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ABSTRACT

This note shows that the egalitarian Dutta and Ray (1989) solution for transferable utility games is self-antidual on the class of exact partition games. By applying a careful antiduality analysis, we derive several new axiomatic characterizations. Moreover, we point out an error in earlier work on antiduality and repair and strengthen several related characterizations on the class of convex games.

1. Introduction

This note focuses on the structure and implications of a specific set of axioms related to the egalitarian Dutta and Ray (1989) solution for transferable utility games. For convex games, the Dutta and Ray (1989) solution prescribes the Lorenz dominating core element. The Dutta and Ray (1989) solution is mainly studied on the class of convex games and several axiomatic characterizations are provided. Already since the work of Dutta (1990), these characterizations are generally based on consistency properties. Whereas Dutta (1990) combined a fixed solution for two-player games with the consistency properties of Davis and Maschler (1965) and Hart and Mas-Colell (1989), these results were turned into full axiomatic characterizations by Klijn et al. (2000). Moreover, Klijn et al. (2000) provided a third characterization on the class of convex games based on an alternative consistency property.

Recently, Llerena and Mauri (2017) introduced the larger class of exact partition games and showed that the Dutta and Ray (1989) solution for these games behaves as for convex games, i.e. it assigns to each such game the Lorenz dominating core element. Dietzenbacher and Yanovskaya (2020) showed that some axiomatic characterizations of Klijn et al. (2000) for convex games can be extended to the class of exact partition games. Moreover, they provided another characterization based on the consistency property of Moulin (1985).

Oishi and Nakayama (2009) introduced an antiduality notion within the context of transferable utility games to structure games and solutions. Two games are antidual if the worth of each coalition in one game equals the worth of its complement minus the worth of the grand coalition in the other game. Two solutions are antidual if one solution assigns to each game the negated set of payoff allocations assigned by the other solution to the antidual game. A solution is self-antidual if it coincides with its antidual. Oishi and Nakayama (2009) showed that some well-known classes of games are antidual to each other and several solutions are self-antidual. Oishi et al. (2016) further developed the antiduality notion and showed that the classes of all games, balanced games, and convex games are each closed under antiduality. Moreover, they introduced antiduality for axioms and characterizations to uncover new structures. Two axioms are antidual if for each two antidual solutions, one property is satisfied by one solution if and only if the other property is satisfied by the other solution. An axiom is self-antidual if for each two antidual solutions, the property is satisfied by one solution if and only if it is satisfied by the other solution. In particular, Oishi et al. (2016) showed that the Dutta and Ray (1989) solution is self-antidual on the class of convex games and revealed new axiomatic characterizations antidual to the results of Klijn et al. (2000). However, these antidual characterizations are invalid due to an incorrect claim about self-antiduality of the axiom equal division stability.

The purpose of this note is twofold. On the one hand, we show that the class of exact partition games is closed under antiduality and that the Dutta and Ray (1989) solution is self-antidual on this class. By applying a careful antiduality analysis, we derive several new axiomatic characterizations of the Dutta and Ray (1989) solution on the class of exact partition games from the results of Dietzenbacher and Yanovskaya (2020). On the other hand, we repair and strengthen the results of Oishi et al. (2016) related to

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the Dutta and Ray (1989) solution on the class of convex games by providing the correct antidual of equal division stability and weakening the corresponding consistency properties.

This note is organized in the following way. Section 2 provides the preliminary notions and notations for transferable utility games and the Dutta and Ray (1989) solution. Section 3 performs an antiduality analysis on the class of exact partition games. Section 4 presents concluding remarks on antiduality in larger classes of games.

2. Preliminaries

Let N be a nonempty and finite set. Denote $2^N = \{S \mid S \subseteq N\}$. For each $x \in \mathbb{R}^N$, we define

$$R_0^x = \emptyset \text{ and } R_k^x = \left\{ i \in N \;\middle|\; \forall j \in N \setminus R_{k-1}^x : x_j \le x_i \right\} \text{ for each } k \in \{1, \ldots, |N|\};$$

$$P_0^x = \emptyset \text{ and } P_k^x = \left\{ i \in N \;\middle|\; \forall j \in N \setminus P_{k-1}^x : x_j \ge x_i \right\} \text{ for each } k \in \{1, \ldots, |N|\}.$$

For each $x \in \mathbb{R}^N$ and each $S \in 2^N \setminus \{\emptyset\}$, $x_S \in \mathbb{R}^S$ denotes $x_S = (x_i)_{i \in S}$, and $x(S) \in \mathbb{R}$ denotes $x(S) = \sum_{i \in S} x_i$.

A *transferable utility game* is a pair (N, v) where N is a nonempty and finite set of *players* and $v : 2^N \to \mathbb{R}$ assigns to each *coalition* $S \in 2^N$ its *worth* $v(S) \in \mathbb{R}$ such that $v(\emptyset) = 0$. Let Γ_{all} denote the class of all games. The *core* of a game (N, v) is given by

$$C(N, v) = \left\{ x \in \mathbb{R}^N \;\middle|\; x(N) = v(N), \forall S \in 2^N : x(S) \ge v(S) \right\}.$$

A game (N, v) is a *balanced game* if $C(N, v) \neq \emptyset$. A game (N, v) is an *exact partition game* (cf. Llerena and Mauri, 2017) if there is $x \in C(N, v)$ such that $x(R_k^x) = v(R_k^x)$ for each $k \in \{1, \ldots, |N|\}$. A game (N, v) is a *convex game* (cf. Shapley, 1971) if $v(S) + v(T) \le v(S \cup T) + v(S \cap T)$ for each pair $S, T \in 2^N$. Let Γ_{bal}, Γ_{exp}, and Γ_{conv} denote the class of all balanced games, exact partition games, and convex games, respectively. We have $\Gamma_{conv} \subset \Gamma_{exp} \subset \Gamma_{bal} \subset \Gamma_{all}$. Throughout this note, (N, v) is the generic notation for a game and Γ is the generic notation for a class of games.

A *solution* σ on Γ assigns to each game $(N, v) \in \Gamma$ a nonempty set of payoff allocations $\sigma(N, v) \subseteq \mathbb{R}^N$ such that $x(N) \le v(N)$ for each $x \in \sigma(N, v)$. The Dutta and Ray (1989) solution assigns to each exact partition game $(N, v) \in \Gamma_{exp}$[1]

$$DR(N, v) = \left\{ x \in C(N, v) \;\middle|\; \forall k \in \{1, \ldots, |N|\} : x(R_k^x) = v(R_k^x) \right\}.$$

This means that for each $x \in DR(N, v)$,

$$x_i = \frac{v(R_k^x) - v(R_{k-1}^x)}{|R_k^x \setminus R_{k-1}^x|} \quad \text{for each } k \in \{1, \ldots, |N|\} \text{ and each } i \in R_k^x;$$

$$x_i = \frac{v(N \setminus P_{k-1}^x) - v(N \setminus P_k^x)}{|P_k^x \setminus P_{k-1}^x|} \quad \text{for each } k \in \{1, \ldots, |N|\} \text{ and each } i \in P_k^x.$$

We have $|DR(N, v)| = 1$ for each $(N, v) \in \Gamma_{exp}$. Throughout this note, σ is the generic notation for a solution.

3. Antiduality

This section performs an antiduality analysis on the class of exact partition games. The antidual of a game is the negated dual game, i.e. the game in which the worth of each coalition equals the worth of its complement minus the worth of the grand coalition in the original game. Two solutions are antidual if one solution assigns to each game the negated set of payoff allocations assigned by the other solution to the antidual game. A solution is self-antidual if it coincides with its antidual.

[1] Llerena and Mauri (2017) showed that this definition is equivalent to the original definition of Dutta and Ray (1989).

Definition 1 (*cf. Oishi and Nakayama, 2009*)**.** The *antidual* of a game (N, v) is the game (N, v^*) defined by

$$v^*(S) = v(N \setminus S) - v(N) \text{ for each } S \in 2^N.$$

Definition 2 (*cf. Oishi and Nakayama, 2009*)**.** Two solutions σ and σ^* on Γ are *antidual* if $(N, v^*) \in \Gamma$ and

$$\sigma(N, v) = -\sigma^*(N, v^*) \text{ for each } (N, v) \in \Gamma.$$

A solution σ on Γ is *self-antidual* if $(N, v^*) \in \Gamma$ and

$$\sigma(N, v) = -\sigma(N, v^*) \text{ for each } (N, v) \in \Gamma.$$

Note that the antidual of the antidual game equals the original game. Moreover, note that each solution on a domain closed under antiduality has at most one antidual. Oishi et al. (2016) showed that the class of all games, the class of balanced games, and the class of convex games are each closed under antiduality. Moreover, they showed that the Dutta and Ray (1989) solution is self-antidual on the class of convex games. We show that the class of exact partition games is also closed under antiduality and that the Dutta and Ray (1989) solution is also self-antidual on this class.

Theorem 1. *The Dutta and Ray (1989) solution is self-antidual on the class of exact partition games.*

Proof. Let $(N, v) \in \Gamma_{exp}$. Let $x \in C(N, v)$ be such that $x(R_k^x) = v(R_k^x)$ for each $k \in \{1, \ldots, |N|\}$. Then

$$-x(N) = -v(N) = v(N \setminus N) - v(N) = v^*(N)$$

and for each $S \in 2^N$,

$$-x(S) = x(N \setminus S) - x(N) \ge v(N \setminus S) - v(N) = v^*(S).$$

This means that $-x \in C(N, v^*)$. Moreover, for each $k \in \{1, \ldots, |N|\}$,

$$-x(R_k^{-x}) = -x(P_k^x) = x(N \setminus P_k^x) - x(N) = v(N \setminus P_k^x) - v(N) = v^*(P_k^x) = v^*(R_k^{-x}).$$

This means that $(N, v^*) \in \Gamma_{exp}$ and $DR(N, v) = -DR(N, v^*)$. Hence, the Dutta and Ray (1989) solution is self-antidual on the class of exact partition games. □

The following punctual properties are satisfied by the Dutta and Ray (1989) on the class of exact partition games.

Efficiency
for each $x \in \sigma(N, v)$, $x(N) = v(N)$.

Feasible richness
for each $x \in \sigma(N, v)$, $x(R_1^x) \le v(R_1^x)$.

Equal division stability
for each $x \in \sigma(N, v)$ and each $S \in 2^N \setminus \{\emptyset\}$, there is $i \in S$ such that $x_i \ge \frac{v(S)}{|S|}$.

Efficiency requires that a solution fully allocates the worth of the grand coalition. Feasible richness requires that the richest players are able to obtain their payoffs by themselves. Equal division stability requires that no coalition is better off by equally dividing the worth among its members.

Remark. A solution on a domain closed under antiduality has an antidual if and only if it satisfies efficiency. If two solutions are antidual, then they both satisfy efficiency. If a solution is self-antidual, then it satisfies efficiency.

Antiduality can also be defined for axioms and characterizations. Two axioms are antidual if for each two antidual solutions, one property is satisfied by one solution if and only if the other property is satisfied by the other solution. An axiom is self-antidual if for each two antidual solutions, the property is satisfied by one solution if and only if it is satisfied by the other solution.

Definition 3 (*cf.* *Oishi et al., 2016*)**.** Two properties for solutions are *antidual* if for each pair of antidual solutions σ and σ^*, one property is satisfied by σ if and only if the other property is satisfied by σ^*. A property for solutions is *self-antidual* if for each pair of antidual solutions σ and σ^*, the property is satisfied by σ if and only if it is satisfied by σ^*.

Clearly, self-antiduality is a self-antidual property. Oishi et al. (2016) claimed that efficiency is self-antidual, showed that feasible richness is antidual to limited poorness, and claimed that equal division stability is self-antidual as well. The following example shows that the latter claim is incorrect.

Example 1. Let $(N, v) \in \Gamma_{conv}$ with $N = \{1, 2, 3\}$ be the convex game given by

$$v(S) = \begin{cases} 2 & \text{if } S \in \{\{1, 2\}, \{1, 2, 3\}\}; \\ 0 & \text{otherwise.} \end{cases}$$

The antidual game $(N, v^*) \in \Gamma_{conv}$ is given by

$$v^*(S) = \begin{cases} 0 & \text{if } S \in \{\emptyset, \{3\}\}; \\ -2 & \text{otherwise.} \end{cases}$$

Let σ and σ^* be two antidual solutions on Γ_{conv} such that σ satisfies equal division stability and $(1, 0, 1) \in \sigma(N, v)$. Then $(-1, 0, -1) \in \sigma^*(N, v^*)$. Since $v^*(\{3\}) > -1$, this means that σ^* does not satisfy equal division stability. Hence, equal division stability is not self-antidual. △

Efficiency is indeed self-antidual, feasible richness is indeed antidual to limited poorness, and equal division stability is in fact dual to the average contribution property. The straightforward proof is omitted.

Limited poorness
for each $x \in \sigma(N, v)$, $x(P_1^x) \geq v(N) - v(N \setminus P_1^x)$.

Average contribution property
for each $x \in \sigma(N, v)$ and each $S \in 2^N \setminus \{\emptyset\}$, there is $i \in S$ such that $x_i \leq \frac{v(N)-v(N\setminus S)}{|S|}$.

Limited poorness requires that the poorest players get at least their joint marginal contribution to the grand coalition. The average contribution property requires that for each coalition, at least one member gets at most the average marginal contribution of this coalition to the grand coalition.

Lemma 1.

(i) Efficiency is self-antidual.
(ii) Feasible richness and limited poorness are antidual.[2]
(iii) Equal division stability and the average contribution property are antidual.

Given two antidual solutions on a domain closed under antiduality and an axiomatic characterization of one solution, the other solution is characterized by the corresponding antidual axioms. To see this, note that the antidual solution satisfies the antidual axioms by definition. It is the unique one, since otherwise the antidual of the third solution would satisfy the antiduals of these antidual axioms, i.e. the original axioms, which contradicts the characterization of the original solution. Moreover, independence of the axioms in the antidual characterization follows from independence of the axioms in the original characterization by the same logic.

We apply this logic to the axiomatic characterizations of the Dutta and Ray (1989) solution on the class of exact partition games provided by Dietzenbacher and Yanovskaya (2020). These characterizations are generally based on consistency properties. Suppose that some players leave with their allocated payoffs and the remaining players reevaluate their payoffs by applying the same solution to a reduced game. The solution is consistent if it assigns to this reduced game the same payoffs for the remaining players as in the original game. However, such reduced game can be defined in many ways and the context of the underlying allocation problem determines which one is most appropriate.

Davis and Maschler (1965) proposed a reduced game in which the worth of a coalition is defined as the maximal joint worth with any subgroup of leaving players subtracted by their payoffs. The corresponding axiom which requires consistent allocations when the richest players leave together, to which we refer as rich-restricted max-consistency, characterizes the Dutta and Ray (1989) solution on the class of exact partition games in combination with feasible richness and equal division stability.

Rich-restricted max-consistency
for each $(N, v) \in \Gamma$ and each $x \in \sigma(N, v)$ with $R_1^x \neq N$,

$(N \setminus R_1^x, v^x_{r\text{-}max}) \in \Gamma$ and $x_{N\setminus R_1^x} \in \sigma(N \setminus R_1^x, v^x_{r\text{-}max})$,

where

$$v^x_{r\text{-}max}(S) = \begin{cases} v(N) - x(R_1^x) & \text{if } S = N \setminus R_1^x; \\ \max_{Q \subseteq R_1^x}\{v(S \cup Q) - x(Q)\} & \text{if } \emptyset \subset S \subset N \setminus R_1^x; \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Theorem (*cf.* *Dietzenbacher and Yanovskaya, 2020*)**.** *The Dutta and Ray (1989) solution is the unique solution on Γ_{exp} satisfying feasible richness, equal division stability, and rich-restricted max-consistency.*

Rich-restricted max-consistency is antidual to poor-restricted max-consistency, which requires consistent allocations based on the same reduced game when the poorest players leave together. The corresponding proof is provided in the Appendix.

Poor-restricted max-consistency
for all $(N, v) \in \Gamma$ and each $x \in \sigma(N, v)$ with $P_1^x \neq N$,

$(N \setminus P_1^x, v^x_{p\text{-}max}) \in \Gamma$ and $x_{N\setminus P_1^x} \in \sigma(N \setminus P_1^x, v^x_{p\text{-}max})$,

where

$$v^x_{p\text{-}max}(S) = \begin{cases} v(N) - x(P_1^x) & \text{if } S = N \setminus P_1^x; \\ \max_{Q \subseteq P_1^x}\{v(S \cup Q) - x(Q)\} & \text{if } \emptyset \subset S \subset N \setminus P_1^x; \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Lemma 2. *Rich-restricted max-consistency and poor-restricted max-consistency are antidual.*

Using antiduality, a characterization of the Dutta and Ray (1989) solution on the class of exact partition games in terms of limited poorness, the average contribution property, and poor-restricted max-consistency follows directly from Theorem 1, Lemmas 1, and 2.

Theorem 2. *The Dutta and Ray (1989) solution is the unique solution on Γ_{exp} satisfying limited poorness, the average contribution property, and poor-restricted max-consistency.*

[2] This fact was proven by Oishi et al. (2016).

Since the Dutta and Ray (1989) solution is also self-antidual on the subclass of convex games, this characterization is also valid on the class of convex games, repairing and strengthening the corresponding axiomatization of Oishi et al. (2016) antidual to the result of Klijn et al. (2000).

Theorem 3. *The Dutta and Ray (1989) solution is the unique solution on Γ_{conv} satisfying limited poorness, the average contribution property, and poor-restricted max-consistency.*

Moulin (1985) proposed a reduced game in which the worth of a coalition is defined as the joint worth with all leaving players subtracted by their payoffs. The corresponding axiom which requires consistent allocations when the richest players leave together, to which we refer as rich-restricted complement-consistency, characterizes the Dutta and Ray (1989) solution on the class of exact partition games in combination with feasible richness and equal division stability.

Rich-restricted complement-consistency
for each $(N, v) \in \Gamma$ and each $x \in \sigma(N, v)$ with $R_1^x \neq N$,

$(N \setminus R_1^x, v_{comp}^x) \in \Gamma$ and $x_{N\setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{comp}^x)$,

where

$$v_{comp}^x(S) = \begin{cases} v(S \cup R_1^x) - x(R_1^x) & \text{if } \emptyset \subset S \subseteq N \setminus R_1^x; \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Theorem (*cf.* *Dietzenbacher and Yanovskaya, 2020*)**.** *The Dutta and Ray (1989) solution is the unique solution on Γ_{exp} satisfying feasible richness, equal division stability, and rich-restricted complement-consistency.*

Rich-restricted complement-consistency is antidual to poor-restricted projection-consistency, which requires consistent allocations when the poorest players leave together based on the reduced game where the worth of a coalition equals its worth in the original game. The corresponding proof is provided in the Appendix.

Poor-restricted projection-consistency
for each $(N, v) \in \Gamma$ and each $x \in \sigma(N, v)$ with $P_1^x \neq N$,

$(N \setminus P_1^x, v_{proj}^x) \in \Gamma$ and $x_{N\setminus P_1^x} \in \sigma(N \setminus P_1^x, v_{proj}^x)$,

where

$$v_{proj}^x(S) = \begin{cases} v(N) - x(P_1^x) & \text{if } S = N \setminus P_1^x; \\ v(S) & \text{if } S \subset N \setminus P_1^x. \end{cases}$$

Lemma 3. *Rich-restricted complement-consistency and poor-restricted projection-consistency are antidual.*

Funaki and Yamato (2001) used projection-consistency in a characterization of the core. Bhattacharya (2004) used projection-consistency in a characterization of the equal division core. Using antiduality, a characterization of the Dutta and Ray (1989) solution on the class of exact partition games in terms of limited poorness, the average contribution property, and poor-restricted projection-consistency follows directly from Theorem 1, Lemmas 1, and 3.

Theorem 4. *The Dutta and Ray (1989) solution is the unique solution on Γ_{exp} satisfying limited poorness, the average contribution property, and poor-restricted projection-consistency.*

Since the Dutta and Ray (1989) solution is also self-antidual on the subclass of convex games, this characterization is also valid on the class of convex games.

Theorem 5. *The Dutta and Ray (1989) solution is the unique solution on Γ_{conv} satisfying limited poorness, the average contribution property, and poor-restricted projection-consistency.*

Klijn et al. (2000) proposed a reduced game in which the worth of a coalition is defined as the joint marginal contribution to the leaving players. The corresponding axiom which requires consistent allocations when the richest players leave together, to which we refer as rich-restricted marginal-consistency, characterizes the Dutta and Ray (1989) solution on the class of exact partition games in combination with equal division stability.

Rich-restricted marginal-consistency
for each $(N, v) \in \Gamma$ and each $x \in \sigma(N, v)$ with $R_1^x \neq N$,

$(N \setminus R_1^x, v_{marg}^x) \in \Gamma$ and $x_{N\setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{marg}^x)$,

where

$v_{marg}^x(S) = v(S \cup R_1^x) - v(R_1^x)$ for all $S \subseteq N \setminus R_1^x$.

Theorem (*cf.* *Dietzenbacher and Yanovskaya, 2020*)**.** *The Dutta and Ray (1989) solution is the unique solution on Γ_{exp} satisfying equal division stability and rich-restricted marginal-consistency.*

Oishi et al. (2016) showed that rich-restricted marginal-consistency is antidual to poor-restricted subgame-consistency, which requires consistent allocations based on the subgame when the poorest players leave together.

Poor-restricted subgame-consistency
for each $(N, v) \in \Gamma$ and each $x \in \sigma(N, v)$ with $P_1^x \neq N$,

$(N \setminus P_1^x, v_{sub}^x) \in \Gamma$ and $x_{N\setminus P_1^x} \in \sigma(N \setminus P_1^x, v_{sub}^x)$,

where

$v_{sub}^x(S) = v(S)$ for all $S \subseteq N \setminus P_1^x$.

Lemma 4 (*cf.* *Oishi et al., 2016*)**.** *Rich-restricted marginal-consistency and poor-restricted subgame-consistency are antidual.*

Using antiduality, a characterization of the Dutta and Ray (1989) solution on the class of exact partition games in terms of the average contribution property and poor-restricted subgame-consistency follows directly from Theorem 1, Lemmas 1, and 4.

Theorem 6. *The Dutta and Ray (1989) solution is the unique solution on Γ_{exp} satisfying the average contribution property and poor-restricted subgame-consistency.*

Since the Dutta and Ray (1989) solution is also self-antidual on the subclass of convex games, this characterization is also valid on the class of convex games, repairing and strengthening the corresponding axiomatization of Oishi et al. (2016) antidual to the result of Klijn et al. (2000).

Theorem 7. *The Dutta and Ray (1989) solution is the unique solution on Γ_{conv} satisfying the average contribution property and poor-restricted subgame-consistency.*

4. Concluding remarks

Using antiduality, this note provides several new axiomatic characterizations of the Dutta and Ray (1989) solution on the class of exact partition games and repairs and strengthens several related characterizations on the subclass of convex games. For other games than exact partition games, existence of the Dutta and Ray (1989) solution is not guaranteed. However, several extensions to the class of balanced games and the class of all games are proposed.

On the class of balanced games, the solution which assigns the Lorenz undominated allocations in the core (cf. Hougaard et al., 2001) is self-antidual. The *Lmin* solution (cf. Arin and Iñarra, 2001) and *Lmax* solution (cf. Arin et al., 2003) are antidual. The least squares solution (cf. Arin et al., 2008) is self-antidual. Future research could use antiduality to further study these solutions.

On the class of all games, neither the equal split-off set (cf. Branzei et al., 2006) nor the procedural egalitarian solution (cf. Dietzenbacher et al., 2017) is self-antidual. However, these solutions are contained in some self-antidual solutions. Future research could further explore this.

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Appendix

Lemma 2. *Rich-restricted max-consistency and poor-restricted max-consistency are antidual.*

Proof. First, we show that $((v^*)^{-x}_{r\text{-}max})^* = v^x_{p\text{-}max}$ and equivalently $((v^*)^{-x}_{p\text{-}max})^* = v^x_{r\text{-}max}$ for each game (N, v) and each $x \in \mathbb{R}^N$ with $x(N) \leq v(N)$ and $P^x_1 \neq N$. If $S = N \setminus P^x_1$, then

$$\begin{aligned}((v^*)^{-x}_{r\text{-}max})^*(S) &= (v^*)^{-x}_{r\text{-}max}((N \setminus P^x_1) \setminus S) - (v^*)^{-x}_{r\text{-}max}(N \setminus P^x_1)\\ &= -(v^*)^{-x}_{r\text{-}max}(N \setminus R^{-x}_1)\\ &= -v^*(N) - x(R^{-x}_1)\\ &= -v(N \setminus N) + v(N) - x(R^{-x}_1)\\ &= v(N) - x(P^x_1)\\ &= v^x_{p\text{-}max}(S).\end{aligned}$$

If $\emptyset \subset S \subset N \setminus P^x_1$, then

$$\begin{aligned}((v^*)^{-x}_{r\text{-}max})^*(S) &= (v^*)^{-x}_{r\text{-}max}((N \setminus P^x_1) \setminus S) - (v^*)^{-x}_{r\text{-}max}(N \setminus P^x_1)\\ &= (v^*)^{-x}_{r\text{-}max}((N \setminus R^{-x}_1) \setminus S) - (v^*)^{-x}_{r\text{-}max}(N \setminus R^{-x}_1)\\ &= \max_{Q \subseteq R^{-x}_1} \{v^*(((N \setminus R^{-x}_1) \setminus S) \cup Q) + x(Q)\}\\ &\quad - v^*(N) - x(R^{-x}_1)\\ &= \max_{Q \subseteq R^{-x}_1} \{v(S \cup (R^{-x}_1 \setminus Q)) - v(N) + x(Q)\}\\ &\quad - v(N \setminus N) + v(N) - x(R^{-x}_1)\\ &= \max_{Q \subseteq P^x_1} \{v(S \cup (P^x_1 \setminus Q)) - x(P^x_1 \setminus Q)\}\\ &= \max_{Q \subseteq P^x_1} \{v(S \cup Q) - x(Q)\}\\ &= v^x_{p\text{-}max}(S).\end{aligned}$$

If $S = \emptyset$, then

$$((v^*)^{-x}_{r\text{-}max})^*(S) = (v^*)^{-x}_{r\text{-}max}((N \setminus P^x_1) \setminus S) - (v^*)^{-x}_{r\text{-}max}(N \setminus P^x_1) = 0 = v^x_{p\text{-}max}(S).$$

Now, let σ and σ^* be two antidual solutions on Γ. Let $(N, v) \in \Gamma$ and let $x \in \sigma(N, v)$ be such that $P^x_1 \neq N$. Assume that σ^* satisfies rich-restricted max-consistency. Then

$$(N \setminus R^{-x}_1, (v^*)^{-x}_{r\text{-}max}) \in \Gamma \text{ and } -x_{N \setminus R^{-x}_1} \in \sigma^*(N \setminus R^{-x}_1, (v^*)^{-x}_{r\text{-}max}).$$

This implies that

$$(N \setminus P^x_1, ((v^*)^{-x}_{r\text{-}max})^*) \in \Gamma \text{ and } x_{N \setminus P^x_1} \in \sigma(N \setminus P^x_1, ((v^*)^{-x}_{r\text{-}max})^*).$$

This means that

$$(N \setminus P^x_1, v^x_{p\text{-}max}) \in \Gamma \text{ and } x_{N \setminus P^x_1} \in \sigma(N \setminus P^x_1, v^x_{p\text{-}max}).$$

Hence, σ satisfies poor-restricted max-consistency. Conversely, assume that σ^* satisfies poor-restricted max-consistency. Then

$$(N \setminus P^{-x}_1, (v^*)^{-x}_{p\text{-}max}) \in \Gamma \text{ and } -x_{N \setminus P^{-x}_1} \in \sigma^*(N \setminus P^{-x}_1, (v^*)^{-x}_{p\text{-}max}).$$

This implies that

$$(N \setminus R^x_1, ((v^*)^{-x}_{p\text{-}max})^*) \in \Gamma \text{ and } x_{N \setminus R^x_1} \in \sigma(N \setminus R^x_1, ((v^*)^{-x}_{p\text{-}max})^*).$$

This means that

$$(N \setminus R^x_1, v^x_{r\text{-}max}) \in \Gamma \text{ and } x_{N \setminus R^x_1} \in \sigma(N \setminus R^x_1, v^x_{r\text{-}max}).$$

Hence, σ satisfies rich-restricted max-consistency. □

Lemma 3. *Rich-restricted complement-consistency and poor-restricted projection-consistency are antidual.*

Proof. First, we show that $((v^*)^{-x}_{comp})^* = v^x_{proj}$ and equivalently $((v^*)^{-x}_{proj})^* = v^x_{comp}$ for each game (N, v) and each $x \in \mathbb{R}^N$ with $x(N) \leq v(N)$ and $P^x_1 \neq N$. If $S = N \setminus P^x_1$, then

$$\begin{aligned}((v^*)^{-x}_{comp})^*(S) &= (v^*)^{-x}_{comp}((N \setminus P^x_1) \setminus S) - (v^*)^{-x}_{comp}(N \setminus P^x_1)\\ &= -(v^*)^{-x}_{comp}(N \setminus R^{-x}_1)\\ &= -v^*(N) - x(R^{-x}_1)\\ &= -v(N \setminus N) + v(N) - x(R^{-x}_1)\\ &= v(N) - x(P^x_1)\\ &= v^x_{proj}(S).\end{aligned}$$

If $\emptyset \subset S \subset N \setminus P^x_1$, then

$$\begin{aligned}((v^*)^{-x}_{comp})^*(S) &= (v^*)^{-x}_{comp}((N \setminus P^x_1) \setminus S) - (v^*)^{-x}_{comp}(N \setminus P^x_1)\\ &= (v^*)^{-x}_{comp}((N \setminus R^{-x}_1) \setminus S) - (v^*)^{-x}_{comp}(N \setminus R^{-x}_1)\\ &= v^*(N \setminus S) + x(R^{-x}_1) - v^*(N) - x(R^{-x}_1)\\ &= v(S) - v(N) - v(N \setminus N) + v(N)\\ &= v(S)\\ &= v^x_{proj}(S).\end{aligned}$$

If $S = \emptyset$, then

$$((v^*)^{-x}_{comp})^*(S) = (v^*)^{-x}_{comp}((N \setminus P^x_1) \setminus S) - (v^*)^{-x}_{comp}(N \setminus P^x_1) = 0 = v(S) = v^x_{proj}(S).$$

Now, let σ and σ^* be two antidual solutions on Γ. Let $(N, v) \in \Gamma$ and let $x \in \sigma(N, v)$ be such that $P^x_1 \neq N$. Assume that σ^* satisfies rich-restricted complement-consistency. Then

$$(N \setminus R^{-x}_1, (v^*)^{-x}_{comp}) \in \Gamma \text{ and } -x_{N \setminus R^{-x}_1} \in \sigma^*(N \setminus R^{-x}_1, (v^*)^{-x}_{comp}).$$

This implies that

$$(N \setminus P^x_1, ((v^*)^{-x}_{comp})^*) \in \Gamma \text{ and } x_{N \setminus P^x_1} \in \sigma(N \setminus P^x_1, ((v^*)^{-x}_{comp})^*).$$

This means that

$$(N \setminus P^x_1, v^x_{proj}) \in \Gamma \text{ and } x_{N \setminus P^x_1} \in \sigma(N \setminus P^x_1, v^x_{proj}).$$

Hence, σ satisfies poor-restricted projection-consistency. Conversely, assume that σ^* satisfies poor-restricted projection-consistency. Then

$$(N \setminus P^{-x}_1, (v^*)^{-x}_{proj}) \in \Gamma \text{ and } -x_{N \setminus P^{-x}_1} \in \sigma^*(N \setminus P^{-x}_1, (v^*)^{-x}_{proj}).$$

This implies that

$$(N \setminus R^x_1, ((v^*)^{-x}_{proj})^*) \in \Gamma \text{ and } x_{N \setminus R^x_1} \in \sigma(N \setminus R^x_1, ((v^*)^{-x}_{proj})^*).$$

This means that

$$(N \setminus R^x_1, v^x_{comp}) \in \Gamma \text{ and } x_{N \setminus R^x_1} \in \sigma(N \setminus R^x_1, v^x_{comp}).$$

Hence, σ satisfies rich-restricted complement-consistency. □

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