all $N \in \mathcal{P}$ and all $\Omega \in \mathbb{R}_+$, there exist $\pi \in \Pi^N$, $g \in G^N$, and $k \in N$ such that for all $e = (R, \Omega) \in \mathcal{E}^N$,*

$$\varphi(e) = \Omega \chi^{f^N_{\pi,g,k}(e)}.$$

Proof. For any fixed coalition $N \in \mathcal{P}$ and any fixed amount of the commodity $\Omega \in \mathbb{R}_+$, the proofs of Lemmas 3.1, 3.2, and Theorem 3.1 are similar as before by replacing the amount 1 by Ω, R by $e = (R, \Omega)$, and $\mathcal{D}^N$ by $\mathcal{E}^N$. This implies Corollary 3.1 ■

Note that *replacement-domination* is trivially satisfied if $|N| = 2$. For later reference, we state this as a corollary.

Corollary 3.2 *Let φ be a rule that satisfies Pareto optimality and strategy-proofness. Then, for all $N \in \mathcal{P}$, $|N| = 2$, and all $\Omega \in \mathbb{R}_+$, there exist $\pi \in \Pi^N$, $g \in G^N$, and $k \in N$*

[9] We also could choose an infinite population of potential agents that is represented by the natural numbers $\mathbb{N}$. However, then we would have to evoke some existence results from social choice theory to prove Theorems 3.2 and 3.3 instead of using constructive proofs. Since this would not only interrupt the flow of the exposition, but also increase the complexity of notation, here we decided to stick to the original constructive version of the proof. However, I wish to thank Ton Storcken for bringing the alternative approach to my attention.

[10] In Klaus, Peters, and Storcken (1996) we restrict the preferences to $[0, 1]$.
Without loss of generality, we may assume that the agents' preference relations are continuous as well.

such that for all $e = (R, \Omega) \in \mathcal{E}^N$,

$$\varphi(e) = \Omega\chi^{f^N_{\pi,g,k}(e)}.$$

The next property we consider is *consistency*. Recall that for an economy $e = (R, \Omega) \in \mathcal{E}^N$ *consistency* describes the stability of a recommended allocation $x \in \mathbb{R}^N_+$ in situations where a group of agents leave with their shares (see also Section 2.2, page 26). Let $M \subseteq N$ and suppose that all agents in $N \backslash M$ leave with the shares assigned to them at x. Then, the shares assigned to the agents in M at x should be equal to the shares they receive for the *reduced economy* $(R_M, \sum_M x_i) \in \mathcal{E}^M$.

Let $M, N \in \mathcal{P}$ such that $M \subseteq N$. For $x \in \mathbb{R}^N$ the *restriction* $(x_i)_{i \in M} \in \mathbb{R}^M$ *of* x *to* M is denoted by x_M.

Consistency: For all $M, N \in \mathcal{P}$ with $M \subseteq N$ and all $e = (R, \Omega) \in \mathcal{E}^N$, $\varphi(e)_M = \varphi(R_M, \sum_M \varphi_i(e))$.

Next, we consider rules that satisfy *Pareto optimality*, *strategy-proofness*, and *consistency*. By Corollary 3.2 it follows that such a rule never assigns a broken allocation to a two-agent economy. Then, by *consistency*, it follows easily that a rule that satisfies the properties listed above always assigns the whole endowment to a single agent. Before we characterize the class of *Pareto optimal*, *strategy-proof*, and *consistent* rules, we show that these properties imply *replacement-domination*.

Lemma 3.3 *Let* φ *be a rule that satisfies Pareto optimality, strategy-proofness, and consistency. Then,* φ *satisfies replacement-domination.*

Proof. Let φ satisfy the properties listed in the lemma. By Corollary 3.2 and feasibility, it follows that for all $N \in \mathcal{P}$ with $|N| \leq 2$ and all $e = (R, \Omega) \in \mathcal{E}^N$,

$$\varphi(e) = \Omega\chi^j, \text{ for some } j \in N. \tag{3.8}$$

Suppose, by contradiction, that φ does not satisfy *replacement-domination*. Then, there exist $S \in \mathcal{P}$, $|S| > 2$, $e = (R, \Omega)$, $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^S$, and $i, j, k \in S$, $j, k \neq i$, such that R is an i-deviation from $\bar{R}$ and

$$\varphi_j(e)\, P_j\, \varphi_j(\bar{e}) \text{ and } \varphi_k(\bar{e})\, P_k\, \varphi_k(e).$$

By *consistency* of φ, this can only hold if $\varphi_i(e) \neq \varphi_i(\bar{e})$. Furthermore, $\varphi_j(e) \neq \varphi_j(\bar{e})$ and $\varphi_k(e) \neq \varphi_k(\bar{e})$. Hence, for two agents $l, m \in \{i, j, k\}$ it must be that either $[\varphi_l(e) \neq 0$ and $\varphi_m(e) \neq 0]$ or $[\varphi_l(\bar{e}) \neq 0$ and $\varphi_m(\bar{e}) \neq 0]$.

Without loss of generality assume that $\varphi_l(e) \neq 0$ and $\varphi_m(e) \neq 0$. Now, consider the reduced economy $\tilde{e} = (R_{\{l,m\}}, \sum_{\{l,m\}} \varphi_i(e)) \in \mathcal{E}^{\{l,m\}}$. Then, by *consistency*, $\varphi(e)_{\{l,m\}} = \varphi(\tilde{e})$. Hence,

$$\varphi_l(\tilde{e}) \neq 0 \text{ and } \varphi_m(\tilde{e}) \neq 0.$$

Since $\tilde{e}$ is a two-agent economy, this is in contradiction to (3.8). ■

As already mentioned above, a rule that satisfies *Pareto optimality*, *strategy-proofness*, and *consistency*, assigns the social endowment to a single agent. Furthermore, the way that agent is selected should be "consistent across populations". Let the endowment Ω be fixed. Then, the selection of the agent can be described as follows. Let $N \in \mathcal{P}$ be a set of agents and $e = (R, \Omega) \in \mathcal{E}^N$ an economy.

By *Pareto optimality* the endowment Ω should, if possible, be allotted to an agent who prefers Ω to 0. Among these agents, if there are several, the choice is made with respect a linear order π_+ on $\mathbb{P}$, the set of potential agents.

We ask the first agent according to π_+ who is a member of N, whether he prefers Ω to 0. If he prefers the endowment, he receives it and we are done. If not, we ask the second agent according to π_+ who is a member of N the same question; *etc.* Hence, the first agent in the order π_+ who is in N and prefers the endowment to nothing receives it.

If no agent in N strictly prefers Ω to 0 and at least one agent in N is indifferent between 0 and Ω, then the endowment is assigned "arbitrarily" to an agent who is indifferent between 0 and Ω. Again, we could make our choice with respect to a fixed order on $\mathbb{P}$ in order to satisfy the required properties. However, in this case, we have more degrees of freedom to specify a "consistent" choice so as to respect *Pareto optimality* and *strategy-proofness* (see Example 3.3).

Finally, if all agents in N strictly prefer 0 to Ω, then Ω is assigned to an agent with respect to a linear order π_- on $\mathbb{P}$. Now, the first agent in the order who is in N receives the endowment.

For a formal description, we introduce the following notation.

Let $\Pi^{\mathbb{P}}$ denote the set of all permutations, or *linear orders*, on $\mathbb{P}$. We call $\mathcal{G} := \{g^N \in G^N\}_{N \in \mathcal{P}}$ a *consistent collection of choice functions* if for all $T' \subseteq T \in \mathcal{P}$ and all $(R, \Omega) \in \mathcal{E}^T$ such that $g^T(R, \Omega) \in T'$, it follows that $g^T(R, \Omega) = g^{T'}(R_{T'}, \Omega)$. By $\mathcal{C}^{\mathbb{P}}$ we denote the class of all consistent collections of choice functions.[11]

Consistent collection of selections $\{f^N_{\pi_+,\mathcal{G},\pi_-}\}_{N \in \mathcal{P}}$: Let $\pi_+, \pi_- \in \Pi^{\mathbb{P}}$ and $\mathcal{G} = \{g^N \in G^N\}_{N \in \mathcal{P}} \in \mathcal{C}^{\mathbb{P}}$. Then, the *consistent collection of selections* $\{f^N_{\pi_+,\mathcal{G},\pi_-}\}_{N \in \mathcal{P}}$ (based on π_+, π_-, and $\mathcal{G}$) is defined as follows. For all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$:

Case 1: If $N_+(e) \neq \emptyset$, then $f^N_{\pi_+,\mathcal{G},\pi_-}(e) = \arg\min\{\pi_+(i) \mid i \in N_+(e)\}$.

Case 2: If $N_+(e) = \emptyset$ and $N_0(e) \neq \emptyset$, then $f^N_{\pi_+,\mathcal{G},\pi_-}(e) = g^N(e)$.

Case 3: If $N_-(e) = N$, then $f^N_{\pi_+,\mathcal{G},\pi_-}(e) = \arg\min\{\pi_-(i) \mid i \in N_-(e)\}$.

Theorem 3.2 *A rule φ satisfies Pareto optimality, strategy-proofness, and consistency if and only if for each $\Omega \in \mathbb{R}_+$ there exist $\pi_+, \pi_- \in \Pi^{\mathbb{P}}$ and $\mathcal{G} \in \mathcal{C}^{\mathbb{P}}$ such that for*

[11] Note that in the description of all *Pareto optimal*, *strategy-proof*, and *consistent* rules, we do not need this notion of consistency for the collection of choice functions in its full strength; consistent choices only have to be made for all economies (R, Ω), $(R_{T'}, \Omega)$ such that $N_+(R, \Omega) = \emptyset$ and $N_0(R_{T'}, \Omega) \neq \emptyset$. However, since the choice functions are not uniquely determined, for notational convenience we require consistency of the collection of choice functions for all economies.

all $N \in \mathcal{P}$ *and all* $e = (R, \Omega) \in \mathcal{E}^N$,

$$\varphi(e) = \Omega\chi^{f^N_{\pi_+,\mathcal{G},\pi_-}(e)}.$$

Before we prove the theorem, we briefly discuss the difference between the rules described in Corollary 3.1 and in the previous theorem.

Let $N \in \mathcal{P}$, $e \in \mathcal{E}^N$ and assume that the rule φ satisfies *Pareto optimality* and *strategy-proofness*. If φ satisfies *consistency*, then it belongs to the class of rules described in Theorem 3.2. By Lemma 3.3, *consistency* implies *replacement-domination* and φ belongs to the class of rules described in Corollary 3.1.

Case 1: $N_+(e) \neq \emptyset$. A rule φ that satisfies *replacement-domination* chooses an agent who prefers Ω to 0 with respect to an "arbitrarily" chosen linear order on N. To the contrary, a rule φ that satisfies *consistency* chooses this agent with respect to a linear order on N that is implied by a fixed linear order on the whole set of potential agents $\mathbb{P}$.

Case 2: $N_+(e) = \emptyset$ and $N_0(e) \neq \emptyset$. A rule φ that satisfies *replacement-domination* "arbitrarily" chooses a single agent who is indifferent between Ω and 0. To the contrary, a rule φ that satisfies *consistency* has to be consistent across populations.

Case 3: $N_-(e) = N$. A rule φ that satisfies *replacement-domination*, "arbitrarily" chooses a single agent who always receives the endowment if nobody else in N wants it. To the contrary, a rule φ that satisfies *consistency* chooses this agent with respect to an linear order on N that is implied by a fixed linear order of the whole set of potential agents $\mathbb{P}$.

Proof. Let $\Omega \in \mathbb{R}_+$. Let $\pi_+, \pi_- \in \Pi^{\mathbb{P}}$ and $\mathcal{G} = \{g^N \in G^N\}_{N\in\mathcal{P}} \in \mathcal{C}^{\mathbb{P}}$ be such that for all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$, $\varphi(e) = \Omega\chi^{f^N_{\pi_+,\mathcal{G},\pi_-}(e)}$. As shown in the proof of Theorem 3.1, φ satisfies *Pareto-optimality* and *strategy-proofness*. In order to establish *consistency*, let $S, T \in \mathcal{P}$ with $S \subseteq T$ and $e = (R, \Omega) \in \mathcal{E}^T$. We have to show that for the reduced economy $\tilde{e} = (R_S, \sum_S \varphi_i(e))$ it follows that $\varphi(e)_S = \varphi(\tilde{e})$. We distinguish the following two cases:

(a) $f^T_{\pi,g,k}(e) \in S$. Then, for all $i \notin S$, $\varphi_i(e) = 0$ and $\tilde{e} = (R_S, \Omega)$. Since $\{f^N_{\pi_+,\mathcal{G},\pi_-}\}_{N\in\mathcal{P}}$ is a consistent collection of choice functions it is based on $\pi_+, \pi_- \in \Pi^{\mathbb{P}}$, and a consistent collection of choice functions $\mathcal{G} = \{g^N \in G^N\}_{N\in\mathcal{P}}$. Hence, $f^T_{\pi,g,k}(e) = f^S_{\pi,g,k}(e)$ and $\varphi(e)_S = \varphi(\tilde{e})$.

(b) $f^T_{\pi,g,k}(e) \notin S$. Hence, for all $i \in S$, $\varphi_i(e) = 0$. So, $\sum_S \varphi_i(e) = 0$ and $\tilde{e} = (R_S, 0)$. Thus, for all $i \in S$, $\varphi_i(\tilde{e}) = 0$ and $\varphi(e)_S = \varphi(\tilde{e})$.

Cases **(a)** and **(b)** together prove the *consistency* of φ.

To prove the remaining part of the theorem let φ satisfy the properties listed in the theorem. By Lemma 3.3 it follows that φ satisfies *replacement-domination*. Then, by Corollary 3.1, for each $N \in \mathcal{P}$ and $\Omega \in \mathbb{R}_+$ there exist $\pi_N \in \Pi^N$, $g_N \in G^N$, and $k_N \in N$ such that for all $e = (R, \Omega) \in \mathcal{E}^N$,

$$\varphi(e) = \Omega\chi^{f^N_{\pi_N,g_N,k_N}(e)}.$$

Definition of $\pi_+ \in \Pi^{\mathbb{P}}$.

Consider the whole set of potential agents $\mathbb{P}$. Then, by *consistency*, $\pi_+ := \pi_{\mathbb{P}+} \in \Pi^{\mathbb{P}}$ is such that for all $N \subseteq \mathbb{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$ with $N_+(e) \neq \emptyset$, $\arg\min\{\pi_+(i) \mid i \in N_+(e)\} = \arg\min\{\pi_N(i) \mid i \in N_+(e)\}$.

Definition of $\mathcal{G} = \{g^N \in G^N\}_{N \in \mathcal{P}} \in \mathcal{C}^{\mathbb{P}}$.

Consider the whole set of potential agents $\mathbb{P}$ and the collection of choice functions $\mathcal{G}_{\mathbb{P}} = \{g^N \in G^N\}_{N \subseteq \mathbb{P}}$. The collection of choice functions $\mathcal{G} = \{\tilde{g}^N \in G^N\}_{N \subseteq \mathbb{P}}$ is defined as follows For all $N \subseteq \mathbb{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$, $\tilde{g}^N(e) = g^N(e)$ if $N_+(e) = \emptyset$ and $N_0(e) \neq \emptyset$ and $\tilde{g}^N(e) = \min\{i \mid i \in N\}$ otherwise. By construction and *consistency*, it follows easily that $\mathcal{G}$ is a consistent subcollection of choice functions.

Definition of $\pi_- \in \Pi^{\mathbb{P}}$.

Next, define $\pi_- \in \Pi^{\mathbb{P}}$ as follows. Let $\pi_-(k_{\mathbb{P}}) = 1$ and $\mathbb{P}^1 := \mathbb{P} \backslash \{k_{\mathbb{P}}\}$. Next, let $\pi_-(k_{\mathbb{P}^1}) = 2$ and $\mathbb{P}^2 := \mathbb{P} \backslash \{k_{\mathbb{P}^1}\}$. Let $\pi_-(k_{\mathbb{P}^2}) = 3$ *etc.* It is clear, how the linear order $\pi_{\mathbb{P}-} \in \Pi^{\mathbb{P}}$ is constructed step by step. Furthermore, by *consistency*, $\pi_- \in \Pi^{\mathbb{P}}$ is such that for all $N \subseteq \mathbb{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$ with $N_-(e) = N$, $\arg\min\{\pi_-(i) \mid i \in N_+(e)\} = k_N$.

Hence, for each $\Omega \in \mathbb{R}_+$, there exist $\pi_+, \pi_- \in \Pi^{\mathbb{P}}$ and $\mathcal{G} = \{\tilde{g}^N \in G^N\}_{N \subseteq \mathbb{P}}$ such that for all $N \subseteq \mathbb{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$, $\varphi(e) = \Omega \chi^{f^N_{\pi_+, \mathcal{G}, \pi_-}(e)}$.[12] ■

Finally, we discuss *population-monotonicity*: if a group of agents leaves the economy, then all remaining agents are affected in the same direction (see also Section 2.2, page 24).

Population-monotonicity: For all $N, M \in \mathcal{P}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^N$, if $M \subseteq N$ and $R_M = \bar{R}$, then either [for all $i \in M$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in M$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

We show that a rule φ that satisfies *Pareto optimality*, *strategy-proofness*, and *population-monotonicity* assigns the whole endowment to a single agent.

Let $\mathcal{B}_\varphi = \{e = (R, \Omega) \in \mathcal{E}^N \mid N \in \mathcal{P}$ and for all $i \in N$, $\varphi_i(e) \neq \Omega \chi^i\}$ be the set of economies that yield a *broken allocation* under φ, *i.e.*, a feasible allocation where no agent obtains the whole endowment.

Lemma 3.4 *Let φ be a rule that satisfies Pareto optimality, strategy-proofness, and population-monotonicity. Then, $\mathcal{B}_\varphi = \emptyset$.*

Proof. Let φ satisfy the properties listed in the lemma and suppose by contradiction that $\mathcal{B}_\varphi \neq \emptyset$.

[12] Note, that for an infinite set of potential agents represented by the natural numbers $\mathbb{N}$, the representation of the rules by means of permutations is not sufficient. However, by changing the notation and by introducing binary relations to describe the selection functions, the result extends to this infinite population setting. To be more specific, one can show that the choice rule that is induced by the rule in Case 1 ($N_+(e) \neq \emptyset$) satisfies the weak axiom of revealed preferences (see Mas-Colell, Whinston, and Green, 1995). Then, it follows that there exists a binary relation that rationalizes this choice rule. A similar argument holds for the choice rule that is induced by the rule in Case 3 ($N_+(e) \neq \emptyset$)

Since $\mathcal{B}_\varphi \neq \emptyset$, there exist $N \in \mathcal{P}$ and $e = (R, \Omega)$ such that for all $i \in N, \varphi_i(e) \neq \Omega\chi^i$. Hence, there exist $i, j \in N$ such that $\varphi_i(e) \neq 0$ and $\varphi_j(e) \neq 0$. By *Lemma 3.1*, either **(a)** $\varphi_i(e) P_i 0$ and $\varphi_j(e) P_j 0$ or **(b)** $\varphi_i(e) P_i \Omega$ and $\varphi_j(e) P_j \Omega$.

Now, let $M = \{i, j\}$ and consider $\bar{e} = (R_M, \Omega) \in \mathcal{E}^M$. By *population-monotonicity*, either $[\varphi_i(e) R_i \varphi_i(\bar{e})$ and $\varphi_j(e) R_j \varphi_j(\bar{e})]$ or $[\varphi_i(\bar{e}) R_i \varphi_i(e)$ and $\varphi_j(\bar{e}) R_j \varphi_j(e)]$.

Since M is a two-agent economy, by Corollary 3.2 it follows that either $[\varphi_i(\bar{e}) = \Omega$ and $\varphi_j(\bar{e}) = 0]$ or $[\varphi_i(\bar{e}) = 0$ and $\varphi_j(\bar{e}) = \Omega]$. This is in contradiction to **(a)** and **(b)**. ■

Now, using the methods of proof developed in Theorem 3.1, Corollary 3.1, and Theorem 3.2, we can derive a characterization of all *Pareto optimal*, *strategy-proof*, and *population-monotonic* rules.

The proof of the following characterization is very similar to the proofs of Theorems 3.1 and 3.2. Therefore we omit it.

Theorem 3.3 *A rule φ satisfies Pareto optimality, strategy-proofness, and population-monotonicity if and only if for each $\Omega \in \mathbb{R}_+$ there exist $\pi_+, \pi_- \in \Pi^{\mathbb{P}}$ and $\mathcal{G} = \{g^N \in G^N\}_{N \in \mathcal{P}}$ such that for all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$,*

$$\varphi(e) = \Omega\chi^{f^N_{\pi_+, \mathcal{G}, \pi_-}(e)}.$$

Note that the only difference between the class of rules described in Theorem 3.3 and 3.2 is that the rules described in Theorem 3.2 are based on collections of choice functions that are in addition consistent. Hence, the class of rules that satisfy *Pareto optimality*, *strategy-proofness*, and *population-monotonicity* is a strict subset of the class of rules that satisfy *Pareto optimality*, *strategy-proofness*, and *consistency*. Furthermore, the class of rules that satisfy *Pareto optimality*, *strategy-proofness*, and *population-monotonicity* strictly contains the class of rules that satisfy *Pareto optimality*, *strategy-proofness*, and *replacement-domination* (see Corollary 3.1).

In the next example, we describe a rule that satisfies all properties we considered until now.

Example 3.2 The following rule $\varphi^{\min}$ satisfies *Pareto optimality, strategy-proofness, replacement-domination, consistency,* and *population-monotonicity*. For all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$:

Case 1 : If $N_+(e) \neq \emptyset$, then $\varphi^{\min}(e) = \Omega\chi^{\min\{i \mid i \in N_+(e)\}}$.

Case 2 : If $N_+(e) = \emptyset$ and $N_0(e) \neq \emptyset$, then $\varphi^{\min}(e) = \Omega\chi^{\min\{i \mid i \in N_0(e)\}}$.

Case 3 : If $N = N_-(e)$, then $\varphi^{\min}(e) = \Omega\chi^{\min\{i \mid i \in N_-(e)\}}$.

By $\mathrm{id} \in \Pi^{\mathbb{P}}$ we denote the identity permutation defined by $\mathrm{id}(i) = i$ for all $i \in \mathbb{P}$. Let $\pi_+ = \pi_- = \mathrm{id} \in \Pi^{\mathbb{P}}$ and define $g^N_{\min} \in G^N$ such that for all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$, $g^N_{\min}(e) = \min\{i \mid i \in N_0(e)\}$ if $N_0(e) \neq \emptyset$ and $g^N_{\min}(e) = \min\{i \mid i \in N\}$ otherwise. Then, $\mathcal{G}' := \{g^N_{\min} \in G^N\}_{N \in \mathcal{P}}$ is a consistent collection of choice functions. Now choosing for all $\Omega \in \mathbb{R}_+$ the consistent collection of selections $\{f^N_{\mathrm{id}, \mathcal{G}', \mathrm{id}}\}_{N \in \mathcal{P}}$, yields for all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$, $\varphi^{\min}(e) = \Omega\chi^{f^N_{\mathrm{id}, \mathcal{G}', \mathrm{id}}(e)}$. ◇

The choice functions $g^N_{\min}$ as defined in Example 3.2 always select agents according to the identity permutation. We say that a collection of choice functions $\mathcal{G} = \{g^N \in G^N\}_{N\in\mathcal{P}}$ is based on a linear order $\pi \in \Pi^{\mathbb{P}}$ if for all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$, $g^N(e) = \arg\min\{\pi(i) \mid i \in N_0(e)\}$ if $N_0(e) \neq \emptyset$. It is easy to show that all collections of choice functions $\mathcal{G} = \{g^N \in G^N\}_{N\in\mathcal{P}}$ that are based on a linear order $\pi \in \Pi^{\mathbb{P}}$ are consistent. However, as the following example shows, the class of consistent collections of choice functions strictly contains the class of choice functions that are based on a linear order $\pi \in \Pi^{\mathbb{P}}$.

Example 3.3 We define the class of collections $\mathcal{G}'' = \{\bar{g}^N \in G^N\}_{N\in\mathcal{P}}$ as follows. For all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$, $\bar{g}^N(e) = \min\{i \mid i \in \arg\min\{d(R_i) \mid i \in N_0(e)\}\}$ if $N_0(e) \neq \emptyset$ and $\bar{g}^N(e) = \min\{i \mid i \in N\}$ otherwise. Note that now we only use the identity permutation to break the tie if there are several agents in $N_0(e)$ with the same smallest dip. ◇

Remark 3.2 By going over the proofs the reader may verify that we do not fully exploit the "richness" of the domain of single-dipped preferences. The results remain true for any sub-domain $\mathcal{D}' \subseteq \mathcal{D}$ of preferences such that for all $x \in (0, \Omega)$, there exist $\bar{R}, \tilde{R} \in \mathcal{D}'$ with $0\,\bar{P}\,\Omega\,\bar{P}\,x$ and $\Omega\,\tilde{P}\,0\,\tilde{P}\,x$. □

3.4 Independence, compatibility, and incompatibility of properties

First we show that the characterizations of Theorem 3.1, Corollary 3.1, Theorems 3.2, and 3.3 are tight, *i.e.*, dropping any of the properties yields alternative rules.

Example 3.4 The following rule $\bar{\varphi}$ satisfies *strategy-proofness, replacement-domination, consistency,* and *population-monotonicity*, but not *Pareto optimality*. For all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$, $\bar{\varphi}(e) = \Omega\chi^{\min\{i \mid i \in N\}}$. ◇

Example 3.5 The following rule $\hat{\varphi}$ satisfies *Pareto optimality, replacement-domination, consistency,* and *population-monotonicity*, but not *strategy-proofness*. For all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$such that $N_+(e) \neq \emptyset$, let $N^1_+(e) = \{i \in N_+(e) \mid d(R_i) = 0\}$ and $N^2_+(e) = \{i \in N_+(e) \mid d(R_i) \neq 0\}$. Then, for all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N$, we define $\hat{\varphi}(e)$ as follows. If $N^1_+(e) \neq \emptyset$, then $\hat{\varphi}(e) = \Omega\chi^{\min\{i \mid i \in N^1_+(e)\}}$, if $N^1_+(e) = \emptyset$ and $N^2_+(e) \neq \emptyset$, then $\hat{\varphi}(e) = \Omega\chi^{\min\{i \mid i \in N^2_+(e)\}}$, and $\hat{\varphi}(e) = \varphi^{\min}(e)$ otherwise. ◇

Example 3.6 The following rule $\tilde{\varphi}$ satisfies *Pareto optimality, strategy-proofness,* but none of the following properties *replacement-domination, consistency,* and *population-monotonicity*. *For* $T = \{1, 2, 3\}$ and $e = (R, 1) \in \mathcal{E}^T$, define $\tilde{\varphi}(e)$ as follows. If $1\,P_3\,0$, then

$$\tilde{\varphi}(e) = \begin{cases} (\frac{1}{2}, \frac{1}{2}, 0) & \text{if } 1\,P_1\,\frac{1}{2}\,P_1\,0 \text{ and } 1\,P_2\,\frac{1}{2}\,P_2\,0, \\ (0, 0, 1) & \text{otherwise.} \end{cases}$$

For all remaining $N \in \mathcal{P}$ and $e = (R, \Omega) \in \mathcal{E}^N$, $\tilde{\varphi}(e) = \varphi^{\min}(e)$. ◇

In the remainder of the section, we explore the compatibility of *Pareto optimality* with other well-known properties for rules we introduced in Chapter 2.

In the model of fair allocation with single-peaked preferences, many rules satisfy *peak-onliness*: the rule only depend on the peaks instead of the complete profile of preferences (see Section 2.2, page 22). *Peak-onliness* not only simplifies the application of a rule to a given problem, it also incorporates a notion of nonmanipulability with respect to intensities of the preferences or the "shape" of the preferences. The natural counterpart of *peak-onliness* in this model is *dip-onliness*. For a profile $R = (R_i)_{i \in N} \in \mathcal{D}^N$, by $d(R) = (d(R_i))_{i \in N}$ we denote the vector of dips.

Dip-onliness: For all $N \in \mathcal{P}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^N$, if $d(R) = d(\bar{R})$, then $\varphi(e) = \varphi(\bar{e})$.

By Lemma 3.1 it follows easily that *Pareto optimality* of an allocation does not only depend on the agents' dips but on the whole preference profile; loosely speaking, *Pareto optimality* depends on the shapes and intensities of the individual preferences. Thus, *dip-onliness* is not compatible with *Pareto optimality*.

The next property we consider represents a standard notion of fairness: *individual rationality from equal division* requires that each agent (weakly) prefers his share to the "equal division share" of the economy (see Section 2.2, page 27).

Individual rationality from equal division: For all $N \in \mathcal{P}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $i \in N$, $\varphi_i(e) \; R_i \; \frac{\Omega}{|N|}$.

Note that for each economy $e \in \mathcal{E}^N$ the equal division allocation $z = (\frac{\Omega}{|N|}, \ldots, \frac{\Omega}{|N|})$ is feasible. If z is not Pareto optimal, then proceeding from z we can easily construct an allocation y that Pareto dominates z and that is *Pareto optimal*. Hence, *Pareto optimal* and *individually rational from equal division* allocations always exist.

Next, we consider the compatibility of *Pareto optimality* with the properties of fairness *anonymity*, *no-envy*, and *equal treatment of equals*.

Anonymity requires that the agents' shares do not depend on their names (see Section 2.2, page 22). *No-envy* states that no agent strictly prefers the share of another agent to his own share (see Section 2.2, page 22). *Equal treatment of equals*, a weakening of *no-envy*, requires that if two agents have the same preference relation, then each of them is indifferent between his share and the other agent's share (see Section 2.2, page 23).

Let $\pi \in \Pi^N$ be a permutation on N and $R \in \mathcal{D}^N$. Then, $R_\pi = (R_{\pi(i)})_{i \in N}$ and $e_\pi = (R_\pi, \Omega)$.

Anonymity: For all $N \in \mathcal{P}$, all $e = (R, \Omega) \in \mathcal{E}^N$, all $\pi \in \Pi^N$, and all $i \in N$, $\varphi_i(e_\pi) = \varphi_{\pi(i)}(e)$.

No-envy: For all $N \in \mathcal{P}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $i, j \in N$, $\varphi_i(e)\, R_i\, \varphi_j(e)$.

Equal treatment of equals: For all $N \in \mathcal{P}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $i, j \in N$, if $R_i = R_j$, then $\varphi_i(e)\, I_i\, \varphi_j(e)$.

By the next example, we show that *Pareto optimality* is generally not compatible with the properties introduced above.

Example 3.7 Let $N = \{1, 2\}$, $\Omega = 10$, and $e = (R, \Omega) \in \mathcal{E}^N$ be such that $R_1 = R_2$ and $d(R_1) = 7$, $0\, I_1\, 9$, and $2\, I_1\, 8$. Suppose the rule φ satisfies *Pareto optimality*. Then, by Lemma 3.1, for $i = 1, 2$, either $\varphi_i(e) \in \{0, 10\}$ or $\varphi_i(e) \in (9, 10)$. Since $\varphi_2(e) = 10 - \varphi_1(e)$, it follows, by feasibility, that $\varphi(e) \in \{(10, 0), (0, 10)\}$ If φ is *anonymous*, then $\varphi(e) = (5, 5)$. If φ satisfies *no-envy* or *equal treatment of equals*, then $\varphi(e) \in \{(5, 5), (2, 8), (8, 2)\}$. However, none of these allocations is *Pareto optimal*. $\diamond$

Since *replication-invariance* (see Section 2.2, page 25) implies *anonymity*, Example 3.7 also shows the incompatibility of *Pareto optimality* and *replication-invariance*.

The next example shows that *Pareto optimality* and *continuity with respect to preferences* are not compatible. In the example we apply *continuity with respect to preferences* in an informal way: continuous changes of the preferences imply continuous changes in the allocations assigned by the rule. For the problem of fair allocation when preferences are single-peaked, *continuity with respect to preferences* was introduced by Sprumont (1991). This definition can easily be adjusted for rules when preferences are single-dipped.

Example 3.8 Let $N = \{1, 2\}$ and $e = ((R_1, R_2), \Omega) \in \mathcal{E}^N$ be such that $0\, I_1\, \Omega$ and $\Omega\, P_2\, 0$. Suppose the rule φ satisfies *Pareto optimality*. Then, $\varphi((R_1, R_2), \Omega) = (0, \Omega)$. Now, change R_2 continuously into R_1 in such a way that in these changes agent 1 still prefers Ω to 0. *Continuity with respect to preferences* implies (in the limit) $\varphi((R_1, R_1), \Omega) = (0, \Omega)$. Next, consider $\bar{e} = ((R_2, R_1), \Omega) \in \mathcal{E}^N$. Then, by the same argument it follows that $\varphi((R_2, R_1), \Omega) = (\Omega, 0)$ and (in the limit) $\varphi((R_1, R_1), \Omega) = (\Omega, 0)$. This is in contradiction to $\varphi((R_1, R_1), \Omega) = (0, \Omega)$. $\diamond$

We conclude this section with a compatibility result concerning the property of *converse consistency* (see Section 2.2, page 28). As already explained in Chapter 2, converse consistency determines the desirability of an allocation on the basis of the desirability of its restrictions to all two-agent reduced economies.

Converse consistency: For all $N \in \mathcal{P}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all feasible allocations $x \in \mathbb{R}^N_+$, if for all $M \subseteq N$ with $|M| = 2$, $\varphi(R_M, \sum_M x_i) = x_M$, then $\varphi(e) = x$.

By Theorem 3.2 it is easy to prove that all rules satisfying *Pareto optimality*, *strategy-proofness*, and *consistency* are *conversely consistent*.

Since in Corollary 3.1 for each set of agents, an arbitrary linear order, choice function, and agent can be selected, it is easy to show that not all rules satisfying *Pareto optimality*, *strategy-proofness*, and *replacement-domination* are *conversely consistent*.

As the following example shows, a *Pareto optimal*, *strategy-proof*, and *conversely consistent* rule does not necessarily always assign the whole amount to a single agent.

Example 3.9 The following allocation rule φ^{cc} satisfies *Pareto optimality*, *strategy-proofness*, and *converse consistency*. Let $T_1 = \{1,2\}$, $T_2 = \{2,3\}$, and $T_3 = \{1,3\}$. For all $e_1 = (R^1, \Omega) \in \mathcal{E}^{T_1}$ such that $N_+(e_1) = T_1$, let $\varphi_1^{cc}(e_1) = \Omega$, for all $e_2 = (R^2, \Omega) \in \mathcal{E}^{T_2}$ such that $N_+(e_2) = T_2$, let $\varphi_2^{cc}(e_2) = \Omega$, and for all $e_3 = (R^3, \Omega) \in \mathcal{E}^{T_3}$ such that $N_+(e_3) = T_3$, let $\varphi_3^{cc}(e_3) = \Omega$. For all remaining economies $e = (R, \Omega) \in \mathcal{E}^N, N \in \mathcal{P}$, let $\varphi^{cc}(e) = \bar{\varphi}(e)$, where $\bar{\varphi}$ is defined in Example 3.6.

The proof that φ^{cc} satisfies *converse consistency* is easy. It is based on the following two cases.

Case 1: The allocation assigned for $N \in \mathcal{P}$ with $|N| > 2$ and $e \in \mathcal{E}^N$ is consistent with the choice for all two-agent economies (if $\{1,2,3\} \cap N \neq \{1,2,3\}$).

Case 2: There exists no allocation that satisfies the hypothesis of *converse consistency*, namely, that the allocation for each reduced two-agent economy equals the restriction of the allocation to this set (this follows from the "cycle" we constructed for the two-agent subsets of $\{1,2,3\}$). $\diamond$

Part II

Fair reallocation

Chapter 4

The uniform reallocation rule[1]

We consider the problem of reallocating the individual endowments of some perfectly divisible commodity among a group of agents with single-peaked preferences. Such reallocation problems may occur when we are concerned with allocation problems, as described in Chapter 2, where preferences might change over time. Consider the following example. A task or a certain amount of work has to be divided among a group of workers. Assume that they are rewarded in proportion to their shares. Then, preferences over individual shares are single-peaked: each worker has a most preferred share, his peak, below which and above which his welfare is decreasing. Suppose now that we solved the allocation problem. If after a certain period of time preferences have changed, then it might be the case that the present allocation can be improved upon by reallocation. In other settings individual endowments are directly given, *e.g.*, in fixed-price exchange economies (see Benassy 1982), or can be interpreted as natural claims or priorities, *e.g.*, investments, as described by Barberà, Jackson, and Neme (1997).

There is a wide literature exploring the situation where the problem is reduced to the allocation of a total or social endowment. As described in Sprumont (1991), rationing in a two-good economy in which prices are in disequilibrium can be interpreted as such an allocation problem with total endowment. For this class of problems the preeminently featured allocation rule is the uniform rule (see Chapter 2). Sprumont (1991) started the axiomatic analysis of this class of problems and gave the first characterizations of the uniform rule (see Theorems 2.1 and 2.2). Since then, a variety of axiomatic studies, which also led to this rule, have been published. We refer the reader to the overview of the most important results given in Chapter 2.

In this chapter we are concerned with the axiomatic analysis of reallocation rules and particularly with the so-called uniform reallocation rule. Here, we explore to what extent the results as achieved for the allocation model can be generalized to the more general reallocation model. First, we extend the model and the properties we introduced in Chapter 2 to the reallocation setting. After exploring the compatibilities and the incompatibilities of this properties, it turns out that, again, a single rule comes out repeatedly as the unique solution satisfying certain combinations of desirable properties:

[1]This chapter is mainly based on Klaus, Peters, and Storcken (1995a,b, 1997b), Klaus (1997b,d).

the uniform reallocation rule.

This chapter is organized as follows. First, in Section 4.1, we give an overview, or more precisely a preview, of the results presented in this chapter. Furthermore, we discuss the parallels and the differences to former allocation results. In Section 4.2 we introduce the reallocation model and the uniform reallocation rule. In Sections 4.3–4.5 we introduce and explore properties for reallocation rules. Furthermore, we derive several characterizations of the uniform reallocation rule and discuss the relation to similar allocation results. We show the independence of the properties in the characterizations in Section 4.6. A discussion of the sensitivity of the results to variations in the model parameters is postponed to Chapter 5.

4.1 A preview of the results

In Section 4.2 we first introduce the reallocation model. In contrast to the allocation problem, where a social endowment has to be divided among a group of agents, the individual endowments have to be reallocated among them. A reallocation rule assigns to each reallocation economy that is determined by a group of agents, individual endowments, and their preference relations a "reallocation" of the resources among the agents of the group. Similar to the allocation model, a central property of reallocation rules is *Pareto optimality*. As examples for *Pareto optimal* reallocation rules, we introduce the proportional and the uniform reallocation rule (see Section 4.2 for the exact definitions).

In Section 4.3.1, similar to the allocation model, we start the analysis of reallocation rules with the well-known properties *strategy-proofness* and *no-envy*. As usual, *strategy-proofness* requires that no agent ever gains by lying about his preference relation. *No-envy* for reallocation rules is formalized in terms of allotment changes rather than in terms of net allotments: no agent strictly prefers another agent's allotment change, or the part of this allotment change that is "feasible" for him,[2] to his own allotment change. It turns out that, similar to the uniform allocation rule (see Theorem 2.2), the uniform reallocation rule is the only rule that satisfies *Pareto optimality*, *no-envy*, and *strategy-proofness* (see Theorem 4.1).

Next, we consider two weakenings of *no-envy*: *weak no-envy* requires that no agent strictly prefers another agent's allotment change that is feasible for him, to his own allotment change, and *equal treatment of equals* requires that if two agents have the same demand, then each of them is indifferent between receiving his allotment change and receiving the part of the other agents' allotment change that is feasible for him. In Corollaries 4.1 and 4.2 we prove that if a reallocation rule satisfies *Pareto optimality*, *strategy-proofness*, and either *weak no-envy* or *equal treatment of equals*, then this rule assigns the uniform reallocation to economies with excess demand.

In Section 4.3.2, we study *endowment strategy-proofness*. If individual endowments are private information, then it might happen that agents manipulate the outcome by

[2]Here, the feasible part of an allotment change yields either a nonnegative allotment or the allotment zero.

only reporting, or exhibiting, a smaller part of their individual endowments. Reallocation rules where no agent can ever profit from withholding a part of his endowment are called *endowment strategy-proof*. First, we prove that the uniform reallocation rule satisfies *endowment strategy-proofness* (see Lemma 4.7). Furthermore, we prove that if a reallocation rule satisfies *Pareto optimality*, *endowment strategy-proofness*, and *weak no-envy*, then this rule assigns the uniform reallocation to economies with excess demand (see Corollary 4.4).

The first property we study in Section 4.3.3 is *demand-onliness*: the agents' allotment changes only depend on the agents' demands. As an extension of Ching's characterization of the uniform allocation rule by *Pareto optimality*, *strategy-proofness*, and *equal treatment of equals* (see Theorem 2.4) we prove that the uniform reallocation rule is the only rule that satisfies *Pareto optimality*, *strategy-proofness*, *equal treatment of equals*, and *demand-onliness* (see Corollary 4.5). Then, it follows easily that the uniform reallocation rule is the only rule that satisfies *Pareto optimality*, *strategy-proofness*, *anonymity*, and *demand-onliness* (see Corollary 4.6). This characterization can be interpreted as an extension of one of Sprumont's characterization of the uniform allocation rule (see Theorem 2.1) to the reallocation model.

In the remainder of Section 4.3.3, we study a strengthening of *demand-onliness*, called *reversibility*. *Reversibility* links the allotment changes of excess demand economies to the allotment changes of certain excess supply economies. To be more precise, consider a situation with demanders, having their peaks above their initial endowments, and suppliers, having their peaks below their initial endowments. Now this situation is reversed by turning demanders into suppliers whose supplies are equal to their former demands and suppliers into demanders in a similar way. Then, *reversibility* requires that the allotment changes of the latter problem are opposite to those of the former. In Theorem 4.2 we show that the uniform reallocation rule is the only rule that satisfies *Pareto optimality*, *equal treatment of equals*, *endowment strategy-proofness*, and *reversibility*.

The last property we introduce in Section 4.3.3 is a weakening of *demand-onliness* that is well-known from the allocation model. *Peak-onliness*, loosely speaking, requires that the reallocation rule only depends on the peaks instead of the complete preference profile. As an extension of Thomson's characterization of the uniform allocation rule by *Pareto optimality*, *no-envy*, and *peak-onliness* (see Theorem 2.3) we prove that the uniform reallocation rule is the only rule that satisfies *Pareto optimality*, *no-envy*, and *peak-onliness* (see Theorem 4.3).

Section 4.4 is devoted to so-called solidarity properties. By "solidarity" we mean that if certain changes in the model parameters (*e.g.*, the population, one agent's preference relation, or one agent's individual endowment) occur, then agents that are not responsible for this change should all (weakly) lose together or (weakly) gain together. Similar to the allocation model the solidarity properties are only compatible with other desirable properties (*e.g.*, *Pareto optimality* and *no-envy*) if they apply to changes that do not turn an economy with excess demand into an economy with excess supply, or *vice versa*. Therefore we concentrate on this "one-sided" version of the solidarity properties.

The first solidarity property we study is *one-sided population-monotonicity*. In the

reallocation model this property describes the impact of merging two economies. We show that the uniform reallocation rule is the only rule that satisfies *Pareto optimality*, *weak no-envy*, and *one-sided population-monotonicity* (see Theorem 4.4). This characterization can be interpreted as an extension of the following characterization (see Theorem 2.6) to the reallocation model: the uniform allocation rule is the only rule that satisfies *Pareto optimality*, *no-envy*, *one-sided population-monotonicity*, and *replication-invariance*, *i.e.*, if an economy is replicated, then also the assigned allocation must be replicated.

A further solidarity property, introduced by Thomson for the allocation model is *one-sided replacement-domination*: if one agent changes his preference relation, then either all remaining agents (weakly) lose or all (weakly) gain. It turns out that in the reallocation model as well as in the allocation model the uniform rule is the only rule that satisfies *Pareto optimality*, *no-envy*, *one-sided replacement-domination*, and *replication-invariance* (see Theorem 4.5 for the uniform reallocation rule and Theorem 2.7 for the uniform allocation rule characterization).

The last solidarity type property we consider is *one-sided endowment-monotonicity*. It describes the change of the solution if certain unilateral changes in the individual endowments occur: if the change of one agent's individual endowment is one-sided, then either all agents (weakly) lose or all (weakly) gain. A stronger notion of *one-sided endowment-monotonicity* is the following *strong one-sided endowment-monotonicity*. If in case of excess demand an agent's individual endowment decreases, then no agent is made better off in the reallocation that follows this one-sided change. Similarly, if in case of excess supply an agent's individual endowment increases, then no agent is made better off in the reallocation that follows this one-sided change.

The two endowment-monotonicity properties we mentioned above can be seen as extensions of the *one-sided resource-monotonicity* as introduced by Thomson for the allocation model. By Theorem 2.5, the uniform allocation rule is the only rule that satisfies *Pareto optimality*, *no-envy*, and *one-sided resource-monotonicity* for a restricted domain of single-peaked, continuous preferences. For the reallocation case, a similar result (see Theorem 4.6) can be deduced: the uniform reallocation rule is the only rule that satisfies *Pareto optimality*, *weak no-envy*, and *one-sided endowment-monotonicity*. In contrast to the allocation model, the proof of Theorem 4.6 is based on different arguments than the proof of Theorem 2.5. Furthermore, Theorem 4.6 remains valid for the whole domain of single-peaked preferences. After showing that *individual rationality* and *strong one-sided endowment-monotonicity* imply *Pareto optimality*, it follows easily that the uniform reallocation rule is the only rule that satisfies *individual rationality*, *strong one-sided endowment-monotonicity*, and *weak no-envy* (see Corollary 4.9).

The next property of the uniform reallocation rule we study is *bilateral consistency*. In the allocation model, *bilateral consistency* requires the following. Consider an allocation situation where all agents except two leave with their assigned shares of the social endowment. Then, the corresponding reduced economy is formed by the remaining agents, their preference relations, and the remaining "social" endowment. If for this reduced economy the shares for the two agents are the same as before, then *bilateral*

consistency is satisfied. Thomson (1994b) provides two characterizations of the uniform rule by means of *Pareto optimality*, *bilateral consistency*, and *continuity* with respect to the social endowment. In the first characterization (see Theorem 2.8) *no-envy* singles out the uniform rule. In the second characterization (see Theorem 2.9) *no-envy* is replaced by *individual rationality from equal division*: no agent, after the allocation, is worse off than in the case of equally dividing the social endowment. In a recent study, Dagan shows that under certain circumstances the continuity property can be skipped (see Theorems 2.11 and 2.12).

Now, consider a reallocation situation where all agents except two leave with their assigned shares of the good. Then, the corresponding reduced economy is formed by the remaining agents, their preference relations, and individual endowments where the leftover (supply or demand) of the departing agents is, up to domain restrictions, added equally to the original individual endowments of the remaining agents. If for this reduced economy the shares for the two agents are the same as before, then *bilateral consistency* is satisfied. As an extension of the results we mentioned above, we show that the uniform reallocation rule is the only rule that satisfies *Pareto optimality*, *bilateral consistency*, and either *individual rationality* or *weak no-envy* (see Theorems 4.9 and 4.10). Further characterizations of the uniform reallocation rule can be obtained by *Pareto optimality*, *bilateral consistency* and extra conditions, for instance *boundedness* (of the outcome) *by endowments and peaks* (see Theorem 4.8), or *endowment strategy-proofness* and the *dummy property*: agents who have their peak as individual endowment do not participate in the reallocation (see Theorem 4.11).

4.2 The model

There is an infinite population of potential agents, indexed by the natural numbers $\mathbb{N}$. Each agent $i \in \mathbb{N}$ is described in terms of an *individual endowment* $\omega_i \in \mathbb{R}_+$ of an infinitely divisible commodity, and a continuous and single-peaked preference relation R_i defined over the non-negative real numbers $\mathbb{R}_+$. Single-peakedness of R_i means that there exists a point $p(R_i) \in \mathbb{R}_+$, called the *peak of agent i*, with the following property: for all $x, y \in \mathbb{R}_+$ with $x < y \leq p(R_i)$ or $x > y \geq p(R_i)$, we have $y \, P_i \, x$.[3] Each preference relation R_i can be described in terms of the *preference function* $r_i \colon \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ that assigns to each point $x \in \mathbb{R}_+$ the point on the other side of $p(R_i)$ that is indifferent to x, if such a point exists, and otherwise 0 or ∞, depending on which side of the peak x is. So, for all $x \in \mathbb{R}_+$ the following holds.

$$\text{If } x \leq p(R_i), \text{ then } r_i(x) \geq p(R_i) \text{ and } \begin{cases} r_i(x) \, I_i \, x & \text{if such a point} \\ & \text{exists, or} \\ r_i(x) = \infty & \text{otherwise.} \end{cases}$$

[3] As usual, $x \, R_i \, y$ is interpreted as "x is weakly preferred to y", and $x \, P_i \, y$ as "x is strictly preferred to y". Furthermore, $x \, I_i \, y$ means that agent i is indifferent between x and y.

$$\text{If } x \geq p(R_i), \text{ then } r_i(x) \leq p(R_i) \text{ and } \begin{cases} r_i(x)\, I_i\, x & \text{if such a point exists, or} \\ r_i(x) = 0 & \text{otherwise.} \end{cases}$$

By $\mathcal{R}$ we denote the class of all continuous, single-peaked preference relations over $\mathbb{R}_+$. By $\mathcal{N}$ we denote the class of non-empty and finite subsets of $\mathbb{N}$. For $N \in \mathcal{N}$, $\mathcal{R}^N$ denotes the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. For profile $R \in \mathcal{R}^N$, $p(R) = (p(R_i))_{i \in N}$ denotes the corresponding *vector of peaks*.

Now, an economy is formalized as follows. Let $N \in \mathcal{N}$ and $\omega = (\omega_i)_{i \in N} \in \mathbb{R}^N_+$ be a vector of individual endowments that have to be reallocated among the agents in N with profile $R \in \mathcal{R}^N$.[4] We call a pair $e = (R, \omega) \in \mathcal{R}^N \times \mathbb{R}^N_+$ an *economy*. Let $\mathcal{E}^N = \mathcal{R}^N \times \mathbb{R}^N_+$ be the class of all economies. A *feasible allocation for* $e = (R, \omega) \in \mathcal{E}^N$, or a *reallocation* for short, is a vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = \sum_N \omega_i$. A *reallocation rule* φ, or a *rule* for short, is a function that assigns to every $N \in \mathcal{N}$ and every $e = (R, \omega) \in \mathcal{E}^N$ a reallocation, denoted by $\varphi(e)$.

Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$. Given $i \in N$, we call $\varphi_i(e)$ the share or *allotment* of agent i and $\Delta\varphi_i(e) := \varphi_i(e) - \omega_i$ his *allotment change*. Agent i is *satiated at* $\varphi(e)$ if $p(R_i) = \varphi_i(e)$ and *non-satiated at* $\varphi(e)$ if $p(R_i) \neq \varphi_i(e)$.

Until now, the "reallocation model" we introduce in this chapter almost equals the "allocation model" as introduced in Chapter 2: the only difference is that we have a vector of individual endowments $\omega \in \mathbb{R}^N_+$ instead of a social endowment $\Omega \in \mathbb{R}_+$. In contrast with the allocation model, not only the individual preferences but also the individual endowments discriminate the agents from each other. Some of these discriminations, for example having a greater endowment than one's peak, turn out to be important in the sequel.

Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$. We call agent $i \in N$ a *demander* if his endowment is strictly less than his peak: he "demands" $p(R_i) - \omega_i$ units of the commodity. For all $i \in N$, we denote this *demand* by $d_i(e)$ and the set of demanders by $D(e)$. We call agent $i \in N$ a *supplier* if his endowment is strictly greater than his peak: he wants to "supply" $\omega_i - p(R_i)$ units of the commodity. For all $i \in N$, we denote this *supply* by $s_i(e)$ and the set of suppliers by $S(e)$. We call agent $i \in N$ a *non-trader* if his endowment is equal to his peak: he favors no trade, *i.e.*, $d_i(e) = s_i(e) = 0$.[5] Let $d(e) := \sum_{i \in D(e)} d_i(e)$ denote *total demand* and $s(e) := \sum_{i \in S(e)} s_i(e)$ *total supply*. Furthermore, let $z(e) := \sum_{i \in N} d_i(e)$ denote *excess demand*. The latter may be positive, zero, or negative. If it is zero, then e is *balanced*. If it is positive, then e is an economy with *excess demand*. If it is negative, then e is an economy with *excess supply*. Note that for all $i \in N$, $d_i(e) = -s_i(e)$ and $z(e) = d(e) - s(e)$.

[4] Note that free disposal of the commodity is not allowed.

[5] Our notion of demanders, suppliers, and non-traders is somewhat different from what it is in "classical exchange economies". There, the types of the agents not only depend on their own characteristics but also on prices. However, for economies with individual endowments and single-peaked preferences derived from fixed-price exchange economies, the notions coincide.

The first property of rules we consider is the well-known requirement of *Pareto optimality*: a reallocation assigned by the rule cannot be changed in such a way that no agent is worse off and some agent is better off.

Because from the context it will be clear if a property is related to an allocation or a reallocation rule, the same names will be used.

Pareto optimality: For all $N \in \mathcal{N}$ and all $e = (R, \omega) \in \mathcal{E}^N$, there is no reallocation $x \in \mathbb{R}^N_+$, such that for all $i \in N$, $x_i \, R_i \, \varphi_i(e)$, and for some $j \in N$, $x_j \, P_j \, \varphi_j(e)$.

Note, that *Pareto optimality* for reallocation rules is the same as *Pareto optimality* for allocation rules as discussed in Chapter 2 (see Section 2.1, page 19). Similarly as before it follows easily that a rule is *Pareto optimal* if and only if it is *same-sided*, that is: for all $N \in \mathcal{N}$ and all $e = (R, \omega) \in \mathcal{E}^N$, either [for all $i \in N$, $\varphi_i(e) \leq p(R_i)$], or [for all $i \in N$, $\varphi_i(e) \geq p(R_i)$].

Examples of *Pareto optimal* reallocations can be derived as follows.

Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ be an economy with excess demand. In this case, it is the set of demanders that are on the long side of the market.[6] One can argue that it is reasonable to reward the agents on the short side of the market, *i.e.*, suppliers and non-traders, with their peaks and to reallocate the so obtained supply *Pareto optimally* among the demanders.[7]

For $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ such that the economy is balanced the only *Pareto optimal* reallocation is the vector of peaks $p(R)$.

Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ be an economy with excess supply. In this case, it is the set of suppliers that are on the long side of the market. Now, one can argue that it is reasonable to let the suppliers *Pareto optimally* supply part of their endowments so that the agents on the short side of the market, *i.e.*, demanders and non-traders, receive their peaks.

In summary, a *Pareto optimal* reallocation can be derived by assigning to the agents on the short side of the market their peaks and by reallocating the difference between the sum of their individual endowments and their peaks, possibly a negative amount, among the remaining agents according to a *Pareto optimal* allocation rule (see Chapter 2). Now, applying the notions of fairness we already considered in Chapter 2 to this (re)allocation yields the following reallocation rules.

A rule that incorporates the proportional principle, and therefore embodies Aristotle's notion of justice, is the proportional (reallocation) rule Pr^r (for the proportional allocation rule, see Section 2.1, page 19). In case of excess demand, suppliers and non-traders receive their peaks. The amount they supply is divided among the remaining agents so that each of them receives in addition to his individual endowment an amount

[6]We follow Benassy's terminology of the "short" and the "long" side of a market, Benassy (1982), page 10: "The "short" side of a market is that where the aggregate volume of desired transactions is smallest. It is thus the demand side if there is excess supply, the supply side if there is excess demand. The other side is the "long" side."

[7]Note that we consider non-traders to be on the short side of the market.

that is proportional to his demand. If the economy is balanced, each agent receives his peak. In case of excess supply, each supplier supplies a part of his endowment that is proportional to his supply in such a way that demanders and non-traders receive their peaks.

Proportional rule Pr^r: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all $j \in N$,

$$Pr_j^r(e) := \begin{cases} \min\{p(R_j), \omega_j + \lambda d_j(e)\} & \text{if } z(e) \geq 0, \\ \max\{p(R_j), \omega_j + \lambda s_j(e)\} & \text{if } z(e) \leq 0, \end{cases}$$

where $\lambda \in [0, 1]$ solves $\sum_N Pr_i^r(e) = \sum_N \omega_i$.

The principle of equal reallocation under the additional assumption of *Pareto optimality* is described by the uniform (*r*eallocation) rule U^r (for the uniform allocation rule, see Section 2.1, page 19). In case of excess demand, suppliers and non-traders receive their peaks. The amount they supply is divided among the remaining agents so that each of them receives in addition to his individual endowment an amount that is as equal as possible, with the agents' peaks as upper bound. If the economy is balanced, each agent receives his peak. In case of excess supply, the reallocation is dual to the excess demand case: all demanders and non-traders receive their peaks. The amount they demand is subtracted from the remaining agents so that each of them diminishes his individual endowment by an amount that is as equal as possible, with the agents' peaks as lower bound.

Benassy (1982), Appendix J, referred to the uniform (*r*eallocation) rule U^r in the slightly different setting of rationing.

Uniform rule U^r: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all $j \in N$,

$$U_j^r(e) := \begin{cases} \min\{p(R_j), \omega_j + \lambda\} & \text{if } z(e) \geq 0, \\ \max\{p(R_j), \omega_j + \lambda\} & \text{if } z(e) \leq 0, \end{cases}$$

where $\lambda \in \mathbb{R}$ solves $\sum_N U_i^r(e) = \sum_N \omega_i$.

So, agents are either satiated or receive the same (maximal or minimal) allotment change. In fact, combined with *Pareto optimality* this determines exactly the uniform rule U^r. For later reference, we state this observation as a lemma.[8] [9]

[8] Schummer and Thomson (1997) note that the uniform reallocation rule assigns the unique Pareto optimal reallocation at which the difference between the smallest and the largest allotment change received by any agent is minimal. Furthermore, they remark that the uniform reallocation rule assigns the unique Pareto optimal reallocation at which the variance of the agents' allotment changes is minimal.

[9] Similar to the uniform allocation rule, the uniform reallocation rule can be seen as a Walrasian solution. Thomson (1995d) states that for exchange economies with a single commodity, the uniform reallocation rule U^r is as a special case of Mas-Colell's (1992) "Walrasian solution with slack".

Lemma 4.1 *Let φ be a Pareto optimal rule. Then $\varphi = U^r$ if and only if for all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all non-satiated agents $j \in N$,*

(a) *if $z(e) > 0$, then $\Delta\varphi_j(e) = \max\{\Delta\varphi_i(e) \mid i \in N\}$ and*

(b) *if $z(e) < 0$, then $\Delta\varphi_j(e) = \min\{\Delta\varphi_i(e) \mid i \in N\}$.*

Examples demonstrating the computation of the uniform rule U^r are given in Chapter 1, Examples 1.5 and 1.6.

It is easy to prove that the uniform reallocation rule U^r is an extension of the uniform allocation rule U in the following sense.

Lemma 4.2 *For all $N \in \mathcal{N}$ and all $(R, \Omega) \in \mathcal{R}^N \times \mathbb{R}_+$,*

$$U(R, \Omega) = U^r\left(R, (\frac{\Omega}{|N|}, \ldots, \frac{\Omega}{|N|})\right).$$

Remark 4.1 In the remainder of this chapter, we are mainly interested in the uniform rule U^r, its properties, its characterizations,[10] and the trade-off between the properties in these characterizations. For further results of the reallocation model, for example other reallocation rules or examples of incompatibility of properties, we refer the reader to Barberà, Jackson, and Neme (1997), Moreno (1996), and Thomson (1995d).

Furthermore, it is worth noting that sometimes allocation rule properties applied to an economy (R, Ω) are equivalent to the corresponding reallocation rule properties applied to the corresponding economy $\left(R, (\frac{\Omega}{|N|}, \ldots, \frac{\Omega}{|N|})\right)$. Hence, using this logical relation between allocation and reallocation rule properties, examples of incompatibility of properties for the reallocation model can be easily constructed from the known examples for the allocation model (see for instance Thomson 1995d, Proposition 3). □

Remark 4.2 In the remainder of this chapter we present several characterizations of the uniform reallocation rule that can be seen as extensions of existing characterizations of the uniform allocation rule. Some of the proofs we present here are very similar to the proofs as given in the allocation setting (this will be indicated). However, we would like to emphasize that, it is not possible to just "translate" all proofs from the "allocation language" into the "reallocation language". The reallocation model contains some important structural differences to the allocation model.

For instance, in the reallocation model we can divide the set of agents into the agents on the long side of the market, *e.g.*, demanders in case of excess demand, and agents on the short side of the market, *e.g.*, suppliers and non-traders in case of excess demand. So, in each proof of a characterization of the uniform reallocation rule it has to be shown that agents on the short side of the market receive their peaks. This step in the proofs does not have a counterpart in the allocation model.

Another very important difference between the "reallocation proofs" and the "allocation proofs" is that in the reallocation model it is not true that in proofs the

[10]Unless mentioned otherwise, the expression "characterization" implies the logical independence of the characterizing properties. For all characterizations appearing in this chapter, however, the discussion of logical independence of the axioms is postponed to Section 4.6.

arguments for excess demand economies and excess supply economies are symmetric. This asymmetry in the reallocation model is related to the following observation. In the reallocation model negative allotment changes can occur, and since we do not allow for negative allotments for these changes, the lower bound zero may be "active", *i.e.*, an agent's allotment reaches zero and therefore cannot be further diminished. Positive allotment changes are not bounded from above. As a consequence, depending on the properties that are involved, one case of a proof—the part of the proof where the lower bound zero may be active—is generally somewhat more complicated than the other case. Therefore, if the proofs for excess demand and excess supply economies are very similar up to this asymmetry, we will concentrate on the part of the proof where the lower bound zero is possibly active. □

4.3 Strategy-proofness, no-envy, and peak-onliness

4.3.1 Strategy-proofness and no-envy

Similar to Sprumont (1991), we start our axiomatic analysis with the well-known property of *strategy-proofness:* no agent ever benefits from misrepresenting his preferences (see also Section 2.2, page 22).[11] Before we give the formal definition, we introduce some notation.

Let $N, M \in \mathcal{N}$ be such that $N \subseteq M$. Let $M \backslash N = \{i \in M \mid i \notin N\}$. For $R \in \mathcal{R}^M$ the *restriction* $(R_i)_{i \in N} \in \mathcal{R}^N$ *of* R *to* N is denoted by R_N. Let $R, \bar{R} \in \mathcal{R}^N$, $i \in N$, and $R_{N \backslash \{i\}} = \bar{R}_{N \backslash \{i\}}$. Then, we call $\bar{R}$ an *i-deviation from* R.

Strategy-proofness: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, all $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R, then $\varphi_j(e)\, R_j\, \varphi_j(\bar{e})$.

As already observed by Benassy (1982) in the context of rationing, the uniform rationing mechanism that corresponds to the uniform reallocation rule U^r is *strategy-proof.*

Lemma 4.3 *The uniform rule U^r satisfies strategy-proofness.*

Proof. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$, and $j \in N$ be such that $\bar{R}$ is a j-deviation from R. In order to prove *strategy-proofness* of U^r, we have to show that

$$U_j^r(e)\, R_j\, U_j^r(\bar{e}). \tag{4.1}$$

Case 1: $z(e) \leq 0$. Hence, $U_j^r(e) = \max\{p(R_j), \omega_j + \lambda\}$, where λ solves $\sum_N U_i^r(e) = \sum_N \omega_i$. If $U_j^r(e) = p(R_j)$, then (4.1). Otherwise, $U_j^r(e) = \omega_j + \lambda > p(R_j)$. Hence, $j \in S(e)$. We distinguish two subcases.

[11] In game theoretical terms a reallocation rule is *strategy-proof* if in the direct revelation game form it is a weakly dominant strategy for each agent to announce his true preference relation.

Case 1.1: $p(\bar{R}_j) < \omega_j + \lambda$. Observe that agent j is still a supplier in the economy $\bar{e}$. Consequently, $S(\bar{e}) = S(e)$ and by feasibility and *same-sidedness*,

$$\begin{aligned} s(\bar{e}) &= \sum_{S(\bar{e})} (\omega_i - p(\bar{R}_i)) \\ &= (\omega_j - p(\bar{R}_j)) + \sum_{S(e)\setminus\{j\}} (\omega_i - p(R_i)) \\ &> -\lambda + \sum_{S(e)\setminus\{j\}} (\omega_i - U_i^r(e)) \\ &= \sum_{S(e)} (\omega_i - U_i^r(e)) \\ &= \sum_{D(e)} (p(R_i) - \omega_i) \\ &= \sum_{D(\bar{e})} (p(\bar{R}_i) - \omega_i) = d(\bar{e}). \end{aligned}$$

Hence, $z(\bar{e}) < 0$. Therefore, $U_j^r(\bar{e}) = \max\{p(\bar{R}_j), \omega_j + \bar{\lambda}\}$, where $\bar{\lambda}$ solves $\sum_N U_i^r(\bar{e}) = \sum_N \omega_i$. Because $p(\bar{R}_j) < \omega_j + \lambda$ and $p(\bar{R}_i) = p(R_i)$ for $i \in N\setminus\{j\}$ it follows that $\bar{\lambda} = \lambda$. Hence, $U_j^r(\bar{e}) = \omega_j + \lambda = U_j^r(e)$. This implies (4.1).

Case 1.2: $p(\bar{R}_j) \geq \omega_j + \lambda$. If $z(\bar{e}) < 0$, then $U_j^r(\bar{e}) = \max\{p(\bar{R}_j), \omega_j + \bar{\lambda}\} \geq \omega_j + \lambda = U_j^r(e) \geq p(R_j)$. If $z(\bar{e}) \geq 0$, then $U_j^r(\bar{e}) = \min\{p(\bar{R}_j), \omega_j + \bar{\lambda}\} \geq \omega_j + \lambda = U_j^r(e) \geq p(R_j)$. This implies (4.1).

Case 2: $z(e) > 0$. Similar to Case 1. ■

It is easy to show, that the proportional rule Pr^r is not *strategy-proof*.

Next, we prove some implications of *Pareto optimality* and *strategy-proofness*. First we show that *Pareto optimality* and *strategy-proofness* imply *own-peak monotonicity*: an agent's allotment is monotonic with respect to his peak.

Own-peak monotonicity: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, all $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R and $p(R_j) \leq p(\bar{R}_j)$, then $\varphi_j(e) \leq \varphi_j(\bar{e})$.

Ching introduced *own-peak monotonicity* for allocation rules and proved that it is implied by *Pareto optimality* and *strategy-proofness* (see Ching 1994, Lemma 1). This result extends to reallocation rules and the structure of our proof is similar to Ching's proof.

Lemma 4.4 *Let the rule φ satisfy Pareto optimality and strategy-proofness. Then φ satisfies own-peak monotonicity.*

Proof. Let φ be a rule that satisfies *Pareto optimality* and *strategy-proofness*. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$, and $j \in N$ be such that $\bar{R}$ is a j-deviation from R and $p(R_j) \leq p(\bar{R}_j)$. In order to prove *own-peak monotonicity* of φ, we have to show that $\varphi_j(e) \leq \varphi_j(\bar{e})$. Suppose, by contradiction, that

$$\varphi_j(e) > \varphi_j(\bar{e}). \tag{4.2}$$

Strategy-proofness for the j-deviation $\bar{R}$ from R implies $\varphi_j(e)\, R_j\, \varphi_j(\bar{e})$. Hence, by (4.2), $p(R_j) > \varphi_j(\bar{e})$. *Strategy-proofness* for the j-deviation R from $\bar{R}$ implies $\varphi_j(\bar{e})\, \bar{R}_j\, \varphi_j(e)$. Hence, by (4.2), $p(\bar{R}_j) < \varphi_j(e)$.

Thus, by $p(R_j) \leq p(\bar{R}_j)$, it follows that $\varphi_j(\bar{e}) < p(R_j) \leq p(\bar{R}_j) < \varphi_j(e)$.

Since $p(R_j) < \varphi_j(e)$, by *same-sidedness* it follows that for all $i \in N$, $p(R_i) \leq \varphi_i(e)$. Similarly, since $p(\bar{R}_j) > \varphi_j(e)$, by *same-sidedness* it follows that for all $i \in N$, $p(\bar{R}_i) \geq \varphi_i(\bar{e})$.

Because for all $i \in N\backslash\{j\}$, $p(R_i) = p(\bar{R}_i)$, it follows that for all $i \in N\backslash\{j\}$, $\varphi_i(\bar{e}) \leq \varphi_i(e)$. Furthermore, by (4.2), $\varphi_j(\bar{e}) < \varphi_j(e)$. Since $\varphi(e)$ and $\varphi(\bar{e})$ are reallocations of economies with the same endowment vector ω, it follows that $\sum_N \omega_i = \sum_N \varphi_i(\bar{e}) < \sum_N \varphi_i(e) = \sum_N \omega_i$. This is a contradiction. ■

An immediate, but important, consequence of *own-peak monotonicity* of a rule φ is *individual peak-onliness:* if an agent changes his preference relation without changing his peak, his allotment does not change.

Individual peak-onliness: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, all $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R and $p(R_j) = p(\bar{R}_j)$, then $\varphi_j(e) = \varphi_j(\bar{e})$.

Next, we show that *Pareto optimality* and *strategy-proofness* imply *uncompromisingness*, a bounded invariance property: if an agent's peak differs from the allotment assigned by the rule, then his allotment does not change if he unilaterally deviates from his original preference relation in such a way that his new peak and his original peak are at the same side of the allotment.

Uncompromisingness: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, all $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R and either $[p(R_j) > \varphi_j(e)$ and $p(\bar{R}_j) \geq \varphi_j(e)]$ or $[p(R_j) < \varphi_j(e)$ and $p(\bar{R}_j) \leq \varphi_j(e)]$, then $\varphi_j(e) = \varphi_j(\bar{e})$.

Uncompromisingness is a well-known property in connection with *strategy-proofness* (see for instance Border and Jordan (1983) in the context of public good economies). Ching considered *uncompromisingness* for allocation rules and proved that it is implied by *Pareto optimality* and *strategy-proofness* (see Ching 1994, Lemma 2). This result extends to reallocation rules and the structure of our proof is similar to Ching's proof.

Lemma 4.5 *Let the rule φ satisfy Pareto optimality and strategy-proofness. Then φ satisfies uncompromisingness.*

Proof. Let φ be a rule that satisfies *Pareto optimality* and *strategy-proofness*. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$, and $j \in N$ be such that $\bar{R}$ is a j-deviation from R.

Case 1: $p(R_j) < \varphi_j(e)$ and $p(\bar{R}_j) \leq \varphi_j(e)$. By *same-sidedness*, it follows that for all $i \in N$, $p(R_i) \leq \varphi_i(e)$. In order to prove *uncompromisingness* of φ, we have to show that $\varphi_j(e) = \varphi_j(\bar{e})$. Suppose, by contradiction, that $\varphi_j(e) \neq \varphi_j(\bar{e})$.

If $\varphi_j(\bar{e}) > \varphi_j(e)$, then it follows that $p(\bar{R}_j) \leq \varphi_j(e) < \varphi_j(\bar{e})$. Hence, in contradiction to *strategy-proofness*, $\varphi_j(e)\, \bar{P}_j\, \varphi_j(\bar{e})$.

If $\varphi_j(\bar{e}) < \varphi_j(e)$, then let $e' = (R', \omega) \in \mathcal{E}^N$, be such that R' is a j-deviation from R with $p(R'_j) = p(R_j)$ and $\varphi_j(\bar{e})\, P'_j\, \varphi_j(e)$. By *individual peak-onliness* it follows that $\varphi_j(e') = \varphi_j(e)$. Hence, for the j-deviation $\bar{R}$ from R' it follows that $\varphi_j(\bar{e})\, P'_j\, \varphi_j(e')$. This is a contradiction to *strategy-proofness*.

Case 2: $p(R_j) > \varphi_j(e)$ and $p(\bar{R}_j) \geq \varphi_j(e)$. Similar to Case 1. ■

The next property we analyze is *no-envy* (see also Section 2.2, page 22). In contrast with the allocation model, agents are now not only characterized by their preferences, but also by their individual endowments. One could interpret these individual endowments as individual rights. In order to incorporate the individual endowments into our notion of *no-envy*, we formulate *no-envy* in terms of trades or allotment changes rather than in terms of net allotments: no agent strictly prefers the allotment change of another agent, or the part of another agent's allotment change that is feasible for him, to his own allotment change.

Let $\alpha \in \mathbb{R}$. Then, $\alpha^+ := \max\{0, \alpha\}$.

No-envy: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all $i, j \in N$, $\varphi_j(e)\, R_j\, (\omega_j + \Delta\varphi_i(e))^+$.

So, agent j only envies agent i if he prefers i's allotment change, or the part of i's allotment change that is feasible for him, to his own allotment change.

The well-known property of *no-envy* can be traced back to Foley (1967) who considered it the context of resource allocation problems. A concept of *no-envy* in terms of allotment changes—called fair net trade—as introduced above was formulated by Schmeidler and Vind (1972) in the more general context of exchange economies.

Lemma 4.6 *The uniform rule U^r satisfies no-envy.*

Proof. Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$. In order to prove *no-envy* of U^r, we have to show that for all $i, j \in N$,

$$U^r_j(e)\, R_j\, (\omega_j + \Delta U^r_i(e))^+. \tag{4.3}$$

Case 1: $z(e) \leq 0$. Hence, $U^r_j(e) = \max\{p(R_j), \omega_j + \lambda\}$, where λ solves $\sum_N U^r_i(e) = \sum_N \omega_i$. If $U^r_j(e) = p(R_j)$, then (4.3). Otherwise, $U^r_j(e) = \omega_j + \lambda > p(R_j)$. Hence, agent j is non-satiated at $U^r(e)$. Then, by Lemma 4.1, for all $i \in N$, $\Delta U^r_j(e) \leq \Delta U^r_i(e)$. Since $U^r_j(e) = \omega_j + \Delta U^r_j(e) \geq 0$, it follows that for all $i \in N$, $\omega_j + \Delta U^r_i(e) \geq 0$. Hence, for all $i \in N$, $p(R_j) < U^r_j(e) = \omega_j + \Delta U^r_j(e) \leq \omega_j + \Delta U^r_i(e) = (\omega_j + \Delta U^r_i(e))^+$. This implies (4.3).

Case 2: $z(e) > 0$. Similar to Case 1. ■

Next, we introduce the slightly weaker notion of *weak no-envy*.

Weak no-envy: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all $i, j \in N$ with $\omega_j + \Delta\varphi_i(e) \geq 0$, $\varphi_j(e)\, R_j\, (\omega_j + \Delta\varphi_i(e))$.

So, agent j only envies agent i if he prefers i's allotment change, added to his endowment, to his own allotment—provided the former is feasible.

It is easy to show that the proportional rule Pr^r satisfies neither *no-envy* nor *weak no-envy*.

No-envy, for allocation problems with single-peaked preferences, is one of the conditions that led Sprumont (1991) to one of his characterizations of the uniform allocation rule: the uniform rule U is the only rule that satisfies *Pareto optimality*, *strategy-proofness*, and *no-envy* (see Theorem 2.2). Ching (1992, Theorem 1) gives a simple and elegant proof of this result. Next, we show that this result extends to reallocation rules.

Theorem 4.1 *The uniform rule U^r is the only rule that satisfies Pareto optimality, strategy-proofness, and no-envy.*

One possibility to prove Theorem 4.1 is to adjust Ching's proof for the allocation model to our reallocation model. The interested reader can find this version of the proof in Appendix B. Here, we present a proof of Theorem 4.1 that is structured similarly to the proof of Ching's characterization of the uniform allocation rule by *Pareto optimality*, *equal treatment of equals*, and *strategy-proofness* (see Theorem 2.4). The advantage of this somewhat longer proof is that by only small adjustments of the proof we can easily derive some related results (see Corollaries 4.1, 4.2, and 4.5).

Proof. The uniform rule U^r is *same-sided* and therefore *Pareto optimal*. As shown in Lemmas 4.3 and 4.6, U^r satisfies *strategy-proofness* and *no-envy*.

To prove the remaining part of the theorem let φ be a rule that satisfies the properties listed in the theorem. By Lemmas 4.4 and 4.5, both φ and U^r satisfy *own-peak monotonicity* and *uncompromisingness*.

Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and suppose, by contradiction, that $\varphi(e) \neq U^r(e)$. Since φ is *Pareto optimal* it follows that either $z(e) > 0$ or $z(e) < 0$.

Case 1: $z(e) > 0$. We call a demander j a *maximal demander* if $j \in \arg\max\{d_i(e) \mid i \in D(e)\}$. We denote the *set of maximal demanders* of an economy $e \in \mathcal{E}^N$ by $D_{\max}(e)$. By $d_{\max} := \max\{d_i(e) \mid i \in D(e)\}$ we denote the *maximal demand*.

Case 1.1: All agents are maximal demanders, *i.e.*, $D_{\max}(e) = N$. Hence, $U^r(e) = \omega$ and for all $i \in N$, $d_i(e) = d_{\max}$.

Since $\varphi(e) \neq U^r(e)$, by Lemma 4.1, there exist agents $j, k \in N$ such that $\varphi_j(e) < p(R_j)$ and $\Delta\varphi_j(e) < \Delta\varphi_k(e)$. *Same-sidedness* implies that $\Delta\varphi_k(e) \leq d_{\max}$. Hence, $\varphi_j(e) = \omega_j + \Delta\varphi_j(e) < \omega_j + \Delta\varphi_k(e) = (\omega_j + \Delta\varphi_k(e))^+ \leq p(R_j)$. Thus, in contradiction to *no-envy*, $(\omega_j + \Delta\varphi_k(e))\, P_j\, \varphi_j(e)$.

Case 1.2: The set of maximal demanders does not equal the set of agents, *i.e.*, $D_{\max}(e) \subsetneq N$.

Since $\varphi(e) \neq U^r(e)$, by Lemma 4.1, there exist agents $j, k \in N$ such that $\varphi_j(e) < p(R_j)$ and $\Delta\varphi_j(e) < \Delta\varphi_k(e)$. Furthermore, by the same argument as in Case 1.1, $j \notin D_{\max}(e)$. Let $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$ such that $\bar{R}$ is a j-deviation from R with $p(\bar{R}_j) = \omega_j + d_{\max}$. We consider the following two subcases.

Case 1.2.1: $\varphi_j(e) < U_j^r(e) \leq p(R_j)$. Since $p(R_j) < p(\bar{R}_j)$, by *uncompromisingness* of φ it follows that $\varphi_j(\bar{e}) = \varphi_j(e)$. *Own-peak monotonicity* of U^r yields $U_j^r(e) \leq U_j^r(\bar{e})$. Hence, $\varphi_j(\bar{e}) < U_j^r(\bar{e})$.

Case 1.2.2: $U_j^r(e) < \varphi_j(e) \leq p(R_j)$. Similar to Case 1.2.1 (exchange the roles of U^r and φ) it follows that $\varphi_j(\bar{e}) > U_j^r(\bar{e})$.

Thus, for the economy $\bar{e}$ it follows that $\varphi_j(\bar{e}) \neq U_j^r(\bar{e})$. Note that the economy $\bar{e}$ has one maximal demander more than the economy e, *i.e.*, $|D_{\max}(\bar{e})| = |D_{\max}(e)| + 1$. By Case 1.1, $D_{\max}(\bar{e}) \subsetneq N$.

Similarly, proceeding from the economy $\bar{e}$, we consider a unilateral change of an agent's preference relation such that the set of maximal demanders increases by one and the uniform reallocation for this economy is unequal to the reallocation assigned by φ, *etc.* Since the set of agents is finite, finally we end up in an economy e' with only maximal demanders and $\varphi(e') \neq U^r(e')$. By Case 1.1, this is a contradiction.

Case 2: $z(e) < 0$. We call an agent $j \in S(e)$ a *maximal supplier* if either $j \in \arg\max\{s_i(e) \mid i \in S(e)\}$ or $p(R_j) = 0$ (here, in contrast to Case 1, the lower bound zero may be active). We denote the *set of maximal suppliers* of an economy $e \in \mathcal{E}^N$ by $S_{\max}(e)$. By $s_{\max} = \max\{s_i(e) \mid i \in S(e)\}$ we denote the *maximal supply*.

Case 2.1: All agents are maximal suppliers, *i.e.*, $S_{\max}(e) = N$. Hence, $U^r(e) = \omega$.

Since $\varphi(e) \neq U^r(e)$, by Lemma 4.1, there exist agents $j, k \in N$ such that $p(R_j) < \varphi_j(e)$ and $\Delta\varphi_j(e) > \Delta\varphi_k(e)$. *Same-sidedness* implies that $\Delta\varphi_k(e) \geq s_{\max}$. If $j \in \arg\max\{s_i(e) \mid i \in S(e)\}$, then $\varphi_j(e) = \omega_j + \Delta\varphi_j(e) > \omega_j + \Delta\varphi_k(e) = (\omega_j + \Delta\varphi_k(e))^+ \geq p(R_j)$. Hence, in contradiction to *no-envy*, $(\omega_j + \Delta\varphi_k(e))^+ \, P_j \, \varphi_j(e)$. If $p(R_j) = 0$, then $\varphi_j(e) = \omega_j + \Delta\varphi_j(e) > (\omega_j + \Delta\varphi_k(e))^+ \geq 0 = p(R_j)$. Hence, in contradiction to *no-envy*, $(\omega_j + \Delta\varphi_k(e))^+ \, P_j \, \varphi_j(e)$.

Case 2.2: The set of maximal suppliers does not equal the set of agents, *i.e.*, $S_{\max}(e) \subsetneq N$.

Since $\varphi(e) \neq U^r(e)$, by Lemma 4.1, there exist agents $j, k \in N$ such that $p(R_j) < \varphi_j(e)$ and $\Delta\varphi_j(e) > \Delta\varphi_k(e)$. Furthermore, by the same argument as in Case 2.1, $j \notin S_{\max}(e)$. Let $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$ such that $\bar{R}$ is a j-deviation from R with $p(\bar{R}_j) = \max\{0, \omega_j - s_{\max}\}$. We consider the following two subcases.

Case 2.2.1: $\varphi_j(e) > U_j^r(e) \geq p(R_j)$. Since $p(R_j) > p(\bar{R}_j)$, by *uncompromisingness* of φ it follows that $\varphi_j(\bar{e}) = \varphi_j(e)$. *Own-peak monotonicity* of U^r yields $U_j^r(e) \geq U_j^r(\bar{e})$. Hence, $\varphi_j(\bar{e}) > U_j^r(\bar{e})$.

Case 2.2.2: $U_j^r(e) > \varphi_j(e) \geq p(R_j)$. Similar to Case 2.1 (exchange the roles of U^r and φ) it follows that $\varphi_j(\bar{e}) < U_j^r(\bar{e})$.

Thus, for the economy $\bar{e}$ it follows that $\varphi_j(\bar{e}) \neq U_j^r(\bar{e})$. Note that the economy $\bar{e}$ has one maximal supplier more than the economy e, *i.e.*, $|S_{\max}(\bar{e})| = |S_{\max}(e)| + 1$. By Case 2.1, $S_{\max}(\bar{e}) \subsetneq N$.

Similarly as in the proof of Case 1 we now construct, step by step, an economy e' with only maximal suppliers and $\varphi(e') \neq U^r(e')$. By Case 2.1, this is a contradiction. ■

It is easy to check that, since the lower bound zero is not active in Case 1 of the previous proof ,we can replace *no-envy* by *weak no-envy*. This implies the following corollary.

Corollary 4.1 *Let φ be a rule that satisfies Pareto optimality, strategy-proofness, and weak no-envy. Then, for all $N \in \mathcal{N}$ and all $e = (R, \omega) \in \mathcal{E}^N$ such that $z(e) \geq 0$, $\varphi(e) = U^r(e)$.*

Next, we introduce an *equal treatment of equals* condition. The notion of *equal treatment of equals* we introduce here for the reallocation model is somewhat different from the equal treatment condition as introduced in Chapter 2 for the allocation model (see Section 2.2, page 23). There, we considered two agents to be equal if they have the same preference relations. However, in the reallocation model, we introduce a notion of "equal agents" in terms of demands rather than in terms of preferences.[12] *Equal treatment of equals* requires that if two agents have the same demand, then each of them is indifferent between his allotment change and the feasible part of the other agent's allotment change.

Equal treatment of equals: For all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $i, j \in N$, if $d_i(e) = d_j(e)$, then $(\omega_i + \Delta\varphi_i(e))\, I_i\, (\omega_i + \Delta\varphi_j(e))^+$.

If φ is in addition *Pareto optimal*, then *equal treatment of equals* implies the stronger equality condition $\omega_i + \Delta\varphi_i(e) = (\omega_i + \Delta\varphi_j(e))^+$. It is easy to check that *equal treatment of equals* is a weakening of *no-envy* and that *equal treatment of equals* and *weak no-envy* are logically independent.

The uniform rule U^r and the proportional rule Pr^r satisfy *equal treatment of equals*.

In the proof of Theorem 4.1, Case 1, we show that if the set of agents consists only of maximal demanders, then the reallocation recommended by the rule in question equals the uniform reallocation. Since in such an economy, all agents have the same demand, we can replace *no-envy* by *equal treatment of equals* without changing the result. Hence, the following holds.

Corollary 4.2 *Let φ be a rule that satisfies Pareto optimality, strategy-proofness, and equal treatment of equals. Then, for all $N \in \mathcal{N}$ and all $e = (R, \omega) \in \mathcal{E}^N$ such that $z(e) \geq 0$, $\varphi(e) = U^r(e)$.*

The following example shows that it is not possible to weaken *no-envy* in Theorem 4.1 to either *weak no-envy* or *equal treatment of equals*.

[12] In Chapter 5, for variations of the reallocation model where preferences are defined over the whole real line, we introduce a stronger equal treatment of equals condition in terms of individual endowments together with preferences rather than in terms of the agents' demands.

Example 4.1 The following rule φ^1 satisfies *Pareto optimality, strategy-proofness, weak no-envy, and equal treatment of equals.* If $T = \{1,2\}$, $e = (R,\omega) \in \mathcal{E}^T$, and $z(e) < 0$, then

$$\varphi^1(e) = \begin{cases} (\omega_2 - p(R_2), p(R_2)) & \text{if } \omega_1 = 0, \\ (p(R_1), \omega_1 - p(R_1) & \text{if } \omega_2 = 0, \\ U^r(e) & \text{otherwise.} \end{cases}$$

For all remaining $N \in \mathcal{N}$ and $e = (R,\omega) \in \mathcal{E}^N$, $\varphi^1(e) = U^r(e)$. ◇

4.3.2 Endowment strategy-proofness

Next, we introduce a notion of *strategy-proofness* that has no counterpart in the allocation model. If individual endowments are private information, it might happen that an agent manipulates the outcome to his advantage by only reporting, or exhibiting, a smaller part of his endowment. An example for a rule that is vulnerable to this kind of strategic behavior is the proportional rule Pr^r. *Endowment strategy-proofness* states that no agent ever benefits from withholding some part of his individual endowment. Before we give the formal definition, we introduce some notation.

Let $N, M \in \mathcal{N}$ be such that $N \subseteq M$. For $\omega \in \mathbb{R}^M_+$ the *restriction* $(\omega_i)_{i\in N} \in \mathbb{R}^N_+$ *of* ω *to* N is denoted by ω_N. Let $\omega, \bar{\omega} \in \mathbb{R}^N_+$ such that for $i \in N$, $\omega_{N\setminus\{i\}} = \bar{\omega}_{N\setminus\{i\}}$. Then, we call $\bar{\omega}$ an *i-deviation from* ω. If in addition $\omega_i > \bar{\omega}_i$, then, we call $\bar{\omega}$ an *i-deviation from* ω *by withholding.*

Endowment strategy-proofness: For all $N \in \mathcal{N}$, all $e = (R,\omega) \in \mathcal{E}^N$, all $\bar{e} = (R,\bar{\omega}) \in \mathcal{E}^N$, and all $j \in N$, if $\bar{\omega}$ is a j-deviation from ω by withholding, then $\varphi_j(e)\, R_j\, (\omega_j + \Delta\varphi_j(\bar{e}))$.

So, if agent j acts strategically and pretends to have the individual endowment $\bar{\omega}_i$ instead of ω_i, then the allotment assigned by the rule equals $\varphi_j(\bar{e})$. If endowments are private information, φ can better be interpreted as a rule that assigns allotment changes. So, j's actual allotment in that situation is $(\omega_i + \Delta\varphi_j(\bar{e}))$.

If agents were also allowed to offer more than their actual possession, then feasibility could cause some agents to obtain a negative allotment. Because our model does not allow such negative allotments and supplies are considered to be real amounts handed out, demands can be faked only by withholding endowment.

The property of *endowment strategy-proofness* can be traced back to Postlewaite (1979), who investigated it for pure exchange economies.[13] In this context, *endowment strategy-proof* mechanisms do not exist (see Postlewaite 1979, Theorem 1). For our model with single-peaked preferences *endowment strategy-proof* rules do exist, *e.g.*, the uniform rule U^r.

Lemma 4.7 *The uniform rule U^r satisfies endowment strategy-proofness.*

[13]In Postlewaite (1979) the manipulation by witholding some part of an individual endowment is called W-manipulability.

Proof. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$, and $j \in N$ be such that $\bar{\omega}$ is a j-deviation from ω by withholding. In order to prove *endowment strategy-proofness* of U^r, we have to show that

$$U_j^r(e)\, R_j\, (\omega_j + \Delta U_j^r(\bar{e})). \tag{4.4}$$

Case 1: $z(e) \leq 0$. Hence, $U_j^r(e) = \max\{p(R_j), \omega_j + \lambda\}$, where λ solves $\sum_N U_i^r(e) = \sum_N \omega_i$. Then, by *same-sidedness*, for all $i \in N$, $U_i^r(e) \geq p(R_i)$. If $U_j^r(e) = p(R_j)$, then (4.4). Otherwise, $j \in S(e)$ and $U_j^r(e) = \omega_j + \lambda > p(R_j)$. We distinguish two cases.

Case 1.1: $z(\bar{e}) \leq 0$. Then, by the definition of U^r, $\Delta U_j^r(e) \leq \Delta U_j^r(\bar{e})$. This implies (4.4).

Case 1.2: $z(\bar{e}) > 0$. Hence, by *same-sidedness*, for all $i \in N$, $U_i^r(\bar{e}) \leq p(R_i)$. By feasibility, it follows that $\sum_N U_i^r(\bar{e}) = \sum_N \bar{\omega}_i = \sum_N \omega_i + \bar{\omega}_j - \omega_j = \sum_N U_i^r(e) + \bar{\omega}_j - \omega_j$. This implies that $U_j^r(\bar{e}) - \bar{\omega}_j = U_j^r(e) - \omega_j + \sum_{N\setminus\{j\}}(U_i^r(e) - U_i^r(\bar{e}))$. Since for all $i \in N\setminus\{j\}$, $U_i^r(e) - U_i^r(\bar{e}) \geq 0$, we have $U_j^r(\bar{e}) - \bar{\omega}_j \geq U_j^r(e) - \omega_j$. Hence, $\Delta U_j^r(e) \leq \Delta U_j^r(\bar{e})$. This implies (4.4).

Case 2: $z(e) > 0$. Similar to Case 1. ■

It is easy to show, that the proportional rule Pr^r is not *endowment strategy-proof*.

The following lemma describes some implications of *endowment strategy-proofness* for the case that a deviation from ω by withholding is a "one-sided change" of the economy: let $N, \bar{N} \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and $\bar{e} = (\bar{R}, \bar{\omega}) \in \mathcal{E}^{\bar{N}}$. If $z(e) \cdot z(\bar{e}) \geq 0$, then $\bar{e}$ is a *one-sided change* of e.

Lemma 4.8 *Let the rule φ satisfy Pareto optimality and endowment strategy-proofness. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, $\bar{e} = (R, \bar{\omega}) \in \mathcal{E}^N$, and $j \in N$ be such that $\bar{e}$ is a one-sided change of e, $\bar{\omega}$ is a j-deviation from ω by withholding, and j is non-satiated at e, i.e., $\varphi_j(e) \neq p(R_j)$.*

(a) *If $z(\bar{e}) \leq 0$, then $\Delta\varphi_j(\bar{e}) \geq \Delta\varphi_j(e)$.*

(b) *If $z(e) \geq 0$, then $\Delta\varphi_j(\bar{e}) \leq \Delta\varphi_j(e)$.*

Proof. Let φ be a rule that satisfies *Pareto optimality* and *endowment strategy-proofness*. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, $\bar{e} = (R, \bar{\omega}) \in \mathcal{E}^N$, and $j \in N$ be as described in the lemma.

(a) Suppose $z(\bar{e}) \leq 0$. Then, $z(e) < 0$. By *same-sidedness*, $\varphi_j(e) > p(R_j)$ and $\varphi_j(\bar{e}) \geq p(R_j)$. By *endowment strategy-proofness*, $\varphi_j(e)\, R_j\, (\omega_j + \Delta\varphi_j(\bar{e}))$. Since $\omega_j > \bar{\omega}_j$, it follows that $\omega_j + \Delta\varphi_j(\bar{e}) \geq \varphi_j(e) > p(R_j)$. Hence, $\Delta\varphi_j(\bar{e}) \geq \Delta\varphi_j(e)$.

(b) Suppose $z(e) \geq 0$. Then, $z(\bar{e}) > 0$. By *same-sidedness*, $\varphi_j(e) < p(R_j)$ and $\varphi_j(\bar{e}) \leq p(R_j)$. By *endowment strategy-proofness*, $\varphi_j(e)\, R_j\, (\omega_j + \Delta\varphi_j(\bar{e}))$. Since $\omega_j > \bar{\omega}_j$, it follows that either (i) $\omega_j + (\varphi_j(\bar{e}) - \bar{\omega}_j) \leq \varphi_j(e)$ or (ii) $\omega_j + (\varphi_j(\bar{e}) - \bar{\omega}_j) \geq r_j(\varphi_j(e))$. Suppose $\omega_j - \bar{\omega}_j < r_j(\varphi_j(e)) - p(R_j)$. Then, $\omega_j - \bar{\omega}_j < r_j(\varphi_j(e)) - \varphi_j(\bar{e})$. Hence, (ii) cannot be the case and (i) implies $\Delta\varphi_j(\bar{e}) \leq \Delta\varphi_j(e)$. Finally, if $\omega_j - \bar{\omega}_j \geq r_j(\varphi_j(e)) - p(R_j)$, we shift ω_j stepwise (with the size of the steps small enough) to $\bar{\omega}_j$ and apply the same argument as above in each step. ■

We have the following consequence of the previous lemma.

Corollary 4.3 *Let the rule φ satisfy Pareto optimality and endowment strategy-proofness. Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ be such that $z(e) > 0$. Then,*

(a) *for all $i \in D(e)$, $\omega_i \leq \varphi_i(e) \leq p(R_i)$ and*

(b) *for all $i \notin D(e)$, $\varphi_i(e) = p(R_i)$.*

Proof. Let φ be a rule that satisfies *Pareto optimality* and *endowment strategy-proofness*. Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ be such that $z(e) > 0$. Then, by *same-sidedness*, for all $i \in N$, $\varphi_i(e) \leq p(R_i)$. Let $j \in N$ and define $\omega(j) \in \mathbb{R}^N_+$ by $\omega(j)_j = 0$ and $\omega(j)_i = \omega_i$ for all $i \in N \setminus \{j\}$. Let $e(j) = (R, \omega(j)) \in \mathcal{E}^N$. Let $i \in N$ and suppose that $\varphi_i(e) < p(R_i)$. Then, by Lemma 4.8, $\Delta\varphi_i(e(i)) \leq \Delta\varphi_i(e)$. Because $\varphi_i(e(i)) \geq 0$, it follows that $\Delta\varphi_i(e(i)) \geq 0$. Hence, $\Delta\varphi_i(e) \geq 0$ and $\omega_i \leq \varphi_i(e) < p(R_i)$. Now, (a) and (b) follow easily. ■

A similar result cannot be obtained for the excess supply case, even if there is an upper bound for the endowments (for instance, if there are finite resources). This is due to the asymmetry in the definition of *endowment strategy-proofness* caused by the requirement that an agent can only deviate by withholding part of his endowment. Therefore, *endowment strategy-proofness* as defined here, has not such a great impact for a rule as one would expect. We will use the next consequence of the previous results in the next section.

Corollary 4.4 *Let φ be a rule that satisfies Pareto optimality, weak no-envy, and endowment strategy-proofness. Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ be such that $z(e) \geq 0$ and for all $i, j \in N$, $p(R_i) = p(R_j)$. Then, $\varphi(e) = U^r(e)$.*

Proof. Let φ be a rule that satisfies *Pareto optimality* and *endowment strategy-proofness*. Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ be such that $z(e) \geq 0$ and for all $i, j \in N$, $p(R_i) = p(R_j)$. Suppose, by contradiction, that $\varphi(e) \neq U^r(e)$. Since φ is *Pareto optimal* it follows that $z(e) > 0$. Furthermore, there exist an agent $j \in N$ such that $\Delta\varphi_j(e) < \Delta U^r_j(e)$. By Corollary 4.3, it follows that $j \in D(e)$ and $\Delta\varphi_j(e) \geq 0$. Similar to the proof of Corollary 4.3, define $\omega(j) \in \mathbb{R}^N_+$ by $\omega(j)_j = 0$ and $\omega(j)_i = \omega_i$ for all $i \in N \setminus \{j\}$. Let $e(j) = (R, \omega(j)) \in \mathcal{E}^N$. Then, by Lemma 4.8, $\Delta\varphi_j(e) \geq \Delta\varphi_j(e(j)) \geq 0$. Because $\Delta U^r_j(e(j)) \geq \Delta U^r_j(e)$, it follows that $\Delta U^r_j(e(j)) > \Delta\varphi_j(e(j))$. Thus, there exists an agent $k \in N$ such that $\Delta U^r_k(e(j)) < \Delta\varphi_k(e(j))$. Since $z(e(j)) > 0$, by Corollary 4.3, $k \in D(e(j))$, $\omega_k \leq \varphi_k(e(j)) \leq p(R_k)$, and $\Delta\varphi_k(e(j)) > \Delta U^r_k(e(j)) \geq \Delta U^r_j(e(j)) > \Delta\varphi_j(e(j))$. Hence, $\omega(j)_j + \Delta\varphi_j(e(j)) < \omega(j)_j + \Delta\varphi_k(e(j)) \leq \varphi_k(e) \leq p(R_k) = p(R_j)$. This contradicts *weak no-envy*. ■

The following example shows that even if we strengthen *weak no-envy* to *no-envy*, the result of Corollary 4.4 does not hold for economies with excess supply.

Example 4.2 The following rule φ^2 satisfies *Pareto optimality, no-envy, and endowment strategy-proofness*. Let $T = \{1, 2\}$ and $e = (R, \omega) \in \mathcal{E}^T$.

If $z(e) < 0$ and $\omega_1 = 0$, then

$$\varphi_1^2(e) = \min\{r_1(0), \omega_2 - p(R_2)\} \text{ and}$$

$$\varphi_2^2(e) = \max\{p(R_2), p(R_2) - r_1(0)\}.$$

If $z(e) < 0$ and $\omega_2 = 0$, then

$$\varphi_1^2(e) = \max\{p(R_1), p(R_1) - r_2(0)\} \text{ and}$$

$$\varphi_2^2(e) = \min\{r_2(0), \omega_1 - p(R_1)\}.$$

For all remaining $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$, $\varphi^2(e) = U^r(e)$. ◇

4.3.3 Demand-onliness, reversibility, and peak-onliness

The next property for rules we introduce in this section is *demand-onliness*: the agents' allotment changes only depend on the agents' demands rather than on the whole economy.

Demand-onliness: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all $\bar{e} = (\bar{R}, \bar{\omega}) \in \mathcal{E}^N$, if for all $i \in N$, $d_i(e) = d_i(\bar{e})$, then for all $i \in N$, $\Delta\varphi_i(e) = \Delta\varphi_i(\bar{e})$.

The uniform rule U^r and the proportional rule Pr^r satisfy *demand-onliness*.

The next corollary states that in Theorem 4.1 we can replace *no-envy* by *equal treatment of equals* and *demand-onliness*.

Corollary 4.5 *The uniform rule U^r is the only rule that satisfies Pareto optimality, strategy-proofness, equal treatment of equals, and demand-onliness.*

We can interpret Corollary 4.5 as an extension of Ching's characterization of the uniform allocation rule by *Pareto optimality*, *equal treatment of equals*, and *strategy-proofness* (see Theorem 2.4) to the reallocation model. In addition to the corresponding reallocation properties, we impose *demand-onliness*. By *demand-onliness* it is sufficient to only consider economies where all agents have identical individual endowments. The remaining part of the proof is then similar to Ching's proof for the allocation model.

Proof. The uniform rule U^r satisfies *Pareto optimality*, *strategy-proofness*, *equal treatment of equals*, and *demand-onliness*.

To prove the remaining part of the corollary let φ be a rule that satisfies the properties listed in the corollary. By *Pareto optimality* and Corollary 4.2, it follows that for all $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ such that $z(e) \geq 0$, $\varphi(e) = U^r(e)$. It remains to prove that for all $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ such that $z(e) < 0$, $\varphi(e) = U^r(e)$. By *demand-onliness* it is sufficient to consider $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ such that for all $i, j \in N$,

$\omega_i = \omega_j$. Then, for all agents $j, k \in N$ it follows that $(\omega_j + \Delta\varphi_k(e))^+ = (\omega_j + \Delta\varphi_k(e))$. Now, the remainder of the proof is similar to the proof of Theorem 4.1 by applying *equal treatment of equals* instead of *no-envy*. ■

The rule φ^1 as defined in Example 4.1 satisfies *Pareto optimality*, *strategy-proofness*, and *equal treatment of equals*, but not *demand-onliness*.

Similar to Theorem 2.4, Sprumont's characterization of the uniform allocation rule by *Pareto optimality*, *strategy-proofness*, and *anonymity* (Theorem 2.1) implies a characterization of the uniform reallocation rule if in addition *demand-onliness* is imposed. *Anonymity* in the reallocation model requires that the agent's allotment changes do not depend on their names (see also Section 2.2, page 22).

Let $N \in \mathcal{N}$. A *permutation* π *on* N is a bijective function $\pi : N \to N$. By Π^N we denote the *set of all permutations on* N. Given $\pi \in \Pi^N$ and $e = (R, \Omega) \in \mathcal{R}^N$, let $R_\pi = (R_{\pi(i)})_{i \in N}$, $\omega_\pi = (\omega_{\pi(i)})_{i \in N}$, and $e_\pi = (R_\pi, \omega_\pi)$.

Anonymity: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, all $\pi \in \Pi^N$, and all $i \in N$, $\varphi_i(e_\pi) = \varphi_{\pi(i)}(e)$.

Lemma 4.9 *Let φ satisfy Pareto optimality, anonymity, and demand-onliness. Then φ satisfies equal treatment of equals.*

Proof. By *demand-onliness* it is sufficient to consider $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ such that for all $i, j \in N$, $\omega_i = \omega_j$. Let $i, j \in N$ be such that $d_i(e) = d_j(e)$. Thus, $p(R_i) = p(R_j)$. By *demand-onliness*, suppose, without loss of generality, that $R_i = R_j$. Let $\pi \in \Pi^N$ be such that $\pi(i) = j$, $\pi(j) = i$, and for all $k \in N \backslash \{i, j\}$, $\pi(k) = k$. Hence, $e = e_\pi$ and by *anonymity*, for all $k \in N$, $\varphi_k(e_\pi) = \varphi_{\pi(k)}(e)$. Thus, $\varphi_i(e) = \varphi_i(e_\pi) = \varphi_j(e)$. Since $\omega_i = \omega_j$, this proves *equal treatment of equals*. ■

The next corollary states that in Theorem 4.1 we can replace *no-envy* by *anonymity* and *demand-onliness*.

Corollary 4.6 *The uniform rule U^r is the only rule that satisfies Pareto optimality, strategy-proofness, anonymity, and demand-onliness.*

Proof. The uniform rule U^r satisfies *Pareto optimality*, *strategy-proofness*, *anonymity*, and *demand-onliness*.

To prove the remaining part of the corollary let φ be a rule that satisfies the properties listed in the corollary. By Lemma 4.9, it follows that φ satisfies *equal treatment of equals*. Thus, by Corollary 4.5, $\varphi = U^r$. ■

The rule φ^1 as defined in Example 4.1 satisfies *Pareto optimality*, *strategy-proofness*, and *anonymity*, but not *demand-onliness*.

The next property we introduce is a strengthening of *demand-onliness*. Consider an economy and a "reversal" of this economy, *i.e.*, demanders in the previous economy are

suppliers with supply equal to their previous demand, non-traders remain non-traders, and suppliers in the previous economy are demanders with demand equal to their previous supply. *Reversibility* states that in such a reversed economy, the allotment changes in one economy equal the reversed allotment changes of the other economy.

Reversibility: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all $\bar{e} = (\bar{R}, \bar{\omega}) \in \mathcal{E}^N$, if for all $i \in N$, $d_i(e) = -d_i(\bar{e})$, then for all $i \in N$, $\Delta\varphi_i(e) = -\Delta\varphi_i(\bar{e})$.

Clearly, by applying *reversibility* two times we obtain *demand-onliness*.

It is easy to check, that the uniform rule U^r and the proportional rule Pr^r satisfy *reversibility*.

Theorem 4.2 *The uniform rule U^r is the only rule that satisfies Pareto optimality, equal treatment of equals, endowment strategy-proofness, and reversibility.*

Proof. As shown in Lemma 4.7, the uniform rule U^r satisfies *endowment strategy-proofness*. Furthermore, U^r satisfies *Pareto optimality, equal treatment of equals*, and *reversibility*.

To prove the remaining part of the theorem, let φ be a rule that satisfies the properties listed in the theorem.

Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and suppose, by contradiction, that $\varphi(e) \neq U^r(e)$. Since φ is *Pareto optimal* it follows that either $z(e) > 0$ or $z(e) < 0$. By *reversibility*, without loss of generality, we can assume that $z(e) > 0$. By *demand-onliness*, without loss of generality, we can assume that for all $i, j \in N$, $p(R_i) = p(R_j)$ and for all $i \in N$, $2p(R_i) - \omega_i \geq 0$. By Corollary 4.3, it follows that

$$\begin{array}{l} \text{for all } i \in D(e),\ \omega_i \leq \varphi_i(e) \leq p(R_i) \text{ and} \\ \text{for all } i \notin D(e),\ \varphi_i(e) = p(R_i) = U_i^r(e). \end{array} \tag{4.5}$$

We denote the set of demanders with zero as individual endowments by $O(e) = \{i \in D(e) \mid \omega_i = 0\}$.

Case 1: $O(e) = D(e)$. Then, for all $i, j \in D(e)$, $d_i(e) = d_j(e)$. By (4.5) and *equal treatment of equals*, $\varphi(e) = U^r(e)$. This is a contradiction.

Case 2: $O(e) \subsetneq D(e)$. By (4.5) and *equal treatment of equals*, it follows that for all $i, j \in O(e)$, $\Delta\varphi_i(e) = \Delta\varphi_j(e)$. Since $\varphi(e) \neq U^r(e)$, there exists $k \in D(e) \backslash O(e)$ such that $\Delta\varphi_k(e) \neq \Delta U_k^r(e)$. We consider the following two subcases.

Case 2.1: $\Delta\varphi_k(e) < \Delta U_k^r(e)$. Similar to the proof of Corollary 4.3 define $\omega(k) \in \mathbb{R}_+^N$ by $\omega(k)_k = 0$ and $\omega(k)_i = \omega_i$ for all $i \in N \backslash \{k\}$. Let $e(k) = (R, \omega(k)) \in \mathcal{E}^N$. Then, by Lemma 4.8, $\Delta\varphi_k(e) \geq \Delta\varphi_k(e(k)) \geq 0$. Furthermore, it follows that $\Delta U_k^r(e(k)) \geq \Delta U_k^r(e)$. Hence, $\Delta U_k^r(e(k)) > \Delta\varphi_k(e(k))$.

Case 2.2: $\Delta\varphi_k(e) > \Delta U_k^r(e)$. Let $e(k)$ be as in Case 2.1. Then, since $U_k^r(e) < p(R_k)$, $\Delta U_k^r(e(k)) = \Delta U_k^r(e)$. Let $\bar{e} = (R, \bar{\omega}) \in \mathcal{E}^N$ and $\bar{e}(k) = (R, \bar{\omega}(k)) \in \mathcal{E}^N$ be such that for all $i \in N$, $d_i(e) = -d_i(\bar{e})$ and $d_i(e(k)) = -d_i(\bar{e}(k))$. Then, by *reversibility*, it follows for all $i \in N$ that $\Delta\varphi_i(e) = -\Delta\varphi_i(\bar{e})$ and $\Delta\varphi_i(e(k)) = -\Delta\varphi_i(\bar{e}(k))$. If $\varphi_k(\bar{e}(k)) = p(R_k)$,

then $\varphi_k(e(k)) = p(R_k)$ and clearly $\Delta\varphi_k(e(k)) \geq \Delta\varphi_k(e)$. If $\varphi_k(\bar{e}(k)) \neq p(R_k)$, since $z(\bar{e}) < 0$, $z(\bar{e}(k)) < 0$. Furthermore, $\bar{\omega}$ is a k-deviation from $\bar{\omega}(k)$ by withholding and Lemma 4.8 implies $\Delta\varphi_k(\bar{e}) \geq \Delta\varphi_k(\bar{e}(k))$. Hence, by *reversibility*, $\Delta\varphi_k(e(k)) \geq \Delta\varphi_k(e)$ and $\Delta\varphi_k(e(k)) > \Delta U_k^r(e(k))$.

Thus, for the economy $e(k)$ it follows that $\varphi_j(e(k)) \neq U_j^r(e(k))$. Note that the economy $e(k)$ has one demanders with zero as individual endowment more than e, *i.e.*, $|O(e(k))| = |O(e)| + 1$. By Case 1, $O(e(k)) \subsetneq D(e(k))$.

Similarly, proceeding from the economy $e(k)$, we consider a unilateral change of an individual endowment by withholding such that the set of demanders with individual endowment zero increases by one and the uniform reallocation for this economy is unequal to the reallocation assigned by φ, *etc.* Since the set of demanders is finite, finally we end up in an economy e' with $D(e') = O(e')$ and $\varphi(e') \neq U^r(e')$. By Case 1, this is a contradiction. ■

Next, we consider the well-known property of *peak-onliness*: the rule only depends on the peaks instead of the complete profile (see also Section 2.2, page 22).

Peak-onliness: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$, if $p(R) = p(\bar{R})$, then $\varphi(e) = \varphi(\bar{e})$.

Many well-known rules are *peak-only*. The uniform allocation rule, the proportional reallocation rule, and the uniform reallocation rule are *peak-only*. Clearly, if a rule takes "intensities" or the shape of the preferences into account, then one would expect that it is apt to be vulnerable to strategic behavior. The following theorem shows that in Theorem 4.1 we can replace *strategy-proofness* by *peak-onliness*.

This characterization of the uniform reallocation rule can be interpreted as an extension of Thomson's characterization of the uniform allocation rule by *Pareto optimality*, *no-envy*, and *peak-onliness* (see Theorem 2.3). The proofs of Theorem 2.3 and the following Theorem 4.3 are similar.[14]

Theorem 4.3 *The uniform rule U^r is the only rule that satisfies Pareto optimality, no-envy, and peak-onliness.*

Proof. By *same-sidedness* and Lemma 4.6 it follows that the uniform rule U^r satisfies *Pareto optimality* and *no-envy*. *Peak-onliness* of U^r follows from its definition.

To prove the remaining part of the theorem let φ be a rule that satisfies the properties listed in the theorem.

Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and suppose, by contradiction, that $\varphi(e) \neq U^r(e)$. Since φ is *Pareto optimal* it follows that either $z(e) > 0$ or $z(e) < 0$.

Case 1: $z(e) < 0$. Since $\varphi(e) \neq U^r(e)$, by Lemma 4.1, there exist agents $j, k \in N$ such that $p(R_j) < \varphi_j(e)$ and $\Delta\varphi_j(e) > \Delta\varphi_k(e)$. Hence, by *no-envy*, $\varphi_j(e) = \omega_j + \Delta\varphi_j(e) > p(R_j) > (\omega_j + \Delta\varphi_k(e))^+$. Let $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$, be such that $\bar{R}$ is a j-deviation from

[14]This result was independently achieved by Moreno (1996), Theorem 1.

R with $p(\bar{R}_j) = p(R_j)$ and $(\omega_j + \Delta\varphi_k(e))^+ \bar{P}_j \varphi_j(e)$. By *peak-onliness* it follows that $\varphi(\bar{e}) = \varphi(e)$ and particularly that $\Delta\varphi_k(e) = \Delta\varphi_k(\bar{e})$. Hence, for the economy $\bar{e}$ and agent j, $(\omega_j + \Delta\varphi_k(\bar{e}))^+ \bar{P}_j \varphi_j(\bar{e})$. This is a contradiction to *no-envy*.

Case 2: $z(e) > 0$. Similar to Case 1. ■

In the proof of Theorem 4.3 we prove, without loss of generality, that for an economy with excess supply the rule φ assigns the uniform reallocation. The proof for the excess demand case is very similar. As we state in the next corollary, in this case we can weaken *no-envy* to *weak no-envy*.

Corollary 4.7 *Let φ be a rule that satisfies Pareto optimality, weak no-envy, and peak-onliness. Then, for all $N \in \mathcal{N}$ and all $e = (R, \omega) \in \mathcal{E}^N$ such that $z(e) \geq 0$, $\varphi(e) = U^r(e)$.*

Proof. Similar to the proof of Corollary 4.1, it is sufficient to adjust the proof of Theorem 4.3 to the excess demand case. Since the lower bound zero is not active (see Remark 4.2) in this case, instead of *no-envy* we can apply *weak no-envy*. ■

It is easy to check that the rule φ^1 as defined in Example 4.1 satisfies *peak-onliness*. Hence, Example 4.1 shows that it is not possible to weaken *no-envy* in Theorem 4.3 to *weak no-envy*. However, as we show in the next corollary, we can weaken *no-envy* in Theorem 4.3 to *weak no-envy* if, at the same time, we strengthen *peak-onliness* to *demand-onliness*.

Corollary 4.8 *The uniform rule U^r is the only rule that satisfies Pareto optimality, weak no-envy, and demand-onliness.*

Proof. The uniform rule U^r satisfies *Pareto optimality*, *weak no-envy*, and *demand-onliness*.

To prove the remaining part of the corollary let φ be a rule that satisfies the properties listed in the corollary. By *Pareto optimality*, *peak-onliness*, and Corollary 4.7, it follows that for all $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ such that $z(e) \geq 0$, $\varphi(e) = U^r(e)$. It remains to prove that for all $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ such that $z(e) < 0$, $\varphi(e) = U^r(e)$. By *demand-onliness* it is sufficient to consider $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ such that for all $i, j \in N$, $\omega_i = \omega_j$. Now, the remainder of the proof is similar to the proof of Theorem 4.3 by applying *weak no-envy* instead of *no-envy*. ■

The rule φ^1 as defined in Example 4.1 satisfies *Pareto optimality*, *weak no-envy*, but not *demand-onliness*.

A summary of the characterizations of the uniform rule U^r by *Pareto optimality*, strategy-proofness properties, equity properties, *etc.*, can be found in Table 4.1. For a comparison with corresponding characterizations of the uniform allocation rule U we refer to Table 2.1.

	Th 4.1	Co 4.1	Co 4.2	Co 4.4	Co 4.5	Co 4.6	Th 4.2	Th 4.3	Co 4.7	Co 4.8
for all $e \in \mathcal{E}^N$	×				×	×	×	×		×
not for $e \in \mathcal{E}^N$ with $z(e) < 0$		×	×	×					×	
Pareto optimality	×	×	×	×	×	×	×	×	×	×
strategy-proofness	×	×	×		×	×				
endowment strategy-proofness				×			×			
no-envy	×							×		
weak no-envy		×		×					×	×
equal treatment of equals			×		×		×			
anonymity						×				
demand-onliness					×	×				×
reversibility							×			
peak-onliness								×	×	

Table 4.1: Characterizations of the uniform rule by *Pareto optimality*, strategy-proofness properties, equity properties, *etc.*

4.4 Solidarity properties

Next we analyze some "solidarity properties" similar to those we introduced in Chapter 2. As already explained in Chapter 2, a solidarity property describes the effect of certain changes in a single parameter of the economy while the other parameters are kept fixed, *e.g.*, the population, the preferences, or the individual endowments. If after such a change, either all agents that are not responsible for the change (weakly) lose together or all (weakly) gain together, then depending on which parameter has been changed, the rule satisfies *population-monotonicity*, *welfare-domination under preference-replacement*, or *endowment-monotonicity*. Thomson (1994a, 1995a, 1997a) shows that these properties, when considered for allocation rules, are generally incompatible with *Pareto optimality* and *no-envy*. However, as already discussed in Chapter 2, these incompatibilities do not occur if the change in the parameter is one-sided, *i.e.*, the change does not turn an economy in excess demand into an economy in excess supply, or *vice versa*. Furthermore, the uniform allocation rule turns out to be essentially the only rule that satisfies any of the one-sided solidarity properties, together with *Pareto optimality* and *no-envy*. In this section we explore how the results as achieved for the allocation model carry through to the reallocation model.

4.4.1 One-sided population-monotonicity

First we consider changes in the population while keeping the other parameters fixed. *Population-monotonicity* requires that after the arrival of new agents either all agents initially present (weakly) lose or all (weakly) gain.

Population-monotonicity: For all $N, \bar{N} \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all $\bar{e} = (\bar{R}, \bar{\omega}) \in \mathcal{E}^{\bar{N}}$, if $N \subseteq \bar{N}$, $R = \bar{R}_N$, and $\omega = \bar{\omega}_N$, then either [for all $i \in N$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$] or [for all $i \in N$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$].

Similar to the allocation model, it can be shown that the properties *Pareto optimality*, *no-envy*, and *population-monotonicity* are generally incompatible. See for instance Moreno (1996), Proposition 1. However, the uniform reallocation rule satisfies *Pareto optimality*, *no-envy*, and the one-sided version of *population-monotonicity*, *i.e.*, if the change in the population is such that the change in the initial economy is one-sided, then either all agents initially present (weakly) lose or all (weakly) gain. Here, we introduce a weaker one-sided version of *population-monotonicity*.

One-sided population-monotonicity states that if two disjoint economies are merged, either both of excess demand or both of excess supply, then all agents of one of the economies (weakly) lose or all (weakly) gain (see also Section 2.2, page 24). Before we give a formal definition, we introduce some notation.

Let $N, \bar{N} \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, $\bar{e} = (\bar{R}, \bar{\omega}) \in \mathcal{E}^{\bar{N}}$, and $N \cap \bar{N} = \emptyset$. Then, $R \cup \bar{R} = (R, \bar{R}) \in \mathcal{R}^{N \cup \bar{N}}$ denotes the preference profile that consists of the preferences of the agents in $N \cup \bar{N}$. Similarly, $\omega \cup \bar{\omega} = (\omega, \bar{\omega}) \in \mathbb{R}_+^{N \cup \bar{N}}$ denotes the endowment vector that consists of the individual endowments of the agents in $N \cup \bar{N}$. Then, the *merged economy* $(R \cup \bar{R}, \omega \cup \bar{\omega}) \in \mathcal{E}^{N \cup \bar{N}}$ is denoted by $e \cup \bar{e}$.

One-sided population-monotonicity: For all $N, \bar{N} \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all $\bar{e} = (\bar{R}, \bar{\omega}) \in \mathcal{E}^{\bar{N}}$, if $N \cap \bar{N} = \emptyset$ and $z(e) \cdot z(\bar{e}) > 0$, then either [for all $i \in N$, $\varphi_i(e \cup \bar{e})\, R_i\, \varphi_i(e)$] or [for all $i \in N$, $\varphi_i(e)\, R_i\, \varphi_i(e \cup \bar{e})$].

Note that $z(e) \cdot z(\bar{e}) > 0$ if, and only if, both problems have excess demand or both problems have excess supply.

Lemma 4.10 *The uniform rule U^r satisfies one-sided population monotonicity.*

Proof. Let $N, \bar{N} \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and $\bar{e} = (\bar{R}, \bar{\omega}) \in \mathcal{E}^{\bar{N}}$ be such that $N \cap \bar{N} = \emptyset$ and $z(e) \cdot z(\bar{e}) > 0$. In order to prove *one-sided population monotonicity* we have to show that either (i) [for all $i \in N$, $\varphi_i(e \cup \bar{e})\, R_i\, \varphi_i(e)$] or (ii) [for all $i \in N$, $\varphi_i(e)\, R_i\, \varphi_i(e \cup \bar{e})$].

Case 1: $z(e), z(\bar{e}) < 0$. Let $e \cup \bar{e} = (R', \omega')$. Let λ, $\bar{\lambda}$ and μ be the minimal allotment changes of U^r for the economies e, $\bar{e}$, and $e \cup \bar{e}$, *i.e.*, for all $i \in N$, $U_i^r(e) = \max\{p(R_i), \omega_i + \lambda\}$, for all $i \in \bar{N}$, $U_i^r(\bar{e}) = \max\{p(\bar{R}_i), \bar{\omega}_i + \bar{\lambda}\}$, and for all $i \in N \cup \bar{N}$, $U_i^r(e \cup \bar{e}) = \max\{p(R'_i), \omega'_i + \mu\}$. Suppose, by contradiction, that there exist agents $j, k \in N$ such that j is strictly better off at $e \cup \bar{e}$ than at e and for k the converse holds.

This can only hold if $j, k \in S(e)$. Hence, j is not satiated at e and k is not satiated at $e \cup \bar{e}$. Hence, $p(R_j) \leq U_j^r(e \cup \bar{e}) < U_j^r(e) = \omega_j + \lambda$ and $p(R_k) \leq U_k^r(e) < U_k^r(e \cup \bar{e}) = \omega_k + \mu$. So, for the corresponding allotment changes it holds that $\Delta U_j^r(e) = \lambda > \Delta U_j^r(e \cup \bar{e}) \geq \mu$ and $\Delta U_k^r(e \cup \bar{e}) = \mu > \Delta U_k^r(e) \geq \lambda$. This is a contradiction. Hence, either (i) or (ii) must hold.

Case 2: $z(e), z(\bar{e}) > 0$. Similar to Case 1. ■

Note that the proportional rule Pr^r and the uniform rule U^r also satisfy the somewhat stronger *one-sided population monotonicity* conditions as considered in Moreno (1996) and Thomson (1995d).[15]

Next, we show that *one-sided population-monotonicity* together with *Pareto optimality* and *weak no-envy* imply *demand-onliness*.

Lemma 4.11 *Let φ satisfy Pareto optimality, weak no-envy, and one-sided population-monotonicity. Then, φ satisfies demand-onliness.*

Proof. Let φ be a rule that satisfies *Pareto optimality*, *weak no-envy*, and *one-sided population-monotonicity*. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and $\bar{e} = (\bar{R}, \bar{\omega}) \in \mathcal{E}^N$ be such that for all $i \in N$, $d_i(e) = d_i(\bar{e})$. In order to prove *demand-onliness* we have to show that for all $i \in N$, $\Delta\varphi_i(e) = \Delta\varphi_i(\bar{e})$.

Case 1: $z(e) \leq 0$. Assume, without loss of generality, that $N = \{1, 2, \ldots, n\}$. Let $N' = \{n+1, \ldots, 2n\}$ and define $e' = (R', \omega') \in \mathcal{E}^{N'}$ such that for all $i \in N$, $\omega'_{i+n} = \bar{\omega}_i$ and $p(R'_{i+n}) = p(\bar{R}_i)$. Clearly, $z(e') < 0$. Hence, $N \cap \bar{N} = \emptyset$ and $z(e) \cdot z(e') > 0$. *Weak no-envy* and *same-sidedness* imply for all $i \in N$,

$$\Delta\varphi_i(e \cup e') = \Delta\varphi_{i+n}(e \cup e'). \tag{4.6}$$

By *one-sided population-monotonicity* and *same-sidedness*,

(i) for all $i \in N$, $\Delta\varphi_i(e) \leq \Delta\varphi_i(e \cup e')$ or
(ii) for all $i \in N$, $\Delta\varphi_i(e) \geq \Delta\varphi_i(e \cup e')$,

and

(iii) for all $i \in N$, $\Delta\varphi_{i+n}(e) \leq \Delta\varphi_{i+n}(e \cup e')$ or
(iv) for all $i \in N$, $\Delta\varphi_{i+n}(e) \geq \Delta\varphi_{i+n}(e \cup e')$.

By *same-sidedness* and *weak no-envy* it follows that either [(i) and (iii)] or [(ii) and (iv)]. By feasibility, this can only hold if for all $i \in N$, $\Delta\varphi_i(e) = \Delta\varphi_i(e \cup e')$. Similarly it follows that for all $i \in N'$, $\Delta\varphi_i(e') = \Delta\varphi_i(e \cup e')$. Hence, by (4.6), for all $i \in N$, $\Delta\varphi_i(e) = \Delta\varphi_{i+n}(e')$. By applying the same argument to the economies $\bar{e}$ and e' instead

[15] The following definition of *one-sided population-monotonicity* is due to Moreno (1996): For all $N, N' \in \mathcal{N}$ with $N' \subset N$, for all $e = (R, \omega) \in \mathcal{R}^N \times \mathbb{R}^N_+$, if either (i) $\sum_N p(R_i) \leq \sum_N \omega_i$ and $\sum_{N'} p(R_i) \leq \sum_{N'} \omega_i$ or (ii) $\sum_N p(R_i) \geq \sum_N \omega_i$ and $\sum_{N'} p(R_i) \geq \sum_{N'} \omega_i$, then for all $i \in N'$, $\varphi_i(e)\, R_i\, \varphi_i(e')$, where $e' = (R_{N'}, \omega_{N'})$, or for all $i \in N'$, $\varphi_i(e')\, R_i\, \varphi_i(e)$,

of e and e' it follows that for all $i \in N$, $\Delta\varphi_i(\bar{e}) = \Delta\varphi_{i+n}(e')$. Hence, for all $i \in N$, $\Delta\varphi_i(e) = \Delta\varphi_{i+n}(e') = \Delta\varphi_i(\bar{e})$.

Case 2: $z(e) > 0$. Similar to Case 1. ■

Thomson (Theorem 2.6) shows that the uniform allocation rule is the only rule that satisfies *Pareto optimality*, *no-envy*, *one-sided population-monotonicity*, and *replication-invariance*. Here, we show that the uniform reallocation rule is the only reallocation rule that satisfies *Pareto optimality*, *weak no-envy*, and *one-sided population-monotonicity*. So, in the following "reallocation version" of Theorem 2.6, *replication-invariance* can be omitted. Moreno (1996) independently proved this theorem. However, he applied a somewhat stronger notion of one-sided population-monotonicity and another proof technique.

Theorem 4.4 [16] *The uniform rule U^r is the only rule that satisfies Pareto optimality, weak no-envy, and one-sided population monotonicity.*

Proof. By *same-sidedness* and Lemmas 4.6 and 4.10 it follows that the uniform rule U^r satisfies *Pareto optimality*, *weak no-envy*, and *one-sided population monotonicity*.

To prove the remaining part of the theorem let φ be a rule that satisfies the properties listed in the theorem.

Then, by Lemma 4.11, φ satisfies *demand-onliness*. Hence, by Corollary 4.8, $\varphi = U^r$. ■

For an alternative proof of Theorem 4.4 we refer the interested reader to Moreno (1996), proof of Theorem 2.

4.4.2 One-sided replacement-domination

Next, we consider a change of one agent's preference relation, keeping the set of agents, the preferences of the remaining agents, and the individual endowments fixed. *Welfare-domination under preference-replacement* requires that either all remaining agents (weakly) lose or all (weakly) gain.

Welfare-domination under preference-replacement, or replacement-domination: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, all $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R, then either [for all $i \in N \backslash \{j\}$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in N \backslash \{j\}$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

In the sequel, we use the shorter phrase of *replacement-domination* for *welfare-domination under preference-replacement*.

[16] Theorem 4.4 is a strengthening of a result of Klaus, Peters, and Storcken (1997b, Theorem 3.2): the uniform rule U^r is the only rule that satisfies *Pareto optimality*, *no-envy*, and *one-sided population-monotonicity*.

Similar to the allocation model, it can be shown that the properties *Pareto optimality*, *weak no-envy*, and *replacement-domination* are generally incompatible. See for instance Thomson (1995d), Proposition 3. However, the uniform reallocation rule satisfies *Pareto optimality*, *no-envy*, and the one-sided version of *replacement-domination*, *i.e.*, if the change of one agent's preference relation is one-sided, then either all remaining agents (weakly) lose or all (weakly) gain (see also Section 2.2, page 24).

One-sided welfare-domination under preference-replacement, or replacement-domination: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, all $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R and $\bar{e}$ is a one-sided change of e, then either [for all $i \in N \backslash \{j\}$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in N \backslash \{j\}$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

Lemma 4.12 *The uniform rule U^r satisfies one-sided replacement-domination.*

Proof. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$ be such that $\bar{R}$ is a j-deviation from R and $\bar{e}$ is a one-sided change of e. In order to prove *one-sided replacement-domination*, we have to show that either (i) [for all $i \in N \backslash \{j\}$, $U_i^r(e)\, R_i\, U_i^r(\bar{e})$] or (ii) [for all $i \in N \backslash \{j\}$, $U_i^r(\bar{e})\, R_i\, U_i^r(e)$].

Case 1: $z(e) \leq 0$ and $z(\bar{e}) \leq 0$. Hence, for all $i \in N$, $U_i^r(e) = \max\{p(R_i), \omega_i + \lambda\}$, where λ solves $\sum_N U_i^r(e) = \sum_N \omega_i$ and $U_i^r(\bar{e}) = \max\{p(\bar{R}_i), \omega_i + \bar{\lambda}\}$, where $\bar{\lambda}$ solves $\sum_N U_i^r(\bar{e}) = \sum_N \omega_i$. If $U_j^r(e) = U_j^r(\bar{e})$, then for all $i \in N \backslash \{j\}$, $U_i^r(e) = U_i^r(\bar{e})$. Otherwise, we distinguish two cases.

Case 1.1: $U_j^r(e) < U_j^r(\bar{e})$. Thus, $p(\bar{R}_j) > p(R_j)$. It follows that $\bar{\lambda} \leq \lambda$. Hence, for all $i \in N \backslash \{j\}$, $U_i^r(e) = \max\{p(R_i), \omega_i + \lambda\} \geq \max\{p(\bar{R}_i), \omega_i + \bar{\lambda}\} = U_i^r(\bar{e}) \geq p(\bar{R}_i)$. This implies (ii).

Case 1.2: $U_j^r(e) > U_j^r(\bar{e})$. Thus, $p(\bar{R}_j) < p(R_j)$. It follows that $\bar{\lambda} \geq \lambda$. Hence, for all $i \in N \backslash \{j\}$, $U_i^r(\bar{e}) = \max\{p(\bar{R}_i), \omega_i + \bar{\lambda}\} \geq \max\{p(R_i), \omega_i + \lambda\} = U_i^r(e) \geq p(\bar{R}_i)$. This implies (i).

Case 2: $z(e) \geq 0$ and $z(\bar{e}) \geq 0$. Similar to Case 1. ■

Next, we show that the uniform rule U^r is the only rule that satisfies *Pareto optimality*, *no-envy*, *one-sided replacement-domination* and the following well-known property of *replication-invariance*.

Replication-invariance states that if an economy is replicated, *i.e.*, the preference profile and the individual endowments are replicated, then also the allocation assigned by the rule must be replicated (see also Section 2.2, page 25). In order to formally describe a replicated problem, we introduce the following notation.

Let $k \in \mathbb{N}$ and $\bar{N}, N_1, \ldots, N_k \in \mathcal{N}$. Then, the collection $\{N_1, \ldots, N_k\}$ constitutes a *partition* of $\bar{N}$ if $\bar{N} = \bigcup_{l \in \{1, \ldots k\}} N_l$ and $N_l \cap N_m = \emptyset$ for all $l, m \in \{1, \ldots k\}$, $l \neq m$.

Let $N, \bar{N} \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and $\bar{e} = (\bar{R}, \bar{\omega}) \in \mathcal{E}^{\bar{N}}$ be such that

(i) there exist $k \in \mathbb{N}$ and $N_1, \ldots, N_k \in \mathcal{N}$ such that $N = N_1$ and $\{N_1, \ldots, N_k\}$ constitutes a partition of $\bar{N}$,

(ii) there exist k bijective functions, indexed by $l \in \{1, \dots k\}$, $b_l \colon N \to N_l$, such that for all $i \in N$, $R_i = \bar{R}_{b_l(i)}$ and $\omega_i = \bar{\omega}_{b_l(i)}$.[17]

Then, $\bar{e}$ is a *k-replica of* e. Let $\bar{e} \in \mathcal{E}^{\bar{N}}$ be a k-replica of $e \in \mathcal{E}^N$ and let $x \in \mathbb{R}^N_+$. If $\bar{x} \in \mathbb{R}^{\bar{N}}_+$ is such that $x_i = \bar{x}_{b_l(i)}$ for the functions b_l, $l \in \{1, \dots k\}$, as described in **(ii)**, then $\bar{x}$ a *k-replica of* x.

Replication-invariance: For all $N, \bar{N} \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all $\bar{e} = (\bar{R}, \bar{\omega}) \in \mathcal{E}^{\bar{N}}$, if $\bar{e}$ is a k-replica of e, then $\varphi(\bar{e})$ is a k-replica of $\varphi(e)$.

It is easy to check, that the uniform rule U^r and the proportional rule Pr^r satisfy *replication-invariance*.

For the allocation model, Thomson establishes a characterization of the uniform allocation rule by *Pareto optimality*, *no-envy*, *one-sided replacement-domination*, and *replication-invariance* (see Theorem 2.7).

Before we prove the "reallocation version" of Theorem 2.7, we show that *Pareto optimality*, *no-envy*, *one-sided replacement-domination*, and *replication-invariance* imply *peak-onliness*.

Lemma 4.13 *Let φ satisfy Pareto optimality, no-envy, one-sided replacement-domination, and replication-invariance. Then, φ satisfies peak-onliness.*

Proof. Let φ be a rule that satisfies *Pareto optimality*, *no-envy*, *one-sided replacement-domination*, and *replication-invariance*. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$ be such that $p(R) = p(\bar{R})$. In order to prove *peak-onliness* we have to show that $\varphi(e) = \varphi(\bar{e})$.

Case 1: $z(e) \leq 0$. Assume, without loss of generality, that $N = \{1, 2, \dots, n\}$. Let $N' = \{n+1, \dots, 2n\}$ and define $e^0 = (R^0, \omega^0) \in \mathcal{E}^{N \cup N'}$ such that for all $i \in N$, $\omega_i = \omega_i^0 = \omega_{n+i}^0$ and $R_i = R_i^0 = R_{i+n}^0$. Clearly, e^0 is a 2-replica of e. Hence, by *replication-invariance*, for all $i \in N$, $\varphi_i(e) = \varphi_i(e^0) = \varphi_{n+i}(e^0)$. Next, we change the economy e^0, step by step, into a 2-replica of $\bar{e}$, denoted by e^{2n}, is such a way that $\varphi(e^0) = \varphi(e^{2n})$. By *replication-invariance*, this completes the proof.

Step 1: Let $R^1 \in \mathcal{R}^{N \cup N'}$ be a 1-deviation from R^0 such that $R_1^1 = \bar{R}_1$. Let $e^1 = (R^1, \omega^0)$ denote this one-sided change of economy e^0. Since $p(R_1^1) = p(R_{n+1}^1)$ and $\omega_1^0 = \omega_{n+1}^0$, by *same-sidedness* and *no-envy*, it follows that $\varphi_1(e^1) = \varphi_{n+1}(e^1)$. Suppose that $\varphi_1(e^1) < \varphi_1(e^0)$. Hence, $\varphi_{n+1}(e^1) < \varphi_{n+1}(e^0)$ and, by *same-sidedness* and *one-sided replacement-domination*, for all $i \in N \backslash \{1\} \cup N'$, $\varphi_i(e^1) \leq \varphi_i(e^0)$. Then, by feasibility, it follows that $\sum_{N \cup N'} \omega_i^0 = \sum_{N \cup N'} \varphi_i(e^1) < \sum_{N \cup N'} \varphi_i(e^0) = \sum_{N \cup N'} \omega_i^0$. This is a contradiction. Similarly, the assumption, $\varphi_1(e^1) > \varphi_1(e^0)$ yields a contradiction. Hence, $\varphi_1(e^1) = \varphi_1(e^0)$. Then, by *same-sidedness* and *one-sided replacement-domination*, for all $i \in N \backslash \{1\} \cup N'$, $\varphi_i(e^1) = \varphi_i(e^0)$. Hence, $\varphi(e^1) = \varphi(e^0)$.

[17] Let $l \in \{1, \dots k\}$ and $i \in N$. Then, agent $b_l(i)$ is the "replica", or the "clone", of agent $i \in N$ in N_l.

Step 2: Next, let $R^2 \in \mathcal{R}^{N \cup N'}$ be a 2-deviation from R^1 such that $R^2_2 = \bar{R}_2$. Let $e^2 = (R^2, \omega^0)$ denote this one-sided change of economy e^1. Similarly as before we deduce that $\varphi(e^2) = \varphi(e^1) = \varphi(e^0)$.

Steps k=3,...,n: For all $k \in \{1, 2, \ldots, n\}$ we let $R^k \in \mathcal{R}^{N \cup N'}$ be a k-deviation from R^{k-1} such that $R^k_k = \bar{R}_k$ and $e^k = (R^k, \omega^0)$. Applying the argument of the previous step, it follows that for all $k \in \{1, 2, \ldots, n\}$, $\varphi(e^k) = \varphi(e^{k-1}) = \varphi(e^0)$.

Step n+1: Next, let $R^{n+1} \in \mathcal{R}^{N \cup N'}$ be a $(n+1)$-deviation from R^n such that $R^2_{n+1} = \bar{R}_1$. Let $e^{n+1} = (R^{n+1}, \omega^0)$ denote this one-sided change of economy e^n. Similarly as before we deduce that $\varphi(e^n) = \varphi(e^0)$.

Steps k=n+2,...,2n: For all $k \in \{n, \ldots, 2n\}$ we let $R^k \in \mathcal{R}^{N \cup N'}$ be a k-deviation from R^{k-1} such that $R^k_k = \bar{R}_k$ and $e^k = (R^k, \omega^0)$. Similarly as before, it follows for all $k \in \{n, \ldots, 2n\}$, $\varphi(e^k) = \varphi(e^{k-1}) = \varphi(e^0)$.

Hence, by construction, e^{2n} is a 2-replica of $\bar{e}$ and $\varphi(e^0) = \varphi(e^{2n})$. By *replication-invariance*, this implies $\varphi(e) = \varphi(\bar{e})$.

Case 2: $z(e) > 0$. Similar to Case 1. ■

Theorem 4.5 *The uniform rule U^r is the only rule that satisfies Pareto optimality, no-envy, one-sided replacement-domination, and replication-invariance.*

Proof. By *same-sidedness* and Lemmas 4.6 and 4.12 it follows that the uniform rule U^r satisfies *Pareto optimality*, *no-envy*, and *one-sided replacement-domination*. Furthermore, the uniform rule U^r is *replication-invariant*.

To prove the remaining part of the theorem let φ be a rule that satisfies the properties listed in the theorem.

Then, by Lemma 4.13, φ satisfies *peak-onliness*. Hence, by Theorem 4.3, $\varphi = U^r$. ■

Another possibility to prove Theorem 4.5 is to adjust Thomson's proof for the allocation model to our reallocation model. The interested reader can find this version of the proof in Appendix B.

Thomson (1995d, Theorem 4) conjectures that the uniform rule U^r is the only rule that satisfies *Pareto optimality*, *weak no-envy*, *one-sided replacement-domination*, and *replication-invariance*. The following example shows that this conjecture is not true and it is not possible to weaken *no-envy* in Theorem 4.5 to either *weak no-envy* or *equal treatment of equals*.

Example 4.3 The following rule φ^3 satisfies *Pareto optimality, weak no-envy, equal treatment of equals, one-sided replacement-domination, and replication-invariance*. If $N \in \mathcal{N}$, and $e = (R, \omega) \in \mathcal{E}^N$ such that $z(e) < 0$ and for all $i \in D(e)$, $\omega_i = 0$, then

$$\varphi^3(e) = \begin{cases} p(R_i) & \text{if } i \notin D(e), \\ U_i(R_{D(e)}, s(e)) & \text{if } i \in D(e). \end{cases}$$

For all remaining $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$, $\varphi^3(e) = U^r(e)$. ◇

4.4.3 One-sided endowment-monotonicity

Finally, we consider a change of one agent's individual endowment, keeping the individual endowments of the remaining agents and the set of agents fixed. *Endowment-monotonicity* requires that either all agents (weakly) lose or all (weakly) gain.

Endowment-monotonicity: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, all $\bar{e} = (R, \bar{\omega}) \in \mathcal{E}^N$, and all $j \in N$, if $\bar{\omega}$ is a j-deviation from ω, then either [for all $i \in N$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in N$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

The following example shows that similar to the allocation model, the properties *Pareto optimality*, *weak no-envy*, and *endowment-monotonicity* are generally incompatible.

Example 4.4 Let the rule φ satisfy *Pareto optimality*, *weak no-envy*, and *endowment-monotonicity*. Let $N = \{1, 2\}$ and $e = (R, \omega) \in \mathcal{E}^N$ such that $p(R) = (0, 4)$, $5\, P_2\, 3$, and $\omega = (1, 2)$. Since $z(e) > 0$, by *same-sidedness* it follows that $\varphi(e) = (0, 3)$. Next, consider $\bar{e} = (R, \bar{\omega}) \in \mathcal{E}^N$ such that $\bar{\omega} = (1, 5)$. Since $z(e) < 0$, by *same-sidedness*, it follows that for $i = 1, 2$, $\Delta\varphi_i(e) \geq -1$. Since $s_1(\bar{e}) = s_2(\bar{e})$, by *weak no-envy* it follows that $\varphi(\bar{e}) = \bar{\omega}$. Since $\bar{\omega}$ is a 2-deviation from ω, in contradiction to *endowment-monotonicity*, $\varphi_1(e)\, P_1\, \varphi_1(\bar{e})$ and $\varphi_2(\bar{e})\, P_2\, \varphi_2(e)$. ◇

However, the uniform reallocation rule satisfies *Pareto optimality*, *weak no-envy*, and the one-sided version of *endowment-monotonicity*, *i.e.*, if the change of one agent's individual endowment is one-sided, then either all agents (weakly) lose or all (weakly) gain (see also Section 2.2, page 24).

One-sided endowment-monotonicity: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, all $\bar{e} = (R, \bar{\omega}) \in \mathcal{E}^N$, and all $j \in N$, if $\bar{\omega}$ is a j-deviation from ω and $\bar{e}$ is a one-sided change of e, then either [for all $i \in N$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in N$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

The proportional rule Pr^r and the uniform rule U^r satisfy *one-sided endowment-monotonicity.*

Lemma 4.14 *The uniform rule U^r satisfies one-sided endowment-monotonicity.*

Proof. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$ be such that $\bar{\omega}$ is a j-deviation from ω and $\bar{e}$ is a one-sided change of e. In order to prove *one-sided endowment-monotonicity*, we have to show that either (i) [for all $i \in N$, $U_i^r(e)\, R_i\, U_i^r(\bar{e})$] or (ii) [for all $i \in N$, $U_i^r(\bar{e})\, R_i\, U_i^r(e)$].

Assume, without loss of generality, that $z(e) \leq 0$, $z(\bar{e}) \leq 0$, and $\bar{\omega}_j > \omega_j$. Hence, for all $i \in N$, $U_i^r(e) = \max\{p(R_i), \omega_i + \lambda\}$, where λ solves $\sum_N U_i^r(e) = \sum_N \omega_i$ and $U_i^r(\bar{e}) = \max\{p(R_i), \bar{\omega}_i + \bar{\lambda}\}$, where $\bar{\lambda}$ solves $\sum_N U_i^r(\bar{e}) = \sum_N \bar{\omega}_i$. By the definition of U^r it follows that $\bar{\lambda} \geq \lambda$. Hence, for all $i \in N$, $p(R_i) \leq U_i^r(e) \leq U_i^r(\bar{e})$. This implies (i).

■

A stronger notion of *one-sided endowment-monotonicity* is the following *strong one-sided endowment-monotonicity*. If in case of excess demand an agent's individual endowments decreases, then no agent is made better off in the reallocation that follows this one-sided change. Similarly, if in case of excess supply an agent's individual endowments increases, then no agent is made better off in the reallocation that follows this one-sided change.

Strong one-sided endowment-monotonicity: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, all $\bar{e} = (R, \bar{\omega}) \in \mathcal{E}^N$, and all $j \in N$, if $\bar{\omega}$ is a j-deviation from ω and $\bar{\omega}_j < \omega_j$, then [if $z(e) \geq 0$, then for all $i \in N$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] and [if $z(\bar{e}) \leq 0$, then for all $i \in N$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

Next, we prove that *Pareto optimality* and *one-sided endowment-monotonicity* together imply *strong one-sided endowment-monotonicity*.

Lemma 4.15 *Let the rule φ satisfy Pareto optimality and one-sided endowment-monotonicity. Then φ satisfies strong one-sided endowment-monotonicity.*

Proof. Let φ be a rule that satisfies *Pareto optimality* and *one-sided endowment-monotonicity*. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, $\bar{e} = (R, \bar{\omega}) \in \mathcal{E}^N$, and $j \in N$ be such that $\bar{\omega}$ is a j-deviation from ω and $\bar{\omega}_j < \omega_j$. Hence, $\sum_{i \in N} \bar{\omega}_i < \sum_{i \in N} \omega_i$. Suppose, without loss of generality, that $z(\bar{e}) \leq 0$. Hence, $z(e) < 0$. Then, by *one-sided endowment-monotonicity*, either (i) [for all $i \in N$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or (ii) [for all $i \in N$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$]. In order to prove *strong one-sided endowment-monotonicity* we have to show that (ii) holds. Suppose, by contradiction, that (i) holds. Then, by *same-sideness*, for all $i \in N$, $p(R_i) \leq \varphi_i(e) \leq \varphi_i(\bar{e})$. So, by feasibility, $\sum_{i \in N} \omega_i = \sum_{i \in N} \varphi_i(e) \leq \sum_{i \in N} \varphi_i(\bar{e}) = \sum_{i \in N} \bar{\omega}_i$. This is in contradiction to $\sum_{i \in N} \bar{\omega}_i < \sum_{i \in N} \omega_i$. ■

The (one-sided) endowment-monotonicity properties introduced above can be seen as extensions of (one-sided) resource-monotonicity introduced for allocation rules (see Chapter 2). Other variations of endowment monotonicity for the reallocation setting are discussed in Thomson (1995d). In Klaus, Peters, and Storcken (1997b) and Klaus (1997b) a slightly stronger notion of (strong) one-sided endowment-monotonicity is applied.

Next, we show that the uniform rule U^r is the only rule that satisfies *Pareto optimality, weak no-envy*, and *one-sided endowment-monotonicity*. This characterization can be seen as an extension of Thomson's characterization of the uniform allocation rule by *Pareto optimality, no-envy*, and *one-sided resource-monotonicity* for the domain of economies with bounded preferences (see Theorem 2.5). However, the proof of this characterization in the reallocation setting is based on a different proof technique and holds for the whole domain of single-peaked preferences.

The stages of the proof of this characterization are as follows. First we show that *Pareto optimality* and *one-sided endowment-monotonicity* together imply "coordinatewise endowment continuity" (see Lemma 4.16). Next, we show that rules that satisfy *Pareto optimality, weak no-envy,* and *one-sided endowment-monotonicity* have the so-called *dummy property*: non-traders receive zero allotment changes (see Lemma 4.17). Then, we show that rules that satisfy *Pareto optimality, one-sided endowment-monotonicity,* and the *dummy property* satisfy *boundedness by endowments and peaks*: each agent's allotment is between his individual endowment and his peak (see Lemma 4.18). Applying these results, the characterization we mentioned above follows easily.

Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, $j \in N$, and $\alpha \in \mathbb{R}_+$. Then, we define the j-deviation $\omega(\alpha, j) \in \mathbb{R}^N_+$ from ω by $\omega(\alpha, j)_j = \alpha$ and $\omega(\alpha, j)_i = \omega_i$ for all $i \in N \backslash \{j\}$. Agent j's individual endowment after the unilateral change is α. Now, continuity of a rule with respect to unilateral endowment changes can be formalized as follows.

Coordinatewise endowment continuity: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, all $j \in N$, and all sequences $\{\alpha^k\}_{k \in \mathbb{N}}$, if $\lim_{k \to \infty} \alpha^k = \omega_j$, then $\lim_{k \to \infty} \varphi(R, \omega(\alpha^k, j)) = \varphi(e)$.

The following lemma states that *Pareto optimality* and *one-sided endowment-monotonicity* together imply *coordinatewise endowment continuity*. A similar result for allocation problems can be found in Thomson (1994a) in the proof of Theorem 2.

Lemma 4.16 *Let the rule φ satisfy Pareto optimality and one-sided endowment-monotonicity. Then φ satisfies coordinatewise endowment continuity.*

Let $x, y \in \mathbb{R}^N$. Then, $x \geq y$ means that for all $i \in N$, $x_i \geq y_i$.

Proof. Let φ be a rule that satisfies *Pareto optimality* and *one-sided endowment-monotonicity*. Then, by Lemma 4.15, φ satisfies *strong one-sided endowment-monotonicity*.

Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, $j \in N$, and $\{\alpha^k\}_{k \in \mathbb{N}}$ be a sequence in $\mathbb{R}_+$ such that $\lim_{k \to \infty} \alpha^k = \omega_j$. Let for all $k \in \mathbb{N}$, $x^k = \varphi(R, \omega(\alpha^k, j))$. In order to prove *coordinatewise continuity* we have to show that $\lim_{k \to \infty} x^k = \varphi(e)$. Suppose, without loss of generality, that $\alpha^1 > \alpha^2 > \alpha^3 > \ldots > \omega_j$. We distinguish two cases.

Case 1: $z(e) \leq 0$. Then, for all $k \in \mathbb{N}$, $z(R, \omega(\alpha^k, j)) < 0$. Hence, by *same-sidedness*, it follows for all $k \in \mathbb{N}$, $\varphi(e) \geq p(R)$ and $x^k \geq p(R)$. Now, by *strong one-sided endowment-monotonicity*, it follows that for all $k \in \mathbb{N}$, $x^k \geq \varphi(e)$ and the sequence $\{x^k\}_{k \in \mathbb{N}}$ is non-increasing. By feasibility, it follows that for all $k \in \mathbb{N}$, $\sum_{i \in N} x^k_i = \sum_{i \in N} \omega(\alpha^k, j)_i$ and $\sum_{i \in N} \varphi_i(e) = \sum_{i \in N} \omega_i$. Because $\omega(\alpha^k, j)_j = \alpha^k$ converges to ω_i, x^t converges to $\varphi(e)$.

Case 2: $z(e) > 0$. Then, there exists $k_0 \in \mathbb{N}$ such that $z(R, \omega(\alpha^{k_0}, j)) > 0$ for all $k \geq k_0$. Without loss of generality, let $k_0 = 1$. The proof proceeds similar to Case 1. ■

Next, we show that rules that satisfy *Pareto optimality, no-envy,* and *one-sided endowment-monotonicity* have the so-called *dummy property*: non-traders receive zero

allotment changes.

Dummy property: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all $i \in N$, if $p(R_i) = \omega_i$, then $\varphi_i(e) = \omega_i$.

Lemma 4.17 *Let the rule φ satisfy Pareto optimality, weak no-envy, and one-sided endowment-monotonicity. Then φ satisfies the dummy property.*

Proof. Let φ be a rule that satisfies *Pareto optimality*, *weak no-envy*, and *one-sided endowment-monotonicity*. Then, by Lemmas 4.15 and 4.16, φ satisfies *strong one-sided endowment-monotonicity* and *coordinatewise endowment continuity*.

Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and $j \in N$ be such that $p(R_j) = \omega_j$. In order to prove the *dummy property* we have to show that $\varphi_j(e) = \omega_j$. Suppose, by contradiction, that $\varphi_j(e) \neq \omega_j$. By *Pareto optimality*, this implies that either $z(e) > 0$ or $z(e) < 0$.

Case 1: $z(e) > 0$. Then, by *same-sidedness*, $\varphi_j(e) < p(R_j) = \omega_j$. Furthermore, by *strong one-sided endowment-monotonicity*, we assume without loss of generality (lower the individual endowments if necessary) that all agents, except agent j, have either maximal demand $d_{\max} = \{p(R_i) - \omega_i \mid i \in N\}$ or zero as individual endowment, *i.e.*, for all $i \in N \backslash \{j\}$, $\omega_i = \max\{p(R_i) - d_{\max}, 0\}$.[18] Since $z(e) > 0$, it follows that $d_{\max} > 0$. Let $\alpha := \omega_j - \varphi_j(e) < 0$. For $0 \leq \varepsilon \leq \alpha$ we consider the endowment vector $\omega(\omega_j - \varepsilon, j) =: \omega^\varepsilon$, *i.e.*, for all $i \in N \backslash \{j\}$, $\omega_i^\varepsilon = \omega_i$ and $\omega_j^\varepsilon = \omega_j - \varepsilon$. Hence, for $0 \leq \varepsilon \leq \alpha$, $\omega_j \geq \omega_j^\varepsilon \geq \varphi_j(e)$. Let for all $i \in N$, $\alpha_i^\varepsilon := \Delta\varphi_i(R, \omega^\varepsilon)$. By *weak no-envy* between j and agents $i \in N \backslash \{j\}$ and *strong one-sided endowment-monotonicity* it follows that for all $i \in N \backslash \{j\}$ and $0 \leq \varepsilon < \alpha$, either $\omega_j^\varepsilon + \alpha_i^\varepsilon > p(R_j)$ or $\omega_j^\varepsilon + \alpha_i^\varepsilon \leq \omega_j^\varepsilon + \alpha_j^\varepsilon = \varphi_j(R, \omega^\varepsilon) \leq \varphi_j(e) < \omega_j^\varepsilon$. Because $p(R_j) - \omega_j^\varepsilon = \varepsilon$ this implies for all $i \in N \backslash \{j\}$ and all $0 \leq \varepsilon < \alpha$, either (i) $\alpha_i^\varepsilon > \varepsilon$ or (ii) $\alpha_i^\varepsilon \leq \alpha_j^\varepsilon \leq \varphi_j(e) - \omega_j^\varepsilon < 0$.

Next, suppose that for some $k \in N \backslash \{j\}$ and some ε, $0 \leq \varepsilon < \alpha$, $\alpha_k^\varepsilon < 0$. Then, by feasibility, $\omega_i^\varepsilon = \omega_i > 0$. Note, that by *same-sidedness*, for all $i \in N$, $\alpha_i^\varepsilon \leq d_{\max} = d_k(R, \omega^\varepsilon)$. Then, by *weak no-envy*, it follows that for all $i \in N \backslash \{k\}$, $\alpha_i^\varepsilon \leq \alpha_k^\varepsilon$. Thus, for all $i \in N$, $\alpha_i^\varepsilon < 0$. This contradicts feasibility. Hence, for all $i \in N \backslash \{j\}$ and all ε, $0 \leq \varepsilon < \alpha$, $\alpha_k^\varepsilon \geq 0$. This implies for all $i \in N \backslash \{j\}$ and all $0 \leq \varepsilon < \alpha$, $\alpha_i^\varepsilon > \varepsilon$.

By *strong one-sided endowment-monotonicity* it follows that for all $i \in N$ and all $0 \leq \tau \leq \varepsilon \leq \alpha$, $\alpha_i^\tau \geq \alpha_i^\varepsilon$. Particularly it follows that $\alpha_j^0 = -\alpha \geq \alpha_j^\alpha$. By *coordinatewise endowment continuity*, for all $i \in N \backslash \{j\}$, $\alpha_i^\alpha \geq \alpha$. This is only possible if $N = \{i, j\}$, $\alpha_i^\alpha = \alpha$ and $\alpha_j^\alpha = -\alpha$. Hence, j envies i.

Case 2: $z(e) < 0$. Similar to Case 1. ■

In the next lemma, we show that *Pareto optimality, one-sided endowment-monotonicity*, and the *dummy property* imply *boundedness by endowments and peaks*: each agent's allotment is between his individual endowment and his peak.

[18] For problems in excess supply, assume that all suppliers have equal maximal supply.

Boundedness by endowments and peaks: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all $i \in N$, either $\omega_i \leq \varphi_i(e) \leq p(R_i)$ or $p(R_i) \leq \varphi_i(e) \leq \omega_i$.

Boundedness by endowments and peaks implies the *dummy property*. It is easy to check that the uniform rule U^r and the proportional rule Pr^r satisfy *boundedness by endowments and peaks*. Furthermore, *boundedness by endowments and peaks* implies *individual rationality*: after the reallocation no agent is worse off than at his individual endowment (see also Section 2.2, page 27).

Individual rationality: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N$, and all $i \in N$, $\varphi_i(e)\, R_j\, \omega_i$.

The following lemma shows that under the assumptions of *Pareto optimality* and *one-sided endowment-monotonicity* the *dummy property* is equivalent to *boundedness by endowments and peaks*.

Lemma 4.18 *Let the rule φ satisfy Pareto optimality, the dummy property, and one-sided endowment-monotonicity. Then φ satisfies boundedness by endowments and peaks.*

Proof. Let φ be a rule that satisfies *Pareto optimality*, the *dummy property*, and *one-sided endowment-monotonicity*. Then, by Lemma 4.15, φ satisfies *strong one-sided endowment-monotonicity*.

Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$.

Case 1: $z(e) \geq 0$. By *same-sidedness*, for all $i \in N$, $\varphi_i(e) \leq p(R_i)$. In order to prove *boundedness by endowments and peaks* we have to show that $\omega_i \leq \varphi_i(e) \leq p(R_i)$. Suppose that there exists a $j \in N$ such that $\varphi_j(e) < p(R_j)$. Now, it is sufficient to prove that $\omega_j \leq \varphi_j(e)$. Consider $\bar{\omega} \in \mathbb{R}^N_+$ such that for all $i \in D(e)$, $\bar{\omega}_i = \omega_i$ and for all $i \notin D(e)$, $\bar{\omega}_i = p(R_i)$. Let $\bar{e} = (R, \bar{\omega}) \in \mathcal{E}^N$. By *strong one-sided endowment-monotonicity* it follows that for all $i \in N$, $\varphi_i(\bar{e}) \leq \varphi_i(e)$. By the *dummy property*, for all $i \notin D(\bar{e})$, $\varphi_i(\bar{e}) = p(R_i)$. Therefore $j \in D(e) = D(\bar{e})$. Now, by *strong one-sided endowment-monotonicity* it is sufficient to prove that $\omega_j = \bar{\omega}_j \leq \varphi_j(\bar{e})$. Suppose to the contrary that $\varphi_j(\bar{e}) < \bar{\omega}_j$.

Consider $\tilde{\omega} = \bar{\omega}(p(R_j), j) \in \mathbb{R}^N_+$, *i.e.*, for all $i \in N \backslash \{j\}$, $\tilde{\omega}_i = \omega_i$ and $\tilde{\omega}_j = p(R_j)$. Let $\tilde{e} = (R, \tilde{\omega}) \in \mathcal{E}^N$. Because $\varphi_j(\bar{e}) < \bar{e}_j$, it follows by *same-sidedness* and feasibility that $D(\bar{e}) \neq \{j\}$. Hence $z(\bar{e}) > 0$. Therefore, by *strong one-sided endowment-monotonicity*, it follows that for all $i \in N$, $\varphi_i(\bar{e}) \leq \varphi_i(\tilde{e})$. By feasibility this yields $\varphi_j(\bar{e}) \leq \varphi_j(\tilde{e}) + p(R_j) - \tilde{\omega}_j < p(R_j) = \tilde{\omega}_j$. This, however, contradicts the *dummy property*. So, $\varphi_j(\bar{e}) \geq \bar{\omega}_j$.

Case 2: $z(e) < 0$. Similar to Case 1. ■

Finally, we prove the characterization of the uniform rule U^r by *Pareto optimality*, *weak no-envy*, and *one-sided endowment-monotonicity*. As already mentioned above,

this result can be interpreted as an extension of Thomson's Theorem 2.5 to the reallocation model.

Theorem 4.6 [19] *The uniform rule U^r is the only rule that satisfies Pareto optimality, weak no-envy, and one-sided endowment-monotonicity.*

Proof. By *same-sidedness* and Lemmas 4.6 and 4.14 it follows that the uniform rule U^r satisfies *Pareto optimality*, *weak no-envy*, and *one-sided endowment-monotonicity*.

To prove the remaining part of the theorem let φ be a rule that satisfies the properties listed in the theorem. Then, by the Lemmas 4.15, 4.16, 4.17, and 4.18, φ satisfies *strong one-sided endowment-monotonicity*, *coordinatewise endowment continuity*, and *boundedness by endowments and peaks*.

Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and suppose, by contradiction, that $\varphi(e) \neq U^r(e)$. Since φ is *Pareto optimal* it follows that either $z(e) > 0$ or $z(e) < 0$.

Case 1: $z(e) > 0$. By *same-sidedness* and *boundedness by endowments and peaks* it follows that for all $i \notin D(e)$, $\varphi_i(e) = p(R_i) = U_i^r(e)$. Then, by Lemma 4.1, there exist agents $j, k \in D(e)$ such that $\varphi_j(e) < p(R_j)$ and $\triangle\varphi_j(e) < \triangle\varphi_k(e)$. By *no-envy*, $\omega_j + \triangle\varphi_k(e) > p(R_j)$. Let $l \in S(e)$ and consider $\omega^\alpha = \omega(\alpha, l) \in \mathbb{R}_+^N$ such that $p(R_l) \leq \alpha \leq \omega_l$ and $\alpha \longrightarrow p(R_l)$. Let $e^\alpha = (R, \omega^\alpha) \in \mathcal{E}^N$. Then, by *coordinatewise endowment continuity*, *strong one-sided endowment-monotonicity*, and *no-envy*, it follows that for $\alpha' = p(R_l)$, $\omega_j^{\alpha'} + \triangle\varphi_k(e^{\alpha'}) > p(R_j)$. Note that l is a non-trader at problem $e^{\alpha'}$. Hence, the set of suppliers has been decreased by one. Repeating this process yields a problem, say $\tilde{e} = (R, \tilde{\omega})$, such that for all $i \in D(e)$, $\tilde{\omega}_i = \omega_i$, for all $i \notin D(e)$, $\tilde{\omega}_i = p(R_i)$, and $\tilde{\omega}_j + \triangle\varphi_k(\tilde{e}) > p(R_j)$. In particular, we have $\triangle\varphi_k(\tilde{e}) > 0$. But then either feasibility or *boundedness by endowments and peaks* is violated.

Case 2: $z(e) < 0$. Similar to Case 1. ■

In the remainder of this section, we derive a characterization of the uniform rule U^r in terms of *individual rationality*, *strong one-sided endowment-monotonicity*, and *weak no-envy* (see Corollary 4.9). First, we show that *individual rationality* and *strong one-sided endowment-monotonicity* imply *Pareto optimality*.

Lemma 4.19 *Let the rule φ satisfy individual rationality and strong one-sided endowment-monotonicity. Then φ satisfies Pareto optimality.*

Proof. Let φ be a rule that satisfies *individual rationality* and *strong one-sided endowment-monotonicity*. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and suppose, by contradiction, that $\varphi(e)$ is not same-sided.

[19] Theorem 4.6 is a strengthening of a result of Klaus, Peters, and Storcken (1997b, Theorem 4.4): the uniform rule U^r is the only rule that satisfies *Pareto optimality*, *no-envy*, and a stronger version of *one-sided endowment-monotonicity*. In Klaus, Peters, and Storcken (1997b), we did not succeed to prove the independence of the properties in the characterization mentioned above. The properties in Theorem 4.6 are independent (see Section 4.6).

Case 1: $z(e) \geq 0$. The proof proceeds as follows. In (i), we show that for e the rule φ assigns to all suppliers and all non-traders their peaks.[20] In (ii), we define a sequence of economies $\{e^q\}_{q\in\mathbb{N}}$ such that for any $q \in \mathbb{N}$ the corresponding allotment $\varphi(e^q)$ is not same-sided. Finally, in (iii), for q large enough we derive a contradiction to feasibility.

(i) Suppose, by contradiction, that for some agent $j \notin D(e)$, $\varphi_j(e) \neq p(R_j)$. By *individual rationality*, $p(R_j)\, P_j\, \varphi_j(e)\, R_j\, \omega_j$.

Now, let $\bar{\omega} = \omega(p(R_j), j) \in \mathbb{R}^N_+$ and $\bar{e} = (R, \bar{\omega}) \in \mathcal{E}^N$. Then, in $\bar{e}$, agent j is a non-trader. Hence, by *individual rationality* in $\bar{e}$, $\varphi_j(\bar{e}) = p(R_j)$. Note that $\bar{\omega}$ is a j-deviation from ω, $\bar{\omega}_j < \omega_j$, and $z(e) \geq 0$. Then, by *strong one-sided endowment-monotonicity* applied to $\bar{e}$ and e, $p(R_j)\, P_j\, \varphi_j(e)\, R_j\, \varphi_j(\bar{e})$. Thus, $\varphi_j(\bar{e}) \neq p(R_j)$. This is a contradiction to *individual rationality*. Hence,

$$\varphi_j(e) = p(R_j) \text{ for all } j \notin D(e). \tag{4.7}$$

(ii) Since $\varphi(e)$ is not same-sided, by (i), there exists a demander $k \in D(e)$ such that $\varphi_k(e) > p(R_k) > \omega_k$. By *individual rationality* in e, there exists a point $a^1 \in \mathbb{R}_+$ such that

$$p(R_k) > a^1 \geq \omega_k \text{ and } \varphi_k(e)\, I_k\, a^1. \tag{4.8}$$

Let $d^1 := p(R_k) - a^1$.

In e, by feasibility and *individual rationality* for all demanders, there exists at least one supplier l. Now, let $\omega^1 \in \mathbb{R}^N_+$ be such that $\omega^1_i = \omega_i$ for $i \in N\backslash\{l\}$ and $\omega^1_l = \omega_l - \min\{\omega_l - p(R_l), d^1\}$. Let $e^1 = (R, \omega^1) \in \mathcal{E}^N$. Note that $D(e) = D(e^1)$. Also, note that ω^1 is a l-deviation from ω, $\omega^1_l < \omega_l$, and $z(e^1) \geq 0$. Then, by *strong one-sided endowment monotonicity* applied to e^1 and e,

$$\varphi_i(e)\, R_i\, \varphi_i(e^1) \text{ for all } i \in N. \tag{4.9}$$

Hence, by (4.8) and (4.9) it follows that for agent k, either

$$\varphi_k(e^1) \geq \varphi_k(e) > p(R_k), \tag{4.10}$$

or

$$p(R_k) > a^1 \geq \varphi_k(e^1) \geq \omega_k. \tag{4.11}$$

If (4.10), then

$$\varphi_k(e^1) > p(R_k). \tag{4.12}$$

If (4.11), then the allotment of agent k moves to the other side of his peak so that by construction $\varphi_k(e) > \varphi_k(e^1) + \min\{\omega_l - p(R_l), d^1\}$.

By (4.7), applied to both e and e^1, and by feasibility,

$$\begin{aligned}\sum_{i\in D(e)} \varphi_i(e) &= \sum_{i\in D(e)} \varphi_i(e^1) + \min\left\{\omega_l - p(R_l), d^1\right\} \\ &= \sum_{\substack{i\in D(e)\\ i\neq k}} \varphi_i(e^1) + \varphi_k(e^1) + \min\left\{\omega_l - p(R_l), d^1\right\}.\end{aligned}$$

[20]This statement holds for arbitrarily chosen problems with excess demand or which are balanced. Similarly, for problems with excess supply, all demanders and non-traders receive their peaks. See also Thomson (1995d), Lemma 4.

Since $\varphi_k(e) > \varphi_k(e^1) + \min\{\omega_l - p(R_l), d^1\}$, in e^1 there exists a demander m such that $\varphi_m(e^1) > \varphi_m(e)$. Hence, by (4.9),

$$\varphi_m(e^1) > p(R_m). \tag{4.13}$$

Since either (4.12) or (4.13) hold, it follows that $\varphi(e^1)$ is not *Pareto optimal*. Now, we repeat the argument, proceeding from e^1 instead of e.

At each step $n \in \mathbb{N}$, the allotment $\varphi(e^n)$ is not same-sided and there exists $i \in D(e^n)$ such that $\varphi_i(e^n) > p(R_i) > a^n \geq \omega_i^n$ and $\varphi_i(e^n)\, I_i\, a^n$. Hence,

$$d^{n+1} = p(R_i) - a^n > 0. \tag{4.14}$$

In step $n+1$ we lower the endowment of some $j \in S(e^n)$ by $\min\{\omega_j^n - p(R_j), d^{n+1}\}$. This procedure yields

$$\begin{aligned} &D(e) = D(e^q) \quad \text{for all } q \in \mathbb{N} \text{ and} \\ &\{\varphi(e^q)\}_{q\in\mathbb{N}}, \quad \text{an infinite sequence of reallocations} \\ &\qquad\qquad\quad \text{which are not } \textit{Pareto optimal}. \end{aligned} \tag{4.15}$$

(iii) Since the set of demanders $D := D(e) = D(e^q)$ is finite, by (4.15) it follows that at least one demander must be several times on the "wrong side" of his peak. Consider such an $i \in D$. Let $s > n$ be such that $\varphi_i(e^s) > p(R_i)$ and $\varphi_i(e^n) > p(R_i)$. Note that $\omega^s \leq \omega^n$ and $z(e^n) \geq 0$. Then, by *strong one-sided endowment monotonicity* applied to $[e^n \text{and } e^{n+1}]$, $[e^{n+1} \text{and } e^{n+2}]$, ...,$[e^{s-1} \text{and } e^s]$ it follows that $\varphi_i(e^n)\, R_i\, \varphi_i(e^{n+1})\, R_i\, \varphi_i(e^{n+2})\, R_i \ldots R_i\, \varphi_i(e^s)$. Thus, $\varphi_i(e^s) \geq \varphi_i(e^n)$. Hence, $a^s \leq a^n$ and therefore

$$d^{s+1} \geq d^{n+1} \text{ for } s > n. \tag{4.16}$$

Because the number of demanders is finite, (4.14) and (4.16) imply that the sequence $\{d^q\}_{q\in\mathbb{N}}$ is bounded from below by a term $d > 0$ of the sequence. So, at each step, the endowment of a supplier is lowered by at least d without this turning him into a demander. Repeating this procedure, all suppliers are made non-traders after a finite number t, $t \in \mathbb{N}$, of steps. For problem e^t, it follows that

$$\varphi_j(e^t) = p(R_j) = \omega_j^t \text{ for } j \notin D(e^t).$$

Then, by feasibility and *individual rationality* in e^t,

$$\varphi_i(e^t) = \omega_i \text{ for } i \in D(e^t).$$

Note that $\varphi(e^t)$ is a term of the sequence $\{\varphi(e^q)\}_{q\in\mathbb{N}}$ which is same-sided and therefore *Pareto optimal*. This contradicts (4.15).

Case 2: $z(e) < 0$. Similar to Case 1. ■

For the next theorem see also Thomson (1995d), Theorem 2.

Theorem 4.7 *On the domain of two-agent economies, $\bigcup_{\substack{N \in \mathcal{N} \\ |N|=2}} \mathcal{E}^N$, the uniform rule U^r is the only rule that satisfies individual rationality and strong one-sided endowment monotonicity.*

Proof. The uniform rule U^r satisfies *individual rationality* and *strong one-sided endowment monotonicity*. To prove the remaining part of the theorem let φ be a rule that satisfies the properties listed in the theorem. Let $N \in \mathcal{N}$, $N = \{i, j\}$, and $e = (R, \omega) \in \mathcal{E}^N$.

Case 1: $U^r(e) = \omega$. Then, by *individual rationality*, $\varphi(e) = \omega = U^r(e)$.

Case 2: $U^r(e) \neq \omega$. This can only occur if agents are of "opposite" types, *i.e.*, N consists of one demander and one supplier. Suppose that $z(e) \geq 0$ and, without loss of generality, let $i \in S(e)$. Then, similarly as in the proof of Lemma 4.18, part (i), it follows that $\varphi_i(e) = p(R_i) = U_i^r(e)$. Thus, $\varphi_j(e) = \omega_j + (\omega_i - p(R_i)) = U_j^r(e)$. For $z(e) < 0$, $\varphi(e) = U^r(e)$ follows similarly. ■

Remark 4.3 Theorem 4.7 only applies to the two-agent case. As already established in the proof, a "real reallocation" only takes place when agents of opposite types exist. In this case one of the agents receives his peak; suppose, without loss of generality, that this is the supplier. Then, his total supply has to be allocated to the demander. For more than two agents, again the agents of one group (e.g., the suppliers and the non-traders) receive their peaks and the total supply of this group is distributed among the demanders. This can be done in different ways. All rules that are *same-sided* and distribute the total supply "monotonically" among the demanders are *individually rational* and *strong one-sided endowment monotonic* for problems with excess demand. A similar statement holds for excess supply. Examples of such rules are the uniform rule U^r, but also the proportional rule Pr^r. □

Corollary 4.9 [21] *The uniform rule U^r is the only rule that satisfies individual rationality, strong one-sided endowment monotonicity, and weak no-envy.*

Proof. The uniform rule U^r satisfies *individual rationality*, *strong one-sided endowment monotonicity*, and *weak no-envy*. To prove the remaining part of the corollary let φ be a rule that satisfies the properties listed in the corollary. By Lemma 4.19, φ is *Pareto optimal*. Then, by Theorem 4.6, $\varphi = U^r$. ■

A summary of the characterizations of the uniform rule U^r by *Pareto optimality* or *individual rationality*, one of the two no-envy properties, and one of the solidarity properties can be found in Table 4.2. For a comparison with corresponding characterizations of the uniform allocation rule U we refer to Table 2.2.

[21] We did not succeed in proving the independence of this characterization. To be more precisely, it is an open question whether *individual rationality* is independent of *weak no-envy* and *strong one-sided endowment-monotonicity*.

	Th 4.4	Th 4.5	Th 4.6	Co 4.9
Pareto optimality	×	×	×	
no-envy		×		
weak no-envy	×		×	×
one-sided population-monotonicity	×			
one-sided replacement-domination		×		
one-sided endowment-monotonicity			×	
strong one-sided endowment-monotonicity				×
replication-invariance		×		
individual rationality				×

Table 4.2: Characterizations of the uniform rule U^r by *Pareto optimality* or *individual rationality*, no-envy properties, and solidarity properties.

4.5 Consistency

In this section we explore a notion of consistency. Consistency essentially describes the stability of a reallocation rule in situations where agents leave with their allotments: by leaving, these agents might create a positive or negative "leftover". Consider the reduced economy where this leftover is divided as equally as possible[22] among the remaining agents. Then, *consistency* states that the agents' allotments for this reduced economy be equal to their allotments for the original economy (see also Section 2.2, page 26). For exchange economies, this notion of consistency can be found in Thomson (1992).

In the sequel, we consider the weaker notion of *bilateral consistency* which applies to situations where all but two agents leave with their allotments. Before we give the formal definition, we introduce some notation.

Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and $x = \varphi(e) \in \mathbb{R}^N_+$ be the reallocation assigned by the rule. Let $\{i, j\} \subsetneq N$, $i \neq j$, and suppose that all agents in $N \backslash \{i, j\}$ leave with the allotments assigned to them at x. By doing so they create a leftover $\sum_{k \in N \backslash \{i,j\}} (\omega_k - x_k)$. Dividing this leftover as equally as possible among i and j yields the *adjusted endowment vector* $\omega(i, j) \in \mathbb{R}^{\{i,j\}}_+$ defined as follows.

If $\Delta\varphi_i(e) \leq \Delta\varphi_j(e)$, then

$$\begin{aligned} \omega(i,j)_j &= \max\{0, \omega_j + \frac{1}{2}(\Delta\varphi_i(e) + \Delta\varphi_j(e))\} \geq 0 \text{ and} \\ \omega(i,j)_i &= \omega_i + (\Delta\varphi_i(e) + \Delta\varphi_j(e)) - (\omega(i,j)_j - \omega_j) \geq 0. \end{aligned}$$

If $\Delta\varphi_i(e) \geq \Delta\varphi_j(e)$, then

$$\omega(i,j)_i = \max\{0, \omega_i + \frac{1}{2}(\Delta\varphi_i(e) + \Delta\varphi_j(e))\} \geq 0 \text{ and}$$

[22] As equal as possible with respect to the domain restrictions.

$$\omega(i,j)_j \;=\; \omega_j + (\Delta\varphi_i(e) + \Delta\varphi_j(e)) - (\omega(i,j)_i - \omega_i) \geq 0.$$

So, endowment adjustments are as close as possible to the mean allotment changes of i and j.[23] We denote the *reduced economy* that is created in this way by $(R_{\{i,j\}}, \omega(i,j)) \in \mathcal{E}^{\{i,j\}}$.

Bilateral consistency: For all $N \in \mathcal{N}$, all $e = (R,\omega) \in \mathcal{E}^N$, and all $i,j \in N$, $i \neq j$, $\varphi(e)_{\{i,j\}} = \varphi_i(R_{\{i,j\}}, \omega(i,j))$.

Bilateral consistency for reallocation rules as introduced above can be seen as an extension of the bilateral consistency condition for allocation rules (see also Section 2.2, page 27). Note that by the definition of the reduced problem, an equity condition is embedded in our notion of *bilateral consistency*.

Next, we show that the uniform rule U^r satisfies *bilateral consistency*.[24]

Lemma 4.20 *The uniform rule U^r satisfies bilateral consistency..*

Proof. Let $N \in \mathcal{N}$, $e = (R,\omega) \in \mathcal{E}^N$, and $i,j \in N$ be such that $i \neq j$. Suppose, without loss of generality, that $\Delta U_j^r(e) \geq \Delta U_i^r(e)$. In order to prove *bilateral consistency*, we have to show that

$$U^r(e)_{\{i,j\}} = U_i^r(R_{\{i,j\}}, \omega(i,j)) \tag{4.17}$$

where $\omega(i,j)_j = \max\{0, \omega_j + \frac{1}{2}(\Delta U_i^r(e) + \Delta U_j^r(e))\}$ and $\omega(i,j)_i = \omega_i + (\Delta U_i^r(e) + \Delta U_j^r(e)) - (\omega(i,j)_j - \omega_j)$.

Case 1: $z(e) \leq 0$.

Case 1.1: $\Delta U_j^r(e) \geq \Delta U_i^r(e) \geq 0$. Hence, $i,j \notin S(e)$ and $U^r(e)_{\{i,j\}} = p(R_{\{i,j\}})$. Since $z(R_{\{i,j\}}, \omega(i,j)) = 0$, by *Pareto optimality* of U^r, it follows that $U^r(R_{\{i,j\}}, \omega(i,j)) = p(R_{\{i,j\}})$. This implies (4.17).

Case 1.2: $\Delta U_j^r(e) \geq 0 > \Delta U_i^r(e)$. Hence, $i \in S(e)$, $j \notin S(e)$, and $U^r(e)_{\{i,j\}} = (\omega_i - p(R_j) + \omega_j, p(R_j))$. Since $z(R_{\{i,j\}}, \omega(i,j)) \leq 0$, $i \in S(R_{\{i,j\}}, \omega(i,j))$, and $j \notin S(R_{\{i,j\}}, \omega(i,j))$ it follows that $U^r(R_{\{i,j\}}, \omega(i,j)) = (\omega_i - p(R_j) + \omega_j, p(R_j))$. This implies (4.17).

Case 1.3: $0 > \Delta U_j^r(e) \geq \Delta U_i^r(e)$. Hence, $i,j \in S(e)$ and $U^r(e)_{\{i,j\}} = (\max\{p(R_i), \omega_i + \lambda\}, \max\{p(R_j), \omega_j + \lambda\})$ where $\lambda \in \mathbb{R}$ solves $\sum_N U_k^r(e) = \sum_N \omega_k$. We distinguish two subcases.

Case 1.3.1: $U_j^r(e) = p(R_j)$. Hence, $U^r(e)_{\{i,j\}} = (\max\{p(R_i), \omega_i + \lambda\}, p(R_j))$ where $\lambda \in \mathbb{R}$ solves $\sum_N U_k^r(e) = \sum_N \omega_k$. Since $z(R_{\{i,j\}}, \omega(i,j)) \leq 0$ and $\Delta U_j^r(e) \geq \Delta U_i^r(e)$, it follows that $j \notin S(R_{\{i,j\}}, \omega(i,j))$. Hence, $U_j^r(R_{\{i,j\}}, \omega(i,j)) = p(R_j)$ and, by feasibility, $U_i^r(R_{\{i,j\}}, \omega(i,j)) = \max\{p(R_i), \omega_i + \lambda\}$ where $\lambda \in \mathbb{R}$ solves $\sum_N U_k^r(e) = \sum_N \omega_k$. This implies (4.17).

[23] By just applying mean allotment changes, negative endowments, which are not admissible in this model, might occur.

[24] The uniform rule U^r also satisfies *consistency* which we have not formalized here.

Case 1.3.2: $U_j^r(e) > p(R_j)$. Hence, $U^r(e)_{\{i,j\}} = (\omega_i+\lambda, \omega_j+\lambda)$. Since $z(R_{\{i,j\}}, \omega(i,j)) \leq 0$ and $\omega(i,j) = (\omega_i + \lambda, \omega_j + \lambda) > p(R_{\{i,j\}})$, it follows that $U^r(R_{\{i,j\}}, \omega(i,j)) = \omega(i,j)$. This implies (4.17).

Case 2: $z(e) > 0$. Similar to Case 1. ■

The next result we prove is a characterization of the uniform rule U^r by *Pareto optimality, bilateral consistency,* and *boundedness by endowments and peaks.*

Theorem 4.8 *The uniform rule U^r is the only rule that satisfies Pareto optimality, bilateral consistency, and boundedness by endowments and peaks.*

Proof. By *same-sidedness* and Lemma 4.20 it follows that the uniform rule U^r satisfies *Pareto optimality* and *bilateral consistency*. Furthermore, the uniform rule U^r satisfies *boundedness by endowments and peaks.*

To prove the remaining part of the theorem let φ be a rule that satisfies the properties listed in the theorem. Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$. If $|N| = 2$, then, by *Pareto optimality* and *boundedness by endowments and peaks*, it follows easily that $\varphi(e) = U^r(e)$. Now, let $|N| \geq 3$ and suppose, by contradiction, that $\varphi(e) \neq U^r(e)$. Since φ is *Pareto optimal* it follows that either $z(e) > 0$ or $z(e) < 0$.

Case 1: $z(e) < 0$. By *boundedness by endowments and peaks* and *same-sidedness*, it follows that for all $i \notin S(e)$, $\varphi_i(e) = p(R_i) = U_i^r(e)$. Hence, by Lemma 4.1, there exist agents $j, k \in S(e)$ such that $\varphi_j(e) > p(R_j)$ and $\Delta\varphi_j(e) > \Delta\varphi_k(e)$. Because of *boundedness by endowments and peaks*, $\Delta\varphi_j(e) \leq 0$. Hence, $\omega(j,k)_j < \varphi_j(e) \leq \omega_j$. By *bilateral consistency* it follows for the reduced economy $(R_{\{j,k\}}, \omega(j,k))$ that $\varphi_j(R_{\{j,k\}}, \omega(j,k)) = \varphi_j(e)$. Hence, in contradiction to *boundedness by endowments and peaks*, $\varphi_j(R_{\{j,k\}}, \omega(j,k)) > p(R_j)$ and $\varphi_j(R_{\{j,k\}}, \omega(j,k)) > \omega(j,k)$.

Case 2: $z(e) > 0$. Similar to Case 1. ■

Since *Pareto optimality, one-sided endowment-monotonicity*, and the *dummy property* imply *boundedness by endowments and peaks* (see Lemma 4.18), Theorem 4.8 implies the following characterization of the uniform rule U^r.

Corollary 4.10 *The uniform rule U^r is the only rule that satisfies Pareto optimality, one-sided endowment-monotonicity, bilateral consistency, and the dummy property.*

The following theorem shows that we can replace *boundedness by endowments and peaks* in Theorem 4.8 by the weaker property of *individual rationality*. This characterization can be seen as an extension of Dagan's characterization of the uniform allocation rule by *Pareto optimality, bilateral consistency,* and *individual rationality from equal division* (see Theorem 2.12).

Theorem 4.9 *The uniform rule U^r is the only rule that satisfies Pareto optimality, bilateral consistency, and individual rationality.*

Proof. The uniform rule U^r satisfies *Pareto optimality* and *bilateral consistency*. Furthermore, the uniform rule U^r satisfies *individual rationality*.

To prove the remaining part of the theorem let φ be a rule that satisfies the properties listed in the theorem. By Theorem 4.8 it is sufficient to prove that φ satisfies *boundedness by endowments and peaks*.

Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$.

Case 1: $z(e) \leq 0$. By *same-sidedness* and *individual rationality* it follows that for all $i \notin D(e)$, $\omega_i \geq \varphi_i(e) \geq p(R_i)$. For all $i \in D(e)$ it follows by *same-sidedness* that $\varphi_i(e) \geq p(R_i)$. Suppose, in contradiction to *boundedness by endowments and peaks*, that there exist $j \in D(e)$ such that $\varphi_j(e) > p(R_j)$. Hence, $|N| \geq 2$ and, by feasibility and *individual rationality* there exists $k \in S(e)$ such that $\Delta\varphi_k(e) < 0$.

Case 1.1: $|N| \geq 3$. Hence, it follows that $\omega(j,k)_j < \varphi_j(e)$. By *bilateral consistency* it follows for the reduced economy $(R_{\{j,k\}}, \omega(j,k))$ that $\varphi_j(R_{\{j,k\}}, \omega(j,k)) = \varphi_j(e)$. So, by *individual rationality*,

$$\omega(i,j)_j \leq p(R_j) < \varphi_j(R_{\{j,k\}}, \omega(j,k)). \tag{4.18}$$

Next, we enlarge the economy $(R_{\{j,k\}}, \omega(j,k))$ by a non-trader $l \neq j,k$. Let $M = \{j,k,l\}$ and consider the endowment vector $\bar{\omega} \in \mathbb{R}^M_+$ such that $\bar{\omega}_{\{j,k\}} = \omega(j,k)$ and $\bar{\omega}_l = \omega(j,k)_j$. Let $\bar{R} \in \mathcal{R}^M$ be such that $\bar{R}_{\{j,k\}} = R_{\{j,k\}}$ and $p(\bar{R}_l) = \bar{\omega}_l$. Let $\bar{e} = (\bar{R}, \bar{\omega}) \in \mathcal{E}^M$. By *individual rationality*, $\varphi_l(\bar{e}) = p(\bar{R}_l) = \bar{\omega}_l$. Thus, $\bar{\omega}(j,k) = \omega(j,k)$. Since $\bar{R}_{\{j,k\}} = R_{\{j,k\}}$ and $\bar{\omega}(j,k) = \omega(j,k)$, by *bilateral consistency*, it follows that $\varphi(\bar{e})_{\{j,k\}} = \varphi(R_{\{j,k\}}, \omega(j,k)) = \varphi(e)$. Next, by *bilateral consistency*, it follows that $\varphi(\bar{e})_{\{j,l\}} = \varphi(\bar{\omega}(j,l), \bar{R}_{\{j,l\}})$. Since $\Delta\varphi_l(\bar{e}) = 0$, $\bar{\omega}_j = \bar{\omega}_l = \omega(j,k)_j$, $\varphi_j(\bar{e}) = \varphi_j(R_{\{j,k\}}, \omega(j,k)) = \varphi_j(e)$, and (4.18),

$$\begin{aligned}
\bar{\omega}(j,l)_j &= \bar{\omega}_j + \frac{1}{2}\left(\Delta\varphi_j(\bar{e}) + \Delta\varphi_l(\bar{e})\right) \\
&= \bar{\omega}_j + \frac{1}{2}\Delta\varphi_j(\bar{e}) \\
&= \bar{\omega}_j + \frac{1}{2}\left(\varphi_j(\bar{e}) - \bar{\omega}_j\right) \\
&= \frac{1}{2}\omega(j,k)_j + \frac{1}{2}\varphi_j(e) < \varphi_j(e).
\end{aligned}$$

By *individual rationality*, $\bar{\omega}(i,l)_j \leq p(R_j) < \varphi_i(e)$. Let $l = l^1$ and $e^1 = (\bar{R}_{\{j,l^1\}}, \bar{\omega}(i,l^1))$. By adding non-traders in this manner we obtain a sequence of economies $\{e^t(j,l^t)\}_{t\in\mathbb{N}}$ such that for all $t \in \mathbb{N}$,

$$\begin{aligned}
\omega^t(j,l^t)_i &\leq p(R_j) < \varphi_j(e) \text{ and} \\
\omega^t(j,l^t)_i &= \left(\frac{1}{2}\right)^t \omega(j,k)_j + \sum_{q=1}^{t}\left(\frac{1}{2}\right)^q \varphi_j(e) \\
&= \left(\frac{1}{2}\right)^t \omega(j,k)_j + \left(1 - \left(\frac{1}{2}\right)^t\right)\varphi_j(e).
\end{aligned}$$

This obviously yields a contradiction for $t \in \mathbb{N}$ large enough. Hence, for all $N \in \mathcal{N}$ such that $|N| \geq 3$ and all $e = (R, \omega) \in \mathcal{E}^N$ it follows that $\varphi(e) = U^r(e)$.

Case 1.2: $|N| = 2$. Hence, $N = \{j, k\}$. We enlarge the economy (R, ω) by a non-trader $l \neq j, k$. Let $M = \{j, k, l\}$ such that $l \notin \{j, k\}$ and consider the endowment vector $\bar{\omega} \in \mathbb{R}_+^M$ such that $\bar{\omega}_{\{j,k\}} = \omega$ and $\bar{\omega}_l = 0$. Let $\bar{R} \in \mathcal{R}^M$ be such that $\bar{R}_{\{j,k\}} = R$ and $\bar{R}_l$ such that $p(\bar{R}_l) = 0$. Let $\bar{e} = (\bar{R}, \bar{\omega}) \in \mathcal{E}^M$. By Case 1, $\varphi(\bar{e}) = U^r(\bar{e})$. Note that $e = (\bar{R}_{\{j,k\}}, \bar{\omega}(j, k))$ and $U^r(\bar{e})_{\{j,k\}} = U^r(e)$. Then, by *bilateral consistency*, it follows that $\varphi(\bar{e})_{\{j,k\}} = \varphi(e)$. Hence, $p(R_j) < \varphi_j(e) = \varphi_j(\bar{e}) = U_j^r(\bar{e}) = U_j^r(e) = p(R_j)$. This is a contradiction.

Case 2: $z(e) > 0$. Similar to Case 1. ■

Since *individual rationality* and *strong one-sided endowment-monotonicity* imply *Pareto optimality* (see Lemma 4.19), Theorem 4.9 implies the following characterization of the uniform reallocation rule. This characterization can be seen as an extension of Sönmez's characterization of the uniform allocation rule by *individual rationality from equal division, one-sided resource-monotonicity*, and *consistency* (see Theorem 2.14).

Corollary 4.11 *The uniform rule U^r is the only rule that satisfies individual rationality, strong one-sided endowment-monotonicity, and bilateral consistency.*

Proof. The uniform rule U^r satisfies *individual rationality, strong one-sided endowment-monotonicity*, and *bilateral consistency*. To prove the remaining part of the corollary let φ be a rule that satisfies the properties listed in the corollary. By Lemma 4.19, φ is *Pareto optimal*. Then, by Theorem 4.9, $\varphi = U^r$. ■

Remark 4.4 Theorem 4.9 and Corollary 4.11 remain true if we restrict the set of potential agents to a subset of $\mathbb{N}$ that contains at least three agents. For a potential set of agents that contains only two agents the theorem and the corollary do not hold, because in that situation bilateral consistency has no bite. □

The following lemma is an extension of a result of Dagan (1996, Lemma 2) for allocation rules.

Lemma 4.21 *Let the rule φ satisfy Pareto optimality, weak no-envy, and bilateral consistency. Then, on the domain of two-agent economies, φ satisfies demand-onliness.*

The proof of Lemma 4.21 is similar to the proof of Dagan's result (1996, Lemma 2).

Proof. Let φ be a rule that satisfies *Pareto optimality, weak no-envy*, and *bilateral consistency*. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and $\bar{e} = (\bar{R}, \bar{\omega}) \in \mathcal{E}^N$ be such that for all $i \in N$, $d_i(e) = d_i(\bar{e})$. In order to prove *demand-onliness* we have to show that for all $i \in N$, $\triangle\varphi_i(e) = \triangle\varphi_i(\bar{e})$.

Case 1: $z(e) \leq 0$. Assume, without loss of generality, that $N = \{1, 2\}$. Let $N' = \{3, 4\}$ and define $e' = (R', \omega') \in \mathcal{E}^{N'}$ such that for all $i \in N$, $\omega'_{i+n} = \bar{\omega}_i$ and

$p(R'_{i+n}) = p(\bar{R}_i)$. Clearly, $z(e') < 0$. Consider the merged economy $\tilde{e} = (\tilde{R}, \tilde{\omega}) = ((R, R'), (\omega, \omega')) \in \mathcal{E}^{N \cup N'}$. *Weak no-envy* and *same-sidedness* imply for all $i \in N$, $\Delta\varphi_i(\tilde{e}) = \Delta\varphi_{i+2}(\tilde{e})$.

Suppose all $i \in N'$ leave the economy $\tilde{e}$. Then, $\tilde{\omega}(1,2) = \omega$. Hence, by *bilateral consistency*, for all $i \in N$, $\Delta\varphi_i(e) = \Delta\varphi_i(\tilde{e}) = \Delta\varphi_{i+2}(\tilde{e})$.

Suppose all $i \in N$ leave the economy $\tilde{e}$. Then, $\tilde{\omega}(3,4) = \omega'$. Hence, by *bilateral consistency*, for all $i \in N'$, $\Delta\varphi_i(e') = \Delta\varphi_i(\tilde{e}) = \Delta\varphi_{i-2}(\tilde{e})$.

Summarizing, we have for all $i \in N$, $\Delta\varphi_i(e) = \Delta\varphi_{i+2}(e')$.

By applying the same argument to the economies $\bar{e}$ and e' instead of e and e' it follows that for all $i \in N$, $\Delta\varphi_i(\bar{e}) = \Delta\varphi_{i+2}(e')$. Hence, for all $i \in N$, $\Delta\varphi_i(e) = \Delta\varphi_{i+2}(e') = \Delta\varphi_i(\bar{e})$.

Case 2: $z(e) > 0$. Similar to Case 1. ■

Remark 4.5 Lemma 4.21 holds if we restrict the set of potential agents to a subset of $\mathbb{N}$ that contains at least four agents. □

The next characterization in this section is obtained by extending a result of Dagan (1996, Lemma 2) to reallocation rules and applying *bilateral consistency* and Corollary 4.8. This characterization can be seen as an extension of Dagan's characterization of the uniform allocation rule by *Pareto optimality, bilateral consistency,* and *no-envy* (see Theorem 2.11).

Theorem 4.10 *The uniform rule U^r is the only rule that satisfies Pareto optimality, weak no-envy, and bilateral consistency.*

Proof. The uniform rule U^r satisfies *Pareto optimality, weak no-envy,* and *bilateral consistency.*

To prove the remaining part of the theorem let φ be a rule that satisfies the properties listed in the theorem. Then, by Lemma 4.21 and bilateral consistency it follows that φ satisfies *demand-onliness*. Hence, by Corollary 4.8, $\varphi = U^r$. ■

Remark 4.6 Lemma 4.21 and Theorem 4.10 remain true if we restrict the set of potential agents to a subset of $\mathbb{N}$ that contains at least four agents. For a potential set of agents that contains only three agents it is not clear whether Lemma 4.21 holds. If the potential set of agents only contains two agents, then the theorem does not hold, because in that situation bilateral consistency has no bite. □

Remark 4.7 In Thomson (1994b) and Dagan (1996) results similar to those described in Theorem 4.9 and Theorem 4.10 are stated for allocation rules. Thomson includes a continuity condition in his characterizations (1994b, Theorem 1, Theorem 2), besides *bilateral consistency, individual rationality from equal division* or *no-envy* respectively. Dagan proves that the results of Thomson remain true without continuity for allocation problems with at least four agents (1996, Theorem 2, Theorem 3). Now, "translating"

the steps of the proofs[25] in Dagan's characterizations into the reallocation setting is almost sufficient to get alternative proofs of Theorem 4.9 and Theorem 4.10. The argument of *converse consistency* (see Dagan 1996, Lemma 4), which completes the proofs of the characterizations, however, has no equivalent in the reallocation setting. By assuming that a rule satisfying the characterizing properties does not equal the uniform reallocation rule, and using *bilateral consistency*, a contradiction which completes the alternative proofs is easily derived. □

The last, and hopefully not least, characterization we present here is the following. The uniform rule U^r is the only rule that satisfies *Pareto optimality*, *endowment strategy-proofness*, *bilateral consistency*, and the *dummy property*.

Theorem 4.11 *The uniform rule U^r is the only rule that satisfies Pareto optimality, bilateral consistency, endowment strategy-proofness, and the dummy property.*

Proof. The uniform rule U^r satisfies the properties listed in the theorem.

To prove the remaining part of the theorem let φ be a rule that satisfies the properties listed in the theorem. By *Pareto optimality, endowment strategy-proofness*, and Corollary 4.3, the rule φ is *bounded by endowments and peaks* on the domain of economies that are either balanced or that have excess demand. Hence, by Theorem 4.8, for all $N \in \mathcal{N}$ and all $e = (R, \omega) \in \mathcal{E}^N$ such that $z(e) \geq 0$, $\varphi(e) = U^r(e)$.

Next, we prove that the rule φ is *bounded by endowments and peaks* on the domain of economies that have excess supply. Then, by Theorem 4.8, we are done.

Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ such that $z(e) < 0$. By *same-sidedness*, for all $i \in N$, $\varphi_i(e) \geq p(R_i)$.

Let $i \in S(e)$ and consider $\omega' \in \mathbb{R}^N_+$ such that $\omega'_k = \omega_k$ for all $k \in N \setminus \{i\}$ and $e'_i = \max\{p(R_i), \omega_i - s(e) + d(e)\}$. Let $e' = (R, \omega') \in \mathcal{E}^N$. So, at e' agent i is either a non-trader or a supplier and demand equals supply. Therefore, $\varphi_i(e') = p(R_i)$. Hence, $\triangle\varphi_i(e') \leq 0$. Since ω' is an i-deviation from ω by withholding, by Lemma 4.8 it follows that $\triangle\varphi_i(e) \leq \triangle\varphi_i(e') \leq 0$. Hence, for all $i \in S(e)$, $\omega_i \geq \varphi_i(e) \geq p(R_i)$.

Let $j \in D(e)$ and suppose, in contradiction to *boundedness by endowments and peaks*, that there exist such that $\varphi_j(e) > p(R_j)$. Then, by feasibility, there exists $i \in S(e)$ and, by the previous step of the proof, $\triangle\varphi_i(e) \leq 0$. By *bilateral consistency*, $\varphi_j(R_{\{i,j\}}, \omega(i,j)) = \varphi_j(e)$.

Since $\triangle\varphi_i(e) \leq 0$, $\omega(i,j)_j < \varphi_j(e)$. We already proved that suppliers obtain an allotment between their peak and their endowment. Therefore, j cannot be a supplier and $\omega(i,j)_i \leq p(R_i) < \varphi_i(e)$.

Now, similarly as in the proof of Theorem 4.9, by introducing a sequence of non-traders $\{l^t\}_{t\in\mathbb{N}}$ we obtain a sequence of economies $\{e^t(j, l^t)\}_{t\in\mathbb{N}}$ such that for all $t \in \mathbb{N}$,

$$\begin{aligned} \omega^t(j, l^t)_i &\leq p(R_j) < \varphi_j(e) \text{ and} \\ \omega^t(j, l^t)_i &= \left(\frac{1}{2}\right)^t \omega(j,k)_j + \sum_{q=1}^{t} \left(\frac{1}{2}\right)^q \varphi_j(e) \end{aligned}$$

[25] Dagan (1996), Lemmas 2, 3, 5 and 6 can be proved in their "reallocation version ".

$$= \left(\frac{1}{2}\right)^t \omega(j,k)_j + \left(1 - \left(\frac{1}{2}\right)^t\right) \varphi_j(e).$$

This obviously yields a contradiction for $t \in \mathbb{N}$ large enough. ■

A summary of the characterizations of the uniform reallocation rule U^r by *Pareto optimality* or *individual rationality*, consistency properties, *etc.* can be found in Table 4.3. For a comparison with corresponding characterizations of the uniform allocation rule U we refer to Tables 2.3 and 2.4.

	Th 4.8	Co 4.10	Th 4.9	Co 4.11	Th 4.10	Th 4.11
Pareto optimality	×	×	×		×	×
bilateral consistency	×	×	×	×	×	×
boundedness by endowments and peaks	×					
dummy property		×				×
individual rationality			×	×		
one-sided endowment-monotonicity		×				
strong one-sided endowment-monotonicity				×		
weak no-envy					×	
endowment strategy-proofness						×

Table 4.3: Characterizations of the uniform rule U^r by *Pareto optimality* or *individual rationality*, *bilateral consistency*, *etc.*

4.6 Proofs of independence

First, we discuss the logical independence of the properties in the characterizations given in Section 4.3. (see Table 4.1). We introduce the following reallocation rules.

Example 4.5 The *no-trade rule* φ^0 is defined as follows. For all $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$, $\varphi^0(e) := \omega$. ◇

Example 4.6 The *hierarchical rule* φ^h is defined as follows.[26] In case of excess demand, the rule satiates all suppliers. Then, the demander with the lowest index is "served" as well as possible, *i.e.*, he receives his peak if that is possible and the total supply

[26]The hierarchical rule is often called serial, or lexicographic, dictatorship.

otherwise. If there is something left, the demander with the second lowest number is served, *etc.* Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$. If $z(e) \geq 0$, then for all $i \in N$,

$$\varphi_i^h(e) := \begin{cases} p(R_i) & \text{if } i \notin D(e), \\ \min\{p(R_i), \omega_i + s(e) - \sum\limits_{\substack{D(e) \\ j<i}} \Delta\varphi_j^h(e)\} & \text{otherwise.} \end{cases}$$

If $z(e) \leq 0$, then $\varphi^h(e)$ is defined similarly. ◇

Example 4.7 The following *maximally satiating rule* $\varphi^{\max}$ satiates as many agents as possible. Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$. If e is in excess demand, then the total supply is distributed among the demanders as follows. First minimal demands are satiated uniformly. If there is some supply left, then the next smallest demands are satiated, and so on. If $z(e) \geq 0$ and, without loss of generality, $D(e) = \{1, 2, \ldots, d_k\}$ such that $d_1(e) = \ldots = d_{t_1}(e) < d_{t_1+1}(e) = \ldots = d_{t_2}(e) < \ldots < d_{t_r}(e) = \ldots = d_k(e)$, then for all $i \in N$,

$$\varphi_i^{\max}(e) := \begin{cases} p(R_i) & \text{if } i \notin D(e), \\ \min\{p(R_i), \omega_i + \frac{1}{t_s - t_{s-1}}(s(e) - \sum\limits_{j \leq t_{s-1}} \Delta\varphi_j(e))\} & \text{if } i \in D(e) \\ & \text{and} \\ & t_{s-1} < i \leq t_s. \end{cases}$$

If $z(e) \leq 0$, then $\varphi^{\max}(e)$ is defined similarly with respect to smallest supplies. ◇

Example 4.8 The following *absorbing agent rule* φ^{ab} is similar to the allocation rule φ^a we introduced in Chapter 2. Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$ be such that $z(e) < 0$ and $D(e) = \{j\}$. Then, for this economy we obtain $\varphi^{ab}(e)$ from the uniform reallocation $U^r(e)$ by letting agent j absorb the amount $a(e) := \min\{s(e), r_j(\omega_j) - p(R_j)\}$ and subtracting this amount uniformly from the uniform allocation shares $U_i^r(e)$ of the suppliers $i \in N \backslash \{j\}$. In order to formalize this, let $R' \in \mathcal{R}$ be a j-deviation from R such that $p(R'_j) = p(R_j) + a(e)$. Then, $\varphi^{ab}(e) := U^r(R', U^r(e))$.

For all other economies, we apply the uniform reallocation rule. So, for all remaining $N \in \mathcal{N}$ and $e = (R, \Omega)$, $\varphi^{ab}(e) := U^r(e)$. ◇

Recall that the reallocation rule φ^1 is defined in Example 4.1, page 73.

In Table 4.4 we show which of the properties introduced in Section 4.3 are satisfied by the rules introduced above (marked in the table by a "+") and which are not (marked in the table by a "−"). In the lower part of the table we indicate for each characterization of Section 4.3 which of the properties in the characterization is not satisfied by the rule while all other characterizing properties are satisfied. The entry "\" means that more than one characterizing property is not satisfied and the rule is

Reallocation rules → / Properties and abbreviations ↓	φ^0	φ^h	$\varphi^{\max}$	φ^{ab}	φ^1
Pareto optimality **Po**	−	+	+	+	+
strategy-proofness **sp**	+	+	−	−	+
endowment strategy-proofness **esp**	+	+	−	+	+
no-envy **ne**	+	−	−	+	−
weak no-envy **wne**	+	−	−	+	+
equal treatment of equals **ete**	+	−	+	+	+
anonymity **an**	+	−	+	+	+
demand-onliness **do**	+	+	+	−	−
reversibility **re**	+	+	+	−	−
peak-onliness **po**	+	+	+	−	+
Theorem 4.1	**Po**	**ne**	\	**sp**	**ne**
Corollary 4.5	**Po**	**ete**	**sp**	\	**do**
Corollary 4.6	**Po**	**an**	**sp**	\	**do**
Theorem 4.2	**Po**	**ete**	**esp**	**re**	**re**
Theorem 4.3	**Po**	**ne**	**ne**	**po**	**ne**
Corollary 4.8	**Po**	**wne**	**wne**	**do**	**do**

Table 4.4: Proofs of independence, Section 4.3

not used in the proof of independence.

Next, we discuss the logical independence of the properties in the characterizations given in Section 4.4. (see Table 4.2). In addition to the previous examples we introduce the following reallocation rules.

Example 4.9 The following *rule* $\tilde{\varphi}^{ab}$ is a variation of the absorbing agent rule φ^{ab} (see Example 4.8). Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$. If $z(e) < 0$ and $D(e) = \{j\}$, then $\tilde{\varphi}^{ab}(e) := \varphi^{ab}(e)$.

If e is a k-replica of an economy $\bar{e} \in \mathcal{E}^{\bar{N}}$ such that $z(\bar{e}) < 0$ and $D(\bar{e}) = \{j\}$, then $\tilde{\varphi}^{ab}(e)$ is a k- replica of $\varphi^{ab}(\bar{e})$.

For all other economies, we apply the uniform reallocation rule. So, for all remaining $N \in \mathcal{N}$ and $e = (R, \Omega)$, $\tilde{\varphi}^{ab}(e) := U^r(e)$. ◇

Example 4.10 The rule $\varphi^{+\lambda}$ is defined as follows. Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$. If e is in excess supply, then the social endowment $\Omega := \sum_N \omega_i$ is divided so that all agents have the same supply with respect to their peaks. So, if $z(e) < 0$, then for all $i \in N$, $\varphi_i^{+\lambda}(e) := p(R_i) + \lambda$,where $\lambda \geq 0$ solves $\sum_N \varphi_i^{+\lambda}(e) = \Omega$.

If $z(e) \geq 0$, then $\varphi^{+\lambda}(e) := U^r(e)$. ◇

In Table 4.5 we show which of the properties introduced in Section 4.4 are satisfied by the rules introduced above (marked in the table by a "+") and which are not (marked in the table by a "−"). In the lower part of the table we indicate for each characterization

of Section 4.4 which of the properties in the characterization is not satisfied by the rule while all other characterizing properties are satisfied. The entry "\" means that more than one characterizing property is not satisfied and the rule is not used in the proof of independence. As already mentioned in Section 4.4, we did not succeed in proving the independence of Corollary 4.9.

Reallocation rules → / Properties and abbreviations ↓	φ^0	φ^h	φ^{ab}	$\bar{\varphi}^{ab}$	$\varphi^{+\lambda}$
Pareto optimality **Po**	−	+	+	+	+
no-envy **ne**	+	−	+	+	−
weak no-envy **wne**	+	−	+	+	−
one-sided population-monotonicity **pm**	+	+	−	−	+
one-sided replacement-domination **rd**	+	+	+	−	+
one-sided endowment-monotonicity **em**	+	+	−	−	+
strong one-sided endowment-monotonicity **sem**	−	+	−	−	+
replication-invariance **ri**	+	−	−	+	+
individual rationality **ir**	+	+	+	+	−
Theorem 4.4	**Po**	**wne**	**pm**	**pm**	**wne**
Theorem 4.5	**Po**	\	**ri**	**rd**	**ne**
Theorem 4.6	**Po**	**wne**	**em**	**em**	**wne**
Corollary 4.9	**sem**	**wne**	**sem**	\	\

Table 4.5: Proofs of independence, Section 4.4

Finally, we discuss the logical independence of the properties in the characterizations given in Section 4.5. (see Table 4.3). In addition to the previous examples, we introduce the following reallocation rule.

Example 4.11 The rule φ^s is defined as follows. Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N$. If e is in excess supply, then all agents except the agent(s) with maximal supply are satiated. Feasibility is adjusted on the account of the agent(s) with maximal supply, $S_{\max}(e) = \arg\max_{i \in S(e)} s_i(e)$. So, if $z(e) < 0$, then for all $i \in N$,

$$\varphi_i^s(e) := \begin{cases} p(R_i) & \text{if } i \notin S_{\max}(e) \text{ and} \\ \omega_i + \lambda & \text{if } i \in S_{\max}(e), \end{cases}$$

where $\lambda \geq 0$ solves $\sum_N \varphi_i^s(e) = \sum_N \omega_i$.

For all other economies, we apply the uniform reallocation rule. So, if $z(e) \geq 0$, then $\varphi^s(e) := U^\tau(e)$. ◇

Reallocation rules → / Properties and abbreviations ↓	φ^0	φ^h	$\varphi^{+\lambda}$	φ^{ab}	φ^s
Pareto optimality **Po**	−	+	+	+	+
bilateral consistency **bc**	+	−	+	−	+
boundedness by endowments and peaks **bep**	+	+	−	−	−
dummy property **dp**	+	+	−	+	+
individual rationality **ir**	+	+	−	+	−
one-sided endowment-monotonicity **em**	+	+	+	−	−
strong one-sided endowment-monotonicity **sem**	−	+	+	−	−
demand-onliness **do**	+	+	+	−	+
weak no-envy **wne**	+	−	−	+	−
endowment strategy-proofness **esp**	+	+	+	+	−
Theorem 4.8	**Po**	**bc**	**bep**	**bep**	**bep**
Corollary 4.10	**Po**	**bc**	**dp**	\	**em**
Theorem 4.9	**Po**	**bc**	**ir**	**bc**	**ir**
Corollary 4.11	**sem**	**bc**	**ir**	\	\
Theorem 4.10	**Po**	\	**wne**	**bc**	**wne**
Theorem 4.11	**Po**	**bc**	**dp**	**bc**	**esp**

Table 4.6: Proofs of independence, Section 4.5

In Table 4.6 we show which of the properties introduced in Section 4.5 are satisfied by the rules introduced above (marked in the table by a "+") and which are not (marked in the table by a "−"). In the lower part of the table we indicate for each characterization of Section 4.5 which of the properties in the characterization is not satisfied by the rule while all other characterizing properties are satisfied. The entry "\" means that more than one characterizing property is not satisfied and the rule is not used in the proof of independence.

Chapter 5

Model variations[1]

As in Chapter 4, we study the problem of reallocating the individual endowments of some perfectly divisible commodity among a group of agents with single-peaked preferences. For corresponding literature, a more detailed explanation of the model, and the connection between allocation and reallocation rules and their properties, we refer the reader to Chapter 4. The aim of this chapter is to study some variations of the model introduced in Chapter 4. We indicate whether the results derived in Chapter 4 remain true under the model variations at hand. Furthermore, we show how some of the results we derived in Chapter 4 can be adjusted to these models.

This chapter is organized as follows. In Section 5.1 the model is extended to include arbitrary, possibly negative, individual endowments and reallocations. Furthermore, the agents' preferences are defined on the whole real line with possibly negative peaks. Hence, in this model, we include the possibility of debts. For this model, we characterize the uniform reallocation rule by *Pareto optimality*, *strategy-proofness*, and a *equal treatment of equals* condition (see Theorem 5.1).

In Section 5.2, we still consider preferences that are defined on the whole real line, but restrict peaks, individual endowments, and reallocations to be non-negative. Hence, debts are not allowed any more although agents still have preferences over debts. For this model, we characterize the uniform reallocation rule by *Pareto optimality*, *strategy-proofness*, an *adjusted equal treatment of equals* condition, and a *reversibility* condition (see Theorem 5.2).

Finally, in Section 5.3, we discuss the following three model variations concerning the preference relations, the individual endowments and reallocations, and the potential set of agents: discontinuity of preferences, boundedness of individual endowments and reallocations, and a finite set of potential agents.

For the model variations mentioned above, most results we derived in Chapter 4 remain true. For each model variation, we indicate which results of Chapter 4 remain true and which do not remain true.

[1]This chapter is mainly based on Klaus, Peters, and Storcken (1995a) and Klaus, Peters, and Storcken (1995b).

5.1 Strategy-proof reallocation with debts

In the reallocation model we consider in Chapter 4 the agents' preferences are defined on the non-negative real line and individual endowments and allotments are bounded from below by zero. This causes an asymmetry between excess demand and excess supply economies we already emphasized in Chapter 4 (see Remark 4.2). In the first model variation we consider in this chapter, we assume that the agents' preferences are single-peaked and defined over the whole real line. Furthermore, individual endowments, peaks, and shares may be negative. Hence, in this model, we include the possibility of debts. In order to keep this chapter as self-contained as possible, we first give a complete description of this reallocation model with debts.

There is an infinite population of potential agents, indexed by the natural numbers $\mathbb{N}$. Each agent $i \in \mathbb{N}$ is described in terms of an *individual endowment* $\omega_i \in \mathbb{R}$ of an infinitely divisible commodity, and a continuous and single-peaked preference relation R_i defined over the real numbers $\mathbb{R}$. Single-peakedness of R_i means that there exists a point $p(R_i) \in \mathbb{R}$, called the *peak of agent* i, with the following property: for all $x, y \in \mathbb{R}$ with $x < y \leq p(R_i)$ or $x > y \geq p(R_i)$, we have $y \, P_i \, x$.

By $\mathcal{R}_{\mathbb{R}}$ we denote the class of all continuous, single-peaked preference relations over $\mathbb{R}$. By $\mathcal{N}$ we denote the class of non-empty and finite subsets of $\mathbb{N}$. For $N \in \mathcal{N}$, $\mathcal{R}_{\mathbb{R}}^N$ denotes the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}_{\mathbb{R}}$. For profile $R \in \mathcal{R}_{\mathbb{R}}^N$, $p(R) = (p(R_i))_{i \in N}$ denotes the *vector of peaks*.

Throughout this section, an economy is formalized as follows. Let $N \in \mathcal{N}$ and $\omega = (\omega_i)_{i \in N} \in \mathbb{R}^N$ be a vector of individual endowments that have to be reallocated among the agents in N with profile $R \in \mathcal{R}_{\mathbb{R}}^N$. We call a pair $e = (R, \omega) \in \mathcal{R}_{\mathbb{R}}^N \times \mathbb{R}^N$ an *economy*. Let $\mathcal{E}_1^N = \mathcal{R}_{\mathbb{R}}^N \times \mathbb{R}^N$ be the class of all these economies. A *feasible allocation for* $e = (R, \omega) \in \mathcal{E}_1^N$, or a *reallocation* for short, is a vector $x \in \mathbb{R}^N$ such that $\sum_N x_i = \sum_N \omega_i$. A *reallocation rule* φ, or a *rule* for short, is a function that assigns to every $N \in \mathcal{N}$ and every $e = (R, \omega) \in \mathcal{E}_1^N$ a reallocation, denoted by $\varphi(e)$.

Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}_1^N$. Given $i \in N$, we call $\varphi_i(e)$ the *allotment* of agent i and $\Delta\varphi_i(e) := \varphi_i(e) - \omega_i$ his *allotment change*. Agent i is *satiated at* $\varphi(e)$ if $p(R_i) = \varphi_i(e)$ and *non-satiated at* $\varphi(e)$ if $p(R_i) \neq \varphi_i(e)$.

We call agent $i \in N$ a *demander* if his endowment is strictly less than his peak: he "demands" $p(R_i) - \omega_i$ units of the commodity. For all $i \in N$, we denote this *demand* by $d_i(e)$ and the set of demanders by $D(e)$. We call agent $i \in N$ a *supplier* if his endowment is strictly greater than his peak: he wants to "supply" $\omega_i - p(R_i)$ units of the commodity. For all $i \in N$, we denote this *supply* by $s_i(e)$ and the set of suppliers by $S(e)$. We call agent $i \in N$ a *non-trader* if his endowment is equal to his peak: he favors no trade, *i.e.*, $d_i(e) = s_i(e) = 0$. Let $d(e) := \sum_{i \in D(e)} d_i(e)$ denote *total demand* and $s(e) := \sum_{i \in S(e)} s_i(e)$ *total supply*. Furthermore, let $z(e) := \sum_{i \in N} d_i(e)$ denote *excess demand*. The latter may be positive, zero, or negative. If it is zero, then e is *balanced*. If it is positive, then e is an economy with *excess demand*. If it is negative, then e is an economy with *excess supply*. Note that for all $i \in N$, $d_i(e) = -s_i(e)$ and $z(e) = d(e) - s(e)$.

An example of a reallocation rule that satisfies many desirable properties is the uniform (reallocation) rule we extensively studied in Chapter 4. The definition for the reallocation model with debts is similar to the definition introduced for the original model of Chapter 4 (see Section 4.2, page 64).

Uniform rule U^r: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}_1^N$, and all $j \in N$,

$$U_j^r(e) := \begin{cases} \min\{p(R_j), \omega_j + \lambda\} & \text{if } z(e) \geq 0, \\ \max\{p(R_j), \omega_j + \lambda\} & \text{if } z(e) \leq 0, \end{cases}$$

where $\lambda \in \mathbb{R}$ solves $\sum_N U_i^r(e) = \sum_N \omega_i$.

So, agents are either satiated or receive the same (maximal or minimal) allotment change.

Let φ be a rule. Similar to Section 4.3, we are interested in the conditions *Pareto optimality* (see Section 4.3, page 63) and *strategy-proofness* (see Section 4.3, page 66).

A standard requirement for rules is *Pareto optimality*: a reallocation assigned by the rule cannot be changed in such a way that no agent is worse off and some agent is better off.

Pareto optimality: For all $N \in \mathcal{N}$ and all $e = (R, \omega) \in \mathcal{E}_1^N$, there is no reallocation $x \in \mathbb{R}^N$, such that for all $i \in N$, $x_i \, R_i \, \varphi_i(e)$, and for some $j \in N$, $x_j \, P_j \, \varphi_j(e)$.

Similarly as before it follows easily that a rule is *Pareto optimal* if and only if it is *same-sided*, that is: for all $N \in \mathcal{N}$ and all $e = (R, \omega) \in \mathcal{E}_1^N$, either [for all $i \in N$, $\varphi_i(e) \leq p(R_i)$], or [for all $i \in N$, $\varphi_i(e) \geq p(R_i)$].

Strategy-proofness ensures that no agent ever benefits from misrepresenting his preferences. Before we give the formal definition, we repeat some notation.

Let $N, M \in \mathcal{N}$ be such that $N \subseteq M$. Let $M \backslash N = \{i \in M \mid i \notin N\}$. For $R \in \mathcal{R}_{\mathbb{R}}^M$ the *restriction* $(R_i)_{i \in N} \in \mathcal{R}_{\mathbb{R}}^N$ *of* R *to* N is denoted by R_N. Let $R, \bar{R} \in \mathcal{R}_{\mathbb{R}}^N$, $i \in N$, and $R_{N \backslash \{i\}} = \bar{R}_{N \backslash \{i\}}$. Then, we call $\bar{R}$ an *i-deviation from R*.

Strategy-proofness: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}_1^N$, all $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}_1^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R, then $\varphi_j(e) \, R_j \, \varphi_j(\bar{e})$.

It is easy to check that the uniform rule U^r satisfies *Pareto optimality* and *strategy-proofness* (see Lemma 4.3).

Next, similar to Section 4.3, we prove some implications of *Pareto optimality* and *strategy-proofness*. First we state that *Pareto optimality* and *strategy-proofness* imply *own-peak monotonicity* (see Section 4.3, page 67): an agent's allotment is monotonic with respect to his peak.

Own-peak monotonicity: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}_1^N$, all $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}_1^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R and $p(R_j) \leq p(\bar{R}_j)$, then $\varphi_j(e) \leq \varphi_j(\bar{e})$.

Lemma 5.1 *Let the rule φ satisfy Pareto optimality and strategy-proofness. Then φ satisfies own-peak monotonicity.*

Proof. Similar to the proof of Lemma 4.4. ■

An immediate, but important, consequence of *own-peak monotonicity* of a rule φ is *individual peak-onliness*: if an agent changes his preference without changing his peak, his allotment does not change.

Individual peak-onliness: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}_1^N$, all $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}_1^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R and $p(R_j) = p(\bar{R}_j)$, then $\varphi_j(e) = \varphi_j(\bar{e})$.

Next, we show that *Pareto optimality* and *strategy-proofness* imply *uncompromisingness*, a bounded invariance property (see Section 4.3, page 68): if an agent's peak differs from the allotment assigned by the rule, then his allotment does not change if he unilaterally deviates from his original preference relation in such a way that his new peak and his original peak are at the same side of the allotment.

Uncompromisingness: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}_1^N$, all $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}_1^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R and either $[p(R_j) > \varphi_j(e)$ and $p(\bar{R}_j) \geq \varphi_j(e)]$ or $[p(R_j) < \varphi_j(e)$ and $p(\bar{R}_j) \leq \varphi_j(e)]$, then $\varphi_j(e) = \varphi_j(\bar{e})$.

Lemma 5.2 *Let the rule φ satisfy Pareto optimality and strategy-proofness. Then φ satisfies uncompromisingness.*

Proof. Let φ be a rule that satisfies *Pareto optimality* and *strategy-proofness*. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}_1^N$, $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}_1^N$, and $j \in N$ be such that $\bar{R}$ is a j-deviation from R. Since, by Lemma 5.1, φ satisfies *own-peak monotonicity* and therefore *individual peak-onliness*, we assume, without loss of generality, that R_j and $\bar{R}_j$ are preference relations that are symmetric around the peaks.

Case 1: $p(R_j) < \varphi_j(e)$ and $p(\bar{R}_j) \leq \varphi_j(e)$. By *same-sidedness*, it follows that for all $i \in N$, $p(R_i) \leq \varphi_i(e)$. In order to prove *uncompromisingness* of φ, we have to show that $\varphi_j(e) = \varphi_j(\bar{e})$. Suppose, by contradiction, that $\varphi_j(e) \neq \varphi_j(\bar{e})$.

If $\varphi_j(\bar{e}) > \varphi_j(e)$, then it follows that $p(\bar{R}_j) \leq \varphi_j(e) < \varphi_j(\bar{e})$. Hence, in contradiction to *strategy-proofness*, $\varphi_j(e) \, \bar{P}_j \, \varphi_j(\bar{e})$.

If $\varphi_j(\bar{e}) < \varphi_j(e)$, then by *strategy-proofness*, $\varphi_j(\bar{e}) \leq \varphi_j(e) - 2p(R_j)$. Hence, by *Pareto optimality* this implies $p(\bar{R}_j) \leq \varphi_j(\bar{e}) \leq \varphi_j(e) - 2p(R_j)$. Hence, for all $e' = (R', \omega) \in \mathcal{E}_1^N$ such that R' is a j-deviation from R and $p(R'_j) > \varphi_j(e) - 2p(R_j)$, we have $\varphi_j(e') = \varphi_j(e)$. By repeating this argument, each time we double the range of peaks $p(R'_j)$ with $\varphi_j(e) = \varphi_j(e')$. Finally, $\varphi_j(e) = \varphi_j(\bar{e})$.

Case 2: $p(R_j) > \varphi_j(e)$ and $p(\bar{R}_j) \geq \varphi_j(e)$. Similar to Case 1. ■

The repetition argument in the proof of Lemma 5.2 can be avoided by taking a sufficiently asymmetric preference R_j instead (see proof of Lemma 4.5).

The next property we analyze is *no-envy*: no agent strictly prefers the allotment change of another agent to his own allotment change. In contrast with the *no-envy* property of Chapter 4 (see Section 4.3, page 69), the sum of an agent's individual endowment and another agent's allotment change is always feasible in this reallocation model with debts. Hence, the extra assumption of feasibility in the definition of *weak no-envy* we introduced in Chapter 4 (see Section 4.3, page 70) is automatically satisfied in this reallocation model with debts.

No-envy: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}_1^N$, and all $i, j \in N$, $\varphi_j(e)\, R_j\, (\omega_j + \Delta\varphi_i(e))$.

The uniform rule U^τ satisfies *no-envy* (see Lemma 4.6).

Note that in all results of Chapter 4 *no-envy* or *weak no-envy* can be replaced in this model by *no-envy* as introduced above.

No-envy implies the following *equal treatment of equals condition*: if the individual endowments and preference relations of two agents are equal up to a translation, then each agent should be indifferent between his own allocation and the translated allocation of the other agent. Before we give the formal definition, we introduce the following notation.

Let $N \in \mathcal{N}$ and $i \in N$. Furthermore, let $R_i \in \mathcal{R}_{\mathbb{R}}$ and $\tau \in \mathbb{R}$. Then, the *translation of R_i by τ*, denoted by $R_i + \tau$, is defined as follows. For all $\alpha, \beta \in \mathbb{R}$, $\alpha\, (R_i + \tau)\, \beta$ if $(\alpha - \tau)\, R_i\, (\beta - \tau)$.

Equal treatment of equals: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}_1^N$, all $i, j \in N$, and all $\tau \in \mathbb{R}$, if $R_j = R_i + \tau$ and $\omega_j = \omega_i + \tau$, then $\varphi_j(e)\, I_j\, \varphi_i(e) + \tau$.

If φ is *Pareto optimal*, then *equal treatment of equals* implies the stronger equality condition $\Delta\varphi_i(e) = \Delta\varphi_j(e)$.

Since *equal treatment of equals* introduced above and the *equal treatment of equals* condition of Chapter 4 (see Section 4.3, page 72) are introduced for different reallocation models, they are not directly comparable. However, one can easily adjust the *equal treatment of equals* condition of Chapter 4 to this model. Then, *equal treatment of equals* we introduced above is a weaker condition (less agents are considered as equals).

The main result of this section is the following characterization of the uniform rule U^τ by *Pareto optimality*, *strategy-proofness*, and *equal treatment of equals*.

Theorem 5.1 *The uniform rule U^τ is on the domain of economies $\bigcup_{N \in \mathcal{N}} \mathcal{E}_1^N$ the only rule that satisfies Pareto optimality, strategy-proofness and equal treatment of equals.*

Proof. The uniform rule U^r satisfies *Pareto optimal*, *strategy-proofness*, and *equal treatment of equals*.

To prove the remaining part of the theorem let φ be a rule that satisfies the properties listed in the theorem. By Lemmas 5.1 and 5.2 both φ and U^r satisfy *own-peak monotonicity* and *uncompromisingness*.

Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}_1^N$, and suppose, by contradiction, that $\varphi(e) \neq U^r(e)$. Since φ is *Pareto optimal* it follows that either $z(e) > 0$ or $z(e) < 0$.

Case 1: $z(e) > 0$. We call a demander j a *maximal demander* if $j \in \arg\max\{d_i(e) \mid i \in D(e)\}$. By $d_{\max} := \max\{d_i(e) \mid i \in D(e)\}$ we denote the *maximal demand*.

Let $m \in \arg\max\{d_i(e) \mid i \in D(e)\}$ and denote the *set of maximal demanders* that have, up to a translation, the same preference relation as agent m by $D^r_{\max}(e)$; $D^r_{\max}(e) := \{i \in D(e) \mid \omega_i - \omega_m = \tau_i \text{ and } R_i = R_m + \tau_i\}$.

Case 1.1: All agents are maximal demanders and have, up to a translation, the same preference relation as agent m, *i.e.*, $D^r_{\max}(e) = N$. Hence, $U^r(e) = \omega$ and for all $i \in N$, $d_i(e) = d_{\max}$. Furthermore, for all $i, j \in N$ we have that $R_j = R_i + \tau_{i,j}$ and $\omega_j = \omega_i + \tau_{i,j}$, where $\tau_{i,j} = \tau_j - \tau_i$. Then, by *equal treatment of equals* it follows that $\varphi_j(e)\, I_j\, \varphi_i(e) + \tau_{i,j}$. Hence, by *Pareto optimality*, for all $i, j \in N$, $\varphi_j(e) = \varphi_i(e) + \tau_{i,j}$. By feasibility and *same-sidedness*, this can only hold if for all $i \in N$, $\varphi_i(e) = \omega_i$. Hence, $\varphi(e) = U^r(e)$. This is in contradiction to our assumption that $\varphi(e) \neq U^r(e)$.

Case 1.2: The set of maximal demanders $D^r_{\max}(e)$ does not equal the set of agents, *i.e.*, $D^r_{\max} \subsetneq N$.

Since for all $i, j \in D^r_{\max}(e)$ we have that $R_j = R_i + \tau_{i,j}$ and $\omega_j = \omega_i + \tau_{i,j}$, where $\tau_{i,j} = \tau_j - \tau_i$, by *equal treatment of equals* and *Pareto optimality* it follows that $\varphi_j(e) = \varphi_i(e) + \tau_{i,j}$. Hence, for all $i, j \in D^r_{\max}(e)$, $\triangle\varphi_j(e) = \triangle\varphi_i(e)$. Since $\varphi(e) \neq U^r(e)$, there exists an agent $j \notin D^r_{\max}$ such that $\varphi_j(e) \neq U^r_j(e)$.

Let $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}_1^N$ such that $\bar{R}$ is a j-deviation from R and $\bar{R}_j = R_m + \tau_j$, where $\tau_j = \omega_j - \omega_m$. We consider the following two subcases.

Case 1.2.1: $\varphi_j(e) < U^r_j(e) \leq p(R_j)$. Since $p(R_j) \leq p(\bar{R}_j)$, by *uncompromisingness* of φ it follows that $\varphi_j(\bar{e}) = \varphi_j(e)$. *Own-peak monotonicity* of U^r yields $U^r_j(e) \leq U^r_j(\bar{e})$. Thus, $\varphi_j(\bar{e}) < U^r_j(\bar{e})$.

Case 1.2.2: $U^r_j(e) < \varphi_j(e) \leq p(R_j)$. Similar to Case 1.2.1 (exchange the roles of U^r and φ) it follows that $\varphi_j(\bar{e}) > U^r_j(\bar{e})$.

Thus, for the economy $\bar{e}$ it follows that $\varphi_j(\bar{e}) \neq U^r_j(\bar{e})$. Note that the economy $\bar{e}$ has one maximal demander with a preference relation that is, up to a translation, the same as agent m's preference relation more than the economy e, *i.e.*, $|D^r_{\max}(\bar{e})| = |D^r_{\max}(e)| + 1$. By Case 1.1, $D^r_{\max}(\bar{e}) \subsetneq N$.

Similarly, proceeding from the economy $\bar{e}$, we consider a unilateral change of an agent's preference relation such that the set of maximal demanders with a preference relation that is, up to a translation, the same as agent m's preference relation increases by one and the uniform reallocation for this economy is unequal to the reallocation assigned by φ, *etc.* Since the set of agents is finite, finally we end up in an economy e' with only maximal demanders who have a preference relation that is, up to a translation,

the same as agent m's preference relation and $\varphi(e') \neq U^r(e')$. By Case 1.1, this is a contradiction.

Case 2: $z(e) < 0$. Similar to Case 1. ■

The no-trade rule φ^0 (see Section 4.6, page 104), the hierarchical rule φ^h (see Section 4.6, page 104), and the proportional rule Pr^r (see Section 4.2, page 64) establish the independence of the properties in Theorem 5.1.

Loosely speaking, we can interpret Theorem 5.1 as a version of the characterization of the uniform rule U^r in Theorem 4.1 by *Pareto optimality*, *strategy-proofness*, and *no-envy* where *no-envy* is replaced by *equal treatment of equals*.

The following results of Chapter 4, which all include the property *endowment strategy-proofness*, cannot be extended to the model we discuss in this section.

- Corollary 4.3,
- Corollary 4.4,
- Theorem 4.2,
- Theorem 4.11.

In the proof of Corollary 4.3, we use the fact that an agent with individual endowment zero cannot get a negative allotment change. This argument is not valid any more for the model at hand. Since the proofs of Corollary 4.4 and Theorems 4.2 and 4.11 are based on Corollary 4.3, these proofs do not extend to this model either.

All results of Chapter 4 that were not mentioned yet, remain, possibly with small adjustments of notation and definitions, valid for the model discussed in this Section. In case of *bilateral consistency* (see Section 4.5, page 98) mean leftover changes for the remaining agents are no longer subject to a non-negativity restriction. The same holds for *no-envy*. This does not affect the proofs.

5.2 Strategy-proof reallocations without debts

In Section 5.1 we introduced the reallocation model where the endowments and allotments of the agents may be negative. In this section a variation of this model is considered.

Consider the situation where preferences are still defined on the real line, but where peaks, endowments, and reallocations are restricted to be non-negative. Hence, debts are not allowed any more although agents still have preferences over debts. As an example, in an exchange economy only non-negative endowments, peaks, and reallocations can occur, although agents may have preferences defined on debts.

The model where we allow the agent's peaks to be negative is essentially the same as the model at hand. All results we derive here can be adjusted to this model by transforming negative peaks to zero and adjusting the definitions of the properties we discuss in the sequel (see Klaus, Peters, and Storcken, 1995b, Section 4.1).

Throughout this section, an economy is formalized as follows. Let $N \in \mathcal{N}$ and $\omega = (\omega_i)_{i \in N} \in \mathbb{R}^N_+$ be a vector of individual endowments that have to be reallocated among the agents in N with profile $R \in \mathcal{R}^N_{\mathbb{R}}$ such that $p(R) \in \mathbb{R}^N_+$. We denote this class of preferences by $\mathcal{R}_{\mathbb{R},\mathbb{R}_+}$. We call a pair $e = (R, \omega) \in \mathcal{R}^N_{\mathbb{R},\mathbb{R}_+} \times \mathbb{R}^N_+$ an *economy*. Let $\mathcal{E}^N_2 = \mathcal{R}^N_{\mathbb{R},\mathbb{R}_+} \times \mathbb{R}^N_+$ be the class of all these economies. A *feasible allocation for* $e = (R, \omega) \in \mathcal{E}^N_2$, or a *reallocation* for short, is a vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = \sum_N \omega_i$. A *reallocation rule* φ, or a *rule* for short, is a function that assigns to every $N \in \mathcal{N}$ and every $e = (R, \omega) \in \mathcal{E}^N_2$ a reallocation, denoted by $\varphi(e)$.

For the model at hand, the definitions of *Pareto optimality* and *strategy-proofness* as introduced in Section 5.1 apply by replacing the set of economies $\mathcal{E}^N_1$ by the set of economies $\mathcal{E}^N_2$. Furthermore, Lemmas 5.1 and 5.2 remain valid. Note that by restricting peaks, individual endowments, and reallocations to be non-negative, we again introduce an asymmetry into the reallocation model (see also Chapter 4, Remark 4.2).

With the next *reversibility* property we introduce a notion of symmetry into the model. Suppose an economy is "reversed" in the sense that each agent's peak and individual endowment are interchanged, and each agent's preference relation is reflected in his peak and translated to his individual endowment. Then, *reversibility* requires that in such a reversed economy the allotment changes in one economy equal the reversed allotment changes of the other economy. To formalize this condition we need the following notation.

Let $R_i \in \mathcal{R}_{\mathbb{R},\mathbb{R}_+}$. Then $R^r_i \in \mathcal{R}_{\mathbb{R},\mathbb{R}_+}$ is the *reflection* of R_i (in the peak $p(R_i)$), if for all $\alpha, \beta \in \mathbb{R}$, $\alpha\, R^r_i\, \beta$ if $(2p(R_i) - \alpha)\, R_i\, (2p(R_i) - \beta)$. By $R^r := (R^r_i)_{i \in N}$ we denote the reflection of $R \in \mathcal{R}_{\mathbb{R},\mathbb{R}_+}$.

Let $N \in \mathcal{N}$ and $e = (R, \omega) \in \mathcal{E}^N_2$. Then, the corresponding *reversed economy* with vector of individual endowments $p(R)$ and preference profile $(R^r - (p(R) - \omega))$ is denoted by e^r. So, in the reversed economy (a "*r*eversal" or "*r*eflection" of the original economy) the role of individual endowments and peaks is interchanged whereby each preference relation is reflected in the peak and translated from the peak to the individual endowment. By this reversal, all agents demand (supply) at e as much as they supply (demand) in the reversed economy e^r. With the following condition we link the outcome of an economy to the outcome of the corresponding reversed economy.

Reversibility: For all $N \in \mathcal{N}$, all $e = (R, \omega) \in \mathcal{E}^N_2$ and the corresponding reversed economies e^r, and all $i \in N$, $\Delta\varphi_i(e) = -\Delta\varphi_i(e^r)$.

It is easy to check, that the uniform rule U^r satisfies *reversibility*.

Since, *reversibility* introduced above and the *reversibility* condition of Chapter 4

(see Section 4.3, page 78) are introduced for different reallocation models, they are not comparable. However, one can easily adjust the *reversibility* condition of Chapter 4 to this model. Then, *reversibility* we introduced above is a weaker condition (for each economy only a single economy is considered to be the reversed economy).

The main result of this section is the following characterization of the uniform rule U^r by *Pareto optimality*, *strategy-proofness*, *equal treatment of equals*, and *reversibility*.

Theorem 5.2 *The uniform rule U^r is on the domain of economies $\bigcup_{N\in\mathcal{N}} \mathcal{E}_2^N$ the only rule that satisfies Pareto optimality, strategy-proofness, equal-treatment of equals, and reversibility.*

Proof. The uniform rule U^r satisfies *Pareto optimal*, *strategy-proofness*, *equal treatment of equals*, and *reversibility*.

To prove the remaining part of the theorem let φ be a rule that satisfies the properties listed in the theorem. By Lemmas 5.1 and 5.2 both φ and U^r satisfy *own-peak monotonicity* and *uncompromisingness*.

Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}_2^N$, and suppose, by contradiction, that $\varphi(e) \neq U^r(e)$.

Since φ is *Pareto optimal* it follows that either $z(e) > 0$ or $z(e) < 0$. By *reversibility* it is sufficient to consider the case $z(e) > 0$. For this case the proof is similar to the proof of Theorem 5.1, Case 1. ■

Comparing Theorem 5.2 with Theorem 5.1, we only added *reversibility* to take care of the "asymmetric" excess supply case. This is done, because by applying the proof technique of the excess demand case, the lower bound zero for the peaks might be tight, *i.e.*, possibly we are not able to shift agents towards the maximal supply, because we hit the lower bound. However, by *reversibility*, the excess supply case is "reversed" to the excess demand case, where the lower bound in not tight. Instead of adding the property of *reversibility*, one may also replace *equal-treatment of equals* by *no-envy*; in other words, Theorem 4.1 also holds in this context.

Remark 5.1 By going over the proofs the reader may verify that our results, in particular Theorems 5.1 and 5.2 remain valid if the domain of single-peaked preferences is replaced by the much smaller domain of all single-peaked preferences that are symmetric around their peaks or that linearly depend only on the distance to the peaks. It is interesting to note that there is a trade-off between this domain restriction and *strategy-proofness*: on the one hand the restriction of the preference domain weakens *strategy-proofness* (less preference profiles can be used in unilateral deviations). On the other hand by the domain restriction extra information about rules is implied, *e.g.*, when preferences are symmetrical they are completely described by their peaks. Consequently, on this domain a rule satisfies *peak-onliness* (see Section 4.3, page 79) by definition. □

Remark 5.2 Comparing our characterization of the uniform reallocation rule with Ching's characterization of the uniform allocation rule (see Theorem 2.4), we observe that the *equal-treatment of equals* condition we impose is not logically equivalent

with Ching's *equal treatment of equals*. In addition, we impose a *reversibility* property which guarantees a symmetrical treatment of excess demand and excess supply economies. Now, imposing *translation invariance*, *i.e.*, translating an economy along the non-negative real line has no impact on the allotment changes, would compensate the asymmetry between excess demand and excess supply in this reallocation setting as well. Then, we can translate any economy to an economy where all agents have identical individual endowments. In this case, *Pareto optimality*, *strategy-proofness*, and *equal-treatment of equals* are logically equivalent to the corresponding properties for allocation economies and the uniform allocation equals the uniform reallocation. As a consequence, Ching's result can be straightforwardly applied and by translation invariance, we obtain the uniform reallocation for the initial economy. With a similar argument, Sprumont's characterizations of the uniform allocation rule (see Theorem 2.1) by *Pareto optimality*, *strategy-proofness*, and *anonymity* directly imply corresponding characterizations of the uniform reallocation rule. □

All results of Chapter 4 that were not mentioned yet, remain, possibly with small adjustments of notation and definitions, valid for the model discussed in this section.

5.3 Further model variations of preferences, endowments, and the set of agents

5.3.1 Discontinuous preferences

The first model variation we consider in this section concerns the domain of the preferences. We assumed until now, that the preferences of the agents are single-peaked and continuous. In fact continuity is only needed to prove Lemma 4.8. Whether the results based on this lemma, Corollaries 4.3 and 4.4, and Theorem 4.2, hold true for the whole class of single-peaked preferences is not yet clear. Continuity is also used in the proof of Lemma 4.19. However, this proof can easily be adjusted.

5.3.2 Bounded peaks, individual endowments, and reallocations

In Chapter 4 we introduced the reallocation model where preferences are defined over the non-negative real line and peaks, endowments, and reallocations are non-negative. In the next model variation, we assume that endowments, peaks, and reallocations are not only bounded from below by zero, but that they are also bounded from above.

Of course, properties like *no-envy*, *weak no-envy*, and *bilateral consistency* must be adapted to this model in order to guarantee that the (adjusted) endowments, which are used in these conditions, are well defined. Then, similarly to Example 4.1, we can construct a counterexample that shows that Corollaries 4.1 and 4.2 do not remain valid.

5.3.3 Finite potential set of agents

Up to now, we assumed that the set of potential agents is infinite. This assumption is crucial for the proofs of Theorems 4.4 and 4.5. As already mentioned in Section 4.5, Theorem 4.9 and Corollary 4.11 only remain true for a set of potential agents that contain at least three agents. Lemma 4.21 and Theorem 4.10 only remain true for a set of potential agents that contain at least three agents. All other results remain true because the proofs of these theorems apply to a fixed number of agents.

Appendix A: Allocation of an indivisible good

The allocation model where an indivisible commodity is allocated among a group of agents is similar to the allocation model with single-dipped preferences we introduced in Chapter 3. In order to keep this appendix self-contained, we introduce the whole model and restate the central properties we explore. However, for more details concerning the motivation and the background of the properties we analyze here, we refer to Chapters 2 and 3.

There is a finite population of potential agents, indexed by $\mathbb{P}$. By $\mathcal{P}$ we denote the class of non-empty subsets of $\mathbb{P}$. An indivisible commodity Ω has to be allocated among a group of agents. Each agent $i \in \mathbb{P}$ is equipped with a preference relation R_i defined over the two alternatives "receiving nothing", denoted by 0, and "receiving the indivisible commodity", denoted by Ω. Hence, for each agent i either $0\, R_i\, \Omega$, or $\Omega\, R_i\, 0$. By $\mathcal{R}_{\{0,\Omega\}}$ we denote the set of preference relations over $\{0, \Omega\}$.

For $N \in \mathcal{P}$, $\mathcal{R}^N_{\{0,\Omega\}}$ denotes the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}_{\{0,\Omega\}}$.

Now, an economy is formalized as follows. Let Ω be the indivisible commodity that has to be allocated among a group of agents $N \in \mathcal{P}$ with profile $R \in \mathcal{R}^N_{\{0,\Omega\}}$.[2] We call such a pair $e = (R, \Omega)$ an *economy*. The set of all these economies is denoted by $\mathcal{E}^N_{\{0,\Omega\}}$.

Remark A.1 The allocation model introduced above is more general than the allocation model introduced by Papai (1996). In Papai (1996) the set of agents is fixed, agents preferences are such that they are not indifferent between 0 and Ω, and there is free disposal of the indivisible comodity. □

Let $N \in \mathcal{P}$ and let $X^N = \{\chi^i\}_{i \in N}$ denote the unit vectors in $\mathbb{R}^N$, where $\chi^i_j = 1$ if $j = i$ and $\chi^i_j = 0$ if $j \neq i$. A *feasible allocation for* $e = (R, \Omega)$ is an allocation of the indivisible good Ω to an agent $i \in N$; we denote this allocation by $\Omega\chi^i$. Let $\Omega X^N = \{\Omega\chi^i\}_{i \in N}$ denote the set of feasible allocations for an economy $e = (R, \Omega)$. An *allocation rule* φ, or a *rule* for short, is a function that assigns to every $N \in \mathcal{P}$ and every $e \in \mathcal{E}^N_{\{0\Omega\}}$ a feasible allocation, denoted $\varphi(e)$. Given $i \in N$, we call $\varphi_i(e)$ the *allotment* of agent i. Note that either $\varphi_i(e) = 0$ or $\varphi_i(e) = \Omega$.

[2]Note that free disposal of the commodity is not allowed.

The first property we are interested in is *Pareto optimality*.

Pareto optimality: For all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N_{\{0,\Omega\}}$, there is no feasible allocation $x \in \mathbb{R}^N_+$, such that for all $i \in N$, $x_i \, R_i \, \varphi_i(e)$, and for some $j \in N$, $x_j \, P_j \, \varphi_j(e)$.

Before we introduce the other properties, we give a simple description of *Pareto optimality* (see alos Lemma 3.1). For this description we use the partition $\{N_+(e), N_0(e), N_-(e)\}$ of the agents we introduced in Chapter 3.

Lemma A.1 A rule φ is *Pareto optimal* if and only if for all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N_{\{0,\Omega\}}$, the following holds.

Case 1 : If $N_+(e) \neq \emptyset$, then

$$\varphi(e) \in \{\Omega\chi^i \mid i \in N_+(e)\}.$$

Case 2 : If $N_+(R) = \emptyset$ and $N_0(R) \neq \emptyset$, then

$$\varphi(e) \in \{\Omega\chi^i \mid i \in N_0(e)\}.$$

Case 3 : If $N_-(R) = N$, then

$$\varphi(e) \in \{\Omega\chi^i \mid i \in N\}.$$

In addition to *Pareto optimality*, we are interested in *strategy-proofness* (see also Section 2.2, page 22) and *replacement-domination* (see also Section 2.2, page 24).

Strategy-proofness: For all $N \in \mathcal{P}$, all $e = (R, \Omega) \in \mathcal{E}^N_{\{0,\Omega\}}$, all $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^N_{\{0,\Omega\}}$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R, then $\varphi_j(e) \, R_j \, \varphi_j(\bar{e})$.

Replacement-domination: For all $N \in \mathcal{P}$, all $e = (R, \Omega) \in \mathcal{E}^N_{\{0,\Omega\}}$, all $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^N_{\{0,\Omega\}}$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R, then either [for all $i \in N \backslash \{j\}$, $\varphi_i(e) \, R_i \, \varphi_i(\bar{e})$] or [for all $i \in N \backslash \{j\}$, $\varphi_i(\bar{e}) \, R_i \, \varphi_i(e)$].

Now, the class of all *Pareto optimal*, *strategy-proof*, and *replacement dominant* rules can be described similarly to the rules described in Chapter 3, Theorem 3.1 and Corollary 3.1. For a more detailed description of the selection mechanism, we refer the reader to Chapter 3.

Corollary A.1 A rule φ satisfies *Pareto optimality*, *strategy-proofness*, and *replacement-domination* if and only if for all $N \in \mathcal{P}$ there exist $\pi \in \Pi^N$, $g \in G^N$, and $k \in N$ such that for all $e = (R, \Omega) \in \mathcal{E}^N_{\{0,\Omega\}}$,

$$\varphi(e) = \Omega\chi^{f^N_{\pi,g,k}(e)}.$$

Remark A.2 Restricting our model to the model of Papai (1996), the class of rules described in Corollary A.1 equals the serial dictatorships described in Papai (1996).

Papai (1996) proved that for her model, the set of *Pareto optimal*, *strategy-proof*, and "*non-bossy*" rules equals the set of serial dictatorships. □

The next property we consider is *consistency* (see also Section 2.2, page 26). Consider an economy $e = (R, \Omega) \in \mathcal{E}^N_{\{0,\Omega\}}$ and let $\varphi(e) = x$. Suppose that a group of agents leave with their allotments at x and that the agents in $M \subseteq N$ are left behind. If $\Omega = x_i$ for $i \in M$, then the reduced economy equals (R_M, Ω). If $\Omega = x_i$ for $i \in N \backslash M$, then the reduced economy equals $(R_M, 0)$. With a slightly abuse of notation, we denote the *reduced economies* by $(R_M, \sum_M x_i) \in \mathcal{E}^M$. Let $M, N \in \mathcal{P}$ such that $M \subseteq N$. For $x \in \Omega X^N$ the *restriction* $(x_i)_{i \in M}$ *of* x *to* M is denoted by x_M.

Consistency: For all $M, N \in \mathcal{P}$ with $M \subseteq N$ and all $e = (R, \Omega) \in \mathcal{E}^N$, $\varphi(e)_M = \varphi(R_M, \sum_M \varphi_i(e))$.

Next, the class of all *Pareto optimal*, *strategy-proof*, and *consistent* rules can be described similarly to the rules described in Chapter 3, Theorem 3.2. For a more detailed description of the selection mechanism, we refer the reader to Chapter 3.

Corollary A.2 A rule φ satisfies *Pareto optimality*, *strategy-proofness*, and *consistency* if and only if there exist $\pi_+, \pi_- \in \Pi^{\mathbb{P}}$ and $\mathcal{G} \in \mathcal{C}^{\mathbb{P}}$ such that for all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N_{\{0,\Omega\}}$,

$$\varphi(e) = \Omega \chi^{f^N_{\pi_+, \mathcal{G}, \pi_-}(e)}.$$

Finally, we consider *population-monotonicity* (see also Section 2.2, page 24).

Population-monotonicity: For all $N, M \in \mathcal{P}$, all $e = (R, \Omega) \in \mathcal{E}^N_{\{0,\Omega\}}$, and all $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^N_{\{0,\Omega\}}$, if $M \subseteq N$ and $R_M = \bar{R}$, then either [for all $i \in M$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in M$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

Finally the class of all *Pareto optimal*, *strategy-proof*, and *population-monotonic* rules can be described similarly to the rules described in Chapter 3, Theorem 3.3. For a more detailed description of the selection mechanism, we refer the reader to Chapter 3.

Corollary A.3 A rule φ satisfies *Pareto optimality*, *strategy-proofness*, and *population-monotonicity* if and only if there exist $\pi_+, \pi_- \in \Pi^{\mathbb{P}}$ and $\mathcal{G} = \{g^N \in G^N\}_{N \in \mathcal{P}}$ such that for all $N \in \mathcal{P}$ and all $e = (R, \Omega) \in \mathcal{E}^N_{\{0\Omega\}}$,

$$\varphi(e) = \Omega \chi^{f^N_{\pi_+, \mathcal{G}, \pi_-}(e)}.$$

The rule described in Example 3.2, interpreted as a rule for our model, satisfies all properties we considered until now. The rules described in Examples 3.1, 3.4, and 3.5, adjusted for our model, establish the independence of the properties in Corollaries A.1, A.2, and A.3.

Appendix B: Two alternative proofs

Alternative proof of Theorem 4.1. The uniform rule U^r is *same-sided* and therefore *Pareto optimal*. As shown in *Lemmas 4.3* and *4.6* it satisfies *strategy-proofness* and *no-envy*.

To prove the remaining part of the theorem let φ be a rule that satisfies *Pareto optimality*, *strategy-proofness*, and *no-envy*. By *Lemma 4.4*, both φ and U^r satisfy *individual peak-onliness*.

Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and suppose, by contradiction, that $\varphi(e) \neq U^r(e)$. Since φ is *Pareto optimal* it follows that either $z(e) > 0$ or $z(e) < 0$. Assume, without loss of generality, that $z(e) < 0$. Since $\varphi(e) \neq U^r(e)$, there exists an agent $j \in N$ such that $p(R_j) < \varphi_j(e)$ and $\triangle\varphi_j(e) > \triangle U^r_j(e)$. Let $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$ be such that $\bar{R}$ is a j-deviation from R such that $p(R_j) = p(\bar{R}_j)$ and $0 \, \bar{P}_j \, \varphi_j(e)$.[3] By *individual peak-onliness* it follows that $\varphi_j(\bar{e}) = \varphi_j(e)$ and $U^r_j(\bar{e}) = U^r_j(e)$. Hence, $\triangle\varphi_j(\bar{e}) > \triangle U^r_j(\bar{e})$. Then, by feasibility, there exist $k \in N \backslash \{j\}$ such that $\triangle\varphi_k(\bar{e}) < \triangle U^r_k(\bar{e})$. By *same-sidedness*, this can only be the case if $p(\bar{R}_k) < U^r_k(\bar{e})$ and $\triangle U^r_k(\bar{e}) = \min\{\triangle U^r_i(\bar{e}) \mid i \in N\}$. Thus, $\triangle\varphi_k(\bar{e}) < \triangle U^r_k(\bar{e}) \leq \triangle U^r_j(\bar{e}) < \triangle\varphi_j(\bar{e})$. So, in contradiction to *no-envy*, $(\omega_j + \triangle\varphi_k(\bar{e}))^+ \, \bar{P}_j \, \varphi_j(\bar{e})$. ■

Alternative proof of Theorem 4.5. By *same-sidedness* and Lemmas 4.6 and 4.12 it follows that the uniform rule satisfies *Pareto optimality*, *no-envy*, and *one-sided replacement-domination*. Furthermore, the uniform rule is *replication-invariant*.

To prove the remaining part of the theorem let φ be a rule that satisfies the properties listed in the theorem. Let $N \in \mathcal{N}$, $e = (R, \omega) \in \mathcal{E}^N$, and suppose, by contradiction, that $\varphi(e) \neq U^r(e)$. Since φ is *Pareto optimal* it follows that either $z(e) > 0$ or $z(e) < 0$. Assume, without loss of generality, that $N = \{1, 2, \ldots, n\}$ and $z(e) < 0$. Since $\varphi(e) \neq U^r(e)$, there exists an agent $j \in N$ such that $p(R_j) < \varphi_j(e)$ and $\triangle\varphi_j(e) > \triangle U^r(e)$.

Let $N' = \{n+1, \ldots, 2n\}$ and define $e' = (R', \omega') \in \mathcal{E}^{N \cup N'}$ such that for all $i \in N$, $\omega_i = \omega'_i = \omega'_{n+i}$ and $R_i = R'_i = R'_{i+n}$. Clearly, e' is a 2-replica of e. Hence, by *replication-invariance*, for all $i \in N$, $\varphi_i(e) = \varphi_i(e') = \varphi_{n+i}(e')$. Particularly, it follows that $p(R_j) < \varphi_j(e')$ and $\triangle\varphi_j(e') > \triangle U^r_j(e')$.

Let $\bar{R} \in \mathcal{R}^{N \cup N'}$ be a j-deviation from R' such that $p(R'_j) = p(\bar{R}_j)$ and $0 \, \bar{P}_j \, \varphi_j(e')$. Let $\bar{e} = (\bar{R}, \omega')$ denote this one-sided change of economy e'. Since $p(\bar{R}_j) = p(\bar{R}_{n+j})$ and $\omega'_j = \omega'_{n+j}$, by *same-sidedness* and *no-envy*, it follows that $\varphi_j(\bar{e}) = \varphi_{n+j}(\bar{e})$. Suppose

[3] In the proof of the excess demand case, let $\bar{e} = (\bar{R}, \omega) \in \mathcal{E}^N$ be such that $\bar{R}$ is a j-deviation from R such that $p(R_j) = p(\bar{R}_j)$ and $\sum_N \omega_i \, \bar{P}_j \, \varphi_j(e)$.

that $\varphi_j(\bar{e}) < \varphi_j(e')$. Hence, $\varphi_{n+j}(\bar{e}) < \varphi_{n+j}(e')$ and, by *same-sidedness* and *one-sided replacement-domination*, for all $i \in N\backslash\{j\} \cup N'$, $\varphi_i(\bar{e}) \leq \varphi_i(e')$. Then, by feasibility, it follows that $\sum_{N\cup N'} \omega_i' = \sum_{N\cup N'} \varphi_i(\bar{e}) < \sum_{N\cup N'} \varphi_i(e') = \sum_{N\cup N'} \omega_i'$. This is a contradiction. Similarly, the assumption, $\varphi_j(\bar{e}) > \varphi_j(e')$ yields a contradiction. Hence, $\varphi_j(\bar{e}) = \varphi_j(e')$. Then, by *same-sidedness* and *one-sided replacement-domination*, for all $i \in N\backslash\{j\} \cup N'$, $\varphi_i(\bar{e}) = \varphi_i(e')$. Hence, $\varphi(\bar{e}) = \varphi(e')$. Then, $\Delta\varphi_j(\bar{e}) > \Delta U_j^r(\bar{e})$ and by feasibility, there exist $k \in N\backslash\{j\} \cup N'$ such that $\Delta\varphi_k(\bar{e}) < \Delta U_k^r(\bar{e})$. By *same-sidedness*, this can only be the case if $p(\bar{R}_k) < U_k^r(\bar{e})$ and $\Delta U_k^r(\bar{e}) = \min\{\Delta U_i^r(\bar{e}) \mid i \in N \cup N'\}$. Thus, $\Delta\varphi_k(\bar{e}) < \Delta U_k^r(\bar{e}) \leq \Delta U_j^r(\bar{e}) < \Delta\varphi_j(\bar{e})$. So, in contradiction to *no-envy*, $(\omega_j + \Delta\varphi_k(\bar{e}))^+ \bar{P}_j \, \varphi_j(\bar{e})$. ■

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Summary in Dutch

Eerlijke Verdeling en Herverdeling: Een Axiomatische Studie

In tal van economische en politieke situaties zijn de preferenties van de betrokken agenten privé-informatie. Afhankelijk van het gekozen beslissingsmechanisme, kunnen agenten een reden hebben om hun ware voorkeuren te verbergen, om zodoende een betere uitkomst te bewerkstelligen, mogelijk ten koste van anderen. Om dit soort gedrag te vermijden kunnen beslissingsmechanismen gekozen worden die "strategy-proof" oftewel niet-manipuleerbaar zijn. Dit houdt in dat geen enkele agent ooit een voordeel kan hebben van het opzettelijk foutief weergeven van zijn preferenties, ongeacht de preferenties die geuit worden door de overige agenten.

Het eerste fundamentele resultaat met betrekking tot niet-manipuleerbaarheid van Gibbard (1973) en Satterthwaite (1975) is dat, als alle preferenties toelaatbaar zijn, in essentie alleen dictatoriale mechanismen niet-manipuleerbaar kunnen zijn. Echter, niet-dictatoriale mechanismen kunnen wel degelijk niet-manipuleerbaar zijn als niet alle preferenties worden toegelaten. Een voorbeeld is de beperking tot de klasse van "single-peaked" preferenties: preferenties zijn single-peaked als alle mogelijke alternatieven op een dusdanige manier geordend kunnen worden, dat er voor iedere agent een uniek optimum is, de piek, en dat zijn voorkeur monotoon daalt in beide richtingen van de piek af.

Verdeel Modellen

Gedurende een lange periode bleven alle bijdragen op het gebied van de analyse van niet-manipuleerbare mechanismen beperkt tot publieke goederen. In dit proefschrift beschouwen we een aantal modellen, waarin een oneindig deelbaar privaat goed verdeeld of herverdeeld wordt onder een aantal agenten. Niet-manipuleerbaarheid is daarbij een centraal thema. Maar we zijn ook geïnteresseerd in zaken als eerlijkheid ("fairness") en solidariteit, bijvoorbeeld "no-envy" of populatie-monotoniciteit. No-envy wil zeggen dat agenten nooit het aandeel van andere agenten prefereren boven hun eigen aandeel. Populatie-monotoniciteit houdt in dat indien er een nieuwe agent opduikt, alle andere agenten daar allemaal (zwak) beter, of allemaal (zwak) slechter van worden. In dit proefschrift betitelen we de analyse van mechanismen die voldoen aan eerlijkheid en andere "gewenste" eigenschappen als "fair allocation" oftewel eerlijke verdeling.

Ook de literatuur over niet-manipuleerbaarheid in economieën met private goederen begint met een onmogelijkheidsstelling. Voor ruileconomieën met twee agenten en twee goederen, toont Hurwicz (1972) aan dat er zelfs voor de beperkte klasse van "trans-

lated" Cobb-Douglas preferenties, in essentie geen verdeelmechanismen bestaan die niet-manipuleerbaar en Pareto optimaal zijn. Net als voor de publieke goederen houdt dit resultaat niet stand voor sommige andere klassen van preferenties, zoals de klasse van single-peaked preferenties. Een typisch voorbeeld is het verdelen van twee goederen in een ruileconomie waarin vraag en aanbod niet in evenwicht zijn (zie Benassy, 1982). In deze context van niet-manipuleerbare ruil beschrijft Benassy (1982) wat nu bekend staat als de uniforme regel. Deze regel verdeelt het totaal over de agenten zo "gelijk mogelijk", door de pieken van de agenten als bovengrenzen te nemen in geval van een tekort en als ondergrenzen in geval van een overschot. De uniforme regel kan ook toegepast worden voor het verdelen van een "social endowment" van een oneindig deelbaar goed over een groep van agenten met single-peaked preferenties, bijvoorbeeld de onderwijslast van een vakgroep over zijn docenten.

Het blijkt dat de uniforme regel aan tal van gewenste eigenschappen voldoet, zoals Pareto optimaliteit en niet-manipuleerbaarheid. Bovendien is het de enige regel die voldoet aan deze eigenschappen en no-envy of anonimiteit (zie Sprumont 1991). Tevens voldoet de uniforme regel aan eigenschappen als consistentie, solidariteit, en monotoniciteit (zie bijvoorbeeld Thomson 1997b). Een conclusie van dit onderzoek is dat de uniforme regel nu geaccepteerd is als de belangrijkste regel voor problemen waarbij een eerlijke verdeling gewenst is en waarbij de preferenties single-peaked zijn.

HOOFDSTUK 2: Eerlijke Verdeling en Single-Peakedness

In Hoofdstuk 2, geven we een overzicht van de meest belangrijke resultaten met betrekking tot de uniforme verdeelregel. Ook geven we antwoorden op twee open vragen gesteld door William Thomson in verband met twee karakteriseringen van de uniforme regel (Thomson 1995a, 1997a): door middel van een voorbeeld geven we een positief antwoord op de vraag of "replication-invariance" in deze karakteriseringen onafhankelijk is van de andere eigenschappen.

Na Hoofdstuk 2 vervolgen we met twee varianten het "klassieke" verdelingsmodel met single-peaked preferenties. In de eerste variant beschouwen we het probleem van eerlijke verdeling voor een klasse van preferenties die omschreven kan worden als "omgekeerd single-peaked", namelijk "single-dipped" preferenties. Preferenties zijn single-dipped als alle mogelijke alternatieven op een dusdanige manier geordend kunnen worden, dat er voor iedere agent een uniek minst geprefereerd punt is, de "dip", en dat zijn voorkeur monotoon stijgt in beide richtingen van de dip af. In de tweede variant gaat het om een verdelingsmodel met single-peaked preferenties, waarbij we aannemen dat een sociaal goed herverdeeld moet worden, in plaats van verdeeld.

HOOFDSTUK 3: Eerlijke Verdeling en Single-Dippedness

Beschouw het verdelen van twee soorten werk met negatieve kruiseffecten, bijvoorbeeld onderwijzen en besturen aan een universiteit: combinaties van beide soorten werk zijn mogelijk minder geprefereerd dan werk van alleen één soort. Zo'n situatie kan geformaliseerd worden als een verdeelprobleem met single-dipped preferenties. Een ander voorbeeld is een twee-goederen ruileconomie met vaste prijzen en strikt quasi-convexe nutsfuncties, welke single-dipped preferenties geven op de budgetlijnen.

De aanname dat preferenties single-dipped in plaats van single-peaked zijn, blijkt bij

problemen van eerlijke verdeling een belangrijke rol te spelen. Het blijkt namelijk dat één van de belangrijkste eigenschappen, namelijk Pareto optimaliteit, nooit samengaat met eigenschappen van eerlijkheid, zoals anonimiteit en no-envy. Bovendien blijkt dat het totale goed aan één enkele agent toegewezen moet worden om te voldoen aan de eigenschappen Pareto optimaliteit, niet-manipuleerbaarheid, aangevuld met of consistentie of een solidariteit eigenschap (of populatie-monotoniciteit of "replacement-domination"). Tevens geven we een volledige karakterisering van alle verdeelregels die voldoen aan de bovenstaande combinaties van eigenschappen. Het is opmerkelijk dat de karakteriseringen ook voor verdeelproblemen met een ondeelbaar goed geldig zijn.

HOOFDSTUKKEN 4 en 5: Eerlijke Herverdeling en Single-Peakedness

In economische situaties zoals de eerder genoemde ruileconomie ontstaan individuele bezittingen op een natuurlijke wijze. Daarom is een logische uitbreiding van het model van eerlijke verdeling met single-peaked preferenties, de introductie van individuele bezittingen. Een andere interpretatie van deze uitbreiding wordt gegeven in Barberà, Jackson, en Neme (1997). Zij beginnen vanuit het oorspronkelijke verdelingsmodel met single-peaked preferenties en laten anonimiteit buiten beschouwing door het introduceren van rechten van agenten en rangordes van agenten, bijvoorbeeld in investeringssituaties.

Net als voor het oorspronkelijke model van eerlijke verdeling, zijn we geïnteresseerd in een axiomatische analyse van eerlijke herverdeling en in het bijzonder in een uitbreiding van de uniforme regel naar het herverdelingsmodel: de uniforme herverdeelregel. Net als de uniforme verdeelregel voldoet de uniforme herverdeelregel aan gewenste eigenschappen zoals Pareto optimaliteit, niet-manipuleerbaarheid, en no-envy in termen van de netto veranderingen van de bezittingen. In Hoofdstukken 4 en 5 geven we antwoord op de vraag in hoeverre de resultaten verkregen voor het verdelingsmodel gegeneraliseerd kunnen worden naar het herverdelingsmodel. In Hoofdstuk 4 introduceren en verkennen we het herverdelingsmodel en eigenschappen voor herverdeelregels. Bovendien leiden we enkele karakteriseringen af voor de uniforme herverdeelregel en gaan we in op de overeenkomsten en de verschillen met verwante verdeelresultaten. In Hoofdstuk 5, tenslotte, bekijken we de gevoeligheid van deze resultaten voor variaties van de model parameters. We laten bijvoorbeeld zien dat de meeste resultaten geldig blijven wanneer we toestaan dat agenten schulden hebben.

Curriculum Vitae

Bettina-Elisabeth Klaus werd geboren op 9 februari 1968 te Koblenz (Duitsland). Van 1978 tot 1987 volgde zij het voorbereidend wetenschappelijk onderwijs aan het Staatliche Gymnasium auf dem Asterstein te Koblenz. Na het behalen van het Abitur begon zij in oktober 1987 met de studie wiskunde aan de Rheinisch-Westfälische Technische Hochschule Aachen (Duitsland). Deze studie werd in februari 1994 met de titel Diplom Mathematikerin afgesloten. Bij het schrijven van haar doctoraalscriptie "Kooperative Spieltheorie: Geometrischer und algebraischer Vergleich von Spielklassen und duale Characterisierungen" werd ze begeleid door Prof.dr. Hans Peters en Dr. Jean Derks.

Per 1 november 1993 trad zij in dienst als assistent in opleiding bij de vakgroep Kwantitatieve Economie van de huidige Universiteit Maastricht, waar zij onder de leiding van Prof.dr. H. Peters en Dr. T. Storcken gedurende vier jaar onderzoek verrichtte op het gebied van de sociale keuzetheorie. De meeste resultaten van dit onderzoek zijn weergegeven in dit proefschift.

Sinds 15 januari 1997 is zij als visiting scholar te gast aan het Department of Economics van de University of Rochester, USA. In het najaar 1997 verzorgde ze daar de cursus "Mathematical Economics" voor Ph.D. studenten.