Cooperative games and mechanisms for division problems

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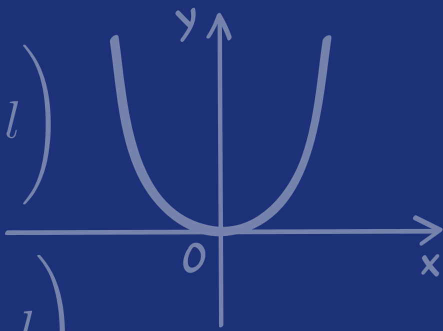

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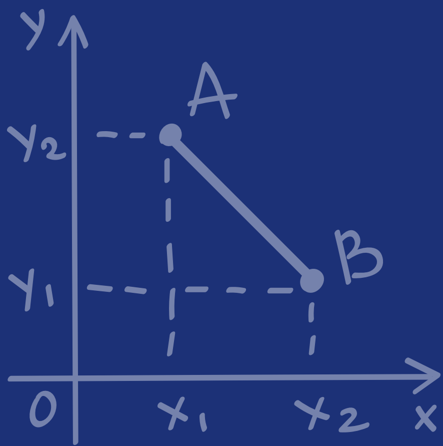

Doctoral thesis

COOPERATIVE GAMES AND MECHANISMS FOR DIVISION PROBLEMS

Doudou Gong

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COOPERATIVE GAMES AND MECHANISMS FOR DIVISION PROBLEMS

Dissertation

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1 Introduction

Game Theory is an extremely valuable subject that studies broad cooperation and conflict among rational and intelligent decision makers, who are also referred to as players or agents. Rationality implies that players choose their strategies so as to maximize their individual payoffs, and intelligence means that players are capable to deduce their best strategies. Since the fundamental book entitled *Theory of Games and Economic Behavior* (von Neumann and Morgenstern, 1944) was published, game theory has become a key ingredient in the areas of Economics, Computer Science, Biology, Engineering, Operations Research and so forth. Generally, game theory is classified into two parts: cooperative game theory and non-cooperative game theory. This thesis has topics from both parts.

In the first two chapters after the Introduction, we focus on cooperative games. In the theory of cooperative games (with transferable utility), players collaborate in coalitions to generate profits. Cooperative game theory analyzes how to allocate profits generated by the grand coalition among the players in a fair way, and provides several significant solution concepts. Among the central solution concepts are the core, the egalitarian core (Arin and Iñarra, 2001), the Shapley value (Shapley, 1953), and the nucleolus (Schmeidler, 1969). These solutions can be studied in a class of cooperative games with the same structure, such as the class of convex games (Shapley, 1971) and the class of balanced games (Bondareva, 1963). On the one hand, the relations of these solutions and equivalent expressions of these solutions for a specific class are explored. On the other hand, inspired by Shapley (1953), these solutions can be characterized using the axiomatic method. Given some desirable properties of solutions, a solution is axiomatically characterized if it uniquely satisfies a combination of independent properties. In this thesis, a new class of cooperative games is introduced, called two-bound core games, where the core is nonempty and can be described by a lower bound and an upper bound on the pre-imputations. We show that the core of each two-bound core game can be described equivalently by the pair of exact core bounds, which are defined by the minimum and maximum individual payoffs within the core. Then, three possible cases are presented to stretch the exact core bounds of a two-bound core game while retaining the core description. We also show that all Davis-Maschler reduced games of two-bound core games with respect to core elements are two-bound core games, and the core of these reduced games can be described by the same pair of bounds. Moreover, new equivalent expressions of the nucleolus and the egalitarian core of two-bound core games in terms of the exact core bounds are provided, and new axiomatic characterizations for the core, the nucleolus and the egalitarian core are given by associated reduced game properties. Interestingly, the egalitarian core for two-bound core games is a single-valued solution.

Next, we focus on non-cooperative games. Non-cooperative game theory analyzes the strategies of players under the condition that binding agreements are not possible. A (weakly) dominant strategy of a player means that playing this strategy is at least as good as any other strategy, regardless of the strategies chosen by the other players. A Nash equilibrium (Nash, 1951) is a profile of strategies where each player's strategy is a best response against the Nash equilibrium strategies of the other players. So, in a Nash equilibrium, each player will not be better off by individually deviating from it. A strong equilibrium (Aumann, 1959) is a strategy profile where no coalition can profitably deviate from it. Note that all dominant strategy equilibria and all strong equilibria are Nash equilibria.

Mechanism design, as a valuable tool to analyze actions among players, can be used to deal with various economic and social issues including division problems. Specifically, a mechanism applied to a division problem induces a non-cooperative game, where individual players act strategically. Therefore, a Nash equilibrium outcome can be regarded as an allocation among players. In this sense, mechanism design can be used to implement allocations for division problems. Conversely, given a mechanism, it can be studied what allocations are obtained in equilibrium for an arbitrary division problem. In this thesis, a few mechanisms are designed to solve two kinds of division problems: bankruptcy problems and division problems with single-dipped preferences.

Based on a sequential partition method, a divide-and-choose mechanism and a divide-and-object mechanism are designed to deal with bankruptcy problems. We show that the unique Nash equilibrium outcome of the corresponding non-cooperative game is the allocation of the constrained equal awards rule. This implies that we give a new non-cooperative interpretation of the constrained equal awards rule for bankruptcy problems.

Then a mechanism, which allocates one unit of an infinitely divisible commodity among agents reporting a number between zero and one, is considered to solve division problems with single-dipped preferences. When the mechanism is anonymous, monotonic, standard and order-preserving, the Pareto optimal Nash and strong equilibria coincide and assign Pareto optimal allocations that are characterized by so-called maximal coalitions: non-involved agents prefer getting zero over an equal coalition share, whereas for agents in the coalition the opposite holds. Pareto optimality means that there is no other allocation which is as good as a desired allocation for everyone and strictly better for at least one player.

Overview

This dissertation is organized as follows.

In Chapter 2, we introduce the new class of *two-bound core games*, where the core can be described by a lower bound and an upper bound on the payoffs of the players. Many classes of games turn out to be two-bound core games. We show that the core of each two-bound core game can be described equivalently by the pair of exact core bounds, and study to what extent the exact core bounds can be stretched while retaining the core description. We provide explicit expressions of the nucleolus for two-bound core games in terms of all pairs of bounds describing the core, using the Talmud rule for bankruptcy problems. We also show that the egalitarian core for two-bound core games is a single-valued solution, and provide an explicit expression in terms of the exact core bounds. Moreover, we study to what extent these expressions are robust against game changes.

In Chapter 3, we study Davis-Maschler reduced games of two-bound core games and show that all these reduced games with respect to core elements are two-bound core games with the same pair of bounds. Based on associated reduced game properties, we axiomatically characterize the core, the nucleolus, and the egalitarian core for two-bound core games.

In Chapter 4, we design two mechanisms for bankruptcy problems based on a sequential partition method. The idea of this method is that claimants gather and partition the estate in a given order. Given the ascending order of claims, we design the *divide-and-choose* mechanism by combining sequential partition with the reversal selection process, and design the *divide-and-object* mechanism by combining sequential partition with the bilateral objection process. For each non-cooperative game induced by our mechanisms for bankruptcy problems, we show that the unique Nash equilibrium outcome is consistent with the allocation of the constrained equal awards rule.

In Chapter 5, a mechanism is designed for division problems with single-dipped preferences, which allocates one unit of an infinitely divisible commodity among agents reporting a number between zero and one. Nash, Pareto optimal Nash, and strong equilibria are analyzed for the games induced by such a mechanism. One of the main results is that when the mechanism is anonymous, monotonic, standard and order-preserving, then the Pareto optimal Nash and strong equilibria coincide and assign Pareto optimal allocations that are characterized by so-called maximal coalitions: non-involved agents prefer getting zero over an equal coalition share, whereas for agents in the coalition the opposite holds.

2
Two-bound core games

Adapted from: Gong, D., B. Dietzenbacher, and H. Peters. *Two-bound core games and the nucleolus*. WorkingPaper 020. Maastricht University, Graduate School of Business and Economics, 2021.

2.1 Introduction

In the theory of cooperative games (with transferable utility), players collaborate in coalitions to generate profits. Cooperative game theory analyzes how to allocate profits generated by the grand coalition among the players in a fair way, and provides several significant solution concepts.

A central solution concept is the core, which consists of all coalitionally stable pre-imputations, that is, no coalition will obtain more by deviating from cooperation in the grand coalition. Bondareva (1963) and Shapley (1967) showed that the core is nonempty if and only if the corresponding cooperative game is balanced. Other important solution concepts are the nucleolus (Schmeidler, 1969), which lexicographically minimizes the excesses of coalitions, and the egalitarian core (Arin and Iñarra, 2001), which assigns all core allocations from which no other core allocation can be obtained by a transfer from a richer to a poorer player. The nucleolus and the egalitarian core select from the core in each balanced game.

Quant et al (2005) studied the class of compromise stable games where the core coincides with the core cover (Tijs and Lipperts, 1982), and provided an explicit expression of the nucleolus for this class using the Talmud rule for bankruptcy problems. The core cover is the set of pre-imputations between a specific pair of bounds. In this chapter, we generalize the approach of Quant et al (2005) to all games where the core equals the set of pre-imputations between an arbitrary pair of bounds, which we call *two-bound core games*.

We show that the core of each two-bound core game can be described equivalently by the pair of exact core bounds (Bondareva and Driessen, 1994), which are defined by the minimum and maximum individual payoffs within the core. Inspired by Quant et al (2005), we provide conditions to check whether a game is a two-bound core game, and describe the extreme points of the core for each such game. All balanced games with at most three players are two-bound core games, but this does not hold for more players.

We study to what extent the exact core bounds of a two-bound core game can be stretched while retaining the core description. It turns out that only three possible cases exist. In the first case, only the lower bounds are decreased for players who obtain their lower exact core bounds when all other players obtain their upper exact core bounds, while keeping all other bounds fixed. In the second case, only the upper bounds are increased for players who obtain their upper exact core bounds when all other players obtain their lower exact core bounds, while keeping all other bounds fixed. In the third case, both the lower bound is decreased and the upper bound is increased for only a single player who obtains the lower exact core bound when all other players obtain their upper exact core bounds and obtains the upper exact core bound when all other players obtain their lower exact core bounds.

In line with Quant et al (2005), we provide an explicit expression of the nucleolus for two-bound core games in terms of the exact core bounds using the Talmud rule. In fact, the nucleolus of these games can be equivalently expressed by each pair of bounds describing the core. Then, we show that the egalitarian core for two-bound core games is single-valued, and we provide an explicit expression of it in terms of the exact core bounds. Finally, we study to what extent these expressions are robust against game changes.

The remainder of this chapter is organized as follows. Section 2.2 introduces preliminary definitions and notation about cooperative games and bankruptcy problems. In Section 2.3, we formally introduce two-bound core games. The nucleolus and the egalitarian core for two-bound core games are studied in Sections 2.4 and 2.5, respectively. Finally, we conclude this chapter with some remarks in Section 2.6.

2.2 Preliminaries

Let N be a nonempty and finite set of *players* and let 2^N be the collection of all subsets of N. An *order* of N is a bijection $\sigma : \{1, \ldots, |N|\} \to N$, where $|N|$ denotes the cardinality of N, and $\sigma(i)$ represents the player at position i. The set of all orders of N is denoted by $\Pi(N)$. Denote by $\mathbb{R}_+$ the set of all non-negative real numbers.

Let $x, y \in \mathbb{R}^N$. We denote $x + y = (x_i + y_i)_{i \in N}$, $x - y = (x_i - y_i)_{i \in N}$, and $\lambda x = (\lambda x_i)_{i \in N}$ for all $\lambda \in \mathbb{R}$. Moreover, $x \geq y$ denotes $x_i \geq y_i$ for all $i \in N$, and $x > y$ denotes $x_i > y_i$ for all $i \in N$. The notations $\leq$ and $<$ are defined analogously. We denote

$$[x, y] = \left\{ z \in \mathbb{R}^N \;\middle|\; x \leq z \leq y \right\}.$$

A *cooperative game with transferable utility* (a *game*, for short) is a pair (N, v), where $v : 2^N \to \mathbb{R}$ is the *characteristic function* with $v(\emptyset) = 0$, representing the *worth* $v(S)$ for each *coalition* $S \subseteq N$ when the players in S cooperate. The set of all games with player set N is denoted by Γ^N. For simplicity, we write $v \in \Gamma^N$ rather than $(N, v) \in \Gamma^N$.

Let $v \in \Gamma^N$. The *pre-imputation set* of v is

$$X(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N) \right\},$$

the *imputation set* of v is

$$I(v) = \{ x \in X(v) \mid \forall i \in N : x_i \geq v(\{i\}) \},$$

and the *core* of v is

$$C(v) = \left\{ x \in X(v) \;\middle|\; \forall S \subseteq N : \sum_{i \in S} x_i \geq v(S) \right\}.$$

Note that $C(v) \subseteq I(v) \subseteq X(v)$, and $C(\lambda v + a) = \lambda C(v) + a$ for all

$\lambda \in \mathbb{R}_+$ and $a \in \mathbb{R}^N$, where $\lambda v + a \in \Gamma^N$ is defined by $(\lambda v + a)(S) = \lambda v(S) + \sum_{i \in S} a_i$ for all $S \subseteq N$.

Bondareva (1963) and Shapley (1967) showed that a game $v \in \Gamma^N$ is *balanced* if and only if $C(v) \neq \emptyset$. The set of all balanced games with player set N is denoted by Γ_b^N. A game $v \in \Gamma^N$ is *convex* (Shapley, 1971) if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$. The set of all convex games with player set N is denoted by Γ_c^N. It is known that $\Gamma_c^N \subseteq \Gamma_b^N \subseteq \Gamma^N$.

A *solution* φ on a domain of games assigns to each game v in this domain a nonempty set $\varphi(v) \subseteq X(v)$. A solution φ on a domain of games is *single-valued* if $|\varphi(v)| = 1$ for each v in this domain. For a single-valued solution φ on a domain of games and a game v in this domain, $\varphi(v)$ is often identified with its unique element.

The *egalitarian core* (Arin and Iñarra, 2001) is the solution EC that assigns to each game $v \in \Gamma^N$ the set

$$EC(v) = \left\{ x \in C(v) \;\middle|\; \forall_{i,j \in N : x_i > x_j} : s_{ij}^x(v) = 0 \right\},$$

where for all $i, j \in N$ with $i \neq j$ and all $x \in \mathbb{R}^N$,

$$s_{ij}^x(v) = \max_{S \in 2^N : i \in S, j \notin S} \left\{ v(S) - \sum_{k \in S} x_k \right\}.$$

The egalitarian core consists of all core elements from which no other core element can be obtained by a transfer from a richer to a poorer player.

The *nucleolus* (Schmeidler, 1969) is the solution η that assigns to each game $v \in \Gamma^N$ with $I(v) \neq \emptyset$ the unique imputation $x \in I(v)$ satisfying $\theta(x) \preccurlyeq \theta(y)$ for all $y \in I(v)$, where $\theta(x) \in \mathbb{R}^{2^{|N|}-2}$ is the vector of excesses $v(S) - \sum_{i \in S} x_i$ for all $S \in 2^N \setminus \{N, \emptyset\}$ arranged in non-increasing order, i.e., $\theta_k(x) \geq \theta_\ell(x)$ for all $1 \leq k < \ell \leq 2^{|N|} - 2$, and $\theta(x) \preccurlyeq \theta(y)$ if there exists $1 \leq t \leq 2^{|N|} - 2$ such that $\theta_t(x) < \theta_t(y)$ and $\theta_k(x) = \theta_k(y)$

for all $1 \leq k < t$, or $\theta(x) = \theta(y)$. It is easy to see that $\eta(v) \in C(v)$ for all $v \in \Gamma_b^N$, and $\eta(\lambda v + a) = \lambda\eta(v) + a$ for all $\lambda \in \mathbb{R}_+$ and $a \in \mathbb{R}^N$.

A *bankruptcy problem* is a triple (N, E, c), where $E \in \mathbb{R}_+$ is the *estate* to be divided and $c \in \mathbb{R}_+^N$ is the vector of *claims* satisfying $\sum_{i \in N} c_i \geq E$. The set of all bankruptcy problems with player set N is denoted by $\mathcal{B}^N$. For simplicity, we write $(E, c) \in \mathcal{B}^N$ rather than $(N, E, c) \in \mathcal{B}^N$.

A bankruptcy rule $f : \mathcal{B}^N \to \mathbb{R}_+^N$ assigns to each bankruptcy problem $(E, c) \in \mathcal{B}^N$ a payoff vector $f(E, c) \in \mathbb{R}_+^N$ such that $\sum_{i \in N} f_i(E, c) = E$ and $f_i(E, c) \leq c_i$ for all $i \in N$. A bankruptcy rule f is *self-dual* (Aumann and Maschler, 1985) if for all $(E, c) \in \mathcal{B}^N$,

$$f(E, c) = c - f\left(\sum_{i \in N} c_i - E, c\right).$$

A bankruptcy rule f is *invariant under claims truncation* if for all $(E, c) \in \mathcal{B}^N$,

$$f(E, c) = f\left(E, (\min\{c_i, E\})_{i \in N}\right).$$

The *Talmud (TAL)* rule assigns to each bankruptcy problem $(E, c) \in \mathcal{B}^N$ and each player $i \in N$,

$$f_i^{TAL}(E, c) = \begin{cases} \min\{c_i/2, \lambda\} \text{ if } \sum\limits_{i \in N} c_i \geq 2E, \\ \max\{c_i/2, c_i - \lambda\} \text{ if } \sum\limits_{i \in N} c_i < 2E, \end{cases}$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} f_i^{TAL}(E, c) = E$.

Aumann and Maschler (1985) showed that the Talmud rule is self-dual and invariant under claims truncation.

The *bankruptcy game* (O'Neill, 1982) $v_{E,c} \in \Gamma^N$ associated to bankruptcy problem $(E, c) \in \mathcal{B}^N$ assigns to each coalition $S \subseteq N$ the residual estate after all other claims have been satisfied, i.e.,

$$v_{E,c}(S) = \max\left\{0, E - \sum_{i \in N \setminus S} c_i\right\}.$$

Curiel et al (1987) showed that bankruptcy games are convex games. Aumann and Maschler (1985) showed that for each bankruptcy problem, the payoff vector assigned by the Talmud rule coincides with the nucleolus of the corresponding bankruptcy game.

2.3 The model

In this section, we introduce two-bound core games, where the core equals the set of pre-imputations between a lower bound and an upper bound. Let $v \in \Gamma^N$. Given $l, u \in \mathbb{R}^N$, the *l,u-efficient set* of v

$$[l, u] \cap X(v)$$

consists of all pre-imputations between *lower bound* l and *upper bound* u, i.e., it is the intersection of the pre-imputation set and the $|N|$-dimensional hypercube restricted by l and u, so it is a convex set. If this set is nonempty, then its extreme points can be described as follows. Similar to Quant et al (2005), we define $m^{l,u,\sigma}(v) \in \mathbb{R}^N$ for all $\sigma \in \Pi(N)$ and all $k \in \{1, \ldots, |N|\}$ by

$$m^{l,u,\sigma}_{\sigma(k)}(v) = \begin{cases} u_{\sigma(k)} \text{ if } \sum_{j=1}^{k} u_{\sigma(j)} + \sum_{j=k+1}^{|N|} l_{\sigma(j)} \leq v(N), \\ l_{\sigma(k)} \text{ if } \sum_{j=1}^{k-1} u_{\sigma(j)} + \sum_{j=k}^{|N|} l_{\sigma(j)} \geq v(N), \\ v(N) - \sum_{j=1}^{k-1} u_{\sigma(j)} - \sum_{j=k+1}^{|N|} l_{\sigma(j)} \text{ otherwise.} \end{cases}$$

Thus, $m^{l,u,\sigma}(v)$ assigns to the first players in σ their upper bound payoffs in such a way that the last players in σ are assigned their lower bound payoffs. The *pivot player* of $m^{l,u,\sigma}(v)$ is the first player in σ who is not assigned the upper bound payoff. If all the players receive their upper bound payoffs, then the last player is the pivot player of $m^{l,u,\sigma}(v)$. These definitions are straightforward generalizations of concepts in Quant et al (2005) to arbitrary lower and upper bounds, which can be used to describe the l,u-efficient set.

Lemma 2.1
Let $v \in \Gamma^N$ and let $l, u \in \mathbb{R}^N$ be such that $[l, u] \cap X(v) \neq \emptyset$. Then

$$[l, u] \cap X(v) = conv \left\{ m^{l,u,\sigma}(v) \;\middle|\; \sigma \in \Pi(N) \right\}.$$

Proof. In view of $m^{l,u,\sigma}(v) \in [l, u] \cap X(v)$ for all $\sigma \in \Pi(N)$, together with the convexity of $[l, u] \cap X(v)$ and $conv\{m^{l,u,\sigma}(v) \mid \sigma \in \Pi(N)\}$, we have

$$conv \left\{ m^{l,u,\sigma}(v) \;\middle|\; \sigma \in \Pi(N) \right\} \subseteq [l, u] \cap X(v).$$

Let $x \in \mathbb{R}^N$ be an arbitrary extreme point of $[l, u] \cap X(v)$, i.e., for each $0 < \lambda < 1$ and all $y, z \in [l, u] \cap X(v)$, $\lambda y + (1 - \lambda)z = x$ implies that $x = y = z$. We claim that there exists at most one player $i \in N$ such that $l_i < x_i < u_i$ and $[x_j = l_j$ or $x_j = u_j$ for all $j \in N \setminus \{i\}]$. Assume, to the contrary, that there exist $i, j \in N$ with $i \neq j$ such that $l_i < x_i < u_i$ and $l_j < x_j < u_j$. Let $0 < \varepsilon < \min\{x_i - l_i, u_i - x_i, x_j - l_j, u_j - x_j\}$, let x' be defined by $x'_i = x_i + \varepsilon$, $x'_j = x_j - \varepsilon$ and $x'_k = x_k$ for all $k \in N \setminus \{i, j\}$, and let x'' be defined by $x''_i = x_i - \varepsilon$, $x''_j = x_j + \varepsilon$ and $x''_k = x_k$ for all $k \in N \setminus \{i, j\}$. Then $x', x'' \in [l, u] \cap X(v)$ and $x = \frac{1}{2}x' + \frac{1}{2}x''$, which contradicts the fact that x is an extreme point of $[l, u] \cap X(v)$.

If $x_i = l_i$ or $x_i = u_i$ for all $i \in N$, then it holds that $x = m^{l,u,\sigma}(v)$ for all $\sigma \in \Pi(N)$ such that $x_{\sigma(k)} = u_{\sigma(k)}$ if and only if $k \leq |\{i \in N \mid x_i = u_i\}|$. If there exists $i \in N$ such that $l_i < x_i < u_i$ and $[x_j = l_j$ or $x_j = u_j$ for all $j \in N \setminus \{i\}]$, then it holds that $x = m^{l,u,\sigma}(v)$ for all $\sigma \in \Pi(N)$ such that $[x_{\sigma(k)} = u_{\sigma(k)}$ if and only if $k \leq |\{j \in N \mid x_j = u_j\}|]$ and $\sigma(|\{j \in N \mid x_j = u_j\}|+1) = i$. Again with the convexity of $[l,u] \cap X(v)$ and $conv\{m^{l,u,\sigma}(v) \mid \sigma \in \Pi(N)\}$, we have $[l,u] \cap X(v) \subseteq conv\{m^{l,u,\sigma}(v) \mid \sigma \in \Pi(N)\}$. $\square$

The l,u-efficient set and the core are both convex subsets of the pre-imputation set. We are interested in l,u-efficient sets that contain the core. Many well-known sets are of this type, such as the imputation set and the core cover (Tijs and Lipperts, 1982).

Example 2.1
Let $v \in \Gamma^N$. Define $l, u \in \mathbb{R}^N$ by

$$l_i = v(\{i\}) \text{ and } u_i = v(N) - \sum_{j \in N \setminus \{i\}} v(\{j\})$$

for all $i \in N$. Then $[l,u] \cap X(v) = I(v)$, so the l,u-efficient set contains the core. $\triangle$

Example 2.2
Let $v \in \Gamma^N$. Define $l, u \in \mathbb{R}^N$ by

$$l_i = v(\{i\}) \text{ and } u_i = v(N) - v(N \setminus \{i\})$$

for all $i \in N$. Then $C(v) \subseteq [l,u] \cap X(v)$, i.e., the l,u-efficient set contains the core. $\triangle$

Example 2.3
Let $v \in \Gamma^N$. Define $l, u \in \mathbb{R}^N$ by

$$l_i = \max_{S \in 2^N : i \in S} \left\{ v(S) - \sum_{j \in S \setminus \{i\}} (v(N) - v(N \setminus \{j\})) \right\} \text{ and}$$

$$u_i = v(N) - v(N \setminus \{i\})$$

for all $i \in N$. Then $[l,u] \cap X(v)$ defines the core cover (Tijs and Lipperts, 1982), which contains the core. Quant et al (2005) defined compromise stable games as games where the core cover coincides with the core. $\triangle$

To check whether a core-containing l,u-efficient set coincides with the core, we only need to verify a specific inequality for each nonempty coalition.

Theorem 2.1
Let $v \in \Gamma_b^N$ and let $l, u \in \mathbb{R}^N$ be such that $C(v) \subseteq [l,u]$. Then $C(v) = [l,u] \cap X(v)$ if and only if for each $S \in 2^N \setminus \{\emptyset\}$,

$$v(S) \leq \max\left\{\sum_{i \in S} l_i, v(N) - \sum_{i \in N \setminus S} u_i\right\}. \tag{2.1}$$

Proof. For the only-if part, assume that $C(v) = [l,u] \cap X(v)$. Then, according to Lemma 2.1, we have $m^{l,u,\sigma}(v) \in C(v)$ for all $\sigma \in \Pi(N)$. Let $S \in 2^N \setminus \{\emptyset\}$ and consider $\sigma^* \in \Pi(N)$ such that $\sigma^*(k) \in N \setminus S$ for all $k \in \{1, \ldots, |N \setminus S|\}$. If the pivot player of $m^{l,u,\sigma^*}(v)$ is an element of $N \setminus S$, then $m_i^{l,u,\sigma^*}(v) = l_i$ for all $i \in S$, so

$$v(S) \leq \sum_{i \in S} m_i^{l,u,\sigma^*}(v) = \sum_{i \in S} l_i.$$

If the pivot player of $m^{l,u,\sigma^*}(v)$ is an element of S, then $m_i^{l,u,\sigma^*}(v) = u_i$ for all $i \in N \setminus S$, so

$$v(S) \leq \sum_{i \in S} m_i^{l,u,\sigma^*}(v) = v(N) - \sum_{i \in N \setminus S} m_i^{l,u,\sigma^*}(v) = v(N) - \sum_{i \in N \setminus S} u_i.$$

Combining these two cases, we obtain expression (2.1).

For the if-part, assume that expression (2.1) holds for all $S \in 2^N \setminus \{\emptyset\}$. We only need to prove that $[l,u] \cap X(v) \subseteq C(v)$. In view of the convexity of the core, together with Lemma 2.1, it suffices to show that $m^{l,u,\sigma}(v) \in C(v)$ for all $\sigma \in \Pi(N)$. For all $S \in 2^N \backslash \{\emptyset\}$ and all $\sigma \in \Pi(N)$,

$$\begin{aligned} v(S) &\leq \max\left\{\sum_{i \in S} l_i, v(N) - \sum_{i \in N \backslash S} u_i\right\} \\ &\leq \max\left\{\sum_{i \in S} m_i^{l,u,\sigma}(v), v(N) - \sum_{i \in N \backslash S} m_i^{l,u,\sigma}(v)\right\} = \sum_{i \in S} m_i^{l,u,\sigma}(v). \end{aligned}$$

Hence, $m^{l,u,\sigma}(v) \in C(v)$ for all $\sigma \in \Pi(N)$. □

Theorem 2.1 generalizes the work of Quant et al (2005), where this result was proven for the specific pair of bounds in Example 2.3. If the l,u-efficient set does not contain the core, then expression (2.1) may hold even when the core does not coincide with the l,u-efficient set. This is shown by the following example.

Example 2.4
Let $N = \{1, 2\}$ and let $v \in \Gamma^N$ be given by $v(\{1\}) = 1$, $v(\{2\}) = 2$ and $v(N) = 4$. Define $l, u \in \mathbb{R}^N$ by $l_1 = u_1 = \frac{3}{2}$ and $l_2 = u_2 = \frac{5}{2}$. It is easy to verify that expression (2.1) holds for each nonempty coalition. However, $C(v) = \{x \in \mathbb{R}^N \mid x_1 + x_2 = 4, x_1 \geq 1, x_2 \geq 2\}$ and $[l, u] \cap X(v) = \{(\frac{3}{2}, \frac{5}{2})\}$. Clearly, $C(v) \neq [l, u] \cap X(v)$. △

We focus on games where the core coincides with some l,u-efficient set. These games are called *two-bound core games*.

Definition 2.1
A game $v \in \Gamma_b^N$ is a *two-bound core game* if there exist $l, u \in \mathbb{R}^N$ such that

$$C(v) = [l, u] \cap X(v).$$

The set of all two-bound core games with player set N is denoted by Γ_t^N. It is worthwhile mentioning that many classical games are two-bound core games. For example, additive games, unanimity games, bankruptcy games (O'Neill, 1982), 1-convex games (Driessen, 1985), big boss games (Muto et al, 1988), clan games (Potters et al, 1989), compromise stable games (Quant et al, 2005) and reasonable stable games (Dietzenbacher, 2018).

It turns out that the core of each two-bound core game can be described by the following specific pair of bounds. Let $v \in \Gamma_b^N$. The *lower exact core bound* is defined by

$$l_i^*(v) = \min_{x \in C(v)} x_i \quad \text{for all } i \in N.$$

The *upper exact core bound* is defined by

$$u_i^*(v) = \max_{x \in C(v)} x_i \quad \text{for all } i \in N.$$

Computational aspects of the lower and upper exact core bounds were studied by Bondareva and Driessen (1994).

Lemma 2.2
A game $v \in \Gamma_b^N$ is a two-bound core game if and only if

$$C(v) = [l^*(v), u^*(v)] \cap X(v).$$

Proof. The if-part follows directly from the definition of two-bound core games. For the only-if part, assume that $C(v) = [l, u] \cap X(v)$ for some $l, u \in \mathbb{R}^N$. Then $l_i \leq l_i^*(v)$ and $u_i \geq u_i^*(v)$ for all $i \in N$, so $[l^*(v), u^*(v)] \subseteq [l, u]$. Together with $C(v) \subseteq [l^*(v), u^*(v)] \cap X(v)$, it follows that $C(v) \subseteq [l^*(v), u^*(v)] \cap X(v) \subseteq [l, u] \cap X(v) = C(v)$. Hence, $C(v) = [l^*(v), u^*(v)] \cap X(v)$. $\square$

All balanced games with at most three players are two-bound core games, but this does not hold for more players.

Proposition 2.1
$\Gamma_t^N = \Gamma_b^N$ *if and only if* $|N| \leq 3$.

Proof. Let $v \in \Gamma_b^N$ with $|N| = 2$. Then it can be seen directly that $v \in \Gamma_t^N$ since $l_i^*(v) = v(\{i\})$ and $u_i^*(v) = v(N) - v(N \setminus \{i\})$ for all $i \in N$, which implies that $v(S) \leq \max\{\sum_{i \in S} l_i^*(v), v(N) - \sum_{i \in N \setminus S} u_i^*(v)\}$ for all $S \in 2^N \setminus \{\emptyset\}$, so Theorem 2.1 applies.

Let $v \in \Gamma_b^N$ with $|N|= 3$. For all $i \in N$,

$$v(\{i\}) \leq l_i^*(v) \leq \max\left\{ l_i^*(v), v(N) - \sum_{j \in N \setminus \{i\}} u_j^*(v) \right\}.$$

For all $S \in 2^N$ with $|S|= 2$,

$$v(S) \leq v(N) - \sum_{i \in N \setminus S} u_i^*(v) \leq \max\left\{ \sum_{i \in S} l_i^*(v), v(N) - \sum_{i \in N \setminus S} u_i^*(v) \right\}.$$

Hence, $v \in \Gamma_t^N$ by Theorem 2.1.

Let $v \in \Gamma_b^N$ with $|N|> 3$ be defined by $v(N) = 3$, $v(\{i,j\}) = 1$ for distinct $i, j \in N$ and $v(S) = 0$ otherwise. Then $l_k^*(v) = 0$ for all $k \in N$, $u_i^*(v) = u_j^*(v) = 3$, and $u_k^*(v) = 2$ for all $k \in N \setminus \{i,j\}$. This implies that

$$v(\{i,j\}) = 1 > 0 + 0 = l_i^*(v) + l_j^*(v)$$

and

$$v(\{i,j\}) = 1 > 3 - 2(|N|-2) = v(N) - \sum_{k \in N \setminus \{i,j\}} u_k^*(v).$$

Hence, $v \notin \Gamma_t^N$ by Theorem 2.1 and Lemma 2.2. $\square$

In what follows next, we study to what extent the exact core bounds of a two-bound core game can be stretched while retaining the core description. It turns out that the exact core bounds can be stretched in only three different ways.

Proposition 2.2
Let $v \in \Gamma_t^N$. If there exist $l, u \in \mathbb{R}^N$ with $[l, u] \neq [l^*(v), u^*(v)]$ such that $C(v) = [l, u] \cap X(v)$, then exactly one of the following cases holds:

(i) $l \leq l^*(v)$ *and* $u = u^*(v)$,

(ii) $l = l^*(v)$ *and* $u \geq u^*(v)$,

(iii) there exists $i \in N$ *such that* $l_i < l_i^*(v)$, $u_i > u_i^*(v)$, *and* $l_j = l_j^*(v)$ *and* $u_j = u_j^*(v)$ *for all* $j \in N \setminus \{i\}$.

Proof. In view of $l \leq l^*(v)$ and $u \geq u^*(v)$, it suffices to prove that if $l \neq l^*(v)$ and $u \neq u^*(v)$, then case (iii) arises. Assume to the contrary that there exist $i, j \in N$ with $i \neq j$ such that $l_i < l_i^*(v)$ and $u_j > u_j^*(v)$. Let $x \in C(v)$. Define x' by $x'_i = x_i - \varepsilon$, $x'_j = x_j + \varepsilon$ and $x'_k = x_k$ for all $k \in N \setminus \{i, j\}$, where $\varepsilon = \min\{x_i - l_i, u_j - x_j\} \geq \min\{l_i^*(v) - l_i, u_j - u_j^*(v)\} > 0$. Then $x' \in [l, u] \cap X(v)$, but $x' \notin C(v)$ in view of $x'_i = l_i < l_i^*(v)$ or $x'_j = u_j > u_i^*(v)$. So, $C(v) \neq [l, u] \cap X(v)$, which is a contradiction. □

Moreover, we show that the first case in Proposition 2.2 arises only if the players whose lower bounds are decreased obtain their lower exact core bounds when all other players obtain their upper exact core bounds. The second case in Proposition 2.2 arises only if the players whose upper bounds are increased obtain their upper exact core bounds when all other players obtain their lower exact core bounds. The third case in Proposition 2.2 arises only if the player whose exact core bounds are stretched obtains the lower exact core bound when all other players obtain their upper exact core bounds and obtains the upper exact core bound when all other players obtain their lower exact core bounds.

Theorem 2.2
Let $v \in \Gamma_t^N$ and let $l, u \in \mathbb{R}^N$. Then the following statements hold:

(i) If $l \leq l^*(v)$ and $u = u^*(v)$, then $C(v) = [l, u] \cap X(v)$ if and only if

$$v(N) = l_i^*(v) + \sum_{j \in N \setminus \{i\}} u_j^*(v) \text{ for all } i \in N \text{ with } l_i < l_i^*(v).$$

(ii) If $l = l^*(v)$ and $u \geq u^*(v)$, then $C(v) = [l, u] \cap X(v)$ if and only if

$$v(N) = u_i^*(v) + \sum_{j \in N \setminus \{i\}} l_j^*(v) \text{ for all } i \in N \text{ with } u_i > u_i^*(v).$$

(iii) If there exists $i \in N$ such that $l_i < l_i^*(v)$, $u_i > u_i^*(v)$, and $l_j = l_j^*(v)$ and $u_j = u_j^*(v)$ for all $j \in N \setminus \{i\}$, then $C(v) = [l, u] \cap X(v)$ if and only if

$$u_i^*(v) + \sum_{j \in N \setminus \{i\}} l_j^*(v) = v(N) = l_i^*(v) + \sum_{j \in N \setminus \{i\}} u_j^*(v). \tag{2.2}$$

Proof. (i) For the only-if part, assume that $C(v) = [l, u] \cap X(v)$, where $l \leq l^*(v)$ and $u = u^*(v)$. We show that $v(N) = l_i^*(v) + \sum_{j \in N \setminus \{i\}} u_j^*(v)$ for all $i \in N$ with $l_i < l_i^*(v)$. Assume, to the contrary, that there exists $i \in N$ with $l_i < l_i^*(v)$ such that $v(N) \neq l_i^*(v) + \sum_{j \in N \setminus \{i\}} u_j^*(v)$. Let $x \in C(v)$ be such that $x_i = l_i^*(v)$. Then we have

$$v(N) = x_i + \sum_{j \in N \setminus \{i\}} x_j < l_i^*(v) + \sum_{j \in N \setminus \{i\}} u_j^*(v).$$

It follows that there exists $j \in N \setminus \{i\}$ such that $x_j < u_j^*(v)$. Define x' by $x'_i = x_i - \varepsilon$, $x'_j = x_j + \varepsilon$ and $x'_k = x_k$ for all $k \in N \setminus \{i, j\}$, where $0 < \varepsilon < \min\{x_i - l_i, u_j^*(v) - x_j\}$. Then $x' \in [l, u] \cap X(v)$, but $x' \notin C(v)$ in view of $x'_i < x_i = l_i^*(v)$. So, $C(v) \neq [l, u] \cap X(v)$, which is a contradiction.

For the if-part, assume that $l \leq l^*(v)$ and $u = u^*(v)$ such that $v(N) = l_i^*(v) + \sum_{j \in N \setminus \{i\}} u_j^*(v)$ for all $i \in N$ with $l_i < l_i^*(v)$. We show that $C(v) = [l, u] \cap X(v)$. In view of $C(v) = [l^*(v), u^*(v)] \cap X(v) \subseteq [l, u] \cap X(v)$, we only need to prove that $[l, u] \cap X(v) \subseteq [l^*(v), u^*(v)] \cap X(v)$. Let $x \in [l, u] \cap X(v)$. Then $x_i \geq l_i = l_i^*(v)$ for all $i \in N$ with $l_i = l_i^*(v)$. For all $i \in N$ with $l_i < l_i^*(v)$,

$$x_i \geq v(N) - \sum_{j \in N \setminus \{i\}} u_j = v(N) - \sum_{j \in N \setminus \{i\}} u_j^*(v) = l_i^*(v).$$

Together with $x \leq u = u^*(v)$, we obtain that $x \in [l^*(v), u^*(v)] \cap X(v)$. Hence, $[l, u] \cap X(v) \subseteq [l^*(v), u^*(v)] \cap X(v)$.

(ii) The proof is analogous to the proof of (i).

(iii) For the only-if part, assume that $C(v) = [l, u] \cap X(v)$, where $l_i < l_i^*(v)$, $u_i > u_i^*(v)$, and $l_j = l_j^*(v)$ and $u_j = u_j^*(v)$ for all $j \in N \setminus \{i\}$. We show that expression (2.2) holds. Assume that $v(N) \neq u_i^*(v) + \sum_{j \in N \setminus \{i\}} l_j^*(v)$ or $v(N) \neq l_i^*(v) + \sum_{j \in N \setminus \{i\}} u_j^*(v)$. Then, analogous to the proofs of (i) and (ii), it follows that $C(v) \neq [l, u] \cap X(v)$, which is a contradiction.

For the if-part, assume that there exists $i \in N$ such that $l_i < l_i^*(v)$, $u_i > u_i^*(v)$, $l_j = l_j^*(v)$ and $u_j = u_j^*(v)$ for all $j \in N \setminus \{i\}$, and expression (2.2) holds. We show that $C(v) = [l, u] \cap X(v)$. In view of $C(v) = [l^*(v), u^*(v)] \cap X(v) \subseteq [l, u] \cap X(v)$, we only need to prove that $[l, u] \cap X(v) \subseteq [l^*(v), u^*(v)] \cap X(v)$. Let $x \in [l, u] \cap X(v)$. Then

$$x_i \geq v(N) - \sum_{j \in N \setminus \{i\}} u_j = v(N) - \sum_{j \in N \setminus \{i\}} u_j^*(v) = l_i^*(v)$$

and $x_j \geq l_j = l_j^*(v)$ for all $j \in N \setminus \{i\}$, so $x \geq l^*(v)$. Similarly,

$$x_i \leq v(N) - \sum_{j \in N \setminus \{i\}} l_j = v(N) - \sum_{j \in N \setminus \{i\}} l_j^*(v) = u_i^*(v)$$

and $x_j \leq u_j = u_j^*(v)$ for all $j \in N \setminus \{i\}$, so $x \leq u^*(v)$. It follows that

$x \in [l^*(v), u^*(v)] \cap X(v)$. Hence, $[l, u] \cap X(v) \subseteq [l^*(v), u^*(v)] \cap X(v)$. □

Proposition 2.2 and Theorem 2.2 directly imply the following result, which shows exactly under which condition two-bound core games can be described by different pairs of bounds.

Corollary 2.1
Let $v \in \Gamma_t^N$. Then there exist $l, u \in \mathbb{R}^N$ with $[l, u] \neq [l^*(v), u^*(v)]$ such that $C(v) = [l, u] \cap X(v)$ if and only if there exists $i \in N$ such that $v(N) = l_i^*(v) + \sum_{j \in N \setminus \{i\}} u_j^*(v)$ or $v(N) = u_i^*(v) + \sum_{j \in N \setminus \{i\}} l_j^*(v)$.

2.4 The nucleolus

In this section, we consider the nucleolus of two-bound core games. Quant et al (2005) provided an explicit expression of the nucleolus for compromise stable games in terms of the pair of bounds in Example 2.3, using the Talmud rule for bankruptcy problems. On the one hand, we extend their approach by providing an explicit expression of the nucleolus for all two-bound core games in terms of the exact core bounds. On the other hand, we show that the nucleolus can be equivalently expressed by each pair of bounds describing the core.

Lemma 2.3
Let $v \in \Gamma_t^N$. Then

$$\begin{aligned}\eta(v) &= l^*(v) + f^{TAL}\left(v(N) - \sum_{i \in N} l_i^*(v), u^*(v) - l^*(v)\right) \\ &= u^*(v) - f^{TAL}\left(\sum_{i \in N} u_i^*(v) - v(N), u^*(v) - l^*(v)\right).\end{aligned}$$

Proof. Define $w \in \Gamma^N$ by $w(S) = v(S) - \sum_{i\in S} l_i^*(v)$ for all $S \in 2^N$. Then $l_i^*(w) = 0$ and $u_i^*(w) = u_i^*(v) - l_i^*(v)$ for all $i \in N$, $C(w) = [l^*(w), u^*(w)] \cap X(w)$, and $\eta(v) = l^*(v) + \eta(w)$.

For each $i \in N$, there exists $x \in C(w)$ such that $x_i = l_i^*(w)$, so

$$0 = l_i^*(w) = x_i = w(N) - \sum_{j\in N\setminus\{i\}} x_i \geq w(N) - \sum_{j\in N\setminus\{i\}} u_j^*(w).$$

Similarly, for each $i \in N$, there exists $x \in C(w)$ such that $x_i = u_i^*(w)$, so

$$u_i^*(w) = x_i = w(N) - \sum_{j\in N\setminus\{i\}} x_i \leq w(N) - \sum_{j\in N\setminus\{i\}} l_j^*(w) = w(N).$$

Define $(E, c) \in \mathcal{B}^N$ by $E = w(N)$ and $c = u^*(w)$. Then $v_{E,c}(N) = E = w(N)$ and for all $i \in N$,

$$\begin{aligned} l_i^*(v_{E,c}) &= \max\left\{0, E - \sum_{j\in N\setminus\{i\}} c_j\right\} = \max\left\{0, w(N) - \sum_{j\in N\setminus\{i\}} u_i^*(w)\right\} \\ &= 0 = l_i^*(w) \end{aligned}$$

and

$$u_i^*(v_{E,c}) = \min\{E, c_i\} = \min\{w(N), u_i^*(w)\} = u_i^*(w).$$

This implies that

$$\begin{aligned} C(v_{E,c}) &= [l^*(v_{E,c}), u^*(v_{E,c})] \cap X(v_{E,c}) \\ &= \left\{x \in \mathbb{R}^N \,\middle|\, \sum_{i\in N} x_i = v_{E,c}(N) \text{ and } l^*(v_{E,c}) \leq x \leq u^*(v_{E,c})\right\} \\ &= \left\{x \in \mathbb{R}^N \,\middle|\, \sum_{i\in N} x_i = w(N) \text{ and } l^*(w) \leq x \leq u^*(w)\right\} \\ &= [l^*(w), u^*(w)] \cap X(w) = C(w). \end{aligned}$$

Potters and Tijs (1994) showed that the nucleoli of two balanced games are equal if their cores are equal and at least one of the two games is convex. Since $v_{E,c}$ is convex, this implies that $\eta(v_{E,c}) = \eta(w)$. Applying self-duality,

$$\begin{aligned}
\eta(v) &= l^*(v) + \eta(w) \\
&= l^*(v) + \eta(v_{E,c}) \\
&= l^*(v) + f^{TAL}(E, c) \\
&= l^*(v) + f^{TAL}(w(N), u^*(w)) \\
&= l^*(v) + f^{TAL}\left(v(N) - \sum_{i \in N} l_i^*(v), u^*(v) - l^*(v)\right) \\
&= u^*(v) - f^{TAL}\left(\sum_{i \in N} u_i^*(v) - v(N), u^*(v) - l^*(v)\right).
\end{aligned}$$

□

The proof of Lemma 2.3 is similar to the proof of Theorem 4.2 of Quant et al (2005). However, as the following example shows, the expression obtained by Quant et al (2005) in terms of the pair of bounds in Example 2.3 is not valid for all two-bound core games.

Example 2.5
Let $v \in \Gamma_t^N$ with $N = \{1, \ldots, n\}$ and $n \geq 4$ be defined by $v(N) = v(\{1,2\}) = v(\{1,3\}) = 1$ and $v(S) = 0$ otherwise. Then $l^*(v) = u^*(v) = (1, 0, \ldots, 0)$ and $C(v) = \{(1, 0, \ldots, 0)\}$, so

$$\eta(v) = (1, 0, \ldots, 0) + f^{TAL}(0, (0, \ldots, 0)) = (1, 0, \ldots, 0).$$

However, v is not a compromise stable game, and $\eta(v)$ cannot be expressed using the lower bound $l = (0, \ldots, 0)$ and the upper bound $u = (1, \ldots, 1)$ from Example 2.3 in view of

$$\eta(v) \neq (0, \ldots, 0) + f^{TAL}(1, (1, \ldots, 1)) = (\frac{1}{n}, \ldots, \frac{1}{n}).$$

△

More generally, the nucleolus of two-bound core games can be equivalently expressed in terms of each pair of bounds describing the core.

Theorem 2.3
Let $v \in \Gamma_t^N$. Then

$$\begin{aligned}\eta(v) &= l + f^{TAL}\left(v(N) - \sum_{i\in N} l_i, u - l\right)\\ &= u - f^{TAL}\left(\sum_{i\in N} u_i - v(N), u - l\right)\end{aligned}$$

for all $l, u \in \mathbb{R}^N$ such that $C(v) = [l, u] \cap X(v)$.

Proof. Let $l, u \in \mathbb{R}^N$ be such that $C(v) = [l, u] \cap X(v)$. If we have $[l, u] = [l^*(v), u^*(v)]$, then the statement follows directly from Lemma 2.3. Suppose that $[l, u] \neq [l^*(v), u^*(v)]$. Then, by Proposition 2.2, exactly one of the following cases holds.

(i) $l \leq l^*(v)$ and $u = u^*(v)$.

By Theorem 2.2, $v(N) = l_i^*(v) + \sum_{j\in N\setminus\{i\}} u_j^*(v)$ for all $i \in N$ with $l_i < l_i^*(v)$. This implies that $\sum_{j\in N} u_j^*(v) - v(N) = u_i^*(v) - l_i^*(v) < u_i - l_i$ for all $i \in N$ with $l_i < l_i^*(v)$. Applying Lemma 2.3, invariance under claims truncation, and self-duality,

$$\begin{aligned}\eta(v) &= u^*(v) - f^{TAL}\left(\sum_{i\in N} u_i^*(v) - v(N), u^*(v) - l^*(v)\right)\\ &= u - f^{TAL}\left(\sum_{i\in N} u_i - v(N), u - l^*(v)\right)\\ &= u - f^{TAL}\left(\sum_{i\in N} u_i - v(N), u - l\right)\end{aligned}$$

$$= l + f^{TAL}\left(v(N) - \sum_{i\in N} l_i, u - l\right).$$

(ii) $l = l^*(v)$ and $u \geq u^*(v)$.

By Theorem 2.2, $v(N) = u_i^*(v) + \sum_{j\in N\setminus\{i\}} l_j^*(v)$ for all $i \in N$ with $u_i > u_i^*(v)$. This implies that $v(N) - \sum_{j\in N} l_j^*(v) = u_i^*(v) - l_i^*(v) < u_i - l_i$ for all $i \in N$ with $u_i > u_i^*(v)$. Applying Lemma 2.3, invariance under claims truncation, and self-duality,

$$\begin{aligned}
\eta(v) &= l^*(v) + f^{TAL}\left(v(N) - \sum_{i\in N} l_i^*(v), u^*(v) - l^*(v)\right) \\
&= l + f^{TAL}\left(v(N) - \sum_{i\in N} l_i, u^*(v) - l\right) \\
&= l + f^{TAL}\left(v(N) - \sum_{i\in N} l_i, u - l\right) \\
&= u - f^{TAL}\left(\sum_{i\in N} u_i - v(N), u - l\right).
\end{aligned}$$

(iii) There exists $i \in N$ such that $l_i < l_i^*(v)$, $u_i > u_i^*(v)$, $l_j = l_j^*(v)$ and $u_j = u_j^*(v)$ for all $j \in N \setminus \{i\}$.

By Theorem 2.2,

$$v(N) - \sum_{j\in N} l_j^*(v) = u_i^*(v) - l_i^*(v) = \sum_{j\in N} u_j^*(v) - v(N).$$

This implies that $v(N) = \frac{1}{2}\sum_{j\in N}(u_j^*(v) + l_j^*(v))$. Then

$$\eta(v) = l^*(v) + f^{TAL}\left(v(N) - \sum_{j\in N} l_j^*(v), u^*(v) - l^*(v)\right)$$

$$= l^*(v) + f^{TAL}\left(\frac{1}{2}\sum_{j\in N}(u_j^*(v) - l_j^*(v)), u^*(v) - l^*(v)\right)$$
$$= l^*(v) + \frac{1}{2}(u^*(v) - l^*(v))$$
$$= \frac{1}{2}(u^*(v) + l^*(v)).$$

Define $(E^*, c^*) \in \mathcal{B}^N$ by $E^* = v(N) - \sum_{j\in N} l_j^*(v)$ and $c^* = u^*(v) - l^*(v)$, and define $(E, c) \in \mathcal{B}^N$ by $E = v(N) - \sum_{j\in N} l_j$ and $c = u - l$. Then

$$E - E^* = \sum_{j\in N} l_j^*(v) - \sum_{j\in N} l_j = l_i^*(v) - l_i > 0,$$
$$c_i - c_i^* = (u_i - u_i^*(v)) + (l_i^*(v) - l_i) > E - E^*,$$

and $c_j = c_j^*$ for all $j \in N \setminus \{i\}$. Moreover, for all $j \in N \setminus \{i\}$,

$$c_i > c_i^* = u_i^*(v) - l_i^*(v) = v(N) - \sum_{k\in N} l_k^*(v) \geq u_j^*(v) - l_j^*(v) = c_j^* = c_j.$$

This implies that $f_i^{TAL}(E, c) = f_i^{TAL}(E^*, c^*) + E - E^* = f_i^{TAL}(E^*, c^*) + l_i^*(v) - l_i$ and $f_j^{TAL}(E, c) = f_j^{TAL}(E^*, c^*)$ for all $j \in N \setminus \{i\}$. Applying Lemma 2.3 and self-duality,

$$\eta(v) = l^*(v) + f^{TAL}\left(v(N) - \sum_{i\in N} l_i^*(v), u^*(v) - l^*(v)\right)$$
$$= l^*(v) + f^{TAL}(E^*, c^*)$$
$$= l + f^{TAL}(E, c)$$
$$= l + f^{TAL}\left(v(N) - \sum_{i\in N} l_i, u - l\right)$$
$$= u - f^{TAL}\left(\sum_{i\in N} u_i - v(N), u - l\right).$$

□

Example 2.6
Let $v \in \Gamma_t^N$ with $N = \{1,2,3\}$ be defined by $v(\{1\}) = v(\{2\}) = 2$, $v(\{3\}) = 4$, $v(\{1,2\}) = 10$, $v(\{1,3\}) = 6$, $v(\{2,3\}) = 12$, and $v(N) = 20$. Then $l^*(v) = (2,2,4)$ and $u^*(v) = (8,14,10)$. Since $l_1^*(v) + u_2^*(v) + l_3^*(v) = v(N) = u_1^*(v) + l_2^*(v) + u_3^*(v)$, Theorems 2.2 and 2.3 imply that

$$\begin{aligned}\eta(v) &= (2,2,4) + f^{TAL}\left(12,(6,12,6)\right) = (8,14,10) - f^{TAL}\left(12,(6,12,6)\right)\\ &= (2,0,4) + f^{TAL}\left(14,(6,14,6)\right) = (8,20,10) - f^{TAL}\left(18,(6,18,6)\right)\\ &= (2,0,4) + f^{TAL}\left(14,(6,20,6)\right) = (8,20,10) - f^{TAL}\left(18,(6,20,6)\right)\\ &= (5,8,7).\end{aligned}$$

The first two expressions are in terms of the lower exact core bounds and the upper exact core bounds. The third expression is based on a decrease of only the lower exact core bound of player 2 to $l_2 = 0$. The fourth expression is based on an increase of only the upper exact core bound of player 2 to $u_2 = 20$. The fifth and sixth expressions are based on a decrease of player 2's lower bound to $l_2 = 0$ and an increase of player 2's upper bound to $u_2 = 20$ simultaneously. In view of $u_1^*(v)+l_2^*(v)+l_3^*(v) < v(N) < l_1^*(v)+u_2^*(v)+u_3^*(v)$ and $l_1^*(v)+l_2^*(v)+u_3^*(v) < v(N) < u_1^*(v) + u_2^*(v) + l_3^*(v)$, the lower and upper exact core bounds of players 1 and 3 cannot be stretched. $\triangle$

2.5 The egalitarian core

In this section, we show that the egalitarian core for two-bound core games is single-valued and we provide an explicit expression of it in terms of the exact core bounds.

To show this result, we first show that the core of a two-bound core game is equal to the core of a particular convex game, where the worth of each coalition is defined by the minimum total payoff of its members in any pre-imputation between the two bounds.

Theorem 2.4
Let $v \in \Gamma_t^N$. Then there exists $\widehat{v} \in \Gamma_c^N$ such that $C(\widehat{v}) = C(v)$.

Proof. Let $l, u \in \mathbb{R}^N$ be such that $C(v) = [l, u] \cap X(v)$. Define $\widehat{v} \in \Gamma^N$ by

$$\widehat{v}(S) = \max\left\{\sum_{i \in S} l_i, v(N) - \sum_{i \in N \setminus S} u_i\right\} \text{ for all } S \in 2^N.$$

By Theorem 2.1, $v(S) \leq \hat{v}(S)$ for all $S \in 2^N \setminus \{\emptyset\}$. Hence $C(\hat{v}) \subseteq C(v)$. We claim that $C(\hat{v}) = C(v)$. Suppose, to the contrary, that there exists $x \in C(v) \setminus C(\hat{v})$. Let $S \in 2^N \setminus \{N, \emptyset\}$ be such that $\sum_{i \in S} x_i < \hat{v}(S)$. If $\hat{v}(S) = \sum_{i \in S} l_i$, then $\sum_{i \in S} x_i < \sum_{i \in S} l_i$, so $x_i < l_i$ for some $i \in S$, contradicting $x \in C(v)$. If $\hat{v}(S) = v(N) - \sum_{i \in N \setminus S} u_i$, then $\sum_{i \in S} x_i < v(N) - \sum_{i \in N \setminus S} u_i$, so $x_i > u_i$ for some $i \in N \setminus S$, contradicting $x \in C(v)$. Hence, $C(v) = C(\hat{v})$, which implies that $\widehat{v} \in \Gamma_t^N$.

For all $S, T \in 2^N$,

$$\begin{aligned}
&\widehat{v}(S) + \widehat{v}(T) \\
= \; &\max\left\{\sum_{i \in S} l_i, v(N) - \sum_{i \in N \setminus S} u_i\right\} + \max\left\{\sum_{i \in T} l_i, v(N) - \sum_{i \in N \setminus T} u_i\right\} \\
= \; &\max\Bigg\{\sum_{i \in S} l_i + \sum_{i \in T} l_i, v(N) + \sum_{i \in S} l_i - \sum_{i \in N \setminus T} u_i, \\
&\qquad v(N) + \sum_{i \in T} l_i - \sum_{i \in N \setminus S} u_i, 2v(N) - \sum_{i \in N \setminus S} u_i - \sum_{i \in N \setminus T} u_i\Bigg\} \\
\leq \; &\max\Bigg\{\sum_{i \in S \cup T} l_i + \sum_{i \in S \cap T} l_i, v(N) + \sum_{i \in S} l_i - \sum_{i \in N \setminus T} u_i + \sum_{i \in S \setminus T} (u_i - l_i), \\
&\qquad v(N) + \sum_{i \in T} l_i - \sum_{i \in N \setminus S} u_i + \sum_{i \in T \setminus S} (u_i - l_i),
\end{aligned}$$

$$
\begin{aligned}
&\qquad 2v(N)-\sum_{i\in N\setminus S}u_i-\sum_{i\in N\setminus T}u_i\Bigg\}\\
&=\max\Bigg\{\sum_{i\in S\cup T}l_i+\sum_{i\in S\cap T}l_i,\, v(N)+\sum_{i\in S\cap T}l_i-\sum_{i\in N\setminus(S\cup T)}u_i,\, v(N)+\\
&\qquad \sum_{i\in S\cap T}l_i-\sum_{i\in N\setminus(S\cup T)}u_i,\, 2v(N)-\sum_{i\in N\setminus(S\cup T)}u_i-\sum_{i\in N\setminus(S\cap T)}u_i\Bigg\}\\
&\le \max\Bigg\{\sum_{i\in S\cup T}l_i,\, v(N)-\sum_{i\in N\setminus(S\cup T)}u_i\Bigg\}+\\
&\quad \max\Bigg\{\sum_{i\in S\cap T}l_i,\, v(N)-\sum_{i\in N\setminus(S\cap T)}u_i\Bigg\}\\
&=\widehat{v}(S\cup T)+\widehat{v}(S\cap T).
\end{aligned}
$$

Hence, $\widehat{v}\in\Gamma_c^N$. □

The following example shows that a convex game may not be a two-bound core game, and a two-bound core game may not be a convex game. Moreover, the egalitarian core of the corresponding game is given as follows.

Example 2.7
Let $N=\{1,2,3,4\}$ and let $w\in\Gamma_c^N$ be given by

$$
w(S)=\begin{cases}2 & \text{if } S=N,\\ 1 & \text{if } S\in\{\{1,2\},\{1,2,3\},\{1,2,4\}\},\\ 0 & \text{otherwise.}\end{cases}
$$

Then we have $l^*(w)=(0,0,0,0)$, $u^*(w)=(2,2,1,1)$. It is easily seen that $(0,0,1,1)\in[l^*(w),u^*(w)]\cap X(w)$, but $(0,0,1,1)\notin C(w)$. Hence, $w\notin\Gamma_t^N$.

Let $N = \{1,2,3\}$ and let $v \in \Gamma_t^N$ be given by

$$v(S) = \begin{cases} 1 & \text{if } S \in \{\{1,2\},\{1,3\},\{1,2,3\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $v(\{1,2\}) + v(\{1,3\}) > v(\{1\}) + v(\{1,2,3\})$. Hence, $v \notin \Gamma_c^N$.

Moreover, $EC(w) = \left\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\right\}$ and $EC(v) = \{(1,0,0)\}$. △

By Theorem 2.4, the core of a two-bound core game is equal to the core of a particular convex game. It can be shown that if two games have equal cores, then the games have equal egalitarian cores. Arin and Iñarra (2001) showed that the egalitarian core is single-valued for convex games. This implies that the egalitarian core for two-bound core games is single-valued. We provide an explicit expression.

Theorem 2.5
The egalitarian core of a two-bound core game $v \in \Gamma_t^N$ is given by

$$EC_i(v) = \begin{cases} l_i^*(v) & \text{if } \lambda \leq l_i^*(v), \\ \lambda & \text{if } l_i^*(v) \leq \lambda \leq u_i^*(v), \\ u_i^*(v) & \text{if } \lambda \geq u_i^*(v), \end{cases}$$

for all $i \in N$, where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} EC_i(v) = v(N)$.

Proof. Let $v \in \Gamma_t^N$. Then it holds that $|EC(v)| = 1$. Define $x \in \mathbb{R}^N$ by $x_i = \min\{\max\{l_i^*(v), \lambda\}, u_i^*(v)\}$ for all $i \in N$, where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} x_i = v(N)$. Then $x \in [l^*(v), u^*(v)] \cap X(v)$, so $x \in C(v)$. Let $i, j \in N$ be such that $x_i > x_j$. Then $x_i = l_i^*(v)$ or $x_j = u_i^*(v)$. Suppose for the sake of contradiction that $s_{ij}^x(v) \neq 0$. Then $s_{ij}^x(v) < 0$, so $v(S) < \sum_{k \in S} x_k$ for all $S \in 2^N$ with $i \in S$ and $j \notin S$. Let $0 < \varepsilon < -s_{ij}^x(v)$. Define $x' \in \mathbb{R}^N$ by $x'_i = x_i - \varepsilon$, $x'_j = x_j + \varepsilon$, and $x'_k = x_k$ for all $k \in N \setminus \{i,j\}$. Then $x' \in C(v)$, which contradicts the definition of $l_i^*(v)$ or $u_j^*(v)$. □

So far, we have studied to what extent the exact core bounds of a two-bound core game can be stretched while retaining the core, nucleolus and egalitarian core descriptions. Instead of stretching the lower and upper bounds, we can also study to what extent these expressions are robust against game changes. It turns out that the worths of coalitions can be increased subject to specific restrictions.

Theorem 2.6
Let $v \in \Gamma_t^N$ and let $l, u \in \mathbb{R}^N$ be such that $C(v) = [l, u] \cap X(v)$. If $w \in \Gamma^N$ is such that $v(S) \leq w(S) \leq \max\{\sum_{i \in S} l_i, v(N) - \sum_{i \in N \setminus S} u_i\}$ for all $S \in 2^N \setminus \{\emptyset\}$, then the following statements hold:

(i) $C(v) = C(w)$,

(ii) $\eta(v) = \eta(w)$.

(iii) $EC(v) = EC(w)$.

Proof. (i) Define $\hat{v} \in \Gamma^N$ by $\hat{v}(S) = \max\{\sum_{i \in S} l_i, v(N) - \sum_{i \in N \setminus S} u_i\}$ for all $S \in 2^N \setminus \{\emptyset\}$. Let $w \in \Gamma^N$ be such that $v(S) \leq w(S) \leq \hat{v}(S)$ for all $S \in 2^N \setminus \{\emptyset\}$. Then $C(\hat{v}) \subseteq C(w) \subseteq C(v)$. By Theorem 2.4, we have $C(v) = C(\hat{v})$. Hence, $C(v) = C(w) = C(\hat{v})$.

(ii) Statement (i) implies that $w \in \Gamma_t^N$ and $C(w) = C(v) = [l, u] \cap X(v) = [l, u] \cap X(w)$. By Theorem 2.3,

$$\begin{aligned} \eta(v) &= l + f^{TAL}\left(v(N) - \sum_{i \in N} l_i, u - l\right) \\ &= l + f^{TAL}\left(w(N) - \sum_{i \in N} l_i, u - l\right) = \eta(w). \end{aligned}$$

(iii) Statement (i) implies that $w \in \Gamma_t^N$ and $l^*(v) = l^*(w)$ and $u^*(v) = u^*(w)$. By Theorem 2.5, it is obvious that $EC(v) = EC(w)$. $\square$

2.6 Concluding remarks

In this chapter, we introduced the large class of two-bound core games and provided explicit expressions of the nucleolus in terms of all pairs of bounds describing the core, using the Talmud rule for bankruptcy problems, and the egalitarian core in terms of the pair of exact core bounds. Other solutions for two-bound core games are directly obtained by replacing the role of the Talmud rule in these expressions by any other bankruptcy rule. Quant et al (2006) studied these extensions from a general point of view and paid particular attention to the specific random arrival rule (O'Neill, 1982). González-Díaz et al (2005) followed a similar approach with a focus on the adjusted proportional rule (Curiel et al, 1987). Future research could study extensions of these and other bankruptcy rules to the class of two-bound core games.

3
Reduced two-bound core games

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3.1 Introduction

Cooperative games describe situations where players collaborate in coalitions and generate profits. A pre-imputation allocates the total profit among all players in the game. The main issue is to select reasonable pre-imputations for each game. Among the central solution concepts are the core, the nucleolus (Schmeidler, 1969), and the egalitarian core (Arin and Iñarra, 2001). The core assigns all pre-imputations that are stable against coalitional deviations. The nucleolus assigns the allocation that lexicographically minimizes the excesses of all coalitions, which is a core allocation whenever the core is nonempty. The egalitarian core assigns all core allocations from which no other core allocation can be obtained by a transfer from a richer to a poorer player.

In the previous chapter, we introduced the class of two-bound core games, where the core is nonempty and can be described by a lower bound and an upper bound on the pre-imputations. Many classes of games turned out to be two-bound core games. We showed that the core of each two-bound core game can be described equivalently by the pair of exact core bounds, and studied to what extent the exact core bounds can be stretched while retaining the core description. We also provided explicit expressions of the nucleolus and the egalitarian core for two-bound core games in terms of the pairs of bounds describing the core.

In this chapter, we study Davis-Maschler reduced games of two-bound core games and show that all these reduced games with respect to core elements are two-bound core games. Moreover, the core of these reduced games can be described by the same pair of bounds. A solution satisfies the bilateral reduced game property (Davis and Maschler, 1965) if each pre-imputation assigned to the original game is consistently assigned to all reduced games with two players. A solution satisfies the converse reduced game property (Davis and Maschler, 1965) if each pre-imputation assigned to all reduced games with two players is assigned to the original game. Using the bilateral reduced game property and the converse reduced game property, we axiomatically characterize the core, the nucleolus, and the egalitarian core for two-bound core games.

The remainder of this chapter is organized as follows. Section 3.2 recalls some definitions and notations for cooperative games. Section 3.3 studies Davis-Maschler reduced games of two-bound core games and axiomatically characterizes the core. Sections 3.4 and 3.5 axiomatically characterize the nucleolus and the egalitarian core, respectively. Section 3.6 shows independence of these axiomatic characterizations.

3.2 Preliminaries

Let N be a nonempty and finite set of *players* and let 2^N be the collection of all subsets of N. For $x, y \in \mathbb{R}^N$, $x \geq y$ denotes $x_i \geq y_i$ for all $i \in N$, and $x > y$ denotes $x_i > y_i$ for all $i \in N$. The notations $\leq$ and $<$ are defined analogously. We denote $x + y = (x_i + y_i)_{i \in N}$, $x - y = (x_i - y_i)_{i \in N}$, $[x, y] = \{z \in \mathbb{R}^N \mid x \leq z \leq y\}$, $\lambda x = (\lambda x_i)_{i \in N}$ for all $\lambda \in \mathbb{R}$, and $x_S = (x_i)_{i \in S}$ for all $S \in 2^N \setminus \{\emptyset\}$.

A *cooperative game with transferable utility* (a *game*, for short) is a pair (N, v), where $v : 2^N \to \mathbb{R}$ is the *characteristic function* assigning to each *coalition* $S \in 2^N$ its *worth*, with $v(\emptyset) = 0$. The set of all games with player set N is denoted by Γ^N. For simplicity, we write $v \in \Gamma^N$ rather than $(N, v) \in \Gamma^N$.

Let $v \in \Gamma^N$. The *pre-imputation set* of v is

$$X(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N) \right\},$$

the *core* of v is

$$C(v) = \left\{ x \in X(v) \;\middle|\; \forall_{S \in 2^N} : \sum_{i \in S} x_i \geq v(S) \right\},$$

and the *egalitarian core* (Arin and Iñarra, 2001) of v is

$$EC(v) = \left\{x \in C(v) \,\middle|\, \forall_{i,j \in N : x_i > x_j} : s_{ij}^x(v) = 0\right\},$$

where for all $i, j \in N$ with $i \neq j$ and all $x \in \mathbb{R}^N$,

$$s_{ij}^x(v) = \max_{S \in 2^N : i \in S, j \notin S} \left\{ v(S) - \sum_{k \in S} x_k \right\}.$$

The set of all games with nonempty core and player set N is denoted by Γ_b^N. A game $v \in \Gamma^N$ is *convex* (Shapley, 1971) if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \in 2^N$. The set of all convex games with player set N is denoted by Γ_c^N. It is known that $\Gamma_c^N \subseteq \Gamma_b^N$.

A game $v \in \Gamma_b^N$ is a *two-bound core game* (see Definition 2.1) if there exist $l, u \in \mathbb{R}^N$ such that $C(v) = [l, u] \cap X(v)$, which is equivalent to $C(v) = [l^*(v), u^*(v)] \cap X(v)$, where $l_i^*(v) = \min_{x \in C(v)} x_i$ and $u_i^*(v) = \max_{x \in C(v)} x_i$ for all $i \in N$. The bounds l^* and u^* were also studied by Bondareva and Driessen (1994). The set of all two-bound core games with player set N is denoted by Γ_t^N.

A *solution* φ on a domain of games assigns to each game v in this domain a nonempty set $\varphi(v) \subseteq X(v)$. Note that $\varphi(v) = \{v(N)\}$ for each game v with one player. A solution φ on a domain of games is *single-valued* if $|\varphi(v)| = 1$ for each v in this domain. For a single-valued solution φ on a domain of games and a game v in this domain, $\varphi(v)$ is often identified with its unique element.

The *nucleolus* η (Schmeidler, 1969) is a single-valued solution that assigns to each game with nonempty core a unique core element. Maschler et al (1971) showed that the nucleolus of a convex game $v \in \Gamma_c^N$ is given by

$$\eta(v) = \{x \in X(v) \mid \forall_{i,j \in N, i \neq j} : s_{ij}^x(v) = s_{ji}^x(v)\}.$$

3.3 Reduced games and axiomatization of the core

In this section, we study Davis-Maschler reduced games of two-bound core games, and axiomatically characterize the core for two-bound core games.

The *reduced game* (Davis and Maschler, 1965) of $v \in \Gamma_t^N$ on $T \in 2^N \setminus \{\emptyset\}$ with respect to $x \in \mathbb{R}^N$, denoted by $v_T^x \in \Gamma^T$, is defined by

$$v_T^x(S) = \begin{cases} v(N) - \sum_{i \in N \setminus T} x_i & \text{if } S = T, \\ \max\limits_{Q \subseteq N \setminus T} \left\{ v(S \cup Q) - \sum\limits_{i \in Q} x_i \right\} & \text{if } S \in 2^T \setminus \{\emptyset, T\}, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

In other words, the worth of a coalition in a reduced game is defined as the maximal surplus in cooperation with any subgroup of players in the original game that are not present in the reduced game. It turns out that all reduced games of two-bound core games with respect to core elements are two-bound core games. Moreover, the core of these reduced games can be described by the same pair of bounds.

Theorem 3.1
Let $v \in \Gamma_t^N$, $T \in 2^N \setminus \{\emptyset\}$, $x \in C(v)$, and let $l, u \in \mathbb{R}^N$ be such that $C(v) = [l, u] \cap X(v)$. Then

$$C(v_T^x) = [l_T, u_T] \cap X(v_T^x).$$

Proof. Let $y \in C(v_T^x)$. Then

$$\sum_{i \in T} y_i + \sum_{i \in N \setminus T} x_i = v_T^x(T) + \sum_{i \in N \setminus T} x_i = v(N) - \sum_{i \in N \setminus T} x_i + \sum_{i \in N \setminus T} x_i = v(N).$$

Let $S \in 2^N$. If $S \cap T = \emptyset$, then

$$\sum_{i \in S \cap T} y_i + \sum_{i \in S \setminus T} x_i = \sum_{i \in S} x_i \geq v(S).$$

If $S \cap T = T$, then

$$\begin{aligned}\sum_{i \in S \cap T} y_i + \sum_{i \in S \setminus T} x_i &= v_T^x(T) + \sum_{i \in S \setminus T} x_i \\ &= v(N) - \sum_{i \in N \setminus T} x_i + \sum_{i \in S \setminus T} x_i = \sum_{i \in S} x_i \geq v(S).\end{aligned}$$

If $S \cap T \notin \{\emptyset, T\}$, then

$$\begin{aligned}\sum_{i \in S \cap T} y_i + \sum_{i \in S \setminus T} x_i &\geq v_T^x(S \cap T) + \sum_{i \in S \setminus T} x_i \\ &= \max_{Q \subseteq N \setminus T} \left\{ v((S \cap T) \cup Q) - \sum_{i \in Q} x_i \right\} + \sum_{i \in S \setminus T} x_i \\ &\geq v(S) - \sum_{i \in S \setminus T} x_i + \sum_{i \in S \setminus T} x_i \\ &= v(S).\end{aligned}$$

This means that $(y, x_{N \setminus T}) \in C(v)$, so $(y, x_{N \setminus T}) \in [l, u] \cap X(v)$, which implies that $y \in [l_T, u_T] \cap X(v_T^x)$. Hence, $C(v_T^x) \subseteq [l_T, u_T] \cap X(v_T^x)$.

Let $y \in [l_T, u_T] \cap X(v_T^x)$. Then $(y, x_{N \setminus T}) \in [l, u] \cap X(v)$, so $(y, x_{N \setminus T}) \in C(v)$. Let $S \in 2^T \setminus \{\emptyset, T\}$. For all $Q \subseteq N \setminus T$,

$$\sum_{i \in S} y_i = \sum_{i \in S} y_i + \sum_{i \in Q} x_i - \sum_{i \in Q} x_i \geq v(S \cup Q) - \sum_{i \in Q} x_i,$$

so

$$\sum_{i \in S} y_i \geq \max_{Q \subseteq N \setminus T} \left\{ v(S \cup Q) - \sum_{i \in Q} x_i \right\} = v_T^x(S).$$

This implies that $y \in C(v_T^x)$. Hence, $[l_T, u_T] \cap X(v_T^x) \subseteq C(v_T^x)$. □

A solution satisfies the *bilateral reduced game property* if the restriction of each pre-imputation assigned to the original game is consistently assigned to all reduced games with two players. A solution satisfies the *converse reduced game property* if each two-player restriction of a pre-imputation is assigned to the corresponding reduced game, then this pre-imputation is assigned to the original game.

Bilateral reduced game property (Davis and Maschler, 1965)
For all $v \in \Gamma_t^N$, all $T \in 2^N$ with $|T|= 2$, and all $x \in \varphi(v)$, we have $v_T^x \in \Gamma_t^T$ and $x_T \in \varphi(v_T^x)$.

Converse reduced game property (Davis and Maschler, 1965)
For all $v \in \Gamma_t^N$ and all $x \in X(v)$, if $v_T^x \in \Gamma_t^T$ and $x_T \in \varphi(v_T^x)$ for all $T \in 2^N$ with $|T|= 2$, then $x \in \varphi(v)$.

By requiring the solution to assign the core to all two-bound core games with two players, Peleg (1986) characterized the core of games using the bilateral reduced game property and the converse reduced game property. We obtain a similar axiomatic characterization of the core for two-bound core games.

Unanimity (Peleg, 1986)
For all $v \in \Gamma_t^N$ with $|N|= 2$, we have $\varphi(v) = \{x \in X(v) \mid \forall_{i \in N} : x_i \geq v(\{i\})\}$.

Theorem 3.2
The core is the unique solution for two-bound core games satisfying unanimity, the bilateral reduced game property, and the converse reduced game property.

Proof. Clearly, the core satisfies unanimity. To prove that the core satisfies the bilateral reduced game property, let $v \in \Gamma_t^N$, let $T \in 2^N$ with $|T|= 2$, let $x \in C(v)$, and let $l, u \in \mathbb{R}^N$ be such that $C(v) = [l, u] \cap X(v)$. By Theorem 3.1, $C(v_T^x) = [l_T, u_T] \cap X(v_T^x)$. In view of

$x_T \in [l_T, u_T] \cap X(v_T^x)$, this implies that $v_T^x \in \Gamma_t^T$ and $x_T \in C(v_T^x)$. Hence, the core satisfies the bilateral reduced game property.

To prove that the core satisfies the converse reduced game property, let $v \in \Gamma_t^N$ and let $x \in X(v)$ be such that $v_T^x \in \Gamma_t^T$ and $x_T \in C(v_T^x)$ for all $T \in 2^N$ with $|T|= 2$. Let $S \in 2^N \setminus \{\emptyset, N\}$ and let $j \in N \setminus S$. For all $i \in S$,

$$x_i \geq v_{\{i,j\}}^x(\{i\}) = \max_{Q \subseteq N \setminus \{i,j\}} \left\{ v(\{i\} \cup Q) - \sum_{k \in Q} x_i \right\} \geq v(S) - \sum_{k \in S \setminus \{i\}} x_k,$$

so $\sum_{i \in S} x_i \geq v(S)$. This implies that $x \in C(v)$. Hence, the core satisfies the converse reduced game property.

To prove uniqueness, let φ be a solution for two-bound core games satisfying unanimity, the bilateral reduced game property, and the converse reduced game property. We show that $\varphi(v) = C(v)$ for all $v \in \Gamma_t^N$. By unanimity, $\varphi(v) = C(v)$ for all $v \in \Gamma_t^N$ with $|N|\leq 2$. Let $v \in \Gamma_t^N$ with $|N|\geq 3$.

Let $x \in \varphi(v)$. By the bilateral reduced game property of φ, $v_T^x \in \Gamma_t^T$ and $x_T \in \varphi(v_T^x)$ for all $T \in 2^N$ with $|T|= 2$, so $x_T \in C(v_T^x)$ for all $T \in 2^N$ with $|T|= 2$. By the converse reduced game property of the core, this implies that $x \in C(v)$. Hence, $\varphi(v) \subseteq C(v)$.

Let $x \in C(v)$. By the bilateral reduced game property of the core, $v_T^x \in \Gamma_t^T$ and $x_T \in C(v_T^x)$ for all $T \in 2^N$ with $|T|= 2$, so $x_T \in \varphi(v_T^x)$ for all $T \in 2^N$ with $|T|= 2$. By the converse reduced game property of φ, this implies that $x \in \varphi(v)$. Hence, $C(v) \subseteq \varphi(v)$. $\square$

3.4 Axiomatization of the nucleolus

In this section, we axiomatically characterize the nucleolus for two-bound core games using the Davis-Maschler reduced game properties. The *nucleolus* η (Schmeidler, 1969) is a single-valued solution that assigns to each game with nonempty core a unique core element. In

Theorem 2.3, we provide an explicit expression of the nucleolus for two-bound core games.

By requiring the solution to assign the nucleolus to all two-bound core games with two players, we obtain an axiomatic characterization of the nucleolus for two-bound core games using the bilateral reduced game property.

Standardness (Aumann and Maschler, 1985)
For all $v \in \Gamma_t^N$ with $|N|= 2$ and all $i \in N$, we have

$$\varphi_i(v) = v(\{i\}) + \frac{1}{2}\left(v(N) - v(\{i\}) - v(N \setminus \{i\})\right).$$

Theorem 3.3
The nucleolus is the unique solution for two-bound core games satisfying standardness and the bilateral reduced game property.

Proof. It is known that the nucleolus satisfies standardness. To prove that the nucleolus satisfies the bilateral reduced game property and the converse reduced game property (used in the uniqueness part), let $v \in \Gamma_t^N$, let $l, u \in \mathbb{R}^N$ be such that $C(v) = [l, u] \cap X(v)$, and let $x \in X(v)$. By Theorem 2.4, there exists $\widehat{v} \in \Gamma_c^N$ such that $C(\widehat{v}) = C(v)$. Theorem 2.6 implies that $\eta(\widehat{v}) = \eta(v)$. Maschler et al (1971) showed that the convexity of $\widehat{v}$ implies that

$$\eta(\widehat{v}) = \left\{x \in X(\widehat{v}) \,\middle|\, \forall_{i,j \in N, i \neq j} : s_{ij}^x(\widehat{v}) = s_{ji}^x(\widehat{v})\right\},$$

where $s_{ij}^x(\widehat{v}) = \max_{S \in 2^N : i \in S, j \notin S}\{\widehat{v}(S) - \sum_{k \in S} x_k\}$ for all $i, j \in N$ with $i \neq j$. For all $i, j \in N$ with $i \neq j$,

$$\begin{aligned} s_{ij}^{x_{\{i,j\}}}(\widehat{v}^x_{\{i,j\}}) &= \widehat{v}^x_{\{i,j\}}(\{i\}) - x_i \\ &= \max_{Q \subseteq N \setminus \{i,j\}} \left\{\widehat{v}(Q \cup \{i\}) - \sum_{k \in Q} x_k\right\} - x_i \end{aligned}$$

$$
\begin{aligned}
&= \max_{S \in 2^N : i \in S, j \notin S} \left\{ \widehat{v}(S) - \sum_{k \in S} x_k \right\} \\
&= s_{ij}^x(\widehat{v}).
\end{aligned}
$$

This implies that $x = \eta(\widehat{v})$ if and only if $\widehat{v}_T^x \in \Gamma_t^T$ and $x_T = \eta(\widehat{v}_T^x)$ for all $T \in 2^N$ with $|T|= 2$. By Theorem 3.1, if $x \in C(v)$ and $C(v) = C(\widehat{v})$, then $C(\widehat{v}_T^x) = C(v_T^x) = [l_T, u_T] \cap X(v_T^x)$ for all $T \in 2^N$ with $|T|= 2$. By Theorem 2.4, $x \in C(v)$ if and only if $v_T^x \in \Gamma_t^T$ and $x_T \in C(v_T^x)$ for all $T \in 2^N$ with $|T|= 2$. Together, this implies that $x = \eta(v)$ if and only if $v_T^x \in \Gamma_t^T$ and $x_T = \eta(v_T^x)$ for all $T \in 2^N$ with $|T|= 2$. Hence, the nucleolus satisfies the bilateral reduced game property and the converse reduced game property.

To prove uniqueness, let φ be a solution for two-bound core games satisfying standardness and the bilateral reduced game property. We show that $\varphi(v) = \eta(v)$ for all $v \in \Gamma_t^N$. By standardness, $\varphi(v) = \eta(v)$ for all $v \in \Gamma_t^N$ with $|N|\leq 2$. Let $v \in \Gamma_t^N$ with $|N|\geq 3$ and let $x \in \varphi(v)$. By the bilateral reduced game property of φ, $v_T^x \in \Gamma_t^T$ and $x_T \in \varphi(v_T^x)$ for all $T \in 2^N$ with $|T|= 2$, so $x_T = \eta(v_T^x)$ for all $T \in 2^N$ with $|T|= 2$. By the converse reduced game property of the nucleolus, this implies that $x = \eta(v)$. Hence, $\varphi(v) = \eta(v)$. $\square$

3.5 Axiomatization of the egalitarian core

In this section, we axiomatically characterize the egalitarian core for two-bound core games using the Davis-Maschler reduced game properties. The egalitarian core consists of all core elements from which no other core element can be obtained by a transfer from a richer to a poorer player. In Theorem 2.5, we provide an explicit expression of the egalitarian core for two-bound core games.

By requiring the solution to assign the egalitarian core to all games with two players, Arin and Iñarra (2001) obtained an axiomatic characterization of the egalitarian core for convex games in conjunction with the bilateral reduced game property and the converse reduced game property. We obtain a similar axiomatic characterization of the egalitarian core for two-bound core games without requiring the converse reduced game property.

Constrained egalitarianism (Dutta, 1990)
For all $v \in \Gamma_t^N$ with $|N|= 2$ and all $i \in N$, we have

$$\varphi_i(v) = \begin{cases} \max\{v(\{i\}), \frac{1}{2}v(N)\} & \text{if } v(\{i\}) \geq v(N \setminus \{i\}), \\ v(N) - \varphi_{N\setminus\{i\}}(v) & \text{if } v(\{i\}) \leq v(N \setminus \{i\}). \end{cases}$$

Theorem 3.4
The egalitarian core is the unique solution for two-bound core games satisfying constrained egalitarianism and the bilateral reduced game property.

Proof. It is known that the egalitarian core satisfies constrained egalitarianism. To prove that the egalitarian core satisfies the bilateral reduced game property and the converse reduced game property (used in the uniqueness part), let $v \in \Gamma_t^N$ and let $x \in X(v)$. By Theorem 3.2, $x \in C(v)$ if and only if $v_T^x \in \Gamma_t^T$ and $x_T \in C(v_T^x)$ for all $T \in 2^N$ with $|T|= 2$. For all $i, j \in N$ with $i \neq j$,

$$\begin{aligned} s_{ij}^{x_{\{i,j\}}}(v^x_{\{i,j\}}) &= v^x_{\{i,j\}}(\{i\}) - x_i \\ &= \max_{Q \subseteq N\setminus\{i,j\}} \left\{ v(Q \cup \{i\}) - \sum_{k \in Q} x_k \right\} - x_i \\ &= \max_{S \in 2^N : i \in S, j \notin S} \left\{ v(S) - \sum_{k \in S} x_k \right\} \\ &= s_{ij}^x(v). \end{aligned}$$

This implies that $x = EC(v)$ if and only if $v_T^x \in \Gamma_t^T$ and $x_T = EC(v_T^x)$ for all $T \in 2^N$ with $|T|= 2$. Hence, the egalitarian core satisfies the bilateral reduced game property and the converse reduced game property.

To prove uniqueness, let φ be a solution for two-bound core games satisfying constrained egalitarianism and the bilateral reduced game property. We show that $\varphi(v) = EC(v)$ for all $v \in \Gamma_t^N$. By constrained egalitarianism, $\varphi(v) = EC(v)$ for all $v \in \Gamma_t^N$ with $|N| \leq 2$. Let $v \in \Gamma_t^N$ with $|N| \geq 3$ and let $x \in \varphi(v)$. By the bilateral reduced game property of φ, $v_T^x \in \Gamma_t^T$ and $x_T \in \varphi(v_T^x)$ for all $T \in 2^N$ with $|T| = 2$, so $x_T = EC(v_T^x)$ for all $T \in 2^N$ with $|T| = 2$. By the converse reduced game property of the egalitarian core, this implies that $x = EC(v)$. Hence, $\varphi(v) = EC(v)$. $\square$

3.6 Concluding remarks

In this chapter, we axiomatically characterized the core, the nucleolus, and the egalitarian core for two-bound core games. In fact, it can be shown that these solutions satisfy the stronger reduced game property which requires that the restriction of each pre-imputation assigned to the original game is consistently assigned to all reduced games (not only with two players), but the weaker bilateral reduced game property suffices in the axiomatic characterizations. To show that the properties in these axiomatic characterizations are independent, we introduce the following additional solutions.

A solution that satisfies unanimity and the converse reduced game property, but does not satisfy the bilateral reduced game property, is the solution $\widehat{X}$, which is for all $v \in \Gamma_t^N$ defined by

$$\widehat{X}(v) = \begin{cases} C(v) & \text{if } |N| \leq 2, \\ X(v) & \text{if } |N| \geq 3. \end{cases}$$

A solution that satisfies unanimity and the bilateral reduced game property, but does not satisfy the converse reduced game property, is the solution $\widehat{C}$, which is for all $v \in \Gamma_t^N$ defined by

$$\widehat{C}(v) = \begin{cases} C(v) & \text{if } |N| \leq 2, \\ \eta(v) & \text{if } |N| \geq 3. \end{cases}$$

A solution that satisfies standardness, but does not satisfy the bilateral reduced game property, is the solution $\widehat{\eta}$, which is for all $v \in \Gamma_t^N$ defined by

$$\widehat{\eta}(v) = \begin{cases} \eta(v) & \text{if } |N| \leq 2, \\ X(v) & \text{if } |N| \geq 3. \end{cases}$$

A solution that satisfies constrained egalitarianism, but does not satisfy the bilateral reduced game property, is the solution $\widehat{EC}$, which is for all $v \in \Gamma_t^N$ defined by

$$\widehat{EC}(v) = \begin{cases} EC(v) & \text{if } |N| \leq 2, \\ X(v) & \text{if } |N| \geq 3. \end{cases}$$

An overview of these solutions, their properties, and the axiomatic characterizations is presented in the following table. Here, $+$ indicates that the rule satisfies the property, $-$ indicates the rule does not satisfy the property, and $*$ indicates the axiomatic characterizations.

	C	η	EC	$\widehat{X}$	$\widehat{C}$	$\widehat{\eta}$	$\widehat{EC}$
unanimity	$+^*$	$-$	$-$	$+$	$+$	$-$	$-$
standardness	$-$	$+^*$	$-$	$-$	$-$	$+$	$-$
constrained egalitarianism	$-$	$-$	$+^*$	$-$	$-$	$-$	$+$
bilateral reduced game property	$+^*$	$+^*$	$+^*$	$-$	$+$	$-$	$-$
converse reduced game property	$+^*$	$+$	$+$	$+$	$-$	$-$	$-$

Hence, the properties in Theorems 3.2, 3.3, and 3.4 are independent.

4
Mechanisms for bankruptcy problems

Adapted from: Gong, D., G. Xu, X. Jin, and L. Gogoi. *A sequential partition method for non-cooperative games of bankruptcy problems*. TOP, 2022, **30**(2), 359-379.

4.1 Introduction

Bankruptcy problems (O'Neill, 1982), also known as estate division problems or claims problems, study the situations where an insufficient estate is allocated among several agents, each of whom has a claim on the estate. The initial approach to solve bankruptcy problems implicitly assumes the presence of an authority, like a court or an arbitrator, who can dispose the estate directly after knowing all claims. Based on this device, several fundamental bankruptcy rules are proposed and studied widely, such as the proportional (PRO) rule, the constrained equal awards (CEA) rule, the constrained equal losses (CEL) rule, the Talmud (TAL) rule, etc. Most relevant work about the bankruptcy rules is devoted to a cooperative and axiomatic explanation, and it can be seen in two recent overviews (Thomson, 2003; Thomson, 2015).

In addition, bankruptcy problems are also dealt with by mechanism design and non-cooperative games. A Nash equilibrium of the non-cooperative game induced by a mechanism and a bankruptcy problem is a good prediction of the strategies of the agents, and therefore the equilibrium outcome can be regarded as an allocation of the estate. From this perspective of view, several interesting approaches have been explored.

A main research line is following Chun (1989), Serrano (1995) and Dagan et al (1997), in which each agent simultaneously announces a payoff vector (called proposal) and then these agents act strategically based on their proposals, or only a special agent proposes and leads the game, while other agents negotiate with the proposer. Chang and Hu (2008) introduced an axiomatization of the f-just rule and designed the corresponding non-cooperative game to interpret this rule. Li and Ju (2014) considered a non-cooperative approach to Talmud solution. Giménez-Gómez (2014) introduced a game that combines the diminishing claims and unanimous concessions procedures, thereby justifying the rules based on averaging. Tsay and Yeh (2019) designed games to strategically justify a class of bankruptcy rules, in which bilateral negotiations are considered.

Other research lines contain: (i) García-Jurado et al (2006) and Ashlagi et al (2012) considered the non-cooperative games in which agents have simpler strategy spaces and the equilibrium achieved based on constraints of rules. (ii) Atlamaz et al (2011) and Peters et al (2019) proposed a non-cooperative approach by allowing agents to put multiple claims on the same part of the estate. (iii) Kıbrıs and Kıbrıs (2013) and Karagözoğlu (2014) constructed a non-cooperative investment game to explain the proportional rule in the economy. (iv) In the aspect of manipulation via merging or splitting claims, the relevant work can be seen in Ju (2003), Moreno-Ternero (2006, 2007), and Ju and Moreno-Ternero (2011).

In this chapter, we consider mechanisms for bankruptcy problems, and explore a sequential partition method for non-cooperative games of bankruptcy problems under the general assumption that the estate and claims are common knowledge, and each agent acts individually. To better introduce this method, we might as well treat the entire estate as a homogeneous cake and each agent takes a morsel of the cake as his payoff. Then, the idea of this method is that agents gather and each of them successively partitions an admissible morsel from the cake in a given order. Here, an agent, in his turn, has known how much each of his predecessors partitions, and his strategy is the portion he partitions, potentially depending on the strategies of his predecessors and expressing the amount he desires (We use he to refer any type of agents).

Obviously, different non-cooperative games are formed by considering different arrival orders and combining the sequential partition with other non-cooperative processes. In this chapter, we mainly focus on the ascending order of claims and study two following games in detail, leaving other non-cooperative games to be explored further.

Combining the sequential partition with the reversal selection processes, we present the *divide-and-choose* game, which is similar to the process in the cake-division problem (Crawford, 1977). First, each agent successively partitions a morsel by announcing a number in ascending order of the claims. Intuitively, the number represents the size of the morsel and implies the portion he asks for himself. Then, each one chooses one of the currently existing morsels as his payoff in reverse order. Note that every morsel can be chosen only once. We show that there is a unique Nash equilibrium outcome, and it coincides with the allocation of the CEA rule, seeing papers (Dagan, 1996; Herrero and Villar, 2002; Yeh, 2006; Yeh, 2008) for results supporting this rule. Although the Nash equilibrium outcome is not a subgame perfect equilibrium outcome for most bankruptcy problems, an approximation of it can be achieved when the game ends. We also show that there is no dominant strategy equilibrium outcome.

Next, we present the *divide-and-object* game by combining the sequential partition with the bilateral objection processes. First, agents successively partition in the same way as the above. Then, an agent can object to another who cuts off a plethoric morsel, where the objection is defined by a bilateral principle. As a result, the initiator receives more rewards and the target is punished severely so as to prevent him from announcing arbitrarily. If there are no objections among agents, each one takes the morsel partitioned by himself. We show that the unique Nash equilibrium outcome is consistent with the allocation of the CEA rule, which is also a subgame perfect equilibrium outcome.

The rest of this chapter is organized as follows. Section 4.2 introduces some basic definitions and notation. The divide-and-choose game and the divide-and-object game are studied in Section 4.3 and Section 4.4, respectively. Section 4.5 concludes with detailed comparisons of a few related non-cooperative games.

4.2 Preliminaries

In this section, we recall the definitions and notation about bankruptcy problems and bankruptcy rules, and introduce some additional definitions and notation.

Let N be a nonempty and finite set of *agents*. A *bankruptcy problem* is a triple (T, E, c), where $T \in 2^N \setminus \{\emptyset\}$, $E \in \mathbb{R}_+$ is the *estate* to be divided and $c \in \mathbb{R}_+^T$ is the vector of *claims* satisfying $\sum_{i \in T} c_i \geq E$. The set of all bankruptcy problems with agent set T is denoted by $\mathcal{B}^T$. For simplicity, we write $(E, c) \in \mathcal{B}^T$ rather than $(T, E, c) \in \mathcal{B}^T$.

A *bankruptcy rule* $f : \mathcal{B}^T \to \mathbb{R}_+^T$ assigns to each bankruptcy problem $(E, c) \in \mathcal{B}^T$ with $T \in 2^N \setminus \{\emptyset\}$ a payoff vector $f(E, c) \in \mathbb{R}_+^T$ such that $\sum_{i \in T} f_i(E, c) = E$ and $f_i(E, c) \leq c_i$ for all $i \in T$.

A bankruptcy rule f satisfies *order-preservation of payoffs* if $f_i(E, c) \leq f_j(E, c)$ for all $(E, c) \in \mathcal{B}^T$ with $T \in 2^N \setminus \{\emptyset\}$ and all $i, j \in T$ with $c_i \leq c_j$.

A bankruptcy rule f satisfies *consistency* if for all $(E, c) \in \mathcal{B}^N$ and $T \in 2^N \setminus \{\emptyset\}$,

$$f_T(E, c) = f\left(\sum_{i \in T} f_i(E, c), c_T\right), \tag{4.1}$$

where $f_T(E, c) = (f_i(E, c))_{i \in T}$ and $c_T = (c_i)_{i \in T}$. A bankruptcy rule f satisfies *bilateral consistency* if expression (4.1) holds for all $T \subseteq N$ with $|T| = 2$.

The *constrained equal awards (CEA)* rule assigns to each bankruptcy problem $(E, c) \in \mathcal{B}^N$ and each agent $i \in N$,

$$f_i^{CEA}(E, c) = \min\{c_i, \lambda\},$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} f_i^{CEA}(E, c) = E$. It assigns the estate as equally as possible provided that no one gets more than his claim.

The *constrained equal losses (CEL)* rule assigns to each bankruptcy problem $(E,c) \in \mathcal{B}^N$ and each agent $i \in N$,

$$f_i^{CEL}(E,c) = \max\{c_i - \lambda, 0\},$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} f_i^{CEL}(E,c) = E$. It assigns the losses $\sum_{i \in N} c_i - E$ as equally as possible provided that no one gets a negative payoff.

The *proportional (PRO)* rule assigns to each bankruptcy problem $(E,c) \in \mathcal{B}^N$ and each agent $i \in N$,

$$f_i^{PRO}(E,c) = \lambda c_i,$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} f_i^{PRO}(E,c) = E$. It assigns the estate in proportion to the claims of agents.

It is well-known that the CEA rule, the CEL rule and the PRO rule satisfy order preservation of payoffs and bilateral consistency (Thomson, 2003; Thomson, 2015).

Let $(E,c) \in \mathcal{B}^N$. In this chapter, we assume that $N = \{1, \ldots, n\}$, $E > 0$ and $c_i > 0$ for all $i \in N$. Moreover, without loss of generality, let $\sum_{i \in N} c_i > E$ and $c_1 \leq c_2 \leq \ldots \leq c_n$. Let $\Pi(N)$ be the set of all permutations of agent set N.

Denote $t_i = \sum_{k=1}^{i} c_k + (n-i)c_i$ for each $i \in N$ and $t_0 = 0$, and denote $T_i = (t_{i-1}, t_i]$ for each $1 \leq i < n$ and $T_n = (t_{n-1}, t_n)$. It is easily seen that $t_{i-1} = t_i$ if and only if $c_{i-1} = c_i$, and it follows that $T_i = \emptyset$. Moreover, we have $T_i \cap T_j = \emptyset$ for distinct $i, j \in N$, and $\cup_{i \in N} T_i = (0, \sum_{i \in N} c_i)$. With these notations, we give the following equivalent expression of the CEA rule.

For each $(E,c) \in \mathcal{B}^N$, there is a unique $l \in N$ such that $E \in T_l$ and for all $i \in N$,

$$f_i^{CEA}(E,c) = \begin{cases} c_i & \text{if } i < l, \\ \frac{E - \sum_{k<l} c_k}{n-l+1} & \text{if } i \geq l. \end{cases} \tag{4.2}$$

4.3 The divide-and-choose game

In this section, we introduce a divide-and-choose mechanism to deal with bankruptcy problems. Given an arbitrary bankruptcy problem $(E, c) \in \mathcal{B}^N$, we can obtain the following divide-and-choose game, denoted by $M(E, c)$. When no confusion arises, the notation $M(E, c)$ is abbreviated to M for convenience.

The divide-and-choose game M:

Stage 1: Each agent $i \in N$ successively announces a number $x_i \in \mathbb{R}_+$ in ascending order of the claims. The number agent i announces can not exceed his claim and the current residual estate, that is,

$$x_i \in \begin{cases} [0, \min\{c_i, E\}] & \text{if } i = 1, \\ \left[0, \min\{c_i, E - \sum_{k=1}^{i-1} x_k\}\right] & \text{if } 2 \leq i \leq n. \end{cases}$$

We call the vector $x = (x_1, \ldots, x_n)$ a *proposal*. Note that a proposal x is not necessarily efficient, i.e., it could be that $\sum_{i=1}^{n} x_i < E$.

Stage 2: Each agent $i \in N$ successively chooses a coordinate of x in reverse order. Note that each coordinate is chosen only once. The game ends with the outcome that every agent receives the amount of estate he chooses.

Since a rational agent will always choose the largest number he can choose, the outcome in M is the payoff vector $x^\sigma = (x^\sigma_{\sigma(1)}, \ldots, x^\sigma_{\sigma(n)})$, where $\sigma \in \Pi(N)$ is such that $x^\sigma_{\sigma(1)} \leq \ldots \leq x^\sigma_{\sigma(n)}$. Here, expression $x^\sigma_i = x_j$ represents that agent i chooses the number announced by agent j. We call $\sigma \in \Pi(N)$ a *feasible permutation* with respect to x if x^σ is such that $x^\sigma_{\sigma(1)} \leq \ldots \leq x^\sigma_{\sigma(n)}$. Therefore, the payoff vector obtained in Stage 2 is directly determined by the proposal in Stage 1. So, an agent's strategy in M can be simplified as his strategy in Stage 1.

In Stage 1, the *strategy* of agent 1, denoted by S_1, is the number he announces, i.e., $S_1 \in [0, \min\{c_i, E\}]$. For $2 \leq i \leq n$, the *strategy* of agent i, denoted by S_i, is a function $S_i : \{(x_1, \ldots, x_{i-1}) \in$

$\mathbb{R}_+^{\{1,\ldots,i-1\}} \mid \forall_{j\in\{1,\ldots,i-1\}} : x_j \leq \min\{c_j, E - \sum_{k=1}^{j-1} x_k\}\} \rightarrow \mathbb{R}_+$ such that $S_i(x_1, \ldots, x_{i-1}) \leq \min\{c_i, E - \sum_{j=1}^{i-1} x_j\}$. For every $i \in N$, we denote by $\mathcal{S}^i$ the set of all strategies of agent i. Denote by $\mathcal{S}^N = \prod_{i\in N} \mathcal{S}^i$ the set of all strategy profiles of M, and $\mathcal{S}^{-i} = \prod_{j\in N\setminus\{i\}} \mathcal{S}^j$ the set of all strategy profiles of all agents other than agent i, where a *strategy profile* of M is denoted by $S_N = (S_i)_{i\in N}$.

Note that a strategy profile is a vector of n functions (when S_1 is regarded as a constant function), yet a proposal is a vector of n numbers. A strategy profile uniquely determines a proposal, not vice versa. The reason is that given an independent variable, the same dependent variable can be obtained in different functions. Moreover, the different strategy profiles which determine the same proposal also determine the same outcome in M. It means that these strategy profiles are equivalent to each other.

For the sake of convenience, we slightly abuse the notation (x) to denote an arbitrary strategy profile which uniquely determines the proposal $x = (x_1, x_2, \ldots, x_n)$. Specifically, a strategy profile (x) represents that the strategy of agent 1 is x_1, the strategy of agent 2 is "when agent 1 announces x_1, I announce x_2; when agent 1 announces other numbers, I announce a feasible number", the strategy of agent 3 is "when agent 1 and agent 2 announce x_1 and x_2 respectively, I announce x_3; when they announce other numbers, I announce a feasible number", and so on. At the end of Stage 1, it derives the proposal $x = (x_1, x_2, \ldots, x_n)$.

In what follows next, we firstly introduce some notation which is prerequisite to define the Nash equilibrium of M, and then we define the dominant strategy equilibrium of M.

For each proposal x and each $x_i \neq y_i \in \left[0, \min\{c_i, E - \sum_{k=1}^{i-1} x_k\}\right]$, we denote by

$$x|y_i = \{z \text{ is a proposal} \mid z_i = y_i \text{ and } z_k = x_k \text{ for all } k < i\}$$

the set of all possible proposals if agent i begins to deviate from x and announces y_i when M proceeds to him. For convenience, we denote $x \nmid x_i = \underset{y_i \neq x_i}{\cup} (x|y_i)$. For each strategy profile $S_N = (S_i)_{i \in N} \in \mathcal{S}^N$, we denote by $u(S_N)$ the outcome of M. Then we have $u(S_N) = x^\sigma$, where x is the corresponding proposal by strategy profile S_N, and $\sigma \in \Pi$ is a feasible permutation of x.

Definition 4.1
Let $(E, c) \in \mathcal{B}^N$. Then a strategy profile $S_N^* \in \mathcal{S}^N$ is a Nash equilibrium of M if for all $i \in N$ and all $S_i \in \mathcal{S}^i$,

$$u_i(S_N^*) \geq u_i(S_i, S_{-i}^*). \tag{4.3}$$

Assume that x and z are the proposals obtained by strategy profiles S_N^* and (S_i, S_{-i}^*), respectively. Then inequality (4.3) equals to $x^\sigma_{\sigma(i)} \geq z^\tau_{\tau(i)}$, where σ and τ are feasible permutations of x and z, respectively. It indicates that agent i cannot receive more by first deviating from his strategy S_i^*, that is, first not announcing x_i. If expression (4.3) holds for all $i \in N$, we say that S_N^* is a Nash equilibrium of M. It can be seen that in M, each strategy profile with the same proposal x is also a Nash equilibrium of M since it results in the same outcome in M. So, we use strategy profile (x) rather than strategy profile S_N^* in the rest of this chapter. The same applies to later Definition 4.3.

Definition 4.2
Let $(E, c) \in \mathcal{B}^N$. Then $S_i^* \in \mathcal{S}^i$ is a dominant strategy for agent i of M if for all $S_i \in \mathcal{S}^i \setminus \{S_i^*\}$ and all $S_{-i} \in \mathcal{S}^{-i}$,

$$u_i(S_i^*, S_{-i}) \geq u_i(S_i, S_{-i}), \tag{4.4}$$

Inequality (4.4) indicates that a dominant strategy of an agent arises when he will always prefer to play a particular strategy over other strategies regardless of other agents' strategies. A dominant strategy equilibrium arises when this holds for all agents.

We show that the Nash equilibrium outcome of M is unique and it coincides with the allocation of the CEA rule. We proceed by two steps. First, we construct a Nash equilibrium in which the outcome is consistent with the allocation of the CEA rule (in Theorem 4.1). Then, we verify the uniqueness (in Theorem 4.2).

Theorem 4.1
Let $(E, c) \in \mathcal{B}^N$. Then each strategy profile $S_N \in \mathcal{S}^N$ with $u(S_N) = f^{CEA}(E, c)$ is a Nash equilibrium of M.

Proof. Let $S_N \in \mathcal{S}^N$, and let $s = f^{CEA}(E, c)$ be the corresponding proposal by S_N. Since the CEA rule satisfies order preservation of payoffs, the permutation $\sigma : \sigma(i) \to i$ for all $i \in N$, is a feasible permutation of s. Hence, we have $s^\sigma = s = f^{CEA}(E, c)$.

To prove the equilibrium, we need to verify that $s^\sigma_{\sigma(i)} \geq z^\tau_{\tau(i)}$ for all $i \in N$ and $(z) \in \mathcal{S}^N$ satisfying $z \in s \nmid s_i$, where σ and τ are feasible permutations with respect to x and z, respectively.

For $i = 1$, we consider two cases.

Case 1. $E \in T_1$. According to expression (4.2), we have $f^{CEA}_j(E, c) = E/n$ for all $j \in N$.

For each $z \in s \nmid s_1$, we argue that $z^\tau_{\tau(1)} \leq E/n$. If not, assume $z^\tau_{\tau(1)} > E/n$. Since $z^\tau_{\tau(j)} \geq z^\tau_{\tau(1)}$ for all $2 \leq j \leq n$, it follows that $\sum_{j \in N} z^\tau_{\tau(j)} > E$, which is a contradiction to our process. Hence, we have $s^\sigma_{\sigma(1)} = f^{CEA}_1(E, c) = E/n \geq z^\tau_{\tau(1)}$.

Case 2. $E \in \bigcup_{2 \leq j \leq n} T_j$. In this case, we have $f^{CEA}_1(E, c) = c_1$ and $f^{CEA}_j(E, c) \geq c_1$ for all $j \geq 2$.

For each $y_1 \neq s_1$, we consider $z \in s|y_1$. Since $y_1 < s_1 = c_1$ and agent 1 is the last one to choose, then he will obtain y_1 if $z_j \geq y_1$ for all $j \geq 2$ and obtain z_l satisfying $z_l = \min_{j \geq 2} z_j$ if $\{j | j \geq 2 \text{ and } z_j < y_1\} \neq \emptyset$. In view of $z_l < y_1$, we conclude that $s^\sigma_{\sigma(1)} = f^{CEA}_1(E, c) = c_1 > y_1 \geq z^\tau_{\tau(1)}$.

Then, for each $2 \leq i \leq n$, we also consider two cases.

Case 1. $E \in \bigcup_{1\leq k\leq i} T_k$. With expression (4.2), we have $f_k^{CEA}(E,c) = c_k$ for all $k < i$ and $f_j^{CEA}(E,c) = \frac{E-\sum_{k=1}^{i-1} c_k}{n-i+1}$ for all $j \geq i$.

Assume that there exists $z \in s \nmid s_i$ such that $s^{\sigma}_{\sigma(i)} < z^{\tau}_{\tau(i)}$. We denote by $\epsilon = z^{\tau}_{\tau(i)} - s^{\sigma}_{\sigma(i)}$ the positive excess. Since $z^{\tau}_{\tau(j)} \geq z^{\tau}_{\tau(i)} = s^{\sigma}_{\sigma(i)} + \epsilon$, together with $z_k = s_k \leq s_i = s^{\sigma}_{\sigma(i)}$, it is impossible that agent j chooses the number announced by agent k. According to order preservation of payoffs of the CEA rule, we have $\tau(k) = k$ for each $k < i$. Thus, it holds that

$$\begin{aligned}\sum_{j\in N} z^{\tau}_{\tau(j)} &= \sum_{k=1}^{i-1} z_k + \sum_{j=i}^{n} z^{\tau}_{\tau(j)} = \sum_{k=1}^{i-1} s_k + \sum_{j=i}^{n} z^{\tau}_{\tau(j)}\\ &\geq \sum_{k=1}^{i-1} s^{\sigma}_{\sigma(k)} + \sum_{j=i}^{n} (s^{\sigma}_{\sigma(i)} + \epsilon) = E + (n-i+1)\epsilon > E,\end{aligned}$$

which violates the process. Hence, we have $s^{\sigma}_{\sigma(i)} \geq z^{\tau}_{\tau(i)}$.

Case 2. $E \in \bigcup_{i<j\leq n} T_j$. As a result, we have $f_k^{CEA}(E,c) = c_k$ for all $k \leq i$, and $f_j^{CEA}(E,c) \geq c_i$ for all $j > i$.

For each $y_i \neq s_i$, we consider $z \in s|y_i$, that is, agent i announces $y_i < s_i = c_i$, together with $s_k = c_k \leq c_i$ for all $k < i$, it follows that $z_k \leq c_i$ for all $1 \leq k \leq i$. Hence, we have $z^{\tau}_{\tau(i)} = \max\{y_i, c_{i-1}\}$ if $z_j \geq \max\{y_i, c_{i-1}\}$ for all $j > i$, and $z^{\tau}_{\tau(i)} \leq \max\{y_i, c_{i-1}\}$ if $\{j > i \mid z_j < \max\{y_i, c_{i-1}\}\} \neq \emptyset$. Thus, we conclude that $s^{\sigma}_{\sigma(i)} = c_i \geq \max\{y_i, c_{i-1}\} \geq z^{\tau}_{\tau(i)}$.

In summary, for all $i \in N$ and $(z) \in \mathcal{S}^N$ satisfying $z \in s \nmid s_i$, we have $s^{\sigma}_{\sigma(i)} \geq z^{\tau}_{\tau(i)}$. Therefore, (s) is a Nash equilibrium of M with outcome $f^{CEA}(E,c)$. $\square$

To show uniqueness, we first introduce Lemma 4.1 to show that in M, there exist an agent i and a step-up proposal x such that x_i is infinitely close to $f_i^{CEA}(E,c)$, which is not affected by his predecessors.

Lemma 4.1
Let $(E,c) \in \mathcal{B}^N$ with $E \in T_i$ for some $1 \leq i \leq n$. Assume that, when M proceeds to i, agent k has announced x_k for each $k < i$. Then, for an arbitrary $\epsilon > 0$, there is a proposal x such that $x_j > x_{j-1} > f_i^{CEA}(E,c) - \epsilon$ for all $j > i$.

Proof. If $i = 1$, we have $T_1 = (0, nc_1]$ and $f_1^{CEA}(E,c) = E/n$. In this case, we construct an $x = (x_1, \ldots, x_n)$ as follows.

We assume that agent 1 announces $x_1 = E/n - \epsilon_1$, where

$$0 < \epsilon_1 < \sum_{j=2}^{n} c_j - \frac{n-1}{n}E, \tag{4.5}$$

then the residual estate satisfies that $\frac{n-1}{n}E + \epsilon_1 < \sum_{j=2}^{n} c_j$.

Then, for each $2 \leq j \leq n-1$, agent j announces $x_j = E/n + \sum_{k=1}^{j-1} \frac{\epsilon_k}{n-k} - \epsilon_j$, where

$$0 < \epsilon_j < \sum_{k=j+1}^{n} c_k - \frac{n-j}{n}E - \sum_{k=1}^{j-1} \frac{n-j}{n-k}\epsilon_k. \tag{4.6}$$

The currently residual estate satisfies that $\frac{n-j}{n}E + \sum_{k=1}^{j-1} \frac{n-j}{n-k}\epsilon_k + \epsilon_j < \sum_{k=j+1}^{n} c_k$. Finally, agent n announces $x_n = E/n + \sum_{k=1}^{n-2} \frac{\epsilon_k}{n-k} + \epsilon_{n-1} = E/n + \sum_{k=1}^{n-1} \frac{\epsilon_k}{n-k}$.

Note that the bounds of $\epsilon_1, \ldots, \epsilon_{n-1}$ are given so as to ensure $x_n \leq c_n$. The same applies to the rest of this proof.

According to the definition of x, it is easy to verify that, for $2 \leq j \leq n-1$, $x_j > x_{j-1}$ if $\epsilon_j < \frac{n-j+2}{n-j+1}\epsilon_{j-1}$, and $x_n > x_{n-1}$.

Therefore, for $\epsilon > 0$, let $\epsilon_1 < \epsilon$ be such hat expression (4.5) holds, and let $\epsilon_j < \frac{n-j+2}{n-j+1}\epsilon_{j-1}$ be such that expression (4.6) holds. Then we conclude that $x_j > x_{j-1} \geq E/n - \epsilon_1 > f_1^{CEA}(E, c) - \epsilon$.

If $2 \leq i \leq n-1$, with expression (4.2), we have $f_j^{CEA}(E, c) = \frac{E - \sum_{k=1}^{j-1} c_k}{n-j+1}$ for all $j \geq i$. Similarly, we construct an x as follows.

Assume that agent i announces $x_i = f_i^{CEA}(E, c) - \epsilon_i$, where

$$0 < \epsilon_i < \sum_{j=i+1}^{n} c_j - \frac{n-i}{n-i+1}(E - \sum_{k=1}^{i-1} c_k), \tag{4.7}$$

for $i < j < n$, agent j announces $x_j = f_j^{CEA}(E, c) + \sum_{k=i}^{j-1} \frac{\epsilon_k}{n-k} - \epsilon_j$, where

$$0 < \epsilon_j < \sum_{k=i+1}^{n} c_k - \frac{n-j}{n-i+1}(E - \sum_{k=1}^{i-1} c_k) - \sum_{k=i}^{j-1} \frac{n-j}{n-k}\epsilon_k, \tag{4.8}$$

and agent n announces $x_n = f_n^{CEA}(E, c) + \sum_{k=i}^{n-1} \frac{\epsilon_k}{n-k}$.

The similar conclusion is that, for $\epsilon > 0$, let $\epsilon_i < \epsilon$ be such taht expression (4.7) holds, and let $\epsilon_j < \frac{n-j+2}{n-j+1}\epsilon_{j-1}$ be such that expression (4.8) holds. Then we have $x_j > x_{j-1} \geq f_i^{CEA}(E, c) - \epsilon_i > f_i^{CEA}(E, c) - \epsilon$.

If $i = n$, we have $f_k^{CEA}(E, c) = c_k$ for each $k < n$ and $f_n^{CEA}(E, c) > c_{n-1}$. For an arbitrary $\epsilon > 0$, assuming $\epsilon_n < \epsilon$, then if agent n announces $x_n = f_n^{CEA}(E, c) - \epsilon_n$, we have $x_n = f_n^{CEA}(E, c) - \epsilon_n > f_n^{CEA}(E, c) - \epsilon$.

Combining these three cases, we obtain this result. $\square$

Theorem 4.2
Let $(E, c) \in \mathcal{B}^N$. Then the unique Nash equilibrium outcome of M is $f^{CEA}(E, c)$.

Proof. We have shown that $s = f^{CEA}(E, c)$ is a Nash equilibrium outcome of M. To prove uniqueness, we assume that a strategy profile (r) satisfying $r = (r_1, \ldots, r_n)$ is an arbitrary Nash equilibrium, then we verify that $r = s$, that is, (r) and (s) are equivalent to each other.

First, we prove that $r^{\tau}_{\tau(1)} \geq s^{\sigma}_{\sigma(1)} = s_1$, where τ and σ are feasible permutations of r and s, respectively. Supposing by contrary that $r^{\tau}_{\tau(1)} < s_1$, then we consider two cases.

Case 1. $E \in T_1$. According to Lemma 4.1, let $\epsilon < s_1 - r^{\tau}_{\tau(1)}$, then there is a proposal x such that $x_j > x_{j-1} > s_1 - \epsilon$ for each $j > 1$. It follows that agent 1′ payoff equals to x_1, where $x_1 > s_1 - \epsilon > r^{\tau}_{\tau(1)}$. In view of $x \in r \nmid r_1$, we obtain a contradiction to the assumption that (r) is a Nash equilibrium. Therefore, $r^{\tau}_{\tau(1)} \geq s_1$.

Case 2. $E \in T_j$, where $1 < j \leq n$. We have $s_1 = c_1$.

If $r_1 < s_1$, it is easy to verify that $s \in r \nmid r_1$ satisfies $s^{\sigma}_{\sigma(1)} > r_1 \geq \min\{r_1, \ldots, r_n\} = r^{\tau}_{\tau(1)}$, which is a contradiction.

If $r_1 = s_1$, with the assumption $r^{\tau}_{\tau(1)} < s_1$ and Lemma 4.1, we construct an x such that

$$x_k = \begin{cases} \frac{1}{2}(r^{\tau}_{\tau(1)} + s_1) & \text{if } k = 1, \\ c_k & \text{if } 1 < k < j, \\ s_j - \epsilon & \text{if } k = j, \\ x_k & \text{if } k > j, \end{cases}$$

where $\epsilon < s_j - c_{j-1}$ and $x_k > x_{k-1}$ for each $k > j$. Then, we have $x_1 \leq \ldots \leq x_n$, which implies that $x \in r \nmid r_1$ is a contradiction to the assumption that (r) is a Nash equilibrium.

Therefore, we have $r^{\tau}_{\tau(1)} \geq s_1$.

Then, for each $2 \leq i \leq n$, we verify that $r^{\tau}_{\tau(i)} \geq s_i$ holds if $r^{\tau}_{\tau(k)} \geq s_k$ for all $1 \leq k < i$. We again consider two cases.

Case 1. $E \in T_k$ and $k < i$. According to expression (4.2), we have $s_i = s_{i-1}$. We conclude immediately that $r^{\tau}_{\tau(i)} \geq r^{\tau}_{\tau(i-1)} \geq s_{i-1} = s_i$.

Case 2. $E \in T_k$ and $k \geq i$. In a way similar to the proof of $r^{\tau}_{\tau(1)} \geq s_1$, supposing conversely that $r^{\tau}_{\tau(i)} < s_i$, we can always construct an $\epsilon > 0$ and an $x \in r \nmid r_i$ such that $r^{\tau}_{\tau(i)} - \epsilon < x_i < \ldots < x_n$. It entails that (x) is a contradiction of the Nash equilibrium (r).

Hence, it holds that $r^{\tau}_{\tau(i)} \geq s_i$ for all $2 \leq i \leq n$.

In conclusion, if (r) is an arbitrary Nash equilibrium, for all $i \in N$, we have $r^{\tau}_{\tau(i)} \geq s_i$. In view of $\sum_{i\in N} r^{\tau}_{\tau(i)} = \sum_{i \in N} s_i = E$, we obtain $r^{\tau}_{\tau(i)} = s_i$, which implies that s is the unique Nash equilibrium outcome of M. From expression (4.2), it is easy to verify that $r_i = t_i$, where $t_i \in \{r^{\tau}_{\tau(j)} | j \in N, r^{\tau}_{\tau(j)} = r^{\tau}_{\tau(i)}\}$. So, we conclude that $r_i = s_i$ for all $i \in N$, which means that the Nash equilibrium outcome can be achieved by an arbitrary equivalent strategy profile (s). $\square$

Theorems 4.1 and 4.2 indicate that it is opportune for agents to follow the proposal s in M. Then, the unique Nash equilibrium outcome of M coincides with the allocation of the CEA rule. Based on this result, we say that the game M gives a non-cooperative interpretation to the CEA rule.

Herein, we present an illustrative example.

Example 4.1
Let $(E, c) \in \mathcal{B}^N$ with $N = \{1, 2\}$, $E = 10$ and $c = (6, 7)$. Then, the corresponding game M includes the following four steps, where steps 1 and 2 are in Stage 1, and steps 3 and 4 are in Stage 2.

Step 1. agent 1 announces $x_1 \in [0, 6]$;

Step 2. agent 2 announces $x_2 \in [0, \min\{7, 10 - x_1\}]$;

Step 3. agent 2 takes x_1 or x_2 as his payoff;

Step 4. agent 1 chooses the other left.

In M, the proposal is $x = (x_1, x_2)$, and the payoff vector is (x_1, x_2) if $x_1 \leq x_2$, and (x_2, x_1) if $x_1 > x_2$. By Theorems 4.1 and 4.2, $s = (s_1, s_2) = (5, 5)$ is the unique Nash equilibrium outcome. $\triangle$

However, in Example 4.1, $s = (5, 5)$ is not a subgame perfect equilibrium outcome. The reason is that if the strategies $S_1 = y_1$ and $S_2(y_1) = y_2$ satisfy the following condition:

$$y_2 = \begin{cases} 7 & \text{if } 0 \leq y_1 < 3, \\ 10 - y_1 & \text{if } 3 \leq y_1 < 6 \text{ and } y_1 \neq 5, \\ 0 & \text{if } y_1 = 5, \end{cases}$$

then agent 1 is better off announcing $y_1 \neq 5$. It means that the strategy $s_1 = 5$ is not optimal for agent 1.

Moreover, s is not a dominant strategy equilibrium outcome by Example 4.1, since $S_1 = 5$ is not a dominant strategy for agent 1. In Example 4.1, when agent 1 announces $s_1 = 5$, consequently the minimal payoff of agent 2 is 5, which equals to s_2. Hence, agent 2 can indifferently announce a number in $[0, 5]$.

Generally, the reason of disappearance of the subgame perfect equilibrium is that an agent can announce arbitrarily, if he believes that he can receive equilibrium payoff even though not announcing it by himself. It corresponds to the bankruptcy problem $(E, c) \in \mathcal{B}^N$ satisfying $\{i, j \in N | f_i^{CEA}(E, c) = f_j^{CEA}(E, c)\} \neq \emptyset$. This motivates the following proposition.

Proposition 4.1
Let $(E, c) \in \mathcal{B}^N$ and let $s = f^{CEA}(E, c)$ be the proposal in M. Then, s is the unique subgame perfect equilibrium outcome of M if $E \in T_n$ and $c_i \neq c_j$ for all $i, j \in N$ with $i \neq j$.

Proof. For each $(E, c) \in \mathcal{B}^N$ with $E \in T_n$ and $c_i \neq c_j$ for all $i \neq j$, we have $f_1^{CEA}(E, c) < \ldots < f_n^{CEA}(E, c)$. Then, we prove that s is a subgame perfect equilibrium outcome by backward induction. For each $2 \leq i \leq n$ and $1 \leq k < i$, when M proceeds to agent i, assuming that x_k is the number agent k announces, we denote

$$P^i := \{k \mid k < i, x_k \neq s_k\}.$$

For agent n, if $P^n = \emptyset$, then agent n is better off announcing $z_n = s_n$ according to Theorem 4.1. If $P^n \neq \emptyset$, agent n will announce

$$z_n = \arg \max_{y_n} \pi_n(x_1, \ldots, x_{n-1}, y_n), \tag{4.9}$$

where $\pi_n(x_1, \ldots, x_{n-1}, y_n)$ denotes the payoffs agent n receives in the end of M with respect to proposal $(x_1, \ldots, x_{n-1}, y_n)$.

For agent $n-1$, if $P^{n-1} = \emptyset$, he knows that if he announces s_{n-1}, agent n will announce s_n. According to Theorem 4.1, it is optimal for agent $n-1$ to announce $z_{n-1} = s_{n-1}$. If $P^{n-1} \neq \emptyset$, agent $n-1$ will announce

$$z_{n-1} = \arg \max_{y_{n-1}} \pi_{n-1}(x_1, \ldots, x_{n-2}, y_{n-1}, z_n),$$

where z_n is given by expression (4.9) such that $x_{n-1} = z_{n-1}$.

We analyze the next agent similarly and this process stops until agent 1. Agent 1 knows that if he announces s_1, it is optimal for each subsequent agent j to announce s_j. Theorem 4.1 has shown that agent 1 is better off announcing s_1. Hence, we conclude that, for each $i \in N$, agent i will announce s_i.

Therefore, s is a subgame perfect equilibrium outcome. It is common knowledge that a subgame perfect equilibrium outcome must be a Nash equilibrium outcome. Together with Theorem 4.2, we conclude that s is the unique subgame perfect equilibrium outcome of M. $\square$

Even though s is not a subgame perfect equilibrium outcome in most bankruptcy problems, according to Lemma 4.1, for an arbitrary $\epsilon > 0$, agent i is capable of receiving payoff more than $s_i - \epsilon$. Therefore, the game M ends with an approximation of s.

In Example 4.1, we have shown that, for each $(E, c) \in \mathcal{B}^N$, $f^{CEA}(E, c)$ is not a dominant strategy equilibrium outcome. Actually, we can verify that there is no dominant strategy equilibrium outcome in M.

Proposition 4.2
Let $(E, c) \in \mathcal{B}^N$. Then $s = f^{CEA}(E, c)$ is not a dominant strategy equilibrium outcome of M.

Proof. We only need to prove that $S_1 = s_1$ is not a dominant strategy for agent 1. We consider two cases.

Case 1. $E \in T_1$. By expression (4.2), we have $s_1 = E/n$.

We consider strategy profile $(S_1, S_2(x_1), \ldots, S_n(x_1, \ldots, x_{n-1})) \in \mathcal{S}^N$, where for each $2 \leq i \leq n$, the strategy of agent i is such that $S_i(x_1, \ldots, x_{i-1}) = x_i$ satisfying

$$x_i = \begin{cases} 0 & \text{if } x_1 = s_1, \\ \frac{x_1}{2} & \text{if } x_1 < s_1, \\ \frac{E - x_1}{n-1} & \text{if } x_1 > s_1. \end{cases}$$

Then, when agent 1 announces $x_1 = s_1$, the outcome is $(0, \ldots, 0, E/n)$; When agent 1 announces $x_1 < s_1$, the outcome is $(\frac{x_1}{2}, \ldots, \frac{x_1}{2}, x_1)$; When agent 1 announces $x_1 > s_1$, the outcome is $(\frac{E-x_1}{n-1}, \ldots, \frac{E-x_1}{n-1}, x_1)$. It is obvious that $S_1 = s_1$ is not a dominant strategy for agent 1.

Case 2. $E \in T_j$ for $1 < j \leq n$. According to expression (4.2) again, we have $s_1 = c_1$.

Similarly, we consider $(S_1, S_2(x_1), \ldots, S_n(x_1, \ldots, x_{n-1})) \in \mathcal{S}^N$, where for each $2 \leq i \leq n$, the strategy of agent i is such that $S_i(x_1, \ldots, x_{i-1}) =$

x_i satisfying

$$x_i = \begin{cases} 0 & \text{if } x_1 = s_1, \\ \frac{x_1}{2} & \text{if } x_1 < s_1. \end{cases}$$

It is easy to verify that when agent 1 announces $x_1 = s_1$, the outcome is $(0, \ldots, 0, c_1)$; and when agent 1 announces $x_1 < s_1$, the outcome is $(\frac{x_1}{2}, \ldots, \frac{x_1}{2}, x_1)$. Therefore, $S_1 = c_1$ is not a dominant strategy.

In conclusion, $S_1 = s_1$ is not a dominant strategy. It follows that $s = f^{CEA}(E, c)$ is not a dominant strategy equilibrium outcome. $\square$

As is known to all, a dominant strategy equilibrium outcome must be a Nash equilibrium outcome. Together with Theorem 4.2 and Proposition 4.2, we conclude that there is no dominant strategy equilibrium outcome in M. The reason is that a dominant strategy of an agent means that the strategy is at least as good as another strategy, regardless of the strategies of the other agents. However, in M, the others can unite in reducing one's payoff by playing special strategies even although these strategies may derive a lose-lose scenario.

In the next section, we introduce another mechanism to deal with bankruptcy problems by substituting the choice procedure in Stage 2 of M to a bargaining procedure. For the same reason, the dominant strategy equilibrium outcome also does not exist in the corresponding non-cooperative games. So we only focus on the Nash equilibrium outcome. It turns out that the unique Nash equilibrium outcome is again consistent with the allocation of the CEA rule, which is also a subgame perfect equilibrium outcome.

4.4 The divide-and-object game

In this section, we introduce a divide-and-object mechanism to solve bankruptcy problems. Given an arbitrary bankruptcy problem $(E, c) \in \mathcal{B}^N$, we can obtain the following divide-and-object game, denoted by

$M'(E,c)$. When no confusion arises, the notation $M'(E,c)$ is abbreviated to M' for convenience.

The divide-and-object game M':

Stage 1: Each agent $i \in N$ successively announces a number $x_i \in \mathbb{R}_+$ in ascending order of the claims. The number agent i announces cannot exceed his claim and the current residual estate, that is,

$$x_i \in \begin{cases} [0, \min\{c_i, E\}] & \text{if } i = 1, \\ \left[0, \min\{c_i, E - \sum_{k=1}^{i-1} x_k\}\right] & \text{if } 2 \leq i \leq n. \end{cases}$$

Similarly, $x = (x_1, \ldots, x_n)$ is called a *proposal*. For distinct $i, j \in N$, an *objection* of i against j with respect to x is denoted by Γ_{ij}, if

$$x_i < \phi_i(x_i + x_j, (c_i, c_j)), \tag{4.10}$$

where ϕ is the following bilateral principle for bankruptcy problem $(E, c) \in \mathcal{B}^N$:

$$\phi(E,c) = \begin{cases} (E/2, E/2) & \text{if } 0 < E < 2c_1, \\ (c_1, E - c_1) & \text{if } 2c_1 \leq E < c_1 + c_2. \end{cases} \tag{4.11}$$

We say that there is no objection for agent i if and only if it holds that for $j \neq i$,

$$\phi_i(x_i + x_j, (c_i, c_j)) = x_i. \tag{4.12}$$

Note that if there is no objection for agent i, then agent i cannot object against any other agent, and any other agent cannot object against him.

Stage 2: All agents try to object against other agents. An agent receives nothing as long as he is objected by others. If an agent has the opportunity to object against more than one agent, he will naturally object against the one who announces the largest number. If more than one agent announce the largest number, the target is chosen randomly. It turns out that for each $i \in N$, one of the following cases will occur:

1. There is no objection for agent i. Agent i receives payoff x_i when the game ends.

2. Γ_{ji} arises. Agent i receives 0 when the game ends.

3. Γ_{ki} does not exist for all $k \in N \setminus \{i\}$, but Γ_{ij} arises. The game ends with the outcome that agent i obtains $\phi_i(x_i + x_j, (c_i, c_j))$.

We state some explanations with regards to the game M'. Firstly, one gets 0 as long as he is objected, so he carefully announces his own number to avoid this severe punishment. Secondly, by expression (4.10), agent i receives more if he can object. It is positive that the excess motivates agents to object against others.

Next, we consider the Nash equilibrium of M'. Much like in M, an agent's strategy in M' can be simplified as his strategy in Stage 1. Note that Stage 1 of two games are the same. When there is no confusion, we continue to use the notation (x) to refer to an arbitrary equivalent strategy profile in M' which uniquely determines the proposal x. The same applies to the other notations.

Definition 4.3
Let $(E, c) \in \mathcal{B}^N$. Then a strategy profile $S_N^* \in \mathcal{S}^N$ is a Nash equilibrium of M' if for all $i \in N$ and all $S_i \in \mathcal{S}^i$,

$$\pi_i(S_N^*) \geq \pi_i(S_i, S_{-i}^*), \tag{4.13}$$

where $\pi_i(S_N^*)$ and $\pi_i(S_i, S_{-i}^*)$ denotes the payoffs agent i receives in the end of M'.

For simplicity, for a strategy profile $(x) \in \mathcal{S}^N$, we write $\pi_i(x) \geq \pi_i(z)$ in expression (4.13), where $(z) \in \mathcal{S}^N$ satisfying $z \in x \nmid x_i$. Similarly with expression (4.3), inequality (4.13) indicates that agent i cannot receive more by first not announcing x_i in M'. If the inequality holds for all

$i \in N$, we say that (x) is a Nash equilibrium of M'. In M, we naturally assume that a rational agent must choose the largest number he can choose. Herein, we assume that a rational agent in M' must not be objected by his predecessors since he has known their strategies before he starts to announce.

In the following, we show that there is a unique Nash equilibrium outcome of M' and it coincides with the allocation of the CEA rule. We proceed by two steps. First, we construct a Nash equilibrium (s) of M', where the outcome equals to the allocation of the CEA rule (in Theorem 4.3). Then, we verify uniqueness (in Theorem 4.4).

To prove these results, we first introduce some lemmas to show how objections form among agents.

Lemma 4.2
Let $(E, c) \in \mathcal{B}^N$ and let x be a proposal in M'. Then for $i, j \in N$ with $i < j$, Γ_{ji} is formed if $x_i > x_j$.

Proof. If $x_i > x_j$, we have $x_i + x_j < 2x_i \leq 2c_i$. According to expression (4.11), it holds that

$$\phi(x_i + x_j, (c_i, c_j)) = (\frac{x_i + x_j}{2}, \frac{x_i + x_j}{2}).$$

Since $\phi_i(x_i + x_j, (c_i, c_j)) = \frac{x_i+x_j}{2} < x_i$, with expression (4.10), we conclude that there is an objection Γ_{ji}. $\square$

Lemma 4.3
Let $(E, c) \in \mathcal{B}^N$ and let x be a proposal in M'. Then agent i will be objected if $x_i > E/n$ for $i = 1$ and $x_i > \frac{E - \sum_{k=1}^{i-1} x_k}{n-i+1}$ for $2 \leq i \leq n$.

Proof. If $x_1 > E/n$, we argue that there is at least an agent $j \geq 2$ such that $x_j < E/n$. On the contrary, assume that $x_j \geq E/n$ for all $2 \leq j \leq n$. Together with $x_1 > E/n$, it holds that $\sum_{i=1}^{n} x_i > E$, which is a contradiction. Hence, we have $x_j < E/n < x_1$. According to Lemma 4.2, Γ_{j1} is formed.

If $x_i > \frac{E-\sum_{k=1}^{i-1} x_k}{n-i+1}$ for $2 \leq i \leq n$, then we can similarly verify that there is at least an agent $j > i$ such that $x_j < \frac{E-\sum_{k=1}^{i-1} x_k}{n-i+1} < x_i$. Using Lemma 4.2, Γ_{ji} is formed. □

Lemma 4.4
Let $(E,c) \in \mathcal{B}^N$ and let x be a proposal in M'. Then for distinct $i, j \in N$, there is an objection Γ_{ji} if $x_i > x_j$ and $x_j < c_j$.

Proof. If $i < j$, according to Lemma 4.2, we conclude that there is an objection Γ_{ji}. If $j < i$, in view of expression (4.11), we have

$$\phi_i(x_j + x_i, (c_j, c_i)) = \begin{cases} \frac{x_j+x_i}{2} & \text{if } x_j + x_i < 2c_j, \\ x_j + x_i - c_j & \text{if } x_j + x_i \geq 2c_j. \end{cases}$$

It is easy to verify that $\phi_i(x_i + x_j, (c_i, c_j)) < x_i$. With expression (4.10), Γ_{ji} is formed immediately. □

Based on Lemmas 4.2, 4.3 and 4.4, we show the main results in M'.

Theorem 4.3
Let $(E,c) \in \mathcal{B}^N$. Then each strategy profile $S_N \in \mathcal{S}^N$ with $\pi(S_N) = f^{CEA}(E,c)$ is a Nash equilibrium of M'.

Proof. Let $S_N \in \mathcal{S}^N$, and let $s = f^{CEA}(E,c)$ be the corresponding proposal of S_N. First, we verify that $\pi(S_N) = \pi(s) = f^{CEA}(E,c)$. Since the CEA rule satisfies bilateral consistency, we have for distinct $i, j \in N$,

$$(s_i, s_j) = (\phi_i(s_i + s_j, (c_i, c_j)), \phi_j(s_i + s_j, (c_i, c_j))).$$

According to expression (4.12), there are no objections among agents. Therefore, the game M' ends with the outcome $\pi(s) = s = f^{CEA}(E,c)$.

Then, we prove that (s) is a Nash equilibrium. To prove this, we only need to prove that $\pi_i(s) \geq \pi_i(z)$ for all $i \in N$ and $(z) \in \mathcal{S}^N$ satisfying $z \in s \nmid s_i$.

For $i = 1$, we consider two cases.

Case 1. $E \in T_1$. With expression (4.2), we have $f_j^{CEA}(E, c) = E/n$ for all $j \in N$.

For each $y_1 \neq s_1$, we consider $z \in s|y_1$. According to Lemma 4.3, agent 1 will be objected if $y_1 > s_1 = E/n$ and $\pi_1(z) = 0$. Hence, we consider $y_1 < s_1$. With Lemma 4.4 and $y_1 < s_1 \leq c_1$, Γ_{1j} arises if $z_j > y_1$ for each $j \geq 2$. To avoid being objected, each subsequent agent j will announce $z_j \leq y_1$. It follows that $\pi_1(z) = y_1 < s_1$ if there is no objection for agent 1, and $\pi_1(z) = 0$ if he is objected. Hence, we have $\pi_1(s) > \pi_1(z)$ for all $z \in s|y_1$.

Case 2. $E \in \underset{2 \leq j \leq n}{\cup} T_j$. In this case, we have $f_1^{CEA}(E, c) = c_1$ and $f_j^{CEA}(E, c) \geq c_1$ for all $j \geq 2$.

For each $y_1 < s_1 = c_1$, we consider $z \in s|y_1$. In view of Lemma 4.4, every subsequent agent j is better off announcing $z_j \leq y_1$ to avoid being objected. With the similar analysis in Case 1, we conclude that $\pi_1(s) > \pi_1(z)$ for all $z \in s|y_1$.

Then, for each $2 \leq i \leq n$, we also consider two cases.

Case 1. $E \in \underset{1 \leq k \leq i}{\cup} T_k$. We have $f_j^{CEA}(E, c) = \frac{E - \sum_{k=1}^{i-1} f_k^{CEA}}{n-i+1}$ for all $j \geq i$.

For each $y_i \neq s_i$, we consider $z \in s|y_i$. With Lemma 4.3, we know that agent i will be objected by his predecessors and receive nothing if $y_i > s_i$. So, we consider $y_i < s_i$. With Lemma 4.2, every subsequent agent j will announce $z_j \leq y_i$ to avoid being objected. So, agent i can only object to his predecessors if possible. Then, we have $\pi_i(z) = 0$ if agent i is objected, $\pi_i(z) = \phi_i(s_{i-1} + y_i, (c_{i-1}, c_i))$ if no one objects to agent i but $\Gamma_{i(i-1)}$ arises, and $\pi_i(z) = y_i$ if there is no objection for agent i. It is easy to verify that $\pi_i(z) < s_i$, and hence, $\pi_i(s) > \pi_i(z)$ for all $z \in s|y_i$.

Case 2. $E \in \underset{i<j\leq n}{\cup} T_j$. We have $f_k^{CEA}(E, c) = c_k$ for all $k \leq i$.

If agent i announces $y_i < s_i = c_i$, similarly, it holds that $\pi_i(s) > \pi_i(z)$ for all $z \in s|y_i$.

In short, we verify that $\pi_i(s) > \pi_i(z)$ for all $i \in N$ and $(z) \in \mathcal{S}^N$ satisfying $z \in s \nmid s_i$. It implies that (s) is a Nash equilibrium of M' with outcome $f^{CEA}(E, c)$. □

Theorem 4.4
Let $(E, c) \in \mathcal{B}^N$. Then the unique Nash equilibrium outcome of M' is $f^{CEA}(E, c)$.

Proof. Without loss of generality, let (r) satisfying $r = (r_1, \ldots, r_n)$ be an arbitrary Nash equilibrium of M'. We denote

$$j = \min\{i \in N | r_i \neq s_i\}.$$

According to Theorem 4.3, there is an (s) satisfying $s \in r|r_i$ such that $\pi_j(r) < \pi_j(s)$, which is a contradiction of the Nash equilibrium (r). So, we conclude that (r) and (s) are equivalent to each other, *i.e.*, $r = s$. It means that s is the unique Nash equilibrium outcome of M'. Obviously, it can be achieved by a strategy profile (s). □

Theorems 4.3 and 4.4 show that the allocation of the CEA rule is the unique Nash equilibrium outcome of M'. Moreover, it is also a subgame perfect equilibrium outcome.

Theorem 4.5
Let $(E, c) \in \mathcal{B}^N$. Then the unique subgame perfect equilibrium outcome of M' is $f^{CEA}(E, c)$.

We omit the proof of Theorem 4.5 since it is similar to the proof of Proposition 4.1. It is an immediate conclusion from Theorems 4.3 and 4.4. In other words, the game M' also gives a non-cooperative interpretation to the CEA rule.

4.5 Concluding remarks

Our games are related to Dagan et al (1997), Chang and Hu (2008), Li and Ju (2014) and Tsay and Yeh (2019). Dagan et al (1997) introduced a game form using bilateral principle to implement consistent bankruptcy rules. In the game, one of the agents with the largest claim proposes an allocation, called a proposal, and other agents make sequential responses to the proposal by either accepting or rejecting it. The agent who accepts the proposal receives the corresponding payoff of it, and the agent who rejects the proposal receives his payoff determined by a predetermined bilateral principle.

Chang and Hu (2008) considered a two-stage game form to implement consistent bankruptcy rules. In Stage 1, each agent proposes an allocation and a permutation of the agents. If all the agents propose the same allocation, the unique allocation is called the proposal and the composition of the permutations decides an agent to the current proposer. Then, the game proceeds to Stage 2; Otherwise, the game ends with a non-positive payoff vector. In Stage 2, the current proposer either accepts the proposal or rejects it. In case of acceptance, the proposal is the outcome of the game. In case of rejection, the proposer chooses an agent to negotiate by a bilateral principle, and the residual agents receive the payoffs specified in the proposal.

Li and Hu (2014) proposed a repetitive two-step game form to implement bankruptcy rules. Herein, we only focus on the game implementing the CEA rule. In the first round, one of the agents with the largest claim divides the estate into n non-negative parts satisfying efficiency in Step 1. In Step 2, each agent successively chooses a part in ascending order of the claims. If the chosen part does not exceed his claim, he receives this part and leaves the game. If an agent firstly chooses the part exceeding his claim, he receives his claim and leaves the game. Then, the next round starts. The residual agents divide the residual estate by repeating Step 1 and Step 2. The game ends until every agent receives a payoff.

Tsay and Yeh (2019) first introduced strategic implementations of three bankruptcy rules for bankruptcy problems with two agents. Then, they considered a three-stage game to implement them. In Stage 1, each agent announces an allocation and a permutation of the agents. The composition of the permutations decides an agent to the coordinator. If all the agents, except possibly for the coordinator, announce the same allocation, the unique allocation is the proposal; Otherwise, the allocation announced by the coordinator is the proposal. In Stage 2, the coordinator either accepts the proposal or rejects it. In case of acceptance, the proposal is the outcome of the game. In case of rejection, the proposer chooses an agent to negotiate. In Stage 3, the coordinator and the chosen agent play a two-person non-cooperative game, and the residual agents receive the payoffs specified in the proposal.

However, our games are different from their games in the following points.

(1) The way of forming a proposal: In Dagan et al (1997) and Li and Ju (2014), one of the agents with the largest claim announces a proposal. In Chang and Hu (2008) and Tsay and Yeh (2019), every agent announces a proposal simultaneously. Unlike them, our games provide a sequential partition method, in which each agent successively announces a number to form a proposal. Specifically, each agent has the right to propose, but only determines a part of the proposal.

(2) Implementing efficiency: In Dagan et al (1997), Chang and Hu (2008) and Tsay and Yeh (2019), a feasible proposal is an allocation (satisfying efficiency). In Li and Ju (2014), it is a division of the estate satisfying efficiency. However, a proposal in our games is not necessarily efficient. The absence of efficiency is fair to guarantee that the

last agent has more strategies than announcing uniquely. Although an efficient proposal is not required in our games, the unique Nash equilibrium outcome satisfies efficiency.

(3) Divide-and-choose processes: Li and Ju (2014) and Tsay and Yeh (2019) also mentioned divide-and-choose games to implement the CEA rule. However, there are essential differences between our game M and their games. In Li and Ju (2014), the partition process is accomplished by one agent, and the selection process is implemented in ascending order of the claims. Furthermore, their game is designed in a repetitive way. In Tsay and Yeh (2019), their game is only defined in two-person bankruptcy problems, in which one agent proposes a division, and the other responds by either accepting or rejecting it. Compared to their games, our game M is designed for all bankruptcy problems. The partition process is accomplished by everyone in ascending order of the claims, and the selection process is implemented in reverse order. Moreover, if we change the ascending order in M to other partition orders, the unique Nash equilibrium outcome disappears.

(4) Negotiation processes: Dagan et al (1997) and Chang and Hu (2008) mentioned bargaining games to implement the CEA rule. Besides the differences in forming a proposal, our game M' has two main differences compared to theirs. First, one agent in M' can negotiate with an arbitrary other agent if an objection arises between them. However, in Dagan et al (1997) and Chang and Hu (2008), a negotiation only arises along with a particular agent. Next, when two agents play a bilateral negotiation, one receives 0 if he is objected in M', and the other receives a payoff decided by the bilateral principle. However, in Dagan et al (1997) and Chang and Hu (2008), two agents, who play a bilateral negotiation, both receive the amounts in this bilateral principle.

Furthermore, the non-cooperative interpretations of the CEL rule and the PRO rule can be given in a similar way.

The CEL rule: As the dual of the CEA rule, the CEL rule can be implemented similarly if for each $(E, c) \in \mathcal{B}^N$, we treat the losses $L =$

$\sum_{i\in N} c_i - E$ as a homogeneous cake and deal with it using the same processes in both M and M'.

In addition, if we define the bilateral principle in expression (4.11) by for each $(E,c) \in \mathcal{B}^N$,

$$\phi(E,c) = \begin{cases} (0, E) & \text{if } 0 < E < c_2 - c_1, \\ (\frac{E+c_1-c_2}{2}, \frac{E-c_1+c_2}{2}) & \text{if } c_2 - c_1 \leq E < c_1 + c_2, \end{cases}$$

then $g^{CEL}(E,c)$ is the unique Nash equilibrium outcome of M'. However, it is no longer a subgame perfect equilibrium outcome. The reason is that one agent may obtain 0 if he follows the proposal $g^{CEL}(E,c)$, which means that he can announce arbitrarily without receiving less. Moreover, in M', the ascending partition order can be changed to an arbitrary partition order, which also leads to the unique Nash equilibrium outcome $g^{CEL}(E,c)$.

The PRO rule: If we define the bilateral principle in expression (4.11) by for each $(E,c) \in \mathcal{B}^N$,

$$\phi(E,c) = (\frac{c_1}{c_1+c_2}E, \frac{c_2}{c_1+c_2}E),$$

then $g^{PRO}(E,c)$ is the unique Nash equilibrium outcome of M'. Moreover, it is also a subgame perfect equilibrium outcome. Similarly, the partition order can be given randomly, which leads to the same equilibrium outcome.

In summary, a sequential partition method for non-cooperative games of bankruptcy problems is given in this chapter, in which a proposal is formed sequentially. It is not only concise but also effective, which is able to interpret a family of bankruptcy rules. We can also combine the sequential partition process with other non-cooperative processes. More interesting games are worth to be explored in the future.

5

Mechanisms for division problems with single-dipped preferences

Adapted from: Gong, D., B. Dietzenbacher, and H. Peters. *Mechanisms for division problems with single-dipped preferences*. WorkingPaper 007. Maastricht University, Graduate School of Business and Economics, 2022.

5.1 Introduction

We consider the problem of allocating one unit of an infinitely divisible commodity among agents with single-dipped preferences. A single-dipped preference has a worst point, the dip, and preference strictly increases in both directions away from the dip. Such a preference may arise from maximizing a strictly quasiconvex utility function on a (budget) line, and reflects that an agent prefers extremes over combinations – for instance, a university employee may prefer either only teaching or only research over a combination of the two.

We take a mechanism design approach: each agent reports a number between zero and one, and a mechanism is a map assigning an allocation of the commodity among the agents, which is evaluated by the agents according to their preferences. Under a number of conditions on mechanisms, we analyze the Nash, Pareto optimal Nash, and strong equilibria for each single-dipped preference profile, and the resulting allocations, in the induced game. Mechanisms are related to (social choice) rules: these assign an allocation to each profile of preferences. In particular, a rule which only depends on the dips of the reported preferences, gives rise to a mechanism.

Almost throughout, we assume that a mechanism is anonymous and monotonic. The latter condition means that if an agent reports a higher or lower number, then that agent's share increases or decreases, if possible. The motivation for this monotonicity requirement is that it provides the agents with ample strategic possibilities to influence their shares – thus, it makes the mechanism highly sensitive to the strategies of the agents.

After preliminaries in Section 5.2, in Section 5.3 we discuss Nash equilibria of games induced by a mechanism and single-dipped preference profiles. The main insight here is that in every Nash equilibrium each agent plays 0 or 1, and we characterize all Nash equilibria (Theorem 5.1). If there are two agents then a Nash equilibrium always exists (Proposition 5.1), but this is no longer true for more than two agents.

In Section 5.4 we consider Pareto optimal Nash equilibria, and we show that an additional condition on a mechanism, namely that when every agent plays 0 or 1, the agents who play 0 receive 0 and the agents who play 1 equally share the commodity, is necessary and sufficient for the existence of a Pareto optimal Nash equilibrium for all games, i.e., all preference profiles. Moreover, in this case the Pareto optimal Nash equilibria are exactly those strategy profiles where agents in a so-called maximal coalition play 1 and the other agents play 0 – 'maximal' means that as many agents as possible (given the restrictions of best reply and Pareto optimality) play 1 and get a positive share. Under the further condition of order-preservation on a mechanism – meaning that playing a higher number than another agent results in obtaining a higher share than that agent – these Pareto optimal Nash equilibria are, moreover, strong equilibria (Aumann, 1959): no coalition can profitably deviate. As a consequence, under the mentioned conditions on a mechanism a selection – denoted by M – of the Pareto social choice correspondence is implemented in strong equilibrium, namely picking the Pareto optimal allocations that are characterized by so-called maximal coalitions: this means that outside agents prefer getting zero over equally sharing the one unit with the agents in the coalition, whereas for agents in the coalition the opposite holds.

Sprumont (1991) shows that under a few natural conditions, the so-called uniform rule is the unique strategy-proof (Gibbard, 1973; Satterthwaite, 1975) rule for division problems with single-peaked preferences – a preference is single-peaked if there is a unique best point, the peak, and preference decreases in both directions away from this peak. Bochet et al (2021) – combining work of Bochet and Sakai (2009) and Thomson (2009) – show that under similar assumptions as ours, equilibria (Nash, Pareto optimal Nash, strong) end up in the allocation assigned by the uniform rule – see Section 5.4.4. While the uniform rule for single-peaked preferences is strategy-proof, we show in Section 5.4.4 that no selection from the implemented correspondence M for single-dipped preferences is strategy-proof.

Single-dipped and single-peaked preferences were already studied by

Inada (1964). For single-dipped preferences in division problems, see Klaus et al (1997), who characterize Pareto optimal allocations (we use their result in Section 5.4), and study strategy-proofness of rules. For strategy-proofness in problems with indivisible goods and single-dipped preferences see Klaus (2001a, 2001b) and Tamura (2022), and for probabilistic rules see Ehlers (2002). Doghmi (2013) shows that Maskin-monotonicity is still a necessary condition for implementation; indeed, it is not difficult to show that the correspondence M is Maskin monotonic.

There is a relatively large literature on single-dipped preferences and public goods (also sometimes called public bads), including Peremans and Storcken (1999), Barberà et al (2012), Bossert and Peters (2014), Öztürk et al (2013, 2014), Manjunath (2014), Ayllón and Caramuta (2016), Tapki (2016), Yamamura (2016), Lahiri et al (2017), and Feigenbaum et al (2020).

5.2 Preliminaries

In this section we introduce allocations, preferences, mechanisms, rules, and equilibria.

5.2.1 Allocations, preferences, mechanisms, and equilibria

For $n \in \mathbb{N}$ with $n \geq 2$, let $N = \{1, \ldots, n\}$ be the set of *agents*. Among these agents one unit of a perfectly divisible good has to be distributed. The set of all *allocations* is denoted by $\mathcal{A} = \left\{x \in [0,1]^N \mid \sum_{i \in N} x_i = 1\right\}$. A subset of agents is also called a *coalition*.

An agent's *preference* is a transitive and complete binary relation R on the interval $[0,1]$. We denote by P *strict* preference, and by I *indifference*: $\alpha P \beta$ if $\alpha R \beta$ and not $\beta R \alpha$, and $\alpha I \beta$ if $\alpha R \beta$ and $\beta R \alpha$, for $\alpha, \beta \in [0,1]$. By $R_N = (R_i)_{i \in N}$ we denote a *profile* of preferences (for N).

An allocation $x \in \mathcal{A}$ is *Pareto optimal* at a preference profile R_N if there is no $x' \in \mathcal{A}$ such that $x'_i R_i x_i$ for all $i \in N$ and $x'_i P_i x_i$ for at least one $i \in N$.

In this chapter we focus on mechanisms in order to select allocations. A *mechanism* is a map $g : [0,1]^N \to \mathcal{A}$. A preference profile R_N and a mechanism g induce a non-cooperative game (R_N, g) as follows. Each agent $i \in N$ has *strategy set* $[0,1]$. A *profile* of strategies $r = (r_i)_{i \in N} \in [0,1]^N$ results in an allocation $g(r) \in \mathcal{A}$, evaluated by each agent i via R_i. A profile r^* is a *Nash equilibrium* of the game (R_N, g) if for all $i \in N$ and $r_i \in [0,1]$,

$$g_i(r^*) R_i g_i(r_i, r^*_{-i}),$$

where $r^*_{-i} = (r^*_j)_{j \in N \setminus \{i\}}$. A Nash equilibrium r^* is a *Pareto optimal Nash equilibrium* in the game (R_N, g) if $g(r^*)$ is Pareto optimal at R_N. A profile r^* is a *strong equilibrium* if there are no $\emptyset \neq S \subseteq N$ and $r'_S \in [0,1]^S$ such that

$$g_i(r'_S, r^*_{N \setminus S}) R_i g_i(r^*) \text{ for all } i \in S$$

and

$$g_j(r'_S, r^*_{N \setminus S}) P_j g_j(r^*) \text{ for some } j \in S,$$

where $r^*_{N \setminus S} = (r^*_i)_{i \in N \setminus S}$.

In most of this chapter we focus on single-dipped preferences. A preference R is *single-dipped* if there is a *dip* $d(R) \in [0,1]$ such that for all $\alpha, \beta \in [0,1]$,

$$\alpha < \beta \leq d(R) \Rightarrow \alpha P \beta \text{ and } \alpha > \beta \geq d(R) \Rightarrow \alpha P \beta.$$

The set of all single-dipped preferences is denoted by $\mathcal{D}$, and $\mathcal{D}^N$ is the set of all single-dipped preference profiles.

A preference R is called *single-peaked* if there is a *peak* $p(R) \in [0,1]$ such that for all $\alpha, \beta \in [0,1]$,

$$p(R) \geq \alpha > \beta \Rightarrow \alpha P \beta \text{ and } p(R) \leq \alpha < \beta \Rightarrow \alpha P \beta.$$

The set of all single-peaked preferences is denoted by $\mathcal{P}$, and $\mathcal{P}^N$ is the set of all single-peaked preference profiles.

5.2.2 Mechanisms versus rules

A mechanism is – indeed – a mechanical device that is used to non-cooperatively determine an allocation, given a strategy profile. A (*social choice*) *rule* is a map φ assigning to each preference profile within a given set, an allocation. If such a rule φ on $\mathcal{D}^N$ or on $\mathcal{P}^N$ depends only on the dips or only on the peaks of a preference profile (i.e., is *dips-only* or *peaks-only*), then it can be identified with a mechanism according to our definition. An agent's strategy can then be interpreted as the agent reporting a dip or peak – not necessarily the true dip or peak. In this sense, the peaks-only rules for single-peaked preference profiles studied in for instance Sprumont (1991) or Bochet et al (2021) can be seen as mechanisms. On the other hand, a property like Pareto optimality makes sense for rules (meaning that they assign a Pareto optimal allocation to each preference profile), but not for mechanisms, which are defined independently of preference profiles. In most of what follows, we impose the following additional conditions on a mechanism g:

- *anonymity*: $g_i(r^\pi) = g_{\pi(i)}(r)$ for all $r \in [0,1]^N$ and every permutation π of N, where $r^\pi = (r_{\pi(i)})_{i \in N}$.
- *monotonicity*: for all $r \in [0,1]^N$, $i \in N$ and $r'_i \in [0,1]$,

$$r'_i > r_i \text{ and } g_i(r) < 1 \Rightarrow g_i(r'_i, r_{-i}) > g_i(r),$$

$$r'_i < r_i \text{ and } g_i(r) > 0 \Rightarrow g_i(r'_i, r_{-i}) < g_i(r),$$

  where (r'_i, r_{-i}) is obtained from r by replacing r_i by r'_i.

The set of all anonymous and monotonic mechanisms is denoted by $\mathcal{G}$.

The monotonicity condition is closely related to the condition of 'strict own-peak monotonicity' in Bochet et al (2021) when the latter is applied to rules that are peaks-only. The difference is that the condition in Bochet et al (2021) allows that an agent i receives 0 when that agent's strategy r_i is positive. Under our monotonicity condition this is not possible (see Lemma 5.2).

We conclude this section with two examples of mechanisms in $\mathcal{G}$.

Example 5.1
Let $N = \{1, 2\}$ and let $g : [0,1]^N \to \mathcal{A}$ be defined by for each $r \in [0,1]^N$,

$$g(r) = \left(\frac{1 + r_1 - r_2}{2}, \frac{1 - r_1 + r_2}{2} \right).$$

Then g is anonymous and monotonic, and thus $g \in \mathcal{G}$. △

Example 5.2
Let $g : [0,1]^N \to \mathcal{A}$ be defined by for each $r \in [0,1]^N$ and $i \in N$,

$$g_i(r) = \begin{cases} \frac{r_i}{\sum_{j \in N} r_j} & \text{if } \sum\limits_{j \in N} r_j \geq 1 \\ 1 - \frac{(n-1)(1-r_i)}{n - \sum_{j \in N} r_j} & \text{if } \sum\limits_{j \in N} r_j \leq 1. \end{cases}$$

This mechanism corresponds to the 'symmetrized proportional rule' in Bochet et al (2021). Again, g is anonymous and monotonic, and therefore $g \in \mathcal{G}$. △

In the next two sections we analyze Nash equilibria, Pareto optimal Nash equilibria, and strong equilibria in games with single-dipped preference profiles, induced by mechanisms in $\mathcal{G}$.

5.3 Nash equilibrium

Before stating the main results, we formulate two elementary lemmas concerning single-dipped preferences and mechanisms, respectively.

The first lemma recalls the well-known fact (Inada, 1964) that if an agent with a single-dipped preference prefers α to β in $[0, 1]$, then this agent prefers α to each γ between α and β. This will be used several times in the sequel.

Lemma 5.1
Let $R \in \mathcal{D}$ and let $\alpha, \beta \in [0, 1]$ with $\alpha R \beta$. Then $\alpha R \gamma$ for all $\gamma \in [0, 1]$ with $\min\{\alpha, \beta\} \leq \gamma \leq \max\{\alpha, \beta\}$.

Proof. If $d(R) \leq \min\{\alpha, \beta\}$, then $d(R) \leq \beta \leq \alpha$, so $\alpha R \gamma$ for all $\beta \leq \gamma \leq \alpha$. If $d(R) \geq \max\{\alpha, \beta\}$, then $d(R) \geq \beta \geq \alpha$, so $\alpha R \gamma$ for all $\alpha \leq \gamma \leq \beta$. If $\min\{\alpha, \beta\} < d(R) < \max\{\alpha, \beta\}$, then we have $\alpha R \gamma$ for all $\min\{\alpha, d(R)\} \leq \gamma \leq \max\{\alpha, d(R)\}$, and $\alpha R \beta R \gamma$ for all $\min\{\beta, d(R)\} \leq \gamma \leq \max\{\beta, d(R)\}$. Therefore, $\alpha R \gamma$ for all $\min\{\alpha, \beta\} \leq \gamma \leq \max\{\alpha, \beta\}$. □

The next lemma shows that a monotonic mechanism assigns 0 to an agent only if his strategy is 0, and assigns 1 to an agent only if his strategy is 1.

Lemma 5.2
Let g be a monotonic mechanism and let $r \in [0, 1]^N$. Then $r_i = 0$ for each $i \in N$ with $g_i(r) = 0$, and $r_i = 1$ for each $i \in N$ with $g_i(r) = 1$.

Proof. For each $i \in N$ with $g_i(r) = 0$, if $r_i \neq 0$, then $g_i(r'_i, r_{-i}) = 0$ for all $0 \leq r'_i < r_i$, which contradicts monotonicity of g. For each $i \in N$ with $g_i(r) = 1$, if $r_i \neq 1$, then $g_i(r'_i, r_{-i}) = 1$ for all $r_i < r'_i \leq 1$, which again contradicts monotonicity of g. □

The following two lemmas are about properties of Nash equilibria for single-dipped preference profiles. We first show that for a monotonic mechanism and a single-dipped preference profile, no agent receives his dip in a Nash equilibrium.

Lemma 5.3
Let $R_N \in \mathcal{D}^N$ and let g be a monotonic mechanism. If a strategy profile $r^* \in [0,1]^N$ is a Nash equilibrium of (R_N, g), then $g_i(r^*) \neq d(R_i)$ for all $i \in N$.

Proof. Let $i \in N$. Assume, to the contrary, that $r^* \in [0,1]^N$ with $g_i(r^*) = d(R_i)$, is a Nash equilibrium of (R_N, g). Then we have $g_i(r_i, r^*_{-i}) = d(R_i)$ for all $r_i \in [0,1]$, which is a contradiction to monotonicity of g. □

Next, we show that, in a Nash equilibrium, an agent's strategy is 0 if his allocation is less than his dip, and is 1 if his allocation is more than his dip .

Lemma 5.4
Let $R_N \in \mathcal{D}^N$, let g be a monotonic mechanism, and let strategy profile $r^* \in [0,1]^N$ be a Nash equilibrium of (R_N, g). Then $r^*_i = 0$ for all $i \in N$ with $g_i(r^*) < d(R_i)$, and $r^*_i = 1$ for all $i \in N$ with $g_i(r^*) > d(R_i)$.

Proof. Let $i \in N$ with $g_i(r^*) < d(R_i)$. If $g_i(r^*) = 0$, then $r^*_i = 0$ by Lemma 5.2. If $g_i(r^*) > 0$ with $r^*_i \neq 0$, then from monotonicity, we have $g_i(r_i, r^*_{-i}) < g_i(r^*) < d(R_i)$ for all $0 \leq r_i < r^*_i$. This implies that $g_i(r_i, r^*_{-i}) P_i g_i(r^*)$, which is a contradiction to the assumption that r^* is a Nash equilibrium. Hence, $r^*_i = 0$.

The case $g_i(r^*) > d(R_i)$ is analogous. □

We now introduce some additional notation for a mechanism $g \in \mathcal{G}$. For each $S \subseteq N$, define $e^S \in \mathbb{R}^N$ by $e^S_i = 1$ for all $i \in S$, and $e^S_j = 0$ for all $j \in N \setminus S$. Then, by anonymity we have $g_i(e^\emptyset) = g_i(e^N) = \frac{1}{n}$ for all $i \in N$, and there exist numbers $p^1(g), \ldots, p^{n-1}(g) \in [0,1]$ such that for each $S \in 2^N \setminus \{\emptyset, N\}$ and $i \in S$,

$$g_i(e^S) = p^s(g),$$

where $s = |S|$. It follows that for each $S \in 2^N \setminus \{\emptyset, N\}$ and $j \in N \setminus S$,

$$g_j(e^S) = q^s(g),$$

where $sp^s(g) + (n - s)q^s(g) = 1$ for all $s = 1, \ldots, n-1$. When no confusion arises, the notations $p^s(g)$ and $q^s(g)$ are abbreviated to p^s and q^s, respectively. For convenience, we denote $p^0 = p^n = q^0 = q^n = \frac{1}{n}$. Then, by monotonicity and Lemma 5.2, it holds that for each $i \in N$ and $S \subseteq N \setminus \{i\}$,

$$p^{s+1} = g_i(e^{S \cup \{i\}}) > g_i(e^S) = q^s.$$

The following theorem characterizes the Nash equilibria in games induced by mechanisms in $\mathcal{G}$.

Theorem 5.1
Let $R_N \in \mathcal{D}^N$, $g \in \mathcal{G}$, and let $r^* \in [0,1]^N$. Then r^* is a Nash equilibrium of (R_N, g) if and only if $r^* = e^S$ for $S \in 2^N$ such that $p^s R_i q^{s-1}$ for all $i \in S$ and $q^s R_j p^{s+1}$ for all $j \in N \setminus S$.

Proof. For the if-part, assume that $r^* = e^S$ for $S \in 2^N$ such that $p^s R_i q^{s-1}$ for all $i \in S$ and $q^s R_j p^{s+1}$ for all $j \in N \setminus S$. We prove that r^* is a Nash equilibrium.

For each $i \in S$, we have $r^*_i = 1$ and $p^s R_i q^{s-1}$, which means that $g_i(1, r^*_{-i}) R_i g_i(0, r^*_{-i})$. With monotonicity, it holds that for all $r_i \in [0,1]$,

$$g_i(0, r^*_{-i}) \leq g_i(r_i, r^*_{-i}) \leq g_i(1, r^*_{-i}).$$

According to Lemma 5.1, we conclude that $g_i(r^*) R_i g_i(r_i, r^*_{-i})$ for all $r_i \in [0,1]$. For each $j \in N \setminus S$, we have $r^*_j = 0$ and $q^s R_j p^{s+1}$, which means that $g_j(0, r^*_{-j}) R_j g_j(1, r^*_{-j})$. From monotonicity and Lemma 5.1 again, it holds that $g_j(r^*) R_j g_j(r_j, r^*_{-j})$ for all $r_j \in [0,1]$. So, $r^* = e^S$ is a Nash equilibrium.

For the only-if part, assume that r^* is a Nash equilibrium. From Lemmas 5.3 and 5.4, we have $r^* = e^S$ for some $S \in 2^N$. In view of

$g_i(r^*)R_i g_i(0, r^*_{-i})$ for all $i \in S$ and $g_j(r^*)R_j g_j(1, r^*_{-j})$ for all $j \in N \setminus S$, it holds that $p^s R_i q^{s-1}$ for all $i \in S$ and $q^s R_j p^{s+1}$ for all $j \in N \setminus S$. $\square$

Theorem 5.1 can also be used to show that a Nash equilibrium does not have to exist, for instance in the following example.

Example 5.3
Let $N = \{1, 2, 3\}$ and let $g \in \mathcal{G}$ satisfy that $p^2 > p^1$. By this assumption and monotonicity, it follows that $q^2 < q^1 < \frac{1}{3} < p^1 < p^2$. Consider $R_N \in \mathcal{D}^N$ such that $q^2 P_1 q^1 P_1 p^2 P_1 p^1 P_1 \frac{1}{3}$, $p^2 P_2 q^2 P_2 q^1 P_2 \frac{1}{3} P_2 p^1$ and $q^1 P_3 p^2$. Then $e^{\emptyset}$ is not a Nash equilibrium in view of $p^1 P_1 \frac{1}{3}$; $e^{\{1\}}$ and $e^{\{3\}}$ are not Nash equilibria in view of $p^2 P_2 q^1$; $e^{\{2\}}$ is not a Nash equilibrium in view of $\frac{1}{3} P_2 p^1$; $e^{\{1,2\}}$ and $e^{\{1,3\}}$ are not Nash equilibria in view of $q^1 P_1 p^2$; $e^{\{2,3\}}$ is not a Nash equilibrium in view of $q^1 P_3 p^2$; and e^N is not a Nash equilibrium in view of $q^2 P_1 \frac{1}{3}$. From Theorem 5.1 it follows that the game (R_N, g) has no Nash equilibrium.

A possible mechanism $g \in \mathcal{G}$ to which this example applies is as follows. For each $r \in [0,1]^N$ and distinct $i, j, k \in N$ let

$$g_i(r) = \frac{8 + 2r_i - r_j - r_k + 2r_i r_j + 2r_i r_k - 4r_j r_k}{24}.$$

Since $g(1,0,0) = (\frac{10}{24}, \frac{7}{24}, \frac{7}{24})$ and $g(1,1,0) = (\frac{11}{24}, \frac{11}{24}, \frac{2}{24})$, we have $q^2 = \frac{2}{24} < q^1 = \frac{7}{24} < \frac{1}{3} < p^1 = \frac{10}{24} < p^2 = \frac{11}{24}$. $\triangle$

We conclude this section by showing that for two agents a Nash equilibrium always exists.

Proposition 5.1
Let $N = \{1, 2\}$, $R_N \in \mathcal{D}^N$ and $g \in \mathcal{G}$. Then the game (R_N, g) has a Nash equilibrium.

Proof. By $p^{s+1} > q^s$ for all $s = 0, 1, \ldots, n-1$, we have $p^1 > \frac{1}{2} > q^1$. We consider three cases.

(a) Suppose that $p^1 P_1 q^1$. Then, by Lemma 5.1, $p^1 R_1 \frac{1}{2}$.

(a1) First suppose that $q^1 R_2 p^1$. Then, according to Lemma 5.1, we have $q^1 R_2 \frac{1}{2}$. It follows that $g_1(1,0) R_1 g_1(0,0)$ and $g_2(1,0) R_2 g_1(1,1)$. With monotonicity and Lemma 5.1 again, it holds that $g_1(1,0) P_1 g_1(r_1,0)$ and $g_2(1,0) P_2 g_2(1,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (1,0)$ is a Nash equilibrium.

(a2) Second, suppose that $p^1 R_2 q^1$. Then, by Lemma 5.1, $p^1 R_2 \frac{1}{2}$.

(a2.1) If $\frac{1}{2} P_1 q^1$ and $\frac{1}{2} P_2 q^1$, then it holds that $g_1(1,1) P_1 g_1(0,1)$ and $g_2(1,1) P_2 g_2(1,0)$. From monotonicity and Lemma 5.1, it holds that $g_1(1,1) R_1 g_1(r_1,1)$ and $g_2(1,1) R_2 g_2(1,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (1,1)$ is a Nash equilibrium.

(a2.2) If $q^1 R_1 \frac{1}{2}$, together with $p^1 R_2 \frac{1}{2}$, then we have $g_1(1,0) R_1 g_1(0,0)$ and $g_2(1,0) R_2 g_2(1,1)$. With monotonicity and Lemma 5.1 again, we have $g_1(1,0) R_1 g_1(r_1,0)$ and $g_2(1,0) R_2 g_2(1,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (1,0)$ is a Nash equilibrium.

(a2.3) If $q^1 R_2 \frac{1}{2}$, then similar to (a2.2), we can prove that $r^* = (0,1)$ is a Nash equilibrium.

(b) Suppose that $p^1 I_1 q^1$. Then, $p^1 R_1 \frac{1}{2}$ and $q^1 R_1 \frac{1}{2}$.

(b1) If $p^1 P_2 q^1$, then $p^1 R_2 \frac{1}{2}$. So, $g_1(0,1) P_1 g_1(1,1)$ and $g_2(0,1) P_2 g_2(0,0)$. With monotonicity and Lemma 5.1, it holds that $g_1(0,1) R_1 g_1(r_1,1)$ and $g_2(0,1) R_2 g_2(0,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (0,1)$ is a Nash equilibrium.

(b2) If $q^1 R_2 p^1$, then $q^1 R_2 \frac{1}{2}$. Similar to (b1), we can prove that $r^* = (1,0)$ is a Nash equilibrium.

(c) Suppose that $q^1 P_1 p^1$. Then, by Lemma 5.1, $q^1 R_1 \frac{1}{2}$.

(c1) First, suppose that $p^1 R_2 q^1$, then similar to (a1), we can verify that $(0,1)$ is a Nash equilibrium.

(c2) Second, suppose that $q^1 P_2 p^1$. Then, $q^1 R_2 \frac{1}{2}$.

(c2.1) If $\frac{1}{2}P_1p^1$ and $\frac{1}{2}P_2p^1$, then it holds that $g_1(0,0)P_1g_1(1,0)$ and $g_2(0,0)P_2g_1(0,1)$. With monotonicity and Lemma 5.1, it holds that $g_1(0,0)R_1g_1(r_1,0)$ and $g_2(0,0)R_2g_2(0,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (0,0)$ is a Nash equilibrium.

(c2.2) If $p^1R_1\frac{1}{2}$, together with $q^1R_2\frac{1}{2}$, we have $g_1(1,0)R_1g_1(0,0)$ and $g_2(1,0)R_2g_1(1,1)$. With monotonicity and Lemma 5.1, it holds that $g_1(1,0)R_1g_1(r_1,0)$ and $g_2(1,0)P_2g_2(1,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (1,0)$ is a Nash equilibrium.

(c2.3) If, finally, $p^1R_2\frac{1}{2}$, then similar to (c2.2), it can be proved $r^* = (0,1)$ is a Nash equilibrium. □

5.4 Pareto optimal Nash equilibrium, strong equilibrium, implementation, and the single-peaked case

In this section, we first consider Pareto optimal Nash equilibria, i.e., Nash equilibria resulting in Pareto optimal allocations. Next, we strengthen this to strong equilibria: no subset of agents can profitably deviate, in the sense that every member is at least as well off, and at least one member is better off. Third, we discuss the related issue of implementation: which social choice correspondence, i.e., multi-valued rule, collects exactly the Pareto optimal Nash equilibria or strong equilibria for a given mechanism? Finally, we compare this to the findings of Bochet et al (2021) for single-peaked preferences.

5.4.1 Pareto optimal Nash equilibrium

Pareto optimal allocations for single-dipped preference profiles were characterized by Klaus et al (1997). For each $R_N \in \mathcal{D}^N$, we denote by $N_+(R_N) = \{i \in N \mid 1P_i0\}$ the set of agents who strictly prefer 1 to 0, by $N_0(R_N) = \{i \in N \mid 0I_i1\}$ the set of agents who are indifferent between 0 and 1, and by $N_-(R_N) = \{i \in N \mid 0P_i1\}$ the set of agents who strictly prefer 0 to 1. The mentioned characterization by Klaus et al (1997) is as follows.

Lemma 5.5
Let $R_N \in \mathcal{D}^N$. An allocation $x \in \mathcal{A}$ is Pareto optimal at R_N if and only if

(i) If $N_+(R_N) \neq \emptyset$, then $x_i = 0$ for every $i \in N \setminus N_+(R_N)$, and for every $i \in N_+(R_N)$ either $x_i = 0$ or $x_i P_i 0$.

(ii) If $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$, then $x = e^{\{i\}}$ for some $i \in N_0(R_N)$.

(iii) If $N_-(R_N) = N$, then for every $i \in N$ either $x_i = 1$ or $x_i P_i 1$.

We first introduce so-called maximal coalitions, which are useful to describe Pareto optimal Nash equilibria.

Definition 5.1
Let $R_N \in \mathcal{D}^N$.

(a) The *sharing index* of an agent $i \in N$ at R_N is the number $m_i(R_N)$ defined by

$$m_i(R_N) = \begin{cases} 0 & \text{if } i \notin N_+(R_N), \\ \max\left\{k \in \{1,\ldots,|N_+(R_N)|\} \;\middle|\; \frac{1}{k} P_i 0\right\} & \text{if } i \in N_+(R_N). \end{cases}$$

(b) A coalition $S \subseteq N$ is a *maximal coalition* at R_N if the following holds.

(i) If $N_+(R_N) \neq \emptyset$, then $S \subseteq N_+(R_N)$ such that $m_i(R_N) \geq |S|$ for every $i \in S$ and $m_j(R_N) \leq |S|$ for every $j \in N \setminus S$.

(ii) If $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$, then $S = \{i\}$ for some $i \in N_0(R_N)$.

(iii) If $N_-(R_N) = N$, and $\{j \in N \mid 1 R_j \frac{1}{n}\} \neq \emptyset$, then $S = \{i\}$ for some $i \in N$ with $1 R_i \frac{1}{n}$.

(iv) If $N_{-}(R_N) = N$, and $\{j \in N \mid 1R_j\frac{1}{n}\} = \emptyset$, then $S = \emptyset$.

The collection of all maximal coalitions at R_N is denoted by $\mathcal{M}(R_N)$.

$\triangle$

The sharing index of an agent $i \in N_{+}(R_N)$ is the maximal size of a coalition of agents strictly preferring one over zero, including i, such that equally sharing the commodity with the members of this coalition is still preferable over receiving 0. For $i \notin N_{+}(R_N)$, this is zero. In Case (i) in (b), a maximal coalition consists of agents who strictly prefer 1 over 0 at R_N. Such a coalition is formed by starting with the agent(s) with maximal sharing index, next adding agent(s) with second maximal sharing index, etc., until the size of the coalition exceeds the sharing indices of the remaining agents. See Example 5.4 for an illustration. In a similar spirit, in Case (ii) in (b), a maximal coalition consists of any arbitrary single agent indifferent between 0 and 1. In Case (iii) in (b), where all agents strictly prefer 0 over 1, a maximal coalition consists of an arbitrary single agent who (weakly) prefers 1 over $\frac{1}{n}$. If there are no such agents, then Case (iv) in (b) applies and the only maximal coalition is the empty coalition.

Example 5.4
Let $N = \{1, 2, 3\}$ and let R_N satisfy $\frac{1}{3}P_1 0$, $\frac{1}{2}P_i 0$ and $0R_i\frac{1}{3}$ for $i = 2, 3$. Then $N_{+}(R_N) = N$, $m_1(R_N) = 3$, and $m_2(R_N) = m_3(R_N) = 2$. To construct a maximal coalition we start with agent 1 and then add either agent 2 or agent 3, to obtain $\{1, 2\}$ and $\{1, 3\}$ as maximal coalitions. Coalition $\{2, 3\}$ is not maximal since $m_1(R_N) = 3 > 2 = |\{2, 3\}|$, and coalition N is not maximal since $m_2(R_N) = 2 < 3 = |N|$. Also singleton coalitions are not maximal: $\{1\}$ is not maximal since $m_2(R_N) = 2 > |\{1\}|$, $\{2\}$ is not maximal since $m_1(R_N) = 3 > |\{2\}|$, and $\{3\}$ is not maximal since $m_1(R_N) = 3 > |\{3\}|$. $\triangle$

The basic reason why maximal coalitions play a role in our analysis is that a member of such a coalition prefers receiving an equal share over receiving 0 and therefore would not deviate and leave the coalition; on the other hand, there is no outside agent who would gain by joining the coalition. This will be made precise in Theorem 5.2.

We first formulate an additional property for a mechanism g.

Standardness
Let $g \in \mathcal{G}$. g is *standard* if $g(e^S) = \frac{1}{|S|}e^S$ for all $S \in 2^N \setminus \{\emptyset\}$.

If g is standard, then $p^s = \frac{1}{s}$ and $q^s = 0$ for each $s = 1, 2, \ldots, n-1$. The mechanisms in Examples 5.1 and 5.2 are standard, but the mechanism in Example 5.3 is not standard.

We show that standardness of a mechanism is a necessary and sufficient condition for all games based on this mechanism to have a Pareto optimal Nash equilibrium.

Lemma 5.6
Let $g \in \mathcal{G}$ and suppose that (R_N, g) has a Pareto optimal Nash equilibrium for every $R_N \in \mathcal{D}^N$. Then g is standard.

Proof. For each $S \in 2^N \setminus \{\emptyset, N\}$, we consider $R_N^S \in \mathcal{D}^N$ such that $d(R_i^S) = 0$ for all $i \in S$ and $d(R_j^S) = 1$ for all $j \in N \setminus S$. Then $N_+(R_N^S) = S$. From Lemmas 5.3 and 5.4, it follows that the only Nash equilibrium in the game (R_N^S, g) is $r^* = e^S$. From Lemma 5.5, we have $g_j(r^*) = 0$ for all $j \in N \setminus N_+(R_N^S) = N \setminus S$. It follows that $g_i(r^*) = \frac{1}{|S|}$ for all $i \in S$. Together with $g(e^N) = \frac{1}{|N|}$, we conclude that $g(e^S) = \frac{1}{|S|}e^S$ for all $S \in 2^N \setminus \{\emptyset\}$. This implies that g is standard. $\square$

Lemma 5.6 says that standardness of the mechanism is a necessary condition for a Pareto optimal Nash equilibrium to exist in every game induced by this mechanism. The sufficiency part follows from the following theorem, which is a main result of this chapter.

Theorem 5.2
Let $R_N \in \mathcal{D}^N$ and let $g \in \mathcal{G}$ be standard. A strategy profile $r^* \in [0,1]^N$ is a Pareto optimal Nash equilibrium of (R_N, g) if and only if $r^* = e^S$ for some $S \in \mathcal{M}(R_N)$.

Proof. For the if-part, let $S \in \mathcal{M}(R_N)$. We prove that $r^* = e^S$ is a Pareto optimal Nash equilibrium.

Case (i): $N_+(R_N) \neq \emptyset$.

Let $i \in S$. Then $r_i^* = 1$ and $g_i(r^*) = \frac{1}{|S|}$. Since $\frac{1}{|S|} \geq \frac{1}{m_i(R_N)}$ and $\frac{1}{m_i(R_N)} P_i 0$, we have that $\frac{1}{|S|} P_i 0$, which implies that $g_i(r^*) P_i g_i(0, r^*_{-i})$. Monotonicity then implies $g_i(0, r^*_{-i}) \leq g_i(r_i, r^*_{-i}) \leq g_i(r^*)$ for all $r_i \in [0,1]$, and by Lemma 5.1, $g_i(r^*) R_i g_i(r_i, r^*_{-i})$ for all $r_i \in [0,1]$.

If $i \in N_+(R_N) \setminus S$, then $r_i^* = 0$ and $g_i(r^*) = 0$. In view of $|S| + 1 > m_i(R_N)$, it holds that $|S| + 1 \geq m_i(R_N) + 1$, i.e., $\frac{1}{|S|+1} \leq \frac{1}{m_i(R_N)+1}$. Together with $0 R_i \frac{1}{m_i(R_N)+1}$, by Lemma 5.1, we have $0 R_i \frac{1}{|S|+1}$, which implies that $g_i(r^*) R_i g_i(1, r^*_{-i})$. With monotonicity and Lemma 5.1 again, we can similarly verify that $g_i(r^*) R_i g_i(r_i, r^*_{-i})$ for all $r_i \in [0,1]$.

For $i \in N \setminus N_+(R_N)$, in view of $S \subseteq N_+(R_N)$, we have $i \in N \setminus S$, $r_i^* = 0$ and $g_i(r^*) = 0$. In view of $0 R_i 1$, by Lemma 5.1, we have $g_i(r^*) R_i g_i(r_i, r^*_{-i})$ for all $r_i \in [0,1]$.

Thus, $g_i(r^*) R_i g_i(r_i, r^*_{-i})$ for all $i \in N$ and $r_i \in [0,1]$, which implies that $r^* = e^S$ is a Nash equilibrium.

Case (ii): $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

Let $S = \{i\}$ with $i \in N_0(R_N)$. Then $g_i(r^*) = 1$ and $g_j(r^*) = 0$ for all $j \in N \setminus \{i\}$. For agent i, in view of $1 R_i 0$, by Lemma 5.1, it holds that $g_i(r^*) R_i g_i(r_i, r^*_{-i})$ for all $r_i \in [0,1]$. For each agent $j \in N \setminus \{i\}$, in view of $0 R_j 1$, by Lemma 5.1 again, we have $g_j(r^*) R_j g_j(r_j, r^*_{-j})$ for all $r_j \in [0,1]$. So, $r^* = e^{\{i\}}$ is a Nash equilibrium.

Case (iii): $N_-(R_N) = N$.

If $S = \{i\}$ for some $i \in N$, then $1R_i\frac{1}{n}$, Then $r_j^* = 0$ and $g_j(r^*) = 0$ for all $j \in N \setminus \{i\}$. In view of $0P_j1$, by Lemma 5.1, we have $g_j(r^*)R_jg_j(r_j, r_{-j}^*)$ for all $r_j \in [0,1]$. With monotonicity, we have $g_i(r_i, r_{-i}^*) \geq \frac{1}{n}$ for all $r_i \in (0,1]$. In view of $1R_i\frac{1}{n}$, by Lemma 5.1 again, we have $g_i(r^*)R_ig_i(r_i, r_{-i}^*)$ for all $r_i \in [0,1]$. So, $r^* = e^{\{i\}}$ is a Nash equilibrium.

If $S = \emptyset$, then $\frac{1}{n}P_i1$ for all $i \in N$. For each $i \in N$, if $r_i > 0$, with monotonicity, we have $g_i(r_i, r_{-i}^*) > g_i(0, r_{-i}^*) = \frac{1}{n}$. Together with $\frac{1}{n}P_i1$, by Lemma 5.1, we have $g_i(r^*)R_ig_i(r_i, r_{-i}^*)$ for all $r_i \in [0,1]$. So, $r^* = e^{\emptyset}$ is a Nash equilibrium.

Combining these three cases, we conclude that for each $S \in \mathcal{M}(R_N)$, $r^* = e^S$ is a Nash equilibrium. Lemma 5.5 implies that $g(r^*)$ is Pareto optimal at R_N.

For the only-if part, assume that r^* is a Pareto optimal Nash equilibrium. From Theorem 5.1, it follows that $r^* = e^S$ for some $S \in 2^N$. We prove that $S \in \mathcal{M}(R_N)$.

Case (i): $N_+(R_N) \neq \emptyset$.

Assume, to the contrary, that $S \notin \mathcal{M}(R_N)$. Let $T \in \mathcal{M}(R_N)$. First, we prove that $|S| = |T|$.

Since $g(e^S)$ and $g(e^T)$ are Pareto optimal at R_N, from Lemma 5.5, we have $\frac{1}{|S|}P_i0$ for all $i \in S$, and $\frac{1}{|T|}P_i0$ for all $i \in T$. Since e^S (by assumption) and e^T (from the if-part) are Nash equilibria of (R_N, g), we have $0R_j\frac{1}{|S|+1}$ for all $j \in N \setminus S$, and $0R_j\frac{1}{|T|+1}$ for all $j \in N \setminus T$. If $|S| < |T|$, then there exists $k \in T \setminus S$ such that $\frac{1}{|T|}P_k0$ and $0R_k\frac{1}{|S|+1}$. However, in view of $|S| < |T|$, we have $|S| + 1 \leq |T|$, i.e., $\frac{1}{|T|} \leq \frac{1}{|S|+1}$. From Lemma 5.1, it follows that $0R_k\frac{1}{|T|}$, which is a contradiction. If $|S| > |T|$, we similarly obtain a contradiction. Thus, $|S| = |T|$.

Then, since $S \notin \mathcal{M}(R_N)$ and $|S| = |T|$, there exist $i \in S$ and $j \in N \setminus S$ such that $m_i(R_N) < m_j(R_N)$. By Lemma 5.5, we have $\frac{1}{|S|}P_i0$. It follows that $|S| \leq m_i(R_N)$. So, $|S| < m_j(R_N)$, i.e., $\frac{1}{|S|+1} \geq \frac{1}{m_j(R_N)}$. In view of $\frac{1}{m_j(R_N)} P_j 0$, we have $\frac{1}{|S|+1} P_j 0$. This implies that $g_j(1, e^S_{-j}) P_j g_j(e^S)$, which contradicts the assumption that e^S is a Nash equilibrium. Thus , $S \in \mathcal{M}(R_N)$.

Case (ii): $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

From Lemma 5.5, $g(e^T)$ is not Pareto optimal for all $T \in 2^N \setminus \mathcal{M}(R_N)$. Thus, $S \in \mathcal{M}(R_N)$.

Case (iii): $N_-(R_N) = N$.

If there exists $i \in N$ such that $1 R_i \frac{1}{n}$, then $e^{\emptyset}$ is not a Pareto optimal Nash equilibrium, hence $S \neq \emptyset$. Since $0 P_j 1$ for all $j \in N$, it follows that e^T is not a Nash equilibrium for each $T \in 2^N$ with $|T| \geq 2$. Hence, $|S| = 1$. For $j \in N$ such that $\frac{1}{n} P_j 1$, it is easily seen that $e^{\{j\}}$ is not a Nash equilibrium. Thus, $S \in \mathcal{M}(R_N)$.

Finally, suppose that $\{i \in N \mid 1 R_i \frac{1}{n}\} = \emptyset$, i.e., $\frac{1}{n} P_i 1$ for all $i \in N$. If $T \neq \emptyset$, then since $0 P_i 1$ and $\frac{1}{n} P_i 1$ for all $i \in N$, it follows that $g_i(e^{T \setminus \{i\}}) P_i g_i(e^T)$ for all $i \in T$, which implies that e^T is not a Nash equilibrium. So, $S = \emptyset \in \mathcal{M}(R_N)$, and the proof of the theorem is complete. □

Theorem 5.2 shows that for a standard mechanism, the Pareto optimal Nash equilibria are those strategy profiles in which all agents in a maximal coalition play 1 and all other agents play 0. Since there exists at least one maximal coalition for every single-dipped preference profile, Lemma 5.6 and Theorem 5.2 imply the result announced earlier.

Corollary 5.1
Let $g \in \mathcal{G}$. There exists a Pareto optimal Nash equilibrium of (R_N, g) for every $R_N \in \mathcal{D}^N$ if and only if g is standard.

The next example shows that for a game based on a standard mechanism, besides Pareto optimal Nash equilibria, there may exist Nash equilibria without Pareto optimal outcomes, or Pareto optimal outcomes, not obtained in any Nash equilibrium.

Example 5.5
Let $N = \{1, 2\}$ and let $g \in \mathcal{G}$ be as in Example 5.1.

(a) Consider $R_N \in \mathcal{D}^N$ such that $1P_1 0 P_1 \frac{1}{2}$ and $0P_2 1 P_2 \frac{1}{2}$. Then, we have $g_1(0,1)P_1 g_1(1,1)$ and $g_2(0,1)P_2 g_2(0,0)$. With monotonicity and Lemma 5.1, it follows that $g_1(0,1)R_1 g_1(r_1,1)$ and $g_2(0,1)R_2 g_2(0,r_2)$ for all $r_1, r_2 \in [0,1]$. So, $e^{\{2\}} = (0,1)$ is a Nash equilibrium. However, $g(e^{\{2\}}) = (0,1)$ is not Pareto optimal at R_N. In fact, Theorem 5.2 implies that the unique Pareto optimal Nash equilibrium is $e^{\{1\}} = (1,0)$.

(b) Consider $R_N \in \mathcal{D}^N$ such that $d(R_1) = d(R_2) = 0$. Then $x = g(\frac{1}{2}, \frac{1}{3}) = (\frac{7}{12}, \frac{5}{12})$ is Pareto optimal at R_N, but there is no $S \in 2^N$ such that $g(e^S) = x$. Thus, Theorem 5.1 implies that there is no Nash equilibrium r^* such that $g(r^*) = x$. In fact, $m_1(R_N) = m_2(R_N) = 2$, and hence the unique maximal coalition is N. From Theorem 5.2 (or direct inspection), the unique Pareto optimal Nash equilibrium is $e^N = (1,1)$. $\triangle$

5.4.2 Strong equilibrium

In this subsection we consider a further strengthening of Pareto optimal Nash equilibrium, namely strong equilibrium (Aumann, 1959): no coalition can profitably deviate. We will show that the Pareto optimal Nash equilibria and strong equilibria coincide if, besides anonymous, monotonic, and standard, the mechanism is order-preserving.[1]

Order-preservation
A mechanism g is *order-preserving* if $g_i(r) \geq g_j(r)$ for all $r \in [0,1]^N$ and $i, j \in N$ with $r_i \geq r_j$.

Theorem 5.3
Let $R_N \in \mathcal{D}^N$ and let $g \in \mathcal{G}$ be standard and order-preserving. If a strategy profile is a Pareto optimal Nash equilibrium of (R_N, g) if and only if it is a strong equilibrium.

[1] A similar condition also occurs in Bochet et al (2021) under the name 'peak order preservation'.

Proof. We start with the only-if part. Let $S \in \mathcal{M}(R_N)$. By Theorem 5.2, it is sufficient to verify that e^S is a strong equilibrium. Assume, to the contrary, that there exist $T \in 2^N \setminus \{\emptyset\}$ and $r_T \in [0,1]^T$ such that $g_i(r_T, e^S_{N\setminus T}) R_i g_i(e^S)$ for all $i \in T$ and $g_j(r_T, e^S_{N\setminus T}) P_j g_j(e^S)$ for some $j \in T$. We consider three cases.

Case (i): $N_+(R_N) \neq \emptyset$.

If $S \cap T \neq \emptyset$, then for each $i \in S \cap T$, it holds that $g_i(r_T, e^S_{N\setminus T}) \geq \frac{1}{|S|}$ in view of $\frac{1}{|S|} P_i 0$ from Theorem 5.2 and $g_i(r_T, e^S_{N\setminus T}) R_i g_i(e^S)$ by assumption. By order-preservation, it follows that for all $j \in S \setminus T$,

$$g_j(r_T, e^S_{N\setminus T}) \geq g_i(r_T, e^S_{N\setminus T}) \geq \frac{1}{|S|}.$$

So, we have $g_i(r_T, e^S_{N\setminus T}) = \frac{1}{|S|} = g_i(e^S)$ for all $i \in S$, and $g_j(r_T, e^S_{N\setminus T}) = 0 = g_j(e^S)$ for all $j \in N \setminus S$, i.e., $g_k(r_T, e^S_{N\setminus T}) I_k g_k(e^S)$ for all $k \in T$, contradicting our assumption.

If $S \cap T = \emptyset$, then we claim that $g_i(r_T, e^S_{N\setminus T}) \leq \frac{1}{|S|+1}$ for each $i \in T$. If not, take $i \in T$ with $g_i(r_T, e^S_{N\setminus T}) > \frac{1}{|S|+1}$. Then for all $j \in S$,

$$g_j(r_T, e^S_{N\setminus T}) \geq g_i(r_T, e^S_{N\setminus T}) > \frac{1}{|S|+1}.$$

It follows that $\sum_{k \in T\cup S} g_k(r_T, e^S_{N\setminus T}) > 1$, which is not possible. In view of $0 R_i \frac{1}{|S|+1}$ from Theorem 5.2, together with Lemma 5.1, we have $g_i(e^S) R_i g_i(r_T, e^S_{N\setminus T})$ for all $i \in T$, which contradicts our assumption.

Case (ii): $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

In this case, $S = \{i\}$ for some $i \in N_0(R_N)$. Then $g(e^S) = e^{\{i\}}$. Since $1 I_i 0$ and $0 R_j 1$ for all $j \in N \setminus \{i\}$, by Lemma 5.1 we have $g_k(e^S) R_i g_k(r_T, e^S_{N\setminus T})$ for all $k \in T$, which is a contradiction to our assumption.

Case (iii): $N_-(R_N) = N$.

If $S = \{i\}$ for some $i \in N$, then $1 R_i \frac{1}{n}$. It follows that $g_j(e^S) = 0$ for all $j \in N \setminus \{i\}$. Since $e^{\{i\}}$ is a Nash equilibrium, it holds that $T \neq \{i\}$. For each $k \in T \setminus \{i\}$, we have $g_k(e^S) R_k g_k(r_T, e^S_{N\setminus T})$ from $0 P_k 1$ and Lemma 5.1. Together with our assumption, it follows that $g_k(r_T, e^S_{N\setminus T}) = g_k(e^S) = 0$ for all $k \in T \setminus \{i\}$. By order-preservation, we have $g_j(r_T, e^S_{N\setminus T}) = 0$ for all $j \in N \setminus \{i\}$. So, $g(r_T, e^S_{N\setminus T}) = g(e^S)$, which is a contradiction.

If $S = \emptyset$, then $\frac{1}{n} P_i 1$ for all $i \in N$. For each $k \in T$, in view of $g_k(r_T, e^S_{N\setminus T}) R_k g_k(e^S)$ and $0 P_k \frac{1}{n} P_k 1$, we have $g_k(r_T, e^S_{N\setminus T}) \leq \frac{1}{n}$. By order-preservation, it holds that $g_j(r_T, e^S_{N\setminus T}) \leq g_k(r_T, e^S_{N\setminus T}) \leq \frac{1}{n}$ for all $j \in N \setminus T$ and $k \in T$. So, $g_k(r_T, e^S_{N\setminus T}) = g_k(e^S) = \frac{1}{n}$ for all $k \in T$, which is a contradiction. This concludes the proof of the only-if part.

For the if-part, suppose that r^* is a strong equilibrium of (R_N, g). Obviously, r^* is a Nash equilibrium. By Theorem 5.1, there is a coalition S such that $r^* = e^S$. Since g is standard, we have $g(e^S) = \frac{1}{|S|} e^S$ if $S \neq \emptyset$. If $S = \emptyset$, then $g(e^S) = \frac{1}{n} e^N$.

If $S = \emptyset$, then, since e^S is a Nash equilibrium, we have $\frac{1}{n} R_i 1$ for all $i \in N$, which implies that $g(e^S) = \frac{1}{n} e^N$ is Pareto optimal.

If $|S| \geq 2$, then, again since e^S is a Nash equilibrium, $\frac{1}{|S|} R_i 0$ for all $i \in S$; in this case, if $x_i R_i g_i(e^S)$ for some $x \in \mathcal{A}$ and all $i \in N$, then in particular $x_i \geq \frac{1}{|S|}$ for all $i \in S$, which implies $x = g(e^S)$ and, thus, $g(e^S)$ is Pareto optimal.

Finally, suppose that $|S| = 1$, say $S = \{n\}$.

If $1 P_n 0$ then clearly $g(e^S) = (0, \ldots, 0, 1)$ is Pareto optimal.

If $1 I_n 0$ and there is some $j \neq n$ with $1 P_j 0$, then $\{j, n\}$ can profitably deviate by $r_j = 1$ and $r_n = 0$, contradicting that e^S is a strong equilibrium; hence, $0 R_j 1$ for all $j \neq n$, so that $g(e^S) = (0, \ldots, 0, 1)$ is Pareto optimal.

If $0P_n1$ and there is some $j \neq n$ with $1R_j0$, then $\{j, n\}$ can profitably deviate by $r_j = 1$ and $r_n = 0$, contradicting that e^S is a strong equilibrium; hence, $0P_j1$ for all $j \neq n$, so that $g(e^S) = (0, \ldots, 0, 1)$ is Pareto optimal. This concludes the proof of the if-part. □

5.4.3 Implementation

In this subsection we reformulate our main results in terms of implementation. A *social choice correspondence* F is a map assigning to each preference profile $R_N \in \mathcal{D}^N$ a nonempty set of allocations. If this set always consists of exactly one allocation, then F is a rule, as defined earlier in Section 5.2. We say that a mechanism *g implements F in Pareto optimal Nash equilibrium* if

$$F(R_N) = \{g(r) \in \mathcal{A} \mid r \text{ is a Pareto optimal Nash equilibrium of } (R_N, g)\}$$

for every preference profile $R_N \in \mathcal{D}^N$. Mechanism *g implements F in strong equilibrium* if

$$F(R_N) = \{g(r) \in \mathcal{A} \mid r \text{ is a strong equilibrium of } (R_N, g)\}$$

for every preference profile $R_N \in \mathcal{D}^N$. For each $S \subseteq N$ define the allocation $\hat{e}^S \in \mathcal{A}$ by

$$\hat{e}^S = \begin{cases} \frac{1}{|S|} e^S & \text{if } S \neq \emptyset \\ (\frac{1}{n}, \ldots, \frac{1}{n}) & \text{if } S = \emptyset. \end{cases}$$

Define the social choice correspondence M on $\mathcal{D}^N$ by

$$M(R_N) = \{\hat{e}^S \in \mathcal{A} \mid S \in \mathcal{M}(R_N)\}$$

for every $R_N \in \mathcal{D}^N$. We now have the following corollary from Theorems 5.2 and 5.3.

Corollary 5.2
Let $g \in \mathcal{G}$. If g is standard, then g implements M in Pareto optimal Nash equilibrium. If g is standard and order-preserving, then g implements M in strong equilibrium.

5.4.4 Single-peaked preferences

The case of single-peaked preferences is extensively studied in Bochet et al (2021). The result that is most closely related to our approach is their Theorem 2, which applies to peaks-only rules – these are equivalent to mechanisms in our sense. Under conditions on rules (mechanisms g), partly similar to ours, they show that the Pareto optimal Nash equilibria and strong equilibria in a game (R_N, g) coincide and result in the uniform allocation, for every $R_N \in \mathcal{P}^N$. An allocation $x \in \mathcal{A}$ is the *uniform allocation at* $R_N \in \mathcal{P}^N$ if there is a $\lambda \in [0,1]$ such that

$$x_i = \begin{cases} \min\{p(R_i), \lambda\} & \text{if } \sum_{i \in N} p(R_i) \geq 1 \\ \max\{p(R_i), \lambda\} & \text{if } \sum_{i \in N} p(R_i) \leq 1. \end{cases}$$

The uniform allocation is the allocation assigned by the uniform rule (single-valued social choice correspondence) U, characterized by Sprumont (1991). At the uniform allocation, either all agents obtain at most their peaks or all agents obtain at least their peaks, or both, and thus the uniform allocation is is indeed Pareto optimal (it is 'same-sided'). The proof of the following proposition is straightforward, and therefore omitted.

Sprumont (1991) shows that the uniform rule is the unique anonymous, Pareto optimal, and strategy-proof rule. Recall that a rule F is strategy-proof if $F_i(R_N)R_iF_i(R'_i, R_{N\setminus\{i\}})$ for every preference profile R_N, agent $i \in N$, and preference R'_i, where preferences are chosen within a specific domain, for instance $\mathcal{P}$ or $\mathcal{D}$. The following example shows that, in the single-dipped case, rules obtained by selecting from

M are not strategy-proof. Let $F\colon \mathcal{D} \to \mathcal{A}$ such that $F(R_N) \in M(R_N)$ for every $R_N \in \mathcal{D}^N$.

Example 5.6
Let $R_N \in \mathcal{D}^N$ such that $0P_i1R_i\frac{1}{n}$ for all $i \in N$. Then we have $M(R_N) = \{\{i\} \mid i \in N\}$, and therefore $F(R_N) = e^{\{j\}}$ for some $j \in N$ (cf. Theorem 5.2). Consider $R'_j \in \mathcal{D}$ such that $0P'_j1$ and $\frac{1}{n}P'_j1$. Then $M(R'_j, R_{-j}) = \{\{i\} \mid i \in N \setminus \{j\}\}$, and therefore $F_j(R'_j, R_{-j}) = 0$, so that $F_j(R'_j, R_{-j})P_jF_j(R_N)$. Hence, F is not strategy-proof. $\triangle$

5.5 Concluding remarks

We have shown that in division problems with single-dipped preferences, the Pareto optimal Nash and strong equilibria of games induced by a fairly general class of mechanisms, result in Pareto optimal allocations characterized by maximal coalitions. A natural extension of our analysis and the analysis in Bochet et al (2021) is to other domains of preferences, notably if both single-dipped and single-peaked preferences in a profile are allowed.

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Impact of the thesis

Division problems are common in the real world, and most relevant work is devoted to find a fair allocation when the allocated resource is insufficient for the agents with specific demands. This thesis focuses on two classes of valuable division problems: bankruptcy problems and division problems with single-dipped preferences. Bankruptcy problems study the situations where an insufficient estate is allocated among several claimants, each of whom has a claim on the estate. For example, when a company goes bankrupt, how should an authority liquidate the residual assets of the company among its creditors. Another example is how to allocate one's heritage among his heirs, when their debts are totally more than the amount of inheritance. Division problems with single-dipped preferences consider the problems of allocating one unit of an infinitely divisible commodity among agents with single-dipped preferences. A single-dipped preference has a worst point, the dip, and preference strictly increases in both directions away from the dip. Such a preference may arise, for instance, when allocating time between two types of work and an agent prefers spending time on only one of the two instead of on a combination – think of research versus teaching at a university. Another example is a two-goods exchange economy with fixed prices and a strictly quasi-convex utility function, which induces a single-dipped preference on the budget line.

This thesis provides new mechanisms to deal with these division problems. Following these mechanisms, some reasonable allocations can be achieved in Nash equilibria of the induced non-cooperative games. In bankruptcy problems, the allocation of the constrained equal awards rule is implemented. In division problems with single-dipped preferences, a selection of the Pareto social choice correspondence is implemented, namely picking the Pareto optimal allocations that are characterized by so-called maximal coalitions: this means that outside agents prefer getting zero over equally sharing the one unit with the agents in the coalition, whereas for agents in the coalition the opposite holds.

Moreover, the new class of two-bound core (cooperative) games is introduced in this thesis, where the core is nonempty and can be described by a lower bound and an upper bound on the allocations. Many games are two-bound core games, including additive games, all balanced games with at most three players, unanimity games, bankruptcy games, 1-convex games, big boss games, clan games, compromise stable games, and reasonable stable games. The core, the nucleolus, and the egalitarian core are studied on this new domain. On the one hand, new expressions of these solutions are provided, which make the calculations of these solutions easier. On the other hand, based on associated reduced game properties, new axiomatic characterizations of these solutions on the new domain are provided.

Summary

This thesis involves cooperative games, non-cooperative games, and mechanism design. We introduce a new class of cooperative games, and study several important solutions in this domain. Then, we design mechanisms for division problems, and consider equilibria in the induced non-cooperative games.

Quant et al (2005) studied the class of compromise stable games where the core coincides with the core cover (Tijs and Lipperts, 1982). The core cover is the set of pre-imputations between a specific pair of bounds. We generalize the approach of Quant et al (2005) to all games where the core equals the set of pre-imputations between an arbitrary pair of bounds, which we call two-bound core games. We show that the core of each two-bound core game can be described equivalently by the pair of exact core bounds (Bondareva and Driessen, 1994), and study to what extent the exact core bounds of a two-bound core game can be stretched while retaining the core description. We provide explicit expressions of the nucleolus (Schmeidler, 1969) and the egalitarian core (Arin and Iñarra, 2001) for two-bound core games in terms of the exact core bounds. We also show that the egalitarian core for two-bound core games is a single-valued solution. Then, we study Davis-Maschler reduced games of two-bound core games. Based on associated reduced game properties, we axiomatically characterize the core, the nucleolus, and the egalitarian core for two-bound core games.

In addition, we design mechanisms to solve bankruptcy problems and division problems with single-dipped preferences. We consider a sequential partition method for bankruptcy problems. The idea of this method is that claimants gather and successively partition the estate in a given order. On the basis of the ascending order of claims, a divide-and-choose mechanism and a divide-and-object mechanism are designed. For each non-cooperative game induced by our mechanisms for bankruptcy problems, we show that the unique Nash equilibrium outcome is consistent with the allocation of the constrained equal awards rule. Then, we consider a mechanism for division problems with single-dipped preferences, which allocates one unit of an infinitely divisible commodity among agents reporting a number between zero and one. Nash, Pareto optimal Nash, and strong equilibria are analyzed for the games induced by our mechanism. We show that when the mechanism is anonymous, monotonic, standard and order-preserving, the Pareto optimal Nash and strong equilibria coincide and assign Pareto optimal allocations that are characterized by so-called maximal coalitions: non-involved agents prefer getting zero over an equal coalition share, whereas for agents in the coalition the opposite holds.

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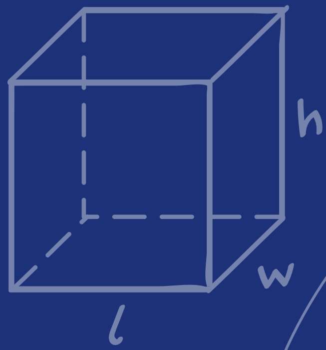
Maastricht
2022-07-11

About the author

Doudou Gong was born in 1995, and grew up in a loving family. He became a student of Northwestern Polytechnical University in Xi'an, China in September 2013, and obtained the degree of Bachelor of Science in June 2017. Three months later, he became a Ph.D. student in School of Mathematics and Statistics, Northwestern Polytechnical University, under the supervision of Prof. dr. Genjiu Xu. From March 2021, he has visited School of Business and Economics, Maastricht University, Maastricht, the Netherlands, as a joint Ph.D. student, under the supervision of Prof. dr. Hans Peters and Dr. Bas Dietzenbacher.

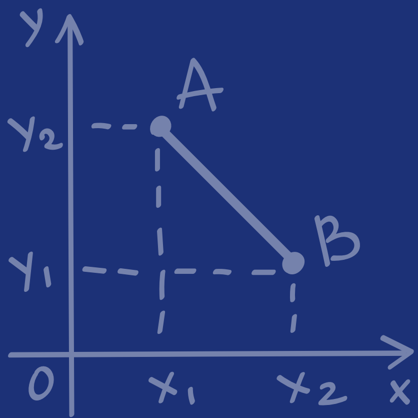
$$g(r) = \left(\frac{1 + r_1 - r_2}{2}, \frac{1 - r_1 + r_2}{2} \right)$$



$$C(v) = [l, u] \cap X(v)$$

$$C(v_T^x) = [l_T, u_T] \cap X(v_T^x)$$

$$M(R_N) = \{\hat{e}^S \in \mathcal{A} \mid S \in \mathcal{M}(R_N)\}$$



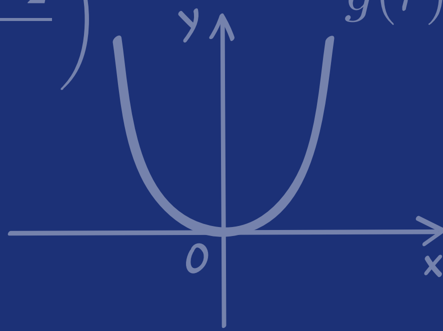
$$C(v) = [l^*(v), u^*(v)] \cap$$

$$\eta(v) = l + f^{TAL} \left(v(N) - \sum_{i \in N}$$

$$\eta(v) = u - f^{TAL} \left(\sum_{i \in N} u_i - v(N) \right.$$

$$g(r) = \left(\frac{1 + r_1 - r_2}{2}, \frac{1 - r_1 + r_2}{2} \right)$$

$$g(r)$$



$$C(v) = [l, u] \cap X(v)$$

$$C(v_T^x) = [l_T, u_T] \cap X(v_T^x)$$

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